Autumn 2012 **Extremal Combinatorics - Sheet 4** E.P. Long

There are also exercises in the notes; these are intended to assist in your understanding of the material and should all be attempted. The examples sheets are unassessed, but you are welcome to hand in your attempts the week after they are handed out for feedback.

1. Show that if $\mathcal{A} \subset \binom{X}{r}$ is a set system and $\mathcal{C} \subset \binom{X}{r}$ is an initial segment of colex of size $|\mathcal{C}| = |\mathcal{A}|$ then $|\partial^t \mathcal{A}| \geq |\partial^t \mathcal{C}|$ whenever $1 \leq t \leq r$. Conclude that if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial^t \mathcal{A}| \geq \binom{k}{r-t}$.

2. Let $\mathcal{C} \subset \mathcal{P}[n]$. Show that if $\mathcal{C}$ is an initial segment...
   - of the cube order on $\mathcal{P}(X)$ then it is i-compressed for each $i \in X$.
   - of the binary order on $\mathcal{P}(X)$ then it is i-binary-compressed for each $i \in X$.

3. The Hamming distance between two sets $A, B \in \mathcal{P}[n]$, is given by $d_H(A, B) := |\{i \in [n] : i \in A \triangle B\}|$. Suppose that $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| > \sum_{i=0}^{t} \binom{n}{i}$ for some $t < n/2$. Show that $\mathcal{A}$ contains two sets at Hamming distance at least $2t + 1$. *(Hint: Adjust the proof of Katona's theorem given in lectures to this setting.)*

4. Suppose that G is a subgraph of the cube graph Q_n, with average degree at least $d \in \mathbb{N}$. Show that $|G| \geq 2^d$. *(Hint: Use the edge isoperimetric inequality for Q_n.)*

5. Write down an intersecting family $\mathcal{A} \subset \mathcal{P}(X)$ of size 2^{n-1} that is not in general isomorphic to one of the two examples given in lectures.

6. Show that any intersecting family $\mathcal{A} \subset \mathcal{P}(X)$ can be extended to one of maximal size. For which r is the analogous result true for intersecting families in $\binom{X}{r}$?

7. A set system $\mathcal{A} \subset \mathcal{P}(X)$ is called an *up-set* if for every $A \in \mathcal{A}$ and every set $B \supset A$, one has $B \in \mathcal{A}$. Show that any intersecting family of size 2^{n-1} is an up-set. Is the converse true?

8. Let $r = 5$ and $t = 3$. What are the sizes of the t-intersecting families $\mathcal{A}_d \subset \binom{X}{r}$, discussed in relation to the theorem of Ahlswede and Khachatrian, for $0 \leq d \leq r - t$? Which is largest for each n?

9. Write down an example showing that equality is possible in the Oddtown Theorem. Can one have equality for a set system with not all the sets of the same size?

10. Neighbouring Oddtown is Eventown, where the mayor also dislikes clubs. Having seen the success of Oddtown's rules on curbing club formation, the mayor decides to impose rules intended to similarly limit the number of clubs. Not wanting to copy the mayor of Oddtown completely, however, Eventown's mayor modifies Oddtown's rules slightly, requiring that each club has an *even* number of members instead of an odd number (while still imposing that each pair of clubs must have an even-sized common membership), and of course no two clubs are allowed to have precisely the same members. Why will this not work as intended?

11. Suppose A is a subset of $\mathbb{R}^n$ in which the pairwise Euclidean distance between distinct points is always one of two values; say $\|x - y\| \in \{d_1, d_2\}$ for every distinct $x, y \in A$. Prove that

$$|A| \leq (n+1)(n+4)/2$$

as follows.

(a) Define

$$F(x,y) = \left(\|x-y\|^2 - d_1^2\right)\left(\|x-y\|^2 - d_2^2\right),$$

a polynomial in $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$, and show that the n-variable polynomials

$$f_a(x) := F(x,a) \in \mathbb{R}[x_1, \dots, x_n], \quad a \in A,$$

are linearly independent. (*Hint: use the 'triangular criterion', or even a simpler variant.*)

(b) Write down a collection of $(n+1)(n+4)/2$ polynomials such that each f_a is a linear combination of these. (*Hint: partially expand out $f_a(x)$ and take polynomials independent of the a_i. The highest degree polynomial you will need to take is $\left(\sum_{i=1}^n x_i^2\right)^2$.*)

(c) Conclude the result.

Please mail me if you have any comments or corrections.

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