Autumn 2012 **Extremal Combinatorics - Sheet 1** E.P. Long

There are also exercises in the notes; these are intended to assist in your understanding of the material and should all be attempted. The examples sheets are unassessed, but you are welcome to hand in your attempts the week after they are handed out for feedback.

1. Determine all the extremal graphs for Mantel's theorem, i.e. all triangle-free graphs on n vertices with $\lfloor \frac{n^2}{4} \rfloor$ edges.
2. In our first proof of Mantel's theorem we showed that
$$\sum_{x \in V} d(x)^2 \leq n\,e$$
for any triangle-free graph on n vertices with e edges. Show that if one replaces the right-hand side with $2(n-1)e$ then the inequality is valid for any graph; one thus 'saves' roughly a factor of 2 for triangle-free graphs. When is this new inequality an equality?
3. By taking the number of triangles t in G into account, generalise the above inequality to one that is valid for any graph and has the right-hand side growing with t.
4. Prove, by using your answer to the previous question or otherwise, that if a graph G on n vertices has e edges then the number of triangles t in G satisfies
$$t \geq \frac{e}{3n}\left(4e - n^2\right).$$
(Thus, if G has $\delta\binom{n}{2}$ edges then the number of triangles is roughly at least $\delta(2\delta - 1)\binom{n}{3}$.) Give some examples of families of graphs for which equality holds.
5. By considering the deletion of any two endvertices of an edge in a triangle-free graph, give an alternative inductive proof of Mantel's theorem. (No calculations involving the floor function should be needed via this route!)
6. Describe $T_r(n)$ for $n \leq r$.
7. Prove that if $n = ar + b$ with $0 \leq b < r$, then $t_r(n) = \left(\frac{r-1}{r}\right)\frac{n^2}{2} - \frac{b(r-b)}{2r}$.
8. What might be the largest graph on n vertices without a 5-cycle C_5 be?

Please mail me if you have any comments or corrections.

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