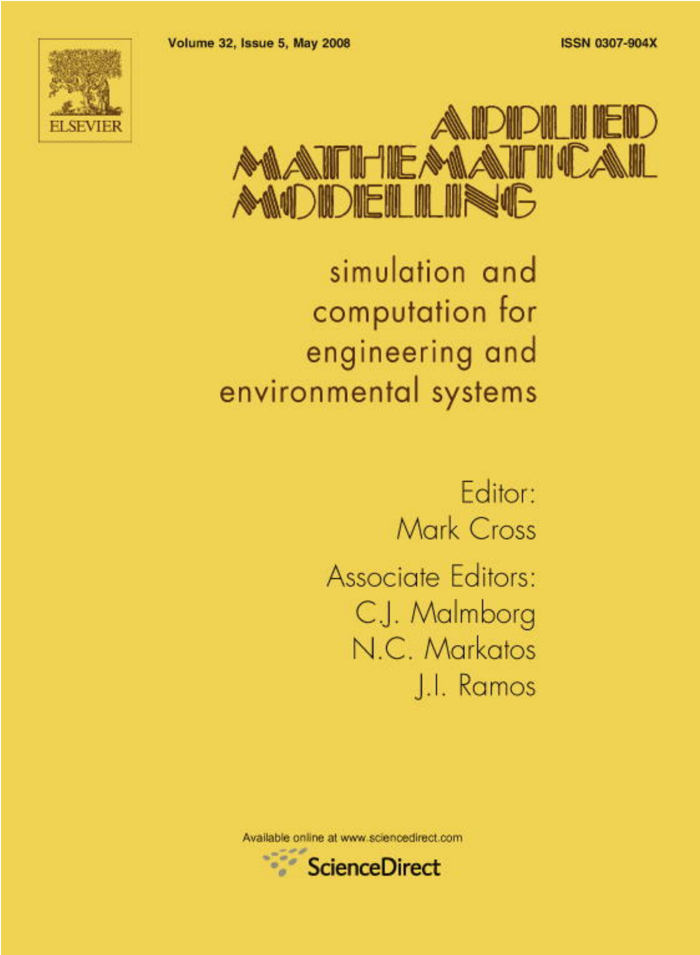
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Modification of basis functions in high order discontinuous Galerkin schemes for advection equation

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Abstract

High order discontinuous Galerkin (DG) discretization schemes are considered for an advection boundary-value problem on 2-D unstructured grids with arbitrary geometry of grid cells. A number of test cases are developed to study the sensitivity of a high order DG scheme to local grid distortion. It will be demonstrated how to modify the formulation of a DG discretization for the advection equation. Our approach allows one to maintain the required accuracy on distorted grids while using a fewer number of basis functions for the solution approximation in order to save computational resources.

Keywords: High order discretization; Unstructured grids; Discontinuous Galerkin; Modified basis functions

1. Introduction

A study of high order discretization schemes is nowadays in the mainstream of research in computational fluid dynamics (CFD). One of such schemes is a discontinuous Galerkin (DG) discretization which has been intensively investigated by many authors for last decades (e.g., see [1] and references therein). One of the advantages of DG schemes is that a compact discretization stencil used in the scheme makes it easy to control adaptation procedures such as h-refinement and p-refinement algorithms on unstructured grids. This feature of a DG discretization is potentially very attractive for modern real-life applications, as using a high order solution approximation on unstructured adaptive grids would allow one to handle flows around complex geometries in an accurate and efficient way [2].

There is a growing understanding in CFD community that coupling a high order discretization with solution grid adaptation could bring great benefits to the flow simulations [3]. However, the properties of a high order discretization on unstructured grids with arbitrary geometry of grid cells still need to be investigated in

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more detail prior to commitment to built a full three-dimensional simulation tool. Despite of the importance of this issue, very little can be found on this topic in the literature [4]. It is well known (e.g., see [5]) that the optimal orders of convergence may be achieved by using high order approximating spaces in DG schemes on grids with regular geometries. Meanwhile, the accuracy of a DG discretization on arbitrary meshes is sensitive to the shape of grid cells and it is difficult to conclude about the convergence rate of the method while dealing with an arbitrary geometry. Thus, there are two issues that should be investigated regarding the impact of geometry on the quality of the DG discretization. First, the question should be answered whether a high order DG scheme maintains an optimal order of convergence while being implemented on grids with arbitrary grid cell geometry. Another topic of research is to study how the accuracy of the discretization can be improved on grids with "bad" cell geometry.

In the present work we consider a simple but illuminating example of an advection boundary-value problem on unstructured grids. We develop a number of test cases to imitate a wrong grid adaptation in a local domain in order to study to what extent the DG scheme is sensitive to local grid distortion. Since one of the requirements to modern CFD codes is automatic grid generation over a computational domain [6], a flaw in the adaptation algorithm may result in a locally distorted grid that can be difficult to recognize and control at a current solution-adaptation iteration. Thus it is strongly desirable to understand the behavior of a discretization scheme on distorted grids, and we believe that the study of the geometry impact on the quality of a discretization is an essential part of development of future CFD codes for real-life applications.

We further discuss how to increase the accuracy of the approximate solution for a given number of degrees of freedom in a DG discretization. It will be demonstrated that taking into account the solution properties in the formulation of a DG discretization allows one to use a fewer number of basis functions for the solution approximation in order to maintain the required accuracy and to save computational resources. Our approach is illustrated by numerical examples.

2. The solution approximation in high order DG schemes

We consider the following advection equation in the domain D:

$$\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 0, \tag{1}$$

where a is a constant advection velocity. Eq. (1) is subject to the boundary condition

$$u(0,y) = u_0(y), \tag{2}$$

where the function $u_0(\xi = y - ax)$ determines the solution $U(x,y)$ to the boundary-value problems (1) and (2)

$$U(x,y) = u_0(y - ax). \tag{3}$$

We are interested in numerical solution $u_h(x,y)$ to the BVP above. The approximate solution $u_h(x,y)$ will be obtained by using a discontinuous Galerkin method [1]. Let us introduce an unstructured computational grid as a set of non-overlapping triangles e_i, $i = 1,2,\dots,N$, in the domain D. The DG discretization scheme defines the approximate solution $u_h(x,y)$ on each grid cell e_i as

$$u_h(x,y) = \sum_{m=0}^{M} u_m\phi_m(x,y), \quad m = 0,1,\dots,M, \ \ x,y \in e_i, \tag{4}$$

where the basis functions are $\phi_m(x,y) = (x - x_{0i})^{\alpha}(y - y_{0i})^{\beta}$, $\alpha + \beta = 0,1,\dots,K$. The functions $\phi_m(x,y)$ are piecewise polynomial, as they are only defined within the grid cell e_i. For a cell-centered DG scheme, x_{0i} and y_{0i} are the coordinates of the grid cell centroid.

In the DG method a weak formulation of the problem is used to find $u_h(x,y)$. The test functions belong to the same approximating space as the basis functions. Eq. (1) is multiplied by test function $\phi_l(x,y)$ and is integrated by parts over the cell e_i. We have

$$-\int\int_{e_i} u(x,y)\left(\frac{\partial\phi_l}{\partial x} + a\frac{\partial\phi_l}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + \oint_{\partial e_i} u(x,y)\phi_l(n_x + an_y)\,\mathrm{d}s = 0, \quad l = 0,1,\dots,M,$$

where $\mathbf{n} = (n_x, n_y)$ is the outward unit normal vector, and the notation ∂e_i is used for the boundary of the cell e_i. Further substitution of the approximate solution (4) into the integrals above gives

$$-\int\int_{e_i} u_h(x,y)\left(\frac{\partial\phi_l}{\partial x} + a\frac{\partial\phi_l}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + \oint_{\partial e_i} h_u(u^-,u^+)\phi_l(n_x + an_y)\,\mathrm{d}s = 0, \quad l = 0,1,\ldots,M, \tag{5}$$

where $h_u(u^-,u^+)$ is the numerical flux that should be defined for the problem. Since the approximate solution $u_h(x,y)$ is discontinuous at any grid edge, we need $h_u(u^-,u^+)$ to approximate the continuous flux $U(x,y)$ at grid interfaces. Let u^- be the solution on the cell e_i and u^+ be the solution on the cell e_j which shares a given edge with the cell e_i. For the advection equation (1), the implementation of a Riemann solver elaborated in order to discretize the flux $U(x,y)$ results in a well-known definition of the upwind flux (e.g., see [7])

$$h_u(u^-,u^+) = \begin{cases} u^-, & \text{if the flow is directed outward from } e_i, \\ u^+, & \text{otherwise.} \end{cases} \tag{6}$$

Finally, substituting the numerical flux (6) into (5) we arrive at a system of algebraic equations of the form

$$\mathbf{R}(\mathbf{u}) = 0, \tag{7}$$

where the vector $\mathbf{R}(\mathbf{u})$ is the residual of the DG method given by (5) on each grid cell, and $\mathbf{u}$ is the solution vector. Taking into account the definition of the numerical flux at $x = 0$, the discretization of boundary conditions is straightforward.

It is well known (e.g., see [5]) that the DG discretization provides an optimal order of convergence on grids with regular geometries. It has been discussed in [5] that the L_2-norm of the solution error can be estimated as $O(h^{K+1})$, where h is a diameter of grid cells. This estimate is not true, however, when arbitrary unstructured grids are considered. Below we demonstrate that the grid geometry may impact on the accuracy of the DG approximation even in the simplest case that a linear scalar BVP is numerically solved over an unstructured grid.

3. High order DG schemes and grid geometry

Let D be the unit square $(x,y) \in [0,1] \times [0,1]$. We generate a computational grid in D as follows. First, a uniform Cartesian grid is generated with a regular distribution of grid nodes given by

$$x_{ij} = ih_x, \quad y_{ij} = jh_y,$$

where $i = 0,\ldots,N_x$, $j = 1,\ldots,N_y$, h_x and h_y are grid step sizes in the x-direction and y-direction, respectively. We then shift grid nodes along the selected line $i = i_0$ as

$$y_{i_0 j} = (jh_y)^\beta \tag{8}$$

for a given value of $0 < \beta \leqslant 1$. This transformation results in a distorted Cartesian grid. Finally, a distorted unstructured grid is obtained by cutting each cell of a structured grid by the diagonal.

The degree of grid distortion is controlled by parameter $\beta \in (0,1]$. Two examples are given in Fig. 1 for an unstructured grid of $N = 722$ cells. A uniform grid ($\beta = 1.0$) is shown in Fig. 1a. A distorted unstructured grid is displayed in Fig. 1b for $\beta = 0.6$.

Our first test case is to study the accuracy of the DG scheme on grids with various β in order to understand to what extent the DG discretization is sensitive to local grid distortion. The exact solution for the test is chosen as

$$U(x,y) = \sin(\pi(y - ax)), \tag{9}$$

where the advection velocity is taken $a = 1.0$. The boundary condition is defined from the solution (9) as

$$u(0,y) \equiv U(0,y) = \sin(\pi y).$$

We compute the solution error $e(x,y)$ on each grid cell e_i as

$$e(x,y) = |U(x,y) - u_h(x,y)|, \quad (x,y) \in e_i. \tag{10}$$

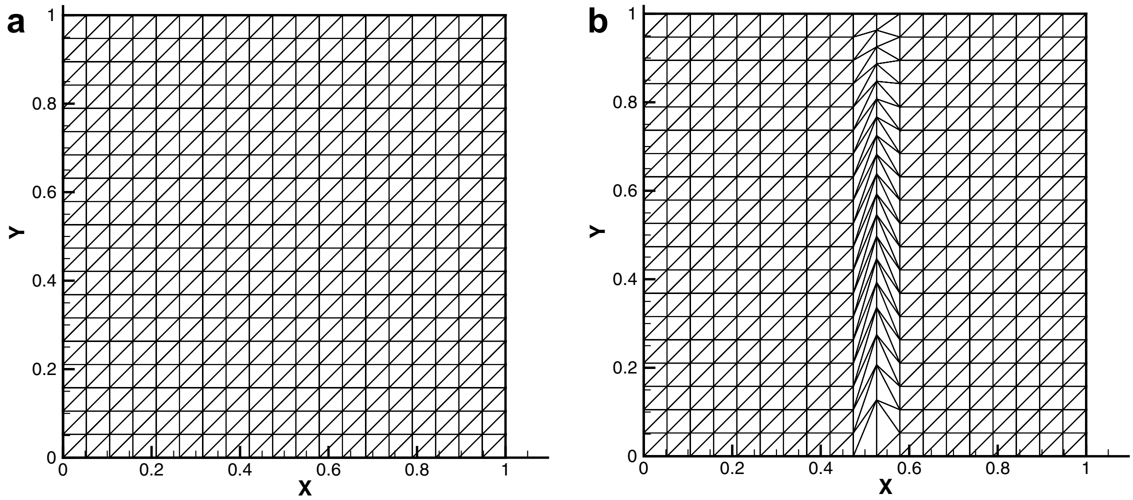

Fig. 1. (a) An unstructured grid with a regular geometry of grid cells, $\beta = 1.0$. (b) An example of a distorted unstructured grid. The distortion parameter is $\beta = 0.6$.

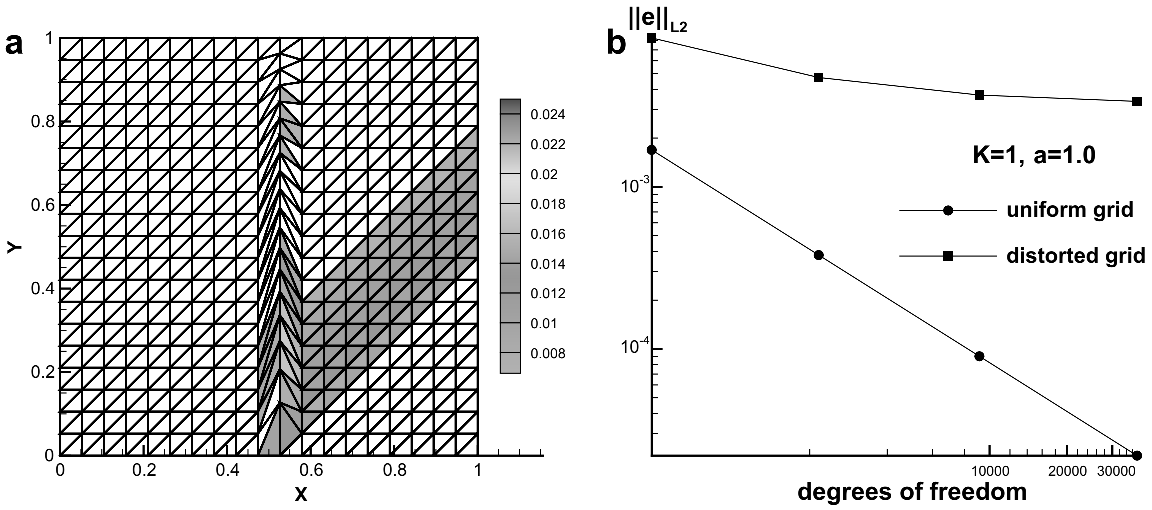

Fig. 2. (a) The function $e(x, y)$ over a distorted grid for the problem (1), (2). The degree of distortion is $\beta = 0.6$, the number of grid cells is $N_c = 722$. A piecewise linear solution approximation is used in a DG scheme. (b) The convergence history for the problem (1), (2) on uniform (circles) and distorted (squares) grids. The L_2-norm of the solution error is shown as a function of the number of the degrees of freedom. The advection velocity is $a = 1.0$.

The solution error distribution over a distorted grid ($\beta = 0.6$) is given in Fig. 2a. The polynomial degree in the DG approximation is taken $K = 1$ and the number of grid cells is $N_c = 722$. It can be seen from the figure that the grid distortion affects the accuracy of the DG discretization. The maximum solution error appears on a distorted strip of grid cells and the error propagates over the domain once the solution front has passed the distorted region.

Let us now measure the solution error on a sequence of grids with various degrees of distortion β. The value $\beta = 1.0$ provides us with a uniform grid, while smaller values of β make the grid more distorted. Hence, one may expect a larger solution error over the grid, as β varies from 1.0 to smaller values. We compute the L_2-norm of the solution error over the grid as

$$\|e\|_{L_2} = \sqrt{\int\int_D (U(x,y) - u_h(x,y))^2 \, dx \, dy}. \tag{11}$$

The error (11) is computed at Gaussian quadrature points used on each grid cell for numerical integration in the DG discretization (5). We also compute the L_∞-norm of the solution error as

Table 1
Numerical solution to the problem (1), (2)

β	1.0	0.8	0.6	0.4
$\|e\|_{L_2}$	0.3796e−3	0.1614e−2	0.4728e−2	0.1783e−1
$\|e\|_{L_\infty}$	0.7592e−3	0.5497e−2	0.2115e−1	0.8361e−1

The results of the piecewise linear DG discretization for various degrees of distortion β. The L_2-norm and the L_∞-norm of the solution error are computed on distorted grids with the number of grid cells $N_c = 722$. The advection velocity is $a = 1.0$.

$$\|e\|_{L_\infty} = \max_{(x,y)\in D} |U(x,y) - u_h(x,y)|. \tag{12}$$

The errors (11) and (12) are given in Table 1 for the DG polynomial degree $K = 1$ (the approximation by piecewise linear functions) on a sequence of distorted grids. The number of grid cells on each distorted grid is $N_c = 722$, and a distorted strip is stationed at $x = 0.5$. The results of the table demonstrate that the solution accuracy decreases on distorted grids as we vary β from 1.0 to 0.4. Thus, one may expect that the grid distortion slows down the convergence rate of the DG approximation. It is important to notice here that a uniform grid refinement that decreases the size of each grid element without transforming its shape will not result in a smaller solution error over the grid, as the accuracy of a DG discretization on distorted grids depends on the geometry rather than the size of grid cells.

The above conclusion is confirmed by the convergence history on distorted grids shown in Fig. 2b. For the convergence test the following refinement procedure has been applied to generate a sequence of distorted grids. The number of grid nodes is doubled in the x-direction and y-direction for each next structured grid in the sequence. The line $i = i_0$ is selected on each grid to provide its location at $x_{i_0 j} = 0.5,\ j = 1, \ldots, N_y$, so that a strip of distorted cells is always stationed in the midpoint of the interval $x \in [0, 1]$. Then the grid nodes are shifted along the selected line according to (8), where the same value of β is applied on each grid in the sequence. Thus we decrease the width of the distorted strip at each refinement step but keep the distortion degree the same.

In our convergence test we compare the L_2-norm of the error on a sequence of uniform unstructured grids ($\beta = 1.0$) with that on distorted grids ($\beta = 0.6$) for a piecewise linear DG approximation. The number of nodes on the initial Cartesian grid is taken as $N_x = 10$ and $N_y = 10$. The error norm is shown in Fig. 2b as a function of the number of the degrees of freedom on a given grid. The results of the convergence test show that while the convergence of the approximate solution on uniform grids is as good as expected, the convergence rate is very poor on distorted grids.

Let us note that poor accuracy of the DG approximation on distorted grids is especially dangerous in numerical solution of non-linear problems on adaptive grids where a grid may become occasionally distorted at a current non-linear iteration as a result of wrong adaptation algorithm. A thorough local grid refinement is then needed to make the grid smoother in the region of the original distortion and to resolve the solution to the required accuracy. That may appear to be a very demanding task from a computational viewpoint, as the intensive grid refinement significantly increases the number of grid cells and, therefore, the number of the degrees of freedom for a DG discretization. Thus our next goal is to try to maintain the better solution accuracy on distorted grids without refining them.

4. The choice of basis functions for the advection equation

In this chapter we discuss how to modify the solution approximation (4) to increase the solution accuracy on distorted grids. The following test case illustrates our approach. Consider a uniform computational grid with regular geometry of grid cells ($\beta = 1.0$) shown in Fig. 1a. Let us vary the advection velocity a and compute the norms (11) and (12) of the solution error on a uniform grid with a given number of grid cells. The results of our computations are given in Table 2 for polynomial degree $K = 1$ in the DG approximation. The number of grid cells is $N_c = 722$, so that we have $N_{\text{dof}} = 2166$ degrees of freedom for a piecewise linear discretization.

Table 2
Numerical solution to the problem (1), (2)

a	1.0	2.0	4.0	6.0	8.0
$\|e\|_{L_2}$	0.3796e−3	0.1951e−2	0.1036e−1	0.2663e−1	0.5016e−1
$\|e\|_{L_\infty}$	0.7592e−3	0.5061e−2	0.2441e−1	0.6238e−1	0.1205

The results of a piecewise linear ($K = 1$) DG discretization on a uniform grid for various advection velocities. The L_2-norm and the L_∞-norm of the solution error are computed over the grid with the number of grid cells $N_c = 722$.

It can be seen from Table 2 that the solution error increases, as we increase the advection velocity in the problem. This result is further illustrated in Fig. 3, where the convergence plots are shown on a sequence of uniform grids for different values of a. The DG discretization provides the same convergence rate for each advection velocity, as we keep the polynomial degree $K = 1$. However, the accuracy of the discretization decreases for bigger values of a on every grid in the sequence.

One important observation about the test case above is that for a particular value of the advection velocity $a = 1.0$, grid edges are aligned with the characteristics of (1). Actually, the characteristics of Eq. (1) are straight lines

$$y(x) = ax + y_0,$$

where y_0 is constant. On the other hand, the diagonals of structured grid cells considered as edges of the unstructured grid are given by the equation

$$y = x + C, \quad C = \text{const},$$

so that the solution (3) remains constant along each grid diagonal for $a = 1.0$. The solution error shown in Table 2 is the smallest in this case. For $a > 1.0$ the flow is skew to the diagonals of structured grid cells and the error is getting bigger. Comparing the results of Table 2 with those in Table 1 we conclude that the effect of making the flow skew to the diagonals of grid cells is similar to the grid distortion. Hence, instead of refining a distorted grid to make it smoother we may try to take the flow direction into account in the formulation of the DG approximation.

For the advection equation (1), the modification of the DG approximation can be done in a very straightforward way. Introducing a characteristic variable ξ as

$$\xi = y - ax$$

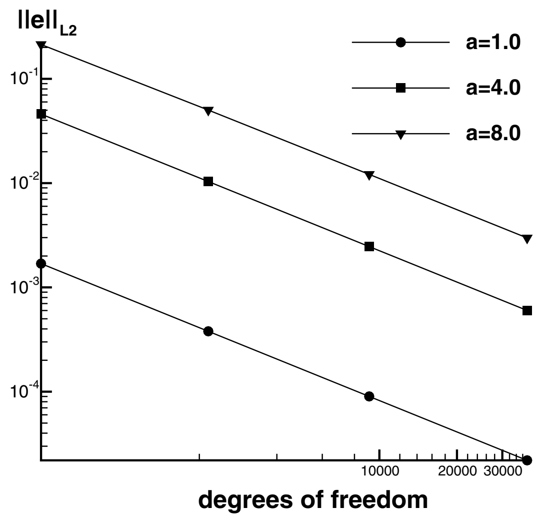

Fig. 3. Numerical solution to the problem (1), (2). Convergence plots for the polynomial degree $K = 1$ on a sequence of uniform grids ($\beta = 1.0$). The L_2-norm of the solution error is shown versus the number of the degrees of freedom for advection velocities $a = 1.0$, $a = 4.0$, and $a = 8.0$.

the solution (3) is $U(x,y) = u_0(\xi)$. Hence, the approximate solution (4) can be expanded as

$$u_h(x,y) = \sum_{k=0}^{K} u_k \tilde{\phi}_k(\xi), \quad x,y \in e_i, \tag{13}$$

where new basis functions

$$\tilde{\phi}_k = (\xi - \xi_0)^k, \quad k = 0,1,\ldots,K, \tag{14}$$

are now "anisotropic" on every grid cell. The coordinate ξ_0 is defined for a given cell e_i as $\xi_0 = y_{0i} - ax_{0i}$, where x_{0i} and y_{0i} are the coordinates of the grid cell centroid.

For the modified basis functions one can expect a smaller solution error for the same number of the degrees of freedom, as the solution is now better resolved by the choice of basis functions. The convergence history for a standard piecewise linear approximation (4) and a modified piecewise linear expansion (13) is shown in Fig. 4. The solution error (11) is computed on a sequence of uniform grids for advection velocity $a = 8.0$. The convergence plots demonstrate that the introduction of the modified basis functions into the problem reduces the solution error for a given number of the degrees of freedom.

Another advantage of the modified basis functions is that their application allows one to reduce the number of the degrees of freedom in the problem, since we consider a set of the basis functions along a single direction. This feature of the modified basis functions is illustrated by the following example. Let us consider the solution error on uniform grids for a set of standard (4) and modified (13) functions in the DG approximation. We want the number of the degrees of freedom to be the same in both cases. In case that we use the expansion (4) we need three basis functions for a piecewise linear solution approximation ($K = 1$). Meanwhile, three basis functions in the expansion (13) provide us with a piecewise quadratic solution reconstruction ($K = 2$). The convergence history on a sequence of uniform grids is shown in Fig. 5 for standard and modified solution expansion. The modified basis functions reconstruct a more accurate solution for a given number of degrees of freedom. Besides, a quadratic approximation results in a faster convergence rate in case that the modified basis functions are employed on each grid cell.

We now validate modified basis functions on distorted grids. It has been discussed in the previous chapter that our goal is to obtain a more accurate solution for the same number of the degrees of freedom. Thus we take a piecewise quadratic solution reconstruction that requires the same number of basis functions as a standard piecewise linear approximation (4). The results of the DG discretization for three basis functions in the expansion (13) are presented for various β in Table 3. It can be seen from the consideration of Tables 1 and 3 that the solution error on distorted grids is much smaller when modified basis functions are employed for the discretization.

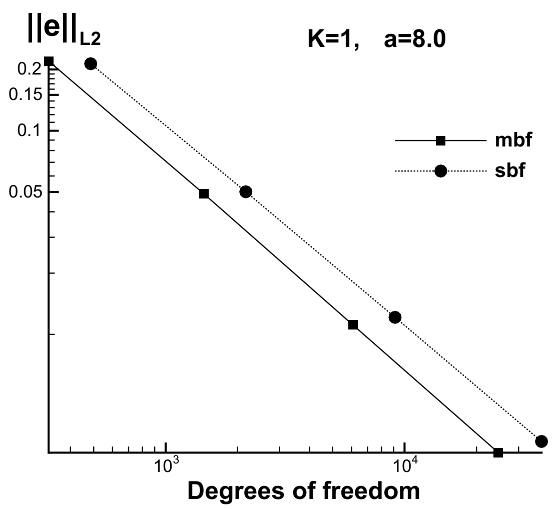

Fig. 4. Numerical solution to the problem (1), (2). Convergence plots on a sequence of uniform grids for advection velocity $a = 8.0$. Standard ("sbf") and modified ("mbf") basis functions are used for the polynomial degree $K = 1$ in the DG approximation.

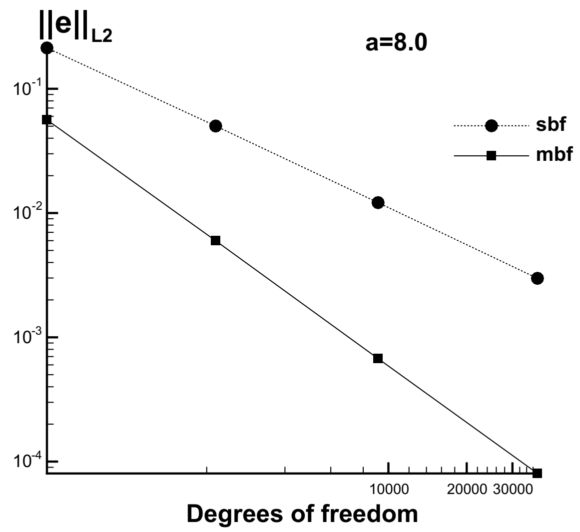

Fig. 5. Convergence plots for the problem (1), (2) on a sequence of uniform grids. Standard ("sbf") and modified ("mbf") basis functions are implemented for the same number of the degrees of freedom in a DG discretization. The advection velocity is $a = 8.0$.

Table 3
Numerical solution to the problem (1), (2)

β	1.0	0.8	0.6	0.4
$\Vert e\Vert_{L_2}$	7.6476e−6	8.1326e−5	3.8777e−4	1.5960e−3
$\Vert e\Vert_{L_\infty}$	1.5300e−5	2.4781e−4	1.7821e−3	8.4768e−3

The results of the modified DG discretization for various degrees of distortion β. The L_2-norm and the L_∞-norm of the solution error are computed on the grid with the number of grid cells $N_c = 722$.

Finally, we demonstrate how modified basis functions for Eq. (1) can be derived from a more general transformation of a coordinate system discussed previously in [8]. Consider a mapping of the (x, y)-plane onto the (ω_1, ω_2)-plane given by

$$\begin{aligned} \omega_1 &= x\cos\delta + y\sin\delta, \\ \omega_2 &= -x\sin\delta + y\cos\delta, \end{aligned} \tag{15}$$

where δ is the angle between the x-axis and the ω_1-axis. Since the transformation (15) is linear, a set of basis functions ϕ_m, $m = 0, 1, \ldots, M$ in the expansion (4) can be replaced by the set η_m, $m = 0, 1, \ldots, M$, where the new basis functions

$$\eta_m(\omega_1, \omega_2) = (\omega_1 - \omega_{1_{0i}})^\alpha(\omega_2 - \omega_{2_{0i}})^\beta, \quad \alpha + \beta = 0, 1, \ldots, K$$

are defined within grid cell e_i. The coordinates $\omega_{1_{0i}}$ and $\omega_{2_{0i}}$ are given by the transformation (15) of the centroid's coordinates x_{0i} and y_{0i}. The approximate solution in the new basis is expanded as

$$\begin{aligned} u_h(\omega_1, \omega_2) &= \sum_{m=0}^{M} u_m\eta_m(\omega_1, \omega_2) \\ &= u_0 + u_1(\omega_1 - \omega_{1_{0i}}) + u_2(\omega_2 - \omega_{2_{0i}}) + u_3(\omega_1 - \omega_{1_{0i}})^2 + u_4(\omega_1 - \omega_{1_{0i}})(\omega_2 - \omega_{2_{0i}}) \\ &\quad + u_5(\omega_2 - \omega_{2_{0i}})^2 + \cdots \end{aligned} \tag{16}$$

Comparing the expansion (16) with that in (13), we conclude that if the direction ω_1 is chosen as

$$\omega_1 = s\xi, \qquad s = \text{const},$$

then the number of basis functions required to reconstruct the solution to given accuracy can be reduced as it has been discussed in the previous chapter.

Let now $U(x,y)$ be a given function and ω_1 be the gradient direction, so that

$$\cos\delta = \frac{(\nabla u)_1}{\|\nabla u\|}, \quad \sin\delta = \frac{(\nabla u)_2}{\|\nabla u\|},$$

where ∇ is a formal notation for the gradient vector, $\nabla = (\nabla_1, \nabla_2) = (\partial/\partial x, \partial/\partial y)$. For the advection equation (1) the solution gradient is

$$\frac{\partial u(x,y)}{\partial x} = -a\frac{\mathrm{d}u_0(\xi)}{\mathrm{d}\xi}, \quad \frac{\partial u(x,y)}{\partial y} = \frac{\mathrm{d}u_0(\xi)}{\mathrm{d}\xi}, \quad \|\nabla u\| = \sqrt{1+a^2}\frac{\mathrm{d}u_0(\xi)}{\mathrm{d}\xi}.$$

Thus the angle δ is defined as follows:

$$\cos\delta = \frac{-a}{\sqrt{1+a^2}}, \quad \sin\delta = \frac{1}{\sqrt{1+a^2}}$$

and we have

$$\begin{aligned} \omega_1 &= s(y-ax) \equiv s\xi, \\ \omega_2 &= s(ay+x), \end{aligned} \tag{17}$$

where the scaling parameter is $s = 1/\sqrt{1+a^2}$. Hence, instead of using the uniform expansion (16) we can reconstruct the solution along the gradient direction using the smaller number of basis functions

$$u_h(\omega_1) = \sum_{m=0}^{M} u_m\eta_m(\omega_1) = u_0 + u_1(\omega_1 - \omega_{1_{0i}}) + u_3(\omega_1 - \omega_{1_{0i}})^2 + \cdots \tag{18}$$

The above consideration gives us the idea of the practical implementation of our approach in case that we cannot derive basis functions from the exact solution. The gradient direction should be defined at each grid cell based on the knowledge of the approximate solution and the gradient vector should be computed. We can then conclude whether the solution grows rapidly in the gradient direction or it remains isotropic over a given grid cell. In case that the solution gradient is large, anisotropic basis functions along the gradient direction can be applied for a DG approximation on the cell.

5. Concluding remarks

We have considered a high order DG discretization of a linear advection boundary-value problem on unstructured grids with arbitrary geometry of grid cells. It has been demonstrated that the high order DG scheme is sensitive to the geometry of a computational grid so that additional computational resources may be required to maintain the accuracy of the numerical solution. It is possible to reduce the computational cost of the discretization by introducing "solution dependent" basis functions into the problem. It has been shown in our work that using anisotropic basis functions allows one to increase the solution accuracy for a given number of the degrees of freedom.

Our study can be considered as a first step on the way of understanding the features of high order discretizations on unstructured grids with arbitrary geometry of grid cells. The future work will be focused on the further numerical validation of anisotropic basis functions. In our consideration of the advection equation we relied heavily upon the knowledge of the analytical solution to the advection BVP. Apparently, we need a more general approach to modify basis functions in the formulation of a DG discretization scheme, as the exact solution is not available in most problems of interest. Thus a direction of the future work will be to design anisotropic basis functions based on the solution gradient evaluation made for the numerical solution. A convection–diffusion equation provides a helpful test case where an analytical solution is still available and the modified basis functions obtained from the numerical solution can be compared with the solution expansion for the exact solution. The results of the convection–diffusion study should be further expanded to non-linear problems where the approximate solution is recomputed at each non-linear iteration.

The systems of equations can present another challenging topic for the future work. A usual approach in a DG method is to use the same basis functions for all variables in a vector equation [5]. Hence, there are a number of questions arising when different basis functions are employed to approximate the vector solution components. For instance, it is not clear if ''mixed'' basis functions are going to produce a well-conditioned discretization matrix and this issue requires further careful study.

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