Spanning triangulations in graphs

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Abstract

We prove that every graph of sufficiently large order n and minimum degree at least $2n/3$ contains a triangulation as a spanning subgraph. This is best possible: for all integers n there are graphs of order n and minimum degree $\lceil 2n/3 \rceil - 1$ without a spanning triangulation.

1 Introduction

With Taraz, we investigated the following extremal problem in [12]: given a function $m = m(n)$, how large does the minimum degree of a graph G of order n have to be in order to guarantee a planar subgraph with at least $m(n)$ edges? Amongst others, we showed that for all positive γ every graph G of sufficiently large order n and minimum degree at least $(2/3 + \gamma)n$ contains a triangulation as spanning subgraph (i.e. a spanning planar subgraph with $3n - 6$ edges). Here we prove that the error term γn can be omitted:

Theorem 1 *There exists an integer n_0 such that every graph G of order $n \geq n_0$ and minimum degree at least $2n/3$ contains a triangulation as a spanning subgraph.*

Our proof of Theorem 1 can easily be extended to obtain a spanning triangulation of an arbitrary surface (see Section 6).

The following example from [12] shows that Theorem 1 is best possible for all integers n which are divisible by 3. Consider the graph G^* which is obtained from two disjoint cliques A and B of order $n/3$ by adding an independent set C of $n/3$ new vertices and joining each of them to all the vertices in the two cliques. So G^* has minimum degree $2n/3 - 1$. Observe that any spanning triangulation in G^* would have two facial triangles T_1 and T_2 which share an edge and are such that T_1 contains a vertex of A and T_2 contains a vertex of B. But this is impossible since every triangle of G^* containing a vertex of A (respectively B) can have at most one vertex outside A (respectively B), namely in C. In Proposition 21 we will extend this example slightly to show that for all n a minimum degree of $\lceil 2n/3 \rceil - 1$ does not ensure a spanning triangulation.

The spanning triangulation guaranteed by Theorem 1 can be found in polynomial time (see Section 6). In other words, the maximum planar subgraph problem (which in a given graph G asks for a planar subgraph with the maximum number of edges) can be solved in polynomial time for graphs G of minimum degree at least $2n/3$. In general this problem was shown to be Max SNP-hard by Călinescu et al. [4], i.e. there exists a positive ε for which there cannot be a polynomial time approximation algorithm with approximation ratio better than $1-\varepsilon$, unless $P = NP$. The best known approximation algorithm has an approximation ratio of $4/9$ [4].

Our proof of Theorem 1 relies on Szemerédi's Regularity lemma, the Blow-up lemma of Komlós, Sárközy and Szemerédi [9] and several ideas which were introduced in [8] by the same authors. (In [8] they proved the related result that every graph of sufficiently large order n and minimum degree at least $2n/3$ contains the square of a Hamilton cycle.)

This paper is organized as follows. In the next section we introduce the necessary definitions and tools, mainly concerning the Regularity lemma and the Blow-up lemma. In Section 3 we give an overview of the proof. In Section 4 we then prove Theorem 1 for the case when G is rather similar to the graph G^* described in the introduction. Based on this, the general case is then dealt with in the Section 5. In the final section, we prove Proposition 21, describe how Theorem 1 can be extended to obtain triangulations of arbitrary surfaces and discuss some open questions.

2 Notation and tools

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order and $\delta(G)$ for its minimum degree. We denote the degree of a vertex $x \in G$ by $d_G(x)$, the set of its neighbours in G by $N_G(x)$ and the set of its neighbours in some $A \subseteq V(G)$ by $N_A(x)$. Given disjoint $A, B \subseteq V(G)$, an *A–B edge* is an edge of G with one endvertex in A and the other in B, the number of these edges is denoted by $e_G(A, B)$ or $e(A, B)$ if this is unambiguous. We write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G. More generally, we write (A, B) for a bipartite graph with vertex classes A and B. Given two disjoint subgraphs H and H' of G, we write $e_G(H, H')$ for $e_G(V(H), V(H'))$. The *common neighbourhood of* a subgraph H of G is the set of all those vertices of G which are joined to all vertices of H. Given a plane graph P, a *facial cycle* of P is a cycle which is the boundary ∂F of some face F of P. P is a *triangulation* if all its faces are bounded by triangles. So by Euler's formula a triangulation has $3n-6$ edges. A 4-*face* of a plane graph is a face bounded by a cycle of length 4.

In our proof, we will build up the spanning triangulation from smaller fragments. For this, we introduce the following notation. Given an integer $k \geq 1$, we call a plane graph P an *almost-triangulation (with k attachment faces)* if precisely k faces of P are bounded by cycles of length at least 4 and all other faces are bounded by triangles. Each face which is not bounded by a triangle is an *attachment face*. Given almost-triangulations P and P' and attachment faces F of P and F' of P' where F' is the outer face of P', we say that a graph H is obtained by *attaching P and P' (via F and F')* if H is (isomorphic to the graph) obtained by inserting P' into F and adding enough edges between ∂F and $\partial F'$ to triangulate the resulting face whose boundary is $\partial F \cup \partial F'$. Note that H is a triangulation if both P and P' had only one attachment face. (We do not allow triangular faces as attachment faces as this turns out to be inconvenient.)

One of our main tools will be the following theorem of Corrádi and Hajnal [5].

Theorem 2 *Every graph R of minimum degree at least $2|R|/3$ contains $\lfloor |R|/3 \rfloor$ disjoint triangles.*

Given $0 < \varepsilon < 1$, a graph G of order n and sets $X \subseteq A \subseteq V(G)$, we say that a bipartition A_1, A_2 of A *splits X ε-fairly* if

$$\big|\, |X \cap A_1| - |X \cap A_2| \,\big| \le \varepsilon n.$$

We say that a bipartition A_1, A_2 of A is *ε-fair* if A as well as the sets $N_A(x)$ are split ε-fairly for every vertex $x \in G$.

Proposition 3 *For each $0 < \varepsilon < 1$ there exists an integer $n_0 = n_0(\varepsilon)$ such that for every graph G of order $n \ge n_0$, for every set $A \subseteq V(G)$ and for every family $\mathcal{X}$ of at most n^2 subsets of A, there exists an ε-fair bipartition of A which splits each set in $\mathcal{X}$ ε-fairly.*

The easy proof of Proposition 3 is based on a standard probabilistic argument. For this, we need the following Chernoff bound (see e.g. [2, Cor. A.14]).

Lemma 4 *Let $X_1, \dots, X_n$ be independent 0-1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \le n$, and let $X := \sum_{i=1}^n X_i$. Then for all $\gamma > 0$ there exists a positive $\beta(\gamma)$ so that*

$$\mathbb{P}(|X - \mathbb{E}X| > \gamma \mathbb{E}X) \le 2\mathrm{e}^{-\beta(\gamma)\mathbb{E}X}.$$

Proof of Proposition 3. Consider a random bipartition A_1, A_2 of A which is obtained by including each vertex of A into A_1 with probability $1/2$ independently of all other vertices in A. Let $\mathcal{X}'$ be the family which consists of A, all the sets in $\mathcal{X}$ as well as all sets of the form $N_A(x)$ $(x \in G)$. We say that a set $X \in \mathcal{X}'$ is *bad* if it is not split ε-fairly. Let $\mathcal{X}^* \subseteq \mathcal{X}'$ be the family of all $X \in \mathcal{X}'$ with $|X| \ge \varepsilon n$. Thus no $X \in \mathcal{X}' \setminus \mathcal{X}^*$ will be bad and for each $X \in \mathcal{X}^*$ we have $n/2 \ge \mathbb{E}(|A_1 \cap X|) = |A \cap X|/2 \ge \varepsilon n/2$. Hence Lemma 4 implies that for each $X \in \mathcal{X}^*$

$$\mathbb{P}(X \text{ is bad}) = \mathbb{P}(\big|\, |A_1 \cap X| - \mathbb{E}(|A_1 \cap X|) \,\big| > \varepsilon n/2) \le 2\mathrm{e}^{-\beta(\varepsilon)\varepsilon n/2}.$$

Therefore, if n is sufficiently large, the expected number of bad sets $X \in \mathcal{X}'$ is at most $|\mathcal{X}'| \cdot \mathrm{e}^{-\beta(\varepsilon)\varepsilon n/2} < 1$. So there is an outcome A_1, A_2 for which no set $X \in \mathcal{X}'$ is bad, i.e. a bipartition A_1, A_2 of A as desired. □

In the remainder of this section we collect all the information we need about the Regularity lemma and the Blow-up lemma. See [11] and [7] respectively for surveys about these. Let us start with some more notation. The *density* of a bipartite graph $G = (A, B)$ is defined to be

$$d(A, B) := \frac{e(A, B)}{|A||B|}.$$

Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A,B) - d(X,Y)| < \varepsilon$. Given $d \in [0,1]$, we say that G is *(ε, d)-super-regular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X,Y) > d$ and, furthermore, if $d_G(a) > d|B|$ for all $a \in A$ and $d_G(b) > d|A|$ for all $b \in B$.

We will often use the following simple fact.

Proposition 5 *Given an ε-regular bipartite graph (A, B) of density $> d$ and a set $X \subseteq A$ with $|X| \geq \varepsilon|A|$, there are less than $\varepsilon|B|$ vertices in B which have at most $(d - \varepsilon)|X|$ neighbours in X.*

We will work with the following degree form of Szemerédi's Regularity lemma which can be easily derived from the classical version. Proofs of the latter are for example included in [3] and [6].

Lemma 6 (Regularity lemma) *For all $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that for every number $d \in [0,1]$ and for every graph G there exist a partition of $V(G)$ into $V_0, V_1, \dots, V_k$ and a spanning subgraph G' of G such that the following holds:*

- $1/\varepsilon \leq k \leq N$,
- $|V_0| \leq \varepsilon|G|$,
- $|V_1| = \cdots = |V_k| =: L$,
- $d_{G'}(x) > d_G(x) - (d + \varepsilon)|G|$ *for all vertices* $x \in G$,
- *for all* $i \geq 1$ *the graph* $G'[V_i]$ *is empty,*
- *for all* $1 \leq i < j \leq k$ *the graph* $(V_i, V_j)_{G'}$ *is ε-regular and has density either* 0 *or* $> d$.

The sets V_i $(i \geq 1)$ are called *clusters*, V_0 is called the *exceptional set*. Given clusters $V_1, \dots, V_k$ and G' as in Lemma 6, the *reduced graph* R is the graph whose vertices are $V_1, \dots, V_k$ and in which V_i is joined to V_j whenever $(V_i, V_j)_{G'}$ is ε-regular and has density $> d$. Thus V_iV_j is an edge of R if and only if G' has an edge between V_i and V_j. Given a graph $H \subseteq R$, we denote by $G(H)$ the subgraph of G which is induced by all those vertices of G that lie in some cluster $V_i \in V(H)$. $G'(H)$ is defined similarly. Given a set $X \subseteq V(R)$, we denote by $G(X)$ the set of all those vertices of G which lie in some cluster $V_i \in X$.

The following well-known proposition shows that, given a subgraph H of R of bounded maximum degree, one can slightly modify the clusters such that each edge of H corresponds to a super-regular graph. An explicit proof is given in [12, Prop. 8].

Proposition 7 *Let H be a subgraph of the reduced graph R with $\Delta(H) \leq \Delta$. Then each vertex V_i of H contains a subset V_i' of size $(1 - \varepsilon\Delta)L$ such that for every edge V_iV_j of H the graph $(V_i', V_j')_{G'}$ is $(\varepsilon/(1 - \varepsilon\Delta), d - (1 + \Delta)\varepsilon)$-super-regular.*

We will also use the next well-known fact. Its proof is quite simple, see e.g. [12, Prop. 9].

Proposition 8 *Suppose that c, d and ε are positive numbers such that $\varepsilon \leq d/2$. Then an application of Lemma 6 to a graph G of minimum degree at least $c|G|$ yields a reduced graph R of minimum degree at least $(c-2d)|R|$.*

A key ingredient of our proof will be the following special case of the Blow-up lemma of Komlós, Sárközy and Szemerédi [9]. It implies that dense 3-partite graphs G behave like complete 3-partite graphs with respect to containing bounded degree graphs as subgraphs if all the 3 bipartite graphs between the vertex classes of G are super-regular.

Lemma 9 (Blow-up lemma) *For all positive d, c and Δ, there are positive numbers $\varepsilon_0 = \varepsilon_0(d, \Delta, c)$ and $\alpha = \alpha(d, \Delta, c) \leq 1/2$ such that the following holds for all $0 < \varepsilon \leq \varepsilon_0$. Given $L \in \mathbb{N}$, let $K_{L,L,L}$ denote the complete 3-partite graph with vertex classes V_1, V_2 and V_3 of size L. Let G be a spanning subgraph of $K_{L,L,L}$ such that for all $1 \leq i < j \leq 3$ the graph $(V_i, V_j)_G$ is (ε, d)-super-regular. Then G contains a copy of every subgraph H of $K_{L,L,L}$ with $\Delta(H) \leq \Delta$. Furthermore, we can additionally require that for vertices $x \in H \subseteq K_{L,L,L}$ lying in V_i their images in the copy of H in G are contained in (arbitrary) given sets $C_x \subseteq V_i$ provided that $|C_x| \geq cL$ for each such x and provided that there are at most αL such vertices x.*

The vertices x in Lemma 9 are said to be *image restricted to C_x*.

Suppose that G is a graph as described in the Blow-up lemma. Then this lemma implies that G contains a triangulation as spanning subgraph. Indeed, all we have to show is that for each $L \in \mathbb{N}$ there is a 3-partite triangulation of bounded maximum degree with vertex classes of size L. But such triangulations can easily be constructed (see e.g. Figure 1 for the case when L is divisible by 4). The following simple corollary shows that much more is true.

Corollary 10 *Given positive numbers d and c, there is a positive number $\varepsilon_0 = \varepsilon_0(d, c)$ and an integer $n_0 = n_0(d, c)$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the following holds. Suppose that G is a 3-partite graph of order $n \geq n_0$ whose vertex classes A, B and C have the same size and such that each of the bipartite graphs $(A, B)_G$, $(A, C)_G$ and $(B, C)_G$ is (ε, d)-super-regular. Furthermore, suppose that X is a set of additional vertices (i.e. $X \cap V(G) = \emptyset$) of size at most $100\varepsilon n$ such that each vertex $x \in X$ has at least cn neighbours in at least two of the vertex classes of G. Finally, suppose that $A' \subseteq A$, $B' \subseteq B$ and $C' \subseteq C$ are sets of size at least cn. Then there exists a triangulation whose vertex set is $V(G) \cup X$ and which has a facial triangle containing a vertex from each of A', B' and C'.*

Proof. Set $\varepsilon_0 := \min\{\varepsilon_0(d, 8, c), \alpha(d, 8, c)/1300\}$, where $\varepsilon_0(d, 8, c)$ and $\alpha(d, 8, c)$ are as defined in the Blow-up lemma. Throughout the proof, we suppose that n is sufficiently large compared with ε for our estimates to hold. Let X_{AB} be the set of all those vertices in X which have at least cn neighbours in both A and

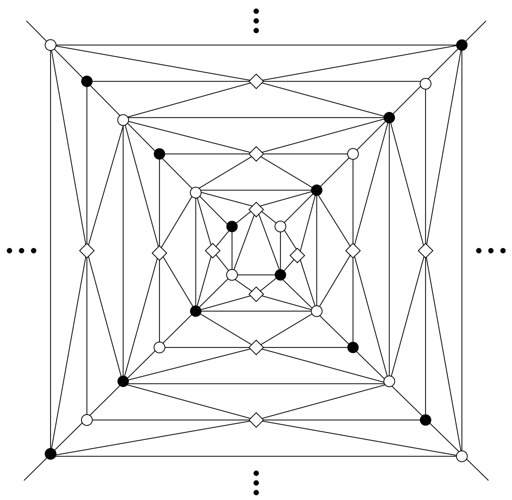

Figure 1: A 3-partite triangulation of maximum degree 8 with vertex classes of equal size

B. Let $X_{BC} \subseteq X \setminus X_{AB}$ be the set of all vertices with at least cn neighbours in both B and C. Set $X_{AC} := X \setminus (X_{AB} \cup X_{BC})$. Thus each $x \in X_{AC}$ has at least cn neighbours in A and at least cn neighbours in C.

Consider any 3-partite almost-triangulation P whose vertex classes are A, B and C and which satisfies the following properties:

- $\Delta(P) \leq 8$.
- P has precisely $|X|$ attachment faces, all of which are bounded by 4-cycles. All these 4-cycles are disjoint. For $|X_{AB}|$ (respectively $|X_{BC}|$; $|X_{AC}|$) attachment faces F of P, the vertices on the boundary ∂F of F lie alternately in A and B (respectively in B and C; in A and C).
- P has a facial triangle D which is disjoint from all the facial 4-cycles of P.

Such almost-triangulations can easily be constructed, for example modify the graph in Figure 1. Assign each vertex $x \in X_{AB}$ to an attachment face F_x of P such that the vertices of ∂F_x lie alternately in A and B and such that the F_x are different for distinct x. Proceed similarly for all $x \in X_{BC} \cup X_{AC}$. The Blow-up lemma implies that G contains a copy of P such that the copy of D in G meets each of A', B' and C' and such that each $x \in X$ is joined to all vertices in the copy of ∂F_x in G. (To achieve the former, when applying the Blow-up lemma we only have to restrict the image of the unique vertex in $V(D) \cap A$ to A', that of the vertex in $V(D) \cap B$ to B' and that of the remaining vertex of D to C'. Similarly, for each $x \in X_{AB}$ (say) we have to restrict the images of the vertices on ∂F_x in A and B to $N_A(x)$ and $N_B(x)$ respectively.) The graph obtained from the copy of P in G by including each vertex x into F_x and joining it to all vertices on ∂F_x is a triangulation as required in the corollary. $\square$

3 Overview of the proof

3.1 Applying the Regularity lemma and covering the reduced graph with triangles

Fix positive constants ε and d with $\varepsilon \ll d \ll 1$. We first apply the Regularity lemma to G to obtain an exceptional set V_0 and clusters $V_1, \dots, V_k$ where $1/\varepsilon \leq k \leq N(\varepsilon)$. Let L and G' be as defined in the Regularity lemma and let R denote the reduced graph. Proposition 8 shows that $\delta(R) \geq (2/3 - 2d)k$. Now Theorem 2 implies that there exists a family $\mathcal{T}$ of disjoint triangles of R such that all but at most $13dk$ vertices of R lie in some triangle from $\mathcal{T}$. (To see this, add $6dk$ new vertices to R and join them to all other vertices. The graph R' thus obtained has minimum degree at least $2|R'|/3$. Theorem 2 shows that R' contains $\lfloor |R'|/3 \rfloor$ disjoint triangles. Then all but at most $12dk + 2$ vertices of R are contained in a triangle lying entirely in R.)

Delete all those vertices from R which do not lie in a triangle from $\mathcal{T}$ and still denote this graph by R. Let $T_1, \dots, T_{k'}$ be the triangles in $\mathcal{T}$. So $|R| = 3k'$ and

$$\delta(R) \geq (2/3 - 2d)k - 13dk \geq (2/3 - 15d)|R|. \tag{1}$$

Clearly, we may assume that $V_1, \dots, V_{3k'}$ are the vertices of R. We add all vertices of G which do not lie in some V_i with $1 \leq i \leq 3k'$ to the exceptional set and still denote this set by V_0. Thus $|V_0| \leq \varepsilon n + 13dkL$. Next we apply Proposition 7 to the union $R^* \subseteq R$ of all the triangles in $\mathcal{T}$ to obtain for each cluster V_i a subset of size $(1 - 2\varepsilon)L =: L'$ such that for all the edges $V_iV_j \in R^*$ the bipartite subgraphs of G' between the corresponding subsets are $(2\varepsilon, d-3\varepsilon)$-super-regular. For each vertex $V_i \in R$, we add all those vertices in V_i which do not lie in the selected subset of V_i to V_0 and still denote this selected subset by V_i and the enlarged exceptional set by V_0. Thus

$$|V_0| \leq \varepsilon n + 13dkL + 2\varepsilon kL \leq 14dn. \tag{2}$$

Note that the bipartite subgraph of G' corresponding to an edge $V_iV_j \in R$ is still 2ε-regular of density $> d - \varepsilon$.

3.2 Gluing together the almost-triangulations corresponding to the triangles T_i

The Blow-up lemma immediately implies that for each triangle $T_i \in \mathcal{T}$ the corresponding subgraph $G(T_i)$ of G contains a spanning triangulation P_i. However, in order to obtain a spanning triangulation in G, we need to 'glue' all these together into a single one. As already indicated in Section 2, to make this 'gluing process' work, some faces of P_i (the attachment faces) will be 4-faces, i.e. P_i will only be an almost-triangulation. We will 'glue' P_i to P_{i+1} by using a very small almost-triangulation $G_i \subseteq G$ which has two attachment 4-faces F_i' and F_i''. F_i' is used to attach G_i to P_i while F_i'' is used to attach it to P_{i+1}. For each $1 \leq i < k'$ we set aside a suitable such graph G_i in advance, i.e. we remove its vertices from the clusters they belong to before applying the Blow-up lemma. In order to attach G_i to P_i, we need to make sure that there are enough edges between the boundary of one of the attachment faces of P_i, F_i say, and the boundary $\partial F_i'$ of F_i'. But if the neighbourhood of $\partial F_i'$ in $G(T_i)$ is large in some sense, then when applying the Blow-up lemma P_i can be chosen so that these ∂F_i–$\partial F_i'$ edges exist. Depending on the (common) neighbourhood of T_i and T_{i+1} in R, there are three ways in which this 'gluing process' will be done (Fig. 5). In Figure 5(a) and (c) the graph G_i is just a 4-cycle, in Figure 5(b) it is the graph obtained from a $K_{2,4}$ by adding two independent edges in the larger vertex class.

3.3 Incorporating the exceptional vertices

The preceding argument gives us a triangulation which contains all vertices of G apart from the exceptional vertices in V_0. These will be incorporated as follows. We assign each exceptional vertex $x \in V_0$ to a triangle T_i such that x has many neighbours in at least two clusters of T_i and such that not too many exceptional vertices are assigned to the same T_i.

For every $i \leq k'$ we now proceed as follows. For the first exceptional vertex x assigned to T_i, we choose a suitable almost-triangulation $Q_x \subseteq G(T_i)$ of bounded order which has two attachment 4-faces F_x and F_x'. All the vertices on the boundary of F_x will be adjacent to x, so F_x can be triangulated by inserting x. For the next exceptional vertex y assigned to T_i we choose a suitable almost-triangulation $Q_y \subseteq G(T_i) - Q_x$ with three attachment 4-faces F_y, F_y' and F_y''. Again, y will be inserted in F_y and we attach Q_y to Q_x via F_y'' and F_x'. We continue in this way until we have dealt with all the exceptional vertices assigned to T_i. In this way we obtain an almost-triangulation Q_i with one attachment 4-face.

Our aim is to apply the Blow-up lemma to find a spanning almost-triangulation in the remainder of $G(T_i)$ which has three suitable attachment 4-faces: one to attach Q_i, one to attach the 'gluing graph' G_i and one to attach G_{i-1}. (If $i = 1$ or $i = k'$ this almost-triangulation will have only two attachment 4-faces. So the graph obtained by carrying out all these attachments will be a spanning triangulation in G.) The only difficulty is that for this the remainder of $G(T_i)$ should remain super-regular. But by choosing the graphs Q_x carefully enough (using a simple probabilistic argument) the super-regularity can also be maintained.

3.4 Extremal case

The strategy described in Section 3.2 breaks down if G is very similar to the graph G^* given in Section 1 which had minimum degree $\lceil 2n/3 \rceil - 1$ but no spanning triangulation. In this case, the vertices of our given graph G can be partitioned into three sets A, B and C of roughly the same size such that the density of the bipartite graph $(A, B)_G$ between A and B is very small. This implies that $G[A]$ and $G[B]$ as well as the bipartite graphs $(A, C)_G$ and $(B, C)_G$ are nearly complete. (If also the maximum degree of G is not much larger than $2n/3$, then the graph $G[C]$ will be nearly empty.) In this extremal case we will find our spanning triangulation in G directly. The structure of G makes it rather easy to find a spanning subgraph consisting of two triangulations (one containing A and part of C and the other B and the remainder of C). The main difficulty is that we may have rather few edges which can be used to glue together these two triangulations into a single one.

4 The extremal case

Given $\varepsilon > 0$ and a partition A, B, C of the vertex set of a graph G, we say that a vertex $x \in A$ (respectively $x \in B$) is *typical* if x has at least $(1-\varepsilon)|A|$ neighbours in A (respectively at least $(1-\varepsilon)|B|$ neighbours in B) and at least $(1-\varepsilon)|C|$ neighbours in C. Similarly, we call a vertex $x \in C$ *typical* if it has at least $(1-\varepsilon)|A|$ neighbours in A and at least $(1-\varepsilon)|B|$ neighbours in B.

Definition 11 Given $\varepsilon > 0$, a partition A, B, C of the vertex set of a graph G is ε*-good* if it satisfies the following properties.

(i) Each vertex of C has at least $3|A|/4$ neighbours in A and at least $3|B|/4$ neighbours in B.

(ii) In each of the classes A, B and C all but at most an ε-fraction of the vertices are typical.

(iii) No vertex in $A \cup B$ has both at least $7|A|/8$ neighbours in A and at least $7|B|/8$ neighbours in B.

(iv) For every vertex $a \in A$ we have $|N_B(a)| \leq |N_A(a)| + \varepsilon n$. Similarly, every vertex $b \in B$ satisfies $|N_A(b)| \leq |N_B(b)| + \varepsilon n$.

(v) $(1-\varepsilon)n/3 \leq |A|, |B|, |C| \leq (1+\varepsilon)n/3$.

Note that if $\delta(G) \geq 2n/3$ and ε is sufficiently small, then conditions (iv) and (v) imply that each vertex $x \in A$ (respectively $x \in B$) has at least $|A|/4$ neighbours in A (respectively at least $|B|/4$ neighbours in B) and (iii) together with (v) imply that x has at least $|C|/10$ neighbours in C.

Throughout the proofs in this section we assume that ε is sufficiently small and that n is sufficiently large for our estimates to hold.

Lemma 12 *For each sufficiently small positive ε there exists a positive $\gamma = \gamma(\varepsilon)$ and an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ whose vertex set can be partitioned into A', B' and C' such that $(1-\gamma)n/3 \leq |A'|, |B'| \leq n/3$ and such that the density of $(A', B')_G$ is at most γ. Then G has an ε-good partition.*

Proof. Set $D := 64/\varepsilon^2$ and $\gamma := 1/D^2$. We will show that the partition A', B', C' can be made into an ε-good partition by moving only a small fraction of the vertices of G into different partition classes. Let A^* be the set of all those vertices in A' which have at least $|B'|/D$ neighbours in B' and define $B^* \subseteq B'$

similarly. Since $|A^*| \cdot |B'|/D \le e_G(A', B') \le \gamma|A'||B'|$ and the same is true for B^*, we have

$$|A^*| \le \frac{|A'|}{D} < \frac{\varepsilon n}{500} \quad \text{and} \quad |B^*| \le \frac{|B'|}{D} < \frac{\varepsilon n}{500}. \tag{3}$$

Note that every vertex in $A' \setminus A^*$ has at least $2n/3 - n/3D$ neighbours in $A' \cup C'$ and thus at least $n/3 - n/3D \ge (1-\varepsilon/2)|C'|$ neighbours in C' and at least $n/3 - 2\gamma n/3 - n/3D \ge (1-\varepsilon/2)|A'|$ neighbours in A'. Thus in A' (respectively in B') only a small fraction of the vertices might become non-typical if we move a few of the vertices of G into different partition classes. We will now show that the same is true for C'. Let C'_A be the set of those vertices in C' which have at most $(1-\varepsilon/2)|A'|$ neighbours in A' and define $C'_B \subseteq C'$ similarly. Then

$$\begin{aligned} |C'_A| \cdot (1-\varepsilon/2)|A'| + |C' \setminus C'_A| \cdot |A'| \ge e_G(A', C') &\ge \left(\frac{n}{3} - \frac{n}{3D}\right) \cdot |A' \setminus A^*| \\ &\overset{(3)}{\ge} \frac{n}{3} \cdot \left(1 - \frac{1}{D}\right)^2 |A'| \end{aligned}$$

and therefore

$$\begin{aligned} |C'_A| &\le \frac{2}{\varepsilon}\left(|C'| - \frac{n}{3} \cdot \left(1 - \frac{2}{D}\right)\right) \le \frac{2}{\varepsilon} \cdot \frac{n}{3}\left(2\gamma + \frac{2}{D}\right) \\ &\le \frac{2}{\varepsilon} \cdot \frac{n}{3} \cdot \frac{2}{D} \cdot \frac{3}{2} \le \frac{\varepsilon n}{32}. \end{aligned} \tag{4}$$

Similarly, $|C'_B| \le \varepsilon n/32$. Let $C^* \subseteq C'$ be the set of all those vertices which either have less than $6|A'|/7$ neighbours in A' or less than $6|B'|/7$ neighbours in B' (or both). (So C^* contains all vertices in C' which, when we move a few of the vertices of G into different partition classes, might come into conflict with condition (i) of Definition 11. Thus we will move each vertex in C^* to either A' or B'.) Note that $C^* \subseteq C'_A \cup C'_B$. Let $C^*_A \subseteq C^*$ be the set of all vertices which have more neighbours in A' than in B' and set $C^*_B := C^* \setminus C^*_A$.

Let $A^*_C \subseteq A'$ be the set of all vertices which have at least $6|A'|/7$ neighbours in A' and at least $6|B'|/7$ neighbours in B'. Note that $A^*_C \subseteq A^*$. Define $B^*_C \subseteq B^*$ similarly. (So $A^*_C \cup B^*_C$ contains all vertices which might come into conflict with condition (iii) of Definition 11 and thus we will move these vertices to C'.) Moreover, note that each vertex in A' which has more neighbours in B' than in A' lies in A^*. Let A^*_B be the set of all those vertices in $A^* \setminus A^*_C$ which have more neighbours in B' than in A' and define $B^*_A \subseteq B^* \setminus B^*_C$ similarly. Finally, set $A := (A' \setminus (A^*_C \cup A^*_B)) \cup C^*_A \cup B^*_A$, $B := (B' \setminus (B^*_C \cup B^*_A)) \cup C^*_B \cup A^*_B$ and $C := (C' \setminus C^*) \cup A^*_C \cup B^*_C$.

Using that all the vertices which we moved into another partition class lie in $A^* \cup B^* \cup C^*$ and that this set has size at most $\varepsilon n/15$, it is not hard to check that the partition A, B, C is ε-good. For example,

$$|C'_A \cup C'_B \cup A^*_C \cup B^*_C| \overset{(3),(4)}{\le} \frac{\varepsilon n}{16} + \frac{\varepsilon n}{250} \le \varepsilon|C|$$

and it is easy to see that all vertices in $C \setminus (C'_A \cup C'_B \cup A^*_C \cup B^*_C)$ have at least $(1-\varepsilon)|A|$ neighbours in A and at least $(1-\varepsilon)|B|$ neighbours in B. So all but at most an ε-fraction of the vertices in C are typical. $\square$

Lemma 13 *For each sufficiently small positive ε there exists an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ which has an ε-good partition A, B, C for which there are two independent A–B edges a_1b_1 and a_2b_2 such that each a_i is typical and such that each b_i has at most $|A|/3$ neighbours in A. Then G contains a triangulation as spanning subgraph.*

Proof. The fact that $\delta(G) \geq 2n/3$ and condition (v) of Definition 11 imply that each b_i has at least $7|C|/12$ neighbours in C. Since both a_1 and a_2 are typical, this implies that there are two typical vertices $c_1, c_2 \in C$ which are both joined to each of a_1, a_2, b_1, b_2. Since each of a_1, a_2, c_1, c_2 is typical, their common neighbourhood in A has size at least $(1 - 4\varepsilon)|A|$ and thus contains two typical vertices a' and a''. Similarly, since each b_i has at least $7|B|/12$ neighbours in B, the common neighbourhood of b_1, b_2, c_1, c_2 contains two typical vertices b' and b''. Set $A' := A \setminus \{a_1, a_2, a', a''\}$, $B' := B \setminus \{b_1, b_2, b', b''\}$ and $C' := C \setminus \{c_1, c_2\}$. Apply Proposition 3 to obtain an ε-fair partition of A' into A'_1 and A'_2, an ε-fair partition of B' into B'_1 and B'_2 and an ε-fair partition of C' into C'_A and C'_B such that firstly, the common neighbourhood of the 4-cycle $a_1a'a_2a'' =: D_a$ in A' as well as that in C' are split ε-fairly and secondly, such that the common neighbourhood of $b_1b'b_2b'' =: D_b$ in B' as well as that in C' are split ε-fairly.

By removing a set X_A of at most $10\varepsilon n$ vertices from $A_1 \cup A_2 \cup C_A$ and a set X_B of at most $10\varepsilon n$ vertices from $B_1 \cup B_2 \cup C_B$, we may assume that $|A_1| = |A_2| = |C_A|$, $|B_1| = |B_2| = |C_B|$ and that all these six sets consist only of typical vertices. Using this, it is easy to see that each of the bipartite graphs $(A_1, A_2)_G$, $(A_1, C_A)_G$ and $(A_2, C_A)_G$ is $(10\varepsilon, 1/2)$-super-regular. Condition (i) of Definition 11 together with the remark after Definition 11 imply that for each $x \in X_A$ its neighbourhood $N_{A'}(x)$ in A' has size at least $|A'|/5$. Since $N_{A'}(x)$ was split ε-fairly, it follows that each vertex $x \in X_A$ has at least $|A_i|/10$ neighbours in A_i $(i = 1, 2)$. Moreover, also the common neighbourhoods of D_a in A_1, A_2 and C_A have sizes at least $|A_1|/10$, $|A_2|/10$ and $|C_A|/10$ respectively. Thus if $\varepsilon \leq \varepsilon_0(1/2, 1/30)/10$ and $n \geq 4n_0(1/2, 1/30)$, where ε_0 and n_0 are as defined in Corollary 10, then Corollary 10 implies that $G[A_1 \cup A_2 \cup C_A \cup X_A]$ contains a spanning triangulation P_A which has a facial triangle that lies in the common neighbourhood of D_a. Similarly, it follows that $G[B_1 \cup B_2 \cup C_B \cup X_B]$ contains a spanning triangulation with one facial triangle lying in the common neighbourhood of D_b. P_A and P_B can be 'glued' together into a spanning triangulation of G by using c_1 and c_2 and the 4-cycles D_a and D_b as well as suitable edges (Fig. 2). $\square$

Lemma 14 *For each sufficiently small positive ε there exists an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ which has an ε-good partition A, B, C for which there are vertices $b_1, b_2 \in B$ and typical vertices $c_1, c_2 \in C$ such that each c_i is joined to both b_1 and b_2 and each b_i has at least $4\varepsilon|A|$ neighbours in A. Then G contains a triangulation as spanning subgraph.*

Proof. Since both c_1 and c_2 are typical, their common neighbourhood in A has size at least $(1 - 2\varepsilon)|A|$. Thus there are typical vertices a_1 and a_2 such that a_1

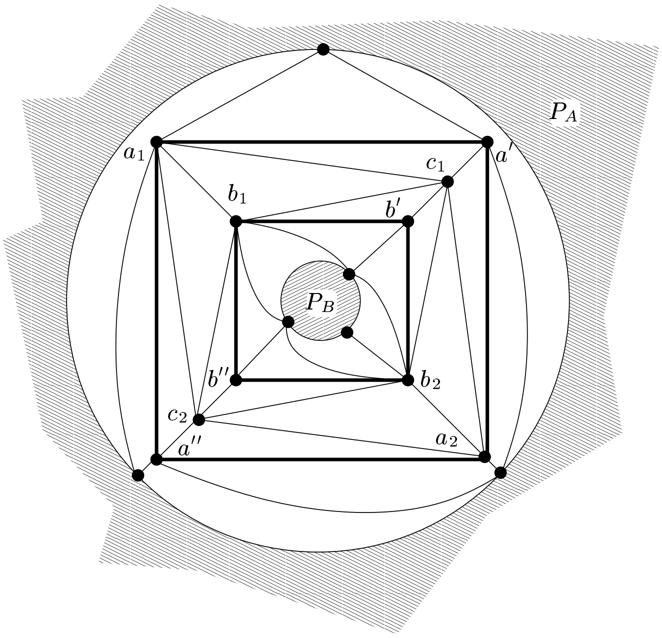

Figure 2: Gluing P_A and P_B together into a spanning triangulation in the proof of Lemma 13

is joined to each of b_1, c_1, c_2 while a_2 is joined to each of b_2, c_1, c_2. Now one can proceed in a similar way as in the proof of Lemma 13. First find two typical vertices $a', a'' \in A$ and six typical vertices $b'_1, b', b'_2, b''_1, b'', b''_2 \in B$ such that $a_1 a' a_2 a'' =: D_a$ is a 4-cycle lying in the common neighbourhood of c_1 and c_2 and such that $b_1 b'_1 b' b'_2 b_2 b''_2 b'' b''_1 =: D_b$ is an 8-cycle in the common neighbourhood of c_1 and c_2 (since b_1 and b_2 might have no common neighbours in B, we cannot take D_b to be a 4-cycle anymore). Similarly as in the proof of Lemma 13, choose disjoint triangulations P_A and P_B such that, firstly, $V(P_A) \cup V(P_B) = V(G) \setminus (V(D_a \cup D_b) \cup \{c_1, c_2\})$, secondly, one of the facial triangles of P_A lies in the common neighbourhood of D_a and thirdly, P_B has a facial triangle which lies in the common neighbourhood of $D_b - \{b_1, b_2\}$ and meets both $N_B(b_1)$ and $N_B(b_2)$. (Since b_1 and b_2 might have no common neighbours in B, this time we cannot guarantee a facial triangle of P_B that lies in the common neighbourhood of the entire cycle D_b.) As c_1 and c_2 are joined to all vertices on D_a and D_b, they can be used to 'glue' P_A and P_B together to obtain a spanning triangulation in G (Fig. 3). $\square$

Corollary 15 *For each sufficiently small positive ε there exists an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ which has an ε-good partition A, B, C for which there are distinct vertices $b_1, b_2, \ldots, b_{11} \in B$ such that each b_i has at least $4\varepsilon|A|$ neighbours in A. Then G contains a triangulation as spanning subgraph.*

Proof. As remarked after Definition 11, each b_i has at least $|C|/10$ neighbours in C. Hence an easy calculation shows that the must be two of the b_i whose

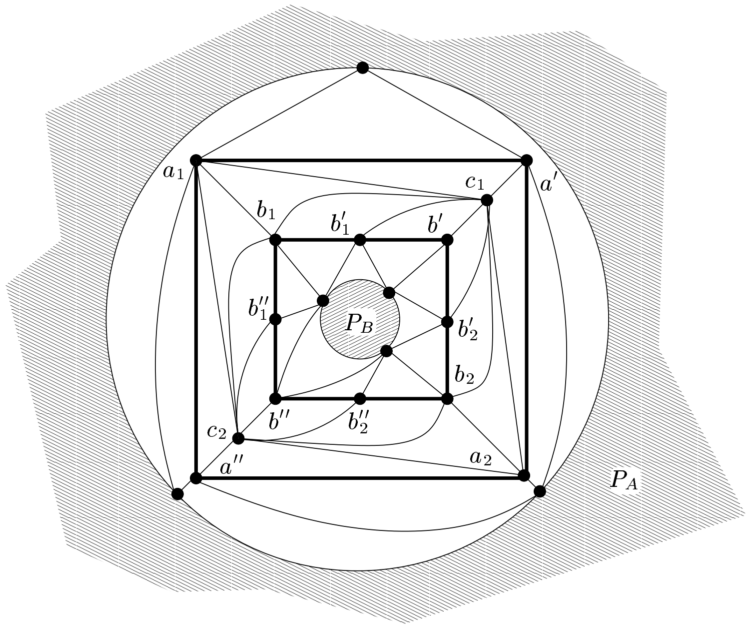

Figure 3: Gluing P_A and P_B together into a spanning triangulation in the proof of Lemma 14

common neighbourhood in C contains at least two typical vertices. So we may assume that there are two typical vertices $c_1, c_2 \in C$ which are joined to both b_1 and b_2. Now the corollary follows immediately from Lemma 14. $\square$

Proposition 16 *Given* $0 < \varepsilon \leq 1$, *every graph* H *of average degree at least* $2+\varepsilon$ *contains a cycle of length at most* $6(\log_2 |H|)/\varepsilon$.

Proof. Note that the average degree of H will not decrease below $2+\varepsilon$ by deleting a vertex of degree one or by deleting all the vertices on a path of length at least $\lceil 2/\varepsilon \rceil - 1 =: \ell$ whose vertices all have degree two in H. So by considering a subgraph of H if necessary, we may assume that the minimum degree of H is at least 2 and that H does not contain paths of length at least ℓ all of whose vertices have degree 2 in H.

Since we may assume that H does not contain a cycle of length at most $\ell+2$, this implies that each vertex x of degree 2 in H lies on a unique path P_x of length at most $\ell+1$ such that all the inner vertices of P_x, but not its endvertices, have degree 2 in H and such that the endvertices of P_x are not joined by an edge. (So P_x is induced.) Moreover, since we may assume that H does not contain a cycle of length at most $2\ell+2$, two such paths P_x and P_y either coincide or meet at most in one of their endpoints. Thus the graph H' obtained from H by replacing each such P_x by an edge has minimum degree at least 3 (and no multiple edges).

It is easy to see that H' contains a cycle C of length at most $2\log_2 |H'| \leq 2\log_2 |H|$. As each edge of C either lies in H or corresponds to some P_x, C corresponds to a cycle in H whose length is at most $(\ell+1)|C|$. $\square$

Lemma 17 *For each sufficiently small positive ε there exists an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ which has an ε-good partition A, B, C for which $G[C]$ contains a cycle D of length at most $(\log_2 n)^2$. Then G contains a triangulation as spanning subgraph.*

Proof. Recall that by condition (i) in Definition 11 the common neighbourhood in A of up to 2 vertices on D has size at least $|A|/2$. Construct a cycle $D_1 = a_1 a_2 \ldots a_{|D|}$ of length $|D|$ which consists of typical vertices from A as follows. Let a_1 be a typical vertex in A which is joined to both d_1 and $d_{|D|}$. For a_2 take a typical vertex in A which is joined to each of a_1, d_1 and d_2 and for a_3 take one which is joined to each of a_2, d_2 and d_3. The last vertex $a_{|D|}$ of D_1 will be a typical vertex in A which is joined to $a_1, a_{|D|-1}, d_{|D|-1}$ and $d_{|D|}$. So the subgraph of G induced by $V(D \cup D_1)$ contains an almost-triangulation with precisely two attachment faces, bounded by D and D_1. Similarly, as now all vertices on D_1 are typical and so every 10 (say) consecutive vertices on D_1 have a large common neighbourhood in A, there is a cycle D_2 of length at most $|D_1|/2$ consisting of typical vertices in A such that the subgraph of G induced by $V(D_1 \cup D_2)$ contains an almost-triangulation with precisely two attachment faces, bounded by D_1 and D_2. Continuing in this fashion, we may assume that for some $i \leq \log_2((\log_2 n)^2)$ the cycle D_i is a triangle. Thus G contains an almost-triangulation P'_A whose vertex set is $V(D) \cup V(D_1) \ldots V(D_i)$ and which has only one attachment face, bounded by D. Note that $|P'_A| \leq (\log_2 n)^3$. Similarly, G contains an almost-triangulation P'_B of order at most $(\log_2 n)^3$ such that all vertices of P'_B either lie on D or are typical vertices from B and such that P'_B has only one attachment face, bounded by D. So $P' := P'_A \cup P'_B$ is a triangulation.

Choose a facial triangle D'_A of P' consisting of typical vertices from A and a facial triangle D'_B consisting of typical vertices from B. Similarly as in the proof of Lemma 13, one can now find disjoint triangulations P_A and P_B with $V(P_A) \cup V(P_B) = V(G - P')$ and such that one facial triangle D_A of P_A lies in the common neighbourhood of the triangle D'_A while one facial triangle D_B of P_B lies in the common neighbourhood of D'_B. Thus the graph obtained from $P_A \cup P_B \cup P'$ by adding suitable D_A–D'_A edges and suitable D_B–D'_B edges is a spanning triangulation in G. $\square$

The next corollary is an immediate consequence of Proposition 16 and Lemma 17.

Corollary 18 *For each sufficiently small positive ε there exists an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ which has an ε-good partition A, B, C for which the average degree of $G[C]$ is at least $2 + \varepsilon$. Then G contains a triangulation as spanning subgraph.* $\square$

Lemma 19 *For each sufficiently small positive ε there exists an integer $n_0 = n_0(\varepsilon)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ which has an ε-good partition A, B, C for which the minimum degree $G[C]$ is at least 2 and for which at least $4\varepsilon|A|$ vertices* *in A and at least $4\varepsilon|B|$ vertices in B are joined to all vertices in C. Then G contains a triangulation as spanning subgraph.*

Proof. Condition (ii) of Definition 11 implies that there is a triangle D'_A that consists of typical vertices in A which are joined to all vertices in C. Similarly, there is a triangle D'_B that consists of typical vertices in B which are joined to all vertices in C. Let D'_C be a cycle in $G[C]$. Choose disjoint triangulations P_A and P_B such that $V(P_A) \cup V(P_B) = V(G - D'_A - D'_B - D'_C)$ and such that one facial triangle D_A of P_A lies in the common neighbourhood of D'_A while one facial triangle D_B of P_B lies in the common neighbourhood of D'_B. The existence of such triangulations can be shown similarly as in the proof of Lemma 13. The only difference is that D'_C might no longer be small compared to C. So if $|D'_C| \geq 10\varepsilon n$ (say), we first choose an ε-fair partition of $C \setminus V(D'_C)$ into C_A and C_B. But instead of doing the same for A and B (as we did in the proof of Lemma 13), we now proceed a little differently: the proofs of Proposition 3 and Lemma 13 immediately show that we can partition $A \setminus V(D'_A)$ into three sets A_0, A_1 and A_2 such that $A_0 \cup C_A$, A_1 and A_2 have roughly the same size and each of the three bipartite graphs between these sets can be made super-regular by removing all vertices which are not typical. by removin all non-typical vertices. Similarly, we split $B \setminus V(D'_B)$ into three sets B_0, B_1 and B_2. Having found P_A and P_B, we can extend $P_A \cup P_B \cup D'_A \cup D'_B \cup D'_C$ to a spanning triangulation of G by adding suitable edges since all vertices of D'_A and D'_B are joined to every vertex on D'_C. $\square$

Lemma 20 *For each sufficiently small positive γ there exists an integer $n_0 = n_0(\gamma)$ such that the following holds. Suppose that G is a graph of order $n \geq n_0$ and minimum degree at least $2n/3$ whose vertex set can be partitioned into A', B', C' such that $(1-\gamma)n/3 \leq |A'|, |B'| \leq n/3$ and such that the density of $(A', B')_G$ is at most γ. Then G contains a triangulation as spanning subgraph.*

Proof. Let ε be sufficiently small to satisfy the requirements of all the previous results in this section. We may assume that $\gamma \leq \gamma(\varepsilon)$ where $\gamma(\varepsilon)$ is as defined in Lemma 12. Apply this lemma to obtain an ε-good partition A, B, C of $V(G)$.

Case 1 $|A| \geq \lfloor n/3 \rfloor$ or $|B| \geq \lfloor n/3 \rfloor$.

We only consider the case when $|B| \geq \lfloor n/3 \rfloor$, the other case is similar. So set $x := |A| - \lfloor n/3 \rfloor$ and $y := |B| - \lfloor n/3 \rfloor$. Thus, by our assumption, $y \geq 0$. Since $\delta(G) \geq 2n/3$, it follows that each vertex in A sends at least

$$\delta_A := \left\lceil \frac{2n}{3} \right\rceil - (|A| - 1 + |C|) = |B| - \lfloor n/3 \rfloor + 1 = y + 1 \geq 1$$

edges to B and that each vertex in B sends at least $\max\{x+1, 0\} =: \delta_B$ edges to A. We call a vertex $a \in A$ *rich* if it has at least $|B|/10$ neighbours in B. Similarly, a vertex $b \in B$ is *rich* if it has at least $|A|/10$ neighbours in A. Let $R_A \subseteq A$ and $R_B \subseteq B$ be the sets of rich vertices in A and B.

Assume first that $|R_B| \leq \delta_A$. For every vertex $a \in A$ choose δ_A edges joining a to B. Note that there are at least $|A|/9$ vertices in A which send at least one of their chosen edges to $B \setminus R_B$. Indeed, suppose that this is not the case. Then $|R_B| = \delta_A$ and each vertex in R_B is joined to all but at most $|A|/9$ vertices in A, and thus, by condition (iv) in Definition 11, to at least $8|B|/9 - 2\varepsilon n > 7|B|/8$ vertices of B, which contradicts condition (iii). So there are at least $|A|/10$ typical vertices in A which send (at least) one of their chosen edges to $B \setminus R_B$. These chosen edges cannot all have the same endvertex in B, since this would be rich. Therefore, there are two independent A–B edges whose endvertices in A are typical and whose endvertices in B are not rich and thus have at most $|A|/10$ neighbours in A. So Lemma 13 implies that G contains a spanning triangulation.

Thus we may assume that $|R_B| > \delta_A$. On the other hand, Corollary 15 shows that we may assume that $|R_B| \leq 10$. Moreover, Lemma 14 implies that we may assume that every two typical vertices in C have at most one common neighbour in R_B. Thus all but at most $\binom{|R_B|}{2} \leq 45$ typical vertices in C have at most one neighbour in R_B. Similarly, if $x \geq 0$ and thus $\delta_B \geq 1$, we may assume that $\delta_B < |R_A| \leq 10$ and that all but at most 45 typical vertices of C have at most one neighbour in R_A.

Suppose that $x \geq 0$ and consider a (typical) vertex $c \in C$ which is joined to at most one vertex in R_B and to at most one vertex in R_A. Then

$$\begin{aligned}\left\lceil \frac{2n}{3} \right\rceil &\leq d_G(c) \leq |N_C(c)| + (|A| - |R_A| + 1) + (|B| - |R_B| + 1) \\ &\leq |N_C(c)| + \left\lfloor \frac{n}{3} \right\rfloor + x - (x+1) + \left\lfloor \frac{n}{3} \right\rfloor + y - (y+1) \qquad (5)\end{aligned}$$

and therefore $|N_C(c)| \geq 2$. The same calculation shows that if $x < 0$ then every (typical) vertex $c \in C$ which is joined to at most one vertex in R_B must have degree at least two in $G[C]$. Thus in both cases all but at most 90 typical vertices in C have degree at least two in $G[C]$ and therefore at most $\varepsilon|C| + 90$ vertices in C have degree less than two in $G[C]$. Since by Corollary 18 we may assume that the average degree of $G[C]$ is less than $2 + \varepsilon$, this implies that at most $3\varepsilon|C| + 180$ vertices in C have degree at least 3 in $G[C]$. (Note that if n is not divisible by 3, then the above calculation already shows that all but at most $\varepsilon|C| + 90$ vertices in C have degree at least 3 in $G[C]$; and thus we are done by Corollary 18 in that case. However, in what follows we will not make use of this fact.)

Call a typical vertex $c \in C$ *useful* if its degree in $G[C]$ is two and if c has at most one neighbour in R_B and, in the case when $x \geq 0$, if in addition c has at most one neighbour in R_A. Thus all but at most $4\varepsilon|C| + 270 \leq 5\varepsilon|C|$ vertices in C are useful. Furthermore, note that inequality (5) (or its analogue for the case when $x < 0$) shows that a useful vertex c must be joined to exactly one vertex in R_B.

Let us now show that $G[C]$ either contains a cycle of length at most 20 (in which case we are done by Lemma 17) or $G[C]$ contains a nontrivial path $Q = c_1 \dots c_2$ of length less than 20 whose endvertices c_1 and c_2 are useful and are joined to the same vertex $b \in R_B$. Indeed, consider a family $\mathcal{P}$ of disjoint subpaths of $G[C]$ which is obtained as follows. For the first path $P_1 \in \mathcal{P}$ take a longest path in $G[C]$. For the second path $P_2 \in \mathcal{P}$ take a longest path in

$G[C]-P_1$. The third path $P_3 \in \mathcal{P}$ is a longest path in $G[C]-P_1-P_2$. Continue in this fashion until all vertices of $G[C]$ lie on some path in $\mathcal{P}$. Consider first a path $P \in \mathcal{P}$ of length at least 20 (if such a path exists). If P does not contain a subpath Q with the required properties, then at most $2|P|/3$ vertices on P can be useful. (To see this, use that $|R_B| \le 10$ and divide P into consecutive subpaths on 20 vertices and a leftover path.) As at most $5\varepsilon|C|$ vertices in C are not useful, this implies that at most $10\varepsilon|C|$ useful vertices from C are covered by paths in $\mathcal{P}$ of length at least 20, i.e. at least $(1-15\varepsilon)|C|$ useful vertices from C are covered by a path in $\mathcal{P}$ of length less than 20. If no such path can be taken for Q, then all these paths contain at most 10 useful vertices. Thus there are at least $(1-15\varepsilon)|C|/10$ paths of length less than 20 in $\mathcal{P}$. Let $\mathcal{P}'$ denote the set of all these paths. By construction of $\mathcal{P}$, each endvertex of a path $P \in \mathcal{P}'$ either has degree one in $G[C]$ or has a second neighbour on P (in which case $G[C]$ contains a cycle of length at most 20, as required) or it is joined to a vertex in C of degree at least 3 (this vertex then lies in the interior of a path from $\mathcal{P}$ that was chosen before P). Since at most $\varepsilon|C|+90$ vertices in C have degree at most one, we may assume that at least $|C|/20$ paths $P \in \mathcal{P}'$ have an endvertex which is joined to some vertex in C of degree at least 3. Thus at least $|C|/20$ edges of $G[C]$ are incident to some vertex in C of degree at least 3, which contradicts the fact that

$$\begin{aligned}\sum_{c\in C:\,|N_C(c)|\ge 3} |N_C(c)| &\le 2e(G[C]) - \sum_{c\in C:\,|N_C(c)|=2} 2 \\ &\le (2+\varepsilon)|C| - 2(1-5\varepsilon)|C| = 11\varepsilon|C|.\end{aligned}$$

So we may assume that there exists a nontrivial path $Q = c_1 \dots c_2$ of length less than 20 whose endvertices c_1 and c_2 are useful and are joined to the same vertex $b \in R_B$. As $|N_A(b)| \ge |A|/10$ and both c_1 and c_2 are useful (and thus typical), there is a typical vertex $a \in A$ which is joined to each of b, c_1 and c_2. Similarly as in the proof of Lemma 17, one can now find planar graphs P_A and P_B satisfying the following properties. Both P_A and P_B are almost-triangulations having only one attachment face. The attachment face of P_A is bounded by the cycle ac_1Qc_2 and that of P_B is bounded by bc_1Qc_2. The vertex set of P_A consists of A, the vertices on Q and of about half of the vertices in $C \setminus V(Q)$. The vertex set of B consists of B, the vertices on Q and of the remainder of $C \setminus V(Q)$. Thus adding the edge ab to $P_A \cup P_B$ yields a spanning triangulation in G (Fig. 4).

Case 2 $|A| < \lfloor n/3 \rfloor$ and $|B| < \lfloor n/3 \rfloor$.

Set $x := \lfloor n/3 \rfloor - |A|$ and $y := \lfloor n/3 \rfloor - |B|$. Thus $x, y \ge 1$. Since $\delta(G) \ge 2n/3$, the minimum degree of $G[C]$ is at least $x + y \ge 2$. Thus Corollary 18 shows that we may assume that $x = y = 1$.

Case 2.1 All but at most $4\varepsilon|A|$ vertices in A have at least one neighbour in B or all but at most $4\varepsilon|B|$ vertices in B have at least one neighbour in A.

Suppose that the former holds, the other case is similar. As in Case 1, call a vertex $b \in B$ *rich* if it has at least $|A|/10$ neighbours in A and denote the set of all rich vertices in B by R_B. Again, it follows that we may assume that

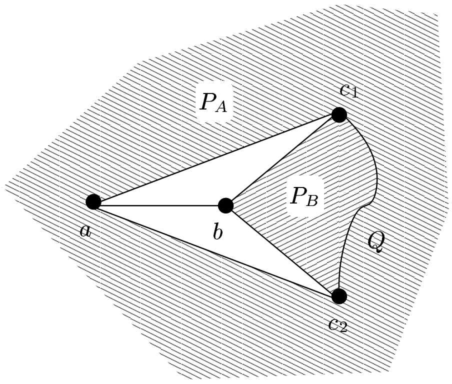

Figure 4: A spanning triangulation obtained by adding the edge ab

$2 \leq |R_B| \leq 10$ and that all but at most 45 typical vertices $c \in C$ have at most one neighbour in R_B. But for each such vertex c we have

$$\left\lceil \frac{2n}{3} \right\rceil \leq d_G(c) \leq |N_C(c)| + |A| + |B| - 1 = |N_C(c)| + 2\left\lfloor \frac{n}{3} \right\rfloor - x - y - 1$$

and so c has at least 3 neighbours in C. This shows that all but at most $\varepsilon|C| + 45$ vertices of C have degree at least 3 in $G[C]$ and so we are done by Corollary 18.

Case 2.2 At least $4\varepsilon|A|$ vertices in A have no neighbours in B and at least $4\varepsilon|B|$ vertices in B have no neighbours in A.

Since $\delta(G) \geq 2n/3$ and $x = y = 1$, each of these vertices is joined to all vertices in C. So we are done by Lemma 19. □

5 Proof of Theorem 1

5.1 Applying the Regularity lemma and covering the reduced graph with triangles

Throughout the proof we assume that the order n of G is sufficiently large for our estimates to hold. Moreover, we will assume that ε, c and d are sufficiently small positive constants such that $\varepsilon \ll d \ll c$. With these parameters ε and d, we proceed exactly as described in Section 3.1 to obtain a set $\mathcal{T} = \{T_1, \ldots, T_{k'}\}$ of triangles covering the (modified) reduced graph R. Thus every edge $V_iV_j \in R$ corresponds to a 2ε-regular bipartite subgraph of G' which has density $> d - \varepsilon$. Moreover, if V_iV_j lies in some triangle from $\mathcal{T}$, then this subgraph is also $(2\varepsilon, d - 3\varepsilon)$-super-regular.

5.2 Choosing the gluing graphs G_i

As outlined in Section 3.2, for all $1 \leq i < k'$ in turn we now choose the graphs G_i that will be used later to glue the almost-triangulation corresponding to the triangle $T_i \in \mathcal{T}$ to that corresponding to T_{i+1}. Given $1 \leq i < k'$, we assign to every vertex $W \in R$ the label $L_i(W) := (x, y)$ where $x := |N_{V(T_i)}(W)|$ and $y := |N_{V(T_{i+1})}(W)|$. So $0 \le x, y \le 3$. We need to distinguish the following cases.

Case 1 There exists a vertex $W \in R - V(T_i \cup T_{i+1})$ with label $L_i(W) = (x, y)$ such that $x, y \ge 2$.

Choose two neighbours A_i and B_i of W in T_i and two neighbours I_{i+1} and J_{i+1} of W in T_{i+1}. Let X_i denote the common neighbourhood of A_i and W in R. So $|X_i| \ge (1/3 - 30d)|R|$ by (1). By making X_i smaller if necessary, we may assume that $|X_i| \le |R|/3$. Similarly, we choose a set Y_{i+1} in the common neighbourhood of I_{i+1} and W with $(1/3 - 30d)|R| \le Y_{i+1} \le |R|/3$.

Case 1.1 X_i and Y_{i+1} are not disjoint.

In this case G_i will be a 4-cycle. Choose a vertex W' in $X_i \cap Y_{i+1}$. Take G_i to be any 4-cycle $v_i v_i' w_i w_i'$ in G with $v_i, w_i \in W$ and $v_i', w_i' \in W'$ and such that the vertices of G_i have at least $(d - 3\varepsilon)^4 L'$ common neighbours in both A_i and I_{i+1} and such that both v_i and w_i have at least $(d - 3\varepsilon)L'$ neighbours in B_i and at least $(d - 3\varepsilon)L'$ neighbours in J_{i+1}. (The existence of such vertices follows from repeated applications of Proposition 5 since each edge of R corresponds to a 2ε-regular subgraph of G' of density $> d - \varepsilon$.) So G_i is a (rather degenerate) almost-triangulation whose attachment faces F_i' and F_i'' are the interior and the exterior of G_i. Recall from Section 3.2 that we wish to apply the Blow-up lemma to find an almost-triangulation P_i in $G(T_i)$ which can be attached to G_i via an attachment 4-face F_i of P_i and F_i'. The above conditions on the neighbourhood of G_i ensure that this can be done as indicated in Figure 5(a) if we restrict the images of two opposite vertices of ∂F_i to the common neighbourhood of G_i in A_i and the images of its remaining two opposite vertices to the neighbourhoods of v_i and w_i in B_i. When applying the Blow-up lemma, the latter can always be guaranteed since by our choice of G_i all these neighbourhoods are sufficiently large.

Case 1.2 X_i and Y_{i+1} are disjoint.

In this case G_i will be the graph obtained from a $K_{2,4}$ by adding two independent edges in the larger vertex class. Note that, since $X_i \cap Y_{i+1} = \emptyset$, Lemma 20 implies that we may assume that there exists an X_i–Y_{i+1} edge $W'W''$ in R. Indeed, if there is no such edge, then the bipartite subgraph of G' between the sets $G(X_i)$ and $G(Y_{i+1})$ is empty. Let γ be the density of the bipartite subgraph of G between $G(X_i)$ and $G(Y_{i+1})$. Then it is easily checked that $G(X_i)$ contains a vertex x with $d_G(x) \ge \gamma n/4$. Since $0 = d_{G'}(x) > d_G(x) - (d+\varepsilon)n$, this implies that $\gamma \le 4(d + \varepsilon)$. Moreover, the sizes of both $G(X_i)$ and $G(Y_{i+1})$ lie between $(1 - 150d)n/3$ and $n/3$ and so if d is sufficiently small we are done by Lemma 20.

Choose vertices $v_i, w_i \in W$, $v_i', w_i' \in W'$ and $v_i'', w_i'' \in W''$ such that both v_i and w_i are joined to each of v_i', v_i'', w_i', w_i'' and such that v_i' is joined to v_i'' and w_i' is joined to w_i''. Furthermore, we choose these vertices so that v_i and w_i have at least $(d - 3\varepsilon)L'$ neighbours in both B_i and J_{i+1}, v_i, w_i, v_i', w_i' have at least $(d - 3\varepsilon)^4 L'$ common neighbours in A_i and so that v_i, w_i, v_i'', w_i'' have at least $(d - 3\varepsilon)^4 L'$ common neighbours in I_{i+1}. (Again, the existence of such vertices follows from repeated applications of Proposition 5.) So v_i and w_j form the

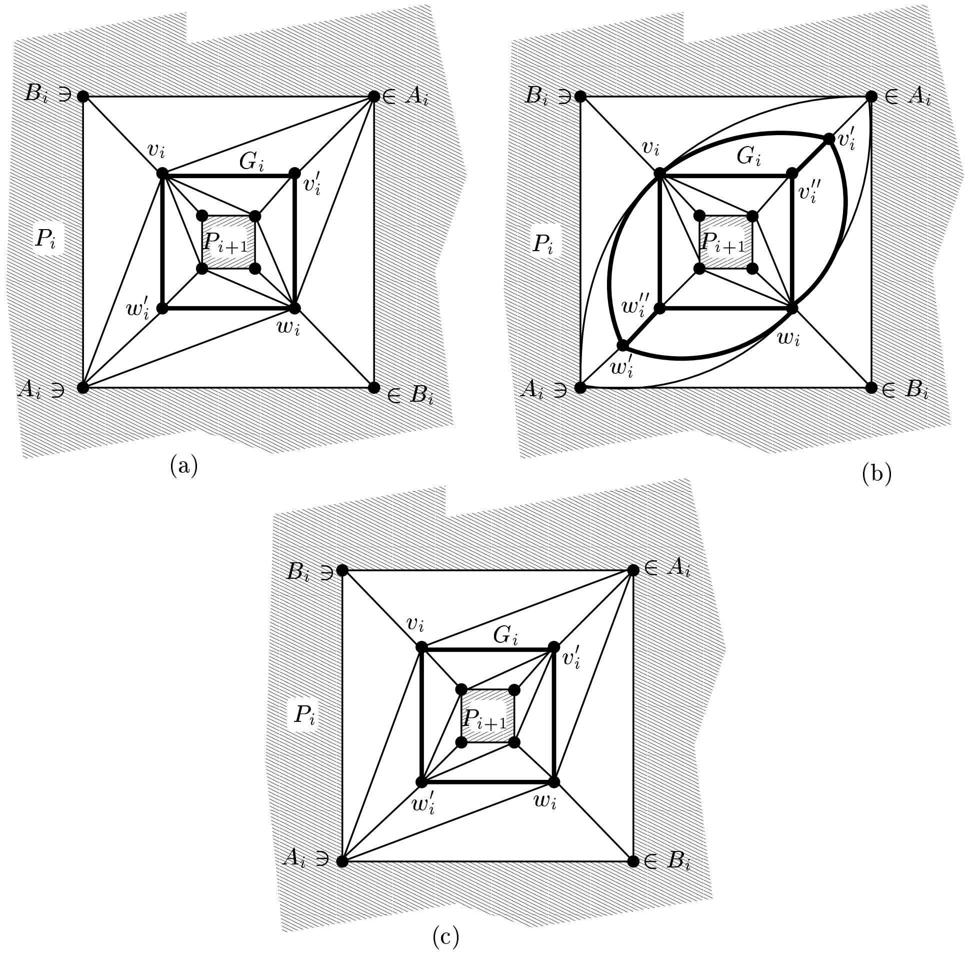

Figure 5: Three ways of gluing together the triangulations P_i and P_{i+1} by using a gluing graph G_i. The graphs G_i are indicated by thick lines.

vertex class of size two of a $K_{2,4}$ in G_i and the independent edges added to the larger vertex class of this $K_{2,4}$ are $v_i'v_i''$ and $w_i'w_i''$. Thus G_i can be viewed as an almost-triangulation whose two attachment faces F_i' and F_i'' are bounded by the 4-cycles $v_iv_i'w_iw_i'$ and $v_iv_i''w_iw_i''$ respectively. Figure 5(b) indicates how G_i will be used in Case 1.2 to glue together the almost triangulations P_i and P_{i+1} corresponding to T_i and T_{i+1} respectively.

Case 2 All vertices W in $R - V(T_i \cup T_{i+1})$ have a label $L_i(W) = (x, y)$ for which either $x \leq 1$ or $y \leq 1$ (or both).

In this case G_i will again be a 4-cycle. Let X_i be the set of all those vertices in $R - V(T_i \cup T_{i+1})$ whose label L_i is $(1, 3)$ and let Y_{i+1} be the set of all those vertices in $R - V(T_i \cup T_{i+1})$ whose label L_i is $(3, 1)$. Then

$$\begin{aligned} 4|R| - 90d|R| - 30 &\overset{(1)}{\leq} 6 \cdot \delta(R) - 2\binom{6}{2} \leq e_R(T_i \cup T_{i+1}, R - T_i - T_{i+1}) \\ &\leq 4(|X_i| + |Y_{i+1}|) + 3(|R| - |X_i| - |Y_{i+1}|) \end{aligned}$$

and therefore

$$|X_i| + |Y_{i+1}| \geq |R| - 100d|R|. \tag{6}$$

On the other hand,

$$\begin{aligned} 2|R| - 45d|R| - 15 &\overset{(1)}{\leq} 3 \cdot \delta(R) - 15 \leq e_R(T_i, R - T_i - T_{i+1}) \\ &\leq |X_i| + 3(|R| - |X_i|) = 3|R| - 2|X_i| \end{aligned}$$

and so $|X_i| \leq |R|/2 + 25d|R|$. Combining this with (6) gives $|Y_{i+1}| \geq |R|/2 - 125d|R|$. Similarly it follows that $|X_i| \geq |R|/2 - 125d|R|$. Since (1) implies that every vertex $x \in X_i$ has more than $|R| - |Y_{i+1}|$ neighbours, this means that R contains an X_i–Y_{i+1} edge $W'W$. Let A_i be the unique neighbour of $W' \in X_i$ in T_i and let B_i be any other vertex of T_i. Let I_{i+1} be the unique neighbour of $W \in Y_{i+1}$ in T_{i+1} and let J_{i+1} be any other vertex of T_{i+1}. Take G_i to be any 4-cycle $v_i v_i' w_i w_i'$ in G with $v_i, w_i \in W$ and $v_i', w_i' \in W'$ and such that the vertices on G_i have at least $(d-3\varepsilon)^4 L'$ common neighbours in both A_i and I_{i+1} and such that furthermore both v_i and w_i have at least $(d-3\varepsilon)L'$ neighbours in B_i while both v_i' and w_i' have at least $(d-3\varepsilon)L'$ neighbours in J_{i+1}. (The existence of such vertices follows from repeated applications of Proposition 5 again.) So G_i is an almost-triangulation whose attachment faces F_i' and F_i'' are the interior and the exterior of G_i. In Case 2 we will use G_i to glue P_i to P_{i+1} as indicated in Figure 5(c).

Since k' does not depend on n, it is not hard to show that for all $1 \leq i < k'$ we can choose the graphs G_i to be disjoint from each other. We then remove all the vertices of G lying in the G_i's from the clusters they belong to. So the size of each cluster now lies between $L' - 2k'$ and L'. We remove further vertices if necessary and add them to the exceptional set to ensure that all the clusters have the same size $L'' \geq L' - 2k'$. We still denote by V_0 the enlarged exceptional set. Thus

$$|V_0| \overset{(2)}{\leq} 14dn + 2k'|R| \leq 15dn. \tag{7}$$

5.3 Incorporating the exceptional vertices

Given an exceptional vertex $x \in V_0$, we say that a triangle T_i is *good for* x if x has at least cL'' neighbours in at least two of the (modified) clusters of T_i. Thus the number N of good triangles for x satisfies

$$3NL'' + (1+2c)L''k' \geq |N_G(x) \setminus V_0| \overset{(7)}{\geq} \frac{2n}{3} - 15dn.$$

Hence

$$N \geq \frac{1}{3L''}\left(\frac{2n}{3} - 15dn - (1+2c)\frac{n}{3}\right) > \frac{n}{12L''}$$

and therefore

$$\frac{|V_0|}{180dL''} \overset{(7)}{<} N.$$

But this shows that we can assign each exceptional vertex $x \in V_0$ greedily to a triangle $T_i \in \mathcal{T}$ such that T_i is good for x and such that to each triangle we assign at most $180dL''$ exceptional vertices. Let E_i be the set consisting of all exceptional vertices assigned to T_i.

Recall that A_i and B_i are two of the vertices of T_i and denote its third vertex by C_i. Moreover, recall that F_i' and F_i'' are the two attachment faces of the gluing graph G_i. For each $1 \le i \le k'$, we will now incorporate the exceptional vertices in E_i as indicated in Section 3.3. However, to ensure that after this process each of the 3 bipartite subgraphs between the vertex classes of the remainder of $G'(T_i)$ is still super-regular, we first apply Proposition 3 to the graph $G'(T_i)$ to obtain a partition A_i', A_i'' of A_i such that firstly $|A_i'| \ge |A_i|/4$ and $|A_i''| \ge |A_i|/4$, secondly, for each vertex $v \in G'(T_i)$ the set $N_{G'}(v) \cap A_i$ is split ε-fairly and, thirdly, for each $v \in E_i \cup V(\partial F_i' \cup \partial F_{i-1}'')$ the set $N_G(v) \cap A_i$ is split ε-fairly. We then apply Proposition 3 to obtain analogous partitions B_i', B_i'' of B_i and C_i', C_i'' of C_i. So in particular, each of the bipartite graphs $(A_i', B_i')_{G'}$, $(A_i', C_i')_{G'}$ and $(B_i', C_i')_{G'}$ is still 10ε-regular and has density $> d - 3\varepsilon$. Let $G_i' := G'[A_i' \cup B_i' \cup C_i']$. Note that for each exceptional vertex $v \in E_i$ at least two of the vertex classes of G_i' contain at least $(cL'' - \varepsilon|G'(T_i)|)/2 \ge cL''/4$ neighbours of v.

Fix an enumeration of the vertices in E_i. Let x be the first vertex in E_i. Applying Proposition 5 repeatedly, it is not hard to find in G_i' a copy Q_x of the graph in Figure 6 such that x is joined to all the vertices on the boundary of the face F_x of Q_x. (The latter can be guaranteed since x has at least $cL''/4$ neighbours in two of the vertex classes of G_i'.) So Q_x is a 3-partite almost-

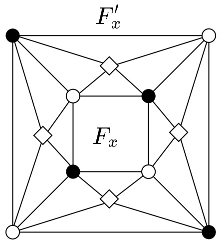

Figure 6: The graph Q_x used to incorporate the exceptional vertex x

triangulation with two attachment 4-faces F_x and F_x'. Moreover, the boundary $\partial F_x'$ of F_x' contains only vertices from two of the vertex classes of G_i'. F_x' will be used to attach Q_x to the graph Q_y which we will choose below to incorporate the second vertex $y \in E_i$. To ensure that this can be done, we choose Q_x such that $\partial F_x'$ has at least $(d - 13\varepsilon)^4 L''/4$ common neighbours in the unique vertex class of G_i' that avoids $\partial F_x'$ and such that every vertex $v \in \partial F_x'$ has at least $(d - 13\varepsilon)L''/4$ neighbours in each of the two vertex classes of G_i' not containing v.

Now we apply Proposition 5 to choose a subgraph Q_y of $G_i' - Q_x$ for the next vertex y in our enumeration of E_i. Q_y will be one of the 3-partite almost-triangulations in Figure 7. Again, Q_y is chosen in such a way that y is joined to all the vertices on ∂F_y. (To ensure this, we take Q_y to be the graph in Figure 7(a) when y has many neighbours in each of the two clusters meeting $\partial F_x'$, and take the graph in Figure 7(b) for Q_y—possibly with the black and

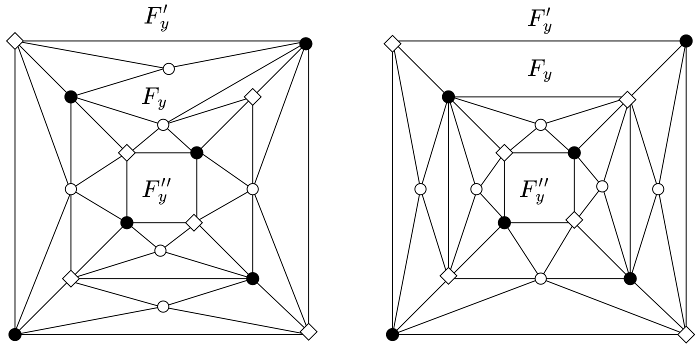

Figure 7: The two possibilities for the graph Q_y which is used to incorporate the exceptional vertex y

the white circular vertices interchanged—otherwise.) Moreover, $\partial F'_y$ will satisfy conditions analogous to $\partial F'_x$. Additionally, this time, Q_y has a third attachment 4-face F''_y which will be used to attach Q_y to Q_x via F''_y and F'_x as shown in Figure 8. (Our choice of $\partial F'_x$ implies that all the necessary $\partial F''_y$–$\partial F'_x$ edges can

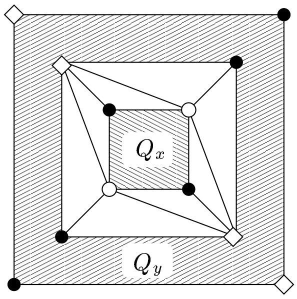

Figure 8: Attaching Q_y to Q_x via F''_y and F'_x

be guaranteed.)

We continue in this fashion until we have dealt with all the vertices in E_i. By attaching the graphs Q_z ($z \in E_i$) to each other, inserting each z into the attachment face F_z of Q_z and joining z to all vertices on ∂F_z we obtain an almost-triangulation Q_i in G'_i which has only one attachment face, namely F'_v, where v is the last vertex in the enumeration of E_i. Note that the definition of G'_i implies that the graph $G''_i := G'(T_i) - Q_i$ is still $(10\varepsilon, d/4)$-super-regular and that the vertices on the 4-cycle $v_i v'_i w_i w'_i$ bounding the attachment face F'_i of the gluing graph G_i still have at least $(d-3\varepsilon)^4 L''/4$ common neighbours in $A_i \cap V(G''_i)$ and that both v_i and w_i have at least $(d-3\varepsilon)L''/4$ neighbours in $B_i \cap V(G''_i)$. The 4-cycle bounding the attachment face F''_{i-1} of G_{i-1} satisfies analogous conditions. Moreover, note that all the graphs Q_z contain the same number of vertices in each cluster of T_i and hence the same applies to G''_i.

5.4 Applying the Blow-up lemma

Having incorporated the exceptional vertices, we can now apply the Blow-up lemma to obtain a spanning almost-triangulation P_1 in G_1'' which has two attachment 4-faces if $E_1 \neq \emptyset$ and only one attachment 4-face otherwise. The boundary of the first attachment 4-face F_1 will have two opposite vertices in the common neighbourhood of the boundary $\partial F_1'$ of the attachment face F_1' of G_1 in $A_1 \cap V(G_1'')$ while its other two vertices lie the neighbourhoods of v_1 and w_1 in $B_1 \cap V(G_1'')$ respectively. Thus P_1 can be attached to G_1 via F_1 and F_1' (see Fig. 5). Similarly, if $E_1 \neq \emptyset$, then the second attachment face of P_1 will be chosen in such a way that it can be used to attach P_1 to Q_1. Again, this is possible since the boundary of the unique attachment face of Q_1 has a large (common) neighbourhood in G_1''.

Next we apply the Blow-up lemma to obtain a spanning almost-triangulation P_2 in G_2'' which has three attachment faces if $E_2 \neq \emptyset$ and only two attachment faces otherwise. Similarly as for P_1, each of of these will be a 4-face, one will be chosen in such a way that it can be used to attach P_2 to G_2 and one to attach P_2 to Q_2 (if $E_2 \neq \emptyset$). The third attachment face will be chosen so that P_2 can be attached to G_1 via this face and F_1''. Continuing in this way, we obtain a spanning triangulation of G. ($P_{k'}$ has two attachment faces if $E_{k'} \neq \emptyset$ and only one attachment face otherwise.) This completes the proof of Theorem 1.

6 Concluding remarks

The example G^* in Section 1 showed that the bound in Theorem 1 is sharp if the order n of the graph G is divisible by 3. The following proposition states that this is in fact the case for all n.

Proposition 21 *For all $n \in \mathbb{N}$ there is a graph G of order n and minimum degree $\lceil 2n/3 \rceil - 1$ which contains no triangulation as a spanning subgraph.*

Proof. Clearly, we may assume that $n \geq 3$. Write $n = 3k+\ell$ where $\ell \in \{0,1,2\}$. If $\ell = 0$ then we can take for G the graph G^* described in Section 1. If $\ell = 1$, we take a similar graph: this time the cliques A and B both have order k and the independent set C has order $k+1$. So we may assume that $\ell = 2$. Also in this case our graph G will again look rather similar to G^*. The cliques A and B both have order k, the set C has size $k+2$, but to ensure that $\delta(G) = 2k+1 = \lceil 2n/3 \rceil - 1$ we now additionally insert a maximal set of independent edges into C and, if k is odd, one extra edge joining the unique vertex in C that is not covered by these independent edges to some other vertex in C. So at most one vertex in C has degree two in $G[C]$ and all other vertices have degree one.

We have to show that G does not contain a spanning triangulation. Suppose on the contrary that P is such a triangulation. Then P must have two facial triangles T_1 and T_2 which share an edge $xy \in G[C]$ and are such that T_1 contains a vertex of A while T_2 contains a vertex of B. We may assume that x has degree 1 in $G[C]$. On the other hand, in P the vertex x has neighbours in both A and

B. Since P is a triangulation, this implies that P contains an A–B edge joining two neighbours of x, a contradiction. $\square$

Note that the graph G in the proof of the $\ell = 2$ case of the proposition already contains the square of a Hamilton cycle if n is large. On the other hand, the proof of that case can be extended to show that one can take for $G[C]$ even a Hamilton path without creating a spanning triangulation. As soon as $G[C]$ contains a Hamilton cycle, a spanning triangulation obviously exists though.

Theorem 1 can easily be extended to arbitrary surfaces:

Theorem 22 *For each surface S there exists an integer $n_0 = n_0(S)$ such that every graph of order $n \geq n_0$ and minimum degree at least $2n/3$ contains some triangulation of S as a spanning subgraph.*

To see this, first note that in every case of the proof of Theorem 1 we apply the Blow-up lemma at least once to find a large planar (almost-) triangulation P similar to the one in Figure 1. (In particular, P is 3-chromatic and its colour classes have equal size.) For an arbitrary surface S, all we have to change is that in one of these applications we choose a 3-chromatic (almost-) triangulation of S with colour classes of equal size instead of P. (Such triangulations of S are easily seen to exist. For example, they too can be obtained by a slight modification of the graph in Figure 1.)

As remarked in the introduction, the proof of Theorem 1 shows that the spanning triangulation can be found in polynomial time. Indeed, both the Regularity lemma and the Blow-up lemma can be implemented in polynomial time (see [1] and [10] respectively). Furthermore, the probabilistic splitting argument in Proposition 3 can be derandomized using standard techniques (see e.g. [2]) and it is easy to see that the cycle of logarithmic length in Proposition 16 can be found in polynomial time using a shortest path algorithm. As the order of the reduced graph R is constant, the remaining steps can also be done in polynomial time.

In the remainder of this section we discuss how Theorem 1 might perhaps be strengthened. Obviously a minimum degree of $2n/3$ will not force every given triangulation P of order n as a subgraph. For example, G might be 3-partite, which implies that we can only hope for triangulations P with chromatic number 3. Of course, we cannot guarantee all of these either, as there are triangulations whose chromatic number is 3 and whose maximum degree is $n-2$. However, in view of our proof of Theorem 1, it might be helpful to restrict one's attention to triangulations P of bounded band-width, as this imposes a linear structure on P. (The *band-width* of a graph H is the smallest integer k for which there exists an enumeration $v_1, \dots, v_{|H|}$ of the vertices of H such that every edge $v_i v_j \in H$ satisfies $|i - j| \leq k$.) Bollobás and Komlós [7] conjectured that for every $\gamma > 0$ and all $r, \Delta \in \mathbb{N}$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that every graph G of order $n \geq n_0$ and minimum degree at least $(1 - \frac{1}{r} + \gamma)n$ contains a copy of every graph H of order n whose chromatic number is at most r, whose maximum degree is at most Δ and whose band-width is at most αn.

This would imply that every sufficiently large graph of minimum degree at least $(2/3+\gamma)n$ contains every 3-chromatic triangulation of bounded band-width. Even in this special case the error term γn cannot be omitted completely: there are 3-chromatic triangulations whose colour classes have different sizes (modify the graph in Figure 1). These obviously do not embed into the complete 3-partite graph whose vertex classes have equal size. However, it might be true that for all integers b there exists a constant $C = C(b)$ such that every graph of order n and minimum degree at least $2n/3 + C$ contains every 3-chromatic triangulation of order n and band-width at most b as a subgraph.

Also, we do not know whether one can strengthen Theorem 1 in the following way. Given n, is there a triangulation P_n of order n which is contained in every graph G of order n and minimum degree at least $2n/3$? When n is divisible by 3, then the preceding arguments show that P_n would have to be 3-chromatic with equal size colour classes. Moreover, P_n would have to contain induced cycles of many different lengths. To see the latter, consider a graph G which is similar to the graph G^* from Section 1. This time the cliques have order $n/3-1$, the independent set C has order $n/3+2$ and we insert a 2-factor into C. Then the proof of Proposition 21 shows that every spanning triangulation of G must contain one of the cycles in $G[C]$.

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