1 OVERVIEW OF LINEAR PROGRAMMING .............................................. 4
  1.1 Definitions .................................................................................................. 5
  1.2 The primal and dual simplex methods .......................................................... 8
  1.3 Duality in linear programming .................................................................... 20

2 INTRODUCTION TO GAME THEORY .................................................... 40
  2.1 Matrix games – solution in pure strategies .................................................. 41
  2.2 Matrix games – solution in mixed strategies ............................................... 62

3 DEFINITIONS AND CLASSIFICATION OF GAMES ............................ 106

4 NON-ANTAGONISTIC GAMES OF TWO PLAYERS ........................... 129
  4.1 Non-cooperative games .............................................................................. 129
  4.2 Cooperative games of two players .............................................................. 160

5 GAMES OF $P$ PLAYERS ($P > 2$) ..................................................... 180
  5.1 Non-cooperative games of $p$ players ...................................................... 181
  5.2 Cooperative games of $p$ players ............................................................. 184
GT: Theory of mathematical models of conflict and cooperation between intelligent rational decision-makers.

Origins of Game Theory: 1944
(J. von Neumann and O. Morgenstern)

Nobel Prize (1994)
"... for their pioneering analysis of equilibria in the theory of non-cooperative games..."

John Nash (A Beautiful Mind)
Richard Selten
John Harsanyi
1 Overview of Linear Programming

1.1 Definitions

**General LP:**

\[
f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \min \text{ or } \max
\]

subject to (s.t.)

\[
a_{i_1}x_1 + a_{i_2}x_2 + \ldots + a_{in}x_n \{ \leq, \geq, = \} b_i \quad (i = 1, \ldots, m)
\]

\[x_1, x_2, \ldots, x_k \geq 0\]

**Example 1.1**

\[3x_1 - 2x_2 \rightarrow \max\]

s.t.

\[2x_1 + 5x_2 \geq 17\]

\[x_1 - x_2 \leq 20\]

\[x_2 \geq 0\]

We may assume w.l.o.g.:

\[c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \min\]

s.t.

\[a_{i_1}x_1 + a_{i_2}x_2 + \ldots + a_{in}x_n = b_i \quad (i = 1, \ldots, m)\]

\[x_1, x_2, \ldots, x_n \geq 0\]
In the vector-matrix notation:

\[
\begin{align*}
  f(x) &= c^T x \rightarrow \text{min} \\
  Ax &= b \\
  x &\geq 0
\end{align*}
\]

(SLP)

\[f \ldots \text{objective function}\]

SLP \ldots LP in standard form

Here

\[A = (a_{ij}) \in \mathbb{R}^{m \times n}\]

\[c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n\]

\[b = (b_1, \ldots, b_m)^T \in \mathbb{R}^m\]

\[x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n\]

\[M = \{x \in \mathbb{R}^n; Ax = b, x \geq 0\} \ldots\]

\[\ldots\ldots\text{set of feasible solutions}\]

\[M^{\text{opt}} = \{x \in M; c^T x \leq c^T z \text{ for all } z \in M\} \ldots\]

\[\ldots\ldots\text{set of optimal solutions}\]

LP in canonical form:

\[
\begin{align*}
  f(x) &= c^T x \rightarrow \text{min} \\
  Ax &\geq b \\
  x &\geq 0
\end{align*}
\]

(CLP)
1.2 The primal and dual simplex methods

*Example* (see PS1):

\[ 4x_1 + 5x_2 + x_3 \rightarrow \text{max} \]

\[ 3x_1 + 2x_2 \leq 10 \]

\[ x_1 + 4x_2 \leq 11 \]

\[ 3x_1 + 3x_2 + x_3 \leq 13 \]

\[ x_1, x_2, x_3 \geq 0 \]

---

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<th>-5</th>
<th>-1</th>
<th>0</th>
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<td>1</td>
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</table>

(continues – see solutions to PS 1)
Simplex Tableau

\[
\begin{array}{cccccc}
-z_0 & \ldots & c_{\ell} - z_{\ell} & \ldots & c_j - z_j & \ldots \\
\hline
x_{10} & & x_{1\ell} & & x_{1j} & \\
\ldots & & \ldots & & \ldots & \\
\hline
x_{k0} & & x_{k\ell} & \times & x_{kj} & \\
\hline
x_{m0} & & x_{m\ell} & & x_{mj} & \\
\hline
z_j = c^T B^{-1} A_j \quad (j = 1, \ldots, n) & & & & & \\
z_0 = c^T B^{-1} b
\end{array}
\]
<table>
<thead>
<tr>
<th></th>
<th>not $\geq 0$</th>
<th>PSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\geq 0$</th>
<th>DSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>not $\geq 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\geq 0$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PSM (apply if 0\textsuperscript{th} col ≥ 0)</td>
<td>DSM (apply if 0\textsuperscript{th} row ≥ 0)</td>
<td></td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>-----------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>(∀j) (c_j - z_j \geq 0) ? (yes → stop, opt)</td>
<td>(∀i) (x_{i0} \geq 0) ? (yes → stop, optimality)</td>
<td></td>
</tr>
<tr>
<td>Take any (ℓ: c_ℓ - z_ℓ &lt; 0)</td>
<td>Take any (k: x_{k0} &lt; 0)</td>
<td></td>
</tr>
<tr>
<td>(∀i) (x_{iℓ} \leq 0) ? (yes → stop, primal unbounded)</td>
<td>(∀j) (x_{kj} \geq 0) ? (yes → stop, dual unbounded → primal infeasible)</td>
<td></td>
</tr>
<tr>
<td>else pivot ((k, ℓ)) where (k) satisfies</td>
<td>else pivot ((k, ℓ)) where (ℓ) satisfies</td>
<td></td>
</tr>
</tbody>
</table>
| \[
\frac{x_{k0}}{x_{kℓ}} = \min \left\{ \frac{x_{i0}}{x_{iℓ}} ; i = 1, ..., m, x_{iℓ} > 0 \right\}
\] | \[
\frac{x_{0ℓ}}{x_{kℓ}} = \max \left\{ \frac{x_{0j}}{x_{kj}} ; j = 1, ..., n, x_{kj} < 0 \right\}
\] |
**Example 1.2 (Dual SM)**

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<th>0</th>
<th>0</th>
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<td>0</td>
<td></td>
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<tr>
<td>-5</td>
<td>0</td>
<td>-11/3</td>
<td>-1/3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>-65/11</td>
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<td>0</td>
<td>3/11</td>
<td>2/11</td>
<td></td>
</tr>
<tr>
<td>50/11</td>
<td>1</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td></td>
</tr>
<tr>
<td>15/11</td>
<td>0</td>
<td>1</td>
<td>1/11</td>
<td>-3/11</td>
<td></td>
</tr>
</tbody>
</table>

\[ x_{10} = -15 < 0, \]
\[ \max\left\{ \frac{1}{-3}, \frac{1}{-1}\right\} = -\frac{1}{3} = x_{01} / x_{11}, \]

**pivot:** \( x_{11} \)

\[ x_{20} = -5 < 0, \]
\[ \max\left\{ \frac{2/3}{-11/3}, \frac{1/3}{-1/3}\right\} = -\frac{2}{11} = x_{02} / x_{22}, \]

**pivot:** \( x_{22} \)

\[ x_{10} = 50/11 > 0 \]
\[ x_{20} = 15/11 > 0 \]

**optimality**
**Fundamental Theorem of LP:**

For every LP one of the following holds:

(i) The LP is infeasible (i.e. $M = \emptyset$),

(ii) The LP has an optimal solution ($M^{opt} \neq \emptyset$),

(iii) $\min_{x \in M} f(x) = -\infty$. 
1.3 Duality in linear programming

*Example 1.3*

\[
\begin{array}{|c|}
\hline
f(x) = 3x_1 - 4x_2 \to \min \\
\text{s.t.} \\
2x_1 - 7x_2 \geq 5 \\
3x_1 + x_2 \geq 7 \\
x_1, x_2 \geq 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|}
\hline
\varphi(\pi) = 5\pi_1 + 7\pi_2 \to \max \\
\text{s.t.} \\
2\pi_1 + 3\pi_2 \leq 3 \\
-7\pi_1 + \pi_2 \leq -4 \\
\pi_1, \pi_2 \geq 0 \\
\hline
\end{array}
\]

“Primal” \hspace{2cm} “Dual”

\[
f(x) = 3x_1 - 4x_2 \geq (2\pi_1 + 3\pi_2)x_1 + (-7\pi_1 + \pi_2)x_2 = \\
= (2x_1 - 7x_2)\pi_1 + (3x_1 + x_2)\pi_2 \geq \\
\geq 5\pi_1 + 7\pi_2 = \varphi(\pi)
\]

\[
\begin{array}{c}
5\pi_1 + 7\pi_2 \to \max \\
\hline
\end{array}
\]

\[
\begin{array}{c}
3x_1 - 4x_2 \to \min \\
\hline
\end{array}
\]

(\textit{real line})
Definition of the dual problem:

<table>
<thead>
<tr>
<th>primal</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = c^T x \to \min )</td>
<td>( \varphi(\pi) = \pi^T b \to \max )</td>
</tr>
<tr>
<td>( a_i^T x = b_i )</td>
<td>( \pi_i \geq 0 )</td>
</tr>
<tr>
<td>( a_i^T x \geq b_i )</td>
<td>( \pi_i \geq 0 )</td>
</tr>
<tr>
<td>( x \in J )</td>
<td>( \pi^T A_j \leq c_j )</td>
</tr>
<tr>
<td>( x \in I )</td>
<td>( \pi^T A_j = c_j )</td>
</tr>
<tr>
<td>( x \in I )</td>
<td>( \pi \geq 0 )</td>
</tr>
<tr>
<td>( x \in J )</td>
<td>( \pi \geq 0 )</td>
</tr>
</tbody>
</table>

\[
A = \begin{pmatrix} a_i^T, i \in I \end{pmatrix} = \begin{pmatrix} A_j, j \in J \end{pmatrix}
\]

**Example 1.1 (modified):**

<table>
<thead>
<tr>
<th>primal</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 3x_1 - 4x_2 \to \min )</td>
<td>( \varphi(\pi) = 5\pi_1 + 7\pi_2 \to \max )</td>
</tr>
<tr>
<td>s.t.</td>
<td>s.t.</td>
</tr>
<tr>
<td>( 2x_1 - 7x_2 \geq 5 )</td>
<td>( 2\pi_1 + 3\pi_2 = 3 )</td>
</tr>
<tr>
<td>( 3x_1 + x_2 = 7 )</td>
<td>( -7\pi_1 + \pi_2 \leq -4 )</td>
</tr>
<tr>
<td>( x \in J )</td>
<td>( \pi \geq 0 )</td>
</tr>
<tr>
<td>( x \in I )</td>
<td>( \pi \geq 0 )</td>
</tr>
<tr>
<td>( x_2 \geq 0 )</td>
<td></td>
</tr>
</tbody>
</table>

23
**Theorem on Symmetry:**

Dual to the dual is the primal.

\(M_P, \ M_P^{\text{opt}}\) ... set of feasible (optimal) solutions to the primal LP

\(M_D, \ M_D^{\text{opt}}\) ... set of feasible (optimal) solutions to the dual LP

**Weak Duality Theorem:**

\[ (\forall x \in M_P) \ (\forall \pi \in M_D) \ c^T x \geq \pi^T b. \]

**Corollary 1:** If \(x \in M_P, \ \pi \in M_D, \ c^T x = \pi^T b\) then

\[ x \in M_P^{\text{opt}}, \ \pi \in M_D^{\text{opt}}. \]

**Corollary 2:**

(a) If \(\min_{x \in M_P} c^T x = -\infty\) then \(M_D = \emptyset\).

(b) If \(\max_{\pi \in M_D} \pi^T b = +\infty\) then \(M_P = \emptyset\).
**Strong Duality Theorem:**

\( M_P^{\text{opt}} \neq \emptyset \) if and only if \( M_D^{\text{opt}} \neq \emptyset \). If an optimal solution exists then \( \min_{x \in M_P} c^T x = \max_{\pi \in M_D} \pi^T b \).

**Corollary 1:** For every primal-dual pair only one of the possibilities in the table below may occur:

<table>
<thead>
<tr>
<th>( M_P^{\text{opt}} \neq \emptyset )</th>
<th>( \min_{x \in M_P} c^T x = -\infty )</th>
<th>( M_P = \emptyset )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_D^{\text{opt}} \neq \emptyset )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \max_{\pi \in M_D} \pi^T b = \infty )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M_D = \emptyset )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Corollary 2:** If both the primal and dual are feasible then both have an optimal solution.
Important primal-dual pairs:

<table>
<thead>
<tr>
<th>“Symmetric pair”</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^T x \to \min$</td>
<td>$\pi^T b \to \max$</td>
</tr>
<tr>
<td>$Ax \geq b$</td>
<td>$\pi^T A \leq c^T$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$\pi \geq 0$</td>
</tr>
<tr>
<td>(CLP)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(SLP)$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^T x \to \min$</td>
<td>$\pi^T b \to \max$</td>
</tr>
<tr>
<td>$Ax = b$</td>
<td>$\pi^T A \leq c^T$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td></td>
</tr>
</tbody>
</table>
| }
Can we find a dual optimal solution in the optimal simplex tableau for the primal problem?

\[
f(x) = c^T x \rightarrow \min \quad \begin{cases} 
Ax \geq b \\
x \geq 0
\end{cases} \quad \text{dual} \quad \begin{cases} 
\varphi(\pi) = \pi^T b \rightarrow \max \\
\pi^T A \leq c^T \\
\pi \geq 0
\end{cases}
\]

First transform: canonical $\rightarrow$ standard

\[
c^T x \rightarrow \min \quad \begin{cases} 
Ax \geq b \\
x \geq 0
\end{cases} = \begin{cases} 
c^T x + 0^T w \rightarrow \min \\
-Ax + I w = -b \\
x \geq 0, \ w \geq 0
\end{cases}
\]

\[
A \quad \text{is} \quad m \times n \quad : \quad (-A, I) \quad \text{is} \quad m \times (n + m)
\]

Columns of \( A' = (-A, I) \): \( A'_1, \ldots, A'_{n+m} \)

Similarly \[
\begin{pmatrix} c \\ 0 \end{pmatrix} = (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n+m})^T
\]
Hence the first and last simplex tableaux have the following form:

<table>
<thead>
<tr>
<th></th>
<th>$c^T$</th>
<th>$0^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-b$</td>
<td>$-A$</td>
<td>$I$</td>
</tr>
<tr>
<td></td>
<td>$c_j - z_j \geq 0$</td>
<td>$0 - z_j \geq 0$</td>
</tr>
</tbody>
</table>

(a) For $j = 1, \ldots, n$: $A'_j = -A_j$

(b) For $k = 1, \ldots, m$: $A'_{n+k} = e_k$

Now $z_j = c^T B^{-1} A'_j \quad (j = 1, \ldots, n + m) \quad (*)$

(a) $z_j = c^T B^{-1} (-A_j) \quad (j = 1, \ldots, n)$

Denote $-c^T B^{-1}$ by $\pi^T$

$c_j \geq z_j = c^T B^{-1} A'_j = -c^T B^{-1} A_j \quad (j = 1, \ldots, n)$
\[ \pi^T A_j \leq c_j \ (j = 1, \ldots, n) \quad \therefore \pi \text{ is dual feas} \ (\geq 0 \text{ will be shown shortly}) \]

and \( f^{\text{opt}} = z_0 = c^T B^{-1}(-b) = \pi^T b = \phi(\pi) \)

\[ \therefore \pi \text{ is dual optimal (by Cor.1 of WDT)} \]

How to find \( \pi \)? Is \( \pi \geq 0 \)?

(b) Since \( A'_{n+k} = e_k \ (k = 1, \ldots, m) \), we have by (*) 

\[ z_{n+k} = c^T B^{-1} e_k = -\pi^T e_k = -\pi_k \ (k = 1, \ldots, m) \]

\[ \therefore \pi_k = -z_{n+k} = c_{n+k} - z_{n+k} \geq 0 \ (k = 1, \ldots, m) \]

**Conclusion:**

If

- an LP in canonical form is transformed to standard form using slack variables, and
- the simplex method is applied

then

an optimal solution to the dual of the original problem can be found using the zeroth row of the optimal simplex tableau in the columns corresponding to the slack variables.
Hence the first and last simplex tableaux have the following form:

<table>
<thead>
<tr>
<th></th>
<th>$c^T$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-b$</td>
<td>$-A$</td>
<td>$I$</td>
</tr>
</tbody>
</table>

$$-z_0 \quad c_j - z_j \geq 0 \quad -z_{n+k} = \pi_k \geq 0$$
Example 1.4 (Ex.1.2 contd)

\[ x_1 + x_2 \rightarrow \min \quad \text{Dual: } 15\pi_1 + 10\pi_2 \rightarrow \max \]

\[
\begin{align*}
3x_1 + x_2 & \geq 15 \\
x_1 + 4x_2 & \geq 10 \\
x_1, x_2 & \geq 0
\end{align*}
\]

\[
\begin{align*}
3\pi_1 + \pi_2 & \leq 1 \\
\pi_1 + 4\pi_2 & \leq 1 \\
\pi_1, \pi_2 & \geq 0
\end{align*}
\]

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<th>0</th>
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<td></td>
</tr>
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<td>1/3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1/3</td>
<td>-1/3</td>
<td>0</td>
<td></td>
</tr>
<tr>
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<td>-11/3</td>
<td>-1/3</td>
<td>1</td>
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<td>3/11</td>
<td>2/11</td>
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</tr>
<tr>
<td>50/11</td>
<td>1</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td></td>
</tr>
<tr>
<td>15/11</td>
<td>0</td>
<td>1</td>
<td>1/11</td>
<td>-3/11</td>
<td></td>
</tr>
</tbody>
</table>

Optimal solution (primal)

\[ x^{\text{opt}} = (50/11, 15/11)^T \]

\[ \pi^{\text{opt}} = (3/11, 2/11)^T \]

\[ f^{\text{min}} = g^{\text{max}} = 65/11 \]
2 Introduction to Game Theory

2.1 Matrix games – solution in pure strategies

*Example 2.1.1* [Winston]:

Two TV networks - audience of 100 million viewers - Network 1 attracts [millions]:

<table>
<thead>
<tr>
<th>Network 2 →</th>
<th>Western</th>
<th>Soap</th>
<th>Comedy</th>
<th>( \text{Row min} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network 1 ↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Western</td>
<td>35</td>
<td>15</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>Soap</td>
<td>45</td>
<td>58</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Comedy</td>
<td>38</td>
<td>14</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>( \text{Col max} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Network 2 attracts all remaining viewers.

Network 1 is choosing

Network 2 is choosing
Slightly modify the previous data

<table>
<thead>
<tr>
<th>Network 2 →</th>
<th>Western</th>
<th>Soap</th>
<th>Comedy</th>
<th>Row min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network 1 ↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Western</td>
<td>35</td>
<td>15</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>Soap</td>
<td>45</td>
<td>58</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Comedy</td>
<td>52</td>
<td>14</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>Col max</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is “solution”?

One possibility:

*A choice of strategies of the two players such that no player can do better by a unilateral change of strategy.*
Conjecture:
There is a “solution” if and only if $v_1 = v_2$.
(We aim to prove this)
Game (in this example):

- Two players (Network 1 and Network 2)
- Three strategies each (Western, Soap, Comedy)
- For any pair of strategies the sum of rewards of the players is the same (100 million)

This is an instance of a two-person constant-sum game (constant here = 100).

We may write the expected numbers of viewers for both networks:

<table>
<thead>
<tr>
<th>Network 1</th>
<th>Western</th>
<th>Soap</th>
<th>Comedy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Western</td>
<td>(35,65)</td>
<td>(15,85)</td>
<td>(60,40)</td>
</tr>
<tr>
<td>Soap</td>
<td>(45,55)</td>
<td>(58,42)</td>
<td>(50,50)</td>
</tr>
<tr>
<td>Comedy</td>
<td>(38,62)</td>
<td>(14,86)</td>
<td>(70,30)</td>
</tr>
</tbody>
</table>
Zero-sum game:

<table>
<thead>
<tr>
<th>Network 2 →</th>
<th>Western</th>
<th>Soap</th>
<th>Comedy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network 1 ↓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Western</td>
<td>(–15,15)</td>
<td>(–35,35)</td>
<td>(10,–10)</td>
</tr>
<tr>
<td>Soap</td>
<td>(–5,5)</td>
<td>(8,–8)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>Comedy</td>
<td>(–12,12)</td>
<td>(–36,36)</td>
<td>(20,–20)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Network 2 →</th>
<th>Western</th>
<th>Soap</th>
<th>Comedy</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Western</td>
<td>–15</td>
<td>–35</td>
<td>10</td>
</tr>
<tr>
<td>Soap</td>
<td>–5</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Comedy</td>
<td>–12</td>
<td>–36</td>
<td>20</td>
</tr>
</tbody>
</table>
Definition of a matrix game:

Given is \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \) ... the payoff matrix

Player I chooses row index \( i \).

Player II chooses column index \( j \).

The reward for Player I is \( a_{ij} \).

The reward for Player II is \( -a_{ij} \).

\((\text{reward} \equiv \text{payoff})\)

Players

- know the payoff matrix,
- are intelligent, i.e. they would be able to make the best decision for them if they knew how their partner plays.

\{1, 2, \ldots, m\} ... the set of (pure) strategies of I

\{1, 2, \ldots, n\} ... the set of (pure) strategies of II
Example 2.1.2

Consider the matrix game

\[
A = (a_{ij}) = \begin{pmatrix}
2 & 1 & 3 \\
4 & 7 & 5 \\
3 & 5 & 9
\end{pmatrix}
\]

Position (2, 1) is a “saddle point”.

Gain-floor: \( v_1 = \max_i \min_j a_{ij} \)
Loss-ceiling: \( v_2 = \min_{j} \max_{i} a_{ij} \)

But \( v_1 = v_2 \) is not always the case: \( \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \)

Recall: \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \)

**Proposition 2.1.1** \( v_1 \leq v_2 \) holds for every matrix game.

**Proof**

If \( v_1 = v_2 \) then we say that the matrix game \( A \) has a solution in pure strategies. The value \( v = v_1 = v_2 \) is called the value of the game (in pure strategies).
Position \((r, s)\) is called a *saddle point* of the matrix \(A = (a_{ij})\) if

\[
a_{is} \leq a_{rs} \leq a_{rj}
\]

hold for all \(i\) and \(j\). Equivalently:

\[
\max_i a_{is} = a_{rs} = \min_j a_{rj}
\]

**Proposition 2.1.2**

If \((r, s)\) is a saddle point then \(v_1 = a_{rs} = v_2\).

**Corollary:** The payoffs at all saddle points are equal.
Theorem 2.1.3

If $v_1 = v_2 = v$ then a saddle point exists (and $a_{rs} = v$ for every saddle point $(r, s)$).

Corollary of Propositions 2.1.2 and 2.1.3:
The matrix game $A$ has a solution in pure strategies if and only if $A$ has a saddle point.

If $(r, s)$ is a saddle point of the matrix $A$ then
$r$ is called an optimal pure strategy of Player I,
s is called an optimal pure strategy of Player II,
$(r, s)$ is called a solution of the matrix game $A$ in pure strategies.
It follows that (then) $v = a_{rs}$.

A matrix may have several saddle points.
Proposition 2.1.4

If both \((r, s)\) and \((k, \ell)\) are saddle points then also \((k, s)\) and \((r, \ell)\) are saddle points.

Remark:

\[ a_{ij} = v_1 = v_2 \text{ and } a_{ij} \text{ is a row minimum} \]

\(\not\exists (i, j) \text{ is a saddle point. For instance} \)

\[
\begin{array}{ccc}
 3 & 3 & 4 \\
 4 & 3 & 2 \\
 4 & 3 & 4 \\
\end{array}
\]
2.2 Matrix games – solution in mixed strategies
Not every matrix has a saddle point, say

\[
A = \begin{pmatrix}
4 & 2 \\
1 & 3
\end{pmatrix}
\]

Let \( A = (a_{ij}) \) be an \( m \times n \) matrix.

A mixed strategy of Player I [Player II] in the game \( A \) is a probability distribution on the set of pure strategies of this player (i.e. on the set \( \{1,2, \ldots, m\} [\{1,2, \ldots, n\}] \)). Equivalently, it is any vector \( x = (x_1, \ldots, x_m)^T \geq 0 \) \( [y = (y_1, \ldots, y_n)^T \geq 0] \) such that

\[
\sum_{i=1}^{m} x_i = 1 \quad \left[ \sum_{j=1}^{n} y_j = 1 \right].
\]
\[ x_i \ldots \text{ probability that Player I chooses strategy } i \]
\[ y_j \ldots \text{ probability that Player II chooses strategy } j \]
\[ x_i y_j = \text{ probability that Player I chooses strategy } i \]
\[ \text{and that Player II chooses strategy } j \]

The pay-off of Player I therefore is

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = x^T A y \quad (\text{notation: } A(x, y)). \]

For \( k \geq 1 \), integer, we denote

\[ S_k = \{ (z_1, \ldots, z_k)^T; z_1, \ldots, z_k \geq 0, \sum z_i = 1 \}. \]

Hence \( A(x, y) \) is a function on \( S_m \times S_n \).

\( z \in S_k \ldots \text{ stochastic vector} \)

\( S_m \ldots \text{ the set of mixed strategies of player I} \)

\( S_n \ldots \text{ the set of mixed strategies of player II} \)

If \( x = e^r \), \( y = e^s \) for some \( r, s \) then \( A(e^r, e^s) = a_{rs} \)

\( \therefore \text{ Mixed strategies are generalisations of pure strategies.} \)
How to play? What is a “solution”?

*Gain-floor and loss-ceiling:*

\[
\text{max min } x^T A y \quad \text{... will be denoted } v_1 \text{ (again)}
\]

\[
\text{min max } x^T A y \quad \text{... will be denoted } v_2 \text{ (again)}
\]

**Proposition 2.2.1** \( v_1 \leq v_2. \)

**Proof:** Problem sheets.

Main theoretical result (mixed strategies only!):  

The Minimax Theorem: \( v_1 = v_2 \) always holds

The proof will show that for every real \( m \times n \) matrix \( A \) a vector \( (x^*, y^*) \in S_m \times S_n \) satisfying

\[
(*) \quad \min_{y \in S_n} x^* y^T A y = v_1 = v_2 = \max_{x \in S_m} x^T A y^*
\]

exists.

\( (x^*, y^*) \quad \text{... solution of the matrix game } A \text{ in mixed strategies (SMS)} \)

\( x^* \quad \text{... optimal mixed strategies of player I} \)

\( y^* \quad \text{... optimal mixed strategies of player II} \)
\[ v_1 = v_2 = v = v(A) \quad \text{... the value of the game (in mixed strategies)} \]

**Proposition** (Generalisation of the saddle point criterion):

\[(x^*, y^*) \in S_m \times S_n \text{ is a solution of the game } A \text{ in mixed strategies if and only if } (\forall x \in S_m)(\forall y \in S_n)\]

\[ x^T Ay^* \leq x^{*T} Ay^* \leq x^{*T} Ay \]

**Corollary** (follows straightforwardly from the definition): \[ v = x^{*T} Ay^* \text{ if } (x^*, y^*) \text{ is a solution of the game } A \text{ in mixed strategies}. \]

Next programme:

1. To prove that (*) indeed always holds.
2. To show how to find a solution in mixed strategies \((x^*, y^*)\).

Recall \[ v_1 = \max_{x \in S_m} \min_{y \in S_n} x^T Ay \]
Lemma 1: \((\forall x \in S_m) \min_{y \in S_n} x^T A y = \min_{y \in S_n} \sum_{j=1,\ldots,n} \sum_{i=1}^{m} x_i a_{ij} \).

Lemma 2: The task of finding 
\[ v_1 = \max_{x \in S_m} \min_{y \in S_n} x^T A y \]
is equivalent to an LP.

Proof: This task is equivalent to

\[
\begin{align*}
z & \rightarrow \text{max} \\
\text{s.t.} & \\
z & \leq \min_{y \in S_n} x^T A y \\
x & \in S_m
\end{align*}
\]
or, equivalently (by Lemma 1):

\[
\begin{align*}
z & \rightarrow \text{max} \\
\text{s.t.} & \\
(2.2.1) & \quad z \leq \min_{y \in S_n} x^T A y = \min_{y \in S_n} \sum_{j=1,\ldots,n} \sum_{i=1}^{m} x_i a_{ij} \\
\sum_{i=1}^{m} x_i & = 1 \\
x_i & \geq 0 \quad (i = 1, \ldots, m)
\end{align*}
\]
But (2.2.1) \( \equiv z \leq \sum_{i=1}^{m} x_i a_{ij} \quad (j = 1, \ldots, n) \)

so, equivalently:

\[
\begin{align*}
z & \rightarrow \text{max} \\
\text{s.t.} \\
z - \sum_{i=1}^{m} x_i a_{ij} & \leq 0 \quad (j = 1, \ldots, n) \\
\sum_{i=1}^{m} x_i & = 1 \\
x_i & \geq 0 \quad (i = 1, \ldots, m)
\end{align*}
\]

\((\text{LP}_1)\)
Lemma 1: \((\forall y \in S_n)\)

\[
\max_{x \in S_m} x^T Ay = \max_{i = 1, \ldots, m} \sum_{j=1}^{n} a_{ij} y_j
\]

Lemma 2: The task of finding

\[
v_2 = \min_{y \in S_n} \max_{x \in S_m} x^T Ay
\]

is equivalent to an LP.

Note: The LP in Lemma 2 is

\[
w \rightarrow \min \left\{ w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 \quad (i = 1, \ldots, m) \right\}
\]

\[
\sum_{j=1}^{n} y_j = 1
\]

\[
y_j \geq 0 \quad (j = 1, \ldots, n)
\]

Theorem 2.2.2 [The Minimax Theorem]: \(v_1 = v_2\).

Proof:

(LP_1), (LP_2) is a pair of primal-dual problems, both are feasible. By Corollary 2 of the Strong Duality Theorem both have an optimal solution and optimal values are equal.■
**Remark:** Minimax Th transforms GT $\rightarrow$ LP.

But the converse is also possible! (Omitted here)

---

**How to find a solution in mixed strategies?**

(A) Using Linear Programming

**Lemma (technical):** If $A = (a_{ij})$, $A' = (a_{ij} + c)$

then $A$ and $A'$ have the same solution in mixed strategies and $v(A') = v(A) + c$.

**Proof:**

$$A'(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} + c)x_i y_j =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j + c \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j =$$

$$= A(x, y) + c \quad \blacksquare$$

**Corollary:** We may assume wlog that $A$ is positive and so $v(A) > 0$. 

---

77
\[
\begin{align*}
    z & \rightarrow \text{max} \\
    \text{s.t.} & \quad \left\{ \begin{array}{l}
    z - \sum_{i=1}^{m} x_i a_{ij} \leq 0 \quad (j = 1, \ldots, n) \\
    \sum_{i=1}^{m} x_i = 1 \\
    x_i \geq 0 \quad (i = 1, \ldots, m)
    \end{array} \right. \\
    & \quad (LP_1)
\end{align*}
\]

Since \( z^{\text{max}} > 0 \) we may use the substitution
\[
    x_i' = \frac{x_i}{z} \quad (i = 1, \ldots, m) \quad \text{yielding} \quad \sum_{i=1}^{m} x_i' = \frac{\sum_{i=1}^{m} x_i}{z} = \frac{1}{z}
\]
Thus LP₁ transforms to
\[
\begin{align*}
\sum_{i=1}^{m} x'_i & \rightarrow \min \\
\sum_{i=1}^{m} a_{ij} x'_i & \geq 1 \quad (j = 1, \ldots, n) \\
x'_i & \geq 0 \quad (i = 1, \ldots, m)
\end{align*}
\]
Equivalently:
\[
A^T x' \geq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}
\]
\[
x' \geq 0
\]
or:
\[
(1, \ldots, 1)x' + (0, \ldots, 0)u \rightarrow \min
\]
\[
-A^T x' + Iu = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}
\]
x' ≥ 0, u ≥ 0
Similarly LP\(_2\) transforms to
\[
\sum_{j=1}^{n} y'_{j} \rightarrow \max
\]
\[
\sum_{j=1}^{n} a_{ij} y'_{j} \leq 1 \quad (i = 1, \ldots, m)
\]
\[
y'_{j} \geq 0 \quad (j = 1, \ldots, n)
\]
where \(y'_{j} = \frac{y_{j}}{w} \quad (j = 1, \ldots, n)\).

Equivalently:
\[
\begin{pmatrix}
1 \\
y'_{1} \quad \vdots \\
y'_{n}
\end{pmatrix}
\rightarrow \max
\]
\[
y'^{T} A^{T} \leq (1, \ldots, 1)
\]
\[
y' \geq 0
\]

Hence LP\(_1\) and LP\(_2\) are a pair of primal-dual problems (symmetric pair).

Due to duality theory there is no need to solve both of them. We only solve LP\(_1\) (say) and then an optimal solution for LP\(_2\) will be found in the optimal ST for LP\(_1\).
Recall that LP₁ is equivalent to

\[(1,...,1)x′ + (0,...,0)u \rightarrow \min\]

\[-A^Tx′ + Iu = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ \vdots \\ -1 \end{pmatrix}\]

\[x′, u \geq 0\]

So in the ST format we have …
Recall that $x'_i = \frac{x_i}{z_{\text{max}}} \text{ and } y'_j = \frac{y_j}{z_{\text{max}}}$.
**Example (see PS 2.2):**

\[
A = \begin{pmatrix}
5 & 4 \\
3 & 6
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-5</td>
<td>-3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-4</td>
<td>-6</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-\frac{1}{5}</td>
<td>0</td>
<td>\frac{2}{5}</td>
<td>\frac{1}{5}</td>
<td>0</td>
</tr>
<tr>
<td>\frac{1}{5}</td>
<td>1</td>
<td>\frac{3}{5}</td>
<td>\frac{1}{5}</td>
<td>0</td>
</tr>
<tr>
<td>-\frac{1}{5}</td>
<td>0</td>
<td>-\frac{18}{5}</td>
<td>-\frac{4}{5}</td>
<td>1</td>
</tr>
<tr>
<td>-\frac{2}{9}</td>
<td>0</td>
<td>0</td>
<td>\frac{1}{9}</td>
<td>\frac{1}{9}</td>
</tr>
<tr>
<td>\frac{1}{6}</td>
<td>1</td>
<td>0</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>\frac{1}{18}</td>
<td>0</td>
<td>1</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>
(B) Domination

Row $i$ of the matrix $A$ is called dominated if

$$(\exists k \neq i)(\forall j) \ a_{ij} \leq a_{kj}$$

Column $j$ is called dominated if

$$(\exists k \neq j)(\forall i) \ a_{ij} \geq a_{ik}$$

Remove dominated rows and columns.

**Example 2.2.1**

$$
\begin{pmatrix}
2 & 1 & 0 & 4 \\
1 & 2 & 5 & 3 \\
4 & 1 & 3 & 2
\end{pmatrix}
\rightarrow
$$
(C) Graphical methods for finding optimal mixed strategies for $m \times 2$ and $2 \times n$ matrices

Suppose $m = 2$ (the case of $n = 2$ similarly).

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \end{pmatrix}$$

$$\nu(A) = A(x^*, y^*) = \max_{x \in S_2} \min_{y \in S_n} x^T Ay = \max_{x \in S_2} \min_{j=1, \ldots, n} \sum_{i=1}^{2} x_i a_{ij} =$$

$$= \max_{x \in S_2} \min_{j=1, \ldots, n} (x_1 a_{1j} + x_2 a_{2j}) =$$

$$= \max_{x \in [0,1]} \min_{j=1, \ldots, n} ((a_{2j} - a_{1j}) x_2 + a_{1j})$$

Set $f_j(x_2) = (a_{2j} - a_{1j}) x_2 + a_{1j} \quad (j = 1, \ldots, n)$

$\therefore f_j(0) = a_{1j}, f_j(1) = a_{2j} \quad (j = 1, \ldots, n)$
Similarly for $n = 2$:

$$\nu(A) = \min_{y_2 \in [0,1]} \max_{i=1,\ldots,m} (a_{i1} + y_2(a_{i2} - a_{i1}))$$

For $g_i(y_2) = a_{i1} + y_2(a_{i2} - a_{i1})$ we have

$$g_i(0) = a_{i1}, g_i(1) = a_{i2} \quad (i = 1, \ldots, m)$$

See the figure below.
Example 2.2.2: Find optimal mixed strategies of both players in the game $A = \begin{pmatrix} 8 & 1 & 2 \\ 0 & 9 & 5 \end{pmatrix}$. Find also the value of the game.
(D) $2 \times 2$ matrix games

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

**Lemma 2.2.3**

If the matrix game $A$ has no solution in pure strategies then neither of the players has a solution in pure strategies.

Denote $d(A) = a_{11} + a_{22} - a_{12} - a_{21}$

**Proposition 2.2.4**

If the matrix game $A$ has no solution in pure strategies then

(a) $d(A) \neq 0$ and

(b) the solutions in mixed strategies for players I and II are given by

$$x_1^* = \frac{a_{22} - a_{21}}{d(A)}, \quad x_2^* = \frac{a_{11} - a_{12}}{d(A)}$$

$$y_1^* = \frac{a_{22} - a_{12}}{d(A)}, \quad y_2^* = \frac{a_{11} - a_{21}}{d(A)}$$
Example 2.2.3

\[ A = \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \]

No SPS!

Essential check

d(A) =

\[ x^* = \]

\[ y^* = \]

v(A) =
How to find $v(A)$ quickly:

If $J = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$ then

$A^* = \text{adj}(A)$ \quad \therefore \quad AA^* = \det(A) \cdot I$

$J^T A^* = (a_{22} - a_{21}, a_{11} - a_{12})$

$A^* J = (a_{22} - a_{12}, a_{11} - a_{21})^T$

\therefore \quad J^T A^* J = d(A)$

$x^*T = \frac{J^T A^*}{J^T A^* J}, \quad y^* = \frac{A^* J}{J^T A^* J}$

**Proposition 2.2.5** \quad $v(A) = \frac{\det(A)}{d(A)}$ if $A$ has no SPS

**Proof** →→→

105
3 Definitions and classification of games

A game has:

• **players**: 1, 2, ..., \( p \)

• non-empty sets of **strategies** for each player:
  \( X_1, X_2, ..., X_p \)

• **pay-off functions** of each player:
  \( \pi_i : X \rightarrow R \ (i = 1, ..., p) \) where
  \( X = X_1 \times X_2 \times ... \times X_p \)

So, \( \pi_i = \pi_i(x_1, ..., x_p) \) where \( x_1, ..., x_p \) are strategies chosen by players 1, 2, ..., \( p \).

The set of players: \( P = \{1, 2, ..., p\} \).

\( \Pi = (\pi_1, ..., \pi_p) = (\pi_1(x_1, ..., x_p), ..., \pi_p(x_1, ..., x_p)) \)

So \( \Pi : X \rightarrow R^p \)

**Definition:** A game of \( p \) players is the triple

\[ \Gamma = < P, X, \Pi >. \]

\( \Gamma \) is called

• **finite** if all \( X_i \) are finite \( (i = 1, ..., p) \)

• **infinite** if it is not finite
• constant-sum if \( \sum_{i=1}^{p} \pi_i(x) = \text{const} \) for all \( x \in X \)

(else general-sum)

• zero-sum if \( \sum_{i=1}^{p} \pi_i(x) = 0 \) for all \( x \in X \)

• antagonistic if \( p = 2 \) and \( \Gamma \) is constant-sum.

Note: If \( p = 2 \) then we usually write

\( X, Y \) instead of \( X_1, X_2 \)

\( x, y \) instead of \( x_1, x_2 \)

\( \pi, \sigma \) instead of \( \pi_1, \pi_2 \)

players I, II instead of players 1, 2; etc...

Example 3.1

\( \Gamma = < \{I, II\}, [-3, 5] \times [1, 8], (x - y, y - x) > \)
Example 3.2 (Battle of the Sexes)

John: prefers TV comedies
Mary: prefers TV tragedies
Both: prefer partner’s choice rather than watching alone (although two TV sets).
“Happiness” rating:
0 = watching either of the programmes alone
1 = watching partner’s choice with partner
5 = watching own choice with partner
Every evening they announce their choice independently of each other.

John’s pay-off:    Mary’s pay-off:

<table>
<thead>
<tr>
<th></th>
<th>Mary</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>T</td>
<td>C</td>
</tr>
<tr>
<td>T</td>
<td></td>
<td>T</td>
</tr>
</tbody>
</table>

As one pay-off “bi-matrix”:

Cooperation?
Example 3.3

John and Peter conceal independently either a 1p or 5p coin in their hands. Then they open and compare. If =, John takes all; if ≠, Peter takes all.

John’s pay-off: Peter’s pay-off:

\[
\begin{array}{c|cc}
\text{Peter} & 1 & 5 \\
\hline
\text{John} & 1 & -1 \\
& 5 & 5
\end{array}
\]

\[
\begin{array}{c|cc}
\text{Peter} & 1 & 5 \\
\hline
\text{John} & -1 & 1 \\
& 5 & -5
\end{array}
\]

The bi-matrix now is:

\[
\left(\begin{array}{cc}
(1,-1) & (-1,1) \\
(-5,5) & (5,-5)
\end{array}\right)
\]

As zero-sum, it can be represented by a single matrix:

\[
\begin{pmatrix}
1 & -1 \\
-5 & 5
\end{pmatrix}
\]

Definition (alternative): A finite antagonistic game is called a matrix game.
Hence an \( m \times n \) matrix game can be written as:
\[ \Gamma = \langle \{ \text{I, II} \}, \{1, \ldots, m\} \times \{1, \ldots, n\}, (A, (c - A)) > \]
where \( c \) is a constant (can be assumed 0, wlog)

**Example 3.4**

Scissors-paper-stone:
win = 1, loss = –1, draw = 0

The payoff matrix for player I (for player II multiply by \(-1\)):

<table>
<thead>
<tr>
<th></th>
<th>Sc</th>
<th>P</th>
<th>St</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sc</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Player I</td>
<td>P</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>St</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \therefore \text{It is a matrix game with an antisymmetric (skew-symmetric) payoff matrix, that is } A = -A^T. \]

**Definition**: A matrix game is called *symmetric* if its payoff matrix is anti-symmetric.
What would be acceptable as a solution of a game? One possibility: stability (equilibrium)

Fundamental concept of game theory:

Let \( \Gamma = < P, X, \Pi > \) be a game. Then \( x^* \in X \) is called an equilibrium point (or equilibrium situation) in \( \Gamma \) (abbr. EP) if

\[
\pi_i (x_1^*, ..., x_{i-1}^*, x_i^*, x_{i+1}^*, ..., x_p^*) \geq \pi_i (x_1^*, ..., x_{i-1}^*, x_i^*, x_{i+1}^*, ..., x_p^*)
\]

for all \( i \in P \) and for all \( x_i \in X_i \).

Is this a generalisation of SPS in matrix games?

An equilibrium point in a matrix game \( \Gamma = <\{ I, \Pi \}, \{1, ..., m\} \times \{1, ..., n\}, (A, -A) > \)

is \((r, s) \in \{1, ..., m\} \times \{1, ..., n\} \) such that

\[ a_{rs} \geq a_{is} \quad \text{for all } i = 1, ..., m \text{ and } \]

\[ -a_{rs} \geq -a_{rj} \quad \text{for all } j = 1, ..., n \]

\[ \therefore \text{ it is a column maximum and a row minimum in } A \text{ at the same time.} \]

\[ \therefore \text{ In matrix games we have:} \]

Equilibrium point \( \equiv \) Saddle point
Equilibrium points in Exs. 3.1-3.4.

**Randomisation:** If there is no EP then instead of choosing strategies we seek probability distributions on the sets of strategies. \( \therefore \) we convert the game \( \Gamma \) to another game of a special form. This game is called a *mixed extension* of \( \Gamma \), notation \( M(\Gamma) \). This game is always infinite.

For instance for the matrix game
\[
\Gamma = < \{I, \Pi\}, \{1, \ldots, m\} \times \{1, \ldots, n\}, (A, -A) >
\]
we have constructed its mixed extension
\[
M(\Gamma) = < \{I, \Pi\}, S_m \times S_n, (A(x, y), -A(x, y)) >
\]
where \( S_k = \{(z_1, \ldots, z_k)^T ; z_1, \ldots, z_k \geq 0, \sum z_i = 1\} \)
\((k = 1, 2, \ldots) \) and
\[
A(x, y) = x^T Ay \quad \text{for} \quad x \in S_m, y \in S_n.
\]

Equilibrium point \( (x^*, y^*) \) in \( M(\Gamma) \) satisfies:
\[
A(x^*, y^*) \geq A(x, y^*) \quad \text{for all} \quad x \in S_m \quad \text{and}
\]
\[
-A(x^*, y^*) \geq -A(x^*, y) \quad \text{for all} \quad y \in S_n.
\]

Hence: EP of a matrix game \( \equiv \) SPS
EP of the mixed extension \( \equiv \) SMS
**Remarks**

1. The mapping \( \Pi : X_1 \times X_2 \times \ldots \times X_p \rightarrow \mathbb{R}^p \) is called a *game in normal form*.

An important special case: \( p = 2 \)

If \( X_1 = \{1, \ldots, m\} \), \( X_2 = \{1, \ldots, n\} \) then the game in *normal form* is a pair \((\pi_1, \pi_2)\) of \( m \times n \) matrices:

\[
\pi_i = \begin{pmatrix}
\pi_i(1,1) & \pi_i(1,2) & \ldots & \ldots \\
\pi_i(2,1) & \pi_i(2,2) & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix} \quad (i = 1, 2)
\]

This is a bi-matrix game (not a matrix game!)

\( \pi_1 \ldots A \)

\( \pi_2 \ldots B \)

If \( B = -A \ldots \) matrix game
2. “\((r, s)\) is an EP” in a bi-matrix game means:

\[
\pi_1: \begin{pmatrix}
 r & & & & & \bullet \to \max \\
 & & & & & \max \to \\
& & & & & \\
\end{pmatrix}
\]

\[
\pi_2: \begin{pmatrix}
 r & \max \to & \bullet \leftarrow \max \\
 & & \\
 & & \\
\end{pmatrix}
\]

In matrix games: \(\pi_2 = -\pi_1\)

\[\therefore \text{row max in } \pi_2 \equiv \text{row min in } \pi_1 (= A)\]
3. Alternative representation of games:

*Games in extensive form* ... using “game-trees”, like (see Example 3.3):

![Game Tree](image)

- John
  - Peter: (1, -1)
  - Peter: (-1, 1)
- Peter: (-5, 5)
- Peter: (5, -5)
• **Finite games of:**

  - **Antagonistic (matrix games)**
    - 2 players
    - Non-Antagonistic (bi-matrix games)
  - \( p \) players (\( p > 2 \))
  - Non-Cooperative
  - Cooperative

• **Infinite games**
Non-antagonistic games of two players

4.1 Non-cooperative games

\[ \Gamma = \langle \{I, \ II\}, \{1, \ldots, m\} \times \{1, \ldots, n\}, (A, B) \rangle \]  
("General-sum games")

\( A, B \) are real \( m \times n \) matrices.

Normal form is a pair of matrices \( \therefore \) these games are also called bi-matrix games.

If \( A = -B \) then we have a zero-sum game \( \therefore \) a matrix game.

**Interpretation:**

\( a_{ij} \ [b_{ij}] = \text{payoff of } I \ [II] \text{ provided that } I \) chooses strategy \( i \) and II chooses strategy \( j \).
Example 4.1.1 [“Market Game”]:

I … a small firm trying to gain control at one of two markets (M1, M2)
II … a large firm controlling the two markets
M1 is twice the size of M2
I attempts to sell a large amount of some commodity on one of the two markets → intensive advertising (IA).
II may take preventive measures (PM).
No resistance → player I will capture the market.
If challenged, player I will certainly lose.

*Question*: Should player I take the risk and try to penetrate one of the markets? Which one?

Each player has two options (strategies): to take an action on exactly one of M1 or M2.
“Action” … IA (player I), PM (player II)
Formally (Ex. 4.1.1 ctd):

Two firms (= players) will act on one of two 
(possibly the same) markets (= strategies) 
They will act at the same time. 
Their choice of market is not known in advance. 
Profit from the first market is double to the 
second market but the loss on the first would 
mean a certain ruin for the first firm.

\[
A = \begin{pmatrix} -10 & 2 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}
\]

Equilibrium point?

(Continued later)
4.1.1 Equilibrium points

Recall that now $\pi_1(i, j) = a_{ij}$ and $\pi_2(i, j) = b_{ij}$ for every $i$ and $j$. By the definition $(r, s)$ is an equilibrium point if

$$\pi_1(r, s) \geq \pi_1(i, s) \text{ for every } i$$

and

$$\pi_2(r, s) \geq \pi_2(r, j) \text{ for every } j.$$ 

Hence an equilibrium point is $(r, s)$ such that

$$a_{rs} \geq a_{is} \text{ for every } i$$

and

$$b_{rs} \geq b_{rj} \text{ for every } j.$$ 

As in matrix games such a position $(r, s)$ is called a solution of the bi-matrix game $(A, B)$ in pure strategies.
Equilibrium points may not exist \( \therefore \) we use \textit{randomisation} (“mixed extension”) again:

\[ M(\Gamma) = \langle \{ \text{I, II} \}, S_m \times S_n, (A(x, y), B(x, y)) \rangle \]

where \( S_k = \{ (z_1, \ldots, z_k)^T; z_1, \ldots, z_k \geq 0, \sum z_i = 1 \} \)

\((k = 1, 2, \ldots)\) and

\[ A(x, y) = x^T Ay, \quad B(x, y) = x^T By \quad \text{for} \quad x \in S_m, \quad y \in S_n. \]

An equilibrium point in \( M(\Gamma) \) is

\((x^*, y^*) \in S_m \times S_n \) satisfying:

\[ A(x^*, y^*) \geq A(x, y^*) \quad \text{for all} \quad x \in S_m \quad \text{and} \quad (*) \]

\[ B(x^*, y^*) \geq B(x^*, y) \quad \text{for all} \quad y \in S_n. \quad (***) \]

If \((x^*, y^*)\) satisfies \((*)\) and \((***)\) then it is called a \textit{solution of the bi-matrix game in mixed strategies}, and \(x^* [y^*]\) is called the vector of \textit{optimal mixed strategies} of player I [II].
**Theorem 4.1.1:**
Every bi-matrix game has a solution in mixed strategies (that is the mixed extension has an equilibrium point).

Proof (omitted here) is based on Brouwer’s fixed point theorem.

**Lemma:**

\[ A(x^*, y^*) \geq A(x, y^*) \] for all \( x \in S_m \) if and only if
\[ A(x^*, y^*) \geq A(e_i, y^*) \] for \( i = 1, 2, \ldots, m \).

**Proof:** → → →
Method for finding SMS in the $2 \times 2$ case

That is $m = 2, n = 2$. Hence

$x^* = (x_1^*, x_2^*)^T, y^* = (y_1^*, y_2^*)^T$

Change of notation (for simplicity):

$x^* = (x, 1 - x)^T, y^* = (y, 1 - y)^T$ where $0 \leq x, y \leq 1$

(a) Solving for player I first:

Proposition 4.1.2:

All solutions to $A(x^*, y^*) \geq A(e_i, y^*)$ for $i = 1, 2$ are $(x, y)$ satisfying:

• $x = 0$ and $y$ satisfying $d(A)y \leq a$,
• $x = 1$ and $y$ satisfying $d(A)y \geq a$,
• $x \in (0, 1)$ and $y$ satisfying $d(A)y = a$,

where $d(A) = a_{11} + a_{22} - a_{12} - a_{21}$, $a = a_{22} - a_{12}$

Proof: $\rightarrow \rightarrow \rightarrow$
• \( d(A) = a = 0 \) \therefore every \((x, y) \in [0, 1] \times [0, 1]\) yields a solution

\[ \begin{array}{c}
1 \\
\hline \\
0 & 1 \\
\hline \\
1 \\
\end{array} \]

• \( d(A) = 0, \ a > 0 \) \therefore \( x = 0, \ y \in [0, 1] \) yield a solution

\[ \begin{array}{c}
1 \\
\hline \\
0 & 1 \\
\hline \\
1 \\
\end{array} \]

• \( d(A) = 0, \ a < 0 \)

\therefore \( x = 1, \ y \in [0, 1] \) yield a solution

\[ \begin{array}{c}
1 \\
\hline \\
0 & 1 \\
\hline \\
1 \\
\end{array} \]
• $d(A) > 0$, $\alpha = \frac{a}{d(A)}$

- $0 \leq \alpha \leq 1$
- $\alpha > 1$
- $\alpha < 0$

• $d(A) < 0$, $\alpha = \frac{a}{d(A)}$

- $0 \leq \alpha \leq 1$
- $\alpha > 1$
- $\alpha < 0$
(b) Now solving for player II:

\[ b = b_{22} - b_{21}, \quad \beta = \frac{b}{d(B)} \]

Typical intersection of the solution sets to (*) and (**):

Hence \( x^* = (\beta, 1 - \beta)^T \), \( y^* = (\alpha, 1 - \alpha)^T \).

This is called the Swastika method.
\[ x^* = (\beta, 1 - \beta)^T, \quad y^* = (\alpha, 1 - \alpha)^T \]

**Proposition 4.1.2**: If \((x, y) = (\beta, \alpha)\) is the unique solution found by the Swastika method for \(x, y \in (0, 1)\) then

\[
A(x^*, y^*) = \frac{\text{det}(A)}{d(A)}, \quad B(x^*, y^*) = \frac{\text{det}(B)}{d(B)}.
\]

**Proof**: \(\rightarrow\) \(\rightarrow\) \(\rightarrow\)

**Remark**: The fractions in the Proposition coincide with the values of the matrix games \(A\) and \(B\) provided that they have no SPS. This is a coincidence, the values \(A(x^*, y^*)\) and \(B(x^*, y^*)\) are given by the above formulas even if these games have an SPS.

Apply to Example 4.1.1 (Market Game) \(\rightarrow\) \(\rightarrow\) \(\rightarrow\)

**Example 4.1.2**: Battle of the Sexes \(\rightarrow\) \(\rightarrow\) \(\rightarrow\)
Stability vs Satisfaction:

Example 4.1.3 (Prisoners’ Dilemma):

Two criminals are arrested for robbery - immediately separated - can’t communicate. Each one is offered the same deal:

- If you confess and implicate the other one then you will serve either 1 year (if the other one doesn’t confess) or 5 years (if the other one does confess).
- If you don’t confess then you will serve either 2 years (if the other one doesn’t confess) or 10 years (if the other one does confess).

Equilibrium point? Is it their best option?

<table>
<thead>
<tr>
<th></th>
<th>$I$</th>
<th>$C$</th>
<th>$NC$</th>
<th>$II$</th>
<th>$C$</th>
<th>$NC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>-5</td>
<td>-1</td>
<td></td>
<td></td>
<td>-5</td>
<td>-10</td>
</tr>
<tr>
<td>$NC$</td>
<td>-10</td>
<td>-2</td>
<td></td>
<td></td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>
Other approaches:

A. Perfect equilibria
B. Pareto optimality
C. Cooperation (if possible)

### 4.1.2 Perfect equilibria

Let \((A, B)\) be a bi-matrix game. Then

\((A, B, \sigma, \tau)\) is called a *perturbation game* if

\[(\sigma, \tau) \in S_m \times S_n, \sigma_i > 0, \tau_j > 0, \sum_i \sigma_i < 1, \sum_j \tau_j < 1\]

and the mixed strategies must satisfy the conditions

\[x_i \geq \sigma_i \quad (i = 1, \ldots, m) \quad \text{and} \quad y_j \geq \tau_j \quad (j = 1, \ldots, n).\]

\(\sigma, \tau\) ... *perturbation vectors*

#### Proposition 4.1.3

Every perturbed game has at least one equilibrium point in mixed strategies.

(Proof similar to that of Theorem 5.1.1)
**Lemma:**

Let \((x, y)\) be an equilibrium point for the perturbed game \((A, B, \sigma, \tau)\) and suppose that row \(k\) satisfies

\[
\sum_j a_{kj} y_j < \max_i \sum_j a_{ij} y_j
\]

Then \(x_k = \sigma_k\) ("if a row is less than optimal as a reply to the other player’s strategy then it must be used as little as possible").

Let \((A, B)\) be a bi-matrix game and \((x^*, y^*)\) be an equilibrium point in mixed strategies. Then \((x^*, y^*)\) is called a **perfect equilibrium** if

(i) there is a sequence of perturbation vectors \((\sigma^k, \tau^k) \to (0, 0)\) \((k = 1, 2, \ldots)\) and

(ii) there is a sequence of equilibrium points \((x^k, y^k)\) to \((A, B, \sigma^k, \tau^k)\) \((k = 1, 2, \ldots)\) such that \((x^k, y^k) \to (x^*, y^*)\) \((k \to \infty)\).
**Proposition 4.1.4**

Every bi-matrix game has at least one perfect equilibrium.

**Example 4.1.4** (Perfect equilibrium sometimes works):

\[
A = \begin{pmatrix}
0 & 0 \\
0 & 1 
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 \\
0 & 1 
\end{pmatrix}
\]

Two equilibrium points, but only one is perfect.

**Example 4.1.5** (Perfect equilibrium sometimes doesn’t work):

\[
A = \begin{pmatrix}
10 & 0 \\
10 & 1 
\end{pmatrix}, \quad B = \begin{pmatrix}
10 & 10 \\
0 & 1 
\end{pmatrix}
\]

Two equilibrium points, one is perfect.
4.1.3 Pareto optimality

Need a new concept of a “solution” for games.

Let \((A, B)\) be a bi-matrix game \(\Gamma\).

The non-cooperative payoff region of \(\Gamma\) is

\[
\pi(\Gamma) = \{(A(x, y), B(x, y)); x \in S_m, y \in S_n\}.
\]

Let \((u, v), (u', v') \in \pi(\Gamma)\). We say that \((u', v')\) dominates \((u, v)\) if \(u \leq u'\) and \(v \leq v'\).

**Notation:** \((u, v) \prec (u', v')\)

**Note:** \((u, v) \prec (u, v)\) for all \(u\) and \(v\).

Let \(S \subseteq \mathbb{R}^2\). Then \((u, v) \in S\) is called Pareto optimal w.r.t. \(S\) if it is not dominated by any other payoff from the set \(S\).

**PO points** … a new solution concept

**Example 4.1.6** (Battle of the Sexes) \(\rightarrow \rightarrow \rightarrow\)
4.2 Cooperative games of two players

4.2.1 Definitions and basic properties

The Battle of the Sexes: the players may speak to each other and agree to flip a coin whether they (both) go for (1, 1) or (2, 2) ["=joint strategy"]]. Probabilities of outcomes may be described using the matrix \[
\begin{pmatrix}
1/2 & 0 \\
0 & 1/2
\end{pmatrix}.
\]
Now they \textit{jointly} decide how each of them will play.

Let \((A, B)\) be an \(m \times n\) bi-matrix game \(\Gamma\). A \textit{joint strategy} (of players I and II) is an \(m \times n\) probability matrix \(Q = (q_{ij})\), that is \(q_{ij} \geq 0\) \((i = 1, \ldots, m, \ j = 1, \ldots, n)\) and \(\sum_{i,j} q_{ij} = 1\).

The expected payoff of player I [II] due to this joint strategy is

\[
\pi_1(Q) = \sum_{i,j} q_{ij}a_{ij} \quad [\pi_2(Q) = \sum_{i,j} q_{ij}b_{ij}].
\]
**Example 4.2.1**

\[(A, B) = \begin{pmatrix} (2,0) & (-1,1) & (0,3) \\ (-2,-1) & (3,-1) & (0,2) \end{pmatrix} \]

\[Q = \begin{pmatrix} 1/8 & 0 & 1/3 \\ 1/4 & 5/24 & 1/12 \end{pmatrix} \text{ ... joint strategy} \]

\[\pi_1(Q) = \]

\[\pi_2(Q) = \]

The **cooperative payoff region** is

\[\gamma(\Gamma) = \{ (\pi_1(Q), \pi_2(Q)) ; Q \text{ is a joint strategy} \} \]

**Battle of the Sexes** → → →

**Proposition 4.2.1:** \[\pi(\Gamma) \subseteq \gamma(\Gamma).\]

*Proof* → → →

163
$S \subseteq \mathbb{R}^n$ is called convex if $\lambda x + (1 - \lambda)y \in S$ for every $x, y \in S$ and $\lambda \in [0, 1]$.

If $S \subseteq \mathbb{R}^n$ then the convex hull of $S$ is the set
\[
\text{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i ; x_1, \ldots, x_k \in S, k \geq 1, \sum_{i} \lambda_i = 1, \lambda_i \geq 0 \right\}.
\]

**Basic properties:**

(a) The intersection of any system of convex sets is convex.

(b) $\text{conv}(S) = \bigcap_{S \subseteq T, T \text{convex}} T$
**Proposition 4.2.2:**

\[ \gamma( \Gamma ) = \text{conv}\{(a_{ij}, b_{ij}) ; i = 1, \ldots, m, j = 1, \ldots, n \} . \]

**Proof:** → → →

**Corollary:** \( \gamma( \Gamma ) \) is convex and compact.

**Example 4.2.2**

\[ A = \begin{pmatrix} 5 & 7 & 1 \\ 1 & 9 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 10 \\ 1 & -2 & 1 \end{pmatrix} \] → → →

4.2.2 Solving a cooperative game

The *maximin value* of player I in an \( m \times n \) bimatrix game \((A, B)\) is the best payoff which can be guaranteed independently of the actions of the other player, that is

\[ v_I = \max_{x \in S_m} \min_{y \in S_n} x^T A y. \]

The same for player II: \( v_{II} = \max_{y \in S_n} \min_{x \in S_m} x^T B y. \)

**Note:**

\( v_{II} \) has a different meaning from \( v_2 \) in Ch. 2

**Proposition 4.2.3:** \( v_I = \nu(A), \ v_{II} = \nu(B^T) \) **Proof:** → → →
An agreement about a joint strategy should satisfy two basic requirements:

(i) The payoff of each player should be at least his/her maximin value,

(ii) The payoff pair should be Pareto optimal.

The bargaining set for a bi-matrix game $\Gamma$ is the set

$B(\Gamma) = \{ (u, v) \in \gamma(\Gamma) ; (u, v) \text{ is Pareto optimal and } u \geq v_1, v \geq v_Ⅱ \}$

*Example 4.2.2 contd: → → →*

*Battle of the Sexes → → →*
Which point in the bargaining set is a fair payoff point [“arbitration pair”]? 

Usually: \((u_0, v_0) = (v_1, v_\Pi)\)
Nash bargaining axioms and arbitration procedure

Arbitration procedure $\Psi$ assigns to any status quo point $(u_0, v_0)$ (usually $(u_0, v_0) = (v_I, v_{II})$) a unique payoff pair $(u^*, v^*)$, called the arbitration pair, in a certain sense fair to both players.

$$(u_0, v_0) \xrightarrow{\Psi} (u^*, v^*) = \Psi(\gamma(\Gamma), (u_0, v_0))$$
Arbitration procedure should satisfy certain basic requirements, the following are Nash’s axioms:

[1] *(Feasibility)*  \((u^*, v^*) \in \gamma( \Gamma)\)

[2] *(Pareto optimality)*  \((u^*, v^*)\) is Pareto optimal w.r.t. \(\gamma( \Gamma)\)

[3] *(Individual rationality)*  \((u^*, v^*) \geq (u_0, v_0)\)

[4] *(Independence of Irrelevant Alternatives)*  
   If \(\gamma' \subseteq \gamma( \Gamma)\), \((u_0, v_0) \in \gamma', (u^*, v^*) \in \gamma'\) then
   \[\Psi(\gamma', (u_0, v_0)) = (u^*, v^*)\]

[5] *(Linearity)*  
   If \(\gamma' = \{(u', v') ; u' = au + b, v' = cv + d, (u, v) \in \gamma( \Gamma)\}\)
   where \(a, c > 0\) then
   \[\Psi(\gamma', (au_0 + b, cv_0 + d)) = (au^* + b, cv^* + d)\]

[6] *(Symmetry)*  
   If \(\gamma( \Gamma)\) is symmetric (that is \((u, v) \in \gamma( \Gamma) \equiv (v, u) \in \gamma( \Gamma)\)) and \(u_0 = v_0\)
   then \(u^* = v^*\)
Nash’s arbitration procedure (NAP):

Assumption: Some \((u_0, v_0) \in \gamma(\Gamma)\) is given.

Notation: \(K = \{(u, v) \in \gamma(\Gamma) ; u > u_0, v > v_0\}\).

Case 1: \(K \neq \emptyset\)

Set \(g(u, v) = (u - u_0)(v - v_0)\) for \((u, v) \in K\).

Theorem 4.2.4: There is a unique \((u^*, v^*) \in K\)
at which \(g(u, v)\) attains its maximum over \(K\).

\[
\Psi(\gamma(\Gamma), (u_0, v_0)) := (u^*, v^*)
\]

Case 2: \(K = \emptyset\)

(i) \(\exists (u_0, v) \in \gamma(\Gamma) , v > v_0 ; \text{then } u^* = u_0 \text{ and } v^* = \max\{ v ; (u_0, v) \in \gamma(\Gamma)\}\)

(ii) \(\exists (u, v_0) \in \gamma(\Gamma) , u > u_0 ; \text{then } v^* = v_0 \text{ and } u^* = \max\{ u ; (u, v_0) \in \gamma(\Gamma)\}\)

(iii) Neither (i) nor (ii) holds; then

\((u^*, v^*) = (u_0, v_0)\)
Theorem 4.2.5 (Nash Arbitration):

Point \((u^*, v^*)\) found by NAP is a unique point satisfying Nash’s axioms [1]-[6].

Proof: \rightarrow \rightarrow \rightarrow

Note:

If \((u_0, v_0) = (v_I, v_{II})\) then NAP is called the Shapley procedure.

Battle of the Sexes \rightarrow \rightarrow \rightarrow

Example 4.2.2 (contd)

Use the Shapley procedure to find the arbitration point and the joint strategy in the cooperative game \((A, B)\), where

\[
A = \begin{pmatrix} 5 & 7 & 1 \\ 1 & 9 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 10 \\ 1 & -2 & 1 \end{pmatrix}.
\]
5 Games of \( p \) players \((p > 2)\)

In this chapter \( P = \{1, 2, \ldots, p\} \) is the set of players.

5.1 Non-cooperative games of \( p \) players

Finite game of 3 players:
\( \Gamma = \langle \{1, 2, 3\}, \{1, \ldots, k\} \times \{1, \ldots, l\} \times \{1, \ldots, m\}, (A, B, C) \rangle \)

where \( A = (a_{qrs}), B = (b_{qrs}), C = (c_{qrs}), q = 1, \ldots, k; \)
\( r = 1, \ldots, l; s = 1, \ldots, m. \)

Mixed extension:
\( M(\Gamma) = \langle \{1, 2, 3\}, S_k \times S_l \times S_m, (A(x, y, z), B(x, y, z), C(x, y, z)) \rangle \)

where
\[
A(x, y, z) = \sum_{q, r, s} a_{qrs} x_q y_r z_s ,
\]
\[
B(x, y, z) = \sum_{q, r, s} b_{qrs} x_q y_r z_s ,
\]
\[
C(x, y, z) = \sum_{q, r, s} c_{qrs} x_q y_r z_s
\]
\((x^*, y^*, z^*) \in S_k \times S_l \times S_m\) is an equilibrium point in mixed strategies if and only if

\[
A(x^*, y^*, z^*) \geq A(x, y^*, z^*) \quad \text{for all} \quad x \in S_k, \\
B(x^*, y^*, z^*) \geq B(x^*, y, z^*) \quad \text{for all} \quad y \in S_l, \\
C(x^*, y^*, z^*) \geq C(x^*, y^*, z) \quad \text{for all} \quad z \in S_m.
\]

Similarly for the game of \(p\) players:

\[\Gamma = < P, X_1 \times X_2 \times \ldots \times X_p, (\pi_1, \pi_2, \ldots, \pi_p) >\]

where \(X_1, X_2, \ldots, X_p\) are finite sets.

**Theorem 5.1.1** (Nash, 1951)

Every finite \(p\)-person game has at least one equilibrium point in mixed strategies.

(Proof similar to that for bi-matrix games.)

It is usually difficult to find equilibrium points.
5.2 Cooperative games of \( p \) players

5.2.1 Definitions and basic properties

Set of players: \( P = \{1, 2, \ldots, p\} \) \( (p > 2) \)

Any subset \( C \subseteq P \) is called a \textit{coalition}.

\( C = P \) \textit{… grand coalition}

\( C = \emptyset \) \textit{… empty coalition}

\( C \neq P \) and \( C \neq \emptyset \) \textit{… proper coalition}

\( P - C \) \textit{… counter coalition}

\textbf{Example 5.2.1}

\( p = 3, \; X_1 = X_2 = X_3 = \{1, 2\} \)

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,1)</td>
<td>(−2,1,2)</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>(1,1,−1)</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>(0,−1,2)</td>
</tr>
<tr>
<td>(1,2,2)</td>
<td>(−1,2,0)</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>(1,−1,1)</td>
</tr>
<tr>
<td>(2,1,2)</td>
<td>(0,0,1)</td>
</tr>
<tr>
<td>(2,2,1)</td>
<td>(1,0,0)</td>
</tr>
<tr>
<td>(2,2,2)</td>
<td>(1,2,−2)</td>
</tr>
</tbody>
</table>
The main goal of game theory is to achieve stability that is fair payoff for all individual players. For this we need to assess the strength of every coalition, called the value of a coalition.

A common way of this assessment is as follows:

If a proper coalition $C$ is formed then the value of $C$, notation $\nu(C)$, is found as the maximin value of player I in the game of two players, player I being the coalition $C$ and player II being the counter-coalition $P - C$. 
Recall that
\[ \Gamma = < P, X_1 \times X_2 \times \ldots \times X_p, (\pi_1, \pi_2, \ldots, \pi_p) > \]
and let us denote \( r_i = |X_i| \) \( (i = 1, \ldots, p) \).

Let \( C \subseteq P, C \neq \emptyset \). The set of pure strategies of \( C \):
\[ \prod_{i \in C} X_i = \{ \sigma_1, \sigma_2, \ldots, \sigma_r \}, \text{ where } r = \prod_{i \in C} r_i. \]

The set of pure strategies of \( P - C \):
\[ \prod_{i \in P - C} X_i = \{ \tau_1, \tau_2, \ldots, \tau_q \}, \text{ where } q = \prod_{i \not\in C} r_i. \]

Payoffs = \( (\sum \text{payoffs in } C, \sum \text{payoffs in } P - C) = (A, B) = ((a_{ij}), (b_{ij})) \) where
\[ a_{ij} = \sum_{k \in C} \pi_k(\sigma_i, \tau_j), \quad b_{ij} = \sum_{k \in P - C} \pi_k(\sigma_i, \tau_j). \]
Example 5.2.1 (contd)

Consider \( C = \{1, 3\} \), thus \( P - C = \{2\} \).

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,1)</td>
<td>(−2,1,2)</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>(1,1,−1)</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>(0,−1,2)</td>
</tr>
<tr>
<td>(1,2,2)</td>
<td>(−1,2,0)</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>(1,−1,1)</td>
</tr>
<tr>
<td>(2,1,2)</td>
<td>(0,0,1)</td>
</tr>
<tr>
<td>(2,2,1)</td>
<td>(1,0,0)</td>
</tr>
<tr>
<td>(2,2,2)</td>
<td>(1,2,−2)</td>
</tr>
</tbody>
</table>

The bi-matrix game between \( C \) and \( P - C \) is

\[
\begin{array}{c|cc}
\text{C} & 1 & 2 \\
\hline
(1,1) & & \\
(1,2) & & \\
(2,1) & & \\
(2,2) & & \\
\end{array}
\]

Recall: \( v_I(A, B) = v(A) \), \( v_{II}(A, B) = v(B^T) \)
The characteristic function of a game $\Gamma$ is the function $v: 2^P \rightarrow R$ satisfying:

(i) If $C$ is a proper coalition then $v(C)$ is the maximin value for $C$ in the bi-matrix game of $C$ and $P - C$,

(ii) $v(\emptyset) = 0$,

(iii) $v(P) = \max \sum_{x \in X} \sum_{i \in P} \pi_i(x)$.

**Example 5.2.1 (contd):**

**Proposition 5.2.1** The inequality

$$v(S \cup T) \geq v(S) + v(T) \quad (\text{superadditivity})$$

holds for all coalitions $S, T \subseteq P$, $S \cap T = \emptyset$.

**Proof:**

**Note:** $\geq$ may be $>$ (which is desirable)
**Proposition 5.2.2**

If $\Gamma$ is a constant-sum game in normal form, with constant $k$, then it is also constant-sum in CFF, that is $v(C) + v(P - C) = k$ for all $C \subseteq P$.

**Note:** The converse to Proposition 5.2.2 may not be true (see exercises).

The characteristic function $v: 2^P \rightarrow R$ has two significant properties:

(P1) $v(\emptyset) = 0$,

(P2) $v(S \cup T) \geq v(S) + v(T)$ for all disjoint coalitions $S, T \subseteq P$.

$(P, v)$ is called a game in the characteristic function form (CFF) if $P = \{1, 2, \ldots, p\}$ and $v: 2^P \rightarrow R$ satisfy (P1) and (P2).

**Convention:** $v(i)$ will stand for $v(\{i\})$. 

195
**Proposition 5.2.3**

Let \((P, v)\) be a game in CFF. Then

\[
v \left( \bigcup_{j=1}^{k} S_j \right) \geq \sum_{j=1}^{k} v(S_j)
\]

for any pairwise disjoint coalitions \(S_1, \ldots, S_k \subseteq P\).

**Proof:** By induction on \(k\) using (P2).

**Corollary:** \(v(S) \geq \sum_{i \in S} v(i)\) for every \(S \subseteq P\).

[In particular, \(v(P) \geq \sum_{i=1}^{p} v(i)\)]

**Convention:** Game \((P, v)\) in CFF will be often abbreviated to “Game \(v\).”
A game \( v \) is called *inessential* if
\[
v(P) = \sum_{i=1}^{p} v(i)
\]
and *essential* if it is not inessential.

*Example 5.2.1 (contd) → → →*

*Proposition 5.2.4*

A game \( v \) is inessential if and only if
\[
v(S) = \sum_{i \in S} v(i) \quad \text{for every } S \subseteq P.
\]

*Proof: → → →*

*Note: Two-person zero-sum game is inessential!*

199
A key question in the theory of cooperative games: How to distribute the payoffs to reflect the contributions of individual players and their willingness to join a coalition.

A vector $x = (x_1, \ldots, x_p)$ is called an **imputation** of the game $v$ if it satisfies

(R1) $x_i \geq v(i)$ for all $i \in P$ [*Individual Rationality*],

(R2) $\sum_{i=1}^{p} x_i = v(P)$ [*Collective Rationality*]

$E(v)$ … the set of all imputations of the game $v$

*Example 5.2.1* (contd) → → →

*Proposition 5.2.5*

(a) Every inessential game $v$ has only one imputation given by $x_i = v(i), \ i \in P$.

(b) Every essential game has an infinite number of imputations.

*Proof:* → → →
Let $v$ be a game, $S \subseteq P$, $S \neq \emptyset$ and $x, y \in E(v)$.

We say that $y$ dominates $x$ over $S$ [notation $y \succ_s x$] if

(Q1) $y_i > x_i$ for all $i \in S$ [pay-rise]

(Q2) $\sum_{i \in S} y_i \leq v(S)$ [feasibility]

*Example 5.2.1* (contd.)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>12</th>
<th>13</th>
<th>23</th>
<th>123</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>−1/3</td>
<td>0</td>
<td>1</td>
<td>4/3</td>
<td>3/4</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\[
\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)_{\{1,2\}} \succ \left( \frac{1}{2}, \frac{1}{2}, 0 \right)_{\{1,3\}}
\]

\[
\left( \frac{2}{3}, 0, \frac{1}{3} \right)_{\{1,3\}} \succ \left( \frac{1}{4}, \frac{12}{3}, \frac{2}{3} \right)_{\{2,3\}} \succ \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)_{\{1,2\}}
\]

∴ Domination is *not* transitive.
Real-life examples:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$i$</th>
<th>$S$</th>
<th>Imputations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company</td>
<td>Employee</td>
<td>Departments/Subsidiaries</td>
<td>Salaries</td>
</tr>
<tr>
<td>Shareholders’ meeting</td>
<td>Shareholder</td>
<td>Interest groups</td>
<td>Dividends</td>
</tr>
<tr>
<td>Country</td>
<td>Citizen</td>
<td>Provinces/political parties</td>
<td>Benefits</td>
</tr>
</tbody>
</table>
What is a “solution” of a game?

5.2.2 Solution concept #1: The core.

The core of a game $v$ is the set of all imputations which are not dominated by any other imputation through any coalition.

$$\text{Core}(v) = \{ x \in E(v) ; (\forall S \subseteq P)(\forall y \in E(v)) \ y \nless_S x \}$$

We need a clear description of cores.

*Theorem 5.2.6*

Let $v$ be a game and $x = (x_1, \ldots, x_p)$ be any real vector. Then $x \in \text{Core}(v)$ if and only if

$$\sum_{i=1}^{p} x_i = v(P) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq P.$$

*Proof:*

*Example 5.2.1* (contd)
Example 5.2.2

Consider the 3-player game whose characteristic function is given by:

\[
\begin{array}{c|cccccccc}
C & \{\} & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\
v(C) & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{2} & 1 \\
\end{array}
\]

Find an imputation in \( \text{Core}(v) \). [But check superadditivity first!]

The core may be empty for some types of games:

**Theorem 5.2.7**

If \( v \) is both essential and constant-sum then

\( \text{Core}(v) = \emptyset. \)

**Proof:** (It will also follow from Theorem 5.2.11)
Strategic equivalence of games

Two games $u, \nu$ of $p$ players are called 
*strategically equivalent* if there exist real 
constants $k > 0$ and $c = (c_1, \ldots, c_p)$ such that 

$$u(S) = kv(S) + \sum_{i \in S} c_i$$

holds for every $S \subseteq P$.

_Notation:_ $u \sim \nu$

_Convention:_ $\sum_{i \in S} c_i = 0$ for $S = \emptyset$.

Proposition 5.2.8

$\sim$ is a relation of equivalence.

_Proof:_ Homework.

**Example 5.2.1** (contd):

<table>
<thead>
<tr>
<th>$C$</th>
<th>${}$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${1,2}$</th>
<th>${1,3}$</th>
<th>${2,3}$</th>
<th>${1,2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu(C)$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$-\frac{1}{3}$</td>
<td>0</td>
<td>1</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{3}{4}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Set $k = 2$, $c = (-1, 0, -1)$

<table>
<thead>
<tr>
<th>$C$</th>
<th>${}$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${1,2}$</th>
<th>${1,3}$</th>
<th>${2,3}$</th>
<th>${1,2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(C)$</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{2}{3}$</td>
<td>$-1$</td>
<td>1</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>
**Proposition 5.2.9**

If \( u \sim v \) with constants \( k > 0, c_1, \ldots, c_p \) and \( c = (c_1, \ldots, c_p) \) then

(i) \( u \) is [in]essential \( \iff \) \( v \) is [in]essential,

(ii) \( x \in E(u) \iff y = kx + c \in E(v) \),

(iii) \( (\forall x, y \in E(u)) (\forall S \subseteq P) \ x \succ_S y \) wrt \( u \iff kx + c \succ_S ky + c \) wrt \( v \),

(iv) \( x \in \text{Core}(u) \iff kx + c \in \text{Core}(v) \),

(v) \( u \) is constant-sum \( \iff \) \( v \) is constant-sum.

**Proof:** \( \rightarrow \rightarrow \rightarrow \)

**Definition:**

A game \( u \) in CFF is said to be in \((0, 1)\)–reduced form if

- \( u(P) = 1 \) and
- \( u(i) = 0 \) for all \( i \in P \).

Note that this concept enables to compare the coalitions on the basis of the “value added” by individual players.
Theorem 5.2.10

Every essential game \( v \) is strategically equivalent to a unique game in \((0, 1)\) – reduced form.

Proof: \( \rightarrow \rightarrow \rightarrow \)

Example 5.2.2 (contd):

Transform it to \((0, 1)\) – reduced form \( \rightarrow \rightarrow \rightarrow \)

Remark 1

Every essential two-person game is strategically equivalent to the game:

\[ v(P) = 1, \ v(1) = v(2) = v(\emptyset) = 0. \]

Remark 2

Every essential three-person constant-sum game is strategically equivalent to the game:

\[ \begin{cases} v(P) = 1 = v(C) & \text{if } |C| = 2, \\ v(1) = v(2) = v(3) = v(\emptyset) = 0. \end{cases} \]

THREE
Example 5.2.1 (contd):
Transform it to (0, 1) – reduced form → → →

Remark 3
Every essential three-person game is strategically equivalent to the game:
\[ v(P) = 1, \ v(1) = v(2) = v(3) = v(\emptyset) = 0, \]
\[ v(\{1, 2\}) = a, \ v(\{1, 3\}) = b, \ v(\{2, 3\}) = c, \] where
\[ 0 \leq a, b, c \leq 1. \]
If $v$ is a game in $(0, 1)$ – reduced form and $v(S)$ is either 0 or 1 for all $S \subseteq P$ then $v$ is called simple (or a voting game). In a voting game, if $v(S) = 1$ then $S$ is called a winning coalition, otherwise it is losing. If $v(P - \{k\}) = 0$ for some $k \in P$, then the player $k$ is called a veto player.

Voting game:

<table>
<thead>
<tr>
<th>$S$</th>
<th>{}</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>12</th>
<th>13</th>
<th>23</th>
<th>123</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(S)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

A winning coalition $C$ in a simple game is called minimal if every proper subset of $C$ is losing.

*Theorem 5.2.11*

Let $v$ be a voting game. Then

Core($v$) $\neq \emptyset$ $\iff$ one of the players is a veto player.

*Proof:* $\rightarrow \rightarrow \rightarrow$

*Example 5.2.1* (contd):

Now clear why the core is $\emptyset$.

*Corollary:* Theorem 5.2.7.
Example 5.2.3 (Lake Wobegon):

The municipal government in LW, Minnesota is run by a City Council (CC) and a Mayor. CC consists of 6 Aldermen and 1 Chairman.

A bill can become a law in two ways:
(a) Majority of CC (chairman voting only in the case of a tie among Aldermen) approves it and the Mayor signs it.
(b) CC passes, the Mayor vetoes it, but at least six of the seven members of CC then vote to override the veto (in this case the Chairman always votes).

Interpret as a cooperative game, find the core, all winning coalitions and all minimal winning coalitions. → → →

Disadvantages of the concept #1:
• Core may be $\emptyset$.
• Core may be too big - difficult to choose a most appropriate solution.
5.2.3 Solution concept #2: Stable sets.

(“von Neumann - Morgenstern solutions”, 1944)

Let $v$ be a game in CFF. A set $S \subseteq E(v)$ is called stable if the following hold:

Axiom 1 [External Stability]

$$(\forall x \in E(v) - S) (\exists y \in S) (\exists C \subseteq P) \ y \succ_c x$$

Axiom 2 [Internal Stability]

$$(\forall x, y \in S) (\forall C \subseteq P) \ x \not\succ_c y$$

Note: If $y \in S$, $S$ stable set, there still may exist $z \not\in S, \ z \succ_c y$ for some $C \subseteq P$. [Remember, transitivity does not hold.]

Stable sets – “standards of behaviour”

(applications in sociology).

Proposition 5.2.12 If a stable set exists in a game $v$ then every stable set contains the core as a subset.

Proof $\rightarrow\rightarrow\rightarrow$
Reminder: In THREE \( v(C) = 1 \) if \( |C| \geq 2 \) and 
\( v(C) = 0 \) if \( |C| \leq 1 \).

**Proposition 5.2.13**

The set \( X = \left\{ \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( \frac{1}{2}, 0, \frac{1}{2} \right), \left( 0, \frac{1}{2}, \frac{1}{2} \right) \right\} \) is stable for THREE.

**Proof:**

**Computational aspect:** This result enables us to find a stable set for any 3-person constant-sum, essential game using the strategic transformation of Remark 2 and Th. 6.2.11.

**Example 5.2.1 (contd)**

Find a stable set using its transformation to THREE.
**Proposition 5.2.14**

Let \( c \) be any constant, \( 0 \leq c \leq \frac{1}{2} \).

Then \( Z_c = \{ (c, x_2, x_3) ; x_2 + x_3 = 1 - c, x_2, x_3 \geq 0 \} \) is a stable set for THREE.

**Proof:**  

Similarly stable sets \( Z_a, Z_b \).

Stable sets for small games.
Proposition 5.2.15

Let \( v \) be a simple game and let \( S \) be a minimal winning coalition. Then every set
\[
X = \{x = (x_1, \ldots, x_p) \in E(v); x_i = 0 \text{ for all } i \in P - S\}
\]
is stable.

Proof:

A game with no stable set? Discovered in 1967 (10 players).
5.2.4 Solution concept #3: The Shapley Value
[Shapley 1953]

SV - a distribution of the payoff of the coalitions to their members reflecting their individual contributions.

Let \( v \) be a game. Then \( C \subseteq P \) is called a carrier for the game \( v \) if

\[
v(S) = v(S \cap C) \quad \text{for all} \quad S \subseteq P.
\]

\( i \notin C, \) \( C \) carrier  ... \( i \) is a \textit{dummy}

\( C(v) \) ... the set of all carriers for \( v \)

\( L_p \) ... the set of all permutations of \( \{1, 2, \ldots, p\} \)

Let \( v \) be a game and \( \pi \in L_p. \) Then \( \pi v \) is the game \( u \) such that

\[
u(\{\pi(i_1), \ldots, \pi(i_s)\}) = v(S)
\]

for every coalition \( S = \{i_1, \ldots, i_s\}. \)
Definition (Shapley axioms): The (Shapley) value of a \( p \)-person game \( v \) is any vector \( \phi[v] = (\phi_1[v], \ldots, \phi_p[v]) \) satisfying the following:

(S1) \( (\forall C \in C(v)) \sum_{i \in C} \phi_i[v] = v(C) \)

(S2) \( (\forall \pi \in L_p) (\forall i \in P) \phi_{\pi(i)}[\pi v] = \phi_i[v] \)

(S3) \( \phi[u + v] = \phi[u] + \phi[v] \) holds for any two games \( u \) and \( v \).

Theorem 5.2.16 (Shapley)

There is a unique function \( \phi \) on the class of all games in CFF satisfying (S1) - (S3).

Theorem 5.2.17 (Computation of the Shapley value) Let \( v \) be a game and \( i \in P \). Then

\[ \phi_i[v] = \sum_{T \subseteq P, i \in T} c(p, t)[v(T) - v(T - \{i\})], \]

where

\[ c(p, t) = \frac{(t-1)! (p-t)!}{p!} \]

and \( t \) is the size of \( T \).

Proof: \( \to \to \to \)

Interpretation of the Shapley value \( \to \to \to \)
\[
\varphi_i[v] = \sum_{T \subseteq P, i \in T} \frac{(t-1)! (p-t)!}{p!} [v(T) - v(T - \{i\})], \quad i \in P
\]

**Example 5.2.1 (contd)**  → → →

**Proposition 5.2.18**

The Shapley value is an imputation.

**Proof:**  → → →

**Shapley value for simple games:**

\[
\varphi_i[v] = \sum \frac{(t-1)! (p-t)!}{p!} \quad \text{where the summation is taken over all winning coalitions } T \text{ such that } i \in T \text{ and } T - \{i\} \text{ is losing.}
\]

**Example 5.2.4**

Find the Shapley value for the corporation of 4 stock holders having respectively 10, 20, 30 and 40 shares of stock. Decision can be made by approval of a simple majority. → → →
**Example 5.2.5**

Same as above but when the shares are 10, 30, 30 and 30. → → →

*Multilinear extension* (MLE) - an alternative way of calculating the Shapley value.

Let \( v \) be a \( p \)-person game with carrier \( P \) (that is there are no dummy players). The *multilinear extension* of \( v \) is the function \( f : [0, 1]^p \to \mathbb{R} \) defined as

\[
f(x_1, \ldots, x_p) = \sum_{S \subseteq P} \left[ \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right] v(S)
\]

**Example 5.2.6**

Find the MLE of THREE. → → →
**Proposition 5.2.19**

For \( i = 1, \ldots, p \) there holds:

\[
\phi_i[v] = \int_0^1 f_i(t, \ldots, t) dt
\]

where

\[
f_i(x_1, \ldots, x_p) = \frac{\partial f(x_1, \ldots, x_p)}{\partial x_i}.
\]

**Example 5.2.7**

Find the Shapley value for the problem of Example 5.2.4 using the MLE.