For which densities are random triangle-free graphs almost surely bipartite?

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Abstract

Denote by $\mathcal{T}(n,m)$ the class of all triangle-free graphs on n vertices and m edges. Our main result is the following sharp threshold, which answers the question for which densities a typical triangle-free graph is bipartite. Fix $\varepsilon > 0$ and let $t_3 = t_3(n) = \frac{\sqrt{3}}{4} n^{3/2}\sqrt{\log n}$. If $n/2 \leq m \leq (1-\varepsilon)t_3$, then almost all graphs in $\mathcal{T}(n,m)$ are not bipartite, whereas if $m \geq (1+\varepsilon)t_3$, then almost all of them are bipartite. For $m \geq (1+\varepsilon)t_3$, this allows us to determine asymptotically the number of graphs in $\mathcal{T}(n,m)$. We also obtain corresponding results for C_ℓ-free graphs, for any cycle C_ℓ of fixed odd length.

1 Introduction and results

Let $\mathcal{G}(n,m)$ denote the set of all graphs with vertex set $[n] := \{1,\dots,n\}$ and m edges and let $G_{n,m}$ be a graph chosen uniformly at random from $\mathcal{G}(n,m)$. The evolution of random graphs (where we begin with $m = 0$ and study the likely properties of $G_{n,m}$ for increasing m) has been the subject of much study (see e.g. [2, 5, 7]). Intimately connected to this problem, but much less understood, is the question of what happens if we restrict our attention to certain subclasses of graphs.

Here, we consider the class of triangle-free graphs, where we denote by $\mathcal{T}(n,m)$ the set of all triangle-free graphs with vertex set $[n]$ and m edges and denote by $T_{n,m}$ a graph chosen uniformly at random from $\mathcal{T}(n,m)$. Information about the typical structure of $T_{n,m}$ also allows us to estimate $|\mathcal{T}(n,m)|$. Since by definition

$$\mathbb{P}[G_{n,m} \text{ is triangle-free}] = \frac{|\mathcal{T}(n,m)|}{|\mathcal{G}(n,m)|},$$

this in turn leads to a more complete picture of the evolution of graphs in $\mathcal{G}(n,m)$. In what follows, we say that *almost all* graphs in $\mathcal{G}(n,m)$ (and similarly for $\mathcal{T}(n,m)$ etc.) have some property if the proportion of graphs in $\mathcal{G}(n,m)$ with this property tends to one as n tends to infinity. We then also say that $G_{n,m}$ has this property *almost surely.*

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To see what results one might expect, consider first the uniform measure on *all* triangle-free graphs on n vertices. Here Erdős, Kleitman and Rothschild [4] proved that *almost all triangle-free graphs are bipartite.* Furthermore, Mantel proved in 1906 that every triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges and moreover that for $m = \lfloor n^2/4 \rfloor$, all graphs in $\mathcal{T}(n,m)$ are bipartite. If $m = o(n)$, then we have a similar result. Indeed, for such m almost surely $G_{n,m}$ contains no cycle at all, and thus almost all elements of $\mathcal{T}(n,m)$ are bipartite. However, the following result by Prömel and Steger shows that there is a range of m where this is not the case.

Theorem 1. [16] *There exist constants c_1, c_2, and c_3 such that*

$$\mathbb{P}\left[T_{n,m} \text{ is bipartite}\right] \to \begin{cases} 1 & \text{if} \quad m = o(n) \\ 0 & \text{if} \quad c_1 n \leq m \leq c_2 n^{3/2} \\ 1 & \text{if} \quad m \geq c_3 n^{7/4} \log n. \end{cases}$$

It is easily seen that the condition $m = o(n)$ is best possible. The main result is the second 1-statement. Prömel and Steger conjectured that the function involving the $n^{7/4}$ term could be replaced by a function growing like $n^{3/2+o(1)}$. Recently, Łuczak proved a related result, which implies that $n^{3/2}$ is the threshold for the property that almost all graphs in $\mathcal{T}(n,m)$ are "almost" bipartite. His proof is based on a sparse version of the Regularity Lemma of Szemerédi.

Theorem 2. [12] *Given $\delta > 0$, there exists a constant $C > 0$ so that almost all triangle-free graphs with n vertices and $m \geq Cn^{3/2}$ edges can be made bipartite by deleting at most δm edges.*

In this paper, we resolve the above conjecture by proving the following sharp threshold for bipartiteness. Quite naturally, it turns out that this threshold is a little larger than that for "almost" bipartiteness. As in Theorem 1, the main result is the second 1-statement.

Theorem 3. *Let*

$$t_3 = t_3(n) = \frac{\sqrt{3}}{4} n^{3/2} \sqrt{\log n}.$$

Then for any $\varepsilon > 0$,

$$\mathbb{P}\left[T_{n,m} \text{ is bipartite}\right] \to \begin{cases} 1 & \text{if} \quad m = o(n) \\ 0 & \text{if} \quad n/2 \leq m \leq (1-\varepsilon)\, t_3 \\ 1 & \text{if} \quad m \geq (1+\varepsilon)\, t_3. \end{cases}$$

In Section 9, we show how the proof of Theorem 3 yields a very natural result on the structure of triangle-free graphs which provides a bridge between Theorems 2 and 3. In [14] we also give a simple argument which shows that in most of the non-bipartite range almost all graphs in $\mathcal{T}_{n,m}$ actually have rather high chromatic number.

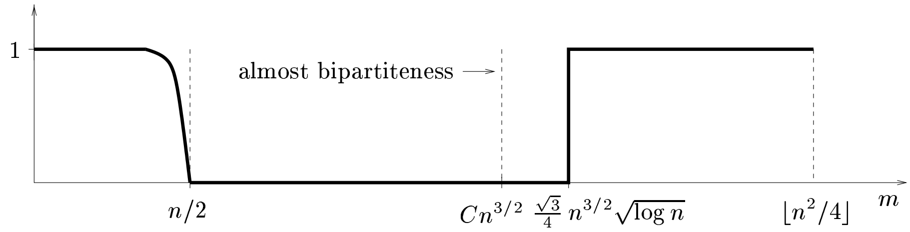

Figure 1: The proportion of triangle-free graphs with n vertices and m edges which are bipartite as $n \to \infty$

We briefly mention some related results. Firstly, a result of Bollobás (see Chapter X of Bollobás [2] or Spencer [18]) implies that

$$m \geq \left(\frac{n^3}{2} \log(\omega n) \right)^{1/2}$$

is a necessary and sufficient condition for $G_{n,m}$ to have diameter two almost surely, where ω is some function tending to infinity arbitrarily slowly. The proof of this result also implies that the threshold function for the property that almost surely every edge of $G_{n,m}$ lies on a triangle is $\Theta(t_3)$. Here, for functions f and g we write $f = \Theta(g)$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$.

Secondly, to prove his lower bound on the Ramsey number $R(3,s)$, Kim [8] showed that there exists a triangle-free graph where the independent sets have size $\mathcal{O}(t_3/n)$. This graph is certainly "far from being bipartite" and it is implicit in his proof that it has $\Theta(t_3)$ edges.

Furthermore, it is worth noting that Theorems 2 and 3 have a deterministic analogue: Erdős et al. [3] proved that a graph in $\mathcal{T}(n,m)$ must exhibit bipartite-like behaviour (it contains a large induced bipartite graph) if and only if $m/n^{3/2} \to \infty$.

As indicated earlier, information about the typical structure of triangle-free graphs allows us to determine asymptotically the number of graphs in $\mathcal{T}(n,m)$ for sufficiently large m. The following theorem (whose proof is based on Corollaries 2.9 and 2.10 of Prömel and Steger [15]) provides a precise asymptotic estimate for the number of bipartite graphs with sufficiently many edges. Together with Theorem 3, this gives us the asymptotic for the number of graphs in $\mathcal{T}(n,m)$ for $m \geq (1+\varepsilon)t_3$, which improves on the bounds implied by Theorem 2 for these m.

Theorem 4. *Almost all bipartite graphs with n vertices and $m \geq n(\log n)^2$ edges have a bipartition where the sizes of the vertex classes differ by at most $2n\sqrt{\log n}/\sqrt{m}$. Moreover, if $m/n^2 \to 0$, the number of such graphs is*

$$(1+o(1))\frac{\sqrt{\pi}}{4}\frac{n}{\sqrt{m}}\binom{n}{\lfloor n/2 \rfloor}\binom{\lfloor n^2/4 \rfloor}{m}.$$

If $m/n^2 \not\to 0$, the number of such graphs is

$$\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\binom{\lfloor n^2/4 \rfloor}{m}\right).$$

For $m = o(n^{4/3})$, the best bounds on $|\mathcal{T}(n,m)|$ are due to Wormald [20]. If $m/n^{4/3} \not\to 0$ and if m is not large enough for Theorem 2 to apply, the best bounds are those which follow from the results in Janson, Łuczak and Ruciński [6].

Finally, it turns out that the proof of Theorem 3 can be extended to work for any odd cycle.

Theorem 5. *Given an odd integer ℓ, let*

$$t_\ell = t_\ell(n) = \left(\frac{\ell}{\ell - 1} \left(\frac{n}{2}\right)^\ell \log n \right)^{1/(\ell-1)}.$$

Then for any $\varepsilon > 0$,

$$\frac{\mathbb{P}\left[G_{n,m} \text{ is bipartite}\right]}{\mathbb{P}\left[G_{n,m} \text{ is } C_\ell\text{-free}\right]} \to \begin{cases} 1 & \text{if} \quad m = o(n) \\ 0 & \text{if} \quad n/2 \le m \le (1-\varepsilon)t_\ell \\ 1 & \text{if} \quad m \ge (1+\varepsilon)t_\ell. \end{cases}$$

This paper is organized as follows. In Section 2, we sketch the proof of the second 1-statement of Theorem 3. In Section 3 we prove Theorem 4. Furthermore, we prove some facts about bipartite graphs which we shall need in Sections 4 and 5. In Section 4, we prove the 0-statement of Theorem 3. In Section 5, we show how to deduce the second 1-statement of Theorem 3 from three lemmas, namely Lemmas 17, 18 and 19. In Section 6 we prove Lemmas 17 and 18. In Section 7 we prove Lemma 19. In Section 8, we show how the proof of Theorem 3 may be extended to yield a proof of the case of forbidding cycles of arbitrary odd length (Theorem 5). Finally, in Section 9, we show how the proof of Theorem 3 can be used to improve on Theorem 2 for $n^{3/2} \ll m \le t_3$.

2 Sketch of the proof

In this section, we outline the proof of the second 1-statement of Theorem 3. We stress that most of the notions and statements are only made precise in the later sections. In contrast to [16] and [12], the main tools used in our proof of the second 1-statement are the correlation inequalities (29) and (30), which are from Janson, Łuczak and Ruciński [6]. In [6], these were applied to give an exponential upper bound on the probability that a random graph $G_{n,p}$ (and thus $G_{n,m}$) is H-free for some fixed graph H, where $G_{n,p}$ denotes a random graph with n vertices and edge probability p. For $m \ge t_3$ and H a triangle, the bound obtained in this way is asymptotically about $\mathrm{e}^{-m/9}$ (note that $\mathrm{e}^{1/9} \approx 1.12$). However, the probability that a random graph $G_{n,m}$ is bipartite (see Corollary 8) is asymptotically only about 2^{-m}. Roughly speaking, the reason for the gap between the two bounds is that the variance of the number of triangles in $G_{n,p}$ is too large.

The outline of our approach is as follows. We say that a graph is *k-bipartite* if k is the minimal number of edges that have to be deleted to make it bipartite. In Lemma 12 we show that the ratio of k-bipartite graphs in $\mathcal{G}(n,m)$ to bipartite graphs in $\mathcal{G}(n,m)$ is at most $f(k)$, where the function $f(k)$ is not much larger than $\binom{m}{k}$. Since every graph is k-bipartite for some $k \le m/2$, we would thus be done if we could show that for all k with $1 \le k \le m/2$,

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free} \mid G_{n,m} \text{ is } k\text{-bipartite}] \le \frac{o(1)}{k^2 f(k)}. \tag{1}$$

Indeed, (1) would imply that

$$\begin{aligned}
\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free}] &= \sum_{k=0}^{m/2} \mathbb{P}[G_{n,m} \text{ is } K_3\text{-free} \mid G_{n,m} \text{ is } k\text{-bip.}]\, \mathbb{P}[G_{n,m} \text{ is } k\text{-bip.}] \\
&\le \mathbb{P}[G_{n,m} \text{ is bipartite}] + \sum_{k\ge 1} \frac{o(1)}{k^2 f(k)} \mathbb{P}[G_{n,m} \text{ is } k\text{-bip.}] \\
&= (1+o(1))\mathbb{P}[G_{n,m} \text{ is bipartite}].
\end{aligned}$$

In other words, the hope would be that by conditioning on the number of edges needed to be deleted to make $G_{n,m}$ bipartite, one can reduce the variance of the number of triangles and thus obtain the desired result.

We now illustrate this idea for the case when $k = 1$ and $m = o(n^2)$. In Proposition 9 we show that the 1-bipartite graphs outnumber the bipartite graphs by a factor of about m. Now consider a random graph chosen as follows. Fix an equitable bipartition into classes A and B, where $a = |A| = \lfloor n/2 \rfloor$ and $b = |B| = \lceil n/2 \rceil$. Fix one edge e with both endpoints in A, and include the edges between A and B independently at random with probability $p = (m-1)/(ab) \sim 4m/n^2$. What is the probability that the edge e can be extended to a triangle using a vertex in B? For each vertex in B the probability that there is a triangle extension using this vertex is p^2. But the triangle extensions are independent and thus the probability that the edge has no extensions is

$$(1-p^2)^b = \mathrm{e}^{-(1+o(1))bp^2} = \mathrm{e}^{-8(1+o(1))m^2/n^3} \begin{cases} \gg 1/m & \text{if } \ m \le (1-\varepsilon)\, t_3 \\ \ll 1/m & \text{if } \ m \ge (1+\varepsilon)\, t_3. \end{cases} \tag{2}$$

Here we write $f(n) \ll g(n)$ if $f(n)/g(n) \to 0$. Note that this does not yet imply (1) for $k = 1$, as the probability model is different. We will also encounter this problem later on, where we will usually prove results in some binomial probability model like the one described above, as these have the advantage of greater independence and tractability, and then use some technical arguments to show that essentially these results are true for the corresponding k-bipartite graphs in $\mathcal{G}(n,m)$.

When $k > 1$, we can no longer assume independence between extensions, so we apply the correlation inequalities (29) and (30). As in the case $k = 1$, we fix some vertex bipartition A, B of the set of n vertices, and fix some graphs G_A and G_B, where the vertex set of G_A is A and that of G_B is B and where these graphs together have k edges. We then apply the correlation inequalities to the bipartite graph between A and B with edge probability $p \sim (m-k)/(ab)$.

However, this yields good upper bounds on the probability that such a graph is triangle-free only if G_A and G_B are close to being regular graphs, as otherwise the variance of the number of triangles is too large. Thus we classify all possible pairs (G_A, G_B) according to how far they are from being regular. This is done by considering the size of the "torsos" – the largest subgraphs of G_A and G_B of sensible maximum degree, see Section 5. In Lemma 12 we prove that the further the pair (G_A, G_B) is from being regular, the fewer k-bipartite graphs there are whose subgraphs induced by A and B equal G_A and G_B. Accordingly, the bounds which we need on the probability that such a graph is triangle-free are allowed to get larger the more irregular G_A and/or G_B are. This tradeoff is modelled by the function $f(k, r)$ introduced in Section 5. Thus we can show in the proofs of Lemmas 17 and 18 that the correlation inequalities are strong enough to cope with moderately irregular graphs too.

However, this scheme does break down when G_A or G_B is much too far from being regular. We deal with these graphs in Section 7, where we will make use of the fact that there are not too many such graphs. Surprisingly, it also turns out that we can actually take advantage of the strong irregularity of G_A, say. Namely, it implies that G_A contains many vertices of high degree (see Proposition 11). It is easily seen that for any k-bipartite graph and any $x \in A$, the degree $d_A(x)$ of x in A is at most the degree $d_B(x)$ in B. This fact implies that A contains many vertices x with large neighbourhood $\Gamma_B(x)$ in B. We then show (see Lemma 22) that with high probability at least half of these neighbourhoods "expand", in the sense that almost all vertices in A are adjacent to some vertex in $\Gamma_B(x)$, i.e. $|\Gamma_A(\Gamma_B(x))| \sim a$. But then the whole graph will contain a triangle unless $\Gamma_A(\Gamma_B(x))$ and $\Gamma_A(x)$ are disjoint, which happens very rarely, since the former set is so large (see the proof of Lemma 19).

Our methods fail if $k \geq \delta m$, where δ is some small constant, as we can then no longer control the variance of the number of triangles. Fortunately however, the result of Łuczak (Theorem 2) means that we can ignore this case. Our methods also fail if $m/n^2 \not\to 0$. However, this case is already covered in Theorem 1 by Prömel and Steger, so we can ignore this possibility too.

For clarity, we will often omit floors and ceilings whenever this does not affect the proof. We will assume throughout that n is sufficiently large for our estimates to hold. All logarithms are base e.

3 Counting bipartite graphs

In this section we prove Theorem 4 and some facts about bipartite graphs which we will need later on. Similar results are also implicit in Prömel and Steger [15]. As the proofs are short, we include them here for completeness.

First we need some definitions. We say that a bipartite graph with vertex set $[n]$ and m edges together with an admissible bipartition of its vertex set is a 2-*coloured* graph (where we consider unordered partitions, i.e. (A, B) and (B, A) give rise to the same 2-coloured graph). We denote the number of these by $\mathrm{Col2}_{n,m}$ and the number of bipartite graphs by $\mathrm{Bip}_{n,m}$. We say that a vertex bipartition is *almost equitable* if the sizes of the vertex classes differ by at most

τ, where

$$\tau = \frac{2n\sqrt{\log n}}{\sqrt{m}}.$$

We denote by $\sum_{A,B}^{eq}$ the sum over all unordered almost equitable bipartitions of $[n]$ and denote by a and b the size of the corresponding vertex classes. For a fixed bipartition with vertex classes A and B, we call any bipartite graph with vertex classes A and B an *AB-graph.* In what follows, we shall make frequent use of the following simple inequalities (see page 5 of [2]), which are valid for $j \leq j + r < i$.

$$\left(1 - \frac{j}{i-r}\right)^r \leq \binom{i-r}{j} \Big/ \binom{i}{j} \leq \mathrm{e}^{-jr/i}. \tag{3}$$

Also we shall need the following crude estimate, which follows from Stirling's formula (see page 4 of [2]).

$$\binom{i}{j} \leq \left(\frac{\mathrm{e}i}{j}\right)^j. \tag{4}$$

Finally, we shall need the following inequality (see page 5 of [2]). For $0 \leq \eta \leq 1/2$,

$$1 - \eta \geq \mathrm{e}^{-(\eta+\eta^2)}. \tag{5}$$

Lemma 6. *For $m \geq n$,*

$$\mathrm{Col2}_{n,m} = (1 + o(1)) \sum_{A,B}^{eq} \binom{ab}{m}.$$

Proof. By inequality (3), the number of two-coloured graphs not counted by the above sum is at most

$$\begin{aligned}
&\sum_{x \geq \tau/2}^{\lfloor n/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor - x} \binom{(\lfloor n/2 \rfloor - x)(\lceil n/2 \rceil + x)}{m} \\
\leq &\binom{n}{\lfloor n/2 \rfloor} \sum_{x \geq \tau/2}^{\lfloor n/2 \rfloor} \binom{\lfloor n^2/4 \rfloor - x^2}{m} \\
\overset{(3)}{\leq} &\binom{n}{\lfloor n/2 \rfloor} \binom{\lfloor n^2/4 \rfloor}{m} \sum_{x \geq \tau/2}^{\lfloor n/2 \rfloor} \mathrm{e}^{-4mx^2/n^2}.
\end{aligned}$$

The result now follows from the fact that the last line is $o(.)$ of the number of 2-coloured graphs with partition size as equal as possible. $\square$

Lemma 7. *Suppose that $m \geq 20n \log n$. Fix a bipartition with vertex classes A and B. The probability that an AB-graph with m edges chosen uniformly at random is disconnected is $o(1/m^2)$.*

Proof. Let $a = |A|$ and $b = |B|$. Let $\sum_{a',b'}$ denote the sum over all ordered pairs of integers a' and b' with $a' + b' \geq 1$ and $a' \leq a/2$ and $b' \leq b/2$. If the AB-graph is disconnected, then there exist sets $A' \subset A$ and $B' \subset B$ with $a' = |A'| \le a/2$ and $b' = |B'| \le b/2$ so that both A' and B' are either (i) connected only to the small set B' or A' on the opposite side or (ii) connected only to the large set on the other side. The probability of (i) is at most

$$\begin{aligned}
&\sum_{a',b'} \binom{a}{a'}\binom{b}{b'}\binom{a'b'+(a-a')(b-b')}{m} \Big/ \binom{ab}{m} \qquad (6)\\
&\le \sum_{a',b'} n^{a'+b'} \binom{ab - a'(b-b') - b'(a-a')}{m} \Big/ \binom{ab}{m}\\
&\overset{(3)}{\le} \sum_{a',b'} \exp\left\{a'\left(\log n - \frac{m}{ab}(b-b')\right) + b'\left(\log n - \frac{m}{ab}(a-a')\right)\right\}\\
&\le \sum_{a',b'} \exp\left\{-a'm/(4a) - b'm/(4b)\right\} = o(1/m^2),
\end{aligned}$$

since $m \ge 20n\log n$. The probability of (ii) is at most that of (i) (as the term $a'b'+(a-a')(b-b')$ in (6) is now replaced by $a'(b-b')+b'(a-a')$ and the latter term is smaller than or equal to the former), and so the result follows. □

Corollary 8. *For $m \ge 20n\log n$,*

$$\mathrm{Bip}_{n,m} = (1+o(1)) \sum_{A,B}^{eq} \binom{ab}{m}. \qquad (7)$$

Moreover, almost all bipartite graphs with n vertices and m edges are uniquely two-colourable.

Proof. Certainly $\mathrm{Bip}_{n,m} \le \mathrm{Col2}_{n,m}$. To prove the lower bound, note that a bipartite graph is uniquely two-colourable if and only if it is connected. But by Lemma 7, the proportion of two-coloured graphs on n vertices and m edges which do not correspond to a uniquely two-colourable graph is $o(1)$, with room to spare. Hence the result now follows by Lemma 6. □

Proof of Theorem 4. Note that Corollary 8 immediately implies the first part of Theorem 4. To prove the "moreover" part, note that Corollary 8 states that it suffices to count the number of 2-coloured graphs with an almost equitable bipartition.

First assume that $m = o(n^2)$. Then the upper bound on the number of these follows as in the proof of Lemma 6 (except that the summation is now over those x with $x \le \tau/2$), using the fact that

$$\begin{aligned}
\sum_{x=0}^{\tau/2} \mathrm{e}^{-4mx^2/n^2} &= (1+o(1)) \int_0^{\tau/2} \mathrm{e}^{-4mx^2/n^2} dx \qquad (8)\\
&= (1+o(1)) \frac{n}{\sqrt{8m}} \int_0^{\sqrt{8\log n}} \mathrm{e}^{-u^2/2} du\\
&= (1+o(1)) \frac{n}{\sqrt{8m}} \sqrt{\pi/2}.
\end{aligned}$$

Here we substituted $u = x\sqrt{8m}/n$ to obtain the density function for the normal distribution. To prove the lower bound, we proceed in the same way as in the proof of the upper bound, except that we now make use of the following two inequalities. Fix some δ with $0 < \delta < 1$ and suppose that $x \le \tau$. Then firstly we have (using $(\lfloor n/2 \rfloor - x)(\lceil n/2 \rceil + x) \ge \lfloor n^2/4 \rfloor - x^2 - x$ in the first line and $m = o(n^2)$ in the second line) that

$$\binom{(\lfloor n/2 \rfloor - x)(\lceil n/2 \rceil + x)}{m} \Big/ \binom{\lfloor n^2/4 \rfloor}{m} \overset{(3)}{\ge} \left(1 - \frac{m}{\lfloor n^2/4 \rfloor - x^2 - x}\right)^{x^2+x}$$
$$\overset{(5)}{\ge} \mathrm{e}^{-4(1+\delta)m(x^2+x)/n^2}$$
$$\ge (1-\delta)\mathrm{e}^{-4(1+\delta)mx^2/n^2},$$

and secondly we have

$$\binom{n}{\lfloor n/2 \rfloor - x} \Big/ \binom{n}{\lfloor n/2 \rfloor} \ge 1 - \delta.$$

This inequality follows since $m \ge n(\log n)^2$ implies that $x \le \tau = o(\sqrt{n})$.

If $m/n^2 \not\to 0$, for the upper bound it suffices to note that (8) still holds with the $1+o(1)$ factor replaced by a sufficiently large constant. For the lower bound, it suffices to note that the number of 2-coloured graphs whose bipartition is as equal as possible is already sufficiently large. $\square$

4 Proof of the 0-statement

We now consider those m which are below the threshold for bipartiteness, but which are not covered by Theorem 1. First observe that the case when $n/2 \le m \le c_1 n$, where c_1 is any fixed constant, follows from results of Erdős and Rényi. Indeed, they showed that in this range, the probability that a random graph $G_{n,m}$ is triangle-free is bounded away from zero (see e.g. Chapter IV in [2]). On the other hand, they noted that for $m \ge n/2$, the probability that $G_{n,m}$ is bipartite tends to zero (see page 57 of [5]). So to prove the 0-statement of Theorem 3, by Theorem 1 it suffices to consider those m with $20n \log n \le m \le (1-\varepsilon)t_3$, where without loss of generality we will assume that $\varepsilon \le 10^{-6}$. Recall also that throughout we assume that n is sufficiently large for our estimates to hold.

A $(2,1)$-*coloured graph* is a graph together with a two-colouring of its vertex set with the property that there is exactly one edge which lies within a colour class and a $(2,1)$-*colourable graph* is one which can be $(2,1)$-coloured. Let $\mathrm{Bip}^{+1}_{n,m}$ denote the number of $(2,1)$-colourable graphs on n vertices with m edges, and let $\mathrm{Col2}^{+1}_{n,m}$ denote the number of $(2,1)$-coloured graphs on n vertices with m edges (where again we consider partitions into unordered parts). Recall that $\mathrm{Bip}_{n,m}$ denotes the number of bipartite graphs on n vertices with m edges.

Throughout, $\sum_{A,B}$ denotes the sum over all unordered bipartitions of n vertices, and we write $a = |A|$ and $b = |B|$.

Proposition 9. *If* $20n \log n \le m = o(n^2)$*, then*

$$\mathrm{Col2}^{+1}_{n,m} = (1 + o(1))m\, \mathrm{Bip}_{n,m}.$$

Proof.

$$\begin{aligned}
\mathrm{Col2}^{+1}_{n,m} =& \sum_{A,B} \binom{ab}{m-1} \left(\binom{a}{2} + \binom{b}{2} \right) \\
=& \frac{m}{2} \sum_{A,B} \frac{a(a-1) + b(b-1)}{ab - m + 1} \binom{ab}{m} \\
=& (1 + o(1))m \sum_{A,B}^{eq} \binom{ab}{m}. \qquad (9)
\end{aligned}$$

The last line follows by separately considering those bipartitions which are almost equitable and those which are not (a calculation as in the proof of Lemma 6 shows that the contribution of the latter is $o(1)$ of that of the almost equitable ones). Now Corollary 8 implies the result. □

Proposition 10. *If* $20n \log n \le m = o(n^2)$*, then the ratio of the number of* $(2,1)$*-coloured graphs which do not correspond to a uniquely* $(2,1)$*-colourable graph to the number of bipartite graphs tends to zero.*

Proof. Fix a partition (A, B) and an edge f within one class and consider a random $(2,1)$-coloured graph G with this partition, with m edges and containing f. Thus $G - f$ is a random AB-graph with $m - 1$ edges. If G is not uniquely $(2,1)$-colourable then either $G - f$ is not uniquely 2-colourable or there exists an edge $e \in E(G)$ between A and B so that deleting e and f yields a bipartite graph which is not uniquely two-colourable. By Lemma 7 the former occurs with probability $o(1/m^2)$ and the latter with probability $o(1/m)$, as there are m possibilities for the edge e.

Thus the number of $(2,1)$-coloured graphs where the corresponding $(2,1)$-colourable graph is not uniquely colourable is $o(\mathrm{Col2}^{+1}_{n,m}/m)$ and hence is $o(\mathrm{Bip}_{n,m})$ by Proposition 9. □

In the following proofs, we shall make use of the following Chernoff bounds (see e.g. Theorem 2.1 in Janson, Łuczak and Ruciński [7]). Let X be the number of successes in N independent coin tosses, each having success probability p'. Then for any $\delta > 0$,

$$\mathbb{P}[X \le (1 - \delta)\mathbb{E}[X]\,] \le \mathrm{e}^{-\delta^2 \mathbb{E}[X]/2}. \qquad (10)$$

$$\mathbb{P}[X \ge (1 + \delta)\mathbb{E}[X]\,] \le \left(\mathrm{e}^{\delta} (1 + \delta)^{-(1+\delta)} \right)^{\mathbb{E}[X]}. \qquad (11)$$

Proof of the 0*-statement of Theorem 3.* As remarked at the beginning of this section, we may assume that $20n \log n \le m \le (1-\varepsilon)t_3$. Fix an almost equitable bipartition into classes A and B, fix an edge e in A and let $p = (1 + \varepsilon^2)4m/n^2$.

Then the probability that a random AB-graph with edge probability p forms no triangle together with the edge e is

$$\left(1-p^2\right)^b \overset{(5)}{\geq} \mathrm{e}^{-(p^2+p^4)b} \geq \mathrm{e}^{-(1+\varepsilon^2)^3 8m^2/n^3} =: \nu.$$

Note that $m\nu \geq (1-\varepsilon)t_3\mathrm{e}^{-(1-\varepsilon)8(t_3)^2/n^3} \to \infty$. The same result holds if we fix an edge in B. Now (10) implies that the number of edges in the bipartite graph is less than $m-1$ with probability at most $\mathrm{e}^{-cm} = o(1/m)$, for some constant c depending only on ε. Since the probability of being triangle-free is monotone decreasing with m, this implies that the probability that an AB-graph, with $m-1$ edges chosen uniformly at random, forms no triangle together with e is certainly at least $\nu/2$. Thus as in the proof of (9), the number of $(2,1)$-coloured graphs not containing a triangle is at least

$$\begin{aligned}\sum_{A,B}^{eq} \binom{ab}{m-1}\left(\binom{a}{2}+\binom{b}{2}\right)\frac{\nu}{2} &= (1+o(1))\frac{\nu}{2}\,m\sum_{A,B}^{eq}\binom{ab}{m} \\ &\overset{(7)}{=} (1+o(1))\frac{\nu}{2}\,m\,\mathrm{Bip}_{n,m}.\end{aligned}$$

The result now follows from Proposition 10 and the fact that $\nu m \to \infty$. □

5 Counting almost bipartite graphs

In this section, we reduce the proof of the second 1-statement of Theorem 3 to proving that certain graphs (which are close to being bipartite) are triangle-free with sufficiently high probability. In this section, up to Lemma 17, we assume that $n(\log n)^2 = o(m)$.

As a preliminary step, we partition the set of graphs with m edges and n vertices. Given any graph G, one of its vertices x and a set of its vertices S, we denote by $d_S(x)$ the number of neighbours of x in S. Given a bipartition of the vertices of G into A and B, we say that the bipartite subgraph spanned by A and B *dominates* the graphs induced by A and B if for each vertex $x \in A$ we have $d_A(x) \leq d_B(x)$ and for each vertex $y \in B$ we have $d_B(y) \leq d_A(y)$. Given (A,B), we say that G is *(k_1,k_2,A,B)-bipartite* if the graph induced by A has exactly k_1 edges, if the one induced by B has exactly k_2 edges and if the bipartite graph spanned by A and B dominates the graphs induced by A and B.

We say that a graph G is *k-bipartite* if k is the minimal number of edges needed to be deleted to make it bipartite. Note that every k-bipartite graph is (k_1,k_2,A,B)-bipartite for some (k_1,k_2,A,B) with $k=k_1+k_2$, but that the converse is not necessarily true. Note also that a $(0,0,A,B)$-bipartite graph is bipartite. Let $a=|A|$ and $b=|B|$ again. We will refer to the subgraph of a (k_1,k_2,A,B)-bipartite graph induced by A as its *(k_1,A)-graph.* Equipped with these definitions, we can now write

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free}] \leq \sum_{A,B} \sum_{k_1+k_2=0}^{m/2} \mathbb{P}[G_{n,m} \text{ is } K_3\text{-free and } (k_1,k_2,A,B)\text{-bip.}]. \tag{12}$$

Recall that the sum $\sum_{A,B}$ is over all unordered bipartitions of $[n]$. We do not have equality in (12) since a graph may well be (k_1,k_2,A,B)-bipartite for several different quadruples (k_1,k_2,A,B).

Now we classify the (k_1,k_2,A,B)-bipartite graphs where k_1 and k_2 are small according to the properties of the subgraphs induced by A and B. Without loss of generality, we assume that $\varepsilon \leq 10^{-6}$, where ε is the constant from Theorem 3. Also assume that $k_1+k_2 \leq \delta_0 m$, where $\log\log 1/\delta_0 = \varepsilon^{-5}$ (recall that m is the total number of edges of the graphs under consideration). For all nonnegative integers r, let

$$k_{1,r} = (1-\varepsilon)^r k_1,$$

let

$$D_1 = \frac{\varepsilon^4 m}{n\log(m/k_1)}, \tag{13}$$

and define r_A by

$$(1-\varepsilon)^{r_A} = \frac{4\log\log(m/k_1)}{\varepsilon^2\log(m/k_1)}. \tag{14}$$

By the assumptions on ε and k_1, r_A is much larger than one, and so the error incurred by treating it as an integer will be negligible. We denote by T_A a largest spanning subgraph of the (k_1,A)-graph of maximum degree at most D_1 and call it a *torso* of the (k_1,A)-graph. For $1 \leq r < r_A$, we say that a (k_1,A)-graph is a $(k_{1,r},A)$*-graph* if any torso has more than $k_{1,r}$ and at most $k_{1,r-1}$ edges. By our restriction on k_1+k_2, D_1 is larger than the average degree of a (k_1,A)-graph, and so a regular (k_1,A)-graph is a $(k_{1,1},A)$-graph and is equal to its torso. A graph has torsos with few edges if it is "far" from being regular. Thus in a sense, the parameter r measures how far a $(k_{1,r},A)$-graph is from being a regular graph. Furthermore note that if $k_1 \leq (1+\varepsilon)D_1$, then any (k_1,A)-graph is a $(k_{1,1},A)$-graph. We call a (k_1,A)-graph a (k_{1,r_A},A)*-graph* if it is not a $(k_{1,r},A)$-graph for any $r < r_A$. We define $k_{2,r}$, D_2, T_B and r_B similarly and then extend these definitions to (k_2,B)-graphs and (k_1,k_2,A,B)-bipartite graphs in the obvious way.

For later reference, we note the main consequences of our definitions in the following proposition. For the next few lemmas (up to and including Proposition 15), we assume that A and B are fixed.

Proposition 11. *For $1 < r \leq r_A$, consider a $(k_{1,r},A)$-graph and one of its torsos, T_A say. Then A contains a set S, which has at most*

$$s = \frac{2k_{1,r-1}}{D_1} \tag{15}$$

vertices, so that each edge not in T_A is incident to a vertex in S. Moreover, the number of $(k_{1,r}, A)$-graphs is at most

$$a^s \binom{as}{k_1 - k_{1,r-1}} \binom{\binom{a}{2}}{k_{1,r-1}}. \tag{16}$$

Proof. Since by definition T_A has at most $k_{1,r-1}$ edges and has maximum degree D_1, there are at most s vertices in A whose degree in T_A is exactly D_1. Let S be the set of these vertices. Every edge not contained in T_A has at least one endvertex in S, since otherwise it would have been included in T_A by the edge-maximality of T_A. To prove (16), note that the above implies that every $(k_{1,r}, A)$-graph can be constructed by first choosing the vertices of S, then $k_1 - k_{1,r-1}$ edges with at least one vertex in S, and then the remaining edges. □

We call the above set S a *spine* of the $(k_{1,r}, A)$-graph (corresponding to T_A). In what follows, we will often use the following crude bounds on s/a, which hold for any a with $n/4 \le a \le n$.

$$\frac{k_1}{\varepsilon^2 m} \overset{(13)}{\le} \frac{k_1}{\varepsilon^2 n D_1 \log(m/k_1)} \overset{(14)}{\le} \frac{s}{a} \le \frac{8k_1}{nD_1} \overset{(13)}{=} \frac{8k_1 \log(m/k_1)}{\varepsilon^4 m}, \tag{17}$$

where we used $k_{1,r-1} \le k_1$ in the upper bound on s/a. In some cases it will suffice to use that the right hand inequality in turn implies that (since $k_1 \le \delta_0 m$ and $a(\log a)^2 = o(m)$) we have

$$s/a \le 1/18 \quad \text{and} \quad s \log a \le k_1. \tag{18}$$

Note also that $s \ge 2$, as a torso of a $(k_{1,r}, A)$-graph has at least D_1 edges if $r > 1$. A simpler definition of a torso would have been to define it to be the subgraph induced by the vertices of degree at most D_1. However the resulting upper bounds on s (and thus of the average degree of the vertices in S) are too large for the calculations in Section 7. For $1 \le k_1 \le \delta_0 m$, let

$$f(k_1, 1) = \mathrm{e}^{k_1 \log(20m/k_1)}.$$

Here and below for convenience we interpret $k_1 \log(m/k_1)$ as zero if $k_1 = 0$. For $1 < r \le r_A$, let

$$f(k_1, r) = \mathrm{e}^{(1+\varepsilon^2)(1-\varepsilon)^{r-1} k_1 \log(m/k_1)} = \mathrm{e}^{(1+\varepsilon^2) k_{1,r-1} \log(m/k_1)}.$$

Note that for such r, $f(k_1, r)$ is large, as we assumed $k_1 \le \delta_0 m$, where $\log\log 1/\delta_0 = \varepsilon^{-5}$.

Lemma 12. *If a, $b \ge n/4$, $m = o(n^2)$, and $k_1 + k_2 \le \delta_0 m$,*

$$\mathbb{P}[G_{n,m} \text{ is } (k_{1,r_1}, k_{1,r_2}, A, B)\text{-bip.}] \le f(k_1, r_1) f(k_2, r_2)\, \mathbb{P}[G_{n,m} \text{ is } (0,0,A,B)\text{-bip.}]$$

Proof. Consider the case $r_1 = r_2 = 1$ first. We bound the number of $(k_{1,1}, k_{2,1}, A, B)$-bipartite graphs by the total number of (k_1, k_2, A, B)-bipartite graphs. For $d \in \mathbb{N}$, let $(c)_d = c(c-1)\dots(c-d+1)$. Since the colour classes are fixed, the ratio of (k_1, k_2, A, B)-bipartite graphs to $(0, 0, A, B)$-bipartite graphs may be bounded by

$$\begin{aligned}
&\binom{\binom{a}{2}}{k_1}\binom{\binom{b}{2}}{k_2}\binom{ab}{m-k_1-k_2}\Big/\binom{ab}{m} \\
&\le \frac{(a^2/2)^{k_1}}{k_1!}\,\frac{(b^2/2)^{k_2}}{k_2!}\,\frac{(ab)_{m-k_1-k_2}}{(m-k_1-k_2)!}\,\frac{m!}{(ab)_m} \\
&\le \binom{m}{k_1;k_2}(a^2/2)^{k_1}(b^2/2)^{k_2}\frac{1}{(ab-m)_{k_1+k_2}} \\
&\le \binom{m}{k_1;k_2}\left(\frac{a}{b}\right)^{k_1}\left(\frac{b}{a}\right)^{k_2} && (19)\\
&\le f(k_1,1)\, f(k_2,1). && (20)
\end{aligned}$$

Here (20) follows from (19) since we have

$$\binom{m}{k_1;k_2} := \binom{m}{k_1}\binom{m-k_1}{k_2} \le \binom{m}{k_1}\binom{m}{k_2} \overset{(4)}{\le} \left(\frac{\mathrm{e}m}{k_1}\right)^{k_1}\left(\frac{\mathrm{e}m}{k_2}\right)^{k_2}. \tag{21}$$

Now (20) follows by substituting this into (19) and using a, $b \ge n/4$.

Now we turn to the case $r_1 > 1$ and $r_2 = 1$. For simplicity, let $k = k_1$ and $k' = k_{1,r_1-1}$. By (16), the ratio of the number of $(k_{1,r_1}, k_{2,1}, A, B)$-bipartite graphs to $(0, 0, A, B)$-bipartite graphs may be bounded by

$$\begin{aligned}
& a^s\binom{as}{k-k'}\binom{\binom{a}{2}}{k'}\binom{\binom{b}{2}}{k_2}\binom{ab}{m-k-k_2}\Big/\binom{ab}{m} \\
\overset{(20)}{\le}\;& f(k,1)f(k_2,1)a^s\binom{as}{k-k'}\binom{\binom{a}{2}}{k'}\Big/\binom{\binom{a}{2}}{k} \\
\le\;& f(k,1)f(k_2,1)\,a^s\frac{(as)^{k-k'}}{(k-k')!}\,\frac{a^{2k'}}{k'!}\,\frac{k!}{(a^2/4)^k} \\
\le\;& f(k,1)f(k_2,1)\,a^s\,8^k\left(\frac{s}{a}\right)^{k-k'}, && (22)
\end{aligned}$$

where in the last line we used $k!/((k-k')!k'!) \le 2^k$. Now use (17) and also the fact that $s\log a \le k$ (see (18)) to see that (22) is at most

$$f(k,1)f(k_2,1)\,(8\mathrm{e})^k\exp\left\{(k-k')\log\left(\frac{8k\log(m/k)}{\varepsilon^4 m}\right)\right\}.$$

Now applying $8\log(m/k)/\varepsilon^4 \ge 1$ to the term involving k' and noting that $\log(8^2\cdot 20\mathrm{e}/\varepsilon^4) \le \log\log(m/k)$, one sees that this is at most

$$\begin{aligned}
&f(k_2,1)\,\exp\left\{2k\log\log(m/k) + k'\log(m/k)\right\} \\
=&f(k,r_1)f(k_2,1)\,\exp\left\{2k\log\log(m/k) - \varepsilon^2 k'\log(m/k)\right\} \\
\le&f(k,r_1)f(k_2,1). && (23)
\end{aligned}$$

In the last line we used the fact that $r_1 \le r_A$ and (14) imply that k/k' is not too large.

Now one may use (23) to prove the general case when $r_1 > 1$ and $r_2 > 1$ in the same way. $\square$

We now prove a lemma and a corollary which we shall need in Section 6 and 7. They will imply that the bound of Lemma 12 is not too far from the truth. For each k_1, let r_1^* denote the largest r_1 with $r_1 \leq r_A$ so that the class of (k_{1,r_1}, A)-graphs is nonempty, and for k_2, define r_2^* similarly.

Lemma 13. *Suppose that $a \geq n/4$, $k_1 \leq \delta_0 m$, and $r_1 \leq r_1^*$. With probability at least $(f(k_1,r_1))^{-2\varepsilon^2}$, a (k_{1,r_1}, A)-graph chosen uniformly at random has maximum degree at most m/n.*

The probability bound in the lemma is far from best possible, but it suffices for our purposes. The proof of Lemma 13 implies that it also holds with A replaced by B and the index "1" replaced by "2".

Proof. The case $r_1 = 1$ is straightforward. Indeed, we are immediately done if $k_1 \leq m/n$. If not, consider a (k_1, A)-graph chosen uniformly at random and for a fixed vertex x in A, let X denote the degree of x in this random graph. (Recall that a (k_1, A)-graph is just a graph with vertex set A and k_1 edges.) Then $\mathbb{E}[X] = 2k_1/a \leq D_1/2$ and moreover, for $0 \leq j < a$,

$$\mathbb{P}[X = j] = \binom{a-1}{j}\binom{\binom{a}{2} - (a-1)}{k_1 - j} \Big/ \binom{\binom{a}{2}}{k_1}.$$

In other words, X is hypergeometrically distributed. Thus we can apply Theorem 2.10. in [7], which says that the Chernoff bounds (10) and (11) hold also for such X. Let Z denote the number of vertices of degree at least D_1. Then applying (11) with $(1+\delta)\mathbb{E}[X] = D_1$ shows that

$$\mathbb{P}[Z > 0] \leq \mathbb{E}[Z] \leq a\mathbb{P}[X \geq D_1] \leq a\eta^{D_1} \to 0,$$

for some fixed $\eta < 1$. Thus almost all (k_1, A)-graphs have maximum degree at most D_1. Any such graph is a $(k_{1,1}, A)$-graph and since $D_1 \leq m/n$, the statement follows.

Next suppose that $r_1 > 1$. As before, we write $k = k_1$ and $k' = k_{1,r_1-1}$ for simplicity. First suppose that $k' \leq D_1/\varepsilon^3$. But since G_A has k edges, the maximum degree of G_A is then at most

$$k \overset{(14)}{\leq} k' \log(m/k) \overset{(13)}{\leq} \frac{\varepsilon^4 m}{n} \cdot \frac{1}{\varepsilon^3} \leq m/n.$$

So we may assume that $k' \geq D_1/\varepsilon^3$. First we prove a lower bound on the number of (k_{1,r_1}, A)-graphs with maximum degree at most m/n. Fix a set S' of $s' = \varepsilon^2 k'/(2D_1)$ vertices in A. By our assumption on k', the rounding error in assuming s' is an integer is negligible. Let $\mathcal{H}_1$ be the set of all graphs with exactly $k - (1-\varepsilon^2)k'$ edges such that each of its edges has one vertex in S' and the other in $A \backslash S'$. Let $\mathcal{H}_1'$ be the set of graphs in $\mathcal{H}_1$ where the vertices in S' all have degree at least D_1 and at most m/n, and where the vertices in $A \backslash S'$ all have degree at most $D_1/2$. Consider a graph chosen uniformly at random from

$\mathcal{H}_1$. Then the expected degree of a vertex in S' is $(k-(1-\varepsilon^2)k')/s'$, and it is easily seen that (using $k/k' \geq 1$ for the lower bound)

$$2D_1 \leq \frac{k-(1-\varepsilon^2)k'}{s'} \leq \frac{kD_1}{\varepsilon^2 k'} \leq \frac{m}{n}\frac{\varepsilon^2}{(1-\varepsilon)^{r_A}\log(m/k)} \overset{(14)}{\leq} \frac{m}{2n}.$$

The expected degree of a vertex in $A\backslash S'$ is at most $k/|A\backslash S'| \leq D_1/4$. Similarly to the case when $r_1 = 1$, by counting the expected number of vertices of too large or too small degree, one sees that $|\mathcal{H}_1'| = (1+o(1))|\mathcal{H}_1|$. Now let $\mathcal{H}_2$ be the set of all graphs with vertex set $A\backslash S'$ and exactly $(1-\varepsilon^2)k'$ edges (recall that $k' \geq D_1/\varepsilon^3$ is large, so we may assume that this is an integer). Let $\mathcal{H}_2'$ be the set of all graphs in $\mathcal{H}_2$ with maximum degree at most $D_1/2$. Again, one can show as before that $|\mathcal{H}_2'| = (1+o(1))|\mathcal{H}_2|$. It is easily seen that all graphs which are the union of a graph in $\mathcal{H}_1'$ and one in $\mathcal{H}_2'$ are (k_{1,r_1}, A)-graphs – any torso of such a graph is constructed by taking all edges from the graph in $\mathcal{H}_2'$ and by taking, for each vertex $x \in S'$, D_1 of the edges incident to x in the graph in $\mathcal{H}_1'$. Thus the above implies that the number of (k_{1,r_1}, A)-graphs with maximum degree at most m/n is at least

$$|\mathcal{H}_1'|\,|\mathcal{H}_2'| = (1+o(1))\binom{s'(a-s')}{k-(1-\varepsilon^2)k'}\binom{\binom{a-s'}{2}}{(1-\varepsilon^2)k'}.$$

The result now follows by comparing this lower bound with the upper bound on the number of (k_{1,r_1}, A)-graphs in (16). The calculation is similar to the proof of the second part of Lemma 12. Note that $s/4 \geq s' \geq \varepsilon^2 s/4$.

$$\begin{aligned}
& |\mathcal{H}_1'|\,|\mathcal{H}_2'| \Big/ \left(a^s\binom{as}{k-k'}\binom{\binom{a}{2}}{k'}\right) \\
\geq\;& \frac{(s'a/2)^{k-(1-\varepsilon^2)k'}\,(a^2/4)^{(1-\varepsilon^2)k'}}{a^{s+2k'}(as)^{k-k'}}\binom{k}{(1-\varepsilon^2)k'}\Big/\binom{k}{k'} \\
\geq\;& a^{-s}\,(\varepsilon^2/8)^k\,(2s'/a)^{\varepsilon^2 k'}\,2^{-k} \\
\overset{(18)}{\geq}\;& (\varepsilon^2/(16\mathrm{e}))^k(\varepsilon^2 s/(2a))^{\varepsilon^2 k'} \\
\geq\;& (k/m)^{3\varepsilon^2 k'/2} \geq f(k,r_1)^{-2\varepsilon^2}.
\end{aligned}$$

The last line follows since (14) implies $k \leq \varepsilon^2 k' \log(m/k)/\log\log(m/k)$, furthermore we have

$$(\varepsilon^2/16\mathrm{e})^{\log(m/k)} = (k/m)^{\log(16\mathrm{e}/\varepsilon^2)} \geq (k/m)^{(\log\log(m/k))/2}$$

and by (17) we have that $\varepsilon^2 s/(2a) \geq k/m$. □

Corollary 14. *Suppose that $|a-b| \leq \varepsilon^2 n$ and that $k_1+k_2 \leq \delta_0 m$. The number of $(k_{1,r_1}, k_{2,r_2}, A, B)$-bipartite graphs is at least $(1+o(1))(f(k_1,r_1)f(k_2,r_2))^{-2\varepsilon^2}$ multiplied with the product of the number of (k_{1,r_1}, A)-graphs, the number of (k_{2,r_2}, B)-graphs and the number of AB-graphs with $m_1 = m-k_1-k_2$ edges.*

Proof. We count only those $(k_{1,r_1}, k_{2,r_2}, A, B)$-bipartite graphs where the AB-graph has minimum degree at least m/n and where its (k_{1,r_1}, A)-graph and its (k_{2,r_2}, B)-graph have maximum degree at most m/n. In an AB-graph with m_1 edges chosen uniformly at random, the vertices in A have average degree $m_1/a \geq (1-\varepsilon)2m/n$ and the vertices in B have average degree $m_1/b \geq (1-\varepsilon)2m/n$. Thus, by counting the expected number of vertices of degree at most m/n, one easily sees that almost all AB-graphs with m_1 edges have minimum degree at least m/n. By Lemma 13, the proportion of (k_{1,r_1}, A)-graphs with maximum degree m/n is at least $f(k_1, r_1)^{-2\varepsilon^2}$ and also the proportion of (k_{2,r_2}, B)-graphs with maximum degree m/n is at least $f(k_2, r_2)^{-2\varepsilon^2}$. □

In addition to Lemma 13, at the end of Section 7 it will be convenient to apply the following explicit (and crude) lower bound. The proof is similar to that of Lemma 13. Recall that $k = k_1$ and $k' = k_{1,r_1-1}$.

Proposition 15. *Suppose $r_1 > 1$. Then the number of (k_{1,r_1}, A)-graphs with maximum degree at most m/n is at least $(sa/8)^k/k!$.*

Proof. Fix a subset S' of A of size $s' = \lfloor k'/D_1 \rfloor = \lfloor s/2 \rfloor$. Choose $k-(k'-s'D_1)$ edges uniformly at random between S' and $A \backslash S'$ and choose the remaining $k'-s'D_1$ edges uniformly at random in $A \backslash S'$. As in the proof of Lemma 13, with probability tending to one a graph obtained in this way has the required properties (in particular, since $k' \geq (1-\varepsilon)k$, the expected degree of a vertex in S' is $D_1 + (k-k')/s' \geq D_1 + \varepsilon k/s' \geq D_1(1+\varepsilon)$. So with probability tending to one, each vertex in S' contributes D_1 edges to any torso). Hence (noting that (18) implies that $\binom{a-s'}{2} \geq s'(a-s')$) the number of (k_{1,r_1}, A)-graphs with maximum degree at most m/n is at least

$$\begin{aligned}(1+o(1))\binom{s'(a-s')}{k-(k'-s'D_1)}\binom{\binom{a-s'}{2}}{k'-s'D_1} &\geq (1+o(1))\binom{s'(a-s')}{k} \\ &\geq (s'a/2)^k/k! \geq (sa/8)^k/k!.\end{aligned}$$

Here we used that $s' = \lfloor s/2 \rfloor$ and $s \geq 2$. □

Next, we prove a simple lemma which implies that in the proof of Theorem 3 we can restrict our attention to vertex partitions of almost equal size.

Lemma 16. *The ratio of the number of (k_1, k_2, A, B)-bipartite graphs with $k_1 + k_2 \leq \varepsilon^5 m$, $|a-b| \geq \varepsilon^2 n$, and m edges to the total number of bipartite graphs with m edges tends to zero as n tends to infinity.*

Proof. The proof is similar to the first part of the proof of Lemma 12. The number of (k_1, k_2, A, B)-bipartite graphs as above is certainly at most

$$\begin{aligned}
&\sum_{A,B\colon |a-b|\ge \varepsilon^2 n}\ \sum_{k_1+k_2\ge 0}^{\varepsilon^5 m} \binom{\binom{a}{2}}{k_1}\binom{\binom{b}{2}}{k_2}\binom{ab}{m-k_1-k_2} \\
\le\ & 2^n \sum_{k_1+k_2\ge 0}^{\varepsilon^5 m} \binom{m}{k_1;k_2} n^{2(k_1+k_2)} \frac{\left((1-\varepsilon^4/4)n^2/4\right)_{m-k_1-k_2}}{(\lfloor n^2/4\rfloor)_m}\binom{\lfloor n^2/4\rfloor}{m} \\
\le\ & 2^n \binom{\lfloor n^2/4\rfloor}{m} \sum_{k_1+k_2\ge 0}^{\varepsilon^5 m} \binom{m}{k_1;k_2} n^{2(k_1+k_2)} \frac{\left(1-\varepsilon^4/4\right)^{m-k_1-k_2}}{(\lfloor n^2/4\rfloor - m)^{k_1+k_2}} \\
\overset{(21)}{\le}\ & 2^n \binom{\lfloor n^2/4\rfloor}{m} \left(\mathrm{e}/\varepsilon^5\right)^{2\varepsilon^5 m}\left(1-\varepsilon^4/4\right)^{m-\varepsilon^5 m} \sum_{k_1+k_2\ge 0}^{\varepsilon^5 m} \frac{n^{2(k_1+k_2)}}{(n^2/5)^{k_1+k_2}} \\
\le\ & 2^n\, m \left\{\left(5\mathrm{e}^2/\varepsilon^{10}\right)^{\varepsilon^5}\left(1-\varepsilon^4/4\right)^{1-\varepsilon^5}\right\}^m \binom{\lfloor n^2/4\rfloor}{m}.
\end{aligned}$$

Since $(1-\varepsilon^4/4)^{1-\varepsilon^5} \le \mathrm{e}^{-\varepsilon^4/8}$, the term in the curly brackets is strictly less than one. The result now follows since the total number of bipartite graphs with m edges is at least $\binom{\lfloor n^2/4\rfloor}{m}$. $\square$

Lemma 12 and Lemma 16 will enable us to deduce Theorem 3 once we have proven the following three lemmas. In each of the following lemmas, we assume that $|a-b| \le \varepsilon^2 n$, $k_1 + k_2 \le \delta_0 m$, $\log\log 1/\delta_0 = \varepsilon^{-5}$, $(1+\varepsilon)t_3 \le m = o(n^2)$, $r_1 \le r_1^*$ and $r_2 \le r_2^*$.

Lemma 17. *If $r_1 < r_A$ and $f(k_1, r_1) \ge f(k_2, r_2)$, then*

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free} \mid G_{n,m} \text{ is } (k_{1,r_1}, k_{2,r_2}, A, B)\text{-bipartite}] \le \left(f(k_1, r_1)\right)^{-1-\varepsilon^2}.$$

Lemma 18. *If $r_1 < r_A$ and $r_2 < r_B$, then*

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free} \mid G_{n,m} \text{ is } (k_{1,r_1}, k_{2,r_2}, A, B)\text{-bip.}] \le \left(f(k_1, r_1)f(k_2,r_2)\right)^{-1-\varepsilon^2}.$$

Lemma 19. *If $r_1 = r_A$ and $f(k_1, r_1) \ge (f(k_2, r_2))^{\varepsilon^2/2}$, then*

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free} \mid G_{n,m} \text{ is } (k_{1,r_1}, k_{2,r_2}, A, B)\text{-bipartite}] \le \left(f(k_1, r_1)\right)^{-4/\varepsilon^2}.$$

Below, we will use the fact that the proofs of Lemmas 17 and 19 imply that they also hold with A and B etc. interchanged.

Proof of the second 1-statement of Theorem 3. (modulo Lemmas 17, 18 and 19) We will assume that $m = o(n^2)$, since the result was proven already by Prömel and Steger [16] for larger m. Furthermore, by Theorem 2 by Łuczak [12] we have that

$$\mathbb{P}[G_{n,m} \not\supseteq K_3] \le (1+o(1))\mathbb{P}[G_{n,m} \not\supseteq K_3 \text{ and } k\text{-bip. for some } k \le \delta_0 m\ ], \quad (24)$$

where without loss of generality we assume that $\varepsilon \leq 10^{-6}$. Since every k-bipartite graph is $(k_{1,r_1}, k_{2,r_2}, A, B)$-bipartite for some A, B, r_1, r_2, and some k_1 and k_2 adding up to k, (24) is at most (see also (12))

$$(1+o(1)) \sum_{A,B} \sum_{k_1+k_2=0}^{\delta_0 m} \sum_{r_1,r_2 \geq 1}^{r_1^*, r_2^*} \mathbb{P}[G_{n,m} \text{ is } (k_{1,r_1}, k_{2,r_2}, A, B)\text{-bipartite}]$$
$$\times \mathbb{P}[G_{n,m} \not\supseteq K_3 \mid G_{n,m} \text{ is } (k_{1,r_1}, k_{2,r_2}, A, B)\text{-bipartite}]. \tag{25}$$

Let $\sum_{A,B}^*$ denote the sum over all bipartitions with $|a-b| \leq \varepsilon^2 n$. By applying Lemma 16 and then Lemma 12, one sees that (25) is at most

$$(1+o(1)) \Big\{ \mathbb{P}[G_{n,m} \text{ is bipartite}] + \sum_{A,B}^{*} \sum_{k_1+k_2=1}^{\delta_0 m} \sum_{r_1,r_2 \geq 1}^{r_1^*, r_2^*} f(k_1, r_1)\, f(k_2, r_2)$$
$$\times \mathbb{P}[G_{n,m} \not\supseteq K_3 \mid G_{n,m} \text{ is } (k_{1,r_1}, k_{2,r_2}, A, B)\text{-bip.}]\, \mathbb{P}[G_{n,m} \text{ is } (0,0,A,B)\text{-bip.}] \Big\}. \tag{26}$$

For those summands with $r_1 < r_A$ and $r_2 < r_B$ we now apply Lemma 18. If $r_1 = r_A$ and $r_2 = r_B$ we apply Lemma 19 if $f(k_1, r_1) \geq f(k_2, r_2)$ and Lemma 19 with A replaced by B otherwise. For those summands with $r_1 = r_A$ and $r_2 < r_B$ we apply Lemma 17 with A replaced by B if $f(k_1, r_1) \leq f(k_2, r_2)^{\varepsilon^2/2}$ and otherwise Lemma 19. Similarly, if $r_1 < r_A$ and $r_2 = r_B$, we apply Lemma 17 if $f(k_2, r_2) \leq f(k_1, r_1)^{\varepsilon^2/2}$ and otherwise Lemma 19, with A replaced by B. Thus we see that (26) is (crudely) at most

$$(1+o(1)) \Big\{ \mathbb{P}[G_{n,m} \text{ is bipartite}]$$
$$+ \sum_{A,B}^{*} \mathbb{P}[G_{n,m} \text{ is } (0,0,A,B)\text{-bip.}] \sum_{k_1+k_2=1}^{\delta_0 m} \sum_{r_1,r_2 \geq 1}^{r_1^*, r_2^*} (f(k_1, r_1) f(k_2, r_2))^{-\varepsilon^2/3} \Big\}.$$

Since it is easily checked that the double sum is $o(1)$, an application of Lemma 6 and Corollary 8 now completes the proof. □

6 Large torsos – Poisson behaviour

This section is devoted to the proofs of Lemmas 17 and 18. In both of the following lemmas, we assume that $|a-b| \leq \varepsilon^2 n$, $k_1 + k_2 \leq \varepsilon^2 m$ and $m \geq (1+\varepsilon) t_3$. Let

$$p = \frac{(1-\varepsilon^2) m_1}{ab} \quad \text{and} \quad m_1 = m - k_1 - k_2$$

and note that $m_1 \geq (1-\varepsilon^2) m$. Fix a (k_{1,r_1}, A)-graph G_A and a (k_{2,r_2}, B)-graph G_B. We consider a random graph $G_{G_A, G_B, p}$ which is obtained by setting $G_{G_A, G_B, p}[A] = G_A$, $G_{G_A, G_B, p}[B] = G_B$ and including the edges between A and B with probability p independently.

Lemma 20. *If in addition to the above conditions, we also have $r_1 < r_A$ and $r_2 < r_B$ as well as*

$$\log f(k_1, r_1) \geq \varepsilon^2 f(k_2, r_2) \text{ and } \log f(k_2, r_2) \geq \varepsilon^2 f(k_1, r_1), \tag{27}$$

then the probability that $G_{G_A,G_B,p}$ is triangle-free is at most

$$(f(k_1, r_1) f(k_2, r_2))^{-1-6\varepsilon^2}.$$

Lemma 21. *Under the conditions stated before Lemma 20 and if $r_1 < r_A$, the probability that $G_{G_A,G_B,p}$ is triangle-free is at most*

$$(f(k_1, r_1))^{-1-6\varepsilon^2}.$$

Lemma 21 also holds if A is replaced by B (and thus also "1" by "2"). Before we prove Lemmas 20 and 21, we show how they imply Lemmas 17 and 18. In the proofs, it will be helpful to keep in our observation from Section 5 that $f(k_i, r_i)$ is a large number.

Proof of Lemma 18. We first transform from the binomial $G_{G_A,G_B,p}$-model to one with a fixed number of edges. Fix a (k_{1,r_1}, A)-graph G_A and a (k_{2,r_2}, B)-graph G_B. Let G_{G_A,G_B,m_1} be a random graph chosen like $G_{G_A,G_B,p}$, but with the difference that the AB-graph is chosen uniformly at random from the set of all AB-graphs with exactly m_1 edges. Alternatively, one may view G_{G_A,G_B,m_1} as a random graph obtained by picking a graph $G_{n,m}$ in $\mathcal{G}(n,m)$ uniformly at random, conditional on $G_{n,m}[A] = G_A$ and $G_{n,m}[B] = G_B$. Let $\mathcal{G}_{G_A,G_B,m_1}$ be the set of such graphs.

By the Chernoff bound (11), the probability that in $G_{G_A,G_B,p}$, the AB-graph has more than m_1 edges is at most $1/2$. The probability of being triangle-free is monotone decreasing with increasing m_1, and so

$$\mathbb{P}[G_{G_A,G_B,p} \text{ is } K_3\text{-free}] \geq \mathbb{P}[G_{G_A,G_B,m_1} \text{ is } K_3\text{-free}]/2.$$

Thus, setting $\eta = (f(k_1, r_1) f(k_2, r_2))^{-1-4\varepsilon^2}$, we have

$$\mathbb{P}[G_{G_A,G_B,m_1} \text{ is } K_3\text{-free}] \leq 2\eta. \tag{28}$$

Here we applied Lemma 20 if (27) holds. In the case that (27) does not hold but $f(k_1, r_1) \geq f(k_2, r_2)$, we applied Lemma 21. In the remaining case where (27) fails but $f(k_1, r_1) \leq f(k_2, r_2)$, we applied Lemma 21 with A and B interchanged (note we are free to apply Lemma 21 either to A or B since we are assuming both $r_1 < r_A$ and $r_2 < r_B$).

Let $\mathcal{A}$ denote the event that a $(k_{1,r_1}, k_{2,r_2}, A, B)$-bipartite graph (chosen uniformly at random) is triangle-free. Also, for every pair G_A, G_B (where G_A is a (k_{1,r_1}, A)-graph and G_B is a (k_{2,r_2}, B)-graph), let $\mathcal{F}_{G_A,G_B}$ denote the event that a random $(k_{1,r_1}, k_{2,r_2}, A, B)$-bipartite graph G satisfies $G[A] = G_A$ and $G[B] = G_B$. Let $\mathcal{D}_{G_A,G_B}$ denote the event that G_{G_A,G_B,m_1} is a $(k_{1,r_1}, k_{2,r_2}, A, B)$-bipartite graph. Note that $\mathcal{D}_{G_A,G_B}$ is exactly the event that the AB-graph in G_{G_A,G_B,m_1} dominates G_A and G_B. Since A and B are fixed, it makes sense to consider the events defined above in the uniform probability measure on $\mathcal{G}(n,m)$. Also note that $\mathbb{P}[\mathcal{A}]$ is exactly the probability we are aiming to bound. Then

$$\begin{aligned}
\mathbb{P}[\mathcal{A}] &= \sum_{G_A,G_B} \mathbb{P}[\mathcal{A} \mid \mathcal{F}_{G_A,G_B}]\,\mathbb{P}[\mathcal{F}_{G_A,G_B}] \\
&= \sum_{G_A,G_B} \mathbb{P}[G_{G_A,G_B,m_1} \text{ is } K_3\text{-free} \mid \mathcal{D}_{G_A,G_B}]\,\mathbb{P}[\mathcal{F}_{G_A,G_B}] \\
&\leq \sum_{G_A,G_B} \frac{\mathbb{P}[G_{G_A,G_B,m_1} \text{ is } K_3\text{-free}]}{\mathbb{P}[\mathcal{D}_{G_A,G_B}]}\,\mathbb{P}[\mathcal{F}_{G_A,G_B}] \\
&\overset{(28)}{\leq} 2\eta \sum_{G_A,G_B} \frac{\mathbb{P}[\mathcal{F}_{G_A,G_B}]}{\mathbb{P}[\mathcal{D}_{G_A,G_B}]}.
\end{aligned}$$

But $\mathbb{P}[\mathcal{D}_{G_A,G_B}]$ is exactly the number of graphs in $\mathcal{G}(n,m)$ dominating G_A and G_B divided by $|\mathcal{G}_{G_A,G_B,m_1}|$. On the other hand, $\mathbb{P}[\mathcal{F}_{G_A,G_B}]$ is exactly the number of graphs in $\mathcal{G}(n,m)$ dominating G_A and G_B divided by the total number of $(k_{1,r_1},k_{2,r_2},A,B)$-bipartite graphs. Thus, writing $\#\{k_{1,r_1},k_{2,r_2},A,B\}$ for the number of these, Corollary 14 implies that

$$\mathbb{P}[\mathcal{A}] \leq 2\eta \sum_{G_A,G_B} \frac{|\mathcal{G}_{G_A,G_B,m_1}|}{\#\{k_{1,r_1},k_{2,r_2},A,B\}} \leq 2\eta(1+o(1))(f(k_1,r_1)f(k_2,r_2))^{2\varepsilon^2},$$

completing the proof. □

The proof of Lemma 17 is the same (except that we do not need Lemma 20 in order to prove the analogue of (28) and in the end, we apply the assumption that $f(k_1,r_1) \geq f(k_2,r_2)$ to the error term $(f(k_2,r_2))^{2\varepsilon^2}$), and is therefore omitted.

To prove Lemmas 20 and 21, we need the following correlation inequalities from [6]. Let $\{I_i\}_{i\in\mathcal{J}}$ be independent 0-1-variables where $\mathcal{J}$ is an arbitrary index set. For every subset α of $\mathcal{J}$, let $I_\alpha = \prod_{i\in\alpha} I_i$. Let $\mathcal{S}$ be a collection of subsets of $\mathcal{J}$ and set $X = \sum_{\alpha\in\mathcal{S}} I_\alpha$. Furthermore set $\mu = \mathbb{E}[X]$ and let $\Delta = \sum_{\alpha\sim\beta} \mathbb{E}[I_\alpha I_\beta]$, where the sum is over all ordered pairs (α,β) of elements of $\mathcal{S}$ so that $\alpha\cap\beta \neq \emptyset$ and $\alpha \neq \beta$. Then

$$\mathbb{P}[X=0] \leq \exp\left\{-\mu + \frac{\Delta}{2}\right\}. \tag{29}$$

If $\Delta > \mu$, then we have

$$\mathbb{P}[X=0] \leq \exp\left\{-\frac{\mu^2}{\mu+\Delta}\right\} \leq \exp\left\{-\frac{\mu^2}{2\Delta}\right\}. \tag{30}$$

Proof of Lemma 20. Fix torsos T_A and T_B of G_A and G_B. In our case, $\mathcal{J}$ is the set of edges between A and B, and $\mathcal{S}$ consists of all those pairs of adjacent edges xz', yz' between A and B so that xy is contained either in T_A or T_B. Thus X is the number of triangles in $G_{G_A,G_B,p}$ with one edge in a torso and the opposite vertex in the opposite vertex class. For any $\alpha \in \mathcal{S}$, we have

$\mathbb{E}[I_\alpha] = p^2$. Certainly $G_{G_A,G_B,p}$ is triangle-free only if X is zero. Furthermore, since $p \geq (1-\varepsilon^2)^2 m/n^2$, we have

$$\mu = \mathbb{E}[X] \geq (bk_{1,r_1} + ak_{2,r_2})p^2 \geq (k_{1,r_1} + k_{2,r_2})\frac{n(1-\varepsilon^2)p^2}{2} \tag{31}$$

$$\geq (k_{1,r_1} + k_{2,r_2})\frac{8(1-\varepsilon^2)^5 m^2}{n^3}. \tag{32}$$

and

$$\Delta = \sum_{xy,yz}\sum_{z'} \mathbb{P}[xz',\, yz',\, zz' \in G_{G_A,G_B,p}] + \sum_{xy,x'y'} 8\,\mathbb{P}[xx',\, yy',\, yx' \in G_{G_A,G_B,p}].$$

Here $\sum_{xy,yz}$ denotes the sum over the set of ordered pairs of edges in the torsos T_A and T_B which are adjacent. Note that the number of summands is at most $k_{1,r_1-1}D_1 + k_{2,r_2-1}D_2$. $\sum_{xy,x'y'}$ denotes the sum over the set of pairs of edges with xy in T_A and $x'y'$ in T_B. The number of summands here is $k_{1,r_1-1}k_{2,r_2-1} \leq k_{1,r_1-1}nD_2$. The factor 8 is due to the fact that there are 8 possible ways of extending two given opposite edges into two ordered adjacent triangles. Without loss of generality we will assume in the remainder of the proof that $k_1 \geq k_2$ (which implies that $D_1 \geq D_2$). Thus

$$\Delta = \left(\sum_{xy,yz} n + \sum_{xy,x'y'} 8\right) p^3 \leq (k_{1,r_1-1}D_1 + k_{2,r_2-1}D_2 + 8k_{1,r_1-1}D_2)np^3$$

$$\leq 9(k_{1,r_1} + k_{2,r_2})D_1 np^3 \overset{(31)}{\leq} 25\mu D_1 p. \tag{33}$$

Suppose first that $\Delta > \mu$. Crudely, we have

$$\begin{aligned}\frac{\mu^2}{2\Delta} &\overset{(33)}{\geq} \frac{\mu}{50D_1p} \overset{(31)}{\geq} (k_{1,r_1} + k_{2,r_2})\frac{np}{101D_1} \geq k_{1,r_1}\frac{n\log(m/k_1)}{102\varepsilon^4 m}\frac{4m}{n} \\ &\geq \frac{4}{\varepsilon^2}\log f(k_1,r_1) \overset{(27)}{\geq} (1+6\varepsilon^2)(\log f(k_1,r_1) + \log f(k_2,r_2)),\end{aligned}$$

and so we are done by correlation inequality (30) in this case. So henceforth assume that $\Delta \leq \mu$. Thus by correlation inequality (29) we will be done whenever we can show that

$$\mu/2 \geq (1+6\varepsilon^2)(\log f(k_1,r_1) + \log f(k_2,r_2)).$$

Now consider the case when $k_1 \geq n^{-3\varepsilon^2/16}m$. This implies

$$\begin{aligned}\mu &\overset{(32)}{\geq} k_{1,r_1}(1-\varepsilon^2)^5(1+\varepsilon)^2\log(n^{3/2}) \geq k_{1,r_1-1}\log(n^{3/2}) \\ &\geq k_{1,r_1-1}\frac{8}{\varepsilon^2}\log(m/k_1) \geq \frac{8}{\varepsilon^2(1+\varepsilon^2)}\log f(k_1,r_1) \\ &\overset{(27)}{\geq} 2(1+6\varepsilon^2)(\log f(k_1,r_1) + \log f(k_2,r_2)).\end{aligned}$$

So henceforth assume that $k_1 \le n^{-3\varepsilon^2/16} m$. Now suppose additionally that $m \le 3t_3$. Then

$$\frac{\Delta}{\mu} \overset{(33)}{\le} \frac{25\varepsilon^4 m}{n \log(m/k_1)} \frac{4m}{n^2} \le 100\varepsilon^4 \frac{9 \cdot 16 (t_3)^2}{3\varepsilon^2 n^3 \log n} \le 1000\varepsilon^2. \tag{34}$$

On the other hand, we have that, using $m \ge (1+\varepsilon)t_3$ in the first line and $m \le 3t_3$ in the third line,

$$\begin{aligned} \mu &\overset{(32)}{\ge} (k_{1,r_1} + k_{2,r_2})(1-\varepsilon^2)^5 (1+\varepsilon)^2 \log(n^{3/2}) \\ &\ge (1+2002\varepsilon^2)(k_{1,r_1-1} + k_{2,r_2-1}) \log(n^{3/2}) \\ &\ge (1+2000\varepsilon^2)(\log f(k_1,r_1) + \log f(k_2,r_2)), \end{aligned} \tag{35}$$

Hence in this case, inequality (34) and correlation inequality (29) give the desired result. If $m \ge 3t_3$, using (32), it is straightforward to check that this implies

$$\mu \ge 4(\log f(k_1,r_1) + \log f(k_2,r_2)).$$

Since we assume that $\Delta \le \mu$, we are again done by correlation inequality (29) as observed above. □

Proof of Lemma 21. The proof here is similar to that of Lemma 20. This time we take X to count only triangles with an edge in A. Thus we can ignore all terms involving k_{2,r_2} (and thus also do not need any assumptions about the relative sizes of the k_i or $f(k_i, r_i)$). □

We remark that the only place in the entire proof of Theorem 3 where we made essential use of the condition $m \ge (1+\varepsilon)t_3$ was to prove (35) when $k_1 + k_2$ is small. We will exploit this in Section 9.

7 Small torsos – expanding neighbourhoods

The aim of this section is to prove Lemma 19. Its proof hinges on the fact that in a $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph, sufficiently large sets in B (which correspond to neighbourhoods of vertices whose degree in G_A is large) will have neighbourhoods of size almost $|A|$ in A with very large probability. We first prove this in a more tractable probability model (Lemma 22).

Let $k = k_1$ and let A' be a given subset of A with $|A'| \ge a - s$, where $s = 2k_{1,r_A-1}/D_1$ as in (15). Recall from Section 5 that $r_1 = r_A \le r_1^*$. Also note that (14) implies that k is a large number. Consider a random $A'B$-graph with edge probability $p_0 = m_0/(a'b)$, where $m_0 \ge m/4$ and $a' = |A'|$. We say that a set $V \subset B$ *expands* if its neighbourhood $\Gamma_{A'}(V)$ in A' contains at least $a - y$ vertices, where

$$y = a / \log^{400/\varepsilon^4}(m/k). \tag{36}$$

Note that this definition makes sense, since one may check (using (17), $k \le \delta_0 m$ where $\log\log 1/\delta_0 = \varepsilon^{-5}$ and $\varepsilon \le 10^{-6}$) that $s \le y/2$ with room to spare. Furthermore we let

$$D_1' = \frac{\varepsilon^6 m}{40 n \log\log(m/k)}.$$

For a sequence $V_1, \ldots, V_h$ of subsets of B we define its *total weight* to be equal to $\sum_{i=1}^{h} |V_i|$.

Lemma 22. *Consider the setting of the previous paragraph. Given $h \leq k$, for $1 \leq i \leq h$ let v_i satisfy $v_i \geq D_1'$. Furthermore, suppose that $\sum_{i=1}^{h} v_i \geq k/4$. Let $V_1, \ldots, V_h$ with $|V_i| = v_i$ be chosen uniformly and independently at random in B and independently of the $A'B$-graph. Then*

$$\mathbb{P}[\textit{the expanding } V_i \textit{ have total weight at most } k/8] \leq f(k, r_A)^{-13/\varepsilon^2}.$$

In later calculations we shall make use of the fact that

$$\log f(k, r_A) \overset{(14)}{<} \frac{5}{\varepsilon^2} k \log\log(m/k). \tag{37}$$

The basic idea of the proof of this lemma is based on that of Lemma 11 in Kohayakawa, Łuczak and Rödl [10]. To prove Lemma 22, we first prove Lemma 23 and Corollary 24, which essentially states that the V_i are likely to contain many large pairwise disjoint subsets. Since the expansion of these sets is independent, Lemma 22 then follows without too much difficulty.

Consider selecting h' subsets W_i of B sequentially. Inductively, we now define whether some W_i is *useful* or not. For $i \geq 1$, let $\mathcal{U}_i$ be the set of those j with $j < i$ where W_j is useful and let

$$u_i = |\bigcup_{j \in \mathcal{U}_i} W_j|.$$

We then say that W_i is *useful* if at least one of the following holds.

- $u_i \geq y'$, where $y' = \frac{b}{\log^{4000/\varepsilon^4}(m/k)}$.
- $|W_i \setminus \bigcup_{j \in \mathcal{U}_i} W_j| \geq |W_i|/2$.

Otherwise call it *not useful*. Thus in particular, W_1 is always useful.

Lemma 23. *Given $h' \leq k$ and a sequence of integers $w_i > 0$, $1 \leq i \leq h'$, suppose that $\sum_{i=1}^{h'} w_i \geq k/8$. Let $W_1, \ldots, W_{h'}$ with $|W_i| = w_i$ be sets chosen uniformly and independently at random in B. Then*

$$\mathbb{P}[\textit{the useful } W_i \textit{ have total weight at most } k/16] \leq f(k, r_A)^{-14/\varepsilon^2}.$$

Proof. Fix $i \geq 1$ and let

$$\eta_i = \left(\frac{\log^{4000/\varepsilon^4}(m/k)}{2\mathrm{e}} \right)^{-|W_i|/2}.$$

Then for any fixed u^* and i, we claim that

$$\mathbb{P}[W_i \text{ is not useful} \,|\, u_i = u^*] \leq \eta_i. \tag{38}$$

Indeed, if $u^* \geq y'$, this follows immediately from the definition. If $u^* < y'$, we apply the Chernoff bound (11), with $X = |W_i \cap \bigcup_{j \in \mathcal{U}_i} W_j|$, $p' = u^*/b$,

$N = |W_i|$, and $1+\delta = b/(2u^*)$, noting that the right hand side of (11) is at most $((1+\delta)/\mathrm{e})^{-(1+\delta)Np'}$. Since the event that W_i is useful depends only on u_i, (38) in turn implies that for any event $\mathcal{A}$ depending only on $W_1, \ldots, W_{i-1}$, we have

$$\mathbb{P}[W_i \text{ is not useful} \mid \mathcal{A}] \le \eta_i.$$

Thus for any fixed $\mathcal{S} \subseteq \{1, \ldots, h'\}$ so that the W_i with $i \in \mathcal{S}$ have total weight at least $k/16$, the probability that none of these are useful is at most

$$\prod_{i\in\mathcal{S}} \eta_i \le \left(\frac{\log^{4000/\varepsilon^4}(m/k)}{2\mathrm{e}}\right)^{-k/32} \overset{(37)}{\le} 2^{-k} f(k, r_A)^{-14/\varepsilon^2}.$$

Now note that if the useful W_i have total weight at most $k/16$, then there must exist a subset $\mathcal{S}$ where the W_i with $i \in \mathcal{S}$ have total weight at least $k/16$ and where none of the W_i is useful. Multiplying by $2^{h'} \le 2^k$ (to account for all possible choices of $\mathcal{S}$) yields the result. $\square$

Corollary 24. *Given $h' \le k$ and a sequence of integers w_i, $1 \le i \le h'$, suppose that $\sum_{i=1}^{h'} w_i \ge k/8$. Let $W_1, \ldots, W_{h'}$ with $|W_i| = w_i$ be sets chosen uniformly and independently at random in B. Let*

$$u = \min\left\{k/16\,,\, y'\right\}.$$

Then with probability at least $1 - f(k, r_A)^{-14/\varepsilon^2}$ there exist (after relabelling) disjoint sets $W_1'', \ldots, W_{h''}''$ of total weight at least $u/2$ so that $W_i'' \subseteq W_i$ and $|W_i''| \ge |W_i|/2$ for all i.

Proof. By the previous lemma, with sufficiently high probability the sequence $\mathcal{W} = \{W_1, \ldots, W_{h'}\}$ contains a sequence $\mathcal{U} = \{U_1, \ldots, U_{h^*}\}$ of useful sets of total weight at least $k/16$. If $k/16 \le y'$, set $h'' = h^*$. Otherwise, let h'' be the smallest number satisfying

$$|\bigcup_{j=1}^{h''} U_j| \ge y'.$$

For each $i \le h''$, let $W_i'' = U_i \setminus \bigcup_{j=1}^{i-1} U_j$ and observe that by the definition of "useful", the W_i'' have the desired properties. $\square$

Proof of Lemma 22. First we calculate the probability that a fixed subsequence $\mathcal{W}' = \{W_1, \ldots, W_{h'}\}$ (of total weight at least $k/8$) of the sequence of random sets $V_1, \ldots, V_h$ contains no expanding sets. By the previous corollary, we may condition on the fact that the W_i contain a sequence $W_1'', \ldots, W_{h''}''$ of sets as in the statement of that corollary. Recall that by definition, the expansion of these sets depends only on the $A'B$ graph and not on the V_i. A set W_i'' is not expanding if there is a set R with $a' - (a-y)$ vertices so that there are no edges between W_i'' and R. Thus given W_i'',

$$\mathbb{P}[W_i'' \text{ is not expanding}] \le \binom{a'}{a-y}(1-p_0)^{(a'-(a-y))|W_i''|}.$$

Now note that

$$\binom{a'}{a-y} \le \binom{a}{a-y} = \binom{a}{y} \overset{(4)}{\le} (\mathrm{e}a/y)^y.$$

Hence we have, using $a' \ge a - s \ge a - y/2$,

$$\begin{aligned}
\mathbb{P}[W_i'' \text{ is not expanding}] &\le \exp\left\{y\big(\log(\mathrm{e}a/y) - p_0|W_i''|/2\big)\right\} \\
&\overset{(36)}{\le} \exp\left\{y\big((400/\varepsilon^4)(1+\log\log(m/k)) - p_0|W_i''|/2\big)\right\} \\
&\le \exp\{-yp_0|W_i''|/4\}, \qquad (39)
\end{aligned}$$

where we used in the last line that $|W_i''| \ge D_1'/2$ and $m \ge t_3$. The disjointness of the W_i'' then implies that

$$\begin{aligned}
\mathbb{P}[\text{none of the } W_i'' \text{ expand}] &\le \mathrm{e}^{-uyp_0/2^3} \le \mathrm{e}^{-uym/(2^5 a' b)} \\
&\overset{(36)}{\le} \mathrm{e}^{-\frac{um}{2^5 b \log^{400/\varepsilon^4}(m/k)}} \\
&\le \max\left\{\mathrm{e}^{-\frac{km}{2^9 b \log^{400/\varepsilon^4}(m/k)}}, \mathrm{e}^{-\frac{m}{2^5 \log^{4400/\varepsilon^4}(m/k)}}\right\} \\
&\overset{(37)}{\le} f(k, r_A)^{-14/\varepsilon^2}.
\end{aligned}$$

In the last line we used the fact that $k/m \le \delta_0$ and $\varepsilon \le 10^{-6}$. Thus in particular

$$\mathbb{P}\left[\text{none of the } W_i \text{ expand}\right] \le f(k, r_A)^{-14/\varepsilon^2}.$$

As in the proof of Lemma 23, the result now follows if we sum over all possibilities of choosing $\mathcal{W}'$, thus multiplying the above probability by $2^h \le 2^k$. □

Now we apply Lemma 22 to show that if G_A is "friendly" and if we let the V_i in Lemma 22 correspond to the neighbourhoods $\Gamma_B(x)$ of the vertices $x \in A$ whose degree in G_A is large, then $\Gamma_A(\Gamma_B(x))$ is likely to be very large. The proof of Lemma 19 is then based on the fact that this severely limits the number of choices for $\Gamma_A(x)$ if we do not want to create a triangle. For convenience let $k = k_1$ and $k' = k_{1,r_A-1}$ again. We say that a (k_{1,r_A}, A)-graph G_A is *friendly* if with probability at least $f(k, r_A)^{-6/\varepsilon^2}$ it is dominated by an AB-graph with $m_1 = m - k - k_2$ edges chosen uniformly at random. For a given set $S' \subset A$, we let $A' = A \backslash S'$. We say that a vertex $x \in S'$ is *S'-expanding* if

$$|\Gamma_{A'}(\Gamma_B(x))| \ge a - y.$$

We say that a set of vertices *S'-expands well* if each of its vertices S'-expands.

Lemma 25. *Fix a friendly (k_{1,r_A}, A)-graph G_A and choose an AB-graph uniformly at random from the set of AB-graphs with m_1 edges dominating G_A. Let S be the lexicographically first spine of G_A and let S' be the subset of S which contains all vertices whose degree in G_A is at least D_1'. With probability at least $1 - f(k, r_A)^{-11/\varepsilon^2}$, S' contains an S'-well expanding set S'' with*

$$\sum_{x \in S''} d_A(x) \ge k/8. \qquad (40)$$

Proof. By Proposition 11, the vertices of S are adjacent to at least $k - k' \geq k/2$ edges of G_A. Recall also that $|S| \leq s = 2k'/D_1$. Thus the average degree in G_A of the vertices in S is at least

$$\frac{k}{2s} \geq \frac{D_1}{5(1-\varepsilon)^{r_A}} \stackrel{(14)}{=} \frac{\varepsilon^6 m}{20n \log\log(m/k)} = 2D_1'. \tag{41}$$

By (41), the vertices in $S \backslash S'$ can be adjacent to at most $|S \backslash S'| D_1' \leq s D_1' \leq k/4$ edges in G_A, and thus the vertices in S' are still adjacent to at least $k/4$ edges in G_A. For any AB-graph with m_1 edges, let

$$s_B = \sum_{x \in S'} d_B(x)$$

and let $s' = |S'|$. Note that by definition, we have $s_B \leq s'b$. Then the proportion of AB-graphs with $s_B \geq m_1/2$ is at most

$$\begin{aligned}
\sum_{s_B \geq m_1/2} \binom{s'b}{s_B} \binom{(a-s')b}{m_1 - s_B} \Big/ \binom{ab}{m_1} &\leq \sum_{s_B \geq m_1/2} \binom{m_1}{s_B} \frac{(s'b)^{s_B} (ab)^{m_1 - s_B}}{(ab/2)^{m_1}} \\
&\leq \sum_{s_B \geq m_1/2} 4^{m_1} (s'/a)^{s_B} \leq m_1 4^{m_1} (s/a)^{m_1/2} \\
&\stackrel{(18)}{\leq} 4^{-m_1} \leq 2^{-m} \stackrel{(37)}{\leq} f(k, r_A)^{-18/\varepsilon^2}.
\end{aligned}$$

Since for any two events $\mathcal{D}$ and $\mathcal{E}$, one has $\mathbb{P}[\mathcal{D}|\mathcal{E}] \leq \mathbb{P}[\mathcal{D}]/\mathbb{P}[\mathcal{E}]$, by the definition of "friendly", the above implies that the proportion of AB-graphs as in the statement of the lemma with $s_B \geq m_1/2$ is at most $f(k, r_A)^{-12/\varepsilon^2}$. So assume that $s_B \leq m_1/2$. Now consider any fixed sequence of degrees $d_B(x)$ for all $x \in S'$, where we require that $d_B(x) \geq d_A(x)$. Recall $A' = A \backslash S'$. Let $\mathbb{P}_{m_2, S'}$ denote the probability measure on the set of all AB-graphs where the $S'B$-graph is uniformly chosen from all $S'B$-graphs which conform to the above degree sequence and the $A'B$-graph is uniformly chosen from the set of all $A'B$-graphs with m_2 edges (independently of the $S'B$-graph), where

$$m_2 = m_1 - s_B = m - k - k_2 - s_B.$$

Note that $m_2 \geq m/4$. Correspondingly, let $\mathbb{P}_{p_2, S'}$ denote the probability measure on the set of all AB-graphs where we choose the $S'B$-graph as above and where the $A'B$-graph has edge probability p_2, where $p_2 = m_2/(a'b)$. Let S'' denote the set of S'-expanding vertices in S'. Let $\mathcal{C}$ denote the event that for all the vertices in A' the number of neighbours in B is at least the number of neighbours in G_A. Clearly, the lemma will follow once we have shown that

$$\mathbb{P}_{m_2, S'} \Big[\sum_{x \in S''} d_A(x) \leq k/8 \mid \mathcal{C} \Big] \leq f(k, r_A)^{-12/\varepsilon^2}. \tag{42}$$

So it remains to prove (42). For this, note that in $\mathbb{P}_{m_2, S'}$, the $A'B$-graph is chosen uniformly at random with m_2 edges and thus has average degree at least $m_2/n \geq m/(4n) \geq 2D_1'$. As in the proof of the case $r_1 = 1$ in Lemma 13, the probability that such a graph has mimimum degree less than D_1' tends to zero (and hence since $A' = A \backslash S'$ the probability that $\mathcal{C}$ holds tends to one). Thus the left hand side of (42) is at most

$$\mathbb{P}_{m_2,S'}\Big[\sum_{x\in S''} d_A(x) \le k/8\Big] / \mathbb{P}_{m_2,S'}[\mathcal{C}] \le \mathbb{P}_{m_2,S'}\Big[\sum_{x\in S''} d_A(x) \le k/8\Big](1+o(1)).$$

Furthermore, by considering the probability under $\mathbb{P}_{p_2,S'}$ that the $A'B$-graph has exactly m_2 edges, we have

$$\begin{aligned}
&\mathbb{P}_{p_2,S'}\Big[\sum_{x\in S''} d_A(x) \le k/8\Big] &(43)\\
&\ge \binom{a'b}{m_2} p_2^{m_2}(1-p_2)^{a'b-m_2}\mathbb{P}_{m_2,S'}\Big[\sum_{x\in S''} d_A(x) \le k/8\Big]\\
&\ge \frac{1}{m_2}\mathbb{P}_{m_2,S'}\Big[\sum_{x\in S''} d_A(x) \le k/8\Big],
\end{aligned}$$

where the last line follows from a Stirling bound (see e.g. pages 4 and 35 of [2]). Write $S' = \{x_1, \ldots, x_h\}$. To bound (43) from above, we apply Lemma 22 with $A' = A\backslash S'$, $h = |S'| \le k$ and where for all i, the V_i are subsets of $\Gamma_B(x_i)$ of size exactly $d_A(x_i)$, chosen uniformly at random in $\Gamma_B(x_i)$. Recall that $d_B(x_i) \ge d_A(x_i) \ge D_1'$ and $\sum_{x\in S'} d_B(x) \ge k/4$. Furthermore note that the definition of $\mathbb{P}_{p_2,S'}$ implies that the sets $\Gamma_B(x)$ are independently and uniformly distributed in B. This in turn implies that the V_i are then also uniformly distributed in B. Thus the conditions of Lemma 22 are satisfied and inequality (42) follows using $m_2 f(k,r_A)^{-13/\varepsilon^2} \le f(k,r_A)^{-12/\varepsilon^2}$. □

Given a friendly (k_{1,r_A}, A)-graph G_A, we say that a (k_{2,r_2}, B)-graph G_B is *friendly with respect to* G_A if with probability at least $f(k,r_A)^{-6/\varepsilon^2}$ it is dominated by an AB-graph chosen uniformly at random from the set of AB-graphs with m_1 edges dominating G_A. In this case, we say that G_A and G_B form a *friendly pair* and that any $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph G with $G[A] = G_A$ and $G[B] = G_B$ is *friendly.* Otherwise we call it an *unfriendly pair* and call any corresponding $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph G *unfriendly.*

Corollary 26. *Fix a friendly pair G_A and G_B where G_A is a (k_{1,r_A}, A)-graph and G_B is a (k_{2,r_2}, B)-graph. Choose a friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph G uniformly at random subject to $G[A] = G_A$ and $G[B] = G_B$. Let S be the lexicographically first spine of G_A and let S' be the subset of S which contains all the vertices whose degree in G_A is at least D_1'. With probability at least $1 - f(k,r_A)^{-5/\varepsilon^2}$, S' contains an S'-well expanding set S'' with*

$$\sum_{x\in S''} d_A(x) \ge k/8. \tag{44}$$

Proof. Consider choosing an AB-graph uniformly at random from the set of AB graphs with m_1 edges dominating G_A. Let $\mathcal{D}$ be the event that it also dominates G_B. Let $\mathcal{E}$ be the event that S' does not contain a set S'' satisfying (44). It suffices to show that $\mathbb{P}[\mathcal{E} \mid \mathcal{D}] \leq f(k, r_A)^{-5/\varepsilon^2}$. But $\mathbb{P}[\mathcal{E} \mid \mathcal{D}] \leq \mathbb{P}[\mathcal{E}]/\mathbb{P}[\mathcal{D}]$ and $\mathbb{P}[\mathcal{D}] \geq f(k, r_A)^{-6/\varepsilon^2}$ as the pair (G_A, G_B) is friendly, while $\mathbb{P}[\mathcal{E}] \leq f(k, r_A)^{-11/\varepsilon^2}$ by Lemma 25. $\square$

The reason why we introduced the notion of a dominating AB-graph in the definition of a $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph was that it makes it easier to prove a result like Corollary 26 – if for instance the AB-graph is chosen uniformly at random from the set of AB-graphs with m_1 edges and if S is a fixed set of vertices in A with $|S| \leq \varepsilon^3 s$, then one may check that even the probability that S is isolated is larger than $f(k, r_A)^{-5/\varepsilon^2}$.

Proof of Lemma 19. As usual, we write k instead of k_1 and k' for k_{1,r_A-1}. We first consider the $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graphs which are unfriendly. Denoting the sum over all unfriendly pairs by $\sum^{\mathrm{unf}}_{G_A,G_B}$, the number of these is

$$\sum^{\mathrm{unf}}_{G_A,G_B} |G_{AB} \colon G_{AB} \text{ is an } AB\text{-graph with } m_1 \text{ edges dominating } G_A \text{ and } G_B| \,.$$

We crudely bound the number of summands by the product of the total number of (k_{1,r_A}, A)-graphs and (k_{2,r_2}, B)-graphs. By definition, each summand is at most the number of AB-graphs multiplied by $f(k, r_A)^{-6/\varepsilon^2}$. Indeed, for each summand, either G_A is unfriendly, in which case we apply the definition preceding Lemma 25. Otherwise G_B is not friendly with respect to G_A and the same bound follows from the definition preceding Corollary 26. Comparing the resulting upper bound with the lower bound on the total number of $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graphs in Corollary 14, one readily sees that the probability that a $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph (chosen uniformly at random) is unfriendly is at most

$$(1 + o(1)) f(k, r_A)^{-6/\varepsilon^2 + 2\varepsilon^2} f(k_2, r_2)^{2\varepsilon^2} \leq f(k, r_A)^{-6/\varepsilon^2 + 5},$$

where the final inequality comes from our assumption made in the statement of the lemma.

Thus it suffices to show that the probability that a friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph is triangle-free is at most $2f(k, r_A)^{-5/\varepsilon^2}$. For a $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph G, let S be the lexicographically first spine in $G[A]$ and let S' be the subset of S containing all those vertices whose degree in G_A is at least D'_1. We say that the AB-graph in G is *S-good* if S' contains an S'-well expanding subset S'' satisfying (44). By Corollary 26, the probability that in a friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph, the AB-graph is not S-good is at most $f(k, r_A)^{-5/\varepsilon^2}$.

Now consider the case that in a friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph the AB-graph is S-good. Consider first fixed sets S, $S' \subseteq S$ and $S'' \subseteq S'$ in A with $|S| \leq s$ and $|S''| = s''$. Let $\mathcal{G}_A$ be the set of all (k_{1,r_A}, A)-graphs G_A whose lexicographically first spine is S, where S' is the subset of S containing the vertices whose degree in G_A is at least D'_1 and where S'' is an S'-well expanding set which satisfies (44). Further consider a fixed AB-graph which is S-good (with the same S'' as above) for at least one (k_{1,r_A}, A)-graph in $\mathcal{G}_A$. We now bound the number of (k_{1,r_A}, A)-graphs in $\mathcal{G}_A$ which could form part of a triangle-free $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph together with the fixed AB-graph. Write $S'' = \{x_1, \dots, x_{s''}\}$. But then, for a vertex $x_i \in S''$, there are at most y possibilities for neighbours of x_i in A which do not produce a triangle together with the AB-graph. For $1 \leq i \leq s''$, let $d_A(x_i)$ be a sequence of nonnegative integers satisfying $d_A(x_i) \leq y$ and adding up to exactly d_A, where $d_A \geq k/8$. Let $\sum_{\mathbf{d_A}}$ denote the sum over all such sequences. By Proposition 11, at least $k - k'$ edges of the (k_{1,r_A}, A)-graph G_A must be incident to S. Thus the number of (k_{1,r_A}, A)-graphs in $\mathcal{G}_A$ not producing a triangle together with the fixed AB-graph is at most

$$
\begin{aligned}
U &:= \sum_{d_A = k/8}^{s''y} \sum_{\mathbf{d_A}} \left(\prod_{i=1}^{s''} \binom{y}{d_A(x_i)} \right) \binom{sa}{k - k' - d_A} \binom{\binom{a}{2}}{k'} \\
&\leq \sum_{d_A = k/8}^{s''y} \binom{s''y}{d_A} \binom{sa}{k - k' - d_A} \binom{\binom{a}{2}}{k'} \\
&\leq a^2 \binom{sy}{k/8} \binom{sa}{7k/8 - k'} \binom{\binom{a}{2}}{k'}. \qquad (45)
\end{aligned}
$$

In the second line we used ($s'' - 1$ times) the fact that $\sum_{j=0}^{d} \binom{t}{j}\binom{v}{d-j} = \binom{t+v}{d}$. In the third line we used $s'' \leq s$ and that the largest summand is the one with $d_A = k/8$. Clearly, the number of choices for S'', S' and S is at most

$$
\sum_{s''=1}^{s} \sum_{|S'|=s''}^{s} \sum_{|S|=|S'|}^{s} \binom{a}{|S|} \binom{|S|}{|S'|} \binom{|S'|}{s''} \leq s^3 a^s \cdot 2^s \cdot 2^s \overset{(18)}{\leq} a^{2s}.
$$

Now let $\#\{k_2, r_2\}$ denote the total number of (k_{2,r_2}, B)-graphs. Then the above implies that the total number of triangle-free friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graphs where the AB-graph is S-good for some S is at most

$$
a^{2s}\, U\, \#\{k_2, r_2\} \binom{ab}{m_1}. \qquad (46)
$$

Similarly as in the "unfriendly" case, we compare this with a lower bound on the total number of friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graphs where the AB-graph is S-good. To this end, note that all (k_{1,r_A}, A)-graphs with maximum degree at most m/n are friendly. Indeed, this follows since (as already noted in the proof of Corollary 14) almost all AB-graphs with m_1 edges have minimum degree at least m/n. This also implies that any (k_{2,r_2}, B)-graph with maximum degree at most m/n forms a friendly pair together with any (k_{1,r_A}, A)-graph with maximum degree at most m/n. Furthermore, note that by Corollary 26 certainly almost all AB-graphs with m_1 edges dominating a friendly pair are S-good. Also by Lemma 13, the number of (k_{2,r_2}, B)-graphs with maximum degree at most m/n is at least $\#\{k_2, r_2\} f(k_2, r_2)^{-2\varepsilon^2}$. Together with Proposition 15, this shows that the number of friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graphs where the AB-graph is S-good is at least

$$(1+o(1))\left((sa/8)^k/k!\right)\#\{k_2,r_2\}f(k_2,r_2)^{-2\varepsilon^2}\binom{ab}{m_1}.$$

Thus together with (46), this shows that the probability that a random friendly $(k_{1,r_A}, k_{2,r_2}, A, B)$-bipartite graph where the AB-graph is S-good is triangle-free is at most

$$\begin{aligned}
& (1+o(1))a^{2s}Uk!\,(sa/8)^{-k}\,f(k_2,r_2)^{2\varepsilon^2} \\
\stackrel{(45)}{\leq}\; & a^{2s+2}\,(sy)^{k/8}(sa)^{7k/8-k'}a^{2k'}\left(\frac{sa}{8}\right)^{-k}\frac{k!}{(k/8)!k'!(7k/8-k')!}f(k_2,r_2)^{2\varepsilon^2} \\
\leq\; & a^{2s+2}\,(a/s)^{k'}\,8^k\,(y/a)^{k/8}\,3^k\,f(k_2,r_2)^{2\varepsilon^2}. \qquad (47)
\end{aligned}$$

In the last line we used that $k!/(x!y!(k-x-y)!) \leq 3^k$. Now use that $s\log a \leq k$ (see (18)), that by (15), $a/s \leq m/k$, to see that (47) is at most

$$\begin{aligned}
& (m/k)^{k'}(24\mathrm{e}^4)^k(y/a)^{k/8}f(k_2,r_2)^{2\varepsilon^2} \\
\stackrel{(14)}{\leq}\; & \exp\left\{k\left(\frac{4}{\varepsilon^2}\log\log(m/k)+\log(24\mathrm{e}^4)+\frac{1}{8}\log(y/a)\right)\right\}f(k_2,r_2)^{2\varepsilon^2} \\
\stackrel{(36)}{\leq}\; & \exp\left\{-\frac{49}{\varepsilon^4}k\log\log(m/k)\right\}f(k_2,r_2)^{2\varepsilon^2} \\
\stackrel{(37)}{\leq}\; & f(k,r_A)^{-6/\varepsilon^2}f(k_2,r_2)^{2\varepsilon^2} \\
\leq\; & f(k,r_A)^{-5/\varepsilon^2},
\end{aligned}$$

where in the last line we used the assumption made in the statement of the lemma. □

8 Odd cycles

The proof of Theorem 5 is quite similar to the one for triangles. However, some of the details are more complicated, which is the reason for presenting a sketch of the necessary modifications to the proof of the triangle case separately.

Again we can restrict our attention to the case where $m = o(n^2)$: Lamken and Rothschild [11] proved that for an odd integer ℓ, almost all C_ℓ-free graphs are bipartite. In Osthus [13], it is then shown how the techniques used in [16] and [11] can be adapted to prove Theorem 5 when $m/n^2 \not\to 0$.

First consider the proof of the 0-statement. We assume without loss of generality that $\varepsilon \leq 10^{-6}/(\ell^2 2^\ell)$. For $n/2 \leq m \leq c_1 n$, where c_1 is fixed, the probability that a random graph $G_{n,m}$ is C_ℓ-free is bounded away from zero, so for these m, we may apply the same reasoning as for triangles.

If $c_1 n \leq m \leq 20n\log n$, the result follows immediately by noting that the number of bipartite graphs on n vertices and m edges is at most $2^n\binom{\lfloor n^2/4\rfloor}{m}$, whereas the following special case of a more general result of Prömel and Steger (see [17]) shows that the number of C_ℓ-free graphs on n vertices and m edges is much larger. We omit the calculations, since they are very similar to the ones for the triangle case in [16]. Let X_ℓ denote the number of ℓ-cycles in $G_{n,m}$. (Note that $\mathbb{E}[X_\ell] = \Theta(m^\ell/n^\ell)$).

Theorem 27. [17] *If $m = o(n^{1+1/(\ell-1)})$, then there exists a positive constant c so that*

$$\mathbb{P}[G_{n,m} \text{ is } C_\ell\text{-free}] \geq \mathrm{e}^{-c\mathbb{E}[X_\ell]}.$$

So assume that $20n \log n \leq m \leq (1-\varepsilon)t_\ell$. As in the case of triangles, we fix an almost equitable bipartition into classes A and B, fix an edge e in A and consider a random AB-graph with edge probability $p = (1+\varepsilon^2)4m/n^2$. Note that $a = (1+o(1))b = (1+o(1))n/2$. We need a lower bound on the probability that the union of such an AB-graph and e contains no ℓ-cycle. But the existence of such cycles is positively correlated, and thus the FKG-inequality (see e.g. Theorem 2.12 in [7]) implies that this probability is at least

$$\left(1 - p^{\ell-1}\right)^{(1+o(1))(n/2)^{\ell-2}} \overset{(5)}{\geq} \mathrm{e}^{-(1+\varepsilon^2)p^{\ell-1}(n/2)^{\ell-2}} := \nu_\ell.$$

By the definition of t_ℓ, we have $m\nu_\ell \geq (1-\varepsilon)t_\ell \mathrm{e}^{-(1-\varepsilon)(t_\ell)^{\ell-1}(2/n)^\ell} \to \infty$, and the rest of the proof goes through as before.

To prove the second 1-statement, we will make use of Theorem 28 below. Łuczak [12] proved that it follows from the odd-cycle case of a conjecture of Kohayakawa, Łuczak and Rödl on the probability of the nonexistence of a fixed subgraph in a certain random graph (Conjecture 8.35 in [7] or Conjecture A in [12]). Based on ideas in [10], Kohayakawa and Kreuter [9] proved a slightly weaker form of the conjecture for the case of cycles of arbitrary length (which was sufficient for their purposes). By using the notion of an $(\ell-2)$-expanding set which is used later on in this section, one can extend their proof to give the full conjecture for cycles of arbitrary length (see Behrisch [1]), which fills the gap in the proof of Theorem 28.

Theorem 28. *Given an odd integer ℓ and $\delta > 0$, there exists a constant $C > 0$ so that almost all C_ℓ-free graphs with n vertices and $Cn^{\ell/(\ell-1)} \leq m \leq n^2/C$ edges can be made bipartite by deleting at most δm edges.*

The proof of the second 1-statement is then the same as for triangles, except that of course we have to prove Lemmas 17, 18 and 19 for odd ℓ-cycles. Lemmas 17 and 18 are derived from Lemmas 21 and 20 as before. We now show how to modify the proof of Lemma 20 so that it applies to ℓ-cycles instead of triangles. $\mathcal{S}$ is now the set of those paths on $\ell-1$ edges (where the edges all have one endpoint in A and one in B) which form an ℓ-cycle together with some edge in T_A or T_B. Thus X now counts the number of ℓ-cycles in $G_{G_A,G_B,p}$ with one edge contained in some torso and the others in the AB-graph. Then, similarly to (31), we have

$$\mu \geq (k_{1,r_1} + k_{2,r_2})((1-\varepsilon^2)n/2)^{\ell-2}p^{\ell-1}.$$

One difference is that Δ now counts the expected number of pairs of ℓ-cycles with at least an edge in common, where the edges in the torso are not necessarily adjacent. Thus

$$\Delta = \sum_{e,e'} \sum_{i=1}^{\ell-2} \mathbb{E}[X_{e,e',i}].$$

Here $\sum_{e,e'}$ denotes the sum over all pairs of edges $\{e, e'\}$ in $T_A \cup T_B$ and $X_{e,e',i}$ denotes the number of ordered pairs of paths α, $\alpha' \in \mathcal{S}$ in $G_{G_A,G_B,p}$ so that $\alpha \cup e$ and $\alpha' \cup e'$ are ℓ-cycles and so that α and α' have exactly i common edges. Now note that if α and α' have i common edges, they must have at least $i+1$ common vertices. Thus one can show that the fact that $pn \to \infty$ implies that the contribution from those terms with $i > 1$ is $o(1)$ of those with $i = 1$. Distinguishing between the cases where e and e' have an endvertex in common or not as in the derivation of (33), one may now check that (assuming without loss of generality that $k_1 \geq k_2$ as before and also with the same summation as before)

$$\begin{aligned} \Delta &\leq (1+\varepsilon)p^{2(\ell-1)-1}\left(\sum_{xy,yz} (n/2)^{2(\ell-2)-1} + \sum_{xy,x'y'} 8\ell (n/2)^{2(\ell-2)-2} \right) \\ &\leq 25\ell\mu D_1 n^{\ell-3} p^{\ell-2}. \end{aligned}$$

Using this, it is easily checked that $\frac{\mu^2}{2\Delta} \geq \frac{4}{\varepsilon^2}\log f(k_1, r_1)$ as before. In the final case analysis, we now distinguish whether $k_1/m \leq n^{-\varepsilon^2\ell/(8(\ell-1))}$ and whether $(m/t_\ell)^{\ell-1} \leq 9$. The analogue of (34) is then that $\Delta/\mu \leq 2000\ell 2^\ell \varepsilon^2$. One can compensate this by showing that (using $\varepsilon \leq 10^{-6}/(\ell^2 2^\ell)$) in (35) the term $2000\varepsilon^2$ can be replaced by $4000\ell 2^\ell \varepsilon^2$.

The proof of Lemma 19 is also similar, except that instead of Lemma 22 we need a lemma which is concerned with $(\ell-2)$-*expanding sets* instead of expanding sets. As in Lemma 22, in what follows, we are given a vertex bipartition into A and B and a subset $A' \subset A$ with $|A'| \geq a - s$. We then consider a random (bipartite) $A'B$-graph with edge probability $p_0 = m_0/(a'b)$, where $m_0 \geq m/4$. We say that a set $V \subset B$ is $(\ell-2)$-*expanding* if its $(\ell-2)$th neighbourhood

$$\Gamma_{A'}(\Gamma_B \Gamma_{A'})^{(\ell-3)/2}(V)$$

contains at least $a - y$ vertices. Thus we are done if we can prove

Lemma 29. *Consider the setting above. Given $h \leq k$, for $1 \leq i \leq h$ let v_i satisfy $v_i \geq D'_1$. Furthermore, suppose that $\sum_{i=1}^h v_i \geq k/4$. Let $V_1, \dots, V_h$ with $|V_i| = v_i$ be chosen uniformly and independently at random in B. Then*

$$\mathbb{P}[\textit{the } (\ell-2)\textit{-expanding } V_i \textit{ have total weight at most } k/8] \leq f(k, r_A)^{-13/\varepsilon^2}.$$

To prove this, we need an auxiliary lemma. For $1 \leq d \leq \ell - 1$, let

$$w_d = \left(\frac{\varepsilon^6 t_\ell}{80n \log\log(m/k)} \right)^d.$$

Let

$$u' = \min\left\{k/32, y'/2\right\}.$$

Lemma 30. *Let d with $2 \leq d \leq \ell - 2$ be an integer. Let $W_{1,d-1}, \dots W_{h',d-1}$ be disjoint sets, all in the same vertex class, satisfying $|W_{i,d-1}| \geq w_{d-1}$ and $\sum_{i=1}^{h'} |W_{i,d-1}| \geq u'$. Consider an $A'B$-graph with edge probability $p_\ell \geq p_0/\ell$. Then with probability at least $1 - f(k, r_A)^{-14/\varepsilon^2}$, there exist (after relabelling) disjoint $W_{1,d}, \dots, W_{h'',d}$, satisfying $|W_{i,d}| \geq w_d$, $\sum_{i=1}^{h''} |W_{i,d}| \geq u'$, and $W_{i,d} \subseteq \Gamma(W_{i,d-1})$.*

The idea of the proof is similar to that of Lemma 23 and Corollary 24. The details are more complicated, but on the other hand, there is more room to spare in the calculations.

Proof of Lemma 30. Since $a' = |A'|$ and $b = |B|$ are approximately equal, without loss of generality we may assume that the $W_{i,d-1}$ are contained in B. Let $W'_{i,d} = \Gamma(W_{i,d-1})$. Inductively, we now define a notion called *d-usefulness*. For all i, let $\mathcal{U}_{i,d}$ be the set of those j with $j < i$ for which the $W'_{j,d}$ are d-useful and let

$$u_i = |\bigcup_{j \in \mathcal{U}_{i,d}} W'_{j,d}|. \tag{48}$$

Then we say that $W'_{i,d}$ is *d-useful* if at least one of the following two possibilities hold.

- $u_i \geq a'/2$.
- $|W'_{i,d} \setminus \bigcup_{j \in \mathcal{U}_{i,d}} W'_{j,d}| \geq \mathbb{E}[|W'_{i,d}|]/2^5$.

Now let $W' = \bigcup_{j \in \mathcal{U}_{i,d}} W'_{j,d}$. Then the probability that $W'_{i,d}$ is d-useful is equal to one if $|W'| \geq a'/2$. So suppose that $|W'| < a'/2$. Suppose also that $p_\ell |W_{i,d-1}| \leq 4$. Then for a vertex $x \in A' \setminus W'$, we have

$$\begin{aligned} p' := \mathbb{P}[x \notin W'_{i,d}] &= (1 - p_\ell)^{|W_{i,d-1}|} \leq (1 - p_\ell)^{|W_{i,d-1}|/4} \\ &\leq 1 - \frac{p_\ell |W_{i,d-1}|}{4} \left(1 - \frac{p_\ell |W_{i,d-1}|}{8}\right) \leq 1 - \frac{p_\ell |W_{i,d-1}|}{8}. \end{aligned}$$

Here we used that for $\tau j \leq 1$, we have $(1-\tau)^j \leq 1 - \tau j + (\tau j)^2/2$. Furthermore, for any x, $y \in A'$ with $x \neq y$, the events that $x \in W'_{i,d}$ and $y \in W'_{i,d}$ are independent. Thus $|W'_{i,d} \setminus W'|$ is binomially distributed with mean

$$\mathbb{E}[|W'_{i,d} \setminus W'|] = |A' \setminus W'|(1 - p') \geq a' p_\ell |W_{i,d-1}|/2^4. \tag{49}$$

On the other hand, $\mathbb{E}[|W'_{i,d}|] = a'(1 - p') \leq a' p_\ell |W_{i,d-1}|$. (The second inequality follows from the fact that the probability $1 - p'$ that $x \in W'_{i,d}$ is at most the expected number of neighbours of x in $W_{i,d-1}$.) Thus by applying the Chernoff bound (10) with $\delta = 1/2$, we have

$$\mathbb{P}[W'_{i,d} \text{ is not } d\text{-useful} \mid |W'| < a'/2] \leq \exp\left\{-a' p_\ell |W_{i,d-1}|/2^7\right\}.$$

Now consider the case $p_\ell |W_{i,d-1}| \geq 4$. If $W'_{i,d}$ is not useful, the trivial inequality $\mathbb{E}[|W'_{i,d}|] \leq a'$ implies that there must be a set V of at least $a'/4$ vertices in A'

which are not adjacent to any vertices in $W_{i,d-1}$. Since there are at most $2^{a'}$ choices for V, in this case we have

$$\begin{aligned}\mathbb{P}[W'_{i,d} \text{ is not } d\text{-useful} \mid |W'| < a'/2] &\le 2^{a'}(1-p_\ell)^{a'|W_{i,d-1}|/4}\\ &\le (\mathrm{e}/2)^{-p_\ell a'|W_{i,d-1}|/4}\\ &\le \mathrm{e}^{-p_\ell a'|W_{i,d-1}|/16}.\end{aligned}$$

Thus for any fixed $\mathcal{W}_{d-1} \subseteq \{1,\dots,h'\}$ where the $W_{i,d-1}$ with $i \in \mathcal{W}_{d-1}$ have total weight at least $u'/2$, the probability that none of the corresponding neighbouring sets $W'_{i,d}$ are d-useful is at most

$$\begin{aligned}\exp\Big\{-\sum_{i\in\mathcal{W}_{d-1}} a'p_\ell|W_{i,d-1}|/2^7\Big\} &\le \exp\left\{-a'p_0u'/(2^8\ell)\right\}\\ &\le 2^{-k}f(k,r_A)^{-14/\varepsilon^2}, \qquad (50)\end{aligned}$$

where we used that $\varepsilon \le 10^{-6}/(\ell^2 2^\ell)$ (with room to spare) to get rid of the $1/\ell$ factor in the exponent. The details of the intermediate calculations are similar to those of the proof of Lemma 22.

Now note that if those $W_{i,d-1}$ whose neighbouring sets $W'_{i,d}$ are useful have total weight at most $u'/2$, then there must exist a subsequence $\mathcal{W}_{d-1} \subseteq \{1,\dots,h'\}$ where the $W_{i,d-1}$ with $i \in \mathcal{W}_{d-1}$ have total weight at least $u'/2$ and where none of $W'_{i,d}$ which correspond to some $W_{i,d-1}$ with $i \in \mathcal{W}_{d-1}$ is useful. Multiplying by $2^{h'} \le 2^k$, (50) implies that with probability at least $1 - f(k,r_A)^{-14/\varepsilon^2}$, no such subsequence $\mathcal{W}_{d-1}$ exists, and so the $W_{i,d-1}$ whose neighbouring sets $W'_{i,d}$ are useful have total weight at least $u'/2$.

Let $\mathcal{W}'_d$ be the set of these useful $W'_{i,d}$. Without loss of generality, assume that the sets are numbered so that $\mathcal{W}'_d = \{W'_{1,d},\dots,W'_{h''',d}\}$ for some $h''' \le h'$. Let $\mathcal{W}''_d = \{W'_{1,d},\dots,W'_{h^*,d}\}$, where $h^* = h'''$ if $u_{h'''} \le a'/2$ (where u_i is defined as in (48)), and otherwise h^* is the smallest number satisfying

$$|\bigcup_{i=1}^{h^*} W'_{i,d}| \ge a'/2.$$

Suppose that $\mathcal{W}''_d$ contains a set $W'_{j,d}$ so that $p_\ell|W_{j,d-1}| \ge 4$ for this j. Then

$$\mathbb{E}[|W'_{j,d}|] = a'(1-p_\ell)^{|W_{j,d-1}|} \ge a'(1-\mathrm{e}^{-p_\ell|W_{j,d-1}|}) \ge a'(1-\mathrm{e}^{-4}) \ge 2^5y'.$$

So setting $h'' = 1$ and $W_{1,d} = W'_{j,d}$ satisfies the requirements of the lemma. So we may assume that $p_\ell|W_{i,d-1}| \le 4$ for all $i \le h^*$. We claim that in this case, the sets in $\mathcal{W}''_d$ have total weight at least $2^5u'$ and satisfy $|W'_{i,d}| \ge 2^5 w_d$ for all $i \le h^*$. Indeed, inequality (49) implies that

$$\mathbb{E}[|W'_{i,d}|] \ge \mathbb{E}[|W'_{i,d}\backslash W'|] \ge a'p_\ell w_{d-1}/2^4 \ge 2^6 w_d.$$

Similarly, $\mathbb{E}[|W'_{i,d}|] \ge 2^7|W_{i,d-1}|$, and so the claim follows from the fact that those $W_{i,d-1}$ whose neighbouring sets $W'_{i,d}$ are useful have total weight at least

$u'/2$ and from the definition of d-usefulness. Now we choose disjoint $W_{i,d}$ from the $W'_{i,d}$ in $\mathcal{W}''_d$ satisfying the assertion of the lemma as in the proof of Corollary 24. $\square$

Proof of Lemma 29. This is similar to the proof of Lemma 22. Again, we first calculate the probability that a fixed subsequence $\mathcal{W} = \{W_1, \dots W_{h'}\}$ of total weight at least $k/8$ of the sequence of sets $V_1, \dots, V_h$ contains no $(\ell-2)$-expanding sets. Since for all i, $|W_i|/2 \geq D'_1/2 \geq w_1$, an application of Corollary 24 tells us that with probability at least $1-f(k,r_A)^{-14/\varepsilon^2}$, we can find a sequence of sets satisfying the conditions of Lemma 30 with $d=2$. Now define p_ℓ by $(1-p_\ell)^\ell = 1-p_0$ and note that an $A'B$-graph with edge probability p_0 can be considered as the union of ℓ independent $A'B$-graphs with edge probability p_ℓ. Note also that $p_\ell \geq p_0/\ell$. Now apply Lemma 30 with $d=2$ to the sets obtained from Corollary 24 and the first of the ℓ random $A'B$-graphs with edge probability p_ℓ. Then with probability at least $1-f(k,r_A)^{-14/\varepsilon^2}$, we obtain sets $W_{i,2}$ which satisfy the conditions of Lemma 30 for $d=3$. Now continue this process, until after $\ell-3$ applications of Lemma 30, with probability at least $1-(\ell-2)f(k,r_A)^{-14/\varepsilon^2}$, we have a sequence of disjoint sets $W_{1,\ell-2}, \dots, W_{h'',\ell-2}$ as in the assertion of Lemma 30 for $d=\ell-2$. The remainder of the proof is now the same as that of Lemma 22, except that we consider a random $A'B$-graph with edge probability p_ℓ again. For all i with $1 \leq i \leq h''$ we have (using $|W_{i,\ell-2}| \geq w_{\ell-2}$)

$$\begin{aligned}\mathbb{P}[\,|\Gamma_{A'}(W_{i,\ell-2})| \leq a-y\,] &\leq \binom{a'}{a-y}(1-p_\ell)^{(a'-(a-y))|W_{i,\ell-2}|} \\ &\leq \exp\{-yp_\ell|W_{i,\ell-2}|/4\}.\end{aligned}$$

The disjointness of the $W_{i,\ell-2}$ then implies that the probability that none of the $W_{i,\ell-2}$ with $1 \leq i \leq h''$ expand is at most $\mathrm{e}^{-yp_\ell u'/4} \leq f(k,r_A)^{-14/\varepsilon^2}$. Putting the above together, the probability that in the union of the above graphs (and thus in the random $A'B$-graph with edge probability p_0) none of the W_i is $(\ell-2)$-expanding is at most $\ell f(k,r_A)^{-14/\varepsilon^2}$. Now note that if the $(\ell-2)$-expanding V_i have total weight at most $k/8$, then there must exist a subsequence $\mathcal{W} = \{W_1, \dots, W_{h'}\}$ of the V_i, where the W_i have total weight at least $k/8$ and where each W_i is not $(\ell-2)$-expanding. The result now follows if we sum over all possibilities of choosing $\mathcal{W}$, thus multiplying the above probability by $2^h \leq 2^k$. $\square$

9 Almost bipartite triangle-free graphs

Spencer (personal communication) asked whether for $m \gg n^{3/2}$, the bounds in Theorem 2 could be improved. Indeed, comparing Theorem 2 and our Theorem 3, it seems natural that as m increases from $n^{3/2}$ to t_3, a random triangle-free graph with m edges should look more and more like a bipartite graph in the sense that fewer and fewer edges are needed to be deleted in order to make it bipartite. The proof of Theorem 3 shows that the following is true.

Theorem 31. *Given $\varepsilon > 0$ and k^* with $0 < k^* = o(m)$, suppose that*

$$m \geq (1+\varepsilon)\frac{1}{\sqrt{8}}n^{3/2}\sqrt{\log(m/k^*)}. \tag{51}$$

Then almost all graphs in $\mathcal{T}_{n,m}$ can be made bipartite by deleting at most k^* edges.

Indeed, the only change in the proof of Theorem 3 that needs to be made is that in the proof of Lemma 20, instead of asking whether $k_1 \geq n^{-3\varepsilon^2/16}m$ (i.e. whether $m/k_1 \leq (n^{3/2})^{\varepsilon^2/8}$), we now ask whether $m/k_1 \leq (m/k^*)^{\varepsilon^2/8}$ and replace all occurences of $\log(n^{3/2})$ by $\log(m/k^*)$. The proof of the lemma then goes through as before. There are only two other places in the proof of Theorem 3 where we used $m \geq t_3$ (and not just that $n(\log n)^2 = o(m)$ or $k \leq \delta_0 m$): in the proof of (39) we used that m^2/n^3 is large compared to $(\log\log(m/k^*))^2$, which remains true. Also, in the derivation of Theorem 3 at the end of Section 5 we used the fact that $n^{3/2} = o(m)$ in order to apply Theorem 2. It is easily seen that this condition is also still satisfied.

By solving (51) for k^*, one obtains the following reformulation.

Corollary 32. *Suppose that $n^{3/2} \ll m \leq t_3$. Then almost all graphs in $\mathcal{T}_{n,m}$ can be made bipartite by deleting at most $m\,\mathrm{e}^{-8(1+o(1))m^2/n^3}$ edges.*

Corollary 32 implies the following upper bound on the number of triangle-free graphs $|\mathcal{T}_{n,m}|$ in terms of the number of bipartite graphs for m in the above range. (With more work, a similar lower bound could also be obtained using the methods of Sections 3 and 4.)

Corollary 33. *Fix $\varepsilon > 0$. Suppose that $n^{3/2} \ll m \leq t_3$ and that $k^* = m\,\mathrm{e}^{-8(1-\varepsilon)m^2/n^3}$. Then*

$$|\mathcal{T}_{n,m}| \leq \binom{m}{k^*}\mathrm{Bip}_{n,m}. \tag{52}$$

Proof. Let $\varepsilon_1 = \varepsilon/2$. By Corollary 32, it suffices to bound the number of the number of k-bipartite graphs with $k \leq k^\diamond$, where $k^\diamond = m\,\mathrm{e}^{-8(1-\varepsilon_1)m^2/n^3}$. Fix a vertex bipartition into A and B with $|a-b| \leq \varepsilon_1^2 n$. As in the proof of the $r_1 = r_2 = 1$ case of Lemma 12 it is easily seen that the ratio of the number of graphs which are at most $k^\diamond$-bipartite with respect to this partition (i.e. which have at most $k^\diamond$ edges inside A and B) to the number of $(0,0,A,B)$-bipartite graphs is

$$\begin{aligned}
\sum_{k=0}^{k^\diamond}\binom{\binom{a}{2}+\binom{b}{2}}{k}\binom{ab}{m-k}\Big/\binom{ab}{m} &\leq \sum_{k=0}^{k^\diamond}\binom{m}{k}\left(\frac{a^2+b^2}{2(ab-m)}\right)^k \\
&\leq k^*\binom{m}{k^*}\frac{(k^*)_{k^*-k^\diamond}}{(m-k^\diamond)_{k^*-k^\diamond}}2^{k^*} \leq \frac{1}{2}\binom{m}{k^*}.
\end{aligned}$$

Here we used that that the condition on m implies that $k^* = o(m)$, $k^\diamond \to \infty$, $k^*/k^\diamond \geq 2$ and that $m = o(ab)$. By Lemma 16, the contribution of the partitions with $|a-b| \geq \varepsilon_1^2 n$ is negligible. An application of Corollary 8 completes the proof. $\square$

Here is a heuristic argument which motivates the function appearing in Corollary 32: the argument leading to (2) suggests that the probability that a k-bipartite graph is triangle-free might be close to $p^* = \mathrm{e}^{-8km^2/n^3}$. The number of triangle-free k-bipartite graphs would then be close to p^* multiplied by the number of k-bipartite graphs. Since the right hand side of (52) gives an upper bound on the number of k-bipartite graphs, one sees that this is much smaller than the number of bipartite graphs when $8km^2/n^3$ is larger than $k \log(m/k)$. Solving for k gives the result.

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References

[1] M. Behrisch, Random graphs with a forbidden short cycle, Diploma thesis, Insitut für Informatik, Humboldt-Universität zu Berlin, in preparation.

[2] B. Bollobás, *Random Graphs*, Academic Press, London, 1985.

[3] P. Erdős, S. Janson, T. Łuczak and J.H. Spencer, A note on triangle-free graphs, in *Random discrete structures*, eds. D. Aldous and R. Pemantle, *IMA Vol.* 76, Springer, New York (1995), 117–120.

[4] P. Erdős, D.J. Kleitman and B.L. Rothschild, Asymptotic enumeration of K_n-free graphs, in *International Colloquium on Combinatorial theory, Atti dei Convegni Lincei 17*, Vol. 2, Rome, 19–27, 1976.

[5] P. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **5** (1960), 17–61.

[6] S. Janson, T. Łuczak and A. Ruciński, An exponential bound for the probability of nonexistence of a specified subgraph in a random graph, in *Random Graphs '87*, eds. M. Karoński, J. Jaworski and A. Ruciński, John Wiley & Sons (1987), 73–87.

[7] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley-Interscience, New York, 2000.

[8] J.H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Struct. Algorithms* **7** (1995), 173–207.

[9] Y. Kohayakawa and B. Kreuter, Threshold functions for asymmetric Ramsey properties involving cycles, *Random Struct. Algorithms* **11** (1997), 245–276.

[10] Y. Kohayakawa, T. Łuczak and V. Rödl, Arithmetic progressions of length three in subsets of a random set, *Acta Arithmetica*, LXXV (1996), 133–163.

[11] E. Lamken and B.L. Rothschild, The numbers of odd-cycle-free graphs, in *Finite and infinite sets*, eds. A. Hajnal, L. Lovász, and V.T. Sós, Colloq. Math. Soc. János Bolyai **37**, North-Holland, Amsterdam (1984), 547–553.

[12] T. Łuczak, On triangle-free random graphs, *Random Struct. Algorithms* **16** (2000), 260–276.

[13] D. Osthus, *On the Evolution of Random Discrete Structures*, Ph.D. Dissertation, Institut für Informatik, Humboldt-Universität zu Berlin, 2000.

[14] D. Osthus, H.J. Prömel and A. Taraz, Almost all graphs with high girth and suitable density have high chromatic number, *Journal of Graph Theory* **37** (2001), 220–226.

[15] H.J. Prömel and A. Steger, Random l-colourable graphs, *Random Struct. Algorithms* **6** (1995), 21–37.

[16] H.J. Prömel and A. Steger, On the asymptotic structure of sparse triangle free graphs, *Journal of Graph Theory* **21** (1996), 137–151.

[17] H.J. Prömel and A. Steger, Counting H-free graphs, *Discr. Math.* **154** (1996), 311–315.

[18] J.H. Spencer, Threshold functions for extension statements, *J. Combin. Theory Ser. A* **53** (1990), 286–305.

[19] A. Steger, unpublished manuscript, Technische Universität München, 1999.

[20] N.C. Wormald, The perturbation method and triangle-free random graphs, *Random Struct. Algorithms* **9** (1996), 253–270.