On a geometrization of the Chung-Lu model for complex networks

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Abstract

We consider a model for complex networks that was introduced by Krioukov et al. [14], where the intrinsic hierarchies of a network are mapped into the hyperbolic plane. Krioukov et al. show that this model exhibits clustering and the distribution of its degrees has a power law tail. We show that **asymptotically** this model locally behaves like the well-known Chung-Lu model in which two nodes are joined independently with probability proportional to the product of some pre-assigned weights whose distribution follows a power law. Using this, we further determine exactly the asymptotic distribution of the degree of an arbitrary vertex.

keywords: mathematical analysis of complex networks, hyperbolic geometry, degree distribution

1 Introduction

The term "*complex networks*" describes a class of large networks which exhibit the following fundamental properties:

1. they are *sparse*, that is, the number of their edges is proportional to the number of nodes;
2. they exhibit the *small world phenomenon*: most pairs of vertices are within a short distance from each other;
3. a significant amount of *clustering* is present. The latter means that two nodes of the network that have a common neighbour are somewhat more likely to be connected with each other;

4. their degree distribution is *scale free*, that is, its tail follows a *power law*. **In** fact, experimental evidence (see [1]) suggests that many networks that emerge in applications have power law degree distribution with exponent between 2 and 3.

The books of Chung and Lu [10] and of Dorogovtsev [11] are excellent references for a detailed discussion of these properties.

Over the last 15 years a number of models have been developed whose aim was to capture these features. Among the first such models is the *preferential attachment model*. This is a class of models of randomly growing graphs whose aim is to capture a basic feature of such networks: nodes which are already popular tend to become more popular as the network grows. It was introduced by Barabási and Albert [2] and subsequently defined and studied rigorously by Bollobás, Riordan and co-authors (see for example [5], [4]).

Another extensively studied model was defined by Chung and Lu [8], [9]. Here every vertex has a weight which effectively corresponds to its expected degree and every two vertices are joined *independently* of every other pair with probability that is proportional to the product of their weights. If these weights follow a power-law distribution, then it turns out that the resulting random graph has power-law degree distribution.

All these models are nonetheless insufficient in the sense that none of them succeeds in incorporating *all* the above features. For example the Chung–Lu model although it exhibits a power law degree distribution, provided that the weights of the vertices are suitably chosen, and average distance of order $O(\log \log N)$, when the exponent of the power law is between 2 and 3 (see [8]) (with N being the number of nodes of the random network) it is locally tree-like around a typical vertex. **That is, for most vertices the following holds: if we fix any given positive integer d, then with high probability the neighbourhood within distance d contains no cycles**. This is also the situation in the Barabási-Albert model. Thus, it seems that there is a "missing link" in these models which is a key ingredient to the process of creating a social network. It seems plausible that the factor which is missing in these models is the *hierarchical structure* of a social network.

Real-world networks consist of heterogeneous nodes, which can be classified into groups. In turn, these groups can be classified into larger groups which, in turn, belong to larger groups and so on. For example, if we consider the network of citations, whose set of nodes is the set of research papers and there is a link from one paper to another if one cites the other, there is a natural classification of the nodes according to the scientific fields each paper belongs to (see for example [6]). In the case of the network of web pages, a similar classification can be considered in terms of the similarity between two web pages. That is, the more similar two web pages are, the more likely it is that there exists a hyperlink between them (see [15]).

This classification can be approximated by tree-like structures representing the hidden hierarchy of the network. The tree-likeness suggests that in fact the geometry of this hierarchy is *hyperbolic*. One of the basic features of a hyperbolic space is that the volume growth is exponential which is also the case, for example, when one considers a k-ary tree, that is, a rooted tree where every vertex has k children. Let us consider the Poincaré disc model. If we place the root of an infinite k-ary tree at the centre of the disc, then the hyperbolic metric provides the necessary room to embed the tree into the disc so that every edge has unit length in the embedding.

Recently Krioukov et al. [14] introduced a model which implements this idea. In this model, a random network is created on the hyperbolic plane (we will see the detailed definition shortly). In particular, Krioukov et al. [14] determined the degree distribution for *large* degrees showing that it is *scale free* and its tail follows a power law, whose exponent is determined by the parameters of the model. Furthermore, they consider the clustering properties of the resulting random network. A numerical approach in [14] suggests that the (local) clustering coefficient[1] is positive and it is determined by one of the parameters of the model.

The aim of our contribution is to show that **when N (the number of vertices of the underlying graph) is large**, then *locally* this model behaves like the Chung-Lu model. In particular, we show that as the size of the network becomes large, the probability that two vertices are joined by an edge is given by the *product* of appropriately defined weights of these vertices. Thereby, we are able to determine the *exact* distribution of the degree of a given vertex and also show that the number of vertices of a given degree is concentrated around its expected value.

1.1 Random graphs on the hyperbolic plane

The most common representations of the hyperbolic space is the upper-half plane representation $\{x + iy \ : \ y > 0\}$ as well as the Poincaré unit disc which is simply the open disc of radius one, that is, $\{(u, v) \in \mathbb{R}^2 \ : \ 1 - u^2 - v^2 > 0\}$. Both spaces are equipped with the hyperbolic metric; in the former case this is $\frac{1}{(\zeta y)^2} dy^2$ whereas in the latter this is $\frac{4}{\zeta^2} \ \frac{du^2+dv^2}{(1-u^2-v^2)^2}$, where ζ is some positive real number. It can be shown that the (Gaussian) curvature in both cases is equal to $-\zeta^2$ and the two spaces are isometric, that is, there exists a bijection between the two spaces which preserves (hyperbolic) distances. In fact, there are more representations of the 2-dimensional hyperbolic space of curvature $-\zeta^2$ which are isometrically equivalent to the above two. We will denote by $\mathbb{H}^2_\zeta$ the class of these spaces.

It can be shown that a circle of radius r around a point has length equal to $\frac{2\pi}{\zeta} \ \sinh \zeta r$ and area equal to $\frac{2\pi}{\zeta^2}(\cosh \zeta r - 1)$.

We are now ready to give the definitions of the two basic models introduced in [14]. Consider the Poincaré disc representation of the hyperbolic plane with curvature $K = -\zeta^2$, for some $\zeta > 0$. For a real number $\nu > 0$, let $N = \nu e^{\zeta R/2}$

[1] This is defined as the average density of the neighbourhoods of the vertices

– thus R is a function of N which grows logarithmically in N. The parameter ν controls the average degree of the random graph. We create a random graph by selecting randomly N points from the disc of radius R centred at the origin O, which we denote by $\mathcal{D}_R$. The distribution of these points is as follows. Assume that a random point u has polar coordinates (r, θ). Then θ is uniformly distributed in $(0, 2\pi]$, whereas the probability density function of r, which we denote by $\rho(r)$, is determined by a parameter $\alpha > 0$ and is equal to

$$\rho(r) = \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1}. \tag{1.1}$$

When $\alpha = \zeta$, then this is the uniform distribution. An alternative way to define this distribution is as follows. Consider the Poincaré disc representation of $\mathbb{H}^2_\alpha$ and let O' be the centre of the disc. Next, consider the disc $\mathcal{D}'_R$ of radius R around O' and select N points within $\mathcal{D}'_R$ uniformly at random. Subsequently, the selected points are projected onto $\mathcal{D}_R$ preserving their polar coordinates. The projections of these points onto $\mathcal{D}_R$ follow the above distribution.

This set of points will be the vertex set of the random graph and we will be denoting this random vertex set by V_N. We will be also treating the vertices as points in the hyperbolic space indistinguishably.

1. *The hard model* Two vertices are joined by an edge, if they are within (hyperbolic) distance R from each other (see Figure 1).

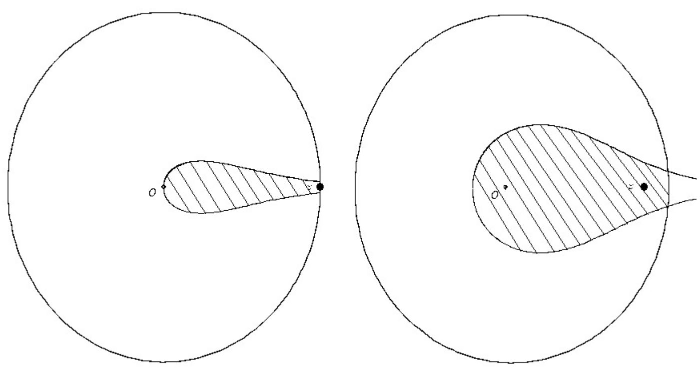

Figure 1: The disc of radius R around v in $\mathcal{D}_R$

2. *The soft model* We join any two distinct vertices u, v with probability

$$p_{u,v} = \frac{1}{\exp\left(\beta\ \frac{\zeta}{2}(d(u,v) - R)\right) + 1},$$

independently of every other pair, where $\beta > 0$ is fixed and $d(u, v)$ is the hyperbolic distance between u and v. We denote the resulting random graph by $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$.

Figure 2 illustrates three examples of the hard model with $N = 500$ vertices on the hyperbolic plane of curvature -1. The parameter α has been set to $3/4, 1$ and 3, respectively. We observe that as α increases, the vertices tend to be located far from the centre of $\mathcal{D}_R$. Also, the closer to the centre a vertex is, the larger its degree is. This is the case since those vertices that are closer to the centre cover a larger area of $\mathcal{D}_R$. We will make these observations precise later.

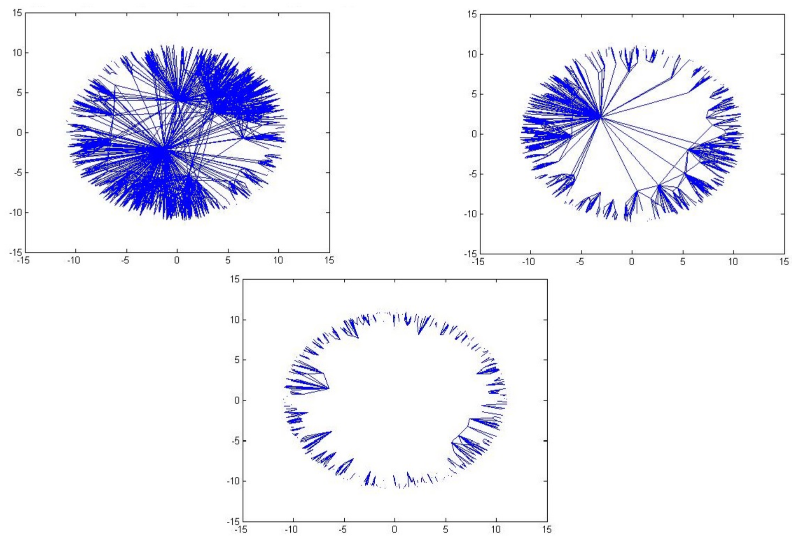

Figure 2: A random network on the hyperbolic plane (**hard model** with $N = 500$, $\zeta = 1$ and $\alpha = 3/4, 1, 3/2$ in a clockwise order)

In some sense, **the soft model can be seen as an approximate version of the hard model**. In the latter, two vertices become adjacent if and only if their hyperbolic distance is at most R. This is *almost* the case in the former model. If $d(u, v) = (1 + \delta)R$, where $\delta > 0$ is some small constant, then $p_{u,v} \to 0$, whereas if $d(u, v) = (1 - \delta)R$, then $p_{u,v} \to 1$, as $N \to \infty$.

Also, the **hard model** can be viewed as a limiting case of the **soft model** as $\beta \to \infty$. Assume that the positions of the vertices in $\mathcal{D}_R$ have been realized. If $u, v \in V_N$ are such that $d(u, v) < R$, then when $\beta \to \infty$ the probability that u and v are adjacent tends to 1; however, if $d(u, v) > R$, then this probability converges to 0 as β grows. Rigorous results for the **hard model** were obtained by Gugelmann et al. [12], regarding their degree distribution as well as the clustering coefficient.

The present paper will focus on the **soft model** and, in particular, on the degree distribution of a given vertex for *all* values of β. The central structural features of the resulting random graph heavily depend on the value of β. In particular, when $\zeta/\alpha < 2$ we shall distinguish between three regimes:

1. $\beta > 1$ the random graph $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$ has constant average degree depending on β, ζ and α.

2. $\beta = 1$ the average degree grows logarithmically in N.

3. $\beta < 1$ the average degree of $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$ grows polynomially in N.

Hence, we focus on the $\beta > 1$ regime, as this gives rise to *sparse* graphs, and in this case we determine the limiting degree distribution of a given vertex *precisely*. For this regime Krioukov et al. [14] argue that the tail of the degree distribution scales as a power law with exponent equal to $2\alpha/\zeta + 1$, for $\zeta/\alpha < 2$. Hence, any exponent greater than 2 can be realized.

The **soft model** has an additional parameter β which turns out to control the clustering coefficient. In particular, in joint work with E. Candellero [7], we show that when $\beta > 1$, the (global) clustering coefficient [2] takes with high probability a specific value that depends only on α, ζ and β. This value is given by a multiple integral of a function that takes only α, ζ and β as parameters. As we shall see in the present paper, the parameter ν can tune the average degree of the resulting network. In other words, given the values of α and ζ which determine the exponent of the power law, the average degree as well as the clustering coefficient can be determined *independently*.

The **soft model** has also a statistical mechanical interpretation. The parameter $\beta > 0$ is interpreted as the inverse of the temperature of a fermionic system where particles correspond to edges. The distance between two points determines the **strength of the interaction** that is incurred by the pair. In particular, **the strength of the interaction among** the pair $\{u, v\}$ is $\omega_{u,v} = \frac{\zeta}{2}\ (d(u,v) - R)$.

An edge between two points corresponds to a particle that "occupies" the pair. In turn, **the Hamiltonian of a graph G on the N points, assuming that their positions on $\mathcal{D}_R$ have been realized, is** $H(G) = \sum_{u,v} \omega_{u,v} e_{u,v}$, where $e_{u,v}$ is the indicator that is equal to 1 if and only if the edge between u and v is present. (Here the sum is over all distinct unordered pairs of vertices.) Each graph G has **probability weight that is equal to** $e^{-\beta H(G)}/Z$, where $Z = \prod_{u,v} \left(1 + e^{-\beta \omega_{u,v}}\right)$ is the normalizing factor also known as the partition function. The analysis of Park and Newman [16] shows that in this distribution the probability that u is adjacent to v is equal to $1/(e^{\beta \omega_{u,v}} + 1)$. Hence in this context, we have $\omega_{u,v} = \frac{\zeta}{2}(d(u,v) - R)$. See also [14] for a more detailed discussion.

Notation Wherever necessary we will abbreviate the standard Landau notation as follows. For two functions f, g on the natural number, we will write $f(N) \asymp g(N)$, meaning that there are positive constants c_1, c_2 such that for all $N \in \mathbb{N}$ we have $c_1 g(N) \leq f(N) \leq c_2 g(N)$. (This is equivalent to $f(N) = \Theta(g(N))$.) Analogously, we write $f(N) \lesssim g(N)$ (resp. $f(N) \gtrsim g(N)$) if there is a positive constant c such that for all $N \in \mathbb{N}$ we have $f(N) \leq cg(N)$ (resp. $f(N) \geq cg(N)$). In the standard Landau notation, this is simply $f(N) = O(g(N))$ and $f(N) = \Omega(g(N))$, respectively. Also, we write $f(N)/g(N) = 1 + o(1)$, **if** $f(N)/g(N) \to 1$ **as** $N \to \infty$.

[2]This is the density of triangles contained in the network as a fraction of paths of length two.

1.2 Some geometric aspects of the two models

It is not very hard to show that setting the threshold distance equal to R is the appropriate choice. The proof of this fact relies on the following lemma which provides a characterization of what it means for two points u, v to have $d(u,v) \leq (1+\delta)R$, for $\delta \in (-1,1)$, in terms of their *relative angle*, which we denote by $\theta_{u,v}$. For this lemma, we need the notion of the *type* of a vertex. For a vertex $v \in V_N$, if r_v is the distance of v from the origin, that is, the radius of v, then we set $t_v = R - r_v$ – we call this quantity the *type* of vertex v.

Lemma 1.1. *Let $\delta \in (-1,1)$ be a real number. For any $\varepsilon > 0$ there exists an $N_0 > 0$ and a $c_0 > 0$ such that for any $N > N_0$ and $u, v \in \mathcal{D}_R$ with $t_u + t_v < (1-|\delta|)R - c_0$ the following hold.*

- *If $\theta_{u,v} < 2(1-\varepsilon)\exp\left(\frac{\zeta}{2}(t_u + t_v - (1-\delta)R)\right)$, then $d(u,v) < (1+\delta)R$.*
- *If $\theta_{u,v} > 2(1+\varepsilon)\exp\left(\frac{\zeta}{2}(t_u + t_v - (1-\delta)R)\right)$, then $d(u,v) > (1+\delta)R$.*

The proof of this lemma can be found in the appendix.

For the sake of simplicity, let us consider temporarily the (modified) hard model, where we assume that two vertices are joined precisely when their hyperbolic distance is at most $(1+\delta)R$. Let $u \in V_N$ be a vertex and assume that $t_u < C$ (in fact, as we shall see shortly, if C is large enough, then most vertices will satisfy this). We will show that if $\delta < 0$, then the expected degree of u, in fact, tends to 0. Let us consider a simple case where δ satisfies $\frac{\zeta}{2\alpha} < 1 - |\delta| < 1$. As we will see later, **with probability that tends to 1 as $N \to \infty$ (*asymptotically almost surely* – a.a.s.)** there are no vertices of type much larger than $\frac{\zeta}{2\alpha}R$. Hence, since $t_u < C$, if N is sufficiently large, then we have $\frac{\zeta}{2\alpha}R < (1-|\delta|)R - t_u - c_0$. By Lemma 1.1, the probability that a vertex v has type at most $\frac{\zeta}{2\alpha}R$ and it is adjacent to u (that is, its hyperbolic distance from u is at most $(1+\delta)R$) is proportional to $e^{\frac{\zeta}{2}(t_u+t_v-(1-\delta)R)}/\pi$. If we average this over t_v we obtain

$$\begin{aligned}
\Pr\left[u \sim v | t_u\right] &\asymp \frac{e^{\zeta t_u/2}}{e^{\frac{\zeta}{2}(1-\delta)R}} \int_0^{\frac{\zeta}{2\alpha}R} e^{\zeta t_v/2} \frac{\alpha \sinh(\alpha(R-t_v))}{\cosh(\alpha R) - 1} dt_v \\
&\lesssim \frac{e^{\zeta t_u/2}}{e^{\frac{\zeta}{2}(1-\delta)R}} \int_0^{R} e^{\zeta t_v/2} \ \frac{e^{\alpha(R-t_v)}}{\cosh(\alpha R) - 1} dt_v \\
&\asymp \frac{e^{\zeta t_u/2}}{e^{\frac{\zeta}{2}(1-\delta)R}} \int_0^{R} e^{(\zeta/2-\alpha)t_v} dt_v \overset{\zeta/\alpha<2}{\asymp} \frac{e^{\zeta t_u/2}}{N^{1-\delta}} \overset{\delta<0}{=} o\left(\frac{1}{N}\right).
\end{aligned}$$

Hence, the probability that there is such a vertex is $o(1)$. In other words, most vertices will have no neighbors.

A similar calculation can actually show that the above probability is $\Omega\left(\frac{e^{\zeta t_u/2}}{N^{1-\delta}}\right)$. Thereby, if $0 < \delta < 1$, then the expected degree of u is of order $N^{1-\delta} \gg \frac{1}{N}$. A more detailed argument can show that the resulting random graph is too dense in the sense that the number of edges is *no longer proportional* to the number of vertices but grows much faster than that.

As a final comment on the type of a vertex, we will show in Lemma 2.1 that the definition of the density function ρ implies that the type of a vertex is approximately exponentially distributed. More precisely, the probability that the type of a vertex exceeds t is approximately equal to $e^{-\alpha t}$. As we shall see, this implies **that a.a.s.** all vertices have their types less than $\frac{\zeta}{2\alpha} R + \omega(N)$, where $\omega(N) \to \infty$ as $N \to \infty$ (cf. Corollary 2.2). A weaker implication of the exponential law is that for any $\varepsilon > 0$ there exists a C such that all but an ε fraction of the vertices have type at most C. That is, all but a small fraction of the vertices have their type bounded by C.

1.3 Results

The main results of this paper describe the degree of a typical vertex of $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$ **when N is large** . For a vertex $u \in V_N$ we let D_u denote the degree of u in $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$. Let F be a cumulative distribution function and let X be a non-negative random variable whose distribution is F. We say that a random variable D follows the *mixed Poisson distribution with mixing distribution F*, if D follows the Poisson distribution with parameter equal to the random variable X. We denote this distribution by $\mathsf{MP}(F)$. If Y_N is a non-negative random variable parameterized by N that takes integral values, we write that $Y_N \xrightarrow{d} D$ to denote that as $N \to \infty$ we have $\Pr[Y_N = k] \to \Pr[D = k]$, for every integer $k \geq 0$.

Theorem 1.2. *If $\beta > 1$ and $\zeta/\alpha < 2$, then*

$$D_u \xrightarrow{d} \mathsf{MP}(F),$$

where $\mathsf{MP}(F)$ denotes a random variable that follows the mixed Poisson distribution with mixing distribution F such that

$$F(t) = 1 - \left(\frac{K}{t}\right)^{2\alpha/\zeta},$$

for any $t \geq K$, where $K = \frac{4\alpha\nu}{2\alpha - \zeta}\ \frac{1}{\beta}\ \csc\left(\frac{\pi}{\beta}\right)$ and $F(t) = 0$ otherwise.

In particular, as k and N grow

$$\Pr\Big[\ D_u = k\ \Big] = \frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta}\ k^{-(2\alpha/\zeta + 1)} + o(1),$$

that is, the degree of vertex u follows a power law with exponent $2\alpha/\zeta + 1$.

We also show a law of large numbers for the fraction of vertices of any given degree in $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$. We write that the random variable $Y_N \sim a$, where a is a real number to denote that for any $\varepsilon > 0$ with probability $\to 1$, as $N \to \infty$, we have $Y_N = (1 \pm \varepsilon)a$.

Theorem 1.3. *For any k let N_k denote the number of vertices of degree k in $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$. Let $\beta > 1$ and $\zeta/\alpha < 2$. For any fixed integer $k \geq 0$, we have*

$$\frac{N_k}{N} \sim \Pr\Big[\mathsf{MP}(F) = k\Big].$$

The distribution of the degree of a given vertex changes abruptly when $\beta \leq 1$.

It is not hard to see that when $\beta < 1$, the random graph $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$ stochastically contains a $\mathcal{G}(N,\nu^\beta/2N^\beta)$ Erdős-Rényi random graph. Indeed, for any two distinct points $u, v \in V_N$ if r_u, r_v denote their radii in $\mathcal{D}_R$, then $d(u,v) \leq r_u + r_v$ – this follows from the triangle inequality. Thereby, we have

$$\beta\,\frac{\zeta}{2}\,(d(u,v) - R) \leq \beta\,\frac{\zeta}{2}\,(r_u + r_v - R) = \beta\,\frac{\zeta}{2}(R - t_u - t_v).$$

Hence, if $R - t_u - t_v \geq 0$, then

$$p_{u,v} \geq \frac{1}{\exp\left(\beta\,\frac{\zeta}{2}(R - t_u - t_v)\right) + 1} \geq \frac{\nu^\beta}{2}\,\left(\frac{e^{\frac{\zeta}{2}(t_u+t_v)}}{N}\right)^\beta.$$

If $R - t_u - t_v < 0$, then $p_{u,v} \geq 1/2$. Hence**, in general,** for any two distinct $u, v \in V_N$, the probability that they are joined is

$$p_{u,v} \geq \frac{1}{2}\,\left\{\left(\nu\,\frac{e^{\frac{\zeta}{2}(t_u+t_v)}}{N}\right)^\beta \wedge 1\right\} \geq \frac{\nu^\beta}{2N^\beta},$$

for N that is large enough. Hence, if $\beta < 1$, the random graph $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$ is a.a.s. connected and, moreover, it has bounded diameter (cf. Theorem 10.10 in [3]).

When $\beta = 1$, our analysis implies that the expected degree of a vertex u of type t_u is proportional to $(R - t_u)e^{\zeta t_u/2}$. Hence, most vertices will have degree that is proportional to R. This will become more explicit at the end of Section 3.

As we stated above, the proof of these results relies on showing that $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$ locally behaves as the Chung-Lu model **when N is large**. This is made precise in Lemma 2.4. In particular, when $\beta > 1$, we show that the probability that two vertices u, v are adjacent is proportional to $\frac{e^{\zeta t_u/2}e^{\zeta t_v/2}}{N}$. Similar expressions, but with different scaling in terms of N, are derived for $\beta \leq 1$ (cf. Lemma 2.4).

1.4 Outline

We begin our analysis in Section 2 with some preliminary results which will be used throughout our proofs. In that section also, we prove the key Lemma 2.4. In Section 3, we prove Theorem 1.2 and we continue in Section 4 with the analysis of the asymptotic correlation of the degrees of finite collections of vertices for $\beta > 1$ and $\zeta/\alpha < 2$ and the proof of Theorem 4.1. Its proof immediately yields

Theorem 1.3 with the use of Chebyschev's inequality.

2 Preliminaries

The next lemma shows that the type of a vertex is essentially exponentially distributed.

Lemma 2.1. *For any* $v \in V_N$ *we have*

$$\Pr\Big[t_v \le x \Big] = 1 - e^{-\alpha x} + O\left(N^{-2\alpha/\zeta}\right),$$

uniformly for all $0 \le x \le R$.

Proof. We use the definition of $\rho(r)$ and write

$$\Pr\Big[t_v \le x \Big] = \alpha \int_{R-x}^{R} \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} dr = \frac{\cosh(\alpha R) - \cosh(\alpha(R-x))}{\cosh(\alpha R) - 1}.$$

We set $u = e^{-\alpha R}$ and $t = e^{-\alpha x}$, whereby the above expression can be re-written as

$$\Pr\Big[t_v \le x \Big] = \frac{u + 1/u - t/u - u/t}{1/u + u - 2}.$$

The latter expression can be further written as follows:

$$\begin{aligned}
\frac{u + 1/u - t/u - u/t}{1/u + u - 2} &= 1 - \frac{t/u + u/t - 2}{1/u + u - 2} \\
&= 1 - \frac{t/u + tu - 2t - tu + 2t + u/t - 2}{1/u + u - 2} \\
&= 1 - t + \frac{tu - 2t - u/t + 2}{1/u + u - 2}.
\end{aligned}$$

For N large enough the last fraction is at most

$$2u(tu + 2) \le 6u,$$

as $tu \le 1$ But $u = O\left(N^{-2\alpha/\zeta}\right)$, whereby

$$\Pr\Big[t_v \le x \Big] = 1 - e^{-\alpha x} + O\left(N^{-2\alpha/\zeta}\right). \tag{2.1}$$

□

Now, let us set $x_0 = \frac{\zeta}{2\alpha} R + \omega(N)$ where $\omega(N) \to \infty$ as $N \to \infty$ but $\omega(N) = o(R)$. The above lemma immediately yields the following corollary.

Corollary 2.2. *If* $\zeta/\alpha < 2$, *then a.a.s all vertices* $v \in V_N$ *have* $t_v \le x_0$.

Proof. Note that $x_0 = \frac{1}{\alpha}\ \log N - \frac{1}{\alpha}\ \log \nu + \omega(N)$. For a vertex $v \in V_N$ applying Lemma 2.1, we have

$$\Pr\Big[t_v > x_0 \Big] = e^{-\alpha x_0} + O\left(N^{-2\alpha/\zeta}\right) = o\left(\frac{1}{N}\right),$$

since $\zeta/\alpha < 2$. The corollary follows from Markov's inequality. $\square$

We will also need an estimate on the distance between two points in the case their relative angle is not too small (this is the typical case).

Lemma 2.3. *Assume that $0 < \zeta/\alpha < 2$ and let $h : \mathbb{N} \to \mathbb{R}^+$ such that $h(N) \to \infty$ as $N \to \infty$. Let u, v be two distinct points in $\mathcal{D}_R$ such that $t_u, t_v \le x_0$, with $\theta_{u,v}$ denoting their relative* **angle**. *Let also $\hat{\theta}_{u,v} := \left(e^{-2\zeta(R-t_u)} + e^{-2\zeta(R-t_v)}\right)^{1/2}$. If*

$$h(N)\hat{\theta}_{u,v} \le \theta_{u,v} \le \pi,$$

then

$$d(u,v) = 2R - (t_u + t_v) + \frac{2}{\zeta}\ \log \sin\left(\frac{\theta_{u,v}}{2}\right) + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right),$$

uniformly for all u, v with $R - t_u - t_v \ge h(N)$.

Proof. Let us set first $r_u = R - t_u$ and $r_v = R - t_v$ - these are the radii of u and v, respectively, in $\mathcal{D}_R$. We begin with the hyperbolic law of cosines:

$$\cosh(\zeta d(u,v)) = \cosh(\zeta r_u)\cosh(\zeta r_v) - \sinh(\zeta r_u)\sinh(\zeta r_v)\cos(\theta_{u,v}).$$

Since $R - t_u - t_v \ge h(N)$, it follows that both $r_u, r_v \to \infty$ as $N \to \infty$. Thus the right-hand side of the above becomes:

$$\begin{aligned}
&\cosh(\zeta r_u)\cosh(\zeta r_v) - \sinh(\zeta r_u)\sinh(\zeta r_v)\cos(\theta_{u,v}) = \\
&\frac{e^{\zeta(r_u+r_v)}}{4}\Big(\left(1 + e^{-2\zeta r_u}\right)\left(1 + e^{-2\zeta r_v}\right) \\
&\qquad\qquad - \left(1 - e^{-2\zeta r_u}\right)\left(1 - e^{-2\zeta r_v}\right)\cos(\theta_{u,v})\Big) \qquad (2.2)\\
&= \frac{e^{\zeta(r_u+r_v)}}{4}\Big(1 - \cos(\theta_{u,v}) + (1 + \cos(\theta_{u,v}))\left(e^{-2\zeta r_u} + e^{-2\zeta r_v}\right) \\
&\qquad\qquad + O\left(e^{-2\zeta(r_u+r_v)}\right)\Big).
\end{aligned}$$

By the convexity of the function $e^{-2\zeta x}$, we have

$$e^{-\zeta(r_u+r_v)} = e^{-2\zeta\frac{r_u+r_v}{2}} \le \frac{1}{2}\left(e^{-2\zeta r_u} + e^{-2\zeta r_v}\right) \le \hat{\theta}^2_{u,v}. \qquad (2.3)$$

Thus, the previous estimate can be written as

$$\cosh(\zeta d(u,v)) = \frac{e^{\zeta(r_u+r_v)}}{4}\left(1-\cos(\theta_{u,v}) + (1+\cos(\theta_{u,v}))\,\hat{\theta}_{u,v}^2 + O\left(\hat{\theta}_{u,v}^4\right)\right).$$

If $\theta_{u,v}$ is bounded away from 0, then clearly $1-\cos(\theta_{u,v})$ dominates the expression in brackets. Now assume that $\theta_{u,v} = o(1)$. It is a basic trigonometric identity that $1-\cos(\theta_{u,v}) = 2\sin^2\left(\frac{\theta_{u,v}}{2}\right)$. Then $1-\cos(\theta_{u,v}) = \frac{\theta_{u,v}^2}{2}(1-o(1))$. But the assumption that $\theta_{u,v} \gg \hat{\theta}_{u,v}$ again implies that also in this case $1-\cos(\theta_{u,v})$ dominates expression in brackets. Thus,

$$\begin{aligned}\cosh(\zeta d(u,v)) &= \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\ (1-\cos(\theta_{u,v}))\ \left(1+O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right)\right) \\ &= \frac{e^{\zeta(2R-(t_u+t_v))}}{2}\ \sin^2\left(\frac{\theta_{u,v}}{2}\right)\ \left(1+O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right)\right).\end{aligned} \tag{2.4}$$

Now, we take logarithms in (2.4) and divide both sides by ζ thus obtaining:

$$d(u,v) + \frac{1}{\zeta}\log\left(1-e^{-2\zeta d(u,v)}\right) = 2R - t_u - t_v + \frac{2}{\zeta}\log\sin\left(\frac{\theta_{u,v}}{2}\right) + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right). \tag{2.5}$$

We now need to give an asymptotic estimate on $e^{-2\zeta d(u,v)}$. We derive this from (2.4) as well. For N large enough, we have

$$e^{\zeta d(u,v)} \geq \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\ \sin^2\left(\frac{\theta_{u,v}}{2}\right) \geq \frac{1}{32}\ e^{\zeta(2R-(t_u+t_v))}\ \theta_{u,v}^2 \overset{(2.3)}{\geq} \frac{1}{32}\ \left(\frac{\theta_{u,v}}{\hat{\theta}_{u,v}}\right)^2,$$

from which it follows that

$$\left|\log\left(1-e^{-2\zeta d(u,v)}\right)\right| = O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^4\right).$$

Substituting this into (2.5) completes the proof of the lemma. $\square$

Let $\hat{p}_{u,v} = \frac{1}{\pi}\int_0^\pi p_{u,v}d\theta$ - this is the probability that two points u and v are connected by an edge, conditional on their types. For an arbitrarily slowly growing function $\omega : \mathbb{N} \to \mathbb{N}$ we define

$$\mathcal{D}_{R,\omega}^{(2)} = \left\{\{u,v\} \in \binom{\mathcal{D}_R}{2}\ : t_v, t_u \leq x_0,\ R - t_u - t_v \geq \omega(N)\right\}.$$

Also, we let $\bar{\mathcal{D}}_{R,\omega}^{(2)}$ denote the complement of this set in $\mathcal{D}_R$. For two points $u, v \in$

$\mathcal{D}_R$ we set $A(t_u, t_v) = \exp\left(\frac{\zeta}{2}\left(R - (t_u + t_v)\right)\right)$. The following lemma gives an asymptotic estimate on $\hat{p}_{u,v}$, for all values of β, in terms of $A(t_u, t_v)$, whenever $\{u, v\} \in \mathcal{D}^{(2)}_{R,\omega}$.

Lemma 2.4. *Let $\beta > 0$. There exists a constant $C_\beta > 0$ such that uniformly for all $u, v \in \mathcal{D}^{(2)}_{R,\omega}$ we have*

$$\hat{p}_{uv} = \begin{cases} (1+o(1))\frac{C_\beta}{A_{u,v}}\ , & \text{if } \beta > 1 \\ \\ (1+o(1))\frac{C_\beta \ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \\ \\ (1+o(1))\frac{C_\beta}{A^\beta_{u,v}}, & \text{if } \beta < 1 \end{cases}.$$

In particular,

$$C_\beta = \begin{cases} \frac{2}{\beta}\ \csc\left(\frac{\pi}{\beta}\right), & \text{if } \beta > 1 \\ \frac{2}{\pi}, & \text{if } \beta = 1 \\ \frac{1}{\sqrt{\pi}}\ \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(1-\frac{\beta}{2}\right)}, & \text{if } \beta < 1 \end{cases}.$$

Proof. Throughout this proof we write $A_{u,v}$ for $A(t_u, t_v)$. Recall that

$$p_{u,v} = \frac{1}{\exp\left(\beta\ \frac{\zeta}{2}(d(u,v) - R)\right) + 1}.$$

We will estimate the integral of $p_{u,v}$ over $\theta_{u,v}$. When $\theta_{u,v}$ is within the range given in Lemma 2.3 we will use the estimate given there. In particular, we shall define $\tilde{\theta}_{u,v} \gg \hat{\theta}_{u,v}$ and split the integral into two parts, namely when $0 \le \theta_{u,v} < \tilde{\theta}_{u,v}$ and when $\tilde{\theta}_{u,v} \le \theta_{u,v} \le \pi$. The parameter $\tilde{\theta}_{u,v}$ is close to $A^{-1}_{u,v}$. Note that the following holds.

Claim 2.5. *If $R - t_u - t_v \ge \omega(N)$, then*

$$A^{-1}_{u,v} \gg \hat{\theta}_{u,v}.$$

We postpone the proof of this claim until later.

Let $\omega(N)$ be slowly enough growing so that in the following definition of $\tilde{\theta}_{u,v}$, we have $\tilde{\theta}_{u,v} \gg \hat{\theta}_{u,v}$, when $\beta \ge 1$, and $\tilde{\theta}_{u,v} = o(A^{-\beta}_{u,v})$, when $\beta < 1$. We set

$$\tilde{\theta}_{u,v} = \begin{cases} \frac{A^{-1}_{u,v}}{\omega(N)}, & \text{if } \beta \ge 1 \\ \omega(N) A^{-1}_{u,v}, & \text{if } \beta < 1 \end{cases}.$$

Thus when $\tilde{\theta}_{u,v} \leq \theta_{u,v} \leq \pi$, we use Lemma 2.3 and write

$$\exp\left(\beta \ \frac{\zeta}{2}(d(u,v) - R)\right) = C \ e^{\beta \frac{\zeta}{2}(R-(t_u+t_v))+\beta \ \log\sin(\theta_{u,v}/2)}, \tag{2.6}$$

where $C = 1 + O\left(\left(\frac{\hat{\theta}_{u,v}}{\tilde{\theta}_{u,v}}\right)^2\right)$. By Claim 2.5 and the choice of the function $\omega(N)$, we have that $C = 1 + o(1)$.

We decompose the integral that gives $\hat{p}_{u,v}$ into two parts which we bound separately.

$$\hat{p}_{u,v} = \frac{1}{\pi} \int_0^{\pi} p_{u,v} d\theta = \frac{1}{\pi} \int_0^{\tilde{\theta}_{u,v}} p_{u,v} d\theta + \frac{1}{\pi} \int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta. \tag{2.7}$$

The first integral can bounded trivially as follows:

$$\int_0^{\tilde{\theta}_{u,v}} p_{u,v} d\theta \leq \tilde{\theta}_{u,v} = \begin{cases} o\left(A_{u,v}^{-1}\right), & \text{if } \beta \geq 1 \\ o\left(A_{u,v}^{-\beta}\right), & \text{if } \beta < 1 \end{cases}. \tag{2.8}$$

We now focus on the second integral in (2.7). We will treat the cases $\beta < 1$ and $\beta \geq 1$ separately, starting with the former one.

$\beta < 1$

Recall that $\tilde{\theta}_{u,v}$ is such that $\tilde{\theta}_{u,v} \gg A_{u,v}^{-1}$. Thus, we write

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta = \int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{C A_{u,v}^{\beta} \sin^{\beta}\left(\frac{\theta}{2}\right) + 1} d\theta = \frac{1+o(1)}{A_{u,v}^{\beta}} \int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right) + \Theta\left(A_{u,v}^{-\beta}\right)} d\theta$$
$$= \frac{1+o(1)}{A_{u,v}^{\beta}} \int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right)} d\theta = \frac{1+o(1)}{A_{u,v}^{\beta}} \int_0^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right)} d\theta. \tag{2.9}$$

Substituting the estimates of (2.8) and (2.9) into (2.7) we obtain

$$\hat{p}_{u,v} = (1+o(1)) \ \frac{1}{\pi} \ \left(\int_0^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right)} d\theta\right) \ \frac{1}{A_{u,v}^{\beta}}, \tag{2.10}$$

uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$. Finally, note that

$$\int_0^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right)} d\theta = 2\int_0^{\pi/2} \frac{1}{\sin^{\beta}(\theta)} d\theta = \sqrt{\pi} \ \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(1-\frac{\beta}{2}\right)}.$$

$\beta \geq 1$

We use the inequality $\sin\theta \leq \theta$, which holds for all $\theta \in [0, \pi]$, and obtain an upper bound on the right-hand side of (2.6).

$$\exp\left(\beta\ \frac{\zeta}{2}(d(u,v)-R)\right) \leq C\ e^{\beta\frac{\zeta}{2}(R-(t_u+t_v))}\ \left(\frac{\theta_{u,v}}{2}\right)^{\beta} = C\ A_{u,v}^{\beta}\ \left(\frac{\theta_{u,v}}{2}\right)^{\beta}.$$

Using this bound we can bound the second integral in (2.7) from below as follows.

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v}d\theta \geq \int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{C\ A_{u,v}^{\beta}\ \left(\frac{\theta}{2}\right)^{\beta}+1}d\theta. \tag{2.11}$$

We perform a change of variable setting $z = C^{1/\beta}\ A_{u,v}\ \frac{\theta}{2}$. Thus with $C' = C^{1/\beta}/2$ the integral on the right-hand side of (2.11) becomes

$$\int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{C\ A_{u,v}^{\beta}\ \left(\frac{\theta}{2}\right)^{\beta}+1}d\theta = \frac{1}{C'}\ \frac{1}{A_{u,v}}\ \int_{C'A_{u,v}\tilde{\theta}_{u,v}}^{C'\pi A_{u,v}} \frac{1}{z^{\beta}+1}dz. \tag{2.12}$$

We now provide an estimate for the integral on the right-hand side of (2.12) for any $\beta \geq 1$ - its proof is elementary and we omit it.

Claim 2.6. *Let $g_1(N)$ and $g_2(N)$ be non-negative real-valued functions on the set of natural numbers, such that $g_1(N) \to 0$ and $g_2(N) \to \infty$ as $N \to \infty$. We have*

$$\int_{g_1(N)}^{g_2(N)} \frac{1}{z^{\beta}+1}dz = \begin{cases} (1+o(1))\int_0^{\infty}\frac{1}{z^{\beta}+1}dz, & \text{if } \beta > 1 \\ \\ (1+o(1))\ln g_2(N), & \text{if } \beta = 1 \end{cases}.$$

We take $g_1(N) = C'A_{u,v}\tilde{\theta}_{u,v} = C'/\omega(N) \to 0$ and $g_2(N) = C'\pi A_{u,v} \to \infty$, since $u, v \in \mathcal{D}_{R,\omega}^{(2)}$, and we obtain through (2.11):

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v}d\theta \geq \begin{cases} (1+o(1))\ \frac{2}{C^{1/\beta}}\ \frac{1}{A_{u,v}}\ \int_0^{\infty}\frac{1}{z^{\beta}+1}dz, & \text{if } \beta > 1 \\ \\ (1+o(1))\ \frac{2}{C^{1/\beta}}\ \frac{\ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \end{cases}. \tag{2.13}$$

To deduce the upper bound we will split the integral into two parts. For an $\varepsilon \in (0,\pi)$, we write

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v}d\theta = \int_{\tilde{\theta}_{u,v}}^{\varepsilon} p_{u,v}d\theta + \int_{\varepsilon}^{\pi} p_{u,v}d\theta. \tag{2.14}$$

We will bound each one of the integrals on the right-hand side separately.

For the first integral we will use (2.6) together with the bound $\sin\theta \geq \theta - \theta^3$, which holds for any $\theta \leq \varepsilon$, provided that the latter is sufficiently small. More specifically, we let $\varepsilon = \varepsilon(N) = 1/\omega(N)$; so $A_{u,v}\varepsilon(N) \to \infty$ as $N \to \infty$. We shall also use $(1-\theta^2)^{\beta} \geq 1 - \beta\theta^2$. Thus (after a change of variable where we replace

$\theta/2$ by θ) for sufficiently large N, we have

$$\int_{\tilde{\theta}_{u,v}}^{\varepsilon} p_{u,v} d\theta \leq 2\int_{\tilde{\theta}_{u,v}/2}^{\varepsilon/2} \frac{1}{CA_{u,v}^{\beta}(\theta-\theta^3)^{\beta}+1} d\theta \leq 2\int_{\tilde{\theta}_{u,v}/2}^{\varepsilon/2} \frac{1}{CA_{u,v}^{\beta}\theta^{\beta}(1-\beta\theta^2)+1} d\theta$$
$$\leq 2\int_{\tilde{\theta}_{u,v}/2}^{\varepsilon/2} \frac{1}{CA_{u,v}^{\beta}\theta^{\beta}(1-\beta\varepsilon^2/4)+1} d\theta.$$

We change the variable in the last integral setting $z = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\theta$. Thus, we obtain:

$$\int_{\tilde{\theta}_{u,v}}^{\hat{\theta}'_{u,v}} p_{u,v} d\theta \leq \frac{2}{\left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}} \int_{B_1}^{B_2} \frac{1}{z^{\beta}+1} dz,$$

where $B_1 = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\tilde{\theta}_{u,v}/2$ and $B_2 = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\varepsilon/2$. We have $B_1 = o(1)$ whereas, $B_2 = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\varepsilon/2 \to \infty$. So, Claim 2.6 yields:

$$\int_{\tilde{\theta}_{u,v}}^{\varepsilon} p_{u,v} d\theta \leq \begin{cases} (1+o(1)) \ \frac{2}{A_{u,v}} \ \int_0^{\infty} \frac{1}{z^{\beta}+1} dz, & \text{if } \beta > 1 \\ \\ (1+o(1)) \ \frac{2\ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \end{cases}, \tag{2.15}$$

uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$. The second integral in (2.14) can be bounded easily.

$$\int_{\varepsilon}^{\pi} p_{u,v} d\theta = \int_{\varepsilon}^{\pi} \frac{1}{CA_{u,v}^{\beta} \ \sin^{\beta}(\theta/2)+1} d\theta = O\left(\frac{1}{(A_{u,v}\varepsilon)^{\beta}}\right) = o(A_{u,v}^{-1}).$$

Hence, uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$ we have

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta \leq \begin{cases} (1+o(1)) \ \frac{2}{A_{u,v}} \ \int_0^{\infty} \frac{1}{z^{\beta}+1} dz, & \text{if } \beta > 1 \\ \\ (1+o(1)) \ \frac{2\ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \end{cases}. \tag{2.16}$$

Thereby, (2.7) together with (2.8) and (2.13),(2.16) yield the lemma for $\beta \geq 1$. Finally, note that when $\beta > 1$ we have $\int_0^{\infty} \frac{1}{z^{\beta}+1} dz = \frac{\pi}{\beta} \csc\left(\frac{\pi}{\beta}\right)$.

We now conclude the proof of the lemma with the proof of Claim 2.5.

Proof of Claim 2.5. We will show that $A_{u,v}^{-1} \gg \hat{\theta}_{u,v}$. We will show that

$$\frac{\zeta}{2}\left(R-(t_u+t_v)\right) + \frac{1}{2}\log\left(e^{-2\zeta(R-t_u)} + e^{-2\zeta(R-t_v)}\right) \leq \frac{-\zeta\omega(N)+1}{2}. \tag{2.17}$$

Adding and subtracting R inside the brackets in the first summand, we obtain:

$$\begin{aligned}&\frac{\zeta}{2}\left(R-(t_u+t_v)\right)+\frac{1}{2}\log\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right)\\&=-\frac{\zeta R}{2}+\frac{1}{2}\left(\zeta(R-t_u)+\zeta(R-t_v)\right)+\frac{1}{2}\log\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right).\end{aligned}$$

For notational convenience, we write $a=\zeta(R-t_u)$ and $b=\zeta(R-t_v)$. Without loss of generality, assume that $a\leq b$. Thus, the above expression is now written as

$$\begin{aligned}&-\frac{\zeta R}{2}+\frac{1}{2}(a+b)+\frac{1}{2}\log\left(e^{-2a}+e^{-2b}\right)\\&=-\frac{\zeta R}{2}+\frac{1}{2}\left[b-a+\log(1+e^{-2(b-a)})\right]\leq-\frac{\zeta R}{2}+\frac{1}{2}\left[b-a+e^{-2(b-a)}\right].\end{aligned}$$

Note that $b-a=\zeta(t_u-t_v)\leq\zeta\left(R-t_v-\omega(N)-t_v\right)\leq\zeta\left(R-\omega(N)\right)$. Therefore,

$$-\frac{\zeta R}{2}+\frac{1}{2}(a+b)+\frac{1}{2}\log\left(e^{-2a}+e^{-2b}\right)\leq\frac{-\zeta\omega(N)+1}{2}\to-\infty.$$

$\square$

$\square$

3 The asymptotic distribution of the degree of a vertex

In this section, we prove Theorem 1.2. Let us fix some $u\in V_N$. We will condition on the position of u in $\mathcal{D}_R$; in particular, we will condition on t_u and the angle θ_u. In this section as well as later, we will be writing $\rho(t)$ for $\alpha\sinh(\alpha(R-t))/(\cosh(\alpha R)-1)$ – this is the density function of the type of a vertex. The re-use of notation at this point should be causing no confusion.

Let us fix first t_u and θ_u such that $t_u\leq x_0$ and $\theta_u\in(0,2\pi]$. (Recall that by Corollary 2.2 we have that a.a.s. $t_u\leq x_0=\zeta R/(2\alpha)+\omega(N)$ for all $u\in V_N$.) Denoting by V_N^u the set $V_N\setminus\{u\}$, we write $D_u=\sum_{v\in V_N^u}I_{uv}$, where I_{uv} is the indicator random variable that is equal to 1 if and only if the edge $\{u,v\}$ is present in $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$. Note that conditional on t_u and θ_u the family $\{I_{uv}\}_{v\in V_N^u}$ is a family of independent and identically distributed random variables.

We begin with the estimation of the expectation of I_{uv} for an arbitrary $v\in V_N^u$ conditional on t_u and θ_u.

Lemma 3.1. *Let $\beta>1$ and $\zeta/\alpha<2$. Let also $u,v\in V_N$ be two distinct vertices. Then uniformly for all $t_u\leq x_0$ and $\theta_u\in(0,2\pi]$, we have*

$$\Pr\Big[I_{uv}=1\mid t_u,\theta_u\Big]\sim K\,\frac{e^{\zeta t_u/2}}{N},$$

where $K = \frac{4\alpha\nu}{2\alpha-\zeta}\ \frac{1}{\beta}\ \csc\left(\frac{\pi}{\beta}\right)$.

Proof. We write $\hat{p}(t_u, t_v)$ for $\hat{p}_{u,v}$ as the latter depends only on t_u and t_v. Hence, we have

$$\begin{aligned}\Pr\Big[I_{uv} = 1 \mid t_u, \theta_u \Big] &= \int_0^R \hat{p}(t_u, t_v)\rho(t_v)dt_v \\ &= \int_0^{R-t_u-\omega(N)} \hat{p}(t_u, t_v)\rho(t_v)dt_v + \int_{R-t_u-\omega(N)}^{R} \hat{p}(t_u, t_v)\rho(t_v)dt_v.\end{aligned} \tag{3.1}$$

The second integral can be bounded as follows.

$$\begin{aligned}&\int_{R-t_u-\omega(N)}^{R} \hat{p}(t_u, t_v)\rho(t_v)dt_v \leq \int_{R-t_u-\omega(N)}^{R} \rho(t_v)dt_v = \frac{\cosh(\alpha(t_u + \omega(N))) - \cosh(0)}{\cosh(\alpha R) - 1} \\ &\sim \frac{e^{\alpha(t_u+\omega(N))}}{\cosh(\alpha R) - 1} \sim \nu^{2\alpha/\zeta}\ \frac{e^{\alpha(t_u+\omega(N))}}{N^{2\alpha/\zeta}} \sim e^{\alpha\omega(N)} \left(\nu\ \frac{e^{\zeta t_u/2}}{N}\right)^{2\alpha/\zeta}.\end{aligned} \tag{3.2}$$

For the first integral we use the estimates obtained in Lemma 2.4. We set $K_1 := (1 + o(1))C_\beta$.
We have:

$$\begin{aligned}&\int_0^{R-t_u-\omega(N)} \hat{p}(t_u, t_v)\rho(t_v)dt_v = K_1 \int_0^{R-t_u-\omega(N)} \frac{1}{A(t_u, t_v)}\ \rho(t_v)dt_v \\ &= K_1 e^{\zeta t_u/2} \int_0^{R-t_u-\omega(N)} e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v \\ &= \alpha K_1 e^{\zeta t_u/2} \int_0^{R-t_u-\omega(N)} e^{-\frac{\zeta}{2}(R-t_v)}\ \frac{\sinh(\alpha(R - t_v))}{\cosh(\alpha R) - 1} dt_v \\ &= \alpha K_1 \frac{e^{\zeta t_u/2}}{\cosh(\alpha R) - 1} \int_{t_u+\omega(N)}^{R} e^{-\frac{\zeta t_v}{2}}\ \sinh(\alpha t_v)dt_v.\end{aligned} \tag{3.3}$$

Note that uniformly for $\omega(N) \leq t_v \leq R$ we have $\sinh(\alpha t_v) = (1 - o(1))e^{\alpha t_v}/2$. Hence the last expression in (3.3) becomes

$$\begin{aligned}&(1 + o(1))\frac{\alpha K_1}{2} \frac{e^{\zeta t_u/2}}{\cosh(\alpha R) - 1} \int_{t_u+\omega(N)}^{R} e^{-\left(\frac{\zeta}{2}-\alpha\right)t_v} dt_v \\ &\overset{\zeta/\alpha<2, t_u\leq x_0}{\sim} (1 + o(1))\frac{2\alpha}{2\alpha - \zeta} K_1 \frac{e^{\zeta t_u/2}}{e^{\alpha R}} e^{-(\zeta/2-\alpha)R} \\ &\sim \frac{2\alpha}{2\alpha - \zeta} \frac{e^{\zeta t_u/2}}{e^{\zeta R/2}} \sim \frac{2\alpha\nu}{2\alpha - \zeta} \frac{e^{\zeta t_u/2}}{N} \\ &\sim K\ \nu\ \frac{e^{\zeta t_u/2}}{N}.\end{aligned}$$

Thus the lemma follows. $\square$

It is clear that the proof of the above lemma yields also the following corollary.

Corollary 3.2. *Let $\beta > 1$ and $0 < \zeta/\alpha < 2$. Then uniformly for all $t_u \leq x_0$, we have*

$$\Pr\Big[I_{uv} = 1 \mid t_u \Big] \sim K \ \frac{e^{\zeta t_u/2}}{N},$$

where K is as in Lemma 3.1.

We close this section with a remark on the $\beta = 1$ case. We use again Lemma 2.4 and we have

$$\begin{aligned}
&\Pr\left[I_{uv} = 1 \mid t_u, \theta_u\right] = \\
&\quad \int_0^{R-t_u-\omega(N)} \hat{p}(t_u,t_v)\rho(t_v)dt_v = K_1 \int_0^{R-t_u-\omega(N)} \frac{\ln A(t_u,t_v)}{A(t_u,t_v)} \ \rho(t_v)dt_v \\
&\quad \asymp \int_0^{R-t_u-\omega(N)} \left(\frac{\zeta}{2}(R-t_u-t_v)\right) e^{-\frac{\zeta}{2}(R-t_u-t_v)} \ \rho(t_v)dt_v = \\
&\quad \asymp \frac{1}{\cosh(\alpha R)-1} \int_0^{R-t_u-\omega(N)} (R-t_u-t_v)\, e^{-\frac{\zeta}{2}(R-t_u-t_v)} \ \sinh(\alpha(R-t_v))dt_v.
\end{aligned} \tag{3.4}$$

As above, uniformly for $0 \leq t_v \leq R - t_u - \omega(N)$ we have $\sinh(\alpha(R-t_v)) = (1-o(1))e^{\alpha(R-t_v)}/2$. Hence the last expression in (3.4) becomes

$$\begin{aligned}
&\asymp \frac{1}{\cosh(\alpha R)} \int_0^{R-t_u-\omega(N)} (R-t_u-t_v)\, e^{-\frac{\zeta}{2}(R-t_u-t_v)+\alpha(R-t_v)}dt_v \\
&= \frac{e^{\alpha t_u}}{\cosh(\alpha R)} \int_0^{R-t_u-\omega(N)} (R-t_u-t_v)\, e^{-\frac{\zeta}{2}(R-t_u-t_v)+\alpha(R-t_u-t_v)}dt_v \\
&= \frac{e^{\alpha t_u}}{\cosh(\alpha R)} \int_0^{R-t_u-\omega(N)} (R-t_u-t_v)\, e^{\left(\alpha-\frac{\zeta}{2}\right)(R-t_u-t_v)}dt_v \\
&= \frac{e^{\alpha t_u}}{\cosh(\alpha R)} \int_{\omega(N)}^{R-t_u} x e^{\left(\alpha-\frac{\zeta}{2}\right)x}dx \\
&\asymp e^{-\alpha(R-t_u)} \int_{\omega(N)}^{R-t_u} x e^{\left(\alpha-\frac{\zeta}{2}\right)x}dx \asymp e^{-\alpha(R-t_u)}(R-t_u)e^{\left(\alpha-\frac{\zeta}{2}\right)(R-t_u)} \asymp (R-t_u)\frac{e^{\zeta t_u/2}}{N}.
\end{aligned}$$

Therefore, the expected degree of a vertex u is $(R-t_u)e^{\zeta t_u/2}$. In particular, if t_u is bounded by a constant, the expected degree grows logarithmically in N.

3.1 The asymptotic distribution of D_u

Let $\hat{D}_u$ be a Poisson random variable with parameter equal to $T_u := \sum_{v\in V_N^u} \Pr\Big[I_{uv} = 1 \mid t_u \Big] \sim K e^{\zeta t_u/2}$. Recall that the *total variation distance* $d_{TV}(X_1, X_2)$ between two non-negative discrete random variables is defined as $\sum_{k=0}^{\infty} |\Pr[X_1 = k] = \Pr[X_2 = k]|$. We prove the following lemma.

Lemma 3.3. *Conditional on the value of t_u, we have*

$$d_{TV}\left(D_u, \hat{D}_u\right) = o(1),$$

uniformly for all $t_u \leq R/2 - \omega(N)$.

Proof. Recall that $D_u = \sum_{v \in V_N^u} I_{uv}$ and conditional on t_u and θ_u this is a sum of independent and identically distributed indicator random variables. Keeping t_u fixed, this is also the case when we average over $\theta_u \in (0, 2\pi]$. Thus conditioning only on t_u, the family $\{I_{uv}\}_{v \in V_N^u}$ is still a family of i.i.d. indicator random variables, whose expected values are given by Lemma 3.1.

By Theorem 2.9 in [13] and Corollary 3.2 we have for N sufficiently large

$$\begin{aligned} d_{TV}(D_u, \hat{D}_u) &\leq \sum_{v \in V_N^u} \Pr\Big[I_{uv} = 1 \mid t_u \Big]^2 \leq 2KN \left(\frac{e^{\zeta R/4 - \zeta \omega(N)/2}}{N} \right)^2 \\ &= 2N e^{-\zeta \omega(N)} \ \frac{1}{N} = o(1). \end{aligned}$$

□

For any integer $k \geq 0$ we have

$$\begin{aligned} \Pr\Big[D_u = k \Big] &= \frac{1}{2\pi} \int_0^R \int_0^{2\pi} \Pr\Big[D_u = k \mid t_u, \theta_u \Big] \rho(t_u) dt_u d\theta_u \\ &= \frac{1}{2\pi} \int_{t_u \leq R/2 - \omega(N)} \int_0^{2\pi} \Pr\Big[D_u = k \mid t_u, \theta_u \Big] \rho(t_u) dt_u d\theta_u \\ &+ \frac{1}{2\pi} \int_{t_u > R/2 - \omega(N)} \int_0^{2\pi} \Pr\Big[D_u = k \mid t_u, \theta_u \Big] \rho(t_u) dt_u d\theta_u. \end{aligned} \tag{3.5}$$

We bound the second integral as follows.

$$\int_{R/2 - \omega(N) < t_u} \Pr\Big[D_u = k \mid t_u \Big] \rho(t_u) dt_u \leq \int_{R/2 - \omega(N) < t_u} \rho(t_u) dt_u = o(1). \tag{3.6}$$

We will use Lemma 3.3 to approximate the first integral.

$$\begin{aligned} \int_{t_u \leq R/2 - \omega(N)} \Pr\Big[D_u = k \mid t_u \Big] \rho(t_u) dt_u &= \int_{t_u \leq R/2 - \omega(N)} \Pr\Big[\hat{D}_u = k \Big] \rho(t_u) dt_u + o(1) \\ &= \int_{t_u \leq R} \Pr\Big[\hat{D}_u = k \Big] \rho(t_u) dt_u + o(1). \end{aligned} \tag{3.7}$$

But recall that $\Pr\Big[\hat{D}_u = k\Big] = \Pr\Big[\mathsf{Po}\,(T_u) = k\Big]$. Let $K_N = (1+o(1))K$ denote the factor of $e^{\zeta t_u/2}$ in the expression of T_u. If $t < K$, then $\Pr\Big[T_u \leq t\Big] \to 0$ as $N \to \infty$. However, for any $t \geq K$ we have

$$\Pr\Big[T_u \leq t\Big] = \Pr\Big[t_u \leq \frac{2}{\zeta}\ \ln\frac{t}{K_N}\Big] = \alpha \int_0^{\frac{2}{\zeta}\ \ln\frac{t}{K_N}} \frac{\sinh(\alpha(R-x))}{\cosh(\alpha R)-1}dx$$
$$= \alpha \int_{R-\frac{2}{\zeta}\ \ln\frac{t}{K_N}}^{R} \frac{\sinh(\alpha x)}{\cosh(\alpha R)-1}dx$$
$$= \frac{1}{\cosh(\alpha R)-1}\left[\cosh(\alpha R) - \cosh\left(\alpha R - \frac{2\alpha}{\zeta}\ \ln\frac{t}{K_N}\right)\right] = 1 - \left(\frac{K}{t}\right)^{\frac{2\alpha}{\zeta}} + o(1).$$

In other words,

$$\Pr\Big[T_u \leq t\Big] \to F(t),\ \text{ as } N \to \infty.$$

Thus,

$$\int_{t_u \leq R/2-\omega(N)} \Pr\Big[D_u = k \mid t_u\Big]\rho(t_u)dt_u \to \Pr\Big[\mathsf{MP}(F) = k\Big],\ \text{ as } N \to \infty.$$

The above together with (3.6) and (3.5) complete the proof of Theorem 1.2.

3.1.1 Power laws

We close this section with a simple calculation proving that $\Pr\Big[\mathsf{MP}(F) = k\Big]$ has power-law behaviour with exponent $2\alpha/\zeta + 1$ as k grows.

Lemma 3.4. *We have*

$$\Pr\Big[\mathsf{MP}(F) = k\Big] \to \frac{2\alpha}{\zeta} K^{2\alpha/\zeta}\ k^{-2\alpha/\zeta-1}\ \textit{ as } k, N \to \infty.$$

Proof. The pdf of $F(t)$ for $t > K$ is equal to $\frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta}/t^{-2\alpha/\zeta-1}$ and equal to 0 otherwise. Thus we have

$$\begin{aligned} \Pr\Big[\mathsf{MP}(F) = k\Big] &= \frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta} \int_K^\infty e^{-t}\ \frac{t^k}{k!}\ t^{-2\alpha/\zeta-1}dt \\ &= \frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta}\ \frac{1}{k!} \int_K^\infty e^{-t}\ t^{k-2\alpha/\zeta-1}dt \\ &= \frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta}\ \frac{1}{k!}\left(\Gamma(k-2\alpha/\zeta) - \int_0^K e^{-t}\ t^{k-2\alpha/\zeta-1}dt\right). \end{aligned} \tag{3.8}$$

Note now that the last integral is $O(K^k)$ and therefore, as $k \to \infty$, we have

$$\frac{\int_0^K e^{-t}\ t^{k+2\alpha/\zeta+1} dt}{k!} = O\left(\left(\frac{Ke}{k}\right)^k\right).$$

Now using the standard asymptotics for the Gamma function we have

$$\Gamma(k - 2\alpha/\zeta) = (1 + o(1))\sqrt{2\pi(k - 2\alpha/\zeta - 1)}\ e^{-k+2\alpha/\zeta+1}\ \ (k - 2\alpha/\zeta - 1)^{k-2\alpha/\zeta-1},$$

and also $k! = (1 + o(1))\sqrt{2\pi k}e^{-k}k^k$. Thus,

$$\begin{aligned}\frac{\Gamma(k - 2\alpha/\zeta)}{k!} &= (1 + o(1))e^{2\alpha/\zeta+1}\left(1 - \frac{2\alpha/\zeta + 1}{k}\right)^{k-2\alpha/\zeta-1}\ \ k^{-(2\alpha/\zeta+1)} \\ &= (1 + o(1))\ k^{-(2\alpha/\zeta+1)}.\end{aligned}$$

Thus, (3.8) now yields as $k \to \infty$

$$\Pr\Big[\,\mathsf{MP}(F) = k\,\Big] = \frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta}\ k^{-(2\alpha/\zeta+1)}(1 + o(1)).$$

□

4 Asymptotic correlations of degrees

In this section, we deal with the correlations of the degrees for the case where $\beta > 1$ and $\zeta/\alpha < 2$. We show that the degrees of any finite collection of vertices are asymptotically independent.

Theorem 4.1. *Let $\beta > 1$ and $\zeta/\alpha < 2$. For any integer $m \geq 2$ and for any collection of m pairwise distinct vertices $v_1, \ldots, v_m$ their degrees $D_{v_1}, \ldots, D_{v_m}$ are asymptotically independent in the sense that for any non-negative integers $k_1, \ldots, k_m$ we have*

$$\left|\Pr\Big[\,D_{v_1} = k_1, \ldots, D_{v_m} = k_m\,\Big] - \Pr\Big[\,D_{v_1} = k_1\,\Big] \cdots \Pr\Big[\,D_{v_m} = k_m\,\Big]\right| = o(1). \tag{4.1}$$

The proof of the above theorem together with Chebyschev's inequality yield the concentration of the number of vertices of any fixed degree and complete the proof of Theorem 1.3.

We proceed with the proof of Theorem 4.1. Let us fix $m \geq 2$ distinct vertices $v_1, \ldots, v_m$ and let $k_1, \ldots, k_m$ be non-negative integers. The proof of Lemma 2.4 suggests that there exists a specific region around each vertex such that if another vertex is located outside it, then the probability that the two vertices are joined becomes much less than the estimate given in Lemma 2.4. In other words, this region is where a vertex is most likely to have its neighbours in – see Figure 3.

Definition 4.2. *For a vertex $v \in V_N$, we let A_v be the set of points $\{w \in \mathcal{D}_R : t_w \le x_0, \theta_{vw} \le \min\{\pi, \tilde{\theta}'_{v,w}\}\}$, where $\tilde{\theta}'_{v,w} := \omega(N) A_{v,w}^{-1}$. We call this the* vital area *of vertex v.*

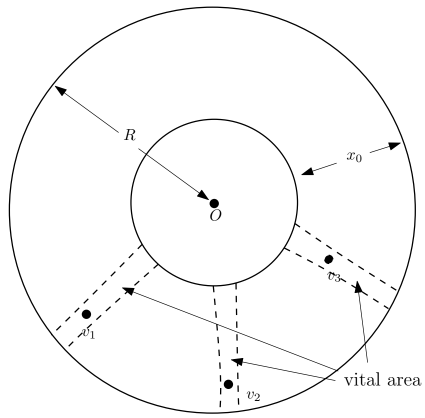

Figure 3: The vital areas of vertices in $\mathcal{D}_R$

We begin our analysis proving that the vital areas A_{v_i}, for $i = 1, \dots, m$, are mutually disjoint with high probability. We let $\mathcal{E}_1$ be this event. Though m is meant to be fixed, the following claim is also valid for m growing as a function of N.

Claim 4.3. *Let $m \ge 2$ be fixed. Then* $\Pr\left[\mathcal{E}_1\right] = 1 - o(1)$.

Proof. Let $i \in \{1, \dots, m\}$ and assume that $t_{v_i} < R(1 - \zeta/(2\alpha)) - 2\omega(N)$. Considering all points $w \in A_{v_i}$ the parameter $\tilde{\theta}'_{v_i,w}$ is maximised when $t_w = x_0$. So let $\tilde{\theta}'_{v_i}$ be this maximum, that is,

$$\tilde{\theta}'_{v_i} = \nu \, \frac{\omega(N)}{N} \exp\left(\frac{\zeta}{2}\left(t_{v_i} + x_0\right)\right) = \omega(N) \, \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \, e^{\zeta\omega(N)/2} \, e^{\zeta t_{v_i}/2}.$$

Observe that our assumption on t_{v_i} implies that

$$e^{\zeta t_{v_i}/2} < e^{\frac{\zeta R}{2}\left(1 - \frac{\zeta}{2\alpha}\right) - \zeta\omega(N)} = N^{1-\zeta/(2\alpha)} e^{-\zeta\omega(N)},$$

whereby $\tilde{\theta}'_{v_i} = o(1)$. Thus for $i \neq j$, with $t_{v_i}, t_{v_j} < R(1 - \zeta/(2\alpha)) - 2\omega(N) =: \hat{x}_0$ we have $A_{v_i} \cap A_{v_j} \neq \emptyset$ if

$$\theta_{v_i v_j} \le \tilde{\theta}'_{v_i} + \tilde{\theta}'_{v_j} = \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \, e^{\zeta\omega(N)/2} \left(e^{\zeta t_{v_i}/2} + e^{\zeta t_{v_j}/2}\right).$$

The probability that this occurs for two given distinct indices i, j is crudely bounded for N large enough using Lemma 2.1 as follows:

$$\frac{1}{\pi}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)}\ e^{\zeta\omega(N)/2} \int_0^{\hat{x}_0} \int_0^{\hat{x}_0} \left(e^{\zeta t_{v_i}/2} + e^{\zeta t_{v_j}/2}\right) \rho(t_{v_i})\rho(t_{v_j}) dt_{v_i} dt_{v_j}$$
$$\leq \frac{2\alpha^2}{\pi}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)}\ e^{\zeta\omega(N)/2} \int_0^{\hat{x}_0} \int_0^{\hat{x}_0} \left(e^{\zeta t_{v_i}/2} + e^{\zeta t_{v_j}/2}\right) e^{-\alpha t_{v_i}} e^{-\alpha t_{v_j}} dt_{v_i} dt_{v_j}$$
$$\leq \frac{4\alpha^2}{\pi}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)}\ e^{\zeta\omega(N)/2} \int_0^{\hat{x}_0} e^{-\left(\alpha - \frac{\zeta}{2}\right)x} dx$$
$$\leq \frac{4\alpha^2}{\pi}\ \frac{2}{2\alpha - \zeta}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)}\ e^{\zeta\omega(N)/2}.$$

Also, by Lemma 2.1 for a vertex v we have

$$\begin{aligned} \Pr\left[t_v \geq R\left(1 - \frac{\zeta}{2\alpha}\right) - 2\omega(N)\right] &= e^{-\alpha R\left(1-\frac{\zeta}{2\alpha}\right) + 2\alpha\omega(N)} + O\left(N^{-2\alpha/\zeta}\right) \\ &= \left(\frac{N}{\nu}\right)^{-\frac{2\alpha}{\zeta}\left(1-\frac{\zeta}{2\alpha}\right)} e^{2\alpha\omega(N)} + O\left(N^{-2\alpha/\zeta}\right) \\ &= \left(\frac{N}{\nu}\right)^{1-\frac{2\alpha}{\zeta}} e^{2\alpha\omega(N)} + O\left(N^{-2\alpha/\zeta}\right). \end{aligned}$$

Thus the probability that there exists a pair of distinct vertices v_i, v_j with $i, j = 1, \ldots, m$ such that $A_{v_i} \cap A_{v_j} \neq \emptyset$ is bounded by

$$m^2 O\left(\frac{\omega(N)}{N^{1-\zeta/(2\alpha)}}\ e^{\zeta\omega(N)/2}\right) + mO\left(N^{1-\frac{2\alpha}{\zeta}} e^{2\alpha\omega(N)}\right) = o(1).$$

□

We assume that $m \geq 2$ is fixed and we condition on the event that $t_{v_i} \leq x_0$ for all $i = 1, \ldots, m$ (which we denote by $\mathcal{T}_1$) as well as on the event $\mathcal{E}_1$. By Corollary 2.2 and Claim 4.3 both events occur with probability $1 - o(1)$.

For a vertex $w \notin \{v_1, \ldots, v_m\}$ we denote by $\mathcal{A}_{v_i}^w$ the event that w is located within A_{v_i} and it is adjacent to v_i. In what follows, we will be omitting the superscript w, whenever this is clear from the context.

Now, let us consider the event that k_i vertices satisfy the event $\mathcal{A}_{v_i}$, for $i = 1, \ldots, m$, whereas all other vertices do not. We denote this event by $\mathcal{A}(k_1, \ldots, k_m)$. Also, for every $i = 1, \ldots, m$ let $\tilde{\mathcal{A}}_{v_i}^w$ be the event that a certain vertex w is located outside A_{v_i} and is adjacent to v_i. We let $\mathcal{B}_1$ be the event $\cup_{w \in V_N^{v_1, \ldots, v_m}} \cup_{i=1}^m \tilde{\mathcal{A}}_{v_i}^w$, that is, the event that there exists a vertex $w \in V_N \setminus \{v_1, \ldots, v_m\}$ which is adjacent to v_i, for some $i = 1, \ldots, m$, but it is located outside A_{v_i}. Thus conditional on $\mathcal{E}_1 \cap \mathcal{T}_1$, if the event $\mathcal{B}_1$ is *not* realized, then the event that vertex v_i has degree k_i, for all $i = 1, \ldots, m$ is realized if and only if $\mathcal{A}(k_1, \ldots, k_m)$ is realized. Using the union bound, we will show that

$$\Pr\Big[\mathcal{B}_1\Big] = o(1). \tag{4.2}$$

(We will show this without any conditioning.) Thereby, we can deduce the following:

$$\begin{aligned}
&\Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m\Big] = \Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&\overset{(4.2)}{=} \Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m \mid \mathcal{E}_1, \mathcal{T}_1, \overline{\mathcal{B}_1}\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1, \overline{\mathcal{B}_1}\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m), \overline{\mathcal{B}_1} \mid \mathcal{E}_1, \mathcal{T}_1\Big] / \Pr\Big[\overline{\mathcal{B}_1} \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m), \overline{\mathcal{B}_1} \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big] - \Pr\Big[\mathcal{A}(k_1, \ldots, k_m), \mathcal{B}_1 \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1).
\end{aligned} \tag{4.3}$$

We will show further that $\Pr\Big[A(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big]$ is asymptotically equal to the product of the probabilities that $D_{v_i} = k_i$, over $i = 1, \ldots, m$.

Lemma 4.4. *Let $\beta > 1$ and $\zeta/\alpha < 2$. Assume that $m \geq 2$ and $k_1, \ldots, k_m \geq 0$ are integers. Then we have*

$$\Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big] \sim \prod_{i=1}^{m} \Pr\Big[D_{v_i} = k_i\Big].$$

Proof. Note that if the positions of $v_1, \ldots, v_m$ have been fixed, then $\cup_{i=1}^m \{\mathcal{A}_{v_i}^w\}_{w \in V_N^{v_1, \ldots, v_m}}$ is an independent family. Thus, assuming that the positions $(t_{v_1}, \theta_{v_1}), \ldots, (t_{v_m}, \theta_{v_m})$ of $v_1, \ldots, v_m$ in $\mathcal{D}_R$ have been exposed so that $\mathcal{E}_1 \cap \mathcal{T}_1$ is realized, we can write

$$\begin{aligned}
&\Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid (t_{v_1}, \theta_{v_1}), \ldots, (t_{v_m}, \theta_{v_m})\Big] = \\
&\qquad \binom{N-m}{k_1 k_2 \cdots N - \sum_{i=1}^m k_i} \left[\prod_{i=1}^{m} \Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]^{k_i}\right] \left(1 - \sum_{i=1}^{m} \Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]\right)^{N - \sum_{i=1}^m k_i}.
\end{aligned} \tag{4.4}$$

We now proceed by giving an estimate for $\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]$. That is, we will calculate the probability that a vertex $w \notin \{v_1, \ldots, v_m\}$ is located within A_{v_i} and it is adjacent to v_i. Setting $x_0' = \min\{x_0, R - t_{v_i} - \omega(N)\}$, we have

$$
\begin{aligned}
\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i} \Big] &= \frac{1}{\pi} \int_0^{x_0} \int_0^{\min\{\pi, \tilde{\theta}'_{v_i,w}\}} p_{v_i,w} \rho(t_w) d\theta dt_w \\
&= \frac{1}{\pi} \int_0^{x_0'} \int_0^{\tilde{\theta}'_{v_i,w}} p_{v_i,w} \rho(t_w) d\theta dt_w + \frac{1}{\pi} \int_{x_0'}^{x_0} \int_0^{\min\{\pi, \tilde{\theta}'_{v_i,w}\}} p_{v_i,w} \rho(t_w) d\theta dt_w .
\end{aligned}
\tag{4.5}
$$

The second integral is bounded as in (3.2). In particular, it is bounded from above by

$$
\int_{R - t_{v_i} - \omega(N)}^{R} p_{v_i,w} \rho(t_w) dt_w = O\left(e^{\alpha \omega(N)} \left(\frac{e^{\zeta t_{v_i}/2}}{N} \right)^{2\alpha/\zeta} \right).
$$

Regarding the first integral, we argue as in (2.8), (2.15) and (2.12). Recall that for $\beta > 1$, we defined $\tilde{\theta}_{v_i,w} = \frac{1}{\omega(N)}\ A_{v_i,w}^{-1}$.

$$
\int_0^{\tilde{\theta}'_{v_i,w}} p_{v_i,w} d\theta = \int_0^{\tilde{\theta}_{v_i,w}} p_{v_i,w} d\theta + \int_{\tilde{\theta}_{v_i,w}}^{\tilde{\theta}'_{v_i,w}} p_{v_i,w} d\theta = \int_{\tilde{\theta}_{v_i,w}}^{\tilde{\theta}'_{v_i,w}} p_{v_i,w} d\theta + o\left(A_{v_i,w}^{-1} \right).
$$

For the first integral, we imitate the calculation in (2.12), expressing $p_{v_i,w}$ using Lemma 2.3 and applying the transformation $z = C^{1/\beta}\ A_{v_i,w}\ \frac{\theta}{2}$. We obtain

$$
\int_{\tilde{\theta}_{v_i,w}}^{\tilde{\theta}'_{v_i,w}} p_{v_i,w} d\theta \sim \frac{C_\beta}{A_{v_i,w}},
$$

where C_β is as in Lemma 2.4 for $\beta > 1$.

Thereby, as in (3.3), the first integral in (4.5) becomes

$$
\frac{1}{\pi} \int_0^{x_0'} \int_0^{\tilde{\theta}'_{v_i,w}} p_{v_i,w} \rho(t_w) d\theta dt_w \sim C_\beta \int_0^{x_0'} A_{v_i,w}^{-1} \rho(t_w) dt_w \sim \frac{2\alpha\nu C_\beta}{2\alpha - \zeta} \frac{e^{\zeta t_{v_i}/2}}{N}.
$$

With $K = 2\alpha\nu C_\beta/(2\alpha - \zeta)$, as it was set in Lemma 3.1 for $\beta > 1$ and substituting the above estimates into (4.5) we obtain

$$
\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i} \Big] = (1 + o(1))\ K\ \frac{e^{\zeta t_{v_i}/2}}{N}. \tag{4.6}
$$

Under the assumption that $t_{v_i} \leq R/2 - \omega(N)$, we have $e^{\zeta t_{v_i}/2}/N = o\left(\frac{1}{N^{1/2}}\right)$. Thus, if $t_{v_i} \leq R/2 - \omega(N)$, for all $i = 1, \ldots, m$

$$
\left(1 - \sum_{i=1}^m \Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i} \Big] \right)^{N - \sum_{i=1}^m k_i} = \exp\left(-(1 + o(1))\ K \sum_{i=1}^m e^{\zeta t_{v_i}/2} \right). \tag{4.7}
$$

Recall now that $T_u = \sum_{v \in V_N^u} \Pr\Big[I_{uv} = 1 \mid t_u \Big]$ and Lemma 3.3 implies that for any integer $k \geq 0$ we have

$$\Pr\Big[D_u = k \mid t_u \Big] = \Pr\Big[\mathrm{Po}(T_u) = k \Big] + o(1).$$

Assume now that the function $\tilde{\omega}(N)$ is such that

i. uniformly for $t_u \leq \tilde{\omega}(N)$ we have $T_u = Ke^{\zeta t_u/2} + o(1)$;

ii. $\tilde{\omega}(N) \leq \min\{R/2 - \omega(N), x_0\}$;

iii. $\tilde{\omega}(N) \to \infty$ as $N \to \infty$.

Substituting the estimates in (4.6) and (4.7) into (4.4) we obtain that uniformly for all $(t_{v_1}, \dots, t_{v_m}) \in [0, \tilde{\omega}(N)]^m$ and all $\theta_{v_1}, \dots, \theta_{v_m} \in (0, 2\pi]$ such that $\mathcal{E}_1 \cap \mathcal{T}_1$ is realized:

$$\Pr\Big[\mathcal{A}(k_1, \dots, k_m) \mid (t_{v_1}, \theta_{v_1}), \dots, (t_{v_m}, \theta_{v_m}) \Big] \sim \prod_{i=1}^{m} \frac{\left(Ke^{\zeta t_{v_i}/2}\right)^{k_i}}{k_i!} \exp\left(-(1+o(1))Ke^{\zeta t_{v_i}/2}\right) \sim \prod_{i=1}^{m} \Pr\Big[D_{v_i} = k_i \mid t_{v_i} \Big], \tag{4.8}$$

by Lemma 3.3. Also, a first moment argument shows that the probability that there exists an index i with $1 \leq i \leq m$ such that $t_{v_i} > \tilde{\omega}(N)$ is $o(1)$. Hence, averaging over all (t_{v_i}, θ_{v_i}), for $i = 1, \dots, m$, on the measure conditional on $\mathcal{E}_1 \cap \mathcal{T}_1$, the lemma follows. $\square$

We conclude the proof of (4.1) with the proof of (4.2).

Lemma 4.5. *For any $\beta > 1$ and any $\zeta/\alpha < 2$ we have*

$$\Pr\Big[\mathcal{B}_1 \Big] = o(1).$$

Proof. For $i \in [m]$, assume that $t_{v_i} < R - \omega(N)$. Corollary 2.2 implies that this holds for all $i \in [m]$ with probability $1 - o(1)$. Now, for a given $w \in V_N^{v_1, \dots, v_m}$, the probability of the event $\tilde{A}_{v_i}^w$, conditional on t_{v_i}, can be bounded as follows:

$$\Pr\Big[\tilde{\mathcal{A}}_{v_i}^w \mid t_{v_i} \Big] \leq \frac{1}{\pi} \int_0^{R - t_{v_i} - \omega(N)} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} p_{v_i,w} \rho(t_w) d\theta dt_w + \int_{R - t_{v_i} - \omega(N)}^{R} p_{v_i,w} \rho(t_w) dt_w. \tag{4.9}$$

(Here $\rho(\cdot)$ denotes the density function of the type of a vertex.) The second integral can be bounded as in (3.2) - uniformly for $t_{v_i} < R - \omega(N)$ we have

$$\int_{R - t_{v_i} - \omega(N)}^{R} p_{v_i,w} \rho(t_w) dt_w = O\left(e^{\alpha \omega(N)} \left(\frac{e^{\zeta t_{v_i}/2}}{N} \right)^{2\alpha/\zeta} \right). \tag{4.10}$$

Regarding the first integral, we use the estimate obtained in Lemma 2.3 to bound the inner integral. With C as in the proof of Lemma 2.4, we have

$$\int_{\tilde{\theta}'_{v_i,w}}^{\pi} p_{v_i,w} d\theta = \int_{\tilde{\theta}'_{v_i,w}}^{\pi} \frac{1}{CA_{v_i,w}^{\beta}\sin^{\beta}\left(\frac{\theta}{2}\right)+1} d\theta \leq \frac{1}{CA_{v_i,w}^{\beta}} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right)} d\theta$$
$$\overset{\sin\left(\frac{\theta}{2}\right)\geq\frac{\theta}{\pi}}{\leq} \frac{\pi^{\beta}}{CA_{v_i,w}^{\beta}} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} \theta^{-\beta} d\theta \leq \frac{\pi^{\beta}}{C(\beta-1)A_{v_i,w}^{\beta}} \tilde{\theta}'^{-\beta+1}_{v_i,w} = O\left(\frac{A_{v_i,w}^{-1}}{\omega(N)^{\beta-1}}\right).$$

Thus, the first integral in (4.9) becomes

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{R-t_{v_i}-\omega(N)} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} p_{v_i,w}\rho(t_w) d\theta dt_w \\
&= O\left(\frac{1}{\omega(N)^{\beta-1}} \frac{e^{\zeta t_{v_i}/2}}{N}\right) \int_0^{R-t_{v_i}-\omega(N)} e^{\zeta t_w/2}\rho(t_w) dt_w \qquad (4.11)\\
&\overset{(3.3)}{=} O\left(\frac{1}{\omega(N)^{\beta-1}} \frac{e^{\zeta t_{v_i}/2}}{N}\right).
\end{aligned}$$

Now, we take the average of each one of the bounds obtained in (4.11) and (4.10), respectively, over t_{v_i}. To this end, we need the following integral, whose simple calculation we omit. We have

$$\int_0^{R-\omega(N)} e^{\lambda t}\rho(t) dt = \begin{cases} \Theta(R) & \text{if } \lambda = \alpha \\ \Theta(1) & \text{if } \lambda = \zeta/2. \end{cases}$$

Thus, the bound in (4.10) is $O\left(e^{\alpha\omega(N)} \frac{R}{N^{2\alpha/\zeta}}\right)$ and that in (4.11) is $O\left(\frac{1}{\omega(N)^{\beta-1}N}\right)$. Since $\zeta/\alpha < 2$, both terms are $o(N^{-1})$. Therefore, the union bound implies that

$$\Pr\Big[\cup_{w\in V_N^{v_1,\ldots,v_m}} \cup_{i=1}^m \tilde{\mathcal{A}}_{v_i}^w\Big] = o(1). \qquad (4.12)$$

□

Thus, the estimates obtained in Lemmas 4.4 and 4.5 substituted in (4.3) imply (4.1).

5 Conclusions

This paper considers a model for complex networks that is based on mapping the intrinsic heterogeneity of a complex network into the hyperbolic plane. **In this mapping, there is an implicit notion of the "weight" or the "importance" of a vertex that is expressed by its type: vertices of high weight, that is, the hubs of a network have high type and vice versa.** This model was introduced by Krioukov et al. [14] as a way to express basic properties of complex networks, such as power law degree distribution and clustering, as a result of the underlying hyperbolic geometry. We have shown that locally this model **asymptotically** reduces to the Chung-Lu model and from that we derive the exact degree distribution.

However, the deeper understanding of this model requires analysis of the global or *long range* properties. One needs to consider the diameter of the components of resulting network as well as the typical distance between two vertices that belong to the same component. Another important direction has to do with the distribution of components. **Is there a component that covers a positive fraction of the vertices, or is the random network fragmented into small components**?

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A Proof of Lemma 1.1

Using (2.2) we obtain

$$\cosh(\zeta d(u,v)) \leq \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(1-\cos(\theta_{u,v})+2\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right)+O\left(e^{-2\zeta(2R-(t_u+t_v))}\right)\right).$$

Since $t_u+t_v < R-c_0$, the last error term is $O(N^{-4})$. Also, it is a basic trigonometric identity that $1-\cos(\theta_{u,v}) = 2\sin^2\left(\frac{\theta_{u,v}}{2}\right)$. The latter is at most $\frac{\theta_{u,v}^2}{2}$. Therefore, the upper bound on $\theta_{u,v}$ yields:

$$\begin{aligned}
&\cosh(\zeta d(u,v)) \leq \\
&\frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(\frac{\theta_{u,v}^2}{2}+2\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right)+O\left(\frac{1}{N^4}\right)\right) \\
&\leq \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(2(1-\varepsilon)^2 e^{\zeta(t_u+t_v-(1-\delta)R)}+2\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right)\right)+O(1) \\
&= (1-\varepsilon)^2\frac{e^{\zeta(1+\delta)R}}{2}+\frac{1}{2}\left(e^{\zeta(t_u-t_v)}+e^{\zeta(t_v-t_u)}\right)+O(1) \\
&< (1-\varepsilon)^2\frac{e^{\zeta(1+\delta)R}}{2}+\varepsilon\frac{e^{\zeta(1+\delta)R}}{2}+O(1) < \frac{e^{\zeta(1+\delta)R}}{2},
\end{aligned}$$

for N sufficiently large and c_0 such that $e^{-c_0} < \frac{1}{2}\varepsilon$, since $t_u + t_v < (1 - |\delta|)R - c_0$ and $t_u, t_v \geq 0$. This implies that $t_u - t_v, t_v - t_u < (1 + \delta)R - c_0$ and, therefore, $\frac{1}{2}\left(e^{\zeta(t_u - t_v)} + e^{\zeta(t_v - t_u)}\right) < \frac{1}{2}\left(e^{\zeta(1+\delta)R - c_0} + e^{\zeta(1+\delta)R - c_0}\right) < \varepsilon \frac{e^{\zeta(1+\delta)R}}{2}$. Also, since $\cosh(\zeta d(u,v)) > \frac{1}{2}e^{\zeta d(u,v)}$, it follows that $d(u,v) < (1+\delta)R$.

To deduce the second part of the lemma, we consider a lower bound on (2.2) using the lower bound on $\theta_{u,v}$:

$$\begin{aligned} \cosh(\zeta d(u,v)) &\geq \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(1 - \cos(\theta_{u,v})\right) + O(1) \\ &\geq \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(1 - \cos\left(2(1+\varepsilon)e^{\frac{\zeta}{2}(t_u+t_v-(1-\delta)R)}\right)\right) + O(1). \end{aligned} \tag{A.1}$$

Using again that $1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$ we deduce that

$$1 - \cos\left(2(1+\varepsilon)e^{\frac{\zeta}{2}(t_u+t_v-(1-\delta)R)}\right) = 2\sin^2\left(\frac{1}{2}\ 4(1+\varepsilon)^2 e^{\zeta(t_u+t_v-(1-\delta)R)}\right).$$

Since $t_u + t_v < (1 - |\delta|)R - c_0$, it follows that $t_u + t_v - (1-\delta)R < -c_0$. So the latter is

$$\sin\left(\frac{1}{2}\ 4(1+\varepsilon)^2 e^{\zeta(t_u+t_v-(1-\delta)R)}\right) > 2\left(1 + \frac{\varepsilon}{2}\right)^2 e^{\zeta(t_u+t_v-(1-\delta)R)},$$

for N and c_0 large enough, using the Taylor's expansion of the sinus function around 0. Substituting this bound into (A.1) we have

$$\cosh(\zeta d(u,v)) \geq \left(1 + \frac{\varepsilon}{2}\right)^2 \frac{e^{\zeta(1+\delta)R}}{2} + O(1).$$

Thus, if $d(u,v) \leq (1+\delta)R$, the left-hand side would be smaller than the right-hand side which would lead to a contradiction.

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