PROOF OF A CONJECTURE OF THOMASSEN ON HAMILTON CYCLES IN HIGHLY CONNECTED TOURNAMENTS

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ABSTRACT. A conjecture of Thomassen from 1982 states that for every k there is an $f(k)$ so that every strongly $f(k)$-connected tournament contains k edge-disjoint Hamilton cycles. A classical theorem of Camion, that every strongly connected tournament contains a Hamilton cycle, implies that $f(1) = 1$. So far, even the existence of $f(2)$ was open. In this paper, we prove Thomassen's conjecture by showing that $f(k) = O(k^2 \log^2 k)$. This is best possible up to the logarithmic factor. As a tool, we show that every strongly $10^4 k \log k$-connected tournament is k-linked (which improves a previous exponential bound). The proof of the latter is based on a fundamental result of Ajtai, Komlós and Szemerédi on asymptotically optimal sorting networks.

1. INTRODUCTION

1.1. Main result.
A *tournament* is an orientation of a complete graph and a Hamilton cycle in a tournament is a (consistently oriented) cycle which contains all the vertices of the tournament. Hamilton cycles in tournaments have a long and rich history. For instance, one of the most basic results about tournaments is Camion's theorem, which states that every strongly connected tournament has a Hamilton cycle [10]. This is strengthened by Moon's theorem [19], which implies that such a tournament is even pancyclic, i.e. contains cycles of all possible lengths. Many related results have been proved; the monograph by Bang-Jensen and Gutin [5] gives an overview which also includes many recent results.

In 1982, Thomassen [22] made a very natural conjecture on how to guarantee not just one Hamilton cycle, but many edge-disjoint ones: he conjectured that for every k there is an $f(k)$ so that every strongly $f(k)$-connected tournament contains k edge-disjoint Hamilton cycles (see also the recent surveys [4, 16]). This turned out to be surprisingly difficult: not even the existence of $f(2)$ was known so far. Our main result shows that $f(k) = O(k^2 \log^2 k)$.

Theorem 1.1. *There exists $C > 0$ such that for all $k \in \mathbb{N}$ with $k \geq 2$ every strongly $Ck^2 \log^2 k$-connected tournament contains k edge-disjoint Hamilton cycles.*

In Proposition 5.1, we describe an example which shows that $f(k) \geq (k - 1)^2/4$, i.e. our bound on the connectivity is asymptotically close to best possible. Thomassen [22] observed that $f(2) > 2$ and conjectured that $f(2) = 3$. He also observed that one cannot weaken the assumption in Theorem 1.1 by replacing strong connectivity with strong edge-connectivity.

To simplify the presentation, we have made no attempt to optimize the value of the constant C. Our exposition shows that one can take $C := 10^{12}$ for $k \geq 20$. Rather than proving Theorem 1.1 directly, we deduce it as an immediate consequence of two further results, which are both of independent interest: we show that every sufficiently highly connected tournament is highly linked (see Theorem 1.3) and show that every highly linked tournament contains many edge-disjoint Hamilton cycles (see Theorem 1.2).

1.2. **Linkedness in tournaments.** Given sets A, B of size k in a strongly k-connected digraph D, Menger's theorem implies that D contains k vertex-disjoint paths from A to B. In a k-linked digraph, we can even specify the initial and final vertex of each such path (see Section 2 for the precise definition).

Theorem 1.2. *There exists $C' > 0$ such that for all $k \in \mathbb{N}$ with $k \geq 2$ every $C'k^2 \log k$-linked tournament contains k edge-disjoint Hamilton cycles.*

The bound in Theorem 1.2 is asymptotically close to best possible, as we shall discuss below. We will show that $C' := 10^7$ works for all $k \geq 20$. (As mentioned earlier, we have made no attempt to optimise the value of this constant.)

It is not clear from the definition that every (very) highly connected tournament is also highly linked. In fact, for general digraphs this is far from true: Thomassen [24] showed that for all k there are strongly k-connected digraphs which are not even 2-linked. On the other hand, he showed that there is an (exponential) function $g(k)$ so that every strongly $g(k)$-connected tournament is k-linked [23]. The next result shows that we can take $g(k)$ to be almost linear in k. Note that this result together with Proposition 5.1 shows that Theorem 1.2 is asymptotically best possible up to logarithmic terms.

Theorem 1.3. *For all $k \in \mathbb{N}$ with $k \geq 2$ every strongly $10^4 k \log k$-connected tournament is k-linked.*

For small k, the constant 10^4 can easily be improved (see Theorem 4.5). The proof of Theorem 1.3 is based on a fundamental result of Ajtai, Komlós and Szemerédi [1, 2] on the existence of asymptotically optimal sorting networks. Though their result is asymptotically optimal, it is not clear whether this is the case for Theorem 1.3. In fact, for the case of (undirected) graphs, a deep result of Bollobás and Thomason [8] states that every $22k$-connected graph is k-linked (this was improved to $10k$ by Thomas and Wollan [21]). Thus one might believe that a similar relation also holds in the case of tournaments:

Conjecture 1.4. *There exists $C > 0$ such that for all $k \in \mathbb{N}$ every strongly Ck-connected tournament is k-linked.*

Similarly, we believe that the logarithmic terms can also be removed in Theorems 1.1 and 1.2:

Conjecture 1.5.

   (i) *There exists $C' > 0$ such that for all $k \in \mathbb{N}$ every $C'k^2$-linked tournament contains k edge-disjoint Hamilton cycles.*
   (ii) *There exists $C'' > 0$ such that for all $k \in \mathbb{N}$ every strongly $C''k^2$-connected tournament contains k edge-disjoint Hamilton cycles.*

Note that Conjectures 1.4 and 1.5(i) together imply Conjecture 1.5(ii).

1.3. **Algorithmic aspects.** Both Hamiltonicity and linkedness in tournaments have also been studied from an algorithmic perspective. Camion's theorem implies that the Hamilton cycle problem (though NP-complete in general) is solvable in polynomial time for tournaments. Chudnovsky, Scott and Seymour [11] solved a long-standing problem of Bang-Jensen and Thomassen [6] by showing that the linkedness problem is also solvable in polynomial time for tournaments. More precisely, for a given tournament on n vertices, one can determine in time polynomial in n whether it is k-linked and if yes, one can produce a corresponding set of k paths (also in polynomial time). Fortune, Hopcroft and Wyllie [13] showed that for general digraphs, the problem is NP-complete even for $k = 2$. We can use the result in [11] to obtain an algorithmic version of Theorem 1.2. More precisely, given a $C'k^2 \log k$-linked tournament on n vertices, one can find k edge-disjoint Hamilton cycles in time polynomial in n (where k is fixed). We discuss this in more detail in Section 9. Note that this immediately results in an algorithmic version of Theorem 1.1.

1.4. **Related results and spanning regular subgraphs.** Proposition 5.1 actually suggests that the 'bottleneck' to finding k edge-disjoint Hamilton cycles is the existence of a k-regular subdigraph: it states that if the connectivity of a tournament T is significantly lower than in Theorem 1.1, then T may not even contain a spanning k-regular subdigraph. There are other results which exhibit this phenomenon: if T is itself regular, then Kelly's conjecture from 1968 states that T itself has a Hamilton decomposition. Kelly's conjecture was proved very recently (for large tournaments) by Kühn and Osthus [17].

Erdős raised a 'probabilistic' version of Kelly's conjecture: for a tournament T, let $\delta^0(T)$ denote the minimum of the minimum out-degree and the minimum in-degree. He conjectured that for almost all tournaments T, the maximum number of edge-disjoint Hamilton cycles in T is exactly $\delta^0(T)$. In particular, this would imply that with high probability, $\delta^0(T)$ is also the degree of a densest spanning regular subdigraph in a random tournament T. This conjecture of Erdős was proved by Kühn and Osthus [18], based on the main result in [17].

It would be interesting to obtain further conditions which relate the degree of the densest spanning regular subdigraph of a tournament T to the number of

edge-disjoint Hamilton cycles in T. For undirected graphs, one such conjecture was made in [15]: it states that for any graph G satisfying the conditions of Dirac's theorem, the number of edge-disjoint Hamilton cycles in G is exactly half the degree of a densest spanning even-regular subgraph of G. An approximate version of this conjecture was proved by Ferber, Krivelevich and Sudakov [12], see e.g. [15, 18] for some related results.

1.5. **Organization of the paper.** The methods used in the current paper are quite different from those used e.g. in the papers mentioned in Section 1.4. A crucial ingredient is the construction of highly structured dominating sets (see Section 3 for an informal description). We believe that this approach will have further applications.

In the next section, we introduce the notation that will be used for the remainder of the paper. In Section 3, we give an overview of the proof of Theorem 1.2. In Sections 4 and 5, we give the relatively short proofs of Theorem 1.3 and Proposition 5.1. In Section 6, we show that given a 'linked domination structure' (as introduced in the proof sketch), we can find a single Hamilton cycle (Lemma 6.7). In Section 7, we show that given several suitable linked domination structures, we can repeatedly apply Lemma 6.7 to find k edge-disjoint Hamilton cycles. In Section 8 we show that any highly linked tournament contains such suitable linked domination structures. Finally, Section 9 contains some concluding remarks.

2. Notation

The digraphs considered in this paper do not have loops and we allow up to two edges between any pair of x, y of distinct vertices, at most one in each direction. A digraph is an *oriented graph* if there is at most one edge between any pair x, y of distinct vertices, i.e. if it does not contain a cycle of length two.

Given a digraph D, we write $V(D)$ for its vertex set, $E(D)$ for its edge set, $e(D) := |E(D)|$ for the number of its edges and $|D|$ for its *order*, i.e. for the number of its vertices. We write $H \subseteq D$ to mean that H is a subdigraph of D, i.e. $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. Given $X \subseteq V(D)$, we write $D - X$ for the digraph obtained from D by deleting all vertices in X, and $D[X]$ for the subdigraph of D induced by X. Given $F \subseteq E(D)$, we write $D - F$ for the digraph obtained from D by deleting all edges in F. We write $V(F)$ for the set of all endvertices of edges in F. If H is a subdigraph of D, we write $D - H$ for $D - E(H)$.

We write xy for an edge directed from x to y. Unless stated otherwise, when we refer to paths and cycles in digraphs, we mean directed paths and cycles, i.e. the edges on these paths and cycles are oriented consistently. Given a path $P = x \ldots y$ from x to y and a vertex z outside P which sends an edge to x, we write zxP for the path obtained from P by appending the edge zx. The *length* of a path or cycle is the number of its edges. We call the terminal vertex of a path P the *head* of P and denote it by $h(P)$. Similarly, we call the initial vertex of a path P the *tail* of P and denote it by $t(P)$. The *interior* $\text{Int}(P)$ of a path

P is the subpath obtained by deleting $t(P)$ and $h(P)$. Thus $\text{Int}(P) = \emptyset$ if P has length at most one. Two paths P and P' are *internally disjoint* if $P \neq P'$ and $V(\text{Int}(P)) \cap V(\text{Int}(P')) = \emptyset$. A *path system* $\mathcal{P}$ is a collection of vertex-disjoint paths. We write $V(\mathcal{P})$ for the set of all vertices lying on paths in $\mathcal{P}$ and $E(\mathcal{P})$ for the set of all edges lying on paths in $\mathcal{P}$. We write $h(\mathcal{P})$ for the set consisting of the heads of all paths in $\mathcal{P}$ and $t(\mathcal{P})$ for the set consisting of the tails of all paths in $\mathcal{P}$. If $v \in V(\mathcal{P})$, we write v^+ and v^- for the successor and predecessor of v on the path in $\mathcal{P}$ containing v. A path system $\mathcal{P}$ is a *path cover* of a directed graph D if every path in $\mathcal{P}$ lies in D and together the paths in $\mathcal{P}$ cover all the vertices of D. If $X \subseteq V(D)$ and $\mathcal{P}$ is a path cover of $D[X]$, we sometimes also say that $\mathcal{P}$ is a path cover of X.

If x is a vertex of a digraph D, then $N_D^+(x)$ denotes the *out-neighbourhood* of x, i.e. the set of all those vertices y for which $xy \in E(D)$. Similarly, $N_D^-(x)$ denotes the *in-neighbourhood* of x, i.e. the set of all those vertices y for which $yx \in E(D)$. We write $d_D^+(x) := |N_D^+(x)|$ for the *out-degree* of x and $d_D^-(x) := |N_D^-(x)|$ for its *in-degree*. We denote the *minimum out-degree* of D by $\delta^+(D) := \min\{d_D^+(x) : x \in V(D)\}$ and the *maximum out-degree* of D by $\Delta^+(D) := \max\{d_D^+(x) : x \in V(D)\}$. We define the *minimum in-degree* $\delta^-(D)$ and the *maximum in-degree* $\Delta^-(D)$ similarly. The *minimum degree* of D is defined by $\delta(D) := \min\{d_D^+(x) + d_D^-(x) : x \in V(D)\}$ and its *minimum semi-degree* by $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$. Whenever $X, Y \subseteq V(D)$ are disjoint, we write $e_D(X)$ for the number of edges of D having both endvertices in X, and $e_D(X, Y)$ for the number of edges of D with tail in X and head in Y. We write $N_D^+(X) := \bigcup_{x \in X} N_D^+(x)$ and define $N_D^-(X)$ similarly. In all these definitions we often omit the subscript D if the digraph D is clear from the context.

A digraph D is *strongly connected* if for all $x, y \in V(D)$, there is a directed path in D from x to y. Given $k \in \mathbb{N}$, we say a digraph is *strongly k-connected* if $|D| > k$ and for every $S \subseteq V(D)$ of size at most $k-1$, $D-S$ is strongly connected. We say a digraph D is *k-linked* if $|D| \geq 2k$ and whenever $x_1, \ldots, x_k, y_1, \ldots, y_k$ are $2k$ distinct vertices of D, there exist vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i is a path from x_i to y_i.

Given a digraph D and sets $X, Y \subseteq V(D)$, we say that X *in-dominates* Y if each vertex in Y is an in-neighbour of some vertex in X. Similarly, we say that X *out-dominates* Y if each vertex in Y is an out-neighbour of some vertex in X.

A tournament T is *transitive* if there exists an ordering $v_1, \ldots, v_n$ of its vertices such that $v_i v_j \in E(T)$ if and only if $i < j$. In this case, we often say that v_1 is the *tail* of T and v_n is the *head* of T.

Given $k \in \mathbb{N}$, we write $[k] := \{1, \ldots, k\}$. We write log for the binary logarithm and $\log^2 n := (\log n)^2$.

3. Sketch of the proof of Theorem 1.2

In this section, we give an outline of the proof of Theorem 1.2. An important idea is the notion of a 'covering edge'. Given a small (pre-determined) set S of vertices in a tournament T, this will mean that it will suffice to find a cycle

covering all vertices of $T - S$. More precisely, let T be a tournament, let $x \in V(T)$, and suppose C is a cycle in T covering $T - x$. If $yz \in E(C)$ and $yx, xz \in E(T)$, then we can replace yz by yxz in C to turn C into a Hamilton cycle. We call yz a *covering edge* for x. More generally, if $S \subseteq V(T)$ and C is a cycle in T spanning $V(T) - S$ such that C contains a covering edge for each $x \in S$, then we can turn C into a Hamilton cycle by using all these covering edges. Note that this idea still works if C covers some part of S. On the other hand, note that S needs to be fixed at the beginning – this is different than in the recently popularized 'absorbing method'.

Another important tool will be the following consequence of the Gallai-Milgram theorem: suppose that G is an oriented graph on n vertices with $\delta(G) \geq n - \ell$. Then the vertices of G can be covered with ℓ vertex-disjoint paths. We use this as follows: suppose we are given a highly linked tournament T and have already found i edge-disjoint Hamilton cycles in T. Then the Gallai-Milgram theorem implies that we can cover the vertices of the remaining oriented graph by a set of $2i$ vertex-disjoint paths. Very roughly, the aim is to link together these paths using the high linkedness of the original tournament T.

To achieve this aim, we introduce and use the idea of 'transitive dominating sets'. Here a transitive out-dominating set A_ℓ has the following properties:

- A_ℓ out-dominates $V(T) \setminus A_\ell$, i.e. every vertex of $V(T) \setminus A_\ell$ receives an edge from A_ℓ.
- A_ℓ induces a transitive tournament in T.

Transitive in-dominating sets B_ℓ are defined similarly.

Now suppose that we have already found i edge-disjoint Hamilton cycles in a highly linked tournament T. Let T' be the oriented subgraph of T obtained by removing the edges of these Hamilton cycles. Suppose that we also have the following 'linked dominating structure' in T', which consists of:

- small disjoint transitive out-dominating sets $A_1, \ldots, A_t$, where $t := 2i+1$;
- small disjoint transitive in-dominating sets $B_1, \ldots, B_t$;
- a set of short vertex-disjoint paths $P_1, \ldots, P_t$, where each P_ℓ is a path from the head b_ℓ of B_ℓ to the tail a'_ℓ of A_ℓ.

Recall that the head of a transitive tournament is the vertex of out-degree zero and the tail is defined analogously. The paths P_ℓ are found at the outset of the proof, using the assumption that the original tournament T is highly linked. (Note that T' need not be highly linked.)

Let A^* denote the union of the A_i and let B^* denote the union of the B_i. Note that $\delta(T' - A^* \cup B^*) \geq n - 1 - 2i = n - t$. So the Gallai-Milgram theorem implies that we can cover the vertices of $T' - A^* \cup B^*$ with t vertex-disjoint paths $Q_1, \ldots, Q_t$. Now we can link up successive paths using the above dominating sets as follows. The final vertex of Q_1 sends an edge to some vertex b in B_2 (since B_2 is in-dominating). Either b is equal to the head b_2 of B_2 or there is an edge in $T'[B_2]$ from b to b_2 (since $T'[B_2]$ is a transitive tournament). Now follow the path P_2 from b_2 to the tail a'_2 of A_2. Using the fact that $T'[A_2]$ is transitive and that A_2 is out-dominating, we can similarly find a path of length at most
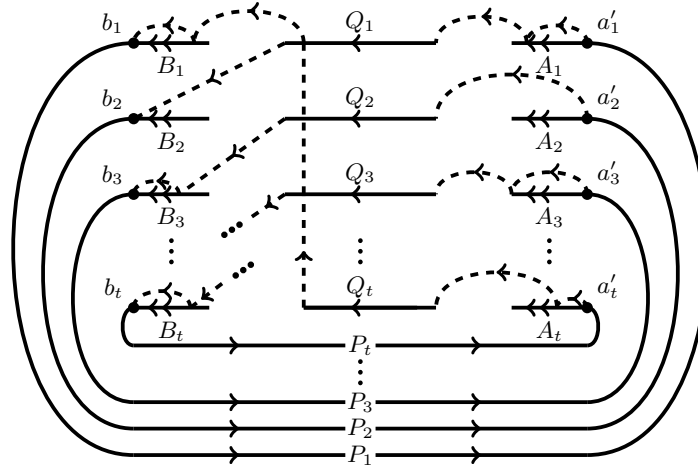
FIGURE 1. Illustrating the paths Q_i and P_i as well as the edges linking them up via the linked domination structure.

two from a_2' to the initial vertex of Q_2. Continuing in this way, we can link up all the paths Q_ℓ and P_ℓ into a single cycle C which covers all vertices outside $A^* \cup B^*$ (and some of the vertices inside $A^* \cup B^*$). The idea is illustrated in Figure 1.

In our construction, we will ensure that the paths P_ℓ contain a set of covering edges for $A^* \cup B^*$. So C also contains covering edges for $A^* \cup B^*$, and so we can transform C into a Hamilton cycle as discussed earlier.

A major obstacle to the above strategy is that in order to guarantee the P_ℓ in $T' - A^* \cup B^*$, we would need the linkedness of T to be significantly larger than $|A^* \cup B^*|$ (and thus larger than $|A_\ell|$). However, there are many tournaments where any in- or out-dominating set contains $\Omega(\log n)$ vertices (consider a random tournament). This leads to a linkage requirement on T which depends on n (and not just on k, as required in Theorem 1.2).

We overcome this problem by considering 'almost dominating sets': instead of out-dominating all vertices outside A_ℓ, the A_ℓ will out-dominate almost all vertices outside A_ℓ. (Analogous comments apply to the in-dominating sets B_ℓ.) This means that we have a small 'exceptional set' E of vertices which are not out-dominated by all of the A_ℓ. The problem with allowing an exceptional set is that if the tail of a path Q_ℓ in our cover is in the exceptional set E, we cannot extend it directly into the out-dominating set A_ℓ as in the above description. However, if we make sure that the A_ℓ include the vertices of smallest in-degree of T, we can deal with this issue. Indeed, in this case we can show that every vertex $v \in E$ has in-degree $d^-(v) > 2|E|$ say, so we can always extend the tail of a path out of the exceptional set if necessary (and then into an almost out-dominating set A_ℓ as before). Unfortunately, we may 'break' one of the paths P_ℓ in the process. However, if we are careful about the place where we break
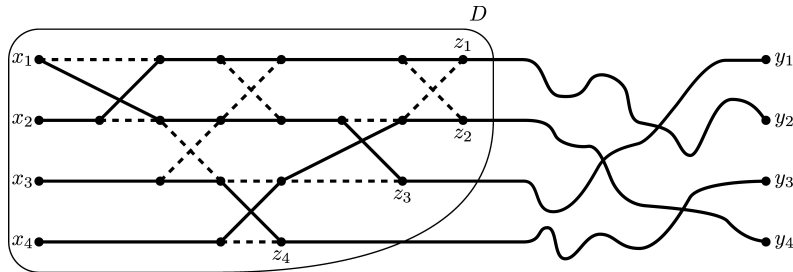
FIGURE 2. Illustrating our construction of a digraph D which corresponds to a sorting network for $k = 4$. D is used to link x_i to y_i. In the notation of the proof of Theorem 1.3, we have $\pi(3) = 1$.

it and construct some 'spare' paths at the outset, it turns out that the above strategy can be made to work.

4. CONNECTIVITY AND LINKEDNESS IN TOURNAMENTS

In this section we give the proof of Theorem 1.3. We will also collect some simple properties of highly linked directed graphs which we will use later on. The proof of Theorem 1.3 is based on an important result of Ajtai, Komlós and Szemerédi [1, 2] on sorting networks. Roughly speaking, the proof idea of Theorem 1.3 is as follows. Suppose that we are given a highly connected tournament T and we want to link an ordered set X of k vertices to a set Y of the same size. Then we construct the equivalent of a sorting network D inside $T - Y$ with 'initial vertices' in X and 'final vertices' in a set Z. The high connectivity of T guarantees an 'unsorted' set of k ZY-paths which avoid the vertices in $D - Z$. One can then extend these paths via D to the appropriate vertices in X. In this way, we obtain paths linking the vertices in X to the appropriate ones in Y. An example is shown in Figure 2.

We now introduce the necessary background on non-adaptive sorting algorithms and sorting networks; see [14] for a more detailed treatment. In a sorting problem, we are given k registers $R_1, \ldots, R_k$, and each register R_i is assigned a distinct element from $[k]$, which we call the *value* of R_i; thus there is some permutation π of $[k]$ such that value i has been assigned to register $R_{\pi(i)}$. Our task is to sort the values into their corresponding registers (so that value i is assigned to R_i) by making a sequence of comparisons: a comparison entails taking two registers and reassigning their values so that the higher value is assigned to the higher register and the lower value to the lower register. A non-adaptive sorting algorithm is a sequence of comparisons specified in advance such that for any initial assignment of k values to k registers, applying the prescribed sequence of comparisons results in every value being assigned to its corresponding register.

Ajtai, Komlós and Szemerédi [1, 2] proved, via the construction of sorting networks, that there exists an absolute constant C' and a non-adaptive sorting algorithm (for k registers and values) that requires $C'k \log k$ comparisons, and this is asymptotically best possible. It is known that we can take $C' := 3050$ [20]

(results of this type are often stated in terms of the *depth* of a sorting network rather than the number of comparisons).

The next theorem is a consequence of the above. Before we can state it, we first need to introduce some notation. A comparison c, which is part of some non-adaptive sorting algorithm for k registers, will be denoted by $c = (s; t)$, where $1 \leq s < t \leq k$, to indicate that c is a comparison in which the values of registers R_s and R_t are compared (and sorted so the higher value is assigned to the higher register).

Theorem 4.1. *Let $C' := 3050$ and $k \in \mathbb{N}$ be such that $k \geq 2$. Then there exist $r \leq C'k \log k$ and a sequence of comparisons $c_1, \ldots, c_r$ satisfying the following property: for any initial assignment of k values to k registers, applying the comparisons in sequence results in register R_i being assigned the value i for all $i \in [k]$.*

We now show how to obtain a structure within a highly connected tournament that simulates the function of a non-adaptive sorting algorithm. Each comparison in the sorting algorithm will be simulated by a 'switch', which we now define. An (a_1, a_2)-*switch* is a digraph D on 5 distinct vertices a_1, a_2, b, b_1, b_2, where either $E(D) = \{a_1 b, b b_1, b b_2, a_2 b_1, a_2 b_2\}$ or $E(D) = \{a_2 b, b b_1, b b_2, a_1 b_1, a_1 b_2\}$. We call b_1 and b_2 the *terminal vertices* of the (a_1, a_2)-switch. Note that for any permutation π of $\{1, 2\}$, there exist vertex-disjoint paths P_1, P_2 of D such that P_i joins a_i to $b_{\pi(i)}$ for $i = 1, 2$.

Proposition 4.2. *Let T be a tournament. Given distinct vertices $a_1, a_2 \in V(T)$, if $d_T^+(a_1), d_T^+(a_2) \geq 7$, then T contains an (a_1, a_2)-switch.*

Proof. We may choose disjoint sets $A_1 \subseteq N_T^+(a_1) \setminus \{a_2\}$ and $A_2 \subseteq N_T^+(a_2) \setminus \{a_1\}$ with $|A_1| = |A_2| = 3$. Consider the bipartite digraph H induced by T between A_1 and A_2. It is easy to check that there exists $b \in A_1 \cup A_2$ with $d_H^+(b) \geq 2$. Let b_1 and b_2 be two out-neighbours of b in H. Now the vertices a_1, a_2, b, b_1, b_2 with suitably chosen edges from T form an (a_1, a_2)-switch (with terminal vertices b_1 and b_2). $\square$

Given $k \in \mathbb{N}$, we write S_k for the set of permutations of $[k]$ and id_k for the identity permutation of $[k]$. The following structural lemma for tournaments is at the heart of the proof of Theorem 1.3. It constructs the equivalent of a sorting network in a tournament of high minimum outdegree.

Lemma 4.3. *Let $C' := 3050$ and $k \in \mathbb{N}$ be such that $k \geq 2$. Let T be a tournament with $\delta^+(T) \geq (3C' + 5)k \log k$, and let $x_1, \ldots, x_k \in V(T)$ be distinct vertices. Then there exists a digraph $D \subseteq T$ and distinct vertices $z_1, \ldots, z_k \in V(D)$ with the following properties:*

(i) *$x_1, \ldots, x_k \in V(D)$.*
(ii) *$|D| \leq (3C' + 1)k \log k$.*
(iii) *For any $\pi \in S_k$, we can find vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i joins $x_{\pi(i)}$ to z_i for all $i \in [k]$.*

Proof. Consider the sorting problem for k registers, and apply Theorem 4.1 to obtain a sequence $c_1, \ldots, c_r$ of $r \leq C'k \log k$ comparisons such that for any $\pi \in S_k$, if value i is initially assigned to register $R_{\pi(i)}$, then applying the comparisons $c_1, \ldots, c_r$ results in every value being assigned to its corresponding register. Given $\pi \in S_k$, we write $\pi_q \in S_k$ for the permutation such that after applying the first q comparisons $c_1, \ldots, c_q$, value i is assigned to register $R_{\pi_q(i)}$ for all i; thus $\pi_r = id_k$.

Let D_0 be the digraph with vertex set $\{x_1, \ldots, x_k\}$ and empty edge set. We inductively construct digraphs $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_r \subseteq T$ and for each D_q we maintain a set $Z_q = \{z_1^q, \ldots, z_k^q\}$ of k distinct *final vertices* such that the following holds:

(a) $|D_q| = 3q + k$.
(b) Whenever $\pi \in S_k$ is a permutation, there exist vertex-disjoint paths $P_1^q, \ldots, P_k^q$ in D_q such that P_i^q joins $x_{\pi(i)}$ to $z_{\pi_q(i)}^q$ for all $i \in [k]$.

Assuming the above statement holds for $q = 0, \ldots, r$, then taking $D := D_r$ with $z_i := z_i^r$ for all $i \in [k]$ proves the lemma. Indeed $|D_r| = 3r + k \leq 3C'k \log k + k \leq (3C' + 1)k \log k$ and $\pi_r = id_k$.

Having already defined D_0, let us describe the inductive step of our construction. Suppose that for some $q \in [r]$ we have constructed $D_{q-1} \subseteq T$ and a corresponding set $Z_{q-1} = \{z_1^{q-1}, \ldots, z_k^{q-1}\}$ of final vertices. Let $s, t \in [k]$ with $s < t$ be such that $c_q = (s; t)$. Define the tournament $T' := T - (V(D_{q-1}) \setminus \{z_s^{q-1}, z_t^{q-1}\})$. Then T' has minimum out-degree at least

$$(3C' + 5)k \log k - |D_{q-1}| \geq (3C' + 5)k \log k - 3r - k \geq 5k \log k - k \geq 7,$$

and so in particular $d_{T'}^+(z_s^{q-1}), d_{T'}^+(z_t^{q-1}) \geq 7$. Thus we may apply Proposition 4.2 to obtain a (z_s^{q-1}, z_t^{q-1})-switch σ in T'. Write b_1, b_2 for the terminal vertices of σ. Now D_q is constructed from D_{q-1} by adding the vertices and edges of σ to D_{q-1}; note that z_s^{q-1} and z_t^{q-1} are precisely the common vertices of D_{q-1} and σ. We define the set $Z_q = \{z_1^q, \ldots, z_k^q\}$ by setting $z_i^q := z_i^{q-1}$ for all $i \neq s, t$ and $z_s^q := b_1$ as well as $z_t^q := b_2$. Note that $z_1^q, \ldots, z_k^q$ are distinct.

Finally we check that conditions (a) and (b) hold for D_q. Condition (a) holds since D_q has exactly 3 more vertices than D_{q-1}. For (b), by induction we may assume that there are vertex-disjoint paths $P_1^{q-1}, \ldots, P_k^{q-1}$ in D_{q-1} such that P_i^{q-1} joins $x_{\pi(i)}$ to $z_{\pi_{q-1}(i)}^{q-1}$ for all $i \in [k]$. Choose vertex-disjoint paths Q_s and Q_t in σ such that

- if c_q swaps values in registers R_s and R_t, then Q_s joins z_s^{q-1} to z_t^q and Q_t joins z_t^{q-1} to z_s^q;
- if c_q does not swap values in registers R_s and R_t, then Q_s joins z_s^{q-1} to z_s^q and Q_t joins z_t^{q-1} to z_t^q.

Now exactly two of the paths from $P_1^{q-1}, \ldots, P_k^{q-1}$ end at z_s^{q-1} and z_t^{q-1}, namely those indexed by $\pi_{q-1}^{-1}(s)$ and $\pi_{q-1}^{-1}(t)$. We extend these two paths using Q_s and

Q_t, and leave all others unchanged to obtain paths $P_1^q, \ldots, P_k^q$. It is straightforward to check that these paths are vertex-disjoint and that P_i joins $x_{\pi(i)}$ to $z_{\pi_q(i)}^q$ for all $i \in [k]$. □

It is now an easy step to prove Theorem 1.3. We will use the following directed version of Menger's Theorem.

Theorem 4.4 (Menger's Theorem). *Suppose D is a strongly k-connected digraph with $A, B \subseteq V(D)$ and $|A|, |B| \geq k$. Then there exist k vertex-disjoint paths in D each starting in A and ending in B.*

Proof of Theorem 1.3. Set $C' := 3050$ and $C := 3C' + 6 < 10^4$. We must show that, given a strongly $Ck \log k$-connected tournament T and distinct vertices $x_1, \ldots, x_k, y_1, \ldots, y_k \in V(T)$, we can find vertex-disjoint paths $R_1, \ldots, R_k$ such that R_i joins x_i to y_i for all $i \in [k]$.

Let $X := \{x_1, \ldots, x_k\}$, $Y := \{y_1, \ldots, y_k\}$ and $T' := T - Y$. Note that T' is strongly $(3C' + 5)k \log k$-connected, and in particular $\delta^+(T'') \geq (3C' + 5)k \log k$. Thus we can apply Lemma 4.3 to T' and $x_1, \ldots, x_k$ to obtain a digraph $D \subseteq T'$ and vertices $z_1, \ldots, z_k \in V(D)$ satisfying properties (i)–(iii) of Lemma 4.3. Let $Z := \{z_1, \ldots, z_k\}$. Since $|D| \leq (3C' + 1)k \log k$, the tournament $T'' := T - (V(D) \setminus Z)$ is strongly k-connected. Therefore, by Theorem 4.4, there exist k vertex-disjoint paths, with each path starting in Z and ending in Y. For each $i \in [k]$, let us assume that $P_{\pi(i)}$ is the path that joins z_i to $y_{\pi(i)}$, where π is some permutation of $[k]$. By Lemma 4.3, we can find vertex-disjoint paths $Q_1, \ldots, Q_k$ in D such that Q_i joins $x_{\pi(i)}$ to z_i. Then the path $R_i := Q_{\pi^{-1}(i)} P_{\pi^{-1}(i)}$ joins x_i to y_i and these paths are vertex-disjoint. □

Batcher [7] (see also [14]) gave a construction of sorting networks which is asymptotically not optimal but which gives better values for small k. More precisely, it uses at most $2k \log^2 k$ comparisons for $k \geq 3$. If we use these as a building block in the proof of Lemma 4.3 instead of the asymptotically optimal ones leading to Theorem 4.1, we immediately obtain the following result which improves Theorem 1.3 for small values of k.

Theorem 4.5. *For all $k \in \mathbb{N}$ with $k \geq 3$, every strongly $12k \log^2 k$-connected tournament is k-linked.*

For $k = 2$, the best bound is obtained by a result of Bang-Jensen [3], who showed that every strongly 5-connected semi-complete digraph is 2-linked.

We will now collect some simple properties of highly linked directed graphs which we will use later on. The first two follow straightforwardly from the definition of linkedness.

Proposition 4.6. *Let $k \in \mathbb{N}$. Then a digraph D is k-linked if and only if $|D| \geq 2k$ and whenever $(x_1, y_1), \ldots, (x_k, y_k)$ are ordered pairs of (not necessarily distinct) vertices of D, there exist internally disjoint paths $P_1, \ldots, P_k$ such that P_i joins x_i to y_i.*

Proposition 4.7. *Let $k, \ell \in \mathbb{N}$ with $\ell < k$, and let D be a k-linked digraph. Let $X \subseteq V(D)$ and $F \subseteq E(D)$ be such that $|X| + 2|F| \leq 2\ell$. Then $D - X - F$ is $(k - \ell)$-linked.*

The next lemma shows that in a sufficiently highly linked digraph we can link given pairs of vertices by vertex-disjoint paths which together do not contain too many vertices.

Lemma 4.8. *Let $k, s \in \mathbb{N}$, and let D be a $2ks$-linked digraph. Let $(x_1, y_1), \ldots, (x_k, y_k)$ be ordered pairs of (not necessarily distinct) vertices in D. Then there exist internally disjoint paths $P_1, \ldots, P_k$ such that P_i joins x_i to y_i for all $i \in [k]$ and $|P_1 \cup \cdots \cup P_k| \leq |D|/s$.*

Proof.   By Proposition 4.6 there exist internally disjoint paths $P_1^1, \ldots, P_k^{2s}$ such that P_i^j joins x_i to y_i for all $i \in [k]$ and all $j \in [2s]$. For any j, the interiors of $P_1^j, \ldots, P_k^j$ contain at least $|P_1^j \cup \cdots \cup P_k^j| - 2k$ vertices. So the disjointness of the paths implies that there is a $j \in [2s]$ with $|P_1^j \cup \cdots \cup P_k^j| - 2k \leq |D|/2s$. The result now follows by setting $P_i := P_i^j$ and noting that $2k \leq |D|/2s$.   □

5. Nearly extremal example

The aim of this section is to prove the following proposition, which shows that the bound on the connectivity in Theorem 1.1 is close to best possible.

Proposition 5.1. *Fix $n, k \in \mathbb{N}$ with $k \geq 2$ and $n > k^2 + k + 2$. There exists a strongly $\lfloor k^2/4 \rfloor$-connected tournament T of order n such that if $D \subseteq T$ is a spanning r-regular subdigraph, then $r \leq k$. In particular, T contains at most k edge-disjoint Hamilton cycles.*

It is easy to see that the above tournament T is also $\Omega(k^2)$-linked. This shows that the bound in Theorem 1.2 has to be at least quadratic in k.

Proof.   Let $\ell \in \mathbb{N}$. We will first describe a tournament $T_\ell = (V_\ell, E_\ell)$ of order $2\ell + 1$ which is strongly ℓ-connected. We then use T_ℓ as a building block to construct a tournament as desired in the proposition.

Let $V_\ell := \{v_0, \ldots, v_{2\ell}\}$ and let E_ℓ consist of the edges $v_i v_{i+t}$ for all $i = 0, \ldots, 2\ell$ and all $t \in [\ell]$, where indices are understood to be modulo $2\ell + 1$. One may think of T_ℓ as the tournament with vertices $v_0, \ldots, v_{2\ell}$ placed in order, clockwise, around a circle, where the out-neighbours of each v_i are the ℓ closest vertices to v_i in the clockwise direction, and the in-neighbours are the ℓ closest vertices in the anticlockwise direction. Note that T_ℓ is regular. Note also that, for any distinct $x, y \in V_\ell$, we can find a path in T_ℓ from x to y by traversing vertices from x to y in clockwise order; this remains true even if we delete any $\ell - 1$ vertices from T_ℓ.

Next we construct a tournament $T_{m,\ell} = (V_{m,\ell}, E_{m,\ell})$ as follows. We take $V_{m,\ell}$ to be the disjoint union of sets $A_\ell := \{a_0, \ldots, a_{2\ell}\}$, $B_\ell := \{b_0, \ldots, b_{2\ell}\}$, and $C_m := \{c_1, \ldots, c_m\}$. The edges of $T_{m,\ell}$ are defined as follows: $T_{m,\ell}[A_\ell]$ and $T_{m,\ell}[B_\ell]$ are isomorphic to T_ℓ (with the natural labelling of vertices), and $T[C_m]$

is a transitive tournament which respects the given order of the vertices in C_m (i.e. $c_i c_j$ is an edge if and only if $i < j$). Each vertex in A_ℓ is an in-neighbour of all vertices in C_m, and each vertex in B_ℓ is an out-neighbour of all vertices in C_m. Finally, a vertex $a_i \in A_\ell$ is an in-neighbour of a vertex $b_j \in B_\ell$ if and only if $i \neq j$. Note that $|T_{m,\ell}| = m + 4\ell + 2$.

Claim 1. *The tournament $T_{m,\ell}$ is strongly ℓ-connected.*

To see that $T_{m,\ell}$ is strongly ℓ-connected, we check that if $S \subseteq V_{m,\ell}$ with $|S| \leq \ell - 1$, then $T_{m,\ell} - S$ is strongly connected. Write A'_ℓ, B'_ℓ and C'_m respectively for $A_\ell \setminus S$, $B_\ell \setminus S$, and $C_m \setminus S$. Note that there is at least one edge of $T_{m,\ell} - S$ from B'_ℓ to A'_ℓ, which we may assume by symmetry to be $b_0 a_0$. Ordering the vertices of $T_{m,\ell}$ as $a_0, \ldots, a_{2\ell}, c_1, \ldots, c_m, b_1, \ldots, b_{2\ell}, b_0$ and removing the vertices of S from this ordering gives a Hamilton cycle in $T_{m,\ell} - S$. Thus $T_{m,\ell} - S$ must be strongly connected. This completes the proof of Claim 1.

Claim 2. *Let $m, \ell \in \mathbb{N}$ be such that $m > \sqrt{4\ell}$. Then for every r-regular spanning subdigraph $D \subseteq T_{m,\ell}$ we have $r \leq \sqrt{4\ell}$.*

Suppose for a contradiction that $D \subseteq T_{m,\ell}$ is an r-regular spanning subdigraph with $r := \lfloor \sqrt{4\ell} \rfloor + 1 > \sqrt{4\ell}$. Since D is regular, we have $e_D(A_\ell, \bar{A}_\ell) = e_D(\bar{A}_\ell, A_\ell)$, where $\bar{A}_\ell := V(D) \setminus A_\ell$. Noting that $r \leq m$, consider the first r vertices $c_1, \ldots, c_r$ of C_m. Since $N_D^-(c_i) \subseteq N_{T_{m,\ell}}^-(c_i) = A_\ell \cup \{c_1, \ldots, c_{i-1}\}$ and $|N_D^-(c_i)| = r$, we have $|N_D^-(c_i) \cap A_\ell| \geq r - i + 1$, so that $e_D(A_\ell, \{c_i\}) \geq r - i + 1$. Thus

$$e_D(\bar{A}_\ell, A_\ell) = e_D(A_\ell, \bar{A}_\ell) \geq e(A_\ell, \{c_1, \ldots, c_r\}) \geq r + \cdots + 1 = \binom{r+1}{2}.$$

But $e_D(\bar{A}_\ell, A_\ell) \leq e_{T_{m,\ell}}(\bar{A}_\ell, A_\ell) = 2\ell + 1$, so $\binom{r+1}{2} \leq 2\ell + 1$. This is easily seen to contradict $r > \sqrt{4\ell}$ for all $\ell \in \mathbb{N}$. This completes the proof of Claim 2.

To prove the proposition, we set $\ell := \lfloor k^2/4 \rfloor$ and $m := n - 4\ell - 2$, and take T to be $T_{m,\ell}$. Thus $|T| = |T_{m,\ell}| = m + 4\ell + 2 = n$. By Claim 1, T is strongly $\lfloor k^2/4 \rfloor$-connected. Since $n > k^2 + k + 2 \geq 4\ell + \sqrt{4\ell} + 2$, we have $m > \sqrt{4\ell}$, so Claim 2 implies that if $D \subseteq T = T_{m,\ell}$ is a spanning r-regular subdigraph, then $r \leq \sqrt{4\ell} \leq k$. $\qquad \square$

6. Finding a single Hamilton cycle in suitable oriented graphs

We first state two simple, well-known facts concerning the degree sequences of tournaments.

Proposition 6.1. *Let T be a tournament on n vertices. Then T contains at least one vertex of in-degree at most $n/2$, and at least one vertex of out-degree at most $n/2$.*

Proposition 6.2. *Let T be a tournament on n vertices and let $d \geq 0$. Then T has at most $2d + 1$ vertices of in-degree at most d, and at most $2d + 1$ vertices of out-degree at most d.*

We will also use the following well-known result due to Gallai and Milgram (see for example [9]). (The *independence number* of a digraph T is the maximal size of a set $X \subseteq V(T)$ such that $T[X]$ contains no edges.)

Theorem 6.3. *Let T be a digraph with independence number at most k. Then T has a path cover consisting of at most k paths.*

The following corollary is an immediate consequence of Theorem 6.3.

Corollary 6.4. *Let T be an oriented graph on n vertices with $\delta(T) \geq n - k$. Then T has a path cover consisting of at most k paths.*

Given a digraph T, we define a *covering edge* for a vertex v to be an edge xy of T such that $xv, vy \in E(T)$. We call xv and vy the *activating edges* of xy. Note that if xy is a covering edge for v and C is a cycle in T containing xy but not v, we can form a new cycle C' with $V(C') = V(C) \cup \{v\}$ by replacing xy with xvy in C. We will see in Section 8 that covering edges are easy to find in strongly 2-connected tournaments.

Recall that, given a path system $\mathcal{P}$, we write $h(\mathcal{P})$ for the set of heads of paths in $\mathcal{P}$ and $t(\mathcal{P})$ for the set of tails of paths in $\mathcal{P}$. If $v \in V(\mathcal{P})$, we write v^+ and v^- respectively for the successor and predecessor of v on the path in $\mathcal{P}$ containing v.

The following lemma allows us to take a path cover $\mathcal{P}$ of a digraph and modify it into a path cover $\mathcal{P}'$ with no heads in some "bad" set I, without adding any heads or tails in $I \cup J$ for some other "bad" set J. Moreover, we can do this without losing any edges in some "good" set $F \subseteq E(\mathcal{P})$, and without altering too many paths in $\mathcal{P}$. In our applications, F will consist of covering edges. We require that every vertex in I has high out-degree.

Lemma 6.5. *Let T be a digraph. Let $I, J \subseteq V(T)$ be disjoint. Let $\mathcal{P} = \mathcal{P}_1 \dot\cup \mathcal{P}_2$ be a path cover of T satisfying $h(\mathcal{P}_2) \cap I = \emptyset$. Let $F \subseteq E(\mathcal{P})$. Suppose $d^+(v) > 3(|I| + |J|) + 2|F|$ for all $v \in I$. Then there exists a path cover $\mathcal{P}'$ of T satisfying the following properties:*

   (i) $h(\mathcal{P}') \cap I = \emptyset$.
   (ii) $h(\mathcal{P}') \cap J = h(\mathcal{P}) \cap J$.
   (iii) $t(\mathcal{P}') \cap (I \cup J) = t(\mathcal{P}) \cap (I \cup J)$.
   (iv) $F \subseteq E(\mathcal{P}')$.
   (v) $|\mathcal{P}'| \leq |\mathcal{P}| + |\mathcal{P}_1|$.
   (vi) $|\mathcal{P}' \cap \mathcal{P}_2| \geq |\mathcal{P}_2| - |\mathcal{P}_1|$.

If in addition $d^+(v) > 3(|I| + |J|) + 2|F| + |V(\mathcal{P}_2)|$ for all $v \in I$, then we may strengthen (vi) to $\mathcal{P}_2 \subseteq \mathcal{P}'$.

Proof. We will use the degree condition on the vertices in I in the hypothesis to repeatedly extend paths with heads in I out of I, breaking other paths in $\mathcal{P}$ as a result. We must ensure that we do not create new paths with endpoints in $I \cup J$ in the process. Let $r := |\mathcal{P}_1|$ and $\mathcal{P}^0 := \mathcal{P}$. We shall find path covers $\mathcal{P}^1, \ldots, \mathcal{P}^r$ of T such that the following properties hold for all $0 \leq i \leq r$:

  (P1) $|h(\mathcal{P}^i) \cap I| \leq r - i$.

(P2) $h(\mathcal{P}^i) \cap J = h(\mathcal{P}) \cap J$.
(P3) $t(\mathcal{P}^i) \cap (I \cup J) = t(\mathcal{P}) \cap (I \cup J)$.
(P4) $F \subseteq E(\mathcal{P}^i)$.
(P5) $|\mathcal{P}^i| \leq |\mathcal{P}| + i$.
(P6) $|\mathcal{P}^i \cap \mathcal{P}_2| \geq |\mathcal{P}_2| - i$.

If this is possible, we may then take $\mathcal{P}' := \mathcal{P}^r$.

By hypothesis, $\mathcal{P}^0$ satisfies (P1)–(P6). So suppose we have found $\mathcal{P}^0, \dots, \mathcal{P}^{i-1}$ for some $i \in [r]$. We then form $\mathcal{P}^i$ as follows. If $|h(\mathcal{P}^{i-1}) \cap I| \leq r - i$, we simply let $\mathcal{P}^i := \mathcal{P}^{i-1}$. Otherwise, let $P \in \mathcal{P}^{i-1}$ be a path with head $v \in I$. We will form $\mathcal{P}^i$ by extending the head v of P and breaking the path in $\mathcal{P}^{i-1}$ which P now intersects into two subpaths. Define

$$X := \{x \in V(T) : \{x^+, x, x^-\} \cap (I \cup J) \neq \emptyset\}.$$

We have

$$d^+(v) > 3(|I| + |J|) + 2|F| \geq |X| + |V(F)| \geq |X \cup V(F)|,$$

and so there exists $w \in N^+(v) \setminus (X \cup V(F))$. Let Q be the path in $\mathcal{P}^{i-1}$ containing w (note that we may have $Q = P$). Split Q into (at most) two paths and an isolated vertex by removing any of the edges w^-w, ww^+ that exist, and let $\mathcal{P}^*$ be the set of paths obtained from $\mathcal{P}^{i-1}$ in this way. Let P^* be the path in $\mathcal{P}^*$ containing v. (Note that $P^* = P$ unless $w \in V(P)$.) We then form $\mathcal{P}^i$ by replacing P^* by P^*vw in $\mathcal{P}^*$.

First suppose $w \in \text{Int}(Q)$. Then $\mathcal{P}^i$ is a path cover of T such that

$$h(\mathcal{P}^i) = (h(\mathcal{P}^{i-1}) \setminus \{v\}) \cup \{w, w^-\} \quad \text{and} \quad t(\mathcal{P}^i) = t(\mathcal{P}^i) \cup \{w^+\}.$$

Since $w \notin X$, we have $w, w^- \notin I$ and hence

$$|h(\mathcal{P}^i) \cap I| = |h(\mathcal{P}^{i-1}) \cap I| - 1 \leq r - i.$$

Thus (P1) holds. Similarly,

$$h(\mathcal{P}^i) \cap J = h(\mathcal{P}^{i-1}) \cap J = h(\mathcal{P}) \cap J,$$
$$t(\mathcal{P}^i) \cap (I \cup J) = t(\mathcal{P}^{i-1}) \cap (I \cup J) = t(\mathcal{P}) \cap (I \cup J),$$

and so (P2) and (P3) hold. By similar arguments, (P1)–(P3) also hold if w is an endpoint of Q. Since $w \notin V(F)$ and $F \subseteq E(\mathcal{P}^{i-1})$ we have $F \subseteq E(\mathcal{P}^i)$ and (P4) holds. (P5) holds too since $|\mathcal{P}^i| \leq |\mathcal{P}^{i-1}| + 1$. Finally, we have altered at most two paths in $\mathcal{P}^{i-1}$. One of these had its head in I, so we have altered at most one path in $\mathcal{P}^{i-1} \cap \mathcal{P}_2$. Thus (P6) holds.

If in addition we have

$$d^+(v) > 3(|I| + |J|) + 2|F| + |V(\mathcal{P}_2)|,$$

then we may use almost exactly the same argument to prove the strengthened version of the result. Instead of choosing $w \in N^+(v) \setminus (X \cup V(F))$, we may choose $w \in N^+(v) \setminus (X \cup V(F) \cup V(\mathcal{P}_2))$. We also strengthen (P6) to the requirement that $\mathcal{P}_2 \subseteq \mathcal{P}^i$. The strengthened (P6) must hold in each step since we now have that $w \notin V(\mathcal{P}_2)$. □

The following analogue of Lemma 6.5 for tails can be obtained by reversing the orientation of each edge of T.

Lemma 6.6. *Let T be a digraph. Let $I, J \subseteq V(T)$ be disjoint. Let $\mathcal{P} = \mathcal{P}_1 \dot\cup \mathcal{P}_2$ be a path cover of T satisfying $t(\mathcal{P}_2) \cap I = \emptyset$. Let $F \subseteq E(\mathcal{P})$. Suppose $d^-(v) > 3(|I| + |J|) + 2|F|$ for all $v \in I$. Then there exists a path cover $\mathcal{P}'$ of T satisfying the following properties:*

(i) $t(\mathcal{P}') \cap I = \emptyset$.
(ii) $t(\mathcal{P}') \cap J = t(\mathcal{P}) \cap J$.
(iii) $h(\mathcal{P}') \cap (I \cup J) = h(\mathcal{P}) \cap (I \cup J)$.
(iv) $F \subseteq E(\mathcal{P}')$.
(v) $|\mathcal{P}'| \leq |\mathcal{P}| + |\mathcal{P}_1|$.
(vi) $|\mathcal{P}' \cap \mathcal{P}_2| \geq |\mathcal{P}_2| - |\mathcal{P}_1|$.

If in addition $d^-(v) > 3(|I| + |J|) + 2|F| + |V(\mathcal{P}_2)|$ for all $v \in I$, then we may strengthen (vi) to $\mathcal{P}_2 \subseteq \mathcal{P}'$.

The following lemma is the main building block of the proof of Theorem 1.2. It will be applied repeatedly to find the required edge-disjoint Hamilton cycles. Roughly speaking, the lemma guarantees a Hamilton cycle provided that we have well-chosen disjoint (almost) dominating sets A_i and B_i which are linked by short paths containing covering edges for all vertices in these dominating sets. (This is the linked dominating structure described in Sections 1 and 3.) An additional assumption is that we have not removed too many edges of our tournament T already. In general, the statement and proof roughly follow the sketch in Section 3, with the addition of a set $X \subseteq V(T)$.

The role of X is as follows. The sets A_i and B_i in the lemma dominate only almost all vertices of T, so we have some small exceptional sets E_A and E_B of vertices which are not dominated. We will use Lemmas 6.5 and 6.6 to extend a certain path system out of these exceptional sets E_A and E_B. For this we need that the vertices in $E_A \cup E_B$ have relatively high in- and out-degree. But T may have vertices which do not satisfy this degree condition. When we apply Lemma 6.7, these problematic vertices will be the elements of X.

Lemma 6.7. *Let $C := 10^6$, $k \geq 20$, $t := 164k$, and $c := \lceil \log 50t + 1 \rceil$. Suppose that T is an oriented graph of order n satisfying $\delta(T) > n - 4k$ and $\delta^0(T) \geq Ck^2$. Suppose moreover that T contains disjoint sets of vertices $A_1, \ldots, A_t$, $B_1, \ldots, B_t$ and X, a matching F, and vertex-disjoint paths $P_1, \ldots, P_t$ such that the following conditions hold, where $A^* := A_1 \cup \cdots \cup A_t$ and $B^* := B_1 \cup \cdots \cup B_t$:*

(i) $2 \leq |A_i| \leq c$ *for all $i \in [t]$. Moreover, $T[A_i]$ is a transitive tournament whose head has out-degree at least $n/3$ in T.*
(ii) *There exists a set $E_A \subseteq V(T) \setminus (A^* \cup B^*)$, such that each A_i out-dominates $V(T) \setminus (A^* \cup B^* \cup E_A)$. Moreover, $|E_A| \leq d^-/40$, where $d^- := \min\{d_T^-(v) : v \in E_A \setminus X\}$.*
(iii) $2 \leq |B_i| \leq c$ *for all $i \in [t]$. Moreover, $T[B_i]$ is a transitive tournament whose tail has in-degree at least $n/3$ in T.*
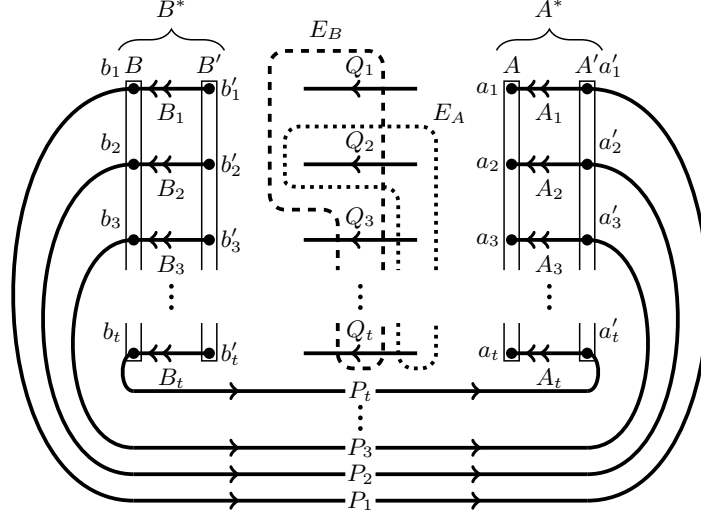
FIGURE 3. Our linked domination structure and path cover at the beginning of the proof of Lemma 7.2.

(iv) *There exists a set $E_B \subseteq V(T) \setminus (A^* \cup B^*)$, such that each B_i in-dominates $V(T) \setminus (A^* \cup B^* \cup E_B)$. Moreover, $|E_B| \leq d^+/40$, where $d^+ := \min\{d_T^+(v) : v \in E_B \setminus X\}$.*

(v) *For all $i \in [t]$, P_i is a path from the head of $T[B_i]$ to the tail of $T[A_i]$ which is internally disjoint from $A^* \cup B^*$. Moreover, $|P_1 \cup \cdots \cup P_t| \leq n/20$.*

(vi) *$F \subseteq E(P_1 \cup \cdots \cup P_t)$ and $V(F) \cap (A^* \cup B^*) = \emptyset$. Moreover, $F = \{e_v : v \in A^* \cup B^*\}$, where e_v is a covering edge for v and $e_v \neq e_{v'}$ whenever $v \neq v'$. In particular, $|F| = |A^* \cup B^*| \leq 2ct$.*

(vii) *We have $X \subseteq V(P_1 \cup \cdots \cup P_t)$, $X \cap (A^* \cup B^*) = \emptyset$ and $|X| \leq 2kt$.*

Then T contains a Hamilton cycle.

Proof. Without loss of generality, suppose that $d^- \leq d^+$. (Otherwise, reverse the orientation of every edge in T.) Write a_i for the head of $T[A_i]$ and a_i' for its tail. Similarly, write b_i for the head of $T[B_i]$ and b_i' for its tail. Let

$$A := \{a_1, \ldots, a_t\}, \quad A' := \{a_1', \ldots, a_t'\}, \quad B := \{b_1, \ldots, b_t\} \text{ and } B' := \{b_1', \ldots, b_t'\}.$$

Thus the sets A, A', B, B' are disjoint, and by condition (v) the paths P_i join B to A'. Let

$$N := V(T) \setminus (A^* \cup B^*), \quad T' := T[N \cup A' \cup B], \quad \text{and} \quad \mathcal{P}_2 := \{P_1, \ldots, P_t\}.$$

By Corollary 6.4, there exists a path cover $\mathcal{P}_1$ of $N \setminus V(\mathcal{P}_2)$ with $|\mathcal{P}_1| \leq 4k$. Then $\mathcal{Q}_1 := \mathcal{P}_1 \dot\cup \mathcal{P}_2$ is a path cover of T'. The situation is illustrated in Figure 3.

Claim. *There exists an oriented graph T'' with $T' \subseteq T'' \subseteq T[V(T') \cup A \cup B']$ and a path cover $\mathcal{Q}$ of T'' such that the following properties hold:*

(Q1) *$F \subseteq E(\mathcal{Q})$.*

(Q2) *$t(\mathcal{Q}) \cap E_A = \emptyset$.*

(Q3) $h(\mathcal{Q}) \cap E_B = \emptyset$.

(Q4) $|\mathcal{Q} \cap \mathcal{P}_2| \geq |\mathcal{Q}_1| - 20k$.

(Q5) If a_i or b_i' is in $V(\mathcal{Q})$, then $P_i \notin \mathcal{Q}$.

(Q6) $|\mathcal{Q}| \leq |\mathcal{Q}_1| + 124k$.

(Q7) No paths in $\mathcal{Q} \setminus \mathcal{P}_2$ have endpoints in $A^* \cup B^*$.

We will prove the claim by applying Lemmas 6.5 and 6.6 repeatedly to improve our current path cover. More precisely, we will construct path covers $\mathcal{Q}_2, \ldots, \mathcal{Q}_6$ such that eventually $\mathcal{Q}_6$ satisfies (Q1)–(Q7). So we can take $\mathcal{Q} := \mathcal{Q}_6$.

In order to be able to apply Lemmas 6.5 and 6.6, we must first bound the degrees of the vertices in T' from below. For all $v \in V(T')$, we have

$$(1) \quad d_{T'}^+(v) \geq d_T^+(v) - |A^* \cup B^*| \overset{(i),(iii)}{\geq} d_T^+(v) - 2ct \geq d_T^+(v) - \frac{\delta^0(T)}{5} \geq \frac{4}{5}d_T^+(v).$$

Similarly,

$$(2) \qquad\qquad d_{T'}^-(v) \geq \frac{4}{5}d_T^-(v)$$

for all $v \in V(T')$.

We will first extend the tails of paths in $\mathcal{Q}_1$ out of E_A. We do this by applying Lemma 6.6 to T' and $\mathcal{Q}_1 = \mathcal{P}_1 \dot{\cup} \mathcal{P}_2$ with $I := E_A \setminus X$, $J := X \cup A' \cup B$ to form a new path cover $\mathcal{Q}_2$ of T' which will satisfy (Q1) and (Q2). By conditions (ii) and (v), no paths in $\mathcal{P}_2$ have endpoints in I. By condition (vi), $F \subseteq E(\mathcal{Q}_1)$. Moreover,

$$\begin{aligned}
3(|I| + |J|) + 2|F| \quad &\leq \quad 3|E_A| + 3|X| + 3|A'| + 3|B| + 2|F| \\
(3) \qquad &\overset{(ii),(vii),(vi)}{\leq} \quad \frac{3}{40}d^- + 6kt + 6t + 4ct < \frac{4}{5}d^-.
\end{aligned}$$

In the final inequality we used the fact that $d^- \geq \delta^0(T) \geq Ck^2$. Thus for all $v \in I$ we have

$$d_{T'}^-(v) \overset{(2)}{\geq} \frac{4}{5}d_T^-(v) \overset{(ii)}{\geq} \frac{4}{5}d^- \overset{(3)}{>} 3(|I| + |J|) + 2|F|.$$

Thus the requirements of Lemma 6.6 are satisfied, and we can apply the lemma to obtain a path cover $\mathcal{Q}_2$ of T'.

Lemma 6.6(iv) implies that $\mathcal{Q}_2$ satisfies (Q1). Moreover, Lemma 6.6(v),(vi) imply that

$$(4) \quad |\mathcal{Q}_2| \leq |\mathcal{Q}_1| + 4k \qquad \text{as well as} \qquad |\mathcal{Q}_2 \cap \mathcal{P}_2| \geq |\mathcal{P}_2| - 4k \geq |\mathcal{Q}_1| - 8k,$$
$$\text{and thus} \qquad |\mathcal{Q}_2 \setminus \mathcal{P}_2| \leq 12k,$$

where we have used that $|\mathcal{Q}_1| = |\mathcal{P}_1| + |\mathcal{P}_2| \leq |\mathcal{P}_2| + 4k$ for the second inequality above. Recall from condition (vii) that $X \subseteq V(\mathcal{P}_2)$ and $X \cap (A^* \cup B^*) = \emptyset$. Thus no paths in $\mathcal{Q}_1$ have endpoints in X. Moreover, since $t(\mathcal{P}_2) = B$ and $h(\mathcal{P}_2) = A'$, no paths in $\mathcal{Q}_1$ have tails in A' or heads in B. Together with Lemma 6.6(i)–(iii) this implies that $\mathcal{Q}_2$ satisfies (Q2) and

(a$_1$) $t(\mathcal{Q}_2) \cap A' = h(\mathcal{Q}_2) \cap B = \emptyset$.

(a$_2$) $h(\mathcal{Q}_2) \cap X = \emptyset$.

We will now extend the heads of paths in $\mathcal{Q}_2$ out of E_B. We do this by applying Lemma 6.5 to T', $(\mathcal{Q}_2 \setminus \mathcal{P}_2) \dot{\cup} (\mathcal{Q}_2 \cap \mathcal{P}_2)$ with $I := E_B \setminus X$, $J :=$ $(E_A \setminus E_B) \cup X \cup A' \cup B$ to form a new path cover $\mathcal{Q}_3$ of T' which will satisfy (Q1)–(Q4). As before, no paths in $\mathcal{P}_2 \supseteq \mathcal{Q}_2 \cap \mathcal{P}_2$ have endpoints in I, and $F \subseteq E(\mathcal{Q}_2)$ by (Q1) for $\mathcal{Q}_2$. Moreover, similarly as in (3) we obtain

$$3(|I| + |J|) + 2|F| \leq 3|E_B| + 3|E_A| + 3|X| + 3|A'| + 3|B| + 2|F|$$

$$\leq \frac{3}{40}d^+ + \frac{3}{40}d^- + 6kt + 6t + 4ct < \frac{4}{5}d^+.$$

(In the final inequality we used our assumption that $d^- \leq d^+$.) Together with (1) this implies that $d_{T'}^+(v) \geq 4d^+/5 > 3(|I| + |J|) + 2|F|$ for all $v \in I$. Thus the requirements of Lemma 6.5 are satisfied, and we can apply the lemma to obtain a path cover $\mathcal{Q}_3$ of T'.

By Lemma 6.5(iv), $\mathcal{Q}_3$ satisfies (Q1). Lemma 6.5(v) implies that

$$(5) \qquad |\mathcal{Q}_3| \leq |\mathcal{Q}_2| + |\mathcal{Q}_2 \setminus \mathcal{P}_2| \overset{(4)}{\leq} |\mathcal{Q}_2| + 12k \overset{(4)}{\leq} |\mathcal{Q}_1| + 16k.$$

Similarly, Lemma 6.5(vi) implies that

$$(6) \qquad |\mathcal{Q}_3 \cap \mathcal{P}_2| \geq |\mathcal{Q}_2 \cap \mathcal{P}_2| - |\mathcal{Q}_2 \setminus \mathcal{P}_2| \overset{(4)}{\geq} |\mathcal{Q}_1| - 20k.$$

So $\mathcal{Q}_3$ satisfies (Q4). Lemma 6.5(iii) and (Q2) for $\mathcal{Q}_2$ together imply that $\mathcal{Q}_3$ satisfies (Q2). Moreover, (a_2) and Lemma 6.5(i),(ii) together imply that no path in $\mathcal{Q}_3$ has its head in $(E_B \setminus X) \cup X \supseteq E_B$ and so $\mathcal{Q}_3$ satisfies (Q3). Finally, (a_1) and Lemma 6.5(ii),(iii) together imply that

(b_1) no paths in $\mathcal{Q}_3$ have tails in A' or heads in B.

We will now extend the paths in $\mathcal{Q}_3 \setminus \mathcal{P}_2$ so that their endpoints lie in $A \cup B'$ rather than $A' \cup B$. More precisely, if $P \in \mathcal{Q}_3 \setminus \mathcal{P}_2$ has head $a_i' \in A'$, then we replace P by $Pa_i'a_i$ (recall that $a_i'a_i \in E(T)$ by condition (i) and $a_i \in A \subseteq V(T) \setminus V(\mathcal{Q}_3)$ by the definition of N). If $P \in \mathcal{Q}_3 \setminus \mathcal{P}_2$ has tail $b_i \in B$, we replace P by $b_i'b_iP$ (recall that $b_i'b_i \in E(T)$ by condition (iii) and $b_i' \in B' \subseteq V(T) \setminus V(\mathcal{Q}_3)$). Let $\mathcal{Q}_4$ be the path system thus obtained from $\mathcal{Q}_3$. Let $T'' := T[V(\mathcal{Q}_4)]$. Then

$$T' \subseteq T'' \subseteq T[V(T') \cup A \cup B'].$$

and $\mathcal{Q}_4$ is a path cover of T'' satisfying (Q1)–(Q4) and such that

$$(7) \qquad |\mathcal{Q}_4| = |\mathcal{Q}_3| \qquad \text{and} \qquad \mathcal{Q}_4 \cap \mathcal{P}_2 = \mathcal{Q}_3 \cap \mathcal{P}_2.$$

Moreover, $h(\mathcal{Q}_4 \setminus \mathcal{P}_2) \cap A' = \emptyset$ and $t(\mathcal{Q}_4 \setminus \mathcal{P}_2) \cap B = \emptyset$. Together with (b_1) this implies that

(c_1) no paths in $\mathcal{Q}_4 \setminus \mathcal{P}_2$ have endpoints in $A' \cup B$.

Moreover, by construction of $\mathcal{Q}_4$, every vertex $a_i \in V(\mathcal{Q}_4) \cap A$ is a head of some path $P \in \mathcal{Q}_4 \setminus \mathcal{P}_2$ and this path P also contains a_i' (so in particular $P_i \notin \mathcal{Q}_4 \cap \mathcal{P}_2$). Similarly, every vertex in $b_i' \in V(\mathcal{Q}_4) \cap B'$ is a tail of some path $P \in \mathcal{Q}_4 \setminus \mathcal{P}_2$ and this path P also contains b_i (in particular $P_i \notin \mathcal{Q}_4 \cap \mathcal{P}_2$). Thus (Q5) as well as the following assertion hold:

(c_2) no paths in $\mathcal{Q}_4$ have heads in B' or tails in A.

We will now extend the tails of paths in $\mathcal{Q}_4 \setminus \mathcal{P}_2$ out of $A^* \cup B^*$. We do this by applying the strengthened form of Lemma 6.6 to T'', $(\mathcal{Q}_4 \setminus \mathcal{P}_2) \dot\cup (\mathcal{Q}_4 \cap \mathcal{P}_2)$ with $I := B'$, $J := E_A \cup E_B \cup A' \cup A \cup B$ to form a new path cover $\mathcal{Q}_5$ of T'' which still satisfies (Q1)–(Q5), and such that no path in $\mathcal{Q}_5 \setminus \mathcal{P}_2$ has endpoints in $A' \cup B' \cup B$. Clearly no paths in $\mathcal{P}_2 \supseteq \mathcal{Q}_4 \cap \mathcal{P}_2$ have tails in I, and $F \subseteq E(\mathcal{Q}_4)$ by (Q1). By condition (iii) we have $d_T^-(v) \geq n/3$ for all $v \in I$. Together with (2) this implies that $d_{T''}^-(v) \geq d_{T'}^-(v) \geq n/4$ for all $v \in I$. Note also that $|V(\mathcal{P}_2)| \leq n/20$ by condition (v). So similarly as in (3), it follows that

$$3(|I| + |J|) + 2|F| + |V(\mathcal{Q}_4 \cap \mathcal{P}_2)|$$
$$\leq 3(|A'| + |A| + |B'| + |B| + |E_A| + |E_B|) + 2|F| + |V(\mathcal{P}_2)|$$
$$\leq 12t + \frac{3}{20}d^+ + 4ct + \frac{n}{20} < \frac{n}{4} \leq d_{T''}^-(v)$$

for all $v \in I$. Thus the requirements of the strengthened form of Lemma 6.6 are satisfied, and we can apply the lemma to obtain a path cover $\mathcal{Q}_5$ of T'' such that $\mathcal{Q}_5 \cap \mathcal{P}_2 \supseteq \mathcal{Q}_4 \cap \mathcal{P}_2$. Note that Lemma 6.6(ii),(iii) imply that the endpoints of $\mathcal{Q}_5 \setminus (\mathcal{P}_2 \cap \mathcal{Q}_4)$ in J are the same as those of $\mathcal{Q}_4 \setminus \mathcal{P}_2$. Together with (c_1) this implies that no paths in $\mathcal{Q}_5 \setminus (\mathcal{P}_2 \cap \mathcal{Q}_4)$ have endpoints in $A' \cup B$. In particular, this means that $\mathcal{Q}_5 \cap \mathcal{P}_2 = \mathcal{Q}_4 \cap \mathcal{P}_2$ and so

(d_1) no paths in $\mathcal{Q}_5 \setminus \mathcal{P}_2$ have endpoints in $A' \cup B$.

Thus (Q5) for $\mathcal{Q}_4$ implies that $\mathcal{Q}_5$ satisfies (Q5) as well. Lemma 6.6(ii)–(iv), (vi) (strengthened) and (Q1)–(Q4) for $\mathcal{Q}_4$ together imply that $\mathcal{Q}_5$ satisfies (Q1)–(Q4). Moreover, Lemma 6.6(v) implies that

$$|\mathcal{Q}_5| \quad \leq \quad |\mathcal{Q}_4| + |\mathcal{Q}_4 \setminus \mathcal{P}_2| \overset{(7)}{=} |\mathcal{Q}_3| + |\mathcal{Q}_3 \setminus \mathcal{P}_2| = 2|\mathcal{Q}_3| - |\mathcal{Q}_3 \cap \mathcal{P}_2|$$
$$(8) \qquad\qquad \overset{(5),(6)}{\leq} \quad |\mathcal{Q}_1| + 52k.$$

By Lemma 6.6(i),(ii) and (c_2), we can also strengthen (d_1) to

(d_2) no paths in $\mathcal{Q}_5 \setminus \mathcal{P}_2$ have endpoints in $A' \cup B' \cup B$ and no paths in $\mathcal{Q}_5$ have tails in A.

Finally, we will extend the heads of paths in $\mathcal{Q}_5 \setminus \mathcal{P}_2$ out of $A^* \cup B^*$. We do this by applying the strengthened form of Lemma 6.5 to T'', $(\mathcal{Q}_5 \setminus \mathcal{P}_2) \dot\cup (\mathcal{Q}_5 \cap \mathcal{P}_2)$ with $I := A$, $J := E_A \cup E_B \cup A' \cup B' \cup B$ to form a new path cover $\mathcal{Q}_6$ of T'' which will satisfy (Q1)–(Q7). Clearly no paths in $\mathcal{P}_2 \supseteq \mathcal{Q}_5 \cap \mathcal{P}_2$ have heads in I, and $F \subseteq E(\mathcal{Q}_5)$ by (Q1). Similarly as before, condition (i) and (1) together imply that

$$3(|I| + |J|) + 2|F| + |V(\mathcal{Q}_5) \cap \mathcal{P}_2| < \frac{n}{4} \leq d_{T''}^+(v)$$

for all $v \in I$. Thus the requirements of the strengthened form of Lemma 6.5 are satisfied, and we can apply the lemma to obtain a path cover $\mathcal{Q}_6$ of T'' such that $\mathcal{Q}_6 \cap \mathcal{P}_2 = \mathcal{Q}_5 \cap \mathcal{P}_2$. (The fact that we have equality follows using a similar argument as in (d_1) above.)

Thus (Q5) for $\mathcal{Q}_5$ implies that $\mathcal{Q}_6$ satisfies (Q5) as well. Lemma 6.5(ii)–(iv), (vi) (strengthened) and (Q1)–(Q4) for $\mathcal{Q}_5$ together imply that $\mathcal{Q}_6$ satisfies

(Q1)–(Q4). Also, by Lemma 6.5(v) we have

$$|\mathcal{Q}_6| \leq |\mathcal{Q}_5| + |\mathcal{Q}_5 \setminus \mathcal{P}_2| = 2|\mathcal{Q}_5| - |\mathcal{Q}_5 \cap \mathcal{P}_2| \overset{(Q4),(8)}{\leq} |\mathcal{Q}_1| + 124k.$$

So (Q6) holds. Moreover, by Lemma 6.5(i)–(iii), (d_2) and the fact that $\mathcal{Q}_6 \cap \mathcal{P}_2 = \mathcal{Q}_5 \cap \mathcal{P}_2$, no paths in $\mathcal{Q}_6 \setminus \mathcal{P}_2$ have endpoints in $A' \cup A \cup B' \cup B$. Since no vertex in $(A^* \cup B^*) \setminus (A' \cup A \cup B' \cup B)$ lies in $V(T'') = V(\mathcal{Q}_6)$, this in turn implies (Q7). So the path system $\mathcal{Q} := \mathcal{Q}_6$ is as required in the claim.

We will now use the fact that each A_i and each B_i is an almost dominating set in order to extend the paths in $\mathcal{Q} \setminus \mathcal{P}_2$ into those A_i and B_i which contain the endpoints of paths in $\mathcal{Q} \cap \mathcal{P}_2$. We then use the paths in $\mathcal{Q} \cap \mathcal{P}_2$ to join these extended paths into a long cycle C covering (at least) N, and with $F \subseteq E(C)$. Finally, we will deploy whatever covering edges we need from F in order to absorb any vertices in $A^* \cup B^*$ not already covered into C.

Let $\mathcal{R} := \mathcal{Q} \setminus \mathcal{P}_2$ and $\mathcal{S} := \mathcal{Q} \cap \mathcal{P}_2$. In order to carry out the steps above, we would like to have $|\mathcal{R}| = |\mathcal{S}|$ to avoid having any paths in $\mathcal{S}$ left over. So we first split the paths in $\mathcal{R}$ until we have exactly $|\mathcal{S}|$ of them. In this process, we wish to preserve (Q1)–(Q3), (Q5) and (Q7). To show that this can be done, first note that by (Q4) and (Q6), we have

$$|\mathcal{R}| = |\mathcal{Q} \setminus \mathcal{P}_2| \leq 144k = t - 20k \leq |\mathcal{Q}_1| - 20k \leq |\mathcal{Q} \cap \mathcal{P}_2| = |\mathcal{S}|.$$

The number of edges in $\mathcal{R}$ which are incident to vertices in $E_A \cup E_B \cup A^* \cup B^*$, or which belong to F, is bounded above by

$$2(|E_A| + |E_B| + |A^*| + |B^*|) + |F| \leq \frac{d^+}{10} + 6ct \leq \frac{n}{4}.$$

On the other hand,

$$|E(\mathcal{R})| = |V(\mathcal{R})| - |\mathcal{R}| \geq (n - |A^* \cup B^*| - |V(\mathcal{P}_2)|) - 144k$$
$$\geq n - 2ct - \frac{n}{20} - 144k \geq \frac{n}{2}.$$

Hence

$$|E(\mathcal{R})| - 2(|E_A| + |E_B| + |A^*| + |B^*|) - |F| \geq \frac{n}{4} > t \geq |\mathcal{S}|.$$

We may therefore form a path cover $\mathcal{R}'$ of $T[V(\mathcal{R})]$ with $|\mathcal{R}'| = |\mathcal{S}|$ by greedily removing edges of paths in $\mathcal{R}$ which are neither incident to $A^* \cup B^* \cup E_A \cup E_B$ nor elements of F. Then $\mathcal{R}' \cup \mathcal{S}$ satisfies (Q1)–(Q3), (Q5) and (Q7).

Next, we extend the paths in $\mathcal{R}'$ into $A^* \cup B^*$ and join them with the paths in $\mathcal{S}$ to form a long cycle C. By relabeling the P_i if necessary, we may assume that $\mathcal{S} = \{P_1, \ldots, P_\ell\}$. Let $R_1, \ldots, R_\ell$ denote the paths in $\mathcal{R}'$ and for each $j \in [\ell]$ let x_j be the tail of R_j and y_j the head of R_j. Recall from (Q2) and (Q7) that $x_j \notin A^* \cup B^* \cup E_A$. Hence by condition (ii) there exists $x'_j \in A_{j-1}$ with $x'_j x_j \in E(T)$, where the indices are understood to be modulo ℓ. Similarly $y_j \notin A^* \cup B^* \cup E_B$ by (Q3) and (Q7), so by condition (iv) there exists $y'_j \in B_j$ with $y_j y'_j \in E(T)$. Let $R'_j := x'_j x_j R_j y_j y'_j$. If $x'_j \neq a'_{j-1}$, then we extend R'_j by

adding the edge $a'_{j-1}x'_j$. Similarly, if $y'_j \neq b_j$ we extend R'_j by adding the edge $y'_j b_j$. In all cases, we still denote the resulting path from a'_{j-1} to b_j by R'_j.

Recall that P_j is a path from b_j to a'_j for all $j \in [\ell]$. Moreover, we have $x'_j, y'_j \notin V(\mathcal{Q} \setminus \mathcal{P}_2) = V(\mathcal{R}')$ for all $j \in [\ell]$. (Indeed, if $x'_j \neq a_j$ this follows since for the oriented graph T'' defined in the claim we have $V(T'') \cap A_i \subseteq \{a_i, a'_i\}$. If $x'_j = a_j$, this follows since $P_j \in \mathcal{Q}$ and so (Q5) implies that $a_j \notin V(\mathcal{Q})$. The argument for y'_j is similar.) Thus $R'_1, \ldots, R'_\ell$ are pairwise vertex-disjoint and internally disjoint from the paths in $\mathcal{S}$. So we can define a cycle C by

$$C := R'_1 P_1 R'_2 P_2 \ldots P_{\ell-1} R'_\ell P_\ell.$$

Note that $N \subseteq V(C)$ since $\mathcal{R}' \cup \mathcal{S}$ is a path cover of T'', and $F \subseteq E(C)$ by (Q1). Recall from condition (vi) that F consists of covering edges e_v for all $v \in A^* \cup B^*$ and that these e_v are pairwise distinct. Thus each e_v lies on C and so neither of the two activating edges of e_v can lie on C. Writing $e_v = x_v y_v$, it follows from these observations that we may form a new cycle C' by replacing $x_v y_v$ by $x_v v y_v$ in C for all $v \in (A^* \cup B^*) \setminus V(C)$. Then C' is a Hamilton cycle of T, as desired. $\square$

7. Finding many edge-disjoint Hamilton cycles in a good tournament

In the proof of Theorem 1.2, we will find the edge-disjoint Hamilton cycles in a given highly-linked tournament by repeatedly applying Lemma 6.7. In each application, we will need to set up all the dominating sets and paths required by Lemma 6.7. The following definition encapsulates this idea. (Recall that $\mathrm{Int}(P)$ denotes the interior of a path P.)

Definition 7.1. We say that a tournament T is (C, k, t, c)-*good* if it contains vertex sets $A_1^1, \ldots, A_k^t$, $B_1^1, \ldots, B_k^t$, $E_{A,1}, \ldots, E_{A,k}$, $E_{B,1}, \ldots, E_{B,k}$, edge sets $F_1, \ldots, F_k$, and paths $P_1^1, \ldots, P_k^t$ such that the following statements hold, where $A_i^* := A_i^1 \cup \cdots \cup A_i^t$, $A^* := A_1^* \cup \cdots \cup A_k^*$, $B_i^* := B_i^1 \cup \cdots \cup B_i^t$, and $B^* := B_1^* \cup \cdots \cup B_k^*$:

(G1) The sets $A_1^1, \ldots, A_k^t$ are disjoint and $2 \leq |A_i^\ell| \leq c$ for all $i \in [k]$ and $\ell \in [t]$. Moreover, each $T[A_i^\ell]$ is a transitive tournament whose head has out-degree at least $2n/5$ in T. Write $A := \{h(T[A_i^\ell]) : i \in [k], \ell \in [t]\}$.

(G2) The sets $B_1^1, \ldots, B_k^t$ are disjoint from each other and from A^*, and $2 \leq |B_i^\ell| \leq c$ for all $i \in [k]$ and $\ell \in [t]$. Moreover, each $T[B_i^\ell]$ is a transitive tournament whose tail has in-degree at least $2n/5$ in T. Write $B' := \{t(T[B_i^\ell]) : i \in [k], \ell \in [t]\}$.

(G3) Write $d_- := \min\{d^-(v) : v \in V(T) \setminus (A \cup B')\}$. Each A_i^ℓ out-dominates $V(T) \setminus (A^* \cup B^* \cup E_{A,i})$. Moreover, $|E_{A,i}| \leq d_-/50$ and $E_{A,i} \cap (A_i^* \cup B_i^*) = \emptyset$ for all $i \in [k]$.

(G4) Write $d_+ := \min\{d^+(v) : v \in V(T) \setminus (A \cup B')\}$. Each B_i^ℓ in-dominates $V(T) \setminus (A^* \cup B^* \cup E_{B,i})$. Moreover, $|E_{B,i}| \leq d_+/50$ and $E_{B,i} \cap (A_i^* \cup B_i^*) = \emptyset$ for all $i \in [k]$.

(G5) Each P_i^ℓ is a path from the head of $T[B_i^\ell]$ to the tail of $T[A_i^\ell]$. For each $i \in [k]$, the paths $P_i^1, \ldots, P_i^t$ are vertex-disjoint and $|P_i^1 \cup \cdots \cup P_k^t| \leq n/20$. For all $i \neq j$ and all $\ell, m \in [t]$, P_i^ℓ and P_j^m are edge-disjoint and

$$V(\mathrm{Int}(P_i^\ell)) \cap (A^* \cup B^*) \subseteq (A \cup B') \setminus (A_i^* \cup B_i^*).$$

(G6) $F_i \subseteq E(P_i^t)$ and $(A \cup B') \setminus (A_i^* \cup B_i^*) \subseteq V(P_i^t)$ for all $i \in [k]$.

(G7) The set $F_1 \cup \cdots \cup F_k$ is a matching in $T - (A^* \cup B^*)$. For all $i \in [k]$ we have $F_i = \{e_v : v \in A_i^* \cup B_i^*\}$, where e_v is a covering edge for v and $e_v \neq e_{v'}$ whenever $v \neq v'$. Moreover, for each $i \in [k]$, let F_i^{act} be the set of activating edges corresponding to the covering edges in F_i. Then $F_i^{\mathrm{act}} \cap E(P_j^\ell) = \emptyset$ for all $i, j \in [k]$ and all $\ell \in [t]$.

(G8) We have $\delta^0(T) \geq Ck^2 \log k$.

For convenience, we collect the various disjointness conditions of Definition 7.1 into a single statement.

(G9)   • The sets $A_1^1, \ldots, A_k^t$, $B_1^1, \ldots, B_k^t$ are disjoint.
      • $(E_{A,i} \cup E_{B,i}) \cap (A_i^* \cup B_i^*) = \emptyset$ for all $i \in [k]$.
      • $F_1 \cup \cdots \cup F_k$ is a matching in $T - (A^* \cup B^*)$.
      • For each $i \in [k]$, the paths $P_i^1, \ldots, P_i^t$ are vertex-disjoint.
      • For all $i \neq j$ and all $\ell, m \in [t]$, P_i^ℓ and P_j^m are edge-disjoint and $V(\mathrm{Int}(P_i^\ell)) \cap (A^* \cup B^*) \subseteq (A \cup B') \setminus (A_i^* \cup B_i^*)$. In particular, $P_i^1, \ldots, P_i^t$ are internally disjoint from $A_i^* \cup B_i^*$.

The next lemma shows that for suitable parameters C, $t = t(k)$ and $c = c(k)$, every (C, k, t, c)-good tournament contains k edge-disjoint Hamilton cycles. In the next section we then show that there exists a constant $C' > 0$ such that any $C'k^2 \log k$-linked tournament is (C, k, t, c)-good (see Lemma 8.7). These two results together immediately imply Theorem 1.2.

As mentioned at the beginning of this section, in order to prove Lemma 7.2 we will apply Lemma 6.7 k times. In the notation for Definition 7.1, our convention is that the sets with subscript i will be used in the ith application of Lemma 6.7 to find the ith Hamilton cycle.

Lemma 7.2. *Let* $C := 10^7$, $k \geq 20$, $t := 164k$, $c := \lceil \log 50t + 1 \rceil$. *Then any* (C, k, t, c)-*good tournament contains* k *edge-disjoint Hamilton cycles.*

Proof.   Let T be a (C, k, t, c)-good tournament, and let $n := |T|$. Let $A_1^1, \ldots, A_k^t$, $B_1^1, \ldots, B_k^t$, $E_{A,1}, \ldots, E_{A,k}$, $E_{B,1}, \ldots, E_{B,k}$, $F_1, \ldots, F_k$, $P_1^1, \ldots, P_k^t$, d_- and d_+ be as in Definition 7.1. (Note that this also implicitly defines sets $A_1^*, \ldots, A_k^*$, A^*, A, $B_1^*, \ldots, B_k^*$, B^*, B', and $F_1^{\mathrm{act}}, \ldots, F_k^{\mathrm{act}}$ as in Definition 7.1.) Our aim is to apply Lemma 6.7 repeatedly to find k edge-disjoint Hamilton cycles. So suppose that for some $i \in [k]$ we have already found edge-disjoint Hamilton cycles $C_1, \ldots, C_{i-1}$ such that the following conditions hold:

(a) $C_1, \ldots, C_{i-1}$ are edge-disjoint from $T[A_j^\ell], T[B_j^\ell]$ and P_j^ℓ for all $i \leq j \leq k$ and all $\ell \in [t]$.

(b) $E(C_1 \cup \cdots \cup C_{i-1}) \cap F_j^{\mathrm{act}} = \emptyset$ for all $i \leq j \leq k$.

Intuitively, these conditions guarantee that none of the edges we will need in order to find $C_i, \ldots, C_k$ are contained in $C_1, \ldots, C_{i-1}$. We have to show that $T - C_1 - \cdots - C_{i-1}$ contains a Hamilton cycle C_i which satisfies (a) and (b) (with i replaced by $i+1$).

Define

$$T_i := T - \left(\bigcup_{j<i} C_j \cup \bigcup_{j>i} F_j^{\mathrm{act}} \right) - \bigcup_{j>i,\ \ell \in [t]} (P_j^\ell \cup T[A_j^\ell] \cup T[B_j^\ell]),$$

$$E'_{A,i} := E_{A,i} \cup \left(\left(\bigcup_{j<i} N_{C_j}^+(A_i^*) \cup \bigcup_{j>i,\ \ell \in [t]} N_{P_j^\ell}^+(A_i^*) \cup A^* \cup B^* \right) \setminus (A_i^* \cup B_i^*) \right),$$

$$E'_{B,i} := E_{B,i} \cup \left(\left(\bigcup_{j<i} N_{C_j}^-(B_i^*) \cup \bigcup_{j>i,\ \ell \in [t]} N_{P_j^\ell}^-(B_i^*) \cup A^* \cup B^* \right) \setminus (A_i^* \cup B_i^*) \right),$$

$$X_i := (A \cup B') \setminus (A_i^* \cup B_i^*).$$

Then it suffices to find a Hamilton cycle C_i of T_i. We will do so by applying Lemma 6.7 to T_i, $A_i^1, \ldots, A_i^t$, $B_i^1, \ldots, B_i^t$, $P_i^1, \ldots, P_i^t$, $E'_{A,i}$, $E'_{B,i}$, F_i and X_i. It therefore suffices to verify that the conditions of Lemma 6.7 hold.

We claim that for each $v \in V(T_i)$, we have

$$(9) \qquad d_{T_i}^+(v) \geq d_T^+(v) - (i-1) - (k-i) - 1 - c > d_T^+(v) - 2k.$$

Indeed, it is immediate that $d_{C_1 \cup \cdots \cup C_{i-1}}^+(v) = i-1$. Since by (G9) for each $j > i$ the paths $P_j^1, \ldots, P_j^t$ are vertex-disjoint, v is covered by at most $k-i$ of the paths $P_{i+1}^1, \ldots, P_k^t$ and hence $d_{P_{i+1}^1 \cup \cdots \cup P_k^t}^+(v) \leq k-i$. Recall from (G7) that $F_1 \cup \cdots \cup F_k$ consists of one covering edge e_v for each $v \in A^* \cup B^*$. Moreover, by (G9) the set $F_1 \cup \cdots \cup F_k$ is a matching in $T - (A^* \cup B^*)$ and $A_1^1, \ldots, A_k^t, B_1^1, \ldots, B_k^t$ are all disjoint. Thus the digraph with edge set $F_1^{\mathrm{act}} \cup \cdots \cup F_k^{\mathrm{act}}$ is a disjoint union of directed paths of length two and therefore has maximum out-degree one. Finally, since $A_1^1, \ldots, A_k^t, B_1^1, \ldots, B_k^t$ are disjoint, v belongs to at most one of $T[A_1^1], \ldots, T[A_k^t], T[B_1^1], \ldots, T[B_k^t]$. Moreover, $\Delta^+(T[A_j^\ell]), \Delta^+(T[B_j^\ell]) \leq c$ for all $j > i$ and all $\ell \in [t]$ by (G1) and (G2). So (9) follows. Similarly, we have

$$(10) \qquad\qquad\qquad\qquad d_{T_i}^-(v) > d_T^-(v) - 2k.$$

In particular, $\delta(T_i) > n - 4k$, as required by Lemma 6.7.

We have $\delta^0(T) > Ck^2$ by (G8), and hence $\delta^0(T_i) > 10^6 k^2$ as required by Lemma 6.7. The disjointness conditions of Lemma 6.7 are satisfied by (G9) and the definition of X_i. Since $V(T_i) = V(T)$, it is immediate that $A_i^1, \ldots, A_i^t, B_i^1, \ldots, B_i^t, X_i \subseteq V(T_i)$. We claim that $P_i^1, \ldots, P_i^t \subseteq T_i$. Indeed, by (a) and (G5), each P_i^ℓ is edge-disjoint from $C_1 \cup \cdots \cup C_{i-1}$ and from P_j^m for all $j > i$ and all $m \in [t]$. By (G7), each P_i^ℓ is edge-disjoint from $F_1^{\mathrm{act}} \cup \cdots \cup F_k^{\mathrm{act}}$. Moreover, by (G5), each P_i^ℓ is edge-disjoint from $T[A_j^m] \cup T[B_j^m]$ for all $j > i$ and all $m \in [t]$. Altogether this implies that $P_i^1, \ldots, P_i^t \subseteq T_i$. We have

$F_i \subseteq E(P_i^t) \subseteq E(T_i)$ by (G6). It therefore suffices to prove that conditions (i)–(vii) of Lemma 6.7 hold.

Condition (v) follows from (G5). Condition (vi) follows from (G6) and (G7). (Note that (G7) implies that $F_i^{\mathrm{act}} \cap F_j^{\mathrm{act}} = \emptyset$ for all $i \neq j$. So (G7), (b) and the definition of T_i imply that $F_i^{\mathrm{act}} \subseteq T_i$.) By (G6) we have $X_i \subseteq V(P_i^t)$ and by (G1) and (G2) we have $|X_i| \leq |A \cup B'| = 2kt$, so condition (vii) holds too.

It therefore remains to verify conditions (i)–(iv). We first check (i). We have $2 \leq |A_i^\ell| \leq c$ by (G1). Moreover, we claim that $T_i[A_i^\ell] = T[A_i^\ell]$ for all $\ell \in [t]$. Indeed, to see this, note that $C_1, \ldots, C_{i-1}$ are edge-disjoint from $T[A_i^\ell]$ by (a); by (G9) for all $j > i$ and all $m \in [t]$ each path P_j^m and each $T[A_j^m]$, $T[B_j^m]$ is edge-disjoint from $T[A_i^\ell]$; by (G7) all edges in F_j^{act} for $j > i$ are incident to a vertex in $A_j^* \cup B_j^*$, and hence by (G9) none of these edges belongs to $T[A_i^\ell]$. Thus $T_i[A_i^\ell] = T[A_i^\ell]$ is a transitive tournament by (G1). Finally, by (G1) the head of each $T[A_i^\ell]$ has out-degree at least $2n/5$ in T, and so by (9) out-degree at least $n/3$ in T_i. Hence condition (i) of Lemma 6.7 is satisfied. A similar argument shows that condition (iii) of Lemma 6.7 is also satisfied.

We will next verify that condition (ii) of Lemma 6.7 holds too. (G9) and the definition of $E'_{A,i}$ together imply that $E'_{A,i} \cap (A_i^* \cup B_i^*) = \emptyset$. By (G3), each A_i^ℓ out-dominates $V(T) \setminus (A^* \cup B^* \cup E_{A,i})$ in T, and hence out-dominates $V(T_i) \setminus (A^* \cup B^* \cup E_{A,i} \cup N_{T-T_i}^+(A_i^*))$ in T_i. However, it follows from (G9) that for all $j > i$ and all $\ell, m \in [t]$, no edge in F_j^{act} has an endpoint in A_i^ℓ and that $A_i^\ell \cap A_j^m = A_i^\ell \cap B_j^m = \emptyset$. Hence by (G9) we have that

$$N_{T-T_i}^+(A_i^*) = \bigcup_{j<i} N_{C_j}^+(A_i^*) \cup \bigcup_{j>i,\ \ell \in [t]} N_{P_j^\ell}^+(A_i^*).$$

It therefore follows from the definitions of $E'_{A,i}$ and T_i that A_i^ℓ out-dominates $V(T_i) \setminus (A_i^* \cup B_i^* \cup E'_{A,i})$ in T_i for all $\ell \in [t]$.

So in order to check that condition (ii) of Lemma 6.7 holds, it remains only to bound $|E'_{A,i}|$ from above. To do this, first note that by (G9), each vertex in A_i^* is contained in at most $k-i$ of the paths $P_{i+1}^1, \ldots, P_k^t$. Moreover, $|E_{A,i}| \leq d_-/50$ by (G3). It therefore follows from the definition of $E'_{A,i}$, (G1) and (G2) that

$$|E'_{A,i}| \leq |E_{A,i}| + \left| \bigcup_{j<i} N_{C_i}^+(A_i^*) \right| + \left| \bigcup_{j>i,\ \ell \in [t]} N_{P_j^\ell}^+(A_i^*) \right| + |A^*| + |B^*|$$

$$\leq \frac{d_-}{50} + (i-1)|A_i^*| + (k-i)|A_i^*| + 2kct \leq \frac{d_-}{50} + kct + 2kct \leq \frac{d_-}{45}.$$

The last inequality follows since $d_- \geq \delta^0(T) \geq Ck^2 \log k$ by (G8). Since $E'_{A,i}$ is disjoint from $A_i^* \cup B_i^*$, we have $E'_{A,i} \setminus X_i = E'_{A,i} \setminus (A \cup B')$. Hence for all $v \in E'_{A,i} \setminus X_i$ we have

$$d_{T_i}^-(v) \overset{(10)}{\geq} d_T^-(v) - 2k \overset{(G3)}{\geq} d_- - 2k \geq \frac{19}{20}d_-$$

and so

$$|E'_{A,i}| \leq \frac{d_-}{45} \leq \frac{1}{40} \min\{d^-_{T_i}(v) : v \in E'_{A,i} \setminus X_i\}.$$

This shows that condition (ii) of Lemma 6.7 is satisfied. The argument that (iv) holds is similar. We may therefore apply Lemma 6.7 to find a Hamilton cycle C_i in T_i as desired. $\qquad\square$

8. Highly-linked tournaments are good

The aim of this section is to prove that any sufficiently highly-linked tournament is (C, k, t, c)-good. We first show that it is very easy to find covering edges for any given vertex – we will use the following lemma to find matchings $F_1, \ldots, F_k$ consisting of covering edges as in Definition 7.1.

Lemma 8.1. *Suppose that T is a strongly 2-connected tournament, and $v \in V(T)$. Then there exists a covering edge for v.*

Proof. Since T is strongly connected and $|T| > 1$, we have $N^+(v), N^-(v) \neq \emptyset$. Since $T - v$ is strongly connected, there is an edge xy from $N^-(v)$ to $N^+(v)$. But then $xv, vy \in E(T)$, so xy is a covering edge for v, as desired. $\qquad\square$

The next lemma will be used to obtain paths $P_1^1, \ldots, P_k^t$ as in Definition 7.1. Recall that we require $F_i \subseteq E(P_i^t)$ and $(A \cup B') \setminus (A_i^* \cup B_i^*) \subseteq V(P_i^t)$ for all $i \in [k]$. We will ensure the latter requirement by first covering $(A \cup B') \setminus (A_i^* \cup B_i^*)$ with few paths and then linking these paths together – hence the form of the lemma.

Lemma 8.2. *Let $s \in \mathbb{N}$, and let T be a digraph. Let $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ be distinct vertices of T, and let $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ be (possibly empty) path systems in $T - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ with $E(\mathcal{Q}_i) \cap E(\mathcal{Q}_j) = \emptyset$ whenever $i \neq j$. Write*

$$(11) \qquad m := k + \sum_{i=1}^{k} |\mathcal{Q}_i| + \left| \bigcup_{i=1}^{k} V(\mathcal{Q}_i) \right|,$$

and suppose that T is $2sm$-linked. Then there exist edge-disjoint paths $P_1, \ldots, P_k \subseteq T$ satisfying the following properties:

   (i) *P_i is a path from x_i to y_i for all $i \in [k]$.*
   (ii) *$Q \subseteq P_i$ for all $Q \in \mathcal{Q}_i$ and all $i \in [k]$.*
   (iii) *$V(P_i) \cap V(P_j) \subseteq V(\mathcal{Q}_i) \cap V(\mathcal{Q}_j)$ for all $i \neq j$.*
   (iv) *$|P_1 \cup \cdots \cup P_k| \leq |T|/s + |V(\mathcal{Q}_1) \cup \cdots \cup V(\mathcal{Q}_k)|$.*

Proof. For all $i \in [k]$, let $a_i^1 \ldots b_i^1, \ldots, a_i^{t_i} \ldots b_i^{t_i}$ denote the paths in $\mathcal{Q}_i$. Let $F \subseteq E(T)$ denote the set of all those edges which form a path of length one in $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_k$. Let

$$T' := T\left[\left(V(T) \setminus \bigcup_{i=1}^{k} V(\mathcal{Q}_i) \right) \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{t_i} \{a_i^j, b_i^j\} \right] - F.$$

Note that $E(T') \cap (E(\mathcal{Q}_1) \cup \cdots \cup E(\mathcal{Q}_k)) = \emptyset$. Define sets $X_1, \ldots, X_k$ of ordered pairs of vertices of T' by

$$X_i := \begin{cases} \{(x_i, a_i^1), (b_i^1, a_i^2), \ldots, (b_i^{t_i-1}, a_i^{t_i}), (b_i^{t_i}, y_i)\}, & \text{if } \mathcal{Q}_i \neq \emptyset, \\ \{(x_i, y_i)\} & \text{if } \mathcal{Q}_i = \emptyset, \end{cases}$$

and let $X := X_1 \cup \cdots \cup X_k$. Let $\ell := 2sm - 2s|X|$. Since $|V(T) \setminus V(T')| + |F| \leq |V(\mathcal{Q}_1) \cup \cdots \cup V(\mathcal{Q}_k)|$ and $|X| = k + \sum_{i=1}^{k} |\mathcal{Q}_i|$, it follows that

$$2\ell = 4s(m - |X|) \stackrel{(11)}{=} 4s \left| \bigcup_{i=1}^{k} V(\mathcal{Q}_i) \right| \geq |V(T) \setminus V(T')| + 2|F|.$$

Thus by Proposition 4.7, T' is $2s|X|$-linked. We may therefore apply Lemma 4.8 to X in order to obtain, for each $i \in [k]$, a path system $\mathcal{P}_i$ whose paths link the pairs in X_i and such that whenever $i \neq j$, we have $E(\mathcal{P}_i) \cap E(\mathcal{P}_j) = \emptyset$ and $V(\mathcal{P}_i) \cap V(\mathcal{P}_j)$ consists of exactly the vertices that lie in a pair in both X_i and X_j. Let P_i be the path obtained from the union of all paths in $\mathcal{P}_i$ and all paths in $\mathcal{Q}_i$. Then $P_1, \ldots, P_k$ are edge-disjoint paths satisfying (i)–(iv). $\qquad \square$

The next lemma shows that given a vertex v in a tournament T, we can find a small transitive subtournament whose head is v and which out-dominates almost all vertices of T.

Lemma 8.3. *Let T be a tournament on n vertices, let $v \in V(T)$, and suppose that $c \in \mathbb{N}$ satisfies $2 \leq c \leq \log d^-(v) - 1$. Then there exist disjoint sets $A, E \subseteq V(T)$ such that the following properties hold:*
   (i) *$2 \leq |A| \leq c$ and $T[A]$ is a transitive tournament with head v.*
   (ii) *A out-dominates $V(T) \setminus (A \cup E)$.*
   (iii) *$|E| \leq (1/2)^{c-1} d^-(v)$.*

The fact that the bound in (iii) depends on $d^-(v)$ is crucial: for instance, we can apply Lemma 8.3 with v being the vertex of lowest in-degree. Then (iii) implies that the 'exceptional set' $|E|$ is much smaller than $d^-(v) \leq d^-(w)$ for any $w \in E$. So while w is not dominated by A directly, it is dominated by many vertices outside E. This will make it possible to cover E by paths whose endpoints lie outside E. (More formally, the lemma is used to ensure (G3), which in turn is used for (Q2) in the proof of Lemma 6.7).

Proof. Let $v_1 := v$. We will find A by repeatedly choosing vertices $v_1, \ldots, v_i$ such that the size of their common in-neighbourhood is minimised at each step. More precisely, let $A_1 := \{v_1\}$. Suppose that for some $i < c$ we have already found a set $A_i = \{v_1, \ldots, v_i\}$ such that $T[A_i]$ is a transitive tournament with head v_1, and such that the common in-neighbourhood E_i of $v_1, \ldots, v_i$ satisfies

$$|E_i| \leq \frac{1}{2^{i-1}} d^-(v).$$

Note that these conditions are satisfied for $i = 1$. Moreover, note that E_i is the set of all those vertices in $T - A_i$ which are not out-dominated by A_i. If

$|E_i| < 4$, then we have

$$
(12) \qquad |E_i| < 4 = \frac{1}{2^{\log d^-(v)-2}} d^-(v) \leq \frac{1}{2^{c-1}} d^-(v),
$$

and so A_i satisfies (i)–(iii). (Note that $|A_i| \geq 2$ since the assumptions imply that $d^-(v) \geq 8$.) Thus in this case we can take $A := A_i$ and $E := E_i$.

So suppose next that $|E_i| \geq 4$. In this case we will extend A_i to A_{i+1} by adding a suitable vertex v_{i+1}. By Proposition 6.1, E_i contains a vertex v_{i+1} of in-degree at most $|E_i|/2$ in $T[E_i]$. Let $A_{i+1} := \{v_1, \ldots, v_{i+1}\}$ and let E_{i+1} be the common in-neighbourhood of $v_1, \ldots, v_{i+1}$. Then $T[A_{i+1}]$ is a transitive tournament with head v_1 and

$$
|E_{i+1}| \leq \frac{1}{2}|E_i| \leq \frac{1}{2^i} d^-(v).
$$

By repeating this construction, either we will find $|E_i| < 4$ for some $i < c$ (and therefore take $A := A_i$ and $E := E_i$) or we will obtain sets A_c and E_c satisfying (i)–(iii). $\qquad \square$

We will also need the following analogue of Lemma 8.3 for in-dominating sets. It immediately follows from Lemma 8.3 by reversing the orientations of all edges.

Lemma 8.4. *Let T be a tournament on n vertices, let $v \in V(T)$, and suppose that $c \in \mathbb{N}$ satisfies $2 \leq c \leq \log d^+(v) - 1$. Then there exist disjoint sets $B, E \subseteq V(T)$ such that the following properties hold:*

   (i) *$2 \leq |B| \leq c$ and $T[B]$ is a transitive tournament with tail v.*
   (ii) *B in-dominates $V(T) \setminus (B \cup E)$.*
   (iii) *$|E| \leq (1/2)^{c-1} d^+(v)$.*

We will now apply Lemma 8.3 repeatedly to obtain many pairwise disjoint small almost-out-dominating sets. We will also prove an analogue for in-dominating sets. These lemmas will be used in order to obtain sets $A_1^1, \ldots, A_k^t$, $B_1^1, \ldots, B_k^t$, $E_{A,1}, \ldots, E_{A,k}$ and $E_{B,1}, \ldots, E_{B,k}$ as in Definition 7.1.

Lemma 8.5. *Let T be a tournament on n vertices, $U \subseteq V(T)$ and $c \in \mathbb{N}$ with $c \geq 2$. Suppose that $\delta^-(T) \geq 2^{c+1} + c|U|$. Then there exist families $\{A_v : v \in U\}$ and $\{E_v : v \in U\}$ of subsets of $V(T)$ such that the following properties hold:*

   (i) *A_v out-dominates $V(T) \setminus (E_v \cup \bigcup_{u \in U} A_u)$ for all $v \in U$.*
   (ii) *$T[A_v]$ is a transitive tournament with head v for all $v \in U$.*
   (iii) *$|E_v| \leq (1/2)^{c-1} d^-(v)$ for all $v \in U$.*
   (iv) *$2 \leq |A_v| \leq c$ for all $v \in U$.*
   (v) *$A_u \cap E_v = \emptyset$ for all $u, v \in U$.*
   (vi) *$A_u \cap A_v = \emptyset$ for all $u \neq v$.*

Proof. We repeatedly apply Lemma 8.3. Suppose that for some $U' \subseteq U$ with $U' \neq U$ we have already found $\{A_u : u \in U'\}$ and $\{E_u' : u \in U'\}$ satisfying (ii)–(vi) (with U' playing the role of U and E_u' playing the role of E_u) such that

   (a) A_v out-dominates $V(T) \setminus (\bigcup_{u \in U'} A_u \cup E_v' \cup U)$ for all $v \in U'$;
   (b) $(\bigcup_{u \in U'} A_u) \cap U = U'$.

Pick $v \in U \setminus U'$. Our aim is to apply Lemma 8.3 to v and

$$T' := T - \left(\bigcup_{u \in U'} A_u \cup (U \setminus \{v\}) \right).$$

Note that $v \in V(T')$ by (b). Moreover,

$$d_{T'}^-(v) \geq \delta^-(T') \overset{\text{(iv)}}{\geq} \delta^-(T) - c|U'| - |U \setminus U'| \geq \delta^-(T) - c|U| \geq 2^{c+1},$$

where the final inequality holds by hypothesis, and so $c \leq \log d_{T'}^-(v) - 1$. Hence we can apply Lemma 8.3 to obtain disjoint sets $A_v, E_v \subseteq V(T')$ as described there. For all $u \in U'$, let $E_u := E_u' \setminus A_v$. Then the collections $\{A_u : u \in U' \cup \{v\}\}$ and $\{E_u : u \in U' \cup \{v\}\}$ satisfy (v) and (vi) (with $U' \cup \{v\}$ playing the role of U). Moreover, (b) holds too (with $U' \cup \{v\}$ playing the role of U'). Conditions (i)–(iii) of Lemma 8.3 imply that (a) holds (with $U' \cup \{v\}$, E_u playing the roles of U', E_u') and that (ii)–(iv) hold (with $U' \cup \{v\}$ playing the role of U).

We continue in this way to obtain sets $\{A_u : u \in U\}$ and $\{E_u : u \in U\}$ which satisfy (ii)–(vi) as well as (a) (with U, E_u playing the roles of U', E_u'). But (a) implies (i) since $\bigcup_{u \in U} A_u \cup U = \bigcup_{u \in U} A_u$ (as $u \in A_u$ by (ii)). $\qquad \square$

The next lemma is an analogue of Lemma 8.5 for in-dominating sets. The proof is similar to that of Lemma 8.5.

Lemma 8.6. *Let T be a tournament on n vertices, $U \subseteq V(T)$ and $c \in \mathbb{N}$ with $c \geq 2$. Suppose that $\delta^+(T) \geq 2^{c+1} + c|U|$. Then there exist families $\{B_v : v \in U\}$ and $\{E_v : v \in U\}$ of subsets of $V(T)$ such that the following properties hold:*

- *(i) B_v in-dominates $V(T) \setminus (E_v \cup \bigcup_{u \in U} B_u)$ for all $v \in U$.*
- *(ii) $T[B_v]$ is a transitive tournament with tail v for all $v \in U$.*
- *(iii) $|E_v| \leq (1/2)^{c-1} d^+(v)$ for all $v \in U$.*
- *(iv) $2 \leq |B_v| \leq c$ for all $v \in U$.*
- *(v) $B_u \cap E_v = \emptyset$ for all $u, v \in U$.*
- *(vi) $B_u \cap B_v = \emptyset$ for all $u \neq v$.*

We will now combine the previous results in order to prove that any sufficiently highly-linked tournament is (C, k, t, c)-good. Note that Lemmas 7.2 and 8.7 together imply Theorem 1.2.

Lemma 8.7. *Let $C := 10^7$, $k \geq 20$, $t := 164k$ and $c := \lceil \log 50t + 1 \rceil$. Then any $Ck^2 \log k$-linked tournament is (C, k, t, c)-good.*

Proof. Let T be a $Ck^2 \log k$-linked tournament, and let $n := |T|$. Note in particular that $\delta^0(T) \geq Ck^2 \log k$ by Proposition 4.6, so (G8) is satisfied. We have to choose $A_1^1, \ldots, A_k^t$, $B_1^1, \ldots, B_k^t$, $E_{A,1}, \ldots, E_{A,k}$, $E_{B,1}, \ldots, E_{B,k}$, $F_1, \ldots, F_k$ and $P_1^1, \ldots, P_k^t$ satisfying (G1)–(G7) of Definition 7.1.

Construct a set $A \subseteq V(T)$ by greedily choosing kt vertices of least possible in-degree in T, and likewise construct a set $B' \subseteq V(T)$ by greedily choosing kt vertices of least possible out-degree in T. Note that by choosing the vertices

in A and B' suitably, we may assume that $A \cap B' = \emptyset$. (Since $n \geq \delta^0(T) \geq 2kt$, this is indeed possible.) Define

$$d_- := \min\{d^-(v) : v \in V(T) \setminus (A \cup B')\},$$
$$d_+ := \min\{d^+(v) : v \in V(T) \setminus (A \cup B')\}.$$

Note that $d^-(a) \leq d_-$ for all $a \in A$ and $d^+(b) \leq d_+$ for all $b \in B'$.

Our first aim is to choose the sets $A_1^1, \ldots, A_k^t$ using Lemma 8.5. Partition A arbitrarily into sets $A_1, \ldots, A_k$ of size t, and write $A_i =: \{a_i^1, \ldots, a_i^t\}$. Since $|B'| = kt \leq \delta^0(T)/2$, we have

$$2^{c+1} + c|A| \leq 400t + ckt \leq \frac{C}{2}k^2 \log k \leq \delta^-(T) - |B'| \leq \delta^-(T - B').$$

Thus we can apply Lemma 8.5 to $T - B'$, A and c in order to obtain almost out-dominating sets $A_i^\ell \ni a_i^\ell$ and corresponding exceptional sets $E_{A,i}^\ell$ as in the statement of Lemma 8.5 (for all $i \in [k]$ and all $\ell \in [t]$). Write $A_i^* := A_i^1 \cup \cdots \cup A_i^t$ and $A^* := A_1^* \cup \cdots \cup A_k^*$.

Let us now verify (G1). By Lemma 8.5(ii), (iv) and (vi), each $T[A_i^\ell]$ is a transitive tournament with head a_i^ℓ, $2 \leq |A_i^\ell| \leq c$, and the sets $A_1^1, \ldots, A_k^t$ are all disjoint. In particular, $A = \{h(A_i^\ell) : i \in [k], \ell \in [t]\}$. We claim in addition that $d^+(a_i^\ell) \geq 2n/5$. Indeed, Proposition 6.2 implies that T has at most $4n/5 + 1$ vertices of out-degree at most $2n/5$, and hence at least $n/5 - 1$ vertices of out-degree at least $2n/5$. Moreover,

$$|A| = kt \leq \frac{Ck^2 \log k}{5} - 1 \leq \frac{n}{5} - 1.$$

So since the vertices of A were chosen to have minimal in-degree in T, it follows that $d^+(a_i^\ell) \geq 2n/5$ for all $i \in [k]$ and all $\ell \in [t]$. Thus (G1) holds.

We will next apply Lemma 8.6 in order to obtain the sets $B_1^1, \ldots, B_k^t$. To do this, we first partition B' arbitrarily into sets $B_1', \ldots, B_k'$ of size t, and write $B_i' =: \{b_i'^1, \ldots, b_i'^t\}$. Since $|A^*| \leq ktc \leq \delta^0(T)/2$, we have

$$2^{c+1} + c|B| \leq 400t + ckt \leq \frac{C}{2}k^2 \log k \leq \delta^+(T) - |A^*| \leq \delta^+(T - A^*).$$

Thus we can apply Lemma 8.6 to $T - A^*$, B' and c in order to obtain almost in-dominating sets $B_i^\ell \ni b_i'^\ell$ and corresponding exceptional sets $E_{B,i}^\ell$ as in the statement of Lemma 8.6 (for all $i \in [k]$ and all $\ell \in [t]$). Write $B_i^* := B_i^1 \cup \cdots \cup B_i^t$ and $B^* := B_1^* \cup \cdots \cup B_k^*$. Similarly as before one can show that (G2) holds.

We now define the exceptional sets $E_{A,i}$ and $E_{B,i}$. For all $i \in [k]$, let

$$E_{A,i} := (E_{A,i}^1 \cup \cdots \cup E_{A,i}^t) \setminus B^* \quad \text{and} \quad E_{B,i} := (E_{B,i}^1 \cup \cdots \cup E_{B,i}^t).$$

Recall from Lemmas 8.5(v) and 8.6(v) that $E_{A,i}^\ell \cap A^* = \emptyset$ and $E_{B,i}^\ell \cap (A^* \cup B^*) = \emptyset$ for all $i \in [k]$ and all $\ell \in [t]$. Thus $E_{A,i} \cap (A_i^* \cup B_i^*) = \emptyset$ and $E_{B,i} \cap (A_i^* \cup B_i^*) = \emptyset$ for all $i \in [k]$. By Lemma 8.5(i), each A_i^ℓ out-dominates $V(T) \setminus (A^* \cup B^* \cup E_{A,i})$.

Lemma 8.5(iii) and the fact that $a_i^\ell \in A$ together imply that

$$(13) \qquad |E_{A,i}| \leq \sum_{\ell=1}^{t} |E_{A,i}^\ell| \leq \sum_{\ell=1}^{t} \frac{1}{2^{c-1}} d^-(a_i^\ell) \leq \frac{t}{2^{c-1}} d_- \leq \frac{d_-}{50},$$

so (G3) holds. Similarly, by Lemma 8.6(i), each B_i^ℓ in-dominates $V(T) \setminus (A^* \cup B^* \cup E_{B,i})$, and as in (13) one can show that $|E_{B,i}| \leq d_+/50$. Thus (G4) holds.

We now use Lemma 8.1 in order to define the sets $F_1, \ldots, F_k$ of covering edges. Recall from (G7) that we require $F_1 \cup \cdots \cup F_k$ to be a matching in $T - (A^* \cup B^*)$. Suppose that for some (possibly empty) subset $V' \subsetneq A^* \cup B^*$ we have defined a set $\{e_v : v \in V'\}$ of independent edges in $T - (A^* \cup B^*)$ such that e_v is a covering edge for v and $e_v \neq e_{v'}$ whenever $v \neq v'$. Pick any vertex $v \in (A^* \cup B^*) \setminus V'$. We will next define e_v. Let T' be the tournament obtained from T by deleting $(A^* \cup B^*) \setminus \{v\}$ as well as the endvertices of the covering edges $e_{v'}$ for all $v' \in V'$. Then

$$|V(T) \setminus V(T')| \leq |A^* \cup B^*| + 2|A^* \cup B^*| \leq 3ktc \leq \frac{C}{2} k^2 \log k,$$

so by Proposition 4.7, T' is still $(Ck^2 \log k/2)$-linked and hence strongly 2-connected. We may therefore apply Lemma 8.1 to find a covering edge e_v for v in T'. Continue in this way until we have chosen e_v for each $v \in A^* \cup B^*$ and let $F_i := \{e_v : v \in A_i^* \cup B_i^*\}$. Then the first part of (G7) holds.

It remains to choose the paths $P_1^1, \ldots, P_k^t$. Recall from (G6) that we need to ensure that $(A \cup B') \setminus (A_i^* \cup B_i^*) \subseteq V(P_i^t)$ for all $i \in [k]$. We could achieve this by incorporating each of these vertices using the high linkedness of T. However, since $|A \cup B'| = 2kt$, a direct application of linkedness would require T to be $\Theta(k^3)$-linked. For each $i \in [k]$, we will therefore first choose a path cover $\mathcal{Q}_i$ of $T[(A \cup B') \setminus (A_i^* \cup B_i^*)]$ consisting of few paths and then use Lemma 8.2 (and thereby the high linkedness of T) to incorporate these paths into P_i^t. This has the advantage that we will only need T to be $\Theta(k^2 \log k)$-linked.

Let us first choose the path covers $\mathcal{Q}_i$ of $T[(A \cup B') \setminus (A_i^* \cup B_i^*)]$. Suppose that for some $j \in [k]$ we have already found path systems $\mathcal{Q}_1, \ldots, \mathcal{Q}_{j-1}$ such that, for each $i < j$, $\mathcal{Q}_i$ is a path cover of $T[(A \cup B') \setminus (A_i^* \cup B_i^*)]$ with $|\mathcal{Q}_i| \leq 2k$, and such that for all $i < i' < j$ the paths in $\mathcal{Q}_i$ are edge-disjoint from paths in $\mathcal{Q}_{i'}$. To choose $\mathcal{Q}_j$, apply Corollary 6.4 to the oriented graph T'' obtained from $T[(A \cup B') \setminus (A_j^* \cup B_j^*)]$ by deleting the edges of all the paths in $\mathcal{Q}_1, \ldots, \mathcal{Q}_{j-1}$. Since $\delta(T'') \geq |T''| - 1 - 2(j-1) \geq |T''| - 2k$, Corollary 6.4 ensures that $|\mathcal{Q}_j| \leq 2k$.

We will now choose $P_1^1, \ldots, P_k^t$. For each $i \in [k]$ and each $\ell \in [t]$, let $a_i'^\ell$ denote the tail of $T[A_i^\ell]$ and b_i^ℓ the head of $T[B_i^\ell]$. Let

$$A' := \{a_i'^\ell : i \in [k], \ \ell \in [t]\} \quad \text{and} \quad B := \{b_i^\ell : i \in [k], \ \ell \in [t]\}.$$

For all $i \in [k]$ and all $\ell \in [t-1]$ let $\mathcal{Q}_i^\ell := \emptyset$. For all $i \in [k]$ let $\mathcal{Q}_i^t$ be the path system consisting of all the edges in F_i (each viewed as a path of length one) and all the paths in $\mathcal{Q}_i$. Let $T''' := T - ((A^* \cup B^*) \setminus (A \cup A' \cup B \cup B'))$. Our aim is to apply Lemma 8.2 with $s := 30$ to T''', the vertices $b_1^1, \ldots, b_k^t, a_1'^1, \ldots, a_k'^t$, and

the path systems $\mathcal{Q}_1^1, \ldots, \mathcal{Q}_k^t$. To verify that T''' is sufficiently highly linked, let m be as defined in (11) and note that

$$m = kt + 3\sum_{i=1}^k |F_i| + \sum_{i=1}^k |\mathcal{Q}_i| + \left|\bigcup_{i=1}^k V(\mathcal{Q}_i)\right| \le kt + 6ckt + 2k^2 + |A \cup B'|$$

$$\le 5kt + 6ckt \le \frac{C}{70}k^2 \log k.$$

Together with the fact that $|T| - |T'''| \le 2ckt$ and Proposition 4.7 this implies that T''' is $2 \cdot 30m$-linked. So we can indeed apply Lemma 8.2 to find edge-disjoint paths P_i^ℓ in T''' (for all $i \in [k]$ and all $\ell \in [t]$) satisfying the following properties:

   (i) P_i^ℓ is a path from b_i^ℓ to $a_i'^\ell$.
   (ii) $Q \subseteq P_i^\ell$ for all $Q \in \mathcal{Q}_i^\ell$.
   (iii) $V(P_i^\ell) \cap V(P_j^m) \subseteq V(\mathcal{Q}_i^\ell) \cap V(\mathcal{Q}_j^m)$ for all $(i, \ell) \ne (j, m)$.
   (iv) We have that

$$|P_1^1 \cup \cdots \cup P_k^t| \le \frac{n}{30} + 2\sum_{i=1}^k |F_i| + \left|\bigcup_{i=1}^k V(\mathcal{Q}_i)\right| = \frac{n}{30} + 2|A^* \cup B^*| + |A \cup B'|$$

$$\le \frac{n}{30} + 4ckt + 2kt \le \frac{n}{20}.$$

Condition (ii) implies that $F_i \subseteq P_i^t$ and $(A \cup B') \setminus (A_i^* \cup B_i^*) = V(\mathcal{Q}_i) \subseteq V(\mathcal{Q}_i^t) \subseteq V(P_i^t)$ for all $i \in [k]$. Thus (G6) holds.

We now prove that (G5) holds. From (iii) and the fact that that $V(\mathcal{Q}_i^\ell) \cap V(\mathcal{Q}_i^m) = \emptyset$ for all $i \in [k]$, $\ell \ne m$, it follows that $P_i^1, \ldots, P_i^t$ are vertex-disjoint for all $i \in [k]$. Together with (i) and (iv) this implies that in order to check (G5), it remains to show that

(14) $V(\mathrm{Int}(P_i^\ell)) \cap (A^* \cup B^*) \subseteq (A \cup B') \setminus (A_i^* \cup B_i^*)$ for all $i \in [k], \ell \in [t]$.

Clearly,

(15) $V(P_i^\ell) \cap (A^* \cup B^*) \subseteq V(T''') \cap (A^* \cup B^*)$

$$= A \cup A' \cup B \cup B' \quad \text{for all } i \in [k], \ell \in [t].$$

By definition, we have $(A' \cup B) \cap V(\mathcal{Q}_j^m) = \emptyset$ for all $j \in [k], m \in [t]$. It therefore follows from (iii) that each vertex in $A' \cup B$ may appear in at most one path P_j^m. However, by (i) each vertex in $A' \cup B$ is an endpoint of P_j^m for some $j \in [k], m \in [t]$. Hence

(16) $V(\mathrm{Int}(P_i^\ell)) \cap (A' \cup B) = \emptyset$ for all $i \in [k], \ell \in [t]$.

Fix $i \in [k], \ell \in [t]$ and take $j \in [k] \setminus \{i\}$. We have $(A \cup B') \cap (A_i^* \cup B_i^*) \cap V(\mathcal{Q}_i^\ell) = \emptyset$, and by (G6) we have $(A \cup B') \cap (A_i^* \cup B_i^*) \subseteq (A \cup B') \setminus (A_j^* \cup B_j^*) \subseteq V(P_j^t)$. Applying (iii) to P_i^ℓ and P_j^t, it therefore follows that

(17) $V(P_i^\ell) \cap (A \cup B') \cap (A_i^* \cup B_i^*) = \emptyset$ for all $i \in [k], \ell \in [t]$.

(15)–(17) now imply (14). Thus (G5) holds.

So it remains to check that the last part of (G7) holds too, i.e. that $F_i^{\mathrm{act}} \cap E(P_j^\ell) = \emptyset$ for all $i, j \in [k]$ and all $\ell \in [t]$. Consider any covering edge $e_v = x_v y_v \in F_i$. Then (G6) implies that x_v and y_v are contained in P_i^t. Moreover, (iii) implies that $V(P_i^t) \cap V(P_j^\ell) \subseteq V(\mathcal{Q}_i^t) \cap V(\mathcal{Q}_j^\ell) \subseteq A \cup B'$ whenever $(i, t) \neq (j, \ell)$. Since $x_v, y_v \notin A \cup B'$, this shows that $x_v v, v y_v \notin E(P_j^\ell)$ whenever $(i, t) \neq (j, \ell)$. But since $e_v \in E(P_i^t)$, we also have $x_v v, v y_v \notin E(P_i^t)$. This completes the proof that T is (C, k, t, c)-good. $\qquad\square$

9. Concluding remarks

9.1. **Eliminating the logarithmic factor.** A natural approach to improve the bound in Theorem 1.2 would be to reduce the parameter c, i.e. to consider smaller 'almost dominating' sets. In particular, if we could choose c independent of k, then we would obtain the (conjectured) optimal bound of $\Theta(k^2)$ for the linkedness. The obstacle to this in our argument is given by (13), which requires that c has a logarithmic dependence on k.

9.2. **Algorithmic aspects.** As remarked in the introduction, the proof of Theorem 1.2 is algorithmic. Indeed, when we apply the assumption of high linkedness to find appropriate paths in the proof of Lemma 8.7 (via Lemma 8.2), we can make use of the main result of [11] that these can be found in polynomial time. Moreover, the proof of the Gallai-Milgram theorem (Theorem 6.3) is also algorithmic (see [9]). These are the only tools we need in the proof, and the proof itself immediately translates into a polynomial time algorithm.

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