On the intersection of infinite matroids

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Abstract

We show that the *infinite matroid intersection conjecture* of Nash-Williams implies the infinite Menger theorem proved by Aharoni and Berger in 2009.

We prove that this conjecture is true whenever one matroid is nearly finitary and the second is the dual of a nearly finitary matroid, where the nearly finitary matroids form a superclass of the finitary matroids.

In particular, this proves the infinite matroid intersection conjecture for finite-cycle matroids of 2-connected, locally finite graphs with only a finite number of vertex-disjoint rays.

1 Introduction

The infinite Menger theorem[1] was conjectured by Erdős in the 1960s and proved recently by Aharoni and Berger [1]. It states that for any two sets of vertices S and T in a connected graph, there is a set of vertex-disjoint S-T-paths whose maximality is witnessed by an S-T-separator picking exactly one vertex form each of these paths.

The complexity of the only known proof of this theorem and the fact that the finite Menger theorem has a short matroidal proof, make it natural to ask whether a matroidal proof of the infinite Menger theorem exists. In this paper, we propose a way to approach this problem by proving that a conjecture of Nash-Williams regarding infinite matroids implies the infinite Menger theorem.

Building on earlier work of Higgs and Oxley, recently, Bruhn, Diestel, Kriesell, Pendavingh and Wollan [7] found axioms for infinite matroids in terms of independent sets, bases, circuits, closure and (relative) rank. These

[1] see Theorem 3.1 below.

axioms allow for duality of infinite matroids as known from finite matroid theory, which settled an old problem of Rado. With these new axioms it is now possible to study which theorems of finite matroid theory have infinite analogues.

Here, we shall look at Edmond's *matroid intersection theorem*, which is a classical result in finite matroid theory [11]. It asserts that *the maximum size of a common independent set of two matroids M_1 and M_2 on a common ground set E is given by*

$$\min_{X \subseteq E} \mathrm{rk}_{M_1}(X) + \mathrm{rk}_{M_2}(E \setminus X), \tag{1}$$

where rk_{M_i} denotes the rank function of the matroid M_i.

In this paper, we consider the following conjecture of Nash-Williams, which first appeared in [2][2] and serves as an infinite analogue to the finite matroid intersection theorem[3].

Conjecture 1.1. [The infinite matroid intersection conjecture]
Any two matroids M_1 and M_2 on a common ground set E have a common independent set I admitting a partition $I = J_1 \cup J_2$ such that $\mathrm{cl}_{M_1}(J_1) \cup \mathrm{cl}_{M_2}(J_2) = E$.

Here, $\mathrm{cl}_M(X)$ denotes the *closure* of a set X in a matroid M; it consists of X and the elements spanned by X in M (see [11]).

1.1 Our results

Aharoni and Ziv [2] proved that Conjecture 1.1 implies the infinite analogues of König's and Hall's theorems. We strengthen this by showing that this conjecture implies the celebrated *infinite Menger theorem* (in the undirected version as stated in Theorem 3.1 below), which is known to imply the infinite analogues of König's and Hall's theorems [9].

Theorem 1.2. *The infinite matroid intersection conjecture for finitary matroids implies the infinite Menger theorem.*

We are able to prove new instances of Conjecture 1.1.[4], see Theorem 1.5 below. Before we can state this theorem, we need to introduce the class of

[2]Historical note: in [2], Nash-Williams's Conjecture is only made for *finitary matroids*, those all of whose circuits are finite.

[3]An alternative notion of infinite matroid intersection was recently proposed by Christian [8].

[4]The methods of this paper are refined in [5], which was submitted to the arxiv half a year after this paper.

‘nearly finitary matroids’. For any matroid M, taking as circuits only the finite circuits of M defines a (finitary) matroid with the same ground set as M. This matroid is called the *finitarization* of M and denoted by M^{fin}.

It is not hard to show that every basis B of M extends to a basis B^{fin} of M^{fin}, and conversely every basis B^{fin} of M^{fin} contains a basis B of M. Whether or not $B^{\text{fin}} \setminus B$ is finite will in general depend on the choices for B and B^{fin}, but given a choice for one of the two, it will no longer depend on the choice for the second one.

We call a matroid M *nearly finitary* if every base of its finitarization contains a base of M such that their difference is finite.

Next, let us look at some examples of nearly finitary matroids. There are three natural extensions to the notion of a finite graphic matroid in an infinite context [7]; each with ground set $E(G)$. The most studied one is the *finite-cycle matroid*, denoted $M_{FC}(G)$, whose circuits are the finite cycles of G. This is a finitary matroid, and hence is also nearly finitary.

The second extension is the *algebraic-cycle matroid*, denoted $M_A(G)$, whose circuits are the finite cycles and double rays of G [7, 6][5].

Proposition 1.3. *$M_A(G)$ is a nearly finitary matroid if and only if G has only a finite number of vertex-disjoint rays.*

The third extension is the *topological-cycle matroid*, denoted $M_C(G)$[6], whose circuits are the topological cycles of G (Thus $M_C^{\text{fin}}(G) = M_{FC}(G)$ for any finitely separable graph G; see Section 6.2 or [6] for definitions).

Proposition 1.4. *Suppose that G is 2-connected and locally finite. Then, $M_C(G)$ is a nearly finitary matroid if and only if G has only a finite number of vertex-disjoint rays.*

Here we prove the following.

Theorem 1.5. *Conjecture 1.1 holds for M_1 and M_2 whenever M_1 is nearly finitary and M_2 is the dual of a nearly finitary matroid.*

Aharoni and Ziv [2] proved that the infinite matroid intersection conjecture is true whenever one matroid is finitary and the other is a countable direct sum of finite-rank matroids. Note that Theorem 1.5 does not imply this result of [2] nor is it implied by it.

Proposition 1.4 and Theorem 1.5 can be used to prove the following.

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[5] $M_A(G)$ is not necessarily a matroid for any G; see [10].

[6] $M_C(G)$ is a matroid for any G; see [6].

Corollary 1.6. *Suppose that G and H are 2-connected, locally finite graphs with only a finite number of vertex-disjoint rays. Then their finite-cycle matroids $M_{FC}(G)$ and $M_{FC}(H)$ satisfy the intersection conjecture.*

Similar results are true for *the algebraic-cycle matroid*, *the topological-cycle matroid*, and their duals.

1.2 An overview of the proof of Theorem 1.5

In finite matroid theory, an exceptionally short proof of the matroid intersection theorem employing the well-known *finite matroid union theorem* [11, 12] is known. The latter theorem asserts[7] that for two finite matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ the set system

$$\mathcal{I}(M_1 \vee M_2) = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\} \tag{2}$$

forms the set of independent sets of their *union matroid* $M_1 \vee M_2$. Throughout, M^* denotes the dual of a matroid M. We prove that this strategy of proof extends to infinite matroids.

Theorem 1.7. *If M_1 and M_2 are matroids on a common ground set E and $M_1 \vee M_2^*$ is a matroid, then Conjecture 1.1 holds for M_1 and M_2.*

Thus in order to prove Conjecture 1.1, it would be enough to prove that the union of any two matroids is a matroid. Unfortunately, this is not true.[8] We provide examples in Section 7. However, we can prove that the union of two nearly finitary matroids is a matroid.

Theorem 1.8. *If M_1 and M_2 are nearly finitary matroids, then $M_1 \vee M_2$ is a nearly finitary matroid.*

Hence Theorem 1.5 follows from combining Theorem 1.8 and Theorem 1.7.

This paper is organized as follows. Additional notation, terminology, and basic lemmas are given in Section 2. In Section 3 we prove Theorem 1.2. In Section 4 we prove Theorem 1.8. In Section 5 we prove Theorem 1.7, and in Section 6 we prove Propositions 1.3 and 1.4 and Corollary 1.6. In Section 7, we construct matroids whose union is not a matroid.

[7] Often the matroid union theorem is complemented by a formula for the rank function of the union. This, however, is implied by the fact that the union is a matroid (as follows from Theorem 1.7 below and results of [5]). This rank formula and its relation to Conjecture 1.1 is studied in [5].

[8] This is not that surprsing as the methods of this paper are much more elementary than those developed by Aharoni and Berger in [1].

2 Preliminaries

Notation and terminology for graphs are that of [9], and for matroids that of [7, 11].

Throughout, G always denotes a graph where $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. We write M to denote a matroid and write $E(M)$, $\mathcal{I}(M)$, $\mathcal{B}(M)$, and $\mathcal{C}(M)$ to denote its ground set, independent sets, bases, and circuits, respectively.

It will be convenient to have a similar notation for set systems. That is, for a set system $\mathcal{I}$ over some ground set E, an element of $\mathcal{I}$ is called *independent*, a maximal element of $\mathcal{I}$ is called a *base* of $\mathcal{I}$, and a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ is called *circuit* of $\mathcal{I}$. A set system is *finitary* if an infinite set belongs to the system provided each of its finite subsets does; with this terminology, M is finitary provided that $\mathcal{I}(M)$ is finitary.

We review the definition of a matroid as this is given in [7]. A set system $\mathcal{I}$ is the set of independent sets of a matroid if it satisfies the following *independence axioms*:

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\lceil \mathcal{I} \rceil = \mathcal{I}$, that is, $\mathcal{I}$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

In [7], an equivalent axiom system to the independence axioms is provided and is called the *circuit axioms system*; this axiom system characterises a matroid in terms of its circuits. Of these circuit axioms, we shall make frequent use of the so called *(infinite) circuit elimination axiom* phrased here for a matroid M:

(C) Whenever $X \subseteq C \in \mathcal{C}(M)$ and $\{C_x \mid x \in X\} \subseteq \mathcal{C}(M)$ satisfies $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \setminus \left(\bigcup_{x \in X} C_x\right)$ there exists a $C' \in \mathcal{C}(M)$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X$.

The following is a well-known fact for finite matroids (see, e.g., [11]), which generalizes easily to infinite matroids.

Lemma 2.1. [7, Lemma 3.11]
Let M be a matroid. Then, $|C \cap C^*| \neq 1$, whenever $C \in \mathcal{C}(M)$ and $C^* \in \mathcal{C}(M^*)$.

3 From infinite matroid intersection to the infinite Menger theorem

In this section, we prove Theorem 1.2; asserting that the infinite matroid intersection conjecture implies the infinite Menger theorem.

Given a graph G and $S, T \subseteq V(G)$, a set $X \subseteq V(G)$ is called an S–T *separator* if $G - X$ contains no S–T path. The infinite Menger theorem reads as follows.

Theorem 3.1 (Aharoni and Berger [1]). *Let G be a connected graph. Then for any $S, T \subseteq V(G)$ there is a set $\mathcal{L}$ of vertex-disjoint S–T paths and an S–T separator $X \subseteq \bigcup_{P \in \mathcal{L}} V(P)$ satisfying $|X \cap V(P)| = 1$ for each $P \in \mathcal{L}$.*

Infinite matroid union cannot be used in order to obtain the infinite Menger Theorem directly via Theorem 1.7 and Theorem 1.2. Indeed, in Section 7 we construct a finitary matroid M and a co-finitary matroid N such that their union is not a matroid. Consequently, one cannot apply Theorem 1.7 to the finitary matroids M and N^* in order to obtain Conjecture 1.1 for them. However, it is easy to see that Conjecture 1.1 is true for these particular M and N^*.

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a connected graph and let $S, T \subseteq V(G)$ be as in Theorem 3.1. We may assume that $G[S]$ and $G[T]$ are both connected. Indeed, an S–T separator with $G[S]$ and $G[T]$ connected gives rise to an S–T separator when these are not necessarily connected. Abbreviate $E(S) := E(G[S])$ and $E(T) := E(G[T])$, let M be the finite-cycle matroid $M_F(G)$, and put $M_S := M/E(S) - E(T)$ and $M_T := M/E(T) - E(S)$; all three matroids are clearly finitary.

Assuming that the infinite matroid intersection conjecture holds for M_S and M_T, there exists a set $I \in \mathcal{I}(M_S) \cap \mathcal{I}(M_T)$ which admits a partition $I = J_S \cup J_T$ satisfying

$$\mathrm{cl}_{M_S}(J_S) \cup \mathrm{cl}_{M_T}(J_T) = E,$$

where $E = E(M_S) = E(M_T)$. We regard I as a subset of $E(G)$.

For the components of $G[I]$ we observe two useful properties. As I is disjoint from $E(S)$ and $E(T)$, the edges of a cycle in a component of $G[I]$ form a circuit in both, M_S and M_T, contradicting the independence of I in either. Consequently,

$$\text{the components of } G[I] \text{ are trees.} \qquad (3)$$

Next, an S-path[9] or a T-path in a component of $G[I]$ gives rise to a circuit of M_S or M_T in I, respectively. Hence,

$$|V(C) \cap S| \leq 1 \text{ and } |V(C) \cap T| \leq 1 \text{ for each component } C \text{ of } G[I]. \quad (4)$$

Let $\mathcal{C}$ denote the components of $G[I]$ meeting both of S and T. Then by (3) and (4) each member of $\mathcal{C}$ contains a unique S–T path and we denote the set of all these paths by $\mathcal{L}$. Clearly, the paths in $\mathcal{L}$ are vertex-disjoint.

In what follows, we find a set X comprised of one vertex from each $P \in \mathcal{L}$ to serve as the required S–T separator. To that end, we show that one may alter the partition $I = J_S \cup J_T$ to yield a partition

$$I = K_S \cup K_T \text{ satisfying } cl_{M_S}(K_S) \cup cl_{M_T}(K_T) = E \text{ and (Y.1-4)}, \quad (5)$$

where (Y.1-4) are as follows.

(Y.1) Each component C of $G[I]$ contains a vertex of $S \cup T$.

(Y.2) Each component C of $G[I]$ meeting S but not T satisfies $E(C) \subseteq K_S$.

(Y.3) Each component C of $G[I]$ meeting T but not S satisfies $E(C) \subseteq K_T$.

(Y.4) Each component C of $G[I]$ meeting both, S and T, contains at most one vertex which at the same time

(a) lies in S or is incident with an edge of K_S, and

(b) lies in T or is incident with an edge of K_T.

Postponing the proof of (5), we first show how to deduce the existence of the required S–T separator from (5). Define a pair of sets of vertices (V_S, V_T) of $V(G)$ by letting V_S consist of those vertices contained in S or incident with an edge of K_S and defining V_T in a similar manner. Then $V_S \cap V_T$ may serve as the required S–T separator. To see this, we verify below that (V_S, V_T) satisfies all of the terms (Z.1-4) stated next.

(Z.1) $V_S \cup V_T = V(G)$;

(Z.2) for every edge e of G either $e \subseteq V_S$ or $e \subseteq V_T$;

(Z.3) every vertex in $V_S \cap V_T$ lies on a path from $\mathcal{L}$; and

(Z.4) every member of $\mathcal{L}$ meets $V_S \cap V_T$ at most once.

[9] A non-trivial path meeting $G[S]$ exactly in its end vertices.

To see (Z.1), suppose v is a vertex not in $S \cup T$. As G is connected, such a vertex is incident with some edge $e \notin E(T) \cup E(S)$. The edge e is spanned by K_T or K_S; say K_T. Thus, $K_T + e$ contains a circle containing e or $G[K_T + e]$ has a T-path containing e. In either case v is incident with an edge of K_T and thus in V_T, as desired.

To see (Z.2), let $e \in cl_{M_T}(K_T) \setminus K_T$; so that $K_T + e$ has a circle containing e or $G[K_T + e]$ has T-path containing e; in either case both end vertices of e are in V_T, as desired. The treatment of the case $e \in cl_{M_S}(K_S)$ is similar.

To see (Z.3), let $v \in V_S \cap V_T$; such is in S or is incident with an edge of K_S, and in T or is incident with an edge in K_T. Let C be the component of $G[I]$ containing v. By (Y.1-4), $C \in \mathcal{C}$, i.e. it meets both, S and T and therefore contains an S–T path $P \in \mathcal{L}$. Recall that every edge of C is either in K_S or K_T and consider the last vertex w of a maximal initial segment of P in $C - K_T$. Then w satisfies (Y.4a), as well as (Y.4b), implying $v = w$; so that v lies on a path from $\mathcal{L}$.

To see (Z.4), we restate (Y.4) in terms of V_S and V_T: each component of $\mathcal{C}$ contains at most one vertex of $V_S \cap V_T$. This clearly also holds for the path from $\mathcal{L}$ which is contained in C.

It remains to prove (5). To this end, we show that any component C of $G[I]$ contains a vertex of $S \cup T$. Suppose not. Let e be the first edge on a $V(C)$–S path Q which exists by the connectedness of G. Then $e \notin I$ but without loss of generality we may assume that $e \in \mathrm{cl}_{M_S}(J_S)$. So in $G[I] + e$ there must be a cycle or an S-path. The latter implies that C contains a vertex of S and the former means that Q was not internally disjoint to $V(C)$, yielding contradictions in both cases.

We define the sets K_S and K_T as follows. Let C be a component of $G[I]$.

1. If C meets S but not T, then include its edges into K_S.
2. If C meets T but not S, then include its edges into K_T.
3. Otherwise (C meets both of S and T) there is a path P from $\mathcal{L}$ in C. Denote by v_C the last vertex of a maximal initial segment of P in $C - J_T$. As C is a tree, each component C' of $C - v_C$ is a tree and there is a unique edge e between v_C and C'. For every such component C', include the edges of $C' + e$ in K_S if $e \in J_S$ and in K_T otherwise, i.e. if $e \in J_T$.

Note that, by choice of v_C, either v_C is the last vertex of P or the next edge of P belongs to J_T. This ensures that K_S and K_T satisfy (Y.4). Moreover, they form a partition of I which satisfies (Y.1-3) by construction. It remains to show that $\mathrm{cl}_{M_S}(K_S) \cup \mathrm{cl}_{M_T}(K_T) = E$.

As $K_S \cup K_T = I$, it suffices to show that any $e \in E \setminus I$ is spanned by K_S in M_S or by K_T in M_T. Suppose $e \in \mathrm{cl}_{M_S}(J_S)$, i.e. $J_S + e$ contains a circuit of M_S. Hence, $G[J_S]$ either contains an e-path R or two disjoint e–S paths R_1 and R_2. We show that $E(R) \subseteq K_S$ or $E(R) \subseteq K_T$ in the former case and $E(R_1) \cup E(R_2) \subseteq K_S$ in the latter.

The path R is contained in some component C of $G[I]$. Suppose $C \in \mathcal{C}$ and v_C is an inner vertex of R. By assumption, the edges preceding and succeeding v_C on R are both in J_S and hence the edges of both components of $C - v_C$ which are met by R plus their edges to v_C got included into K_S, showing $E(R) \subseteq K_S$. Otherwise $C \notin \mathcal{C}$ or $C \in \mathcal{C}$ but v_C is no inner vertex of R. In both cases the whole set $E(R)$ got included into K_S or K_T.

We apply the same argument to R_1 and R_2 except for one difference. If $C \notin \mathcal{C}$ or $C \in \mathcal{C}$ but v_C is no inner vertex of R_i, then $E(R_i)$ got included into K_S as R_i meets S.

Although the definitions of K_S and K_T are not symmetrical, a similar argument shows $e \in \mathrm{cl}_{M_S}(K_S) \cup \mathrm{cl}_{M_T}(K_T)$ if e is spanned by J_T in M_T. $\square$

Note that the above proof requires only that Conjecture 1.1 holds for finite-cycle matroids.

4 Union

In this section, we prove Theorem 1.8. The main difficulty in proving this theorem is the need to verify that given two nearly finitary matroids M_1 and M_2, that the set system $\mathcal{I}(M_1 \vee M_2)$ satisfies the axioms (IM) and (I3).

To verify the (IM) axiom for the union of two nearly finitary matroids we shall require the following theorem, proved below in Section 4.2.

Proposition 4.1. *If M_1 and M_2 are finitary matroids, then $M_1 \vee M_2$ is a finitary matroid.*

To verify (IM) for the union of finitary matroids we use a compactness argument (see Section 4.2). More specifically, we will show that $\mathcal{I}(M_1 \vee M_2)$ is a finitary set system whenever M_1 and M_2 are finitary matroids. It is then an easy consequence of Zorn's lemma that all finitary set systems satisfy (IM).

The verification of axiom (I3) is dealt in a joint manner for both matroid families. In the next section we prove the following.

Proposition 4.2. *The set system $\mathcal{I}(M_1 \vee M_2)$ satisfies* (I3) *for any two matroids M_1 and M_2.*

Indeed, for finitary matroids, Proposition 4.2 is fairly simple to prove. We, however, require this proposition to hold for nearly finitary matroids as well. Consequently, we prove this proposition in its full generality, i.e., for any pair of matroids. In fact, it is interesting to note that the union of infinitely many matroids satisfies (I3); though the axiom (IM) might be violated as seen in Observation 4.10).

At this point it is insightful to note a certain difference between the union of finite matroids to that of finitary matroids in a more precise manner. By the finite matroid union theorem if M admits two disjoint bases, then the union of these bases forms a base of $M \vee M$. For finitary matroids the same assertion is false.

Claim 4.3. *There exists an infinite finitary matroid M with two disjoint bases whose union is not a base of the matroid $M \vee M$ as it is properly contained in the union of some other two bases.*

Proof. Consider the infinite one-sided ladder with every edge doubled, say H, and recall that the bases of $M_F(H)$ are the ordinary spanning trees of H. In Figure 1, (B_1, B_2) and (B_3, B_4) are both pairs of disjoint bases of $M_F(H)$. However, $B_3 \cup B_4$ properly covers $B_1 \cup B_2$ as it additionally contains the leftmost edge of H □

Clearly, a direct sum of infinitely many copies of H gives rise to an infinite sequence of unions of disjoint bases, each properly containing the previous one. In fact, one can construct a (single) matroid formed as the union of two nearly finitary matroids that admits an infinite properly nested sequence of unions of disjoint bases.

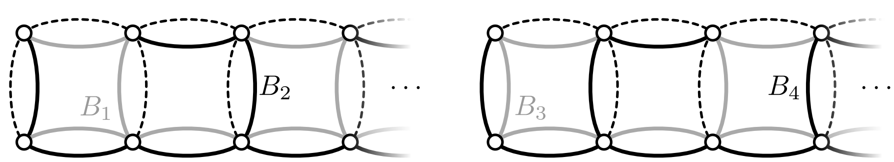

Figure 1: The disjoint bases B_1 and B_2 on the left are properly covered by the bases B_3 and B_4 on the right.

4.1 Exchange chains and the verification of axiom (I3)

In this section, we prove Proposition 4.2. Throughout this section M_1 and M_2 are matroids. It will be useful to show that the following variant of (I3) is satisfied.

Proposition 4.4. *The set $\mathcal{I} = \mathcal{I}(M_1 \vee M_2)$ satisfies the following.*

(I3') For all $I, B \in \mathcal{I}$ where B is maximal and all $x \in I \setminus B$ there exists $y \in B \setminus I$ such that $(I + y) - x \in \mathcal{I}$.

Observe that unlike in (I3), the set I in (I3') may be maximal.

We begin by showing that Proposition 4.4 implies Proposition 4.2.

Proof of Proposition 4.2 from Proposition 4.4. Let $I \in \mathcal{I}$ be non-maximal and $B \in \mathcal{I}$ be maximal. As I is non-maximal there is an $x \in E \setminus I$ such that $I + x \in \mathcal{I}$. We may assume $x \notin B$ or the assertion follows by (I2). By (I3'), applied to $I + x$, B, and $x \in (I + x) \setminus B$ there is $y \in B \setminus (I + x)$ such that $I + y \in \mathcal{I}$. □

We proceed to prove Proposition 4.4. The following notation and terminology will be convenient. A circuit of M which contains a given set $X \subseteq E(M)$ is called an *X-circuit*.

By a *representation of* a set $I \in \mathcal{I}(M_1 \vee M_2)$, we mean a pair (I_1, I_2) where $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$ such that $I = I_1 \cup I_2$.

For sets $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$, and elements $x \in I_1 \cup I_2$ and $y \in E(M_1) \cup E(M_2)$ (possibly in $I_1 \cup I_2$), a tuple $Y = (y_0 = y, \ldots, y_n = x)$ with $y_i \neq y_{i+1}$ for all i is called an *even* (I_1, I_2, y, x)*-exchange chain*[10] (or *even* (I_1, I_2, y, x)*-chain*) of *length* n if the following terms are satisfied.

(X1) For an even i, there exists a $\{y_i, y_{i+1}\}$-circuit $C_i \subseteq I_1 + y_i$ of M_1.

(X2) For an odd i, there exists a $\{y_i, y_{i+1}\}$-circuit $C_i \subseteq I_2 + y_i$ of M_2.

If $n \geq 1$, then (X1) and (X2) imply that $y_0 \notin I_1$ and that, starting with $y_1 \in I_1 \setminus I_2$, the elements y_i alternate between $I_1 \setminus I_2$ and $I_2 \setminus I_1$; the single exception being y_n which can lie in $I_1 \cap I_2$.

By an *odd exchange chain* (or *odd chain*) we mean an even chain with the words 'even' and 'odd' interchanged in the definition. Consequently, we say *exchange chain* (or *chain*) to refer to either of these notions. Furthermore, a subchain of a chain is also a chain; that is, given an (I_1, I_2, y_0, y_n)-chain $(y_0, \ldots, y_n)$, the tuple $(y_k, \ldots, y_l)$ is an (I_1, I_2, y_k, y_l)-chain for $0 \leq k \leq l \leq n$.

Lemma 4.5. *If there exists an (I_1, I_2, y, x)-chain, then $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$ where $I := I_1 \cup I_2$. Moreover, if $x \in I_1 \cap I_2$, then $I + y \in \mathcal{I}(M_1 \vee M_2)$.*

[10]Some authors call them *augmenting paths*

Remark. In the proof of Lemma 4.5 chains are used in order to alter the sets I_1 and I_2; the change is in a single element. Nevertheless, to accomplish this change, exchange chain of arbitrary length may be required; for instance, a chain of length four is needed to handle the configuration depicted in Figure 2.

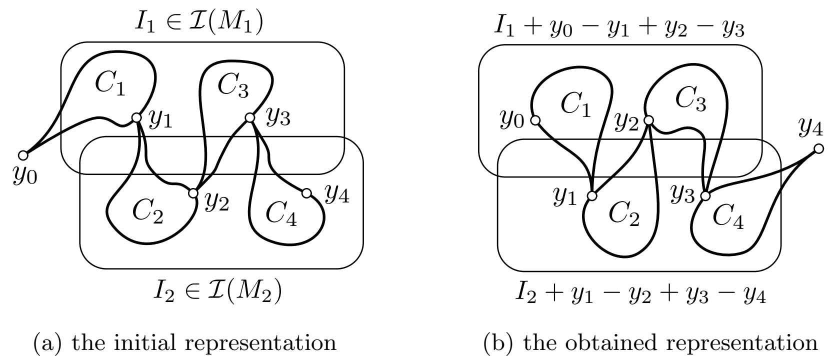

Figure 2: An even exchange chain of length 4.

Next, we prove Lemma 4.5.

Proof of Lemma 4.5. The proof is by induction on the length of the chain. The statement is trivial for chains of length 0. Assume $n \geq 1$ and that $Y = (y_0, \ldots, y_n)$ is a shortest (I_1, I_2, y, x)-chain. Without loss of generality, let Y be an even chain. If $Y' := (y_1, \ldots, y_n)$ is an (odd) (I_1', I_2, y_1, x)-chain where $I_1' := (I_1 + y_0) - y_1$, then $((I_1' \cup I_2) + y_1) - x \in \mathcal{I}(M_1 \vee M_2)$ by the induction hypothesis and the assertion follows, since $(I_1' \cup I_2) + y_1 = (I_1 \cup I_2) + y_0$. If also $x \in I_1 \cap I_2$, then either $x \in I_1' \cap I_2$ or $y_1 = x$ and hence $n = 1$. In the former case $I + y \in \mathcal{I}(M_1 \vee M_2)$ follows from the induction hypothesis and in the latter case $I + y = I_1' \cup I_2 \in \mathcal{I}(M_1 \vee M_2)$ as $x \in I_2$.

Since I_2 has not changed, (X2) still holds for Y', so to verify that Y' is an (I_1', I_2, y_1, x)-chain, it remains to show $I_1' \in \mathcal{I}(M_1)$ and to check (X1). To this end, let C_i be a $\{y_i, y_{i+1}\}$-circuit of M_1 in $I_1 + y_i$ for all even i. Such exist by (X1) for Y. Notice that any circuit of M_1 in $I_1 + y_0$ has to contain y_0 since $I_1 \in \mathcal{I}(M_1)$. On the other hand, two distinct circuits in $I_1 + y_0$ would give rise to a circuit contained in I_1 by the circuit elimination axiom applied to these two circuits, eliminating y_0. Hence C_0 is the unique circuit of M_1 in $I_1 + y_0$ and $y_1 \in C_0$ ensures $I_1' = (I_1 + y_0) - y_1 \in \mathcal{I}(M_1)$.

To see (X1), we show that there is a $\{y_i, y_{i+1}\}$-circuit C_i' of M_1 in $I_1' + y_i$ for every even $i \geq 2$. Indeed, if $C_i \subseteq I_1' + y_i$, then set $C_i' := C_i$; else, C_i

contains an element of $I_1 \setminus I_1' = \{y_1\}$. Furthermore, $y_{i+1} \in C_i \setminus C_0$; otherwise $(y_0, y_{i+1}, \ldots, y_n)$ is a shorter (I_1, I_2, y, x)-chain for, contradicting the choice of Y. Applying the circuit elimination axiom to C_0 and C_i, eliminating y_1 and fixing y_{i+1}, yields a circuit $C_i' \subseteq (C_0 \cup C_i) - y_1$ of M_1 containing y_{i+1}. Finally, as I_1' is independent and $C_i' \setminus I_1' \subseteq \{y_i\}$ it follows that $y_i \in C_i'$. □

We shall require the following. For $I_1 \in \mathcal{I}(M_1)$, $I_2 \in \mathcal{I}(M_2)$, and $x \in I_1 \cup I_2$, let

$$A(I_1, I_2, x) := \{a \mid \text{there exists an } (I_1, I_2, a, x)\text{-chain}\}.$$

This has the property that

$$\text{for every } y \notin A, \text{ either } I_1 + y \in \mathcal{I}(M_1) \text{ or the unique circuit } C_y \text{ of } M_1 \text{ in } I_1 + y \text{ is disjoint from } A. \tag{6}$$

To see this, suppose $I_1 + y \notin \mathcal{I}(M_1)$. Then there is a unique circuit C_y of M_1 in $I_1 + y$. If $C_y \cap A = \emptyset$, then the assertion holds so we may assume that $C_y \cap A$ contains an element, a say. Hence there is an (I_1, I_2, a, x)-chain $(y_0 = a, y_1, \ldots, y_{n-1}, y_n = x)$. As $a \in I_1$ this chain must be odd or have length 0, that is, $a = x$. Clearly, $(y, a, y_1, \ldots, y_{n-1}, x)$ is an even (I_1, I_2, y, x)-chain, contradicting the assumption that $y \notin A$.

Next, we prove Proposition 4.4.

Proof of Proposition 4.4. Let $B \in \mathcal{I}(M_1 \vee M_2)$ maximal, $I \in \mathcal{I}(M_1 \vee M_2)$, and $x \in I \setminus B$. Recall that we seek a $y \in B \setminus I$ such that $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$. Let (I_1, I_2) and (B_1, B_2) be representations of I and B, respectively. We may assume $I_1 \in \mathcal{B}(M_1|I)$ and $I_2 \in \mathcal{B}(M_2|I)$. We may further assume that for all $y \in B \setminus I$ the sets $I_1 + y$ and $I_2 + y$ are dependent in M_1 and M_2, respectively, for otherwise it holds that $I + y \in \mathcal{I}(M_1 \vee M_2)$ so that the assertion follows. Hence, for every $y \in (B \cup I) \setminus I_1$ there is a circuit $C_y \subseteq I_1 + y$ of M_1; such contains y and is unique since otherwise the circuit elimination axiom applied to these two circuits eliminating y yields a circuit contained in I_1, a contradiction.

If $A := A(I_1, I_2, x)$ intersects $B \setminus I$, then the assertion follows from Lemma 4.5. Else, $A \cap (B \setminus I) = \emptyset$, in which case we derive a contradiction to the maximality of B. To this end, set (Figure 3)

$$B_1' := (B_1 \setminus b_1) \cup i_1 \qquad \text{where} \quad b_1 := B_1 \cap A \quad \text{and} \quad i_1 := I_1 \cap A$$

$$B_2' := (B_2 \setminus b_2) \cup i_2 \qquad \text{where} \quad b_2 := B_2 \cap A \quad \text{and} \quad i_2 := I_2 \cap A$$

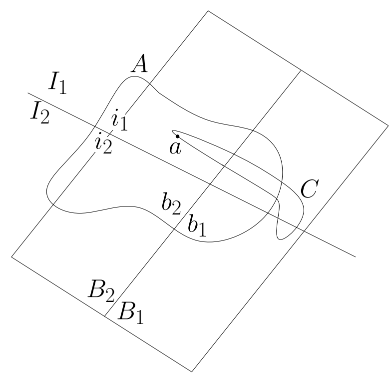

Figure 3: The independent sets I_1, at the top, and I_2, at the bottom, the bases B_1, on the right, and B_2, on the left, and their intersection with A.

Since A contains x but is disjoint from $B \setminus I$, it holds that $(b_1 \cup b_2) + x \subseteq i_1 \cup i_2$ and thus $B + x \subseteq B_1' \cup B_2'$. It remains to verify the independence of B_1' and B_2' in M_1 and M_2, respectively.

Without loss of generality it is sufficient to show $B_1' \in \mathcal{I}(M_1)$. For the remainder of the proof 'independent' and 'circuit' refer to the matroid M_1. Suppose for a contradiction that the set B_1' is dependent, that is, it contains a circuit C. Since i_1 and $B_1 \setminus b_1$ are independent, neither of these contain C. Hence there is an element $a \in C \cap i_1 \subseteq A$. But $C \setminus I_1 \subseteq B_1 \setminus A$ and therefore no C_y with $y \in C \setminus I_1$ contains a by (6). Thus, applying the circuit elimination axiom on C eliminating all $y \in C \setminus I_1$ via C_y fixing a, yields a circuit in I_1, a contradiction. □

Since in the proof of Proposition 4.4 the maximality of B is only used in order to avoid the case that $B + x \in \mathcal{I}(M_1 \vee M_2)$, one may prove the following slightly stronger statement.

Corollary 4.6. *For all $I, J \in \mathcal{I}(M_1 \vee M_2)$ and $x \in I \setminus J$, if $J + x \notin \mathcal{I}(M_1 \vee M_2)$, then there exists $y \in J \setminus I$ such that $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$.*

Next, the proof of Proposition 4.4, shows that for any maximal representation (I_1, I_2) of I there is $y \in B \setminus I$ such that exchanging finitely many elements of I_1 and I_2 gives a representation of $(I + y) - x$.

For subsequent arguments, it will be useful to note the following corollary. Above we used chains whose last element is fixed. One may clearly use chains whose first element is fixed. If so, then one arrives at the following.

Corollary 4.7. *For all $I, J \in \mathcal{I}(M_1 \vee M_2)$ and $y \in J \setminus I$, if $I + y \notin \mathcal{I}(M_1 \vee M_2)$, then there exists $x \in I \setminus J$ such that $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$.*

4.2 Finitary matroid union

In this section, we prove Proposition 4.1. In view of Proposition 4.2, it remains to show that $\mathcal{I}(M_1 \vee M_2)$ satisfies (IM) whenever M_1 and M_2 are finitary matroids.

The verification of (IM) for countable finitary matroids can be done using König's infinity lemma. Here, in order to capture matroids on any infinite ground set, we employ a topological approach. See [4] for the required topological background needed here.

We recall the definition of the product topology on $\mathcal{P}(E)$. The usual base of this topology is formed by the system of all sets

$$C(A, B) := \{X \subseteq E \mid A \subseteq X, B \cap X = \emptyset\},$$

where $A, B \subseteq E$ are finite and disjoint. Note that these sets are closed as well. Throughout this section, $\mathcal{P}(E)$ is endowed with the product topology and *closed* is used in the topological sense only.

We show that Proposition 4.1 can easily be deduced from Proposition 4.8 and Lemma 4.9, presented next.

Proposition 4.8. *Let $\mathcal{I} = \lceil \mathcal{I} \rceil \subseteq \mathcal{P}(E)$. The following are equivalent.*

4.8.1. $\mathcal{I}$ is finitary;

4.8.2. $\mathcal{I}$ is compact, in the subspace topology of $\mathcal{P}(E)$.

A standard compactness argument can be used in order to prove 4.8.1. Here, we employ a slightly less standard argument to prove 4.8.2 as well. Note that as $\mathcal{P}(E)$ is a compact Hausdorff space, assertion 4.8.2 is equivalent to the assumption that $\mathcal{I}$ is closed in $\mathcal{P}(E)$, which we use quite often in the following proofs.

Proof of Proposition 4.8. To deduce 4.8.2 from 4.8.1, we show that $\mathcal{I}$ is closed. Let $X \notin \mathcal{I}$. Since $\mathcal{I}$ is finitary, X has a finite subset $Y \notin \mathcal{I}$ and no superset of Y is in $\mathcal{I}$ as $\mathcal{I} = \lceil \mathcal{I} \rceil$. Therefore, $C(Y, \emptyset)$ is an open set containing X and avoiding $\mathcal{I}$ and hence $\mathcal{I}$ is closed.

For the converse direction, assume that $\mathcal{I}$ is compact and let X be a set such that all finite subsets of X are in $\mathcal{I}$. We show $X \in \mathcal{I}$ using the finite intersection property[11] of $\mathcal{P}(E)$. Consider the family $\mathcal{K}$ of pairs (A, B) where $A \subseteq X$ and $B \subseteq E \setminus X$ are both finite. The set $C(A, B) \cap \mathcal{I}$ is closed for every $(A, B) \in \mathcal{K}$, as $C(A, B)$ and $\mathcal{I}$ are closed. If $\mathcal{L}$ is a finite subfamily of $\mathcal{K}$, then

$$\bigcup_{(A,B)\in\mathcal{L}} A \in \bigcap_{(A,B)\in\mathcal{L}} \left(C(A, B) \cap \mathcal{I}\right).$$

As $\mathcal{P}(E)$ is compact, the finite intersection property yields

$$\left(\bigcap_{(A,B)\in\mathcal{K}} C(A, B)\right) \cap \mathcal{I} = \bigcap_{(A,B)\in\mathcal{K}} \left(C(A, B) \cap \mathcal{I}\right) \neq \emptyset.$$

However, $\bigcap_{(A,B)\in\mathcal{K}} C(A, B) = \{X\}$. Consequently, $X \in \mathcal{I}$, as desired. □

Lemma 4.9. *If $\mathcal{I}$ and $\mathcal{J}$ are closed in $\mathcal{P}(E)$, then so is $\mathcal{I} \vee \mathcal{J}$.*

Proof. Equipping $\mathcal{P}(E)\times\mathcal{P}(E)$ with the product topology, yields that Cartesian products of closed sets in $\mathcal{P}(E)$ are closed in $\mathcal{P}(E)\times\mathcal{P}(E)$. In particular, $\mathcal{I} \times \mathcal{J}$ is closed in $\mathcal{P}(E) \times \mathcal{P}(E)$. In order to prove that $\mathcal{I} \vee \mathcal{J}$ is closed, we note that $\mathcal{I} \vee \mathcal{J}$ is exactly the image of $\mathcal{I} \times \mathcal{J}$ under the union map

$$f : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{P}(E), \quad f(A, B) = A \cup B.$$

It remains to check that f maps closed sets to closed sets; which is equivalent to showing that f maps compact sets to compact sets as $\mathcal{P}(E)$ is a compact Hausdorff space. As continuous images of compact spaces are compact, it suffices to prove that f is continuous, that is, to check that the pre-images of subbase sets $C(\{a\}, \emptyset)$ and $C(\emptyset, \{b\})$ are open as can be seen here:

$$\begin{aligned} f^{-1}(C(\{a\}, \emptyset)) &= (C(\{a\}, \emptyset) \times \mathcal{P}(E)) \cup (\mathcal{P}(E) \times C(\{a\}, \emptyset)) \\ f^{-1}(C(\emptyset, \{b\})) &= C(\emptyset, \{b\}) \times C(\emptyset, \{b\}). \end{aligned}$$

□

Next, we prove Proposition 4.1.

Proof of Proposition 4.1. By Proposition 4.2 it remains to show that the union $\mathcal{I}(M_1) \vee \mathcal{I}(M_2)$ satisfies (IM). As all finitary set systems satisfy (IM), by Zorn's lemma, it is sufficient to show that $\mathcal{I}(M_1 \vee M_2)$ is finitary. By Proposition 4.8, $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$ are both compact and thus closed in

[11] *The finite intersection property* ensures that an intersection over a family $\mathcal{C}$ of closed sets is non-empty if every intersection of finitely many members of $\mathcal{C}$ is.

$\mathcal{P}(E)$, yielding, by Lemma 4.9, that $\mathcal{I}(M_1) \vee \mathcal{I}(M_2)$ is closed in $\mathcal{P}(E)$, and thus compact. As $\mathcal{I}(M_1) \vee \mathcal{I}(M_2) = \lceil \mathcal{I}(M_1) \vee \mathcal{I}(M_2) \rceil$, Proposition 4.8 asserts that $\mathcal{I}(M_1) \vee \mathcal{I}(M_2)$ is finitary, as desired. □

We conclude this section with the following observation.

Observation 4.10. *A countable union of finitary matroids need not be a matroid.*

Proof. We show that for any integer $k \geq 1$, the set system

$$\mathcal{I} := \bigvee_{n \in \mathbb{N}} U_{k,\mathcal{R}}$$

is not a matroid, where here $U_{k,\mathcal{R}}$ denotes the k-uniform matroid with ground set $\mathcal{R}$.

Since a countable union of finite sets is countable, we have that the members of $\mathcal{I}$ are the countable subsets of $\mathcal{R}$. Consequently, the system $\mathcal{I}$ violates the (IM) axiom for $I = \emptyset$ and $X = \mathcal{R}$. □

Above, we used the fact that the members of $\mathcal{I}$ are countable and that the ground set is uncountable. One can have the following more subtle example, showing that a countable union of finite matroids need not be a matroid.

Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be disjoint countable sets, and for $n \in \mathbb{N}$, set $E_n := \{a_1, \ldots, a_n\} \cup \{b_n\}$. Then $\bigvee_{n \in \mathbb{N}} U_{1,E_n}$ is an infinite union of finite matroids and fails to satisfy (IM) for $I = A$ and $X = A \cup B = E(M)$.

4.3 Nearly finitary matroid union

In this section, we prove Theorem 1.8.

For a matroid M, let $\mathcal{I}^{\text{fin}}(M)$ denote the set of subsets of $E(M)$ containing no finite circuit of M, or equivalently, the set of subsets of $E(M)$ which have all their finite subsets in $\mathcal{I}(M)$. We call $M^{\text{fin}} = (E(M), \mathcal{I}^{\text{fin}}(M))$ the *finitarization* of M. With this notation, a matroid M is *nearly finitary* if it has the property that

$$\text{for each } J \in \mathcal{I}(M^{\text{fin}}) \text{ there exists an } I \in \mathcal{I}(M) \text{ such that } |J \setminus I| < \infty. \quad (7)$$

For a set system $\mathcal{I}$ (not necessarily the independent sets of a matroid) we call a maximal member of $\mathcal{I}$ a *base* and a minimal member subject to not being in $\mathcal{I}$ a *circuit*. With these conventions, the notions of *finitarization* and *nearly finitary* carry over to set systems.

Let $\mathcal{I} = \lceil \mathcal{I} \rceil$. The finitarization $\mathcal{I}^{\text{fin}}$ of $\mathcal{I}$ has the following properties.

1. $\mathcal{I} \subseteq \mathcal{I}^{\text{fin}}$ with equality if and only if $\mathcal{I}$ is finitary.
2. $\mathcal{I}^{\text{fin}}$ is finitary and its circuits are exactly the finite circuits of $\mathcal{I}$.
3. $(\mathcal{I}|X)^{\text{fin}} = \mathcal{I}^{\text{fin}}|X$, in particular $\mathcal{I}|X$ is nearly finitary if $\mathcal{I}$ is.

The first two statements are obvious. To see the third, assume that $\mathcal{I}$ is nearly finitary and that $J \in \mathcal{I}^{\text{fin}}|X \subseteq \mathcal{I}^{\text{fin}}$. By definition there is $I \in \mathcal{I}$ such that $J \setminus I$ is finite. As $J \subseteq X$ we also have that $J \setminus (I \cap X)$ is finite and clearly $I \cap X \in \mathcal{I}|X$.

Proposition 4.11. *The pair $M^{\text{fin}} = (E, \mathcal{I}^{\text{fin}}(M))$ is a finitary matroid, whenever M is a matroid.*

Proof. By construction, the set system $\mathcal{I}^{\text{fin}} = \mathcal{I}(M^{\text{fin}})$ satisfies the axioms (I1) and (I2) and is finitary, implying that it also satisfies (IM).

It remains to show that $\mathcal{I}^{\text{fin}}$ satisfies $(I3)$. By definition, a set $X \subseteq E(M)$ is not in $\mathcal{I}^{\text{fin}}$ if and only if it contains a finite circuit of M.

Let $B, I \in \mathcal{I}^{\text{fin}}$ where B is maximal and I is not, and let $y \in E(M) \setminus I$ such that $I + y \in \mathcal{I}^{\text{fin}}$. If $I + x \in \mathcal{I}^{\text{fin}}$ for any $x \in B \setminus I$, then we are done.

Assuming the contrary, then $y \notin B$ and for any $x \in B \setminus I$ there exists a finite circuit C_x of M in $I + x$ containing x. By maximality of B, there exists a finite circuit C of M in $B + y$ containing y. By the circuit elimination axiom (in M) applied to the circuits C and $\{C_x\}_{x \in X}$ where $X := C \cap (B \setminus I)$, there exists a circuit

$$D \subseteq \left(C \cup \bigcup_{x \in X} C_x \right) \setminus X \subseteq I + y$$

of M containing $y \in C \setminus \bigcup_{x \in X} C_x$. The circuit D is finite, since the circuits C and $\{C_x\}$ are; this contradicts $I + y \in \mathcal{I}^{\text{fin}}$. □

Proposition 4.12. *For arbitrary matroids M_1 and M_2 it holds that*

$$\mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}}) = \mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}})^{\text{fin}} = \mathcal{I}(M_1 \vee M_2)^{\text{fin}}.$$

Proof. By Proposition 4.11, the matroids M_1^{fin} and M_2^{fin} are finitary and therefore $M_1^{\text{fin}} \vee M_2^{\text{fin}}$ is a finitary as well, by Proposition 4.1. This establishes the first equality.

The second equality follows from the definition of finitarization provided we show that the finite members of $\mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}})$ and $\mathcal{I}(M_1 \vee M_2)$ are the same.

Since $\mathcal{I}(M_1) \subseteq \mathcal{I}(M_1^{\text{fin}})$ and $\mathcal{I}(M_2) \subseteq \mathcal{I}(M_2^{\text{fin}})$ it holds that $\mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}}) \supseteq \mathcal{I}(M_1 \vee M_2)$. On the other hand, a finite set $I \in \mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}})$ can be written as $I = I_1 \cup I_2$ with $I_1 \in \mathcal{I}(M_1^{\text{fin}})$ and $I_2 \in \mathcal{I}(M_2^{\text{fin}})$ finite. As I_1 and I_2 are finite, $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$, implying that $I \in \mathcal{I}(M_1 \vee M_2)$. $\square$

With the above notation a matroid M is nearly finitary if each base of M^{fin} contains a base of M such that their difference is finite. The following is probably the most natural manner to construct nearly finitary matroids (that are not finitary) from finitary matroids.

For a matroid M and an integer $k \geq 0$, set $M[k] := (E(M), \mathcal{I}[k])$, where

$$\mathcal{I}[k] := \{I \in \mathcal{I}(M) \mid \exists J \in \mathcal{I}(M) \text{ such that } I \subseteq J \text{ and } |J \setminus I| = k\}.$$

Proposition 4.13. *If* $\operatorname{rk}(M) \geq k$*, then* $M[k]$ *is a matroid.*

Proof. The axiom (I1) holds as $\operatorname{rk}(M) \geq k$; the axiom (I2) holds as it does in M. For (I3) let $I', I \in \mathcal{I}(M[k])$ such that I' is maximal and I is not. There is a set $F' \subseteq E(M) \setminus I'$ of size k such that, in M, the set $I' \cup F'$ is not only independent but, by maximality of I', also a base. Similarly, there is a set $F \subseteq E(M) \setminus I$ of size k such that $I \cup F \in \mathcal{I}(M)$.

We claim that $I \cup F$ is non-maximal in $\mathcal{I}(M)$ for any such F. Suppose not and $I \cup F$ is maximal for some F as above. By assumption, I is contained in some larger set of $\mathcal{I}(M[k])$. Hence there is a set $F^+ \subseteq E(M) \setminus I$ of size $k+1$ such that $I \cup F^+$ is independent in M. Clearly $(I \cup F) \setminus (I \cup F^+) = F \setminus F^+$ is finite, so Lemma 4.14 implies that

$$\left|F^+ \setminus F\right| = \left|(I \cup F^+) \setminus (I \cup F)\right| \leq \left|(I \cup F) \setminus (I \cup F^+)\right| = \left|F \setminus F^+\right|.$$

In particular, $k + 1 = |F^+| \leq |F| = k$, a contradiction.

Hence we can pick F such that $F \cap F'$ is maximal and, as $I \cup F$ is non-maximal in $\mathcal{I}(M)$, apply (I3) in M to obtain a $x \in (I' \cup F') \setminus (I \cup F)$ such that $(I \cup F) + x \in \mathcal{I}(M)$. This means $I + x \in \mathcal{I}(M[k])$. And $x \in I' \setminus I$ follows, as $x \notin F'$ by our choice of F.

To show (IM), let $I \subseteq X \subseteq E(M)$ with $I \in \mathcal{I}(M[k])$ be given. By (IM) for M, there is a $B \in \mathcal{I}(M)$ which is maximal subject to $I \subseteq B \subseteq X$. We may assume that $F := B \setminus I$ has at most k elements; for otherwise there is a superset $I' \subseteq B$ of I such that $|B \setminus I'| = k$ and it suffices to find a maximal set containing $I' \in \mathcal{I}(M[k])$ instead of I.

We claim that for any $F^+ \subseteq X \setminus I$ of size $k+1$ the set $I \cup F^+$ is not in $\mathcal{I}(M[k])$. For a contradiction, suppose it is. Then in $M|X$, the set $B = I \cup F$ is a base and $I \cup F^+$ is independent and as $(I \cup F) \setminus (I \cup F^+) \subseteq F \setminus F^+$ is finite, Lemma 4.14 implies

$$\left|F^+ \setminus F\right| = \left|(I \cup F^+) \setminus (I \cup F)\right| \leq \left|(I \cup F) \setminus (I \cup F^+)\right| = \left|F \setminus F^+\right|.$$

This means $k+1 = |F^+| \leq |F| = k$, a contradiction. So by successively adding single elements of $X \setminus I$ to I as long as the obtained set is still in $\mathcal{I}(M[k])$ we arrive at the wanted maximal element after at most k steps. □

We conclude this section with a proof of Theorem 1.8. To this end, we shall require following two lemmas.

Lemma 4.14. *Let M be a matroid and $I, B \in \mathcal{I}(M)$ with B maximal and $B \setminus I$ finite. Then, $|I \setminus B| \leq |B \setminus I|$.*

Proof. The proof is by induction on $|B \setminus I|$. For $|B \setminus I| = 0$ we have $B \subseteq I$ and hence $B = I$ by maximality of B. Now suppose there is $y \in B \setminus I$. If $I + y \in \mathcal{I}$ then by induction

$$|I \setminus B| = |(I + y) \setminus B| \leq |B \setminus (I + y)| = |B \setminus I| - 1$$

and hence $|I \setminus B| < |B \setminus I|$. Otherwise there exists a unique circuit C of M in $I + y$. Clearly C cannot be contained in B and therefore has an element $x \in I \setminus B$. Then $(I + y) - x$ is independent, so by induction

$$|I \setminus B| - 1 = |((I + y) - x) \setminus B| \leq |B \setminus ((I + y) - x)| = |B \setminus I| - 1,$$

and hence $|I \setminus B| \leq |B \setminus I|$. □

Lemma 4.15. *Let $\mathcal{I} \subseteq \mathcal{P}(E)$ be a nearly finitary set system satisfying (I1), (I2), and the following variant of (I3):*

- *(*) For all $I, J \in \mathcal{I}$ and all $y \in I \setminus J$ with $J + y \notin \mathcal{I}$ there exists $x \in J \setminus I$ such that $(J + y) - x \in \mathcal{I}$.*

Then $\mathcal{I}$ satisfies (IM).

Proof. Let $I \subseteq X \subseteq E$ with $I \in \mathcal{I}$. As $\mathcal{I}^{\text{fin}}$ satisfies (IM) there is a set $B^{\text{fin}} \in \mathcal{I}^{\text{fin}}$ which is maximal subject to $I \subseteq B^{\text{fin}} \subseteq X$ and being in $\mathcal{I}^{\text{fin}}$. As $\mathcal{I}$ is nearly finitary, there is $J \in \mathcal{I}$ such that $B^{\text{fin}} \setminus J$ is finite and we may assume that $J \subseteq X$. Then, $I \setminus J \subseteq B^{\text{fin}} \setminus J$ is finite so that we may choose a J minimizing $|I \setminus J|$. If there is a $y \in I \setminus J$, then by (*) we have $J + y \in \mathcal{I}$ or there is an $x \in J \setminus I$ such that $(J + y) - x \in \mathcal{I}$. Both outcomes give a set containing more elements of I and hence contradicting the choice of J.

It remains to show that J can be extended to a maximal set B of $\mathcal{I}$ in X. For any superset $J' \in \mathcal{I}$ of J, we have $J' \in \mathcal{I}^{\text{fin}}$ and $B^{\text{fin}} \setminus J'$ is finite as it is a subset of $B^{\text{fin}} \setminus J$. As $\mathcal{I}^{\text{fin}}$ is a matroid, Lemma 4.14 implies

$$|J' \setminus B^{\text{fin}}| \leq |B^{\text{fin}} \setminus J'| \leq |B^{\text{fin}} \setminus J|.$$

Hence, $|J' \setminus J| \leq 2|B^{\text{fin}} \setminus J| < \infty$. Thus, we can greedily add elements of X to J to obtain the wanted set B after finitely many steps. □

Next, we prove Theorem 1.8.

Proof of Theorem 1.8. By Proposition 4.4, in order to prove that $M_1 \vee M_2$ is a matroid, it is sufficient to prove that $\mathcal{I}(M_1 \vee M_2)$ satisfies (IM). By Corollary 4.7 and Lemma 4.15 it remains to show that $\mathcal{I}(M_1 \vee M_2)$ is nearly finitary.

So let $J \in \mathcal{I}(M_1 \vee M_2)^{\text{fin}}$. By Proposition 4.12 we may assume that $J = J_1 \cup J_2$ with $J_1 \in \mathcal{I}(M_1^{\text{fin}})$ and $J_2 \in \mathcal{I}(M_2^{\text{fin}})$. By assumption there are $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$ such that $J_1 \setminus I_1$ and $J_2 \setminus I_2$ are finite. Then $I = I_1 \cup I_2 \in \mathcal{I}(M_1 \vee M_2)$ and the assertion follows as $J \setminus (I_1 \cup I_2) \subseteq (J_1 \setminus I_1) \cup (J_2 \setminus I_2)$ is finite. □

5 From infinite matroid union to infinite matroid intersection

In this section, we prove Theorem 1.7.

Proof of Theorem 1.7. Our starting point is the well-known proof from finite matroid theory that matroid union implies a solution to the matroid intersection problem. With that said, let $B_1 \cup B_2^* \in \mathcal{B}(M_1 \vee M_2^*)$ where $B_1 \in \mathcal{B}(M_1)$ and $B_2^* \in \mathcal{B}(M_2^*)$, and let $B_2 = E \setminus B_2^* \in \mathcal{B}(M_2)$. Then, put $I = B_1 \cap B_2$ and note that $I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)$. We show that I admits the required partition.

For an element $x \notin B_i$, $i = 1, 2$, we write $C_i(x)$ to denote the fundamental circuit of x into B_i in M_i. For an element $x \notin B_2^*$, we write $C_2^*(x)$ to denote the fundamental circuit of x into B_2^* in M_2^*. Put $X = B_1 \cap B_2^*$, $Y = B_2 \setminus I$, and $Z = B_2^* \setminus X$, see Figure 4.

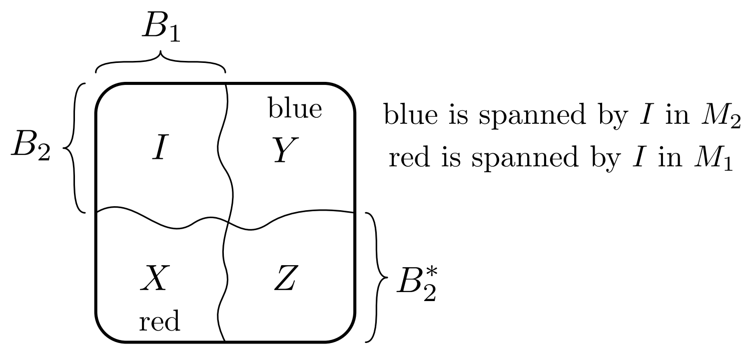

Figure 4: The sets X, Y, and Z and their colorings.

Observe that

$$cl_{M_1}(I) \cup cl_{M_2}(I) = E = I \cup X \cup Y \cup Z. \tag{8}$$

To see (8), note first that

$$X \subseteq cl_{M_2}(I). \tag{9}$$

Clearly, no member of X is spanned by I in M_1. Assume then that $x \in X$ is not spanned by I in M_2 so that there exists a $y \in C_2(x) \cap Y$. Then, $x \in C_2^*(y)$, by Lemma 2.1. Consequently, $B_1 \cup B_2^* \subsetneq B_1 \cup (B_2^* + y - x) \in \mathcal{I}(M_1 \vee M_2^*)$; contradiction to the maximality of $B_1 \cup B_2^*$, implying (9).

By a similar argument, it holds that

$$Y \subseteq cl_{M_1}(I). \tag{10}$$

To see that

$$Z \subseteq cl_{M_1}(I) \cup cl_{M_2}(I), \tag{11}$$

assume, towards contradiction, that some $z \in Z$ is not spanned by I neither in M_1 nor in M_2 so that there exist an $x \in C_1(z) \cap X$ and a $y \in C_2(z) \cap Y$. Then $B_1 - x + z$ and $B_2 - y + z$ are bases and thus $B_1 \cup B_2^* \subsetneq (B_1 - x + z) \cup (B_2^* - z + y)$; contradiction to the maximality of $B_1 \cup B_2^*$. Assertion (8) is proved.

The problem of finding a suitable partition $I = J_1 \cup J_2$ can be phrased as a (directed) graph coloring problem. By (8), each $x \in E \setminus I$ satisfies

$C_1(x) - x \subseteq I$ or $C_2(x) - x \subseteq I$. Define $G = (V, E)$ to be the directed graph whose vertex set is $V = E \setminus I$ and whose edge set is given by

$$E = \{(x, y) : C_1(x) \cap C_2(y) \cap I \neq \emptyset\}. \tag{12}$$

Recall that a *source* is a vertex with no incoming edges and a *sink* is a vertex with no outgoing edges. As $C_1(x)$ does not exist for any $x \in X$ and $C_2(y)$ does not exist for any $y \in Y$, it follows that

$$\text{the members of } X \text{ are sinks and those of } Y \text{ are sources in } G. \tag{13}$$

A 2-coloring of the vertices of G, by say blue and red, is called *divisive* if it satisfies the following:

(D.1) I spans all the blue elements in M_1;

(D.2) I spans all the red elements in M_2; and

(D.3) $J_1 \cap J_2 = \emptyset$ where $J_1 := (\bigcup_{x \text{ blue}} C_1(x)) \cap I$ and $J_2 := (\bigcup_{x \text{ red}} C_2(x)) \cap I$.

Clearly, if G has a divisive coloring, then I admits the required partition.

We show then that G admits a divisive coloring. Color with blue all the sources. These are the vertices that can only be spanned by I in M_1. Color with red all the sinks, that is, all the vertices that can only be spanned by I in M_2. This defines a partial coloring of G in which all members of X are red and those of Y are blue. Such a partial coloring can clearly be extended into a divisive coloring of G provided that

$$G \text{ has no } (y, x)\text{-path with } y \text{ blue and } x \text{ red.} \tag{14}$$

Indeed, given (14) and (13), color all vertices reachable by a path from a blue vertex with the color blue, color all vertices from which a red vertex is reachable by a path with red, and color all remaining vertices with, say, blue. The resulting coloring is divisive.

It remains to prove (14). We show that the existence of a path as in (14) contradicts the following property:
Suppose that M and N are matroids and $B \cup B'$ is maximal in $\mathcal{I}(M \vee N)$. Let $y \notin B \cup B'$ and let $x \in B \cap B'$. Then, (by Lemma 4.5)

$$\textit{there exists no } (B, B', y, x)\textit{-chain;} \tag{15}$$

(in fact, the contradiction in the proofs of (9),(10), and (11) arose from simple instances of such forbidden chains).

Assume, towards contradiction, that P is a (y,x)-path with y blue and x red; the intermediate vertices of such a path are not colored since they are not a sink nor a source. In what follows we use P to construct a $(B_1, B_2^*, y_0, y_{2|P|})$-chain $(y_0, y_1, \ldots, y_{2|P|})$ such that $y_0 \in Y$, $y_{2|P|} \in X$, all odd indexed members of the chain are in $V(P) \cap Z$, and all even indexed elements of the chain other than y_0 and $y_{2|P|}$ are in I. Existence of such a chain would contradict (15).

Definition of y_0. As y is pre-colored blue then either $y \in Y$ or $C_2(y) \cap Y \neq \emptyset$. In the former case set $y_0 = y$ and in the latter choose $y_0 \in C_2(y) \cap Y$.

Definition of $y_{2|P|}$. In a similar manner, x is pre-colored red since either $x \in X$ or $C_1(x) \cap X \neq \emptyset$. In the former case, set $y_{2|P|} = x$ and in the latter case choose $y_{2|P|} \in C_1(x) \cap X$.

The remainder of the chain. Enumerate $V(P) \cap Z = \{y_1, y_3, \ldots, y_{2|P|-1}\}$ where the enumeration is with respect to the order of the vertices defined by P. Next, for an edge $(y_{2i-1}, y_{2i+1}) \in E(P)$, let $y_{2i} \in C_1(y_{2i-1}) \cap C_2(y_{2i+1}) \cap I$; such exists by the assumption that $(y_{2i-1}, y_{2i+1}) \in E$. As $y_{2i+1} \in C_2^*(y_{2i})$ for all relevant i, by Lemma 2.1, the sequence $(y_0, y_1, y_2, \ldots, y_{2|P|})$ is a $(B_1, B_2^*, y_0, y_{2|P|})$-chain in $\mathcal{I}(M_1 \vee M_2^*)$.

This completes our proof of Theorem 1.7. □

Note that in the above proof, we do not use the assumption that $M_1 \vee M_2^*$ is a matroid; in fact, we only need that $\mathcal{I}(M_1 \vee M_2^*)$ has a maximal element.

6 The graphic nearly finitary matroids

In this section we prove Propositions 1.3 and 1.4 yielding a characterization of the graphic nearly finitary matroids.

For a connected graph G, a maximal set of edges containing no finite cycles is called an *ordinary spanning tree*. A maximal set of edges containing no finite cycles nor any double ray is called an *algebraic spanning tree*. These are the bases of $M_F(G)$ and $M_A(G)$, respectively. We postpone the discussion about $M_C(G)$ to Section 6.2.

To prove Propositions 1.3 and 1.4, we require the following theorem of Halin [9, Theorem 8.2.5].

Theorem 6.1 (Halin 1965)**.** *If an infinite graph G contains k disjoint rays for every $k \in \mathbb{N}$, then G contains infinitely many disjoint rays.*

6.1 The nearly finitary algebraic-cycle matroids

The purpose of this subsection is to prove Proposition 1.3.

Proof of Proposition 1.3. Suppose that G has k disjoint rays for every integer k; so that G has a set $\mathcal{R}$ of infinitely many disjoint rays by Theorem 6.1. We show that $M_A(G)$ is not nearly finitary.

The edge set of $\bigcup \mathcal{R} = \bigcup_{R \in \mathcal{R}} R$ is independent in $M_A(G)^{\text{fin}}$ as it induces no finite cycle of G. Therefore there is a base of $M_A(G)^{\text{fin}}$ containing it; such induces an ordinary spanning tree, say T, of G. We show that

$$T - F \text{ contains a double ray for any finite edge set } F \subseteq E(T). \tag{16}$$

This implies that $E(T) \setminus I$ is infinite for every independent set I of $M_A(G)$ and hence $M_A(G)$ is not nearly finitary. To see (16), note that $T - F$ has $|F| + 1$ components for any finite edge set $F \subseteq E(T)$ as T is a tree and successively removing edges always splits one component into two. So one of these components contains infinitely many disjoint rays from $\mathcal{R}$ and consequently a double ray.

Suppose next, that G has at most k disjoint rays for some integer k and let T be an ordinary spanning tree of G, that is, $E(T)$ is maximal in $M_A(G)^{\text{fin}}$. To prove that $M_A(G)$ is nearly finitary, we need to find a finite set $F \subseteq E(T)$ such that $E(T) \setminus F$ is independent in $M_A(G)$, i.e. it induces no double ray of G. Let $\mathcal{R}$ be a maximal set of disjoint rays in T; such exists by assumption and $|\mathcal{R}| \leq k$. As T is a tree and the rays of $\mathcal{R}$ are vertex-disjoint, it is easy to see that T contains a set F of $|\mathcal{R}| - 1$ edges such that $T - F$ has $|\mathcal{R}|$ components which each contain one ray of $\mathcal{R}$. By maximality of $\mathcal{R}$ no component of $T - F$ contains two disjoint rays, or equivalently, a double ray. $\square$

6.2 The nearly finitary topological-cycle matroids

In this section we prove Proposition 1.4 that characterizes the nearly finitary topological-cycle matroids. Prior to that, we first define these matroids. To that end we shall require some additional notation and terminology on which more details can be found in [6].

An *end of* G is an equivalence class of rays, where two rays are *equivalent* if they cannot be separated by a finite edge set. In particular, two rays meeting infinitely often are equivalent. Let the *degree* of an end ω be the size of a maximal set of vertex-disjoint rays belonging to ω, which is well-defined [9]. We say that a double ray *belongs to* an end if the two rays which arise from the removal of one edge from the double ray belong to that end; this does not depend on the choice of the edge. Such a double ray is an example of a *topological cycle*[12]

For a graph G the topological-cycle matroid of G, namely $M_C(G)$, has $E(G)$ as its ground set and its set of circuits consists of the finite and topological cycles. In fact, every infinite circuit of $M_C(G)$ induces at least one double ray; provided that G is locally finite [9].

A graph G has only finitely many disjoint rays if and only if G has only finitely many ends, each with finite degree. Also, note that

$$\text{every end of a 2-connected locally finite graph has degree at least 2.} \quad (17)$$

Indeed, applying Menger's theorem inductively, it is easy to construct in any k-connected graph for any end ω a set of k disjoint rays of ω.

Now we are in a position to start the proof of Proposition 1.4.

Proof of Proposition 1.4. If G has only a finite number of vertex-disjoint rays then $M_A(G)$ is nearly finitary by Proposition 1.3. Since $M_A(G)^{\text{fin}} = M_C(G)^{\text{fin}}$ and $\mathcal{I}(M_A(G)) \subseteq \mathcal{I}(M_C(G))$, we can conclude that $M_C(G)$ is nearly finitary as well.

Now, suppose that G contains k vertex-disjoint rays for every $k \in \mathbb{N}$. If G has an end ω of infinite degree, then there is an infinite set $\mathcal{R}$ of vertex-disjoint rays belonging to ω. As any double ray containing two rays of $\mathcal{R}$ forms a circuit of $M_C(G)$, the argument from the proof of Proposition 1.3 shows that $M_C(G)$ is not nearly finitary.

Assume then that G has no end of infinite degree. There are infinitely many disjoint rays, by Theorem 6.1. Hence, there is a countable set of ends $\Omega = \{\omega_1, \omega_2, \dots\}$.

We inductively construct a set $\mathcal{R}$ of infinitely many vertex-disjoint double rays, one belonging to each end of Ω. Suppose that for any integer $n \geq 0$ we have constructed a set $\mathcal{R}_n$ of n disjoint double rays, one belonging to each of the ends $\omega_1, \dots, \omega_n$. Different ends can be separated by finitely many vertices so there is a finite set S of vertices such that $\bigcup \mathcal{R}_n$ has no vertex in the component C of $G - S$ which contains ω_{n+1}. Since ω_{n+1} has degree 2 by (17), there are two disjoint rays from ω_{n+1} in C an thus also a double ray D belonging to ω_{n+1}. Set $\mathcal{R}_{n+1} := \mathcal{R}_n \cup \{D\}$ and $\mathcal{R} := \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$.

As $\bigcup \mathcal{R}$ contains no finite cycle of G, it can be extended to an ordinary spanning tree of G. Removing finitely many edges from this tree clearly

[12] Formally, the topological cycles of G are those subgraphs of G which are homeomorphic images of S^1 in the Freudenthal compactification $|G|$ of G. However, the given example is the only type of topological cycle which shall be needed for the proof.

leaves an element of $\mathcal{R}$ intact. Hence, the edge set of the resulting graph still contains a circuit of $M_C(G)$. Thus, $M_C(G)$ is not nearly finitary in this case as well. □

In the following we shall propose a possible extension of Theorem 6.1 to matroids. We call a matroid M *k-nearly finitary* if every base of its finitarization contains a base of M such that their difference has size at most k. Note that saying 'at most k' is not equivalent to saying 'equal to k', consider for example the algebraic-cycle matroid of the infinite ladder. We conjecture the following.

Conjecture 6.2. *Every nearly finitary matroid is k-nearly finitary for some k.*

We remark that Propositions 1.3 and 1.4 above are special cases of this conjecture. In the proof of Proposition 1.4 we used Theorem 6.1. In fact it is not difficult to show that Proposition 1.4 and Theorem 6.1 are equivalent. In particular, Conjecture 6.2 implies Theorem 6.1.

6.3 Graphic matroids and the intersection conjecture

By Theorem 1.5, the intersection conjecture is true for $M_C(G)$ and $M_{FC}(H)$ for any two graphs G and H since the first is co-finitary and the second is finitary. Using also Proposition 1.4, we obtain the following.

Corollary 6.3. *Suppose that G and H are 2-connected, locally finite graphs with only a finite number of vertex-disjoint rays. Then, $M_C(G)$ and $M_C(H)$ satisfy the intersection conjecture.* □

Using Proposition 1.3 instead of Proposition 1.4, we obtain the following.

Corollary 6.4. *Suppose that G and H are graphs with only a finite number of vertex-disjoint rays. Then, $M_A(G)$ and $M_A(H)$ satisfy the intersection conjecture if both are matroids.* □

With a little more work, the same is also true for $M_{FC}(G)$, see Corollary 1.6.

Proof of Corollary 1.6. First we show that $(((M_C(G)^{\text{fin}})^*)^{\text{fin}})^* = M_C(G)$ if G is locally finite. Indeed, then $M_C(G)^{\text{fin}} = M_{FC}(G)$, $M_{FC}(G)^*$ is the matroids whose circuits are the finite and infinite bonds of G, and its finitarization has as its circuits the finite bonds of G. And the dual of this matroid is $M_C(G)$, see [7] for example.

Having showed that $(((M_C(G)^{\rm fin})^*)^{\rm fin})^* = M_C(G)$ if G is locally finite, we next show that if $M_C(G)$ is nearly finitary, then so is $M_{FC}(G)^*$. For this let B be a base of $M_{FC}(G)^*$ and B' be a base of $(M_{FC}(G)^*)^{\rm fin}$. Then $B' \setminus B = (E \setminus B) \setminus (E \setminus B')$. Now $E \setminus B$ is a base of $M_{FC}(G) = M_C(G)^{\rm fin}$ and by the above $E \setminus B'$ is a base of $M_C(G)$. Since $M_C(G)$ is nearly finitary, $B' \setminus B$ is finite, yielding that $M_{FC}(G)^*$ is nearly finitary.

As $M_{FC}(G)^*$ is nearly finitary and $M_{FC}(H)$ is finitary, $M_{FC}(H)$ and $M_{FC}(G)$ satisfy the intersection conjecture by Theorem 1.5. □

A similar argument shows that if G and H are are 2-connected, locally finite graphs with only a finite number of vertex-disjoint rays, then one can also prove that $M_{FC}(G)^*$ and $M_{FC}(H)^*$ satisfy the intersection conjecture. Similar results are true for $M_C(G)^*$ or $M_A(G)^*$ in place of $M_{FC}(G)^*$.

7 Union of arbitrary infinite matroids

In this section, we show that there exists infinite matroids M and N whose union is not a matroid.

In Claim 7.1, we treat the relatively simpler case in which M is finitary and N is co-finitary and both have uncountable ground sets. Second, then, in Claim 7.2, we refine the argument as to have M both finitary and co-finitary and N co-finitary and both on countable ground sets.

Claim 7.1. *There exists a finitary matroid M and a co-finitary matroid N such that $\mathcal{I}(M \vee N)$ is not a matroid.*

Proof. Set $E = E(M) = E(N) = \mathbb{N} \times \mathcal{R}$. Next, put $M := \bigoplus_{n\in\mathbb{N}} M_n$, where $M_n := U_{1,\{n\}\times\mathcal{R}}$. The matroid M is finitary as it is a direct sum of 1-uniform matroids. For $r \in \mathcal{R}$, let N_r be the circuit matroid on $\mathbb{N} \times \{r\}$; set $N := \bigoplus_{r\in\mathcal{R}} N_r$. As N is a direct sum of circuits, it is co-finitary. (see Figure 5).

We show that $\mathcal{I}(M \vee N)$ violates the axiom (IM) for $I = \emptyset$ and $X = E$; so that $\mathcal{I}(M \vee N)$ has no maximal elements. It is sufficient to show that a set $J \subseteq E$ belongs to $\mathcal{I}(M \vee N)$ if and only if it contains at most countably many circuits of N. For if so, then for any $J \in \mathcal{I}(M \vee N)$ and any circuit $C = \mathbb{N} \times \{r\}$ of N with $C \nsubseteq J$ (such a circuit exists) we have $J \cup C \in \mathcal{I}(M \vee N)$.

The point to observe here is that every independent set of M is countable, (since every such set meets at most one element of M_n for each $n \in \mathbb{N}$), and that every independent set of N misses uncountably many elements of E (as any such set must miss at least one element of N_r for each $r \in \mathcal{R}$).

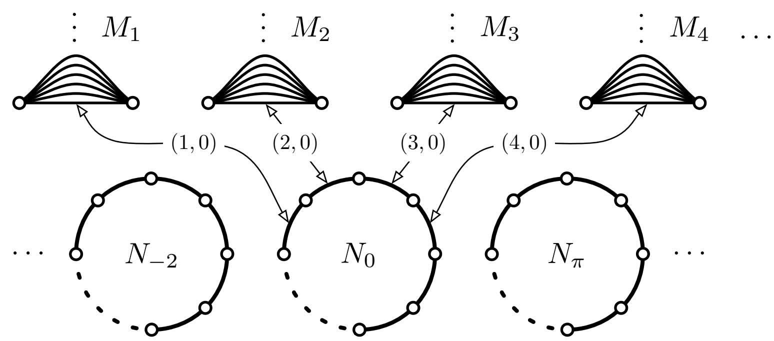

Figure 5: $M = \bigoplus_{n\in\mathbb{N}} M_n$ and $N = \bigoplus_{r\in\mathcal{R}} N_r$.

Suppose $J \subseteq E$ contains uncountably many circuits of N. Since each independent set of N misses uncountably many elements of E, every set $D = J \setminus J_N$ is uncountable whenever $J_N \in \mathcal{I}(J)$. On the other hand, since each independent set of M is countable, we have that $D \notin \mathcal{I}(M)$. Consequently, $J \notin \mathcal{I}(M \vee N)$, as required.

We may assume then that $J \subseteq E$ contains only countably many circuits of N, namely, $\{C_{r_1}, C_{r_2}, \ldots\}$. Now the set $J_M = \{(i, r_i) : i \in \mathbb{N}\}$ is independent in M; consequently, $J \setminus J_M$ is independent in N; completing the proof. □

We proceed with matroids on countable ground sets.

Claim 7.2. *There exist a matroid M that is both finitary and co-finitray, and a co-finitary matroid N whose common ground is countable such that $\mathcal{I}(M \vee N)$ is not a matroid.*

Proof. For the common ground set we take $E = (\mathbb{N} \times \mathbb{N}) \cup L$ where $L = \{\ell_1, \ell_2, \ldots\}$ is countable and disjoint to $\mathbb{N} \times \mathbb{N}$. The matroids N and M are defined as follows. For $r \in \mathbb{N}$, let N_r be the circuit matroid on $\mathbb{N} \times \{r\}$. Set N to be the matroid on E obtained by adding the elements of L to the matroid $\bigoplus_{r\in\mathbb{N}} N_r$ as loops. Next, for $n \in \mathbb{N}$, let M_n be the 1-uniform matroid on $(\{n\} \times \{1, 2, \ldots, n\}) \cup \{\ell_n\}$. Let M be the matroid obtained by adding to the matroid $\bigoplus_{n\in\mathbb{N}} M_n$ all the members of $E \setminus E(\bigoplus_{n\in\mathbb{N}} M_n)$ as loops

We show that $\mathcal{I}(M \vee N)$ violates the axiom (IM) for $I = \mathbb{N} \times \mathbb{N}$ and $X = E$. It is sufficient to show that

(a) $I \in \mathcal{I}(M \vee N)$; and that

(b) every set J satisfying $I \subset J \subseteq E$ is in $\mathcal{I}(M \vee N)$ if and only if it misses infinitely many elements of L.

To see that $I \in \mathcal{I}(M \vee N)$, note that the set $I_M = \{(n, n) \mid n \in \mathbb{N}\}$ is independent in M and meets each circuit $\mathbb{N} \times \{r\}$ of N. In particular, the set $I_N := (\mathbb{N} \times \mathbb{N}) \setminus I_M$ is independent in N, and therefore $I = I_M \cup I_N \in \mathcal{I}(M \vee N)$.

Let then J be a set satisfying $I \subseteq J \subseteq E$, and suppose, first, that $J \in \mathcal{I}(M \vee N)$. We show that J misses infinitely many elements of L.

There are sets $J_M \in \mathcal{I}(M)$ and $J_N \in \mathcal{I}(N)$ such that $J = J_M \cup J_N$. As J_N misses at least one element from each of the disjoint circuits of N in I, the set $D := I \setminus J_N$ is infinite. Moreover, we have that $D \subseteq J_M$, since $I \subseteq J$. In particular, there is an infinite subset $L' \subseteq L$ such that $D + l$ contains a circuit of M for every $\ell \in L'$. Indeed, for every $e \in D$ is contained in some M_{n_e}; let then $L' = \{\ell_{n_e} : e \in D\}$ and note that $L' \cap J = \emptyset$. This shows that J_M and L' are disjoint and thus J and L' are disjoint as well, and the assertion follows.

Suppose, second, that there exists a sequence $i_1 < i_2 < \ldots$ such that J is disjoint from $L' = \{\ell_{i_r} : r \in \mathbb{N}\}$. We show that the superset $E \setminus L'$ of J is in $\mathcal{I}(M \vee N)$. To this end, set $D := \{(i_r, r) \mid r \in \mathbb{N}\}$. Then, D meets every circuit $\mathbb{N} \times \{r\}$ of N in I, so that the set $J_N := \mathbb{N} \times \mathbb{N} \setminus D$ is independent in N. On the other hand, D contains a single element from each M_n with $n \in L'$. Consequently, $J_M := (L \setminus L') \cup D \in \mathcal{I}(M)$ and therefore $E \setminus L' = J_M \cup J_N \in \mathcal{I}(M \vee N)$. □

While the union of two finitary matroids is a matroid, by Proposition 4.1, the same is not true for two co-finitary matroids.

Corollary 7.3. *The union of two co-finitary matroids is not necessarily a matroid.*

Since two matroids M and N^* satisfy Conjecture 1.1 by Theorem 1.7 if the union of M and N is a matroid, it seems worth investigating where the boundaries of this approach are. In particular, we have the following question. Is the class of nearly finitary matroids the largest class containing the fintary matroids that is closed under taking (finite) unions in the following sense?

Question 7.4. *Is there for every non-nearly finitary matroid M a finitary matroid N such that the union of M and N is not a matroid?*

In [3] we prove that this conjecture is true for any matroid M such that the finitarization of M has an independent set I containing only countably many M-circuits such that I has no finite subset meeting all of these circuits.

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