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Algorithms of Solution Reconstruction on Unstructured Grids in Computational Aerodynamics: Impact on Aircraft Design at the Boeing Company

Natalia Petrovskaya

Abstract We describe work that demonstrated the benefits achieved when the mathematical and computational aspects of a fluid dynamics problem were brought together to work on real-world aerodynamic applications. The research into solution reconstruction on adaptive grids was required by The Boeing Company in order to help them to design an efficient and accurate discretization of the governing equations that have to be solved numerically for the generation of aerodynamic data for various flow regimes. While earlier insight into the solution reconstruction problem was purely based on empirical intuition, research conducted by the author under a contract with Boeing has resulted in the development of the necessary synthetic judgement in which the importance of accurate reconstruction on unstructured grids has been fully recognised by the CFD researchers at Boeing and has helped them to make an informed decision on the choice of a discretization method in their CFD code. Efficient use of CFD in the design of new aircraft has allowed The Boeing Company to further strengthen their core operations, improve their execution and competitiveness and leverage their international advantage.

Introduction

The overall significance of computational fluid dynamics (CFD) in the aircraft design process is now well-established. Among other commercial companies CFD is widely used at Boeing where its application has "revolutionised the process of aerodynamic design" [1], joining the wind tunnel and flight test as primary tools. The resulting financial savings to the Boeing Company were estimated in [1] as "tens of millions of dollars" over a twenty year period. CFD also provided added value by achieving design solutions that would not otherwise be achievable, as well as shortening the design development process by reducing or eliminating the need to build successive prototypes.

N. Petrovskaya (✉)
School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK
e-mail: n.b.petrovskaya@bham.ac.uk

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Project engineers at Boeing (and elsewhere) use commercial codes to undertake CFD analyses. These codes take many years to design and validate, are applied to various real-life engineering tasks where appropriate during their development phase and are then released allowing decades of use across Boeing and a wider aerodynamics community. For instance, development work on the TRANAIR computational tool began in 1984 with useful results published in 1989 and on-going development in the 1990s. The CFD codes are used very extensively; TRANAIR was run more than 70,000 times between 1989 and 2004, with about 90 users in Boeing only [1]. The code was heavily applied in the design of aircraft such as the Boeing 777, one of the company's best-selling products. Following the success of TRANAIR, Boeing began the process of developing their next-generation computational code in 1998 to meet the needs of modern aircraft design process. The ultimate purpose of the new code has been formulated as to allow the generation of aerodynamic data for various flow regimes about realistic complex geometries in a timely and affordable manner. This highly challenging and ambitious goal has placed substantially increased demands on the solution methodology and resources required for the design of a reliable and accurate CFD toolkit.

One of key requirements in the design of a modern computational aerodynamics code is the use of adaptive grids whenever it is possible in computation. Adaptive computational grids are opposite to grids with the fixed number of grid points, as the adaptive grid has to be refined several times along with the numerical solution to provide accurate simulation of aerodynamic flow. Grid refinement allows for better accuracy on a final grid and adaptive grid techniques also offer great potential in computational savings. However, adaptive grids have not seen widespread use in computational aerodynamics due to various computational issues, inadequate solution accuracy estimation on initial grids being one of them.

One difficulty arising when adaptive grids are employed in the problem is that they have highly anisotropic geometry in the boundary layer close to an airfoil and solution discretization can degrade to unacceptable accuracy on highly stretched meshes at the beginning of grid adaptation process when a computational grid is not perfectly fitted to the solution. In particular, least-squares (LS) technique intensively exploited in computational aerodynamics gives very inaccurate results when it is used for solution reconstruction on anisotropic grids. Solution reconstruction is an essential part of many discretization methods and when Boeing engineers and researchers started working on a new CFD code it became clear to them that a detailed investigation of a solution reconstruction procedure on anisotropic grids was required. Based on her earlier work as a research consultant for Boeing, the author was asked by the CFD research team to investigate the reconstruction problem in depth. In the present chapter we briefly discuss implementation of the LS method for aerodynamical applications and explain the findings of the LS study made by the author for The Boeing Company.

Least-Squares Reconstruction on Anisotropic Grids

Let a computational grid G be generated in a two-dimensional domain. The grid G can be considered as a collection of points $P_i = (x_i, y_i), i = 1, 2, \ldots, N_G$, selected according to some computational rule and supported with a data structure specified in the problem (i.e., grid edges, grid cells, boundary edges and so on). An example of an irregular computational grid generated around an airfoil is shown in Fig. 1a. The number N_G of grid nodes on an adaptive computational grid typically used in aerodynamical applications is $N_G \sim 10^7$.

We assume that a function $U(x, y)$ (the solution function) is available at any grid node P_i. Given the values $U_1, U_2, \ldots, U_{N_G}$ at grid nodes, the solution function $U(x, y)$ has to be reconstructed at edge midpoints with reasonable accuracy. For this purpose a reconstruction stencil is defined and local LS approximation of the function $U(x, y)$ is done over the stencil points. An example of the reconstruction stencil at edge midpoint p is shown in Fig. 1b.

In the LS problem local numbering of stencil points is used. The edge midpoint p is re-denoted as P_0 and is called a central reconstruction node. The other stencil points are numbered as P_i, $i = 1, \ldots, N$. Clearly the number N of stencil points can be different for two different central nodes as N depends on the geometry of a computational grid. Similarly the values of the solution function at stencil points are re-numbered as $(U_1, U_2, \ldots, U_N)$. The number N of points belonging to a local reconstruction stencil is $N \sim 10 - 20$.

A weighted LS approximation requires that the data $\mathbf{U} = (U_1, U_2, \ldots, U_N)$ should be fitted to the function

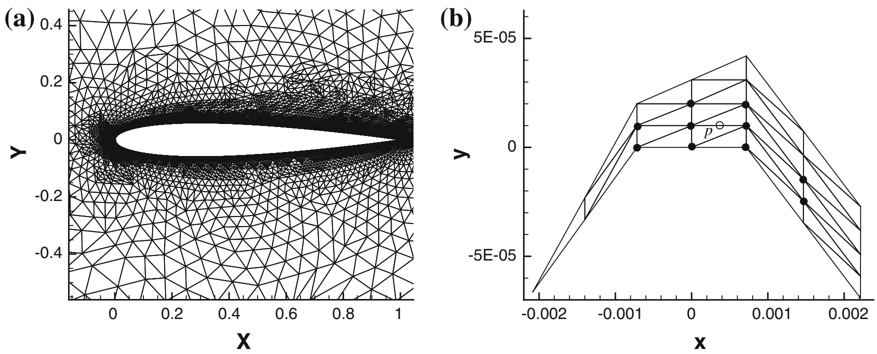

Fig. 1 **a** An unstructured computational grid generated about an airfoil. **b** A reconstruction stencil for LS approximation at a given edge midpoint. Grid points used to reconstruct the solution at point p shown as an open circle in the figure are shown as closed black circles. Note a different length scale along the x and y-axes

$$u_{LS}(x, y) = \sum_{k=0}^{M} u_k \phi_k(x, y), \quad M < N, \tag{1}$$

where $\mathbf{u} = (u_0, u_1, u_2, \ldots, u_M)$ are fitting parameters, and $\phi_k(x, y),\ k = 0, \ldots, M$, are polynomial basis functions. The unknown parameters $\{u_k\}$ are determined by seeking the minimum of the merit function,

$$F^2 = \sum_{i=1}^{N} w(P_0, P_i)\,[U(P_i) - u_{LS}(P_i)]^2\,,$$

where $w(P_0, P_i)$ is the weight function that should be specified in the problem. The solution of the above minimization problem is defined by the design matrix $\mathbf{A} : A_{ij} = \phi_j(P_i),\ i = 1, \ldots, N,\ \ j = 0, \ldots, M$ and can formally be written as $\mathbf{u} = \mathbf{A}_{wls}^{-1}\mathbf{b}_{wls}$, where $\mathbf{A}_{wls} = \mathbf{A}^T\mathbf{W}\mathbf{A}, \quad \mathbf{b}_{wls} = \mathbf{A}^T\,\mathbf{W}\mathbf{U}$, and a diagonal weight matrix $\mathbf{W}$ is given by

$$W_{ij} = \begin{cases} w(P_0, P_i), & i = j, \\ 0, & otherwise, \end{cases} \qquad i, j = 1, 2, \ldots, N. \tag{2}$$

Once a function $u_{LS}(x, y)$ has been reconstructed, we can define its value at the point P_0 from (1). The next edge midpoint is then taken and the reconstruction procedure is repeated.

Earlier insight into the reconstruction problem, made by researchers at Boeing and NASA, attributed poor accuracy of the LS method on irregular stretched grids to the impact of distant points on the results of LS reconstruction (see Fig. 1b). Thus the following weight function widely employed in aerodynamic applications was selected for reconstruction (1)–(2):

$$w(P_0, P_i) \equiv w(r_{0i}) = r_{0i}^{-q},\ q = 0, 1, 2, \ldots, \tag{3}$$

where r_{0i} is the Euclidian distance between P_0 and P_i, $i = 1, 2 \ldots, N$, and q is an integer polynomial degree. Any $q > 0$ provides inverse distance weighting used to mitigate the impact of remote stencil points on the results of LS approximation.

In many applied problems weighting (3) allows users to improve the accuracy of reconstruction. However, the study made in [3] revealed that inverse distance weight-

Table 1 The reconstruction error (4) for LS approximation with various degrees q of polynomial weight function (3)

q	0	1	2	4	8
e^f_{max}	1.27282×10^{-3}	1.09508×10^{-3}	1.08304×10^{-3}	1.14044×10^{-3}	1.38461×10^{-3}
e^b_{max}	1.38595	1.52966	1.72857	2.18609	198.303

The maximum error e^f_{max} is computed in a far field and the maximum error e^b_{max} is computed in a 'boundary layer' sub-domain near the airfoil

n.b.petrovskaya@bham.ac.uk

ing of stencil points was not efficient in practical aerodynamic computations on highly anisotropic adaptive grids. The first findings of the study of solution reconstruction on stretched grids are summarised in Table 1. The validation of the accuracy of a LS approximation has been made from comparison of the accurate solution $U(x, y)$ available in the test cases and a reconstructed solution $u_{LS}(x, y)$ taken at the same point (x, y). The maximum error has been computed as

$$e^b_{max} = \max_{(x,y)\in D_b} e(x, y), \quad e^f_{max} = \max_{(x,y)\in D_f} e(x, y), \tag{4}$$

where $e(x, y) = |U(x, y) - u_{LS}(x, y)|$. A 'boundary layer' region D_b in (4) is defined as a computational sub-domain near the airfoil where a highly anisotropic grid is generated, while a 'far field' region D_f is a computational sub-domain far away from the airfoil where the grid is almost isotropic.

It can be seen from the table that the weight function (3) results in accurate solution approximation when a LS procedure is applied in the far field. However, the choice of (3) is not efficient in the domain D_b, as weighting of stencil points further increases the maximum error of the reconstruction. In particular, weighting with $q = 2$, which appears to be optimal in the far field, does not provide an acceptable reconstruction error near the wall.

It became clear from the results above that further insight into the problem was required. The further study of the reconstruction problem revealed that another class of distant points may appear in the reconstruction stencil. Those points called 'numerically distant points' have been defined as stencil points that are remote in the data space [4]. While recognition of geometrically distant points is a straightforward task, it is difficult to detect numerically distant points in the stencil, as their definition depends essentially on the solution function $U(x, y)$. Such points can be located close to the point P_0, but the function $U(x, y)$ measured at a numerically distant point still has a big data error that affects the accuracy of reconstruction.

Numerically distant points cannot be eliminated from the stencil by inverse distance weighting as they are not remote points in a geometric domain. Another approach is required and the numerically distant points have to be weighted in the data space in order to remove them from the reconstruction stencil. Thus the following definition of the weight function $w(P_0, P_i)$ in (3) has been suggested

$$\tilde{r}^2_{0i} = r^T_{0i}|\mathbf{H}|_{P_i} r_{0i}, \quad \text{and} \quad w(P_0, P_i) = \tilde{r}_{0i}^{-q}, \; q = 0, 1, 2, \ldots \tag{5}$$

where the matrix $\mathbf{H}$ depends on the solution function $U(x, y)$ (see [4] for more details). A novel reconstruction algorithm has been designed and a research code has been written to handle numerically distant points in the reconstruction procedure.

Conclusions

The research on the LS reconstruction has had impact in the following ways:

1. It was demonstrated that, in two and three dimensions, LS reconstruction on stencils with irregular geometry can cause severe problems with accuracy of a numerical solution. This is especially true for unstructured viscous grids with high aspect ratio grid cells and wide disparities in cell sizes and shapes, as well as for under-resolved curved boundaries. For grids of 10^6-10^7 nodes used in CFD computations it is unlikely that anomalous reconstructions would not arise and a disastrous reconstruction can feed on itself yielding worse and worse grids in the grid adaptation procedure. Hence the Boeing CFD team identified the solution reconstruction procedure on unstructured grids as one of critical tasks associated with the design of a solver for computational toolkits in modern CFD [2].
2. Numerous cases have been documented where a higher order LS algorithm originally considered as a potentially more accurate algorithm in comparison with interpolation yielded reconstructed values much less accurate than any values being interpolated. Those cases helped CFD researchers at Boeing to admit that higher order solution reconstruction can be dangerous on unstructured viscous grids unless the solution latent features are resolved [2]. That in turn made the impact on the choice of a baseline discretization scheme used in the Boeing solver.
3. The research on numerically distant points revealed true nature of a large reconstruction error. It was suggested that a large error is inevitable on coarse grids where the solution is not well resolved, no matter what the grid cell aspect ratio is. Hence Boeing researchers admitted the need for a careful choice of the initial grid when a grid refinement algorithm is concerned. The low accuracy of reconstruction may affect a solution on the initial coarse grid and this issue must be taken into account when a solution grid adaptation algorithm is designed [2].

As a result of the LS reconstruction study the importance of the reconstruction problem has been fully acknowledged by the Boeing CFD team and that issue was taken into account and implemented while designing a new computational toolkit. Boeing's subsequent and current codes have been improved and these benefits are being extended to cover further aspects of aircraft design.

Finally, it is worth noting here that the work on approximation on coarse grids is being continued by the author and further mathematical insight into a general problem of quantifying uncertainty of approximation has recently been provided [5]. The problem of accurate solution approximation from sparse data arises in many practical applications and the study made for The Boeing Company strongly influenced the author's current interest in this difficult yet fascinating research topic.

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n.b.petrovskaya@bham.ac.uk

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