PATH DECOMPOSITIONS OF TOURNAMENTS

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Abstract. In 1976, Alspach, Mason, and Pullman conjectured that any tournament T of even order can be decomposed into exactly $\mathrm{ex}(T)$ paths, where $\mathrm{ex}(T) := \frac{1}{2}\sum_{v\in V(T)} |d_T^+(v) - d_T^-(v)|$. We prove this conjecture for all sufficiently large tournaments. We also prove an asymptotically optimal result for tournaments of odd order.

1. Introduction

Path and cycle decomposition problems have a long history. For example, the Walecki construction [15], which goes back to the 19^{th} century, gives a decomposition of the complete graph of odd order into Hamilton cycles (see also [2]). A version of this for (regular) directed tournaments was conjectured by Kelly in 1968 and proved for large tournaments in [10]. Beautiful open problems in the area include the Erdős-Gallai conjecture which asks for a decomposition of any graph into linearly many cycles and edges. The best bounds for this are due to Conlon, Fox, and Sudakov [6]. Another famous example is the linear arboricity conjecture, which asks for a decomposition of a d-regular graph into $\lceil \frac{d+1}{2} \rceil$ linear forests. The latter was resolved asymptotically by Alon [1] and the best current bounds are due to Lang and Postle [13].

1.1. **Background.**

The problem of decomposing digraphs into paths was first explored by Alspach and Pullman [4], who provided sharp bounds for the minimum number of paths needed in path decompositions of digraphs. (Throughout this paper, in a digraph, for any two vertices $u \neq v$, we allow a directed edge uv from u to v as well as a directed edge vu from v to u, whereas in an oriented graph we allow at most one directed edge between any two distinct vertices.) Given a digraph D, define the *path number of* D, denoted by $\mathrm{pn}(D)$, as the minimum integer k such that D can be decomposed into k paths. Alspach and Pullman [4] proved that, for any oriented graph D on n vertices, $\mathrm{pn}(D) \leq \frac{n^2}{4}$, with equality holding for transitive tournaments. O'Brien [16] showed that the same bound holds for digraphs on at least 4 vertices.

The path number of digraphs can be bounded below by the following quantity. Let D be a digraph and $v \in V(D)$. Define the *excess* at v as $\mathrm{ex}_D(v) := d_D^+(v) - d_D^-(v)$. Let $\mathrm{ex}_D^+(v) := \max\{0, \mathrm{ex}_D(v)\}$ and $\mathrm{ex}_D^-(v) := \max\{0, -\mathrm{ex}_D(v)\}$ be the *positive excess* and *negative excess* at v, respectively. Then, as observed in [4], if $d_D^+(v) > d_D^-(v)$, a path decomposition of D contains at most $d_D^-(v)$ paths which have v as an internal vertex, and thus at least $d_D^+(v) - d_D^-(v) = \mathrm{ex}_D^+(v)$ paths starting at v. Similarly, a path decomposition will contain at least $\mathrm{ex}_D^-(v)$ paths ending at v. Thus, the *excess* of D, defined as

$$\mathrm{ex}(D) := \sum_{v\in V(D)} \mathrm{ex}_D^+(v) = \sum_{v\in V(D)} \mathrm{ex}_D^-(v) = \frac{1}{2}\sum_{v\in V(D)} |\,\mathrm{ex}_D(v)|,$$

provides a natural lower bound for the path number of D, i.e. any digraph D satisfies

$$\mathrm{pn}(D) \geq \mathrm{ex}(D). \tag{1.1}$$

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It was shown in [4] that equality is satisfied for acyclic digraphs. A digraph satisfying equality in (1.1) is called *consistent*. Clearly, not all digraphs are consistent (e.g. regular digraphs have excess 0). However, Alspach, Mason, and Pullman [3] conjectured in 1976 that tournaments of even order are consistent.

Conjecture 1.1 (Alspach, Mason, and Pullman). *Let $n \in \mathbb{N}$ be even. Then, any tournament T on n vertices satisfies $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

This conjecture is discussed also e.g. in the Handbook of Combinatorics [5]. Moreover, it is listed as being of "high importance" in the list of problems on the "open problem garden" website.

Note that the results of Alspach and Pullman [4] mentioned above imply that Conjecture 1.1 holds for tournaments of excess $\frac{n^2}{4}$. Moreover, as observed by Lo, Patel, Skokan, and Talbot [14], Conjecture 1.1 for tournaments of excess $\frac{n}{2}$ is equivalent to Kelly's conjecture on Hamilton decompositions of regular tournaments. Recently, Conjecture 1.1 was verified in [14] for sufficiently large tournaments of sufficiently large excess. Moreover, they extended this result to tournaments of odd order n whose excess is at least $n^{2-\frac{1}{18}}$.

Theorem 1.2 ([14]). *The following hold.*

(a) *There exists $C \in \mathbb{N}$ such that, for any tournament T of even order n, if $\mathrm{ex}(T) \geq Cn$, then $\mathrm{pn}(T) = \mathrm{ex}(T)$.*
(b) *There exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, if T is a tournament on n vertices satisfying $\mathrm{ex}(T) \geq n^{2-\frac{1}{18}}$, then $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

1.2. **New results.** Building on the results and methods of [10, 14], we prove Conjecture 1.1 for large tournaments.

Theorem 1.3. *There exists $n_0 \in \mathbb{N}$ such that, for any even $n \geq n_0$, any tournament T on n vertices satisfies $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

In fact, our methods are more general and allow us determine the path number of most tournaments of odd order, whose behaviour turns out to be more complex. As mentioned above, not every digraph is consistent.

Let D be a digraph. Let $\Delta^0(D)$ denote the largest semidegree of D, that is $\Delta^0(D) := \max\{d^+(v), d^-(v) \mid v \in V(D)\}$. Note that $\Delta^0(D)$ is a natural lower bound for $\mathrm{pn}(D)$ as every vertex $v \in V(D)$ must be in at least $\max\{d^+(v), d^-(v)\}$ paths. This leads to the notation of the *modified excess* of a digraph D, which is defined as

$$\widetilde{\mathrm{ex}}(D) := \max\{\mathrm{ex}(D), \Delta^0(D)\}.$$

This provides a natural lower bound for the path number of any digraph D.

Fact 1.4. *Any digraph D satisfies $\mathrm{pn}(D) \geq \widetilde{\mathrm{ex}}(D)$.*

Observe that, by Theorem 1.2(b), equality holds for large tournaments of excess at least $n^{2-\frac{1}{18}}$. However, note that equality does not hold for regular digraphs. (Here a digraph is r-regular if for every vertex, both its in and outdegree equal r.) Indeed, by considering the number of edges, one can show that any path decomposition of an r-regular digraph will contain at least $r+1$ paths. Thus, any regular digraph satisfies $\mathrm{pn}(D) \geq \widetilde{\mathrm{ex}}(D) + 1$. Alspach, Mason, and Pullman [3] conjectured that equality holds for regular tournaments. We verify this conjecture for sufficiently large tournaments.

Theorem 1.5. *There exists $n_0 \in \mathbb{N}$ such that any regular tournament T on $n \geq n_0$ vertices satisfies $\mathrm{pn}(T) = \frac{n+1}{2} = \widetilde{\mathrm{ex}}(T) + 1$.*

In fact, our argument also applies to regular oriented graphs of large degree.

Theorem 1.6. *For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if D is an r-regular oriented graph on $n \geq n_0$ vertices satisfying $r \geq \left(\frac{3}{8} + \varepsilon\right) n$, then $\mathrm{pn}(D) = r + 1 = \widetilde{\mathrm{ex}}(D) + 1$.*

More generally, we will see that Theorem 1.6 can be extended to regular digraphs of linear degree which are "robust outexpanders" (see Theorem 4.36).

There also exist non-regular tournaments for which equality does not hold in Fact 1.4. Indeed, let $\mathcal{T}_{\rm apex}$ be the set of tournaments T on $n \geq 5$ vertices for which there exists a partition $V(T) = V_0 \cup \{v_+\} \cup \{v_-\}$ such that $T[V_0]$ is a regular tournament on $n-2$ vertices (and so n is odd), $N_T^+(v_+) = V_0 = N_T^-(v_-)$, $N_T^-(v_+) = \{v_-\}$, and $N_T^+(v_-) = \{v_+\}$. The tournaments in $\mathcal{T}_{\rm apex}$ are called *apex tournaments*. We show that any sufficiently large tournament $T \in \mathcal{T}_{\rm apex}$ satisfies $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T) + 1$ (see Theorem 4.35). Denote by $\mathcal{T}_{\rm reg}$ the class of regular tournaments and let $\mathcal{T}_{\rm excep} \coloneqq \mathcal{T}_{\rm apex} \cup \mathcal{T}_{\rm reg}$. The tournaments in $\mathcal{T}_{\rm excep}$ are called *exceptional*. We conjecture that the tournaments in $\mathcal{T}_{\rm excep}$ are the only ones which do not satisfy equality in Fact 1.4.

Conjecture 1.7. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \notin \mathcal{T}_{\rm excep}$ on $n \geq n_0$ vertices satisfies* $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$.

We prove an approximate version of this conjecture (see Corollary 1.9). Moreover, in Theorem 1.8, we prove Conjecture 1.7 exactly unless n is odd and T is extremely close to being a regular tournament (in the sense that the number of vertices of nonzero excess is $o(n)$, the excess of each vertex is $o(n)$, and the total excess is $\frac{n}{2} \pm o(n)$).

Theorem 1.8. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. If T is a tournament on $n \geq n_0$ vertices such that $T \notin \mathcal{T}_{\rm excep}$ and*

(a) *$\widetilde{\mathrm{ex}}(T) \geq \frac{n}{2} + \beta n$, or*
(b) *$N^+(T), N^-(T) \geq \beta n$, where $N^+(T) \coloneqq |\{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ and $N^-(T) \coloneqq |\{v \in V(T) \mid \mathrm{ex}_T^-(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$,*

then $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$.

In Section 7, we will derive Theorem 1.3 (i.e. the exact solution when n is even) from Theorem 1.8. This will make use of the fact that $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$ for n even (see Proposition 4.22). We will also derive an approximate version of Conjecture 1.7 from Theorem 1.8.

Corollary 1.9. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any tournament T on $n \geq n_0$ vertices,* $\mathrm{pn}(T) \leq \widetilde{\mathrm{ex}}(T) + \beta n$.

Note that Theorem 1.8(b) corresponds to the case where linearly many different vertices can be used as endpoints of paths in an optimal decomposition. Indeed, let T be a tournament and $\mathcal{P}$ be a path decomposition of T. Then, as mentioned above, each $v \in V(T)$ must be the starting point of at least $\mathrm{ex}_T^+(v)$ paths in $\mathcal{P}$. Thus, for any tournament T, $N^+(T)$ is the maximum number of distinct vertices which can be a starting point of a path in a decomposition of T of size $\widetilde{\mathrm{ex}}(T)$ and similarly for $N^-(T)$ and the ending points of paths.

One can show that almost all large tournaments satisfy $\mathrm{ex}(T) = n^{\frac{3}{2}+o(1)}$. Indeed, consider a tournament T on n vertices, where the orientation of each edge is chosen uniformly at random, independently of all other orientations. For the upper bound on $\mathrm{ex}(T)$, one can simply apply a Chernoff bound to show that for a given vertex v and $\varepsilon > 0$, we have $\mathrm{ex}_T^+(v) \leq n^{\frac{1}{2}+\varepsilon}$ with probability $1 - o\left(\frac{1}{n}\right)$. The result follows by a union bound over all vertices. For the lower bound, let X denote the number of vertices v with $d_T^-(v) \in \left[\frac{n}{2} - 2\sqrt{n}, \frac{n}{2} - \sqrt{n}\right]$. Then it is easy to see that, for large enough n, we have $\mathbb{E}[X] \geq \frac{n}{10^4}$, say. Moreover, Chebyshev's inequality can be used to show that, with probability $1-o(1)$, we have $X \geq \frac{n}{2 \cdot 10^4}$, again with room to spare. Thus, Theorem 1.8 implies the following.

Corollary 1.10. *As $n \to \infty$, the proportion of tournaments on n vertices satisfying* $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$ *tends to* 1.

Note that the case when n is even already follows from Theorem 1.2(a). Corollary 1.10 is an analogue of a result of [11], which states that almost all sufficiently large tournaments T contain $\delta^0(T) \coloneqq \min\{d_T^+(v), d_T^-(v) \mid v \in V(T)\}$ edge-disjoint Hamilton cycles and which proved a conjecture of Erdős (see [18]).

Finally, we will see in Section 9 that our methods give a short proof of (a stronger version of) a result of Osthus and Staden [17], which guarantees an approximate decomposition of regular "robust outexpanders" of linear degree into Hamilton cycles and which was used as a tool in the proof of Kelly's conjecture [10].

1.3. **Organisation of the paper.** This paper is organised as follows. In Section 2, we give a proof overview of Theorem 1.8. Notation will be introduced in Section 3, while tools and preliminary results will be collected in Section 4. Moreover, exceptional tournaments will be considered in Section 4.7. Sections 5–8 will be devoted to the proof of Theorem 1.8. Theorem 1.3 and Corollary 1.9 are derived from Theorem 1.8 in Section 7. Finally, in Section 9, we discuss Hamilton decompositions of robust outexpanders and conclude with a remark about Conjecture 1.7.

2. Proof overview

2.1. **Robust outexpanders.** Our proof of Theorem 1.8 will be based on the concept of robust outexpanders. Roughly speaking, a digraph D is called a *robust outexpander* if, for any set $S \subseteq V(D)$ which is neither too small nor too large, there exist significantly more than $|S|$ vertices with many inneighbours in S. (Robust outexpanders will be defined formally in Section 4.1.) Any (almost) regular tournament is a robust outexpander and we will use that this property is inherited by random subdigraphs. The main result of [10] states that any regular robust outexpander of linear degree has a Hamilton decomposition (see Theorem 4.9). We can apply this to obtain an optimal path decomposition in the following setting. Let D be a digraph on n vertices, $0 < \eta < 1$, and suppose that $X^+ \cup X^- \cup X^0$ is a partition of $V(D)$ such that $|X^+| = |X^-| = \eta n$ and the following hold.

(†) *Each $v \in X^0$ satisfies $d_D^\pm(v) = \eta n$; each $v \in X^+$ satisfies $d_D^+(v) = \eta n$ and $d_D^-(v) = \eta n - 1$; and each $v \in X^-$ satisfies $d_D^+(v) = \eta n - 1$ and $d_D^+(v) = \eta n$.*

Then, the digraph D' obtained from D by adding a new vertex v with $N_{D'}^\pm(v) = X^\pm$ is ηn-regular. Thus, if D is a robust outexpander, then there exists a decomposition of D' into Hamilton cycles. This induces a decomposition $\mathcal{P}$ of D into ηn Hamilton paths, where each vertex in X^+ is the starting point of exactly one path in $\mathcal{P}$ and each vertex in X^- is the ending point of exactly one path in $\mathcal{P}$. This is formalised in Corollary 4.10. (A similar observation was already made and used in [14].) Our main strategy will be to reduce our tournaments to a digraph of the above form. This will be achieved as follows.

2.2. **Simplified approach for well behaved tournaments.** Let $\beta > 0$ and fix additional constants such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \gamma \ll \eta \ll \beta$. Let T be a tournament on $n \geq n_0$ vertices. Note that by Theorem 1.2, we may assume that $\widetilde{\mathrm{ex}}(T) \leq \varepsilon^2 n^2$. Moreover, for simplicity, we first also assume that each $v \in V(T)$ satisfies $|\mathrm{ex}_T(v)| \leq \varepsilon n$ (i.e. T is almost regular), $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, and both $|\{v \in V(T) \mid \mathrm{ex}_T^\pm(v) > 0\}| \geq \eta n$. In Section 2.3, we will explain how the argument can be generalised if any of these conditions is not satisfied.

Firstly, since T is almost regular, it is a robust outexpander and so we can fix a random spanning subgraph $\Gamma \subseteq T$ of density γ such that Γ is a robust outexpander and $T \setminus \Gamma$ is almost regular. The digraph Γ will serve two purposes. Firstly, its robust outexpansion properties will be used to construct an approximate path decomposition of T. Secondly, provided few edges of Γ are used throughout this approximate decomposition, it will guarantee that the leftover (consisting of all of those edges of T not covered by the approximate path decomposition) is still a robust outexpander, as required to complete our decomposition of T in the way described in Section 2.1.

Fix $X^\pm \subseteq \{v \in V(T) \mid \mathrm{ex}_T^\pm(v) > 0\}$ of size ηn and denote $X^0 := V(T) \setminus (X^+ \cup X^-)$. Our goal is then to find an approximate path decomposition $\mathcal{P}$ of T such that $|\mathcal{P}| = \widetilde{\mathrm{ex}}(T) - \eta n$ and such that the leftover $D := T \setminus \bigcup \mathcal{P}$ satisfies the degree conditions in (†). Thus, it suffices that $\mathcal{P}$ satisfies the following.

(i) Each $v \in X^+$ is the starting point of exactly $\mathrm{ex}_T^+(v)-1$ paths in $\mathcal{P}$, while each $v \in V(T)\setminus X^+$ is the starting point of exactly $\mathrm{ex}_T^+(v)$ paths in $\mathcal{P}$. Similarly, each $v \in X^-$ is the ending point of exactly $\mathrm{ex}_T^-(v) - 1$ paths in $\mathcal{P}$, while each $v \in V(T) \setminus X^-$ is the ending point of exactly $\mathrm{ex}_T^-(v)$ paths in $\mathcal{P}$.

(ii) Each $v \in V \setminus (X^+ \cup X^-)$ is the internal vertex of exactly $\frac{(n-1)-|\mathrm{ex}_T(v)|}{2} - \eta n$ paths in $\mathcal{P}$, while each $v \in X^+ \cup X^-$ is the internal vertex of exactly $\frac{(n-1)-|\mathrm{ex}_T(v)|}{2} - \eta n + 1$ paths in $\mathcal{P}$.

Indeed, (i) ensures that $|\mathcal{P}| = \mathrm{ex}(T) - \eta n$ and each vertex has the desired excess in D, namely $\mathrm{ex}_D(v) = \pm 1$ if $v \in X^\pm$ and $\mathrm{ex}_D(v) = 0$ otherwise. In addition, (ii) ensures that the degrees in D satisfy (†).

Recall that, by assumption, T is almost regular. Thus, in a nutshell, (i) and (ii) state that we need to construct edge-disjoint paths with specific endpoints and such that each vertex is covered by about $(\frac{1}{2} - \eta)n$ paths. To ensure the latter, we will in fact approximately decompose T into about $(\frac{1}{2} - \eta)n$ spanning sets of internally vertex-disjoint paths. To ensure the former, we will start by constructing $(\frac{1}{2} - \eta)n$ auxiliary digraphs on $V(T)$ such that, for each $v \in V(T)$, the total number of edges starting (and ending) at v is the number of paths that we want to start (and end, respectively) at v. These auxiliary digraphs will be called layouts. These layouts are constructed in Section 5. Then, it will be enough to construct, for each layout L, a spanning set of paths $\mathcal{P}_L$, called a *spanning configuration of shape* L, such that each path $P \in \mathcal{P}_L$ corresponds to some edge $e \in E(L)$ and such that the starting and ending points of P equal those of e.

These spanning configurations will be constructed one by one as follows. At each stage, given a layout L, fix an edge $uv \in E(L)$. Then, using the robust outexpanding properties of (the remainder of) Γ, find short internally vertex-disjoints paths with endpoints corresponding to the endpoints of the edges in $L \setminus \{uv\}$. Denote by $\mathcal{P}'_L$ the set containing these paths. Then, it only remains to construct a path from u to v spanning $V(T) \setminus V(\mathcal{P}'_L)$. We achieve this as follows.

Let D' and Γ' be obtained from (the remainders of) $(T \setminus \Gamma) - V(\mathcal{P}'_L)$ and $\Gamma - V(\mathcal{P}'_L)$ by merging the vertices u and v into a new vertex w whose outneighbourhood is the outneighbourhood of u and whose inneighbourhood is the inneighbourhood of v. Then, observe that a Hamilton cycle of $D' \cup \Gamma'$ corresponds to a path from u to v of T which spans $V(T) \setminus V(\mathcal{P}'_L)$. We construct a Hamilton cycle of $D' \cup \Gamma'$ as follows. Of course, one can simply use the fact that Γ' is a robust expander to find a Hamilton cycle. However, if we proceed in this way, then the robust outexpanding property of Γ' might be destroyed before constructing all the desired spanning configurations.

So instead we construct a Hamilton cycle with only few edges in Γ' as follows. Using the fact that $T \setminus \Gamma$ is almost regular, we first find an almost spanning linear forest F in D' which has few components. Then, we use the robust outexpanding properties of Γ' to tie up F into a Hamilton cycle C of $D' \cup \Gamma'$. This construction of spanning configurations (respecting the layouts formed in Section 5) is carried out in Section 6.

2.3. **General tournaments.** For a general tournament T, we adapt the above argument as follows. Let W be the set of vertices $v \in V(T)$ such that $|\mathrm{ex}_T(v)| > \varepsilon n$. If $W \neq \emptyset$, then T is no longer almost regular and we cannot proceed as above. However, since $\mathrm{ex}(T) \leq \varepsilon^2 n^2$, $|W|$ is small. Thus, we can start with a cleaning procedure which efficiently decreases the excess and degree at W by taking out few edge-disjoint paths. The corresponding proof is deferred until Section 8, as it is quite involved and carrying out the other steps first helps to give a better picture of the overall argument. Then, we apply the above argument to (the remainder of) $T - W$. We incorporate all remaining edges at W in the approximate decomposition by generalising the concept of a layout introduced above.

If $|\{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}| < \eta n$ but $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, say, then we cannot choose $X^+ \subseteq \{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}$ of size ηn. We circumvent this problem as follows. Select a small set of vertices W_A such that $\sum_{v \in W_A} \mathrm{ex}_T^+(v) \geq \eta n$ and let A be a set of ηn edges such that the following hold. Each edge in A starts in W_A and ends in $V(T) \setminus W_A$. Moreover, each $v \in W_A$ is the starting point of at most $\mathrm{ex}_T^+(v)$ edges in A and each $v \in V(T) \setminus W_A$ is the ending point of at most one edge in A. We will call A an *absorbing set of starting edges*. Then, let the ending points of the edges in A play the role of X^+ and add the vertices in W_A to W so that, at the end of the approximate decomposition, the only remaining edges at W_A are the edges in A. Thus, in the final decomposition step, we can use the edges in A to extend the paths starting at X^+ into paths starting in W_A. (See Section 4.5 for details.) If $|\{v \in V(T) \mid \mathrm{ex}_T^-(v) > 0\}| < \eta n$, then we proceed analogously.

If $\widetilde{\mathrm{ex}}(T) > \mathrm{ex}(T)$, then not all paths will "correspond" to some excess. For simplicity, we will choose which additional endpoints to use at the beginning and artificially add excess to those vertices. This then enables us to proceed as if $\mathrm{ex}(T) = \widetilde{\mathrm{ex}}(T)$. More precisely, we will choose a set $U^* \subseteq \{v \in V(T) \mid \mathrm{ex}_T(v) = 0\}$ of size $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ and we will treat the vertices in U^* in the same way as we treat those with $\mathrm{ex}_T^\pm(v) = 1$. Note that selecting additional endpoints in this way maximises the number of distinct endpoints, which will enable us to choose $X^\pm \subseteq \{v \in V(T) \mid \mathrm{ex}_T^\pm(v) > 0\} \cup U^*$ when $N^\pm(T) = |\{v \in V(T) \mid \mathrm{ex}_T^\pm(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) \geq \eta n$ and use absorbing edges otherwise, i.e. if condition (b) fails in Theorem 1.8. More details of this approach are given in Section 4.6.

3. Notation

We denote by $\mathbb{N}$ the set of natural numbers (including 0). Let $a, b, c \in \mathbb{R}$. We write $a = b \pm c$ if $b - c \leq a \leq b + c$. For simplicity, we use hierarchies instead of explicitly calculating the values of constants for which our statements hold. More precisely, if we write $0 < a \ll b \ll c \leq 1$ in a statement, we mean that there exist non-decreasing functions $f\colon (0,1] \longrightarrow (0,1]$ and $g\colon (0,1] \longrightarrow (0,1]$ such that the statement holds for all $0 < a, b, c \leq 1$ satisfying $b \leq f(c)$ and $a \leq g(b)$. Hierarchies with more constants are defined in a similar way. We assume large numbers to be integers and omit floors and ceilings, provided this does not affect the argument.

In general, a statement $\mathcal{C}^\pm$ will mean that both statements $\mathcal{C}^+$ and $\mathcal{C}^-$ hold simultaneously. If used in the form that $\mathcal{C}^\pm$ is the statement "$\mathcal{A}^\pm$ implies $\mathcal{B}^\pm$", the convention means that "$\mathcal{A}^+$ implies $\mathcal{B}^+$" and "$\mathcal{A}^-$ implies $\mathcal{B}^-$".

A *digraph* D is a directed graph without loops which contains, for any distinct vertices u and v of D, at most two edges between u and v, at most one in each direction. A digraph D is called an *oriented graph* if it contains, for any distinct vertices u and v of D, at most one edge between u and v; that is, D can be obtained by orienting the edges of an undirected graph.

Let G be a (di)graph. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. We say G is *non-empty* if $E(G) \neq \emptyset$. Let $u, v \in V(G)$ be distinct. If G is undirected, then we write uv for an edge between u and v. If G is directed, then we write uv for an edge directed from u to v, where u and v are called the *starting* and *ending points* of the edge uv, respectively. Let $A, B \subseteq V(G)$ be disjoint. Denote $E_A(G) \coloneqq \{e \in E(G) \mid V(e) \cap A \neq \emptyset\}$. Moreover, we write $G[A, B]$ for the undirected graph with vertex set $A \cup B$ and edge set $\{ab \in E(G) \mid a \in A, b \in B\}$ and $e(A, B) \coloneqq |E(G[A, B])|$.

Given $S \subseteq V(G)$, we write $G[S]$ for the sub(di)graph of G induced on S and $G - S$ for the (di)graph obtained from G by deleting all vertices in S. Given $E \subseteq E(G)$, we write $G \setminus E$ for the (di)graph obtained from G by deleting all edges in E. Similarly, given a sub(di)graph $H \subseteq G$, we write $G \setminus H \coloneqq G \setminus E(H)$. If F is a set of non-edges of G, then we write $G \cup F$ for the (di)graph obtained by adding all edges in F. Given a (di)graph H, if G and H are edge-disjoint, then we write $G \cup H$ for the (di)graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Assume G is an undirected graph. For any $v \in V(G)$, we write $N_G(v)$ for the *neighbourhood* of v in G and $d_G(v)$ for the *degree* of v in G. Given $S \subseteq V(G)$, we denote $N_G(S) \coloneqq \bigcup_{v \in S} N_G(v)$.

Let D be a digraph. Let $v \in V(D)$. We write $N_D^+(v)$ and $N_D^-(v)$ for the *outneighbourhood* and *inneighbourhood* of v in D, respectively, and define the *neighbourhood* of v in D as $N_D(v) \coloneqq N_D^+(v) \cup N_D^-(v)$. We denote by $d_D^+(v)$ and $d_D^-(v)$ the *outdegree* and *indegree* of v in D, respectively, and define the *degree* of v in D as $d_D(v) \coloneqq d_D^+(v) + d_D^-(v)$. Denote $d_D^{\min}(v) \coloneqq \min\{d_D^+(v), d_D^-(v)\}$

and $d_D^{\max}(v) \coloneqq \max\{d_D^+(v), d_D^-(v)\}$. If $d_D^+(v) \neq d_D^-(v)$, then define

$$N_D^{\min}(v) \coloneqq \begin{cases} N_D^+(v) & \text{if } d_D^{\min} = d_D^+(v), \\ N_D^-(v) & \text{if } d_D^{\min} = d_D^-(v), \end{cases} \quad \text{and} \quad N_D^{\max}(v) \coloneqq \begin{cases} N_D^+(v) & \text{if } d_D^{\max} = d_D^+(v), \\ N_D^-(v) & \text{if } d_D^{\max} = d_D^-(v). \end{cases}$$

The *minimum semidegree* of D is defined as $\delta^0(D) \coloneqq \min\{d_D^{\min}(v) \mid v \in V(D)\}$ and, similarly, $\Delta^0(D) \coloneqq \max\{d_D^{\max}(v) \mid v \in V(D)\}$ is called the *maximum semidegree* of D. Define the *minimum degree* and *maximum degree* of D by $\delta(D) \coloneqq \min\{d_D(v) \mid v \in V(D)\}$ and $\Delta(D) \coloneqq \max\{d_D(v) \mid v \in V(D)\}$, respectively. Given $S \subseteq V(D)$, we denote $N_D^\pm(S) \coloneqq \bigcup_{v \in S} N_D^\pm(v)$ and $N_D(S) \coloneqq \bigcup_{v \in S} N_D(v)$.

Let D be a digraph on n vertices. We say D is r*-regular* if, for any $v \in V(D)$, $d_D^+(v) = d_D^-(v) = r$. We say D is *regular* if it is r-regular for some $r \in \mathbb{N}$. Let $\varepsilon, \delta > 0$. We say D is (δ, ε)*-almost regular* if, for each $v \in V(D)$, both $d_D^+(v) = (\delta \pm \varepsilon)n$ and $d_D^-(v) = (\delta \pm \varepsilon)n$.

Let A and B be multisets. The *support* of A is the set $S(A) \coloneqq \{a \mid a \in A\}$. For each $a \in S(A)$, we denote by $\mu_A(a)$ the *multiplicity* of a in A. For any $a \notin S(A)$, we define $\mu_A(a) \coloneqq 0$. We write $A \cup B$ for the multiset with support $S(A \cup B) \coloneqq S(A) \cup S(B)$ and such that, for each $a \in S(A \cup B)$, $\mu_{A \cup B}(a) \coloneqq \mu_A(a) + \mu_B(b)$. We denote by $A \setminus B$ the multiset with support $S(A \setminus B) \coloneqq \{a \in S(A) \mid \mu_A(a) > \mu_B(a)\}$ and such that, for each $a \in S(A \setminus B)$, $\mu_{A \setminus B}(a) \coloneqq \mu_A(a) - \mu_B(a)$. We say A is a *submultiset* of B, denoted $A \subseteq B$, if $S(A) \subseteq S(B)$ and, for each $a \in S(A)$, $\mu_A(a) \leq \mu_B(a)$.

By a *multidigraph*, we mean a directed graph where we allow multiple edges but no loops. All the notation and definitions introduced thus far extend naturally to multidigraphs, with unions/differences of edge sets now interpreted as multiset unions/differences. In a multidigraph, two instances of an edge are considered to be distinct. In particular, given a multidigraph D, we say $D_1, D_2 \subseteq D$ are *edge-disjoint* submultidigraphs of D if, for any $e \in E(D)$, $\mu_{E(D_1)}(e) + \mu_{E(D_2)}(e) \leq \mu_{E(D)}$.

In this paper, all paths and cycles are directed, with edges consistently oriented. The *length* of a path P, denoted by $e(P)$, is the number of edges it contains. A path on one vertex, i.e. a path of length 0 is called *trivial*. Let $P = v_1v_2 \dots v_\ell$ be a path. We say v_1 is the *starting point* of P and v_ℓ is the *ending point* of P. We say v is an *endpoint* of a path P if v is the starting or ending point of P. We say $v_2, \dots, v_{\ell-1}$ are *internal vertices* of P. We write $V^+(P) = \{v_1\}$, $V^-(P) = \{v_\ell\}$, and $V^0(P) = \{v_2, \dots, v_{\ell-1}\}$. We say that a path P is a (u, v)*-path* if $V^+(P) = \{u\}$ and $V^-(P) = \{v\}$. Given $1 \leq i < j \leq \ell$, we denote $v_iPv_j \coloneqq v_iv_{i+1} \dots v_j$. A *linear forest* is a set of pairwise vertex-disjoint paths.

Similarly, given a (multi)set $\mathcal{P}$ of paths, we write $V^+(\mathcal{P})$ for the set of vertices which are the starting point of a path in $\mathcal{P}$. Similarly, we write $V^-(\mathcal{P})$ for the set of vertices which are the ending point of a path in $\mathcal{P}$ and $V^0(\mathcal{P})$ for the set of vertices which are an internal vertex of a path in $\mathcal{P}$. (Note that $V^\pm(\mathcal{P})$ and $V^0(\mathcal{P})$ are sets and not multisets.)

Given an (auxiliary) directed edge xy and a path P, we say P has *shape* xy if P is an (x, y)-path. Similarly, let E be a (multi)set of (auxiliary) directed edges and $\mathcal{P}$ be a (multi)set of paths. We say $\mathcal{P}$ has *shape* E if there exists a bijection $\phi : E \longrightarrow \mathcal{P}$ such that, for each $xy \in E$, $\phi(xy)$ is an (x, y)-path.

For convenience, a (multi)set $\mathcal{P}$ of paths will sometimes be viewed as the (multi)digraph consisting of their union. In particular, given a (multi)set $\mathcal{P}$ of paths, we write $V(\mathcal{P})$ for the set of vertices of $\mathcal{P}$ and $E(\mathcal{P})$ for the (multi)set of edges of $\mathcal{P}$, i.e. $V(\mathcal{P})$ is the set $\bigcup_{P \in \mathcal{P}} V(P)$ and $E(\mathcal{P})$ is the (multi)set $\bigcup_{P \in \mathcal{P}} E(P)$. (Note that $V(\mathcal{P})$ is a set and not a multiset.) For any $v \in V(\mathcal{P})$, we write $d_\mathcal{P}^\pm(v)$ and $\mathrm{ex}_\mathcal{P}^\pm(v)$ for the in/outdegree and positive/negative excess of v in $\mathcal{P}$ when viewed as a multidigraph, i.e. $d_\mathcal{P}^\pm(v) \coloneqq d_{\bigcup \mathcal{P}}^\pm(v)$ and $\mathrm{ex}_\mathcal{P}^\pm(v) \coloneqq \mathrm{ex}_{\bigcup \mathcal{P}}^\pm(v)$. We define $d_\mathcal{P}(v)$ and $\mathrm{ex}_\mathcal{P}(v)$ similarly, and denote $D \setminus \mathcal{P} \coloneqq D \setminus \bigcup \mathcal{P}$.

Let D and D' be digraphs and $uv \in E(D)$. We say D' is obtained from D by *subdividing* uv, if $V(D') = V(D) \cup \{w\}$, for some $w \notin V(D)$, and $E(D') = (E(D) \setminus \{uv\}) \cup \{uw, wv\}$. We say D' is a *subdivision* of D if D' is obtained by successively subdividing some edges of D. Let P be a (u, v)-path satisfying $V^0(P) \cap V(D) = \emptyset$. We say D' is obtained from D by *subdividing* uv *into* P, if $V(D') = V(D) \cup V^0(P)$ and $E(D') = (E(D) \setminus \{uv\}) \cup E(P)$. Similarly, given an induced (u,v)-path $P \subseteq D$, we say D' is obtained from D by *contracting the path P into an edge uv* if $V(D') = V(D) \setminus V^0(P)$ and $E(D') = (E(D) \setminus E(P)) \cup \{uv\}$.

Let D be a digraph. A *decomposition* of D is set $\mathcal{D}$ of non-empty edge-disjoint subdigraphs of D such that every edge of D is in one of these subdigraphs. A *(Hamilton) path decomposition* of D is a decomposition $\mathcal{P}$ of D such that each subdigraph $P \in \mathcal{P}$ is a (Hamilton) path of D. Similarly, a *(Hamilton) cycle decomposition* of D is a decomposition $\mathcal{C}$ of D such that each subdigraph $C \in \mathcal{C}$ is a (Hamilton) cycle of D. By a *Hamilton decomposition* of D, we mean a Hamilton cycle decomposition of D.

4. Preliminaries

In this section, we introduce some tools which will be used throughout the rest of the paper.

4.1. **Robust outexpanders.** Let D be a digraph on n vertices. Given $S \subseteq V(D)$, the *ν-robust outneighbourhood of S* is the set $RN^+_{\nu,D}(S) := \{v \in V(D) \mid |N^-_D(v) \cap S| \geq \nu n\}$. We say that D is a *robust (ν,τ)-outexpander* if, for any $S \subseteq V(D)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$, $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$.

In this section we state some useful properties of robust outexpanders. First, observe that the next fact follows immediately from the definition.

Fact 4.1. *Let D be a robust (ν,τ)-outexpander. Then, for any $\nu' \leq \nu$ and $\tau' \geq \tau$, D is a robust (ν',τ')-outexpander.*

The following lemma states that robust outexpansion is preserved when few edges are removed and/or few vertices are removed and/or added.

Lemma 4.2. *Let $0 < \varepsilon \leq \nu \leq \tau \leq 1$. Let D be a robust (ν,τ)-outexpander on n vertices.*

(a) *If D' is obtained from D by removing at most εn in/out edges at each vertex, then D' is a robust $(\nu-\varepsilon,\tau)$-outexpander.*
(b) *If D' is obtained from D by adding or removing at most εn vertices, then D' is a robust $(\nu-\varepsilon,2\tau)$-outexpander.*

One can easily show that the τ-parameter of robust outexpansion can be decreased when the minimum semidegree is large. This will enable us to state some results of [9, 10] with slightly adjusted parameters.

Lemma 4.3. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$. Assume D is a robust $(\nu, \frac{\delta}{2})$-outexpander on n vertices satisfying $\delta^0(D) \geq \delta n$. Then D is a robust (ν,τ)-outexpander.*

The next result states that oriented graphs of sufficiently large minimum semidegree are robust outexpanders.

Lemma 4.4 ([10, Lemma 13.1])**.** *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \varepsilon \leq 1$. Let D be oriented graph on n vertices with $\delta^0(D) \geq (\frac{3}{8}+\varepsilon)n$. Then D is a robust (ν,τ)-outexpander.*

The next lemma follows easily from the definition of robust outexpansion and states that robust outexpanders of linear minimum semidegree have small diameter.

Lemma 4.5 ([9, Lemma 6.6])**.** *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$. Let D be a robust (ν,τ)-outexpander on n vertices with $\delta^0(D) \geq \delta n$. Then, for any $x, y \in V(D)$, D contains an (x,y)-path of length at most ν^{-1}.*

Corollary 4.6. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$. Let D be a robust (ν,τ)-outexpander on n vertices. Suppose $\delta^0(D) \geq \delta n$ and let $S \subseteq V(D)$ be such that $|S| \leq \varepsilon n$. Let $k \leq \nu^3 n$ and $x_1, \dots, x_k, x_1', \dots, x_k'$ be (not necessarily distinct) vertices of D. Let $X := \{x_1, \dots, x_k, x_1', \dots, x_k'\}$. Then, there exist internally vertex-disjoint paths $P_1, \dots, P_k \subseteq D$ such that, for each $i \in [k]$, P_i is an (x_i, x_i')-path of length at most $2\nu^{-1}$ and $V^0(P_i) \subseteq V(D) \setminus (X \cup S)$.*

We will use the fact that robust outexpanders of linear minimum degree contain Hamilton paths from any fixed vertex x to any vertex $y \neq x$. This immediately follows from the result that such digraphs contain a Hamilton cycle (by contracting x and y to a single vertex z whose outneighbourhood is that of x and whose inneighbourhood is that of y). The Hamiltonicity of such digraphs was first proven in [8, 12].

Lemma 4.7. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$. Let D be a robust (ν, τ)-outexpander on n vertices with $\delta^0(D) \geq \delta n$. Then, for any distinct $x, y \in V(D)$, D contains a Hamilton (x, y)-path.*

Using Lemmas 4.2, 4.3, and 4.7 and Corollary 4.6, one can prove the following corollary.

Corollary 4.8. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$ and $k \leq \nu^3 n$. Let D be a digraph and $P_1, \dots, P_k \subseteq D$ be vertex-disjoint paths. For each $i \in [k]$, denote by v_i^+ and v_i^- the starting and ending points of P_i, respectively. Let $V' \coloneqq V(D) \setminus \bigcup_{i \in [k]} V(P_i)$ and $S \subseteq V'$. Suppose that $D' \coloneqq D[V' \setminus S]$ is a robust (ν, τ)-outexpander on n vertices satisfying $\delta^0(D') \geq \delta n$. Assume that for each $i \in [k-1]$, $|N_D^+(v_i^-) \cap (V' \setminus S)| \geq 2k$ and $|N_D^-(v_{i+1}^+) \cap (V' \setminus S)| \geq 2k$. Then, the following hold.*

- (a) *There exists a (v_1^+, v_k^-)-path $Q \subseteq D$ of length at most $2\nu^{-1}k + \sum_{i \in [k]} e(P_i)$ such that, for each $i \in [k]$, $P_i \subseteq Q$ and $V(Q) \setminus \bigcup_{i \in [k]} V(P_i) \subseteq V' \setminus S$.*
- (b) *There exists a (v_1^+, v_k^-)-Hamilton path Q' of $D - S$ such that, for each $i \in [k]$, $P_i \subseteq Q'$.*
- (c) *There exists a Hamilton cycle C of $D - S$ such that, for each $i \in [k]$, $P_i \subseteq C$.*

The main result of [10] states that regular robust outexpanders of linear degree can be decomposed into Hamilton cycles. Note that this implies Kelly's conjecture on Hamilton decompositions of regular tournaments.

Theorem 4.9 ([10, Theorem 1.2])**.** *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$ and $r \geq \delta n$. Suppose D is an r-regular robust (ν, τ)-outexpander on n vertices. Then D has a Hamilton decomposition.*

The following result is a consequence of Theorem 4.9 and will be used to complete our path decompositions (recall the proof overview). In particular, this implies that any digraph D satisfying (†) from Section 2.1 is consistent, i.e. $\mathrm{pn}(D) = \mathrm{ex}(D)$.

Corollary 4.10. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$ and $r \geq \delta n$. Suppose D is a robust (ν, τ)-outexpander on n vertices with a vertex partition $V(D) = X^+ \cup X^- \cup X^* \cup X^0$ such that $|X^+ \cup X^*| = |X^- \cup X^*| = r$ and, for all $v \in V(D)$, the following hold.*

$$\mathrm{ex}_D(v) = \begin{cases} \pm 1 & \text{if } v \in X^\pm, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad d_D(v) = \begin{cases} 2r-1 & \text{if } v \in X^\pm, \\ 2r-2 & \text{if } v \in X^*, \\ 2r & \text{otherwise}. \end{cases}$$

Then, $\mathrm{pn}(D) = r$.

The proof is very similar to [14, Theorem 4.7], but we include it here for completeness.

Proof. By Fact 4.1 and Lemma 4.3, we may assume that $\tau \ll \delta$.

Note that $\mathrm{pn}(D) \geq r$. Indeed, if $X^* = V(D)$, then D is $(r-1)$-regular; otherwise, $\Delta^0(D) = r$ and so, by Fact 1.4, $\mathrm{pn}(D) \geq \widetilde{\mathrm{ex}}(D) \geq r$. Thus, it is enough to find a path decomposition of D of size r.

Let D' be obtained from D by adding a new vertex v with $N_{D'}^\pm(v) \coloneqq X^\pm \cup X^*$. Then, by Lemma 4.2, D' is a r-regular robust $(\frac{\nu}{2}, 2\tau)$-outexpander. Applying Theorem 4.9 with $D', \frac{\delta}{2}, \frac{\nu}{2}$, and 2τ playing the roles of D, δ, ν, and τ yields a Hamilton decomposition of D'. This induces a path decomposition of D of size r, as desired. □

4.2. **Probabilistic estimates.** In this section, we introduce a Chernoff-type bound and derive several easy probabilistic lemmas which will be used in the approximate decomposition step.

Let X be a random variable. We write $X \sim \mathrm{Bin}(n, p)$ if X follows a binomial distribution with parameters n and p. Let $N, n, m \in \mathbb{N}$ be such that $\max\{n, m\} \leq N$. Let Γ be a set of size N and $\Gamma' \subseteq \Gamma$ be of size m. Recall that X has a *hypergeometric distribution with parameters N, n,* *and* m if $X = |\Gamma_n \cap \Gamma'|$, where Γ_n is a random subset of Γ with $|\Gamma_n| = n$ (i.e. Γ_n is obtained by drawing n elements of Γ without replacement). We will denote this by $X \sim \mathrm{Hyp}(N, n, m)$.

We will use the following Chernoff-type bound.

Lemma 4.11 (see e.g. [7, Theorem 2.1 and Theorem 2.10])**.** *Assume $X \sim \mathrm{Bin}(n, p)$ or $X \sim \mathrm{Hyp}(N, n, m)$. Then, for any $0 < \varepsilon \leq 1$, the following hold.*

(a) $\mathbb{P}[X \leq (1-\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$.

(b) $\mathbb{P}[X \geq (1+\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$.

Using Lemma 4.11, it is easy to see that robust outexpansion is preserved with high probability when taking random edge-slices (see e.g. the proof of [11, Lemma 3.2(ii)]).

Lemma 4.12. *Let $0 < \frac{1}{n} \ll \nu \ll \tau, \gamma \leq 1$. Let D a robust (ν, τ)-outexpander on n vertices. Suppose Γ is obtained from D by taking each edge independently with probability γ. Then, with probability at least $1 - \exp(-\nu^3 n^2)$, Γ is a robust $(\frac{\gamma\nu}{2}, \tau)$-outexpander.*

Similarly, using Lemma 4.11, it is easy to see that the property of being almost regular is preserved when a random edge-slice is taken.

Lemma 4.13. *Let $0 < \frac{1}{n} \ll \varepsilon, \gamma \ll \delta \leq 1$. Let D be a (δ, ε)-almost regular digraph on n vertices. Let Γ be obtained from D by taking each edge independently with probability $\frac{\gamma}{\delta}$. Then, with probability at least $1 - \frac{1}{n}$, Γ is (γ, ε)-almost regular and $D \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular.*

Let D be a digraph on n vertices. We say D is an *(ε, p)-robust (ν, τ)-outexpander* if D is a robust (ν, τ)-outexpander and, for any integer $k \geq \varepsilon n$, if $S \subseteq V(D)$ is a random subset of size k, then $D[S]$ is a robust (ν, τ)-outexpander with probability at least $1 - p$. Note that the following analogue of Fact 4.1 holds for this new notion of robust outexpansion.

Fact 4.14. *Let D be a (ε, p)-robust (ν, τ)-outexpander. Then, for any $\varepsilon' \geq \varepsilon$, $p' \geq p$, $\nu' \leq \nu$, and $\tau' \geq \tau$, D is a (ε', p')-robust (ν', τ')-outexpander.*

Moreover, by Lemma 4.2, the following holds.

Lemma 4.15. *Let $0 < \varepsilon \leq \nu \leq \tau \leq 1$. Let D be a (ε, p)-robust (ν, τ)-outexpander on n vertices. If D' is obtained from D by removing at most εn in/out edges at each vertex, then D' is a $(\sqrt{\varepsilon}, p)$-robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander.*

We will see in the concluding remarks that any robust outexpander is in fact (ε, p)-robust (for some suitable parameters). However, our method for showing this requires the regularity lemma and so, for brevity, we will not prove this result. In this paper, we work with almost regular tournaments. Thus, it will be enough to use the next lemma, which shows that (ε, p)-robustness is easily inherited from almost regular robust outexpanders of sufficiently large minimum semidegree.

Lemma 4.16. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \frac{3}{7} \leq \delta \leq 1$. Let D be a (δ, ε)-almost regular oriented graph on n vertices. Then, there exists a (γ, ε)-almost regular spanning subdigraph Γ of D which is an (ε, n^{-3})-robust (ν, τ)-outexpander and such that $D \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular.*

Proof. Let Γ be obtained from D by taking each edge independently with probability $\frac{\gamma}{\delta}$. By Lemma 4.13, with probability at least $1 - \frac{1}{n}$, Γ is (γ, ε)-almost regular and $D \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular.

By Lemma 4.4, D is a robust $(2\gamma^{-1}\nu, \tau)$-outexpander. Therefore, by Lemma 4.12, Γ is a robust (ν, τ)-outexpander with probability at least $1 - \exp(-8\gamma^{-3}\nu^3 n^2)$.

Assume $S \subseteq V(D)$ is such that $|S| \geq \varepsilon n$ and $D[S]$ is a robust $(2\nu^{-1}\gamma, \tau)$-outexpander. Then, by Lemma 4.12, $\Gamma[S]$ is a robust (ν, τ)-outexpander with probability at least $1 - \exp(-8\gamma^{-3}\nu^3\varepsilon^2 n^2)$. Therefore, the probability that $\Gamma[S]$ is a robust (ν, τ)-outexpander for each such S is at least $1 - 2^n \exp(-8\gamma^{-3}\nu^3\varepsilon^2 n^2)$.

Thus, by a union bound, there exists a (γ, ε)-almost regular $\Gamma \subseteq D$ which is a robust (ν, τ)-outexpander and such that $D \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular and, for each $S \subseteq V(D)$ with $|S| \geq \varepsilon n$, if $D[S]$ is a robust $(2\gamma^{-1}\nu, \tau)$-outexpander then $\Gamma[S]$ is also a robust (ν, τ)-outexpander.

It now suffices to check that for any integer $k \geq \varepsilon n$, if $S \subseteq V(D)$ is chosen uniformly among the subsets of $V(D)$ of size k, then $D[S]$ is a robust $(2\gamma^{-1}\nu, \tau)$-outexpander with probability at least $1 - n^{-3}$. Fix an integer $k \geq \varepsilon n$ and let $S \subseteq V(D)$ satisfy $|S| = k$. Then, for any $v \in V(D)$, $\mathbb{E}[d^{\pm}_{D[S]}(v)] = (\delta \pm \varepsilon)|S|$ and, by Lemma 4.11,

$$\mathbb{P}\left[d^{\pm}_{D[S]}(v) < \left(\frac{3}{8} + \gamma\right)|S|\right] \leq \mathbb{P}\left[d^{\pm}_{D[S]}(v) < \frac{9}{10}\mathbb{E}[d^{\pm}_{D[S]}(v)]\right] \leq \exp\left(-\varepsilon^2 n\right).$$

Therefore, by Lemma 4.4, $D[S]$ is a robust $(2\gamma^{-1}\nu, \tau)$-outexpander with probability at least $1 - n\exp(-\varepsilon^2 n) \geq 1 - n^{-3}$. This completes the proof. □

The following result is an easy consequence of Lemma 4.11.

Lemma 4.17. *Let $0 < \frac{1}{n} \ll \frac{1}{k}, \varepsilon, \delta \ll 1$. Let D be a (δ, ε)-almost regular digraph on n vertices. Let $n_1, \dots, n_k \in \mathbb{N}$ be such that $\sum_{i \in [k]} n_i = n$ and, for each $i \in [k]$, $n_i = \frac{n}{k} \pm 1$. Assume $V_1, \dots, V_k$ is a random partition of $V(D)$ such that, for each $i \in [k]$, $|V_i| = n_i$. Then, with probability at least $1 - n^{-1}$, the following holds. For each $i \in [k]$ and $v \in V(D)$, $|N^{\pm}_D(v) \cap V_i| = (\delta \pm 2\varepsilon)\frac{n}{k}$.*

4.3. **Some tools for finding matchings.** In this section, we record two easy consequences of Hall's theorem which will enable us to construct matchings.

Proposition 4.18. *Let G be a bipartite graph on vertex classes A and B with $|A| \leq |B|$. Suppose that, for each $a \in A$, $d_G(a) \geq \frac{|B|}{2}$ and, for each $b \in B$, $d_G(b) \geq |A| - \frac{|B|}{2}$. Then, G contains a matching covering A.*

Proposition 4.19. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \delta \leq 1$. Let G be a bipartite graph on vertex classes A and B such that $|A|, |B| = (1 \pm \varepsilon)n$. Suppose that, for each $v \in V(G)$, $d_G(v) = (\delta \pm \varepsilon)n$. Then, G contains a matching of size at least $\left(1 - \frac{3\varepsilon}{\delta}\right)n$.*

4.4. **Partial path decompositions.** In this section, we introduce some notation which will be convenient to work with path decompositions, as well some basic properties about the excess of digraphs which will be used throughout the remainder of the paper.

Fact 4.20. *Any tournament $T \notin \mathcal{T}_{\mathrm{reg}}$ on n vertices satisfies $\widetilde{\mathrm{ex}}(T) \geq \left\lceil \frac{n}{2} \right\rceil$.*

The (in)equalities below follow immediately from the definitions.

Fact 4.21. *Let D be a digraph and $v \in V(D)$.*

(a) $\widetilde{\mathrm{ex}}(D) \geq \Delta^0(D) \geq \frac{\Delta(D)}{2} \geq \frac{\delta(D)}{2}$.
(b) $d^{\min}_D(v) = \frac{d_D(v) - |\mathrm{ex}_D(v)|}{2}$.
(c) $d^{\max}_D(v) = \frac{d_D(v) + |\mathrm{ex}_D(v)|}{2}$.
(d) $\widetilde{\mathrm{ex}}(D) \geq \Delta^0(D) \geq d^{\max}_D(v) = d^{\min}_D(v) + |\mathrm{ex}_D(v)|$.

The following proposition shows that, if n is even, then $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$. This will be used in Section 7 to derive Theorem 1.3 from Theorem 1.8.

Proposition 4.22. *Let T be a tournament of even order n. Then, $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$.*

Proof. It is easy to see that each $v \in V(T)$ satisfies $\mathrm{ex}_T(v) \neq 0$. Let $v \in V(T)$ be such that $d^{\max}_T(v) = \Delta^0(T)$. Thus,

$$\mathrm{ex}(T) = \frac{1}{2}\sum_{u \in V(T)} |\mathrm{ex}_T(u)| \geq \frac{n - 1 + |\mathrm{ex}_T(v)|}{2} \overset{\text{Fact 4.21(c)}}{=} d^{\max}_T(v) = \Delta^0(T),$$

so $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, as desired. □

Given a digraph D and $S \subseteq V(D)$, denote $\mathrm{ex}^{\pm}_D(S) := \sum_{v \in S} \mathrm{ex}^{\pm}_D(v)$. Then, observe that the following holds.

Fact 4.23. *Let D be a digraph, $V := V(D)$, and $S \subseteq V$. Then, $\mathrm{ex}(D) = \mathrm{ex}^{\pm}_D(V) = \mathrm{ex}^{\pm}_D(S) + \mathrm{ex}^{\pm}_D(V \setminus S)$.*

Denote $U^{\pm}(D) := \{v \in V(D) \mid \mathrm{ex}^{\pm}_D(v) > 0\}$ and $U^0(D) := \{v \in V(D) \mid \mathrm{ex}_D(v) = 0\}$.

Proposition 4.24. *Any oriented graph D satisfies $|U^0(D)| \geq \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.*

Proof. Assume for a contradiction that there exists an oriented graph D such that $|U^0(D)| < \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Then, note that $\widetilde{\mathrm{ex}}(D) = \Delta^0(D)$ and let $v \in V$ be such that $d_D^{\max}(v) = \Delta^0(D)$. Assume without loss of generality that $v \in U^+(D)$. Then, $d_D^+(v) = \widetilde{\mathrm{ex}}(D) > \mathrm{ex}(D)$. By Fact 4.23, $\mathrm{ex}(D) \geq \mathrm{ex}_D^+(v) + |U^+(D)| - 1$ and so $|U^+(D)| \leq \mathrm{ex}(D) - \mathrm{ex}_D^+(v) + 1$. Moreover, by assumption, we have $|U^0(D)| \leq d_D^+(v) - \mathrm{ex}(D)$. Therefore, by Facts 4.21(b) and 4.21(c), we have $\mathrm{ex}(D) \geq |U^-(D)| = n - |U^+(D)| - |U^0(D)| \geq n - (\mathrm{ex}(D) - \mathrm{ex}_D^+(v) + 1) - (d_D^+(v) - \mathrm{ex}(D)) = n - 1 - d_D^-(v) \geq d_D^+(v)$, a contradiction. □

This motivates the following definition. Let D be a digraph. A set $\mathcal{P}$ of edge-disjoint paths of D is called a *partial path decomposition of D* if the following hold.

(P1) Any vertex $v \in V(D) \setminus U^0(D)$ is the starting point of at most $\mathrm{ex}_D^+(v)$ paths in $\mathcal{P}$ and the ending point of at most $\mathrm{ex}_D^-(v)$ paths in $\mathcal{P}$.

(P2) Any vertex $v \in U^0(D)$ is the starting point of at most one path in $\mathcal{P}$ and the ending point of at most one path in $\mathcal{P}$.

(P3) There are at most $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ vertices $v \in U^0(D)$ such that v is an endpoint of a path in $\mathcal{P}$, that is, $|U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))| \leq \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.

Recall from Theorem 1.8 that

$$N^{\pm}(D) = |U^{\pm}(D)| + \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D).$$

Note that if $\mathcal{P}$ is a partial path decomposition of D, then there are at most $N^+(D)$ distinct vertices which are the starting point of a path in $\mathcal{P}$ and at most $N^-(D)$ distinct vertices which are the ending point of a path in $\mathcal{P}$.

Proposition 4.25. *Let D be a digraph and $\mathcal{P}$ be a partial path decomposition of D. Then,*

(a) $\mathrm{ex}(D) - |\mathcal{P}| \leq \mathrm{ex}(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}| \leq \widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D)$.

(b) *If* $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$, *then* $\mathrm{ex}(D \setminus \mathcal{P}) = \mathrm{ex}(D) - |\mathcal{P}|$.

Proof. If $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$, then by (P1) and (P2), $\mathrm{ex}(D \setminus \mathcal{P}) = \mathrm{ex}(D) - |\mathcal{P}|$ and so, since $\Delta^0(D \setminus \mathcal{P}) \leq \Delta^0(D)$, (a) and (b) hold.

We may therefore assume that $\widetilde{\mathrm{ex}}(D) = \Delta^0(D) \neq \mathrm{ex}(D)$. We show (a) is satisfied ((b) holds vacuously). Clearly, $\widetilde{\mathrm{ex}}(D) - |\mathcal{P}| = \Delta^0(D) - |\mathcal{P}| \leq \Delta^0(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D \setminus \mathcal{P})$.

We now show that $\mathrm{ex}(D) - |\mathcal{P}| \leq \mathrm{ex}(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. Let k be the number of paths in $\mathcal{P}$ which start in $U^0(D)$ and let S be the set of vertices $v \in U^0(D)$ such that no path in $\mathcal{P}$ starts at v but v is the ending point of path in $\mathcal{P}$. Note that by definition of a partial path decomposition, $k + |S| \leq \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Moreover, observe that, for each $v \in S$, $\mathrm{ex}^+_{D \setminus \mathcal{P}}(v) = 1$ and, for each $v \in (U^0(D) \setminus S) \cup U^-(D)$, $\mathrm{ex}^+_{D \setminus \mathcal{P}}(v) = 0$. For each $v \in U^+(D)$, let $n^+_{\mathcal{P}}(v)$ be the number of paths in $\mathcal{P}$ which start at v. Then, by the definition of a partial path decomposition, for each $v \in U^+(D)$, $\mathrm{ex}^+_{D \setminus \mathcal{P}}(v) = \mathrm{ex}^+_D(v) - n^+_{\mathcal{P}}(v)$. Thus, $\mathrm{ex}(D \setminus \mathcal{P}) = \sum_{v \in V(D)} \mathrm{ex}^+_{D \setminus \mathcal{P}}(v) = \mathrm{ex}(D) - (|\mathcal{P}| - k) + |S|$ and, therefore, $\mathrm{ex}(D) - |\mathcal{P}| \leq \mathrm{ex}(D \setminus \mathcal{P}) \leq \mathrm{ex}(D) - |\mathcal{P}| + (\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)) = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$, as desired.

Finally, note that, since $\mathrm{ex}(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}| \leq \widetilde{\mathrm{ex}}(D) = \Delta^0(D)$ and $\Delta^0(D \setminus \mathcal{P}) \leq \Delta^0(D)$, we have $\widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D)$. □

Let D be a digraph on n vertices. We say that a partial path decomposition $\mathcal{P}$ of D is *good* if $\widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. We say that a path decomposition $\mathcal{P}$ of D is *perfect* if $|\mathcal{P}| = \widetilde{\mathrm{ex}}(D)$.

Our path decompositions will be constructed in stages. One can easily show from the definitions that partial and perfect path decompositions can be combined to form larger (partial) path decompositions.

Fact 4.26. *Let $k \in \mathbb{N} \setminus \{0\}$ and D be a digraph. Denote $D_0 := D$.*

(a) *Suppose that, for each $i \in [k]$, $\mathcal{P}_i$ is a good partial path decomposition of D_{i-1} with $V^{\pm}(\mathcal{P}_i) \cap U^0(D_{i-1}) \subseteq U^0(D)$, and $D_i := D_{i-1} \setminus \mathcal{P}_i$. Let $\mathcal{P} := \bigcup_{i \in [k]} \mathcal{P}_i$. Suppose that for* *each $v \in U^0(D)$ there exists at most one path in $\mathcal{P}$ which starts at v and at most one path in $\mathcal{P}$ which ends at v. Suppose that $|U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))| \leq \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Then, $\mathcal{P}$ is a good partial path decomposition of D of size $|\mathcal{P}| = \sum_{i\in[k]} |\mathcal{P}_i|$.*

(b) *Suppose that, for each $i \in [k-1]$, $\mathcal{P}_i$ is a good partial path decomposition of D_{i-1} and $D_i \coloneqq D_{i-1} \setminus \mathcal{P}_i$. Suppose that $\mathcal{P}_k$ is a perfect path decomposition of D_{k-1}. Then, $\mathcal{P} \coloneqq \bigcup_{i\in[k]} \mathcal{P}_i$ is a perfect path decomposition of D of size $|\mathcal{P}| = \sum_{i\in[k]} |\mathcal{P}_i|$.*

We will also need the following result.

Proposition 4.27. *Let $0 < \frac{1}{n} \ll \eta \ll 1$. Let D be an oriented graph on n vertices satisfying $\mathrm{ex}(D) \geq (1-21\eta)n$. Then, the following hold.*

(a) *For each $\diamond \in \{+,-\}$, if there exists $v \in V$ such that $d_D^\diamond(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then*
  (i) *$\widetilde{\mathrm{ex}}(D) \leq (1+22\eta)n$; and*
  (ii) *$\mathrm{ex}_D^\diamond(v) \geq (1-86\eta)n$.*

(b) *Suppose that $\mathcal{P}$ is a partial path decomposition of D such that $|\mathcal{P}| \leq 22\eta n$ and, for each $\diamond \in \{+,-\}$ and $v \in V(D)$ satisfying $d_D^\diamond(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, $v \in V^\diamond(P) \cup V^0(P)$ for each $P \in \mathcal{P}$. Then, $\mathcal{P}$ is good.*

Proof. For (a), let $\diamond \in \{+,-\}$ and assume $v \in V$ satisfies $d_D^\diamond(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$. Note that, since $d_D^\diamond(v) \leq n$, $\widetilde{\mathrm{ex}}(D) \leq (1+22\eta)n$. Moreover, $d_D^\diamond(v) = d_D^{\max}(v)$, since otherwise $d_D(v) \geq 2d_D^\diamond(v) \geq 2\,\widetilde{\mathrm{ex}}(D) - 44\eta n \geq 2\,\mathrm{ex}(D) - 44\eta n > n$, a contradiction. Thus, by Fact 4.21(c), $\mathrm{ex}_D^\diamond(v) \geq 2\,\widetilde{\mathrm{ex}}(D) - 44\eta n - d_D(v) \geq 2\,\mathrm{ex}(D) - 44\eta n - n \geq (1-86\eta)n$.

Let $\mathcal{P}$ be as in (b). By Proposition 4.25, $\mathrm{ex}(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}| \leq \widetilde{\mathrm{ex}}(D \setminus \mathcal{P})$. Therefore, it only remains to show that $\Delta^0(D \setminus \mathcal{P}) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$.

Let $v \in V(D)$ and assume without loss of generality that $d_D^+(v) \geq d_D^-(v)$, i.e. that $v \in U^+(D) \cup U^0(D)$. If $d_D^+(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n > \frac{2n}{3}$, then $v \in U^+(D)$ and so, by assumption and since $\mathcal{P}$ is a partial path decomposition of D, $d_{D\setminus\mathcal{P}}^-(v) \leq d_{D\setminus\mathcal{P}}^+(v) = d_D^+(v) - |\mathcal{P}| \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. If $d_D^+(v) \leq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $d_{D\setminus\mathcal{P}}^{\max}(v) \leq d_D^+(v) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. Therefore, $\Delta^0(D\setminus\mathcal{P}) \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$, as desired. □

4.5. **Absorbing edges.** We will now introduce the concept of absorbing edges. As discussed in the proof overview, the goal is to complete our path decomposition by applying Corollary 4.10. But, it will not always be possible to find enough distinct vertices to serve as endpoints. To solve this problem, we will set aside some edges in order to extend the paths obtained from Corollary 4.10 to suitable endpoints.

Definition 4.28. *Let D be a digraph. Let $W, V' \subseteq V(D)$ be disjoint. An* absorbing set of (W,V')-starting edges (for D) *is a set $A \subseteq E(D)$ of edges with starting point in W and ending point in V' such that, for each $w \in W$, at most $\mathrm{ex}_D^+(w)$ edges in A start at w, and, for each $v \in V'$, at most one edge in A ends at v. Similarly, an* absorbing set of (V',W)-ending edges (for D) *is a set $A \subseteq E(D)$ of edges with starting point in V' and ending point in W such that, for each $w \in W$, at most $\mathrm{ex}_D^-(w)$ edges in A end at w, and, for each $v \in V'$, at most one edge in A starts at v.*

Let D be a digraph, let $W, V' \subseteq V(D)$ be disjoint, and $A^\pm \subseteq E(D)$. Suppose that A^+ and A^- are absorbing sets of (W,V')-starting and (V',W)-ending edges for D. Denote $A \coloneqq A^+ \cup A^-$. Then, a partial path decomposition $\mathcal{P}$ of D is *consistent* with A^+ and A^- if $\mathcal{P} \subseteq D \setminus A$ and each $v \in W$ is the starting point of at most $\mathrm{ex}_D^+(v) - d_A^+(v)$ paths in $\mathcal{P}$ and the ending point of at most $\mathrm{ex}_D^-(v) - d_A^-(v)$ paths in $\mathcal{P}$.

We will need the following observation.

Fact 4.29. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose that $A^+, A^- \subseteq E(D)$ are absorbing sets of (W,V')-starting and (V',W)-ending edges for D. Denote $A \coloneqq A^+ \cup A^-$. Suppose $\mathcal{P}$ is a partial path decomposition of D which is consistent with A^+ and A^-. Denote $D' \coloneqq D \setminus \mathcal{P}$. Then, A^+ and A^- are absorbing sets of (W,V')-starting and (V',W)-ending edges for D'.*

The following corollary shows how absorbing edges are used to the extend paths obtained from Corollary 4.10.

Corollary 4.30. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$ and $r \geq \delta n$. Suppose that D is a digraph with a vertex partition $V(D) = W \cup V'$ such that $D[V']$ is a robust (ν, τ)-outexpander on n vertices. Suppose that $A^+, A^- \subseteq E(D)$ are absorbing sets of (W, V')-starting and (V', W)-ending edges such that $|A^\pm| \leq r$. Denote $A \coloneqq A^+ \cup A^-$. Suppose furthermore that there exists a partition $V' = X^+ \cup X^- \cup X^* \cup X^0$ such that $|X^\pm \cup X^* \cup A^\pm| = r$, $V(A^\pm) \cap V' \subseteq X^\mp \cup X^0$, and, for each $v \in V(D)$, the following hold.*

$$\mathrm{ex}_D(v) = \begin{cases} d_A(v) & \text{if } v \in W, \\ \pm 1 & \text{if } v \in X^\pm, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_D(v) = \begin{cases} d_A(v) & \text{if } v \in W, \\ 2r-1 & \text{if } v \in X^\pm, \\ 2r-2 & \text{if } v \in X^*, \\ 2r & \text{otherwise.} \end{cases}$$

Then, $\mathrm{pn}(D) = r$.

Proof. Observe that $\mathrm{pn}(D) \geq r$. Indeed, if $X^* = V'$, then $D[V']$ is $(r-1)$-regular and $A^+ \cup A^- = \emptyset$. Hence, W is a set of isolated vertices and, therefore, $\mathrm{pn}(D) = \mathrm{pn}(D[V']) \geq r$. Otherwise, $\mathrm{pn}(D) \geq \Delta^0(D) = r$. Thus, it suffices to find a path decomposition of D of size r.

Let

$$\begin{aligned} Y^\pm &\coloneqq (X^\pm \cup (V(A^\pm) \cap V')) \setminus (X^\mp \cup V(A^\mp)) = (X^\pm \cup (V(A^\pm) \cap X^0)) \setminus V(A^\mp), \\ Y^* &\coloneqq X^* \cup (V(A^+) \cap V(A^-)) \cup (X^+ \cap V(A^-)) \cup (X^- \cap V(A^+)), \text{ and} \\ Y^0 &\coloneqq X^0 \setminus (V(A^+) \cup V(A^-)). \end{aligned}$$

Then, observe that Y^+, Y^-, Y^*, and Y^0 are all pairwise disjoint and form a partition of V'. Moreover, $|Y^\pm \cup Y^*| = |X^\pm \cup X^* \cup (V(A^\pm) \cap V')| = r$ and, for each $v \in V'$, the following hold.

$$\mathrm{ex}_{D[V']}(v) = \begin{cases} \pm 1 & \text{if } v \in Y^\pm, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_{D[V']}(v) = \begin{cases} 2r-1 & \text{if } v \in Y^\pm, \\ 2r-2 & \text{if } v \in Y^*, \\ 2r & \text{otherwise.} \end{cases} \tag{4.1}$$

Thus, we can apply Corollary 4.10 with $D[V'], Y^\pm, Y^*$, and Y^0 playing the roles of $D, X^\pm, X^*$, and X^0 to obtain a path decomposition $\mathcal{P}$ of $D[V']$ of size r. Note that each vertex in $Y^+ \cup Y^*$ is the starting point of exactly one path in $\mathcal{P}$ and each vertex in $Y^- \cup Y^*$ is the ending point of exactly one path in $\mathcal{P}$. Indeed, by (4.1) and since $|\mathcal{P}| = r$, $V^\pm(\mathcal{P}) \subseteq Y^\pm \cup Y^*$. Moreover, each vertex in $Y^\pm \cup Y^*$ is the starting/ending point of at most one path in $\mathcal{P}$. Thus, since $|Y^\pm \cup Y^*| = r$, each vertex in $Y^\pm \cup Y^*$ is the starting/ending point of exactly one path in $\mathcal{P}$.

Denote $\mathcal{P} \coloneqq \{P_1, \dots, P_r\}$ and, for each $i \in [r]$, let $v_i^\pm$ denote the starting/ending point of P_i. We use A^+ to absorb the paths starting at $V(A^+) \cap V'$ as follows. For each $i \in [r]$, if $v_i^+ \notin V(A^+)$, let $P_i^+ \coloneqq P_i$; otherwise, denote by $w_i^+ v_i^+$ the unique edge in A^+ which is incident to v_i^+ and let $P_i^+ \coloneqq w_i^+ v_i^+ P_i v_i^-$. Then, absorb the paths ending in $V(A^-) \cap V'$ similarly. For each $i \in [r]$, if $v_i^- \notin V(A^-)$, let $P_i^- \coloneqq P_i^+$; otherwise, denote by $v_i^- w_i^-$ the unique edge in A^- which is incident to v_i^- and let P_i^- be obtained by concatenating P_i^+ and $v_i^- w_i^-$. Then, $\mathcal{P}' \coloneqq \{P_i^- \mid i \in [r]\}$ is a path decomposition of D of size r, as desired. □

4.6. **Auxiliary excess function.** Once we have chosen absorbing edges, we need to ensure that (i) these edges are not used for other purposes and (ii) not too many paths have endpoints in W. Moreover, recall from Section 2.3 that if $\widetilde{\mathrm{ex}}(T) > \mathrm{ex}(T)$, then some vertices v will have to be used as starting/ending points of paths more than $\mathrm{ex}_T^\pm(v)$ times. For simplicity, we will choose in advance which vertices will be used as these additional endpoints. When choosing these endpoints, we aim to maximise the total number of distinct endpoints available at each step of our decomposition (this is helpful when finding suitable $X^\pm$ for Corollary 4.30 in the case where we are not using absorbing edges). In other words, this means that we will initially select a set U^* of $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ (distinct) vertices in $U^0(T)$ as additional endpoints of paths. We then treat each vertex in U^* as if it has positive and negative excess both equal to one. The following concept of an auxiliary excess function (as defined in (4.2)) encapsulates all this – it also incorporates the constraints given by (i) and (ii) above. It will enable us to easily keep track of how many paths remain to be chosen and which vertices can be used as endpoints.

Let D be a digraph and $U^* \subseteq U^0(D)$ satisf $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Let $A^+, A^- \subseteq E(D)$ be absorbing sets of (W, V')-starting and (V', W)-ending edges. Note that, by Definition 4.28, $(V(A) \cap W) \cap U^* = \emptyset$. For each $v \in V(D)$, define

$$\widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v) := \begin{cases} 1 & \text{if } v \in U^*, \\ \mathrm{ex}^{\pm}_D(v) - d^{\pm}_A(v) & \text{if } v \in V(A) \cap W, \\ \mathrm{ex}^{\pm}_D(v) & \text{otherwise.} \end{cases} \tag{4.2}$$

Then, define $\widetilde{U}^{\pm}_{U^*,W,A}(D) := \{v \in V(D) \mid \widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v) > 0\}$ and $\widetilde{U}^0_{U^*,W,A}(D) := V(D) \setminus (\widetilde{U}^+_{U^*,W,A}(D) \cup \widetilde{U}^-_{U^*,W,A}(D))$. For each $S \subseteq V(D)$, denote $\widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(S) := \sum_{v \in S} \widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v)$. Denote $\widetilde{\mathrm{ex}}^{\pm}_{U^*,W,A}(D) := \sum_{v \in V} \widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v)$. Note that

$$\widetilde{\mathrm{ex}}^{\pm}_{U^*,W,A}(D) = \mathrm{ex}(D) + |U^*| - |A^{\pm}| = \widetilde{\mathrm{ex}}(D) - |A^{\pm}|. \tag{4.3}$$

Finally, a set $\mathcal{P}$ of edge-disjoint paths of D is a *(U^*, W, A)-partial path decomposition of D* if $\mathcal{P} \subseteq D \setminus A$ and each $v \in V(D)$ is the starting point of at most $\widetilde{\mathrm{ex}}^+_{D,U^*,W,A}(v)$ paths in $\mathcal{P}$ and the ending point of at most $\widetilde{\mathrm{ex}}^-_{D,U^*,W,A}(v)$ paths in $\mathcal{P}$, i.e. if $\mathcal{P}$ is a partial path decomposition of D which is consistent with A^+ and A^- and such that, for each $v \in U^0(D)$, if v is an endpoint of a path in $\mathcal{P}$, then $v \in U^*$. For simplicity, when A and W are clear from the context, they will be omitted in the subscripts of the above notation.

Note that, by (4.2), this auxiliary excess function designates which vertices are still available to use as endpoints and, by (4.3), it indicates how many paths we are still allowed to take. For these reasons, fixing U^* at the beginning will prove very useful in Section 8, even though it is not necessary and may look cumbersome at first glance.

Note that the analogue of Fact 4.23 holds for this auxiliary excess function.

Fact 4.31. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose $A^+, A^- \subseteq E(D)$ are absorbing sets of (W, V')-starting and (V', W)-ending edges for D. Let $V := V(D)$ and let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Then, for any $S \subseteq V$, $\widetilde{\mathrm{ex}}^{\pm}_{U^*}(D) = \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(S) + \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(V \setminus S)$.*

In the final path decomposition of D, each vertex in U^* will be used as an endpoint precisely twice (once as a starting point and once as an ending point). Thus, after removing a (U^*, W, A)-partial path decomposition $\mathcal{P}$ from D, we update U^* with $U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. The following fact states that, by updating the auxiliary excess function in this way, the correct amount of excess is allocated.

Fact 4.32. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose $A^+, A^- \subseteq E(D)$ are absorbing sets of (W, V')-starting and (V', W)-ending edges for D. Denote $A := A^+ \cup A^-$. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Suppose $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D. Let $U^{**} := U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Then, $\widetilde{\mathrm{ex}}^{\pm}_{U^{**}}(D \setminus \mathcal{P}) = \widetilde{\mathrm{ex}}^{\pm}_{U^*}(D) - |\mathcal{P}|$.*

We will need the following observation.

Proposition 4.33. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose $A^+, A^- \subseteq E(D)$ are absorbing sets of (W, V')-starting and (V', W)-ending edges for D. Denote $A := A^+ \cup A^-$. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Suppose $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of D. Let $U^{**} := U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Then, $|U^{**}| = \widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) - \mathrm{ex}(D \setminus \mathcal{P})$.*

Proof. We have

$$\begin{aligned} |U^{**}| &\overset{(4.3)}{=} \widetilde{\mathrm{ex}}^+_{U^{**}}(D \setminus \mathcal{P}) - \mathrm{ex}(D \setminus \mathcal{P}) + |A^+| \overset{\text{Fact 4.32}}{=} (\widetilde{\mathrm{ex}}^+_{U^*}(D) - |\mathcal{P}|) - \mathrm{ex}(D \setminus \mathcal{P}) + |A^+| \\ &\overset{(4.3)}{=} (\widetilde{\mathrm{ex}}(D) - |A^+|) - |\mathcal{P}| - \mathrm{ex}(D \setminus \mathcal{P}) + |A^+| = \widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) - \mathrm{ex}(D \setminus \mathcal{P}). \end{aligned}$$

□

Finally, by Proposition 4.33, the following analogue of Fact 4.26(a) holds.

Fact 4.34. *Let $k \in \mathbb{N} \setminus \{0\}$. Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose $A^+, A^- \subseteq E(D)$ are absorbing sets of (W, V')-starting and (V', W)-ending edges for D. Denote $A := A^+ \cup A^-$. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| \leq \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Denote $D_0 := D$ and $U_0^* := U^*$. Suppose that, for each $i \in [k]$, $\mathcal{P}_i$ is a good (U_{i-1}^*, W, A)-partial path decomposition of D_{i-1}, $D_i := D_{i-1} \setminus \mathcal{P}_i$, and $U_i^* := U_{i-1}^* \setminus (V^+(\mathcal{P}_i) \cup V^-(\mathcal{P}_i))$. Let $\mathcal{P} := \bigcup_{i \in [k]} \mathcal{P}_i$. Then, $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of D of size $|\mathcal{P}| = \sum_{i \in [k]} |\mathcal{P}_i|$.*

4.7. **Exceptional tournaments.** Recall the definition of the class $\mathcal{T}_{\mathrm{excep}} = \mathcal{T}_{\mathrm{reg}} \cup \mathcal{T}_{\mathrm{apex}}$ of exceptional tournaments from Section 1. The main purpose of this section is to prove Theorem 1.6 as well as the following result.

Theorem 4.35. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \in \mathcal{T}_{\mathrm{excep}}$ on $n \geq n_0$ vertices satisfies $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T) + 1$.*

By Lemma 4.4, Theorem 1.6 (and thus also Theorem 4.35 in the case when $T \in \mathcal{T}_{\mathrm{reg}}$) is an immediate corollary of the following result.

Theorem 4.36. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$ and $r \geq \delta n$. Let D be a r-regular digraph on n vertices. Assume D is a robust (ν, τ)-outexpander. Then, $\mathrm{pn}(D) = \widetilde{\mathrm{ex}}(D) + 1 = r + 1$.*

Proof. Clearly, $\widetilde{\mathrm{ex}}(D) = r$. Let $P := v_1 \dots v_{r+1}$ be a path of D. Then, by Lemma 4.2, $D \setminus P$ is a robust $(\frac{\nu}{2}, \tau)$-outexpander. Let $X^+ := \{v_{r+1}\}$, $X^- := \{v_1\}$, $X^* := \{v_2, \dots, v_r\}$, and $X^0 := V(D) \setminus (X^+ \cup X^- \cup X^*)$. Applying Corollary 4.10 with $D \setminus P$ and $\frac{\nu}{2}$ playing the roles of D and ν completes the proof. □

In order to prove Theorem 4.35 for $T \in \mathcal{T}_{\mathrm{apex}}$, we need the following result.

Proposition 4.37. *Any $T \in \mathcal{T}_{\mathrm{apex}}$ on n vertices satisfies $\mathrm{pn}(T) \geq \widetilde{\mathrm{ex}}(T) + 1 = n - 1$.*

Proof. Denote by $v_\pm \in V(T)$ the unique vertices such that $v_\pm \in U^\pm(T)$ and $V_0 := V(T) \setminus \{v_+, v_-\} = U^0(T)$. Thus $v^- v^+ \in E(T)$ and one can easily verify that $\widetilde{\mathrm{ex}}(T) = n - 2$. We show that $\mathrm{pn}(T) \geq n - 1$. Indeed, let $P \subseteq T$ be a path containing the edge $v_- v_+$. It suffices to show that $\mathrm{pn}(T \setminus P) \geq n - 2$.

Let v be the starting point of P. Observe that, since $v_- v_+ \in E(P)$, we have $v \neq v_+$. If $v = v_-$, then $\mathrm{ex}(D \setminus P) \geq \mathrm{ex}_{D \setminus P}^-(v_-) = n - 2$; otherwise, $v \in U^0(T)$ and so $\mathrm{ex}(D \setminus P) \geq \mathrm{ex}_{D \setminus P}^-(v_-) + \mathrm{ex}_{D \setminus P}^-(v) = (n-3) + 1 = n - 2$. Thus, we have shown that $\mathrm{ex}(T \setminus P) \geq n - 2$. By (1.1), $T \setminus P$ cannot be decomposed into fewer than $n-2$ paths. Therefore, $\mathrm{pn}(T) \geq 1 + (n-2) = \widetilde{\mathrm{ex}}(T) + 1$. □

Proof of Theorem 4.35. By Lemma 4.4 and Theorem 4.36, we may assume that $T \in \mathcal{T}_{\mathrm{apex}}$. Fix additional constants such that $0 < \frac{1}{n_0} \ll \nu \ll \tau \ll 1$. Let $T \in \mathcal{T}_{\mathrm{apex}}$ be a tournament on $n \geq n_0$ vertices. By Proposition 4.37, $\mathrm{pn}(T) \geq \widetilde{\mathrm{ex}}(T) + 1 = n - 1$. Thus, it suffices to find a path decomposition of T of size $n - 1$.

Let $v_\pm \in V(T)$ denote the unique vertices such that $v_\pm \in U^\pm(T)$. Let $V' := U^0(T)$ and $W := V(T) \setminus V' = \{v_+, v_-\}$. Let $v_1, \dots, v_{n-2}$ be an enumeration of V', $\ell := \frac{n-1}{2}$, and $r := \frac{n-3}{2}$. Since $T[V']$ is a regular tournament on $n - 2$ vertices, Lemma 4.4 implies that $T[V']$ is a robust (ν, τ)-outexpander. Thus, by Lemma 4.7, we may assume without loss of generality that $v_1 \dots v_\ell$ is a path in $T[V']$. Let $\mathcal{P} := \{v_- v_+, v_+ v_1 \dots v_\ell v_-, v_+ v_{\ell+1} v_-, \dots, v_+ v_{n-2} v_-\}$ and $D := T \setminus \mathcal{P}$. Note that $|\mathcal{P}| = r + 2$. Moreover, $d_D(v_\pm) = \ell - 1 = r$ and, for each $i \in [n-2]$, $d_D(v_i) = n - 3 = 2r$. Let $A^+ := \{v_+ v_i \mid 2 \leq i \leq \ell\}$ and $A^- := \{v_i v_- \mid 1 \leq i \leq \ell - 1\}$. Define $X^+ := X^- := X^* := \emptyset$, and $X^0 := \{v_i \mid i \in [n-2]\}$. Note that $|X^\pm \cup X^* \cup A^\pm| = \ell - 1 = r$. Thus, we can apply Corollary 4.30 with $n - 2$ and $\frac{1}{4}$ playing the roles of n and δ to obtain a path decomposition $\mathcal{P}'$ of D of size r. Then, $\mathcal{P} \cup \mathcal{P}'$ is a path decomposition of size $r + 2 + r = n - 1$, as desired. □

We will need the following observation about tournaments in $\mathcal{T}_{\mathrm{apex}}$.

Fact 4.38. *A tournament T satisfies $|U^+(T)| = |U^-(T)| = 1$, $e(U^-(T), U^+(T)) = 1$ and $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) < 2$ if and only if $T \in \mathcal{T}_{\mathrm{apex}}$.*

5. Constructing layouts

As mentioned in the proof overview, in order to reduce the excess and the vertex degrees at the correct rate, we will approximately decompose our digraphs into sets of paths. To do so, we will start by constructing auxiliary multidigraphs called *layouts* which will prescribe the "shape" of the structures in our approximate decomposition.

For example, suppose that we would like to find a Hamilton (v_+, v_-)-path which contains a fixed edge $f = u_+u_-$. We can view this as the task of finding two paths of shapes v_+u_+ and u_-v_-, respectively, that are vertex-disjoint and cover all remaining vertices. (Recall from Section 3 that, given an (auxiliary) edge uv, a (u, v)-path has shape uv.) We now generalise this approach to *layouts*, which will tell us the shapes of paths required, the set F of fixed edges to be included, and the vertices to be avoided by these paths. The "spanning" extension of a layout will be called a *spanning configuration*. To ensure that the spanning configuration has a suitable path decomposition, we will define a layout to consist of a (multi)set of paths rather than a multiset of edges.

We will be working with multidigraphs. Let V be a vertex set. We say (L, F) is a *layout* if the following hold.

(L1) L is a multiset consisting of paths on V and isolated vertices.

(L2) $F \subseteq E(L)$.

(L3) $E(L) \setminus F \neq \emptyset$.

Conditions (L1)–(L3) can be motivated as follows. Firstly, the construction of the paths that we find in the approximate decomposition step will be based on the robust outexpansion property. But the robust outexpander that we work with will only span $V' = V \setminus W$ so we cannot automatically incorporate the set of edges F incident to W in the approximate decomposition step (here W will be the "exceptional set" defined as in Section 2.3). As indicated in the above example, we will allocate the edges in F to some given paths in advance. Replacing the edges of L which are not in F as in the above example now gives a spanning configuration (defined formally below) with endpoints induced by the endpoints of the paths in L and which contains all edges in F. In particular, each path of the spanning configuration contains exactly the edges in F which were in the corresponding path in L. Because we need to construct (almost) spanning structures, we need $E(L) \setminus F$ to be non-empty, which is the reason for (L3). Secondly, since we have already covered some edges and not all vertices have the same excess, we do not actually want our structures to be completely spanning, but want them to avoid a suitable small set of vertices. This is why we allow paths of length 0 in (L1).

A multidigraph $\mathcal{H}$ on V is a *spanning configuration of shape* (L, F) if $\mathcal{H}$ can be decomposed into internally vertex-disjoint paths $\{P_e \mid e \in E(L)\}$ such that each P_e has shape e; $P_f = f$ for all $f \in F$; and $V^0(\bigcup_{e\in E(L)}\{P_e\}) = V \setminus V(L)$. (Note that the last equality implies that the isolated vertices of L remain isolated in $\mathcal{H}$.) There is a natural bijection between a path Q in L and the path $P_Q \coloneqq \bigcup\{P_e \mid e \in E(Q)\}$ in $\mathcal{H}$ (note that this bijection is not necessarily unique since if e has multiplicity more than 1 in L, then there are different ways to define P_e). A path decomposition $\mathcal{P}$ of $\mathcal{H}$ consisting of all such P_Q for all the paths $Q \in L$ is said to be *induced* by (L, F). (Note that each path in $\mathcal{P}$ is non-trivial.) Thus, our above example is a spanning configuration of shape $(\{v_+u_+u_-v_-\}, \{u_+u_-\})$, where $(\{v_+u_+u_-v_-\}, \{u_+u_-\})$ is a layout.

For $W \subseteq V$, we say that a layout (L, F) is W*-exceptional* if $E_W(L) \subseteq F$. (Recall that $E_W(L)$ denotes the set of all those edges of L which have at least one endpoint in W.) Let $W \subseteq V$ and (L, F) be a W-exceptional layout. A multidigraph $\mathcal{H}$ on V is a W*-exceptional spanning configuration of shape* (L, F) if $\mathcal{H}$ can be decomposed into internally vertex-disjoint paths $\{P_e \mid e \in E(L)\}$ such that each P_e has shape e; $P_f = f$ for all $f \in F$; and $V^0(\bigcup_{e\in E(L)}\{P_e\}) = V \setminus (V(L) \cup W)$. Note that the last equality implies that the vertices in $W \setminus V(L)$ are isolated in $\mathcal{H}$. Thus, roughly speaking, a W-exceptional spanning configuration of shape (L, F) is one such that all "additional" edges (i.e. those edges of $\mathcal{H}$ that are not in F) are disjoint from W.

Let D be a digraph on V. When we refer to a spanning configuration of shape (L, F) in D, we mean that this configuration is contained in the multidigraph $D \cup F$ (as F may not be in D). Let $\mathcal{F}$ be a multiset of edges on V and $(L_1, F_1), \dots, (L_\ell, F_\ell)$ be layouts, where $F_i \subseteq \mathcal{F}$ for each $i \in [\ell]$. We would like the union of their spanning configurations to form a good partial path decomposition of $D \cup \mathcal{F}$. For this, these layouts will need to satisfy the following property. Let $U^* \subseteq U^0(D \cup \mathcal{F})$ be such that $|U^*| \leq \widetilde{\mathrm{ex}}(D \cup \mathcal{F}) - \mathrm{ex}(D \cup \mathcal{F})$ and define the multiset L by $L \coloneqq \bigcup_{i \in [\ell]} L_i$. We say $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are *U^*-path consistent with respect to $(D, \mathcal{F})$*, if $\bigcup_{i \in [\ell]} F_i \subseteq \mathcal{F}$ (counting multiplicity) and the following hold.

- For any $v \in V \setminus U^0(D \cup \mathcal{F})$, v is the starting point of at most $\mathrm{ex}^+_{D\cup\mathcal{F}}(v)$ non-trivial paths in L and the ending point of at most $\mathrm{ex}^-_{D\cup\mathcal{F}}(v)$ non-trivial paths in L.
- For any $v \in U^*$, L contains at most one non-trivial path starting at v and at most one non-trivial path ending at v.
- If $v \in U^0(D \cup \mathcal{F}) \setminus U^*$, then v is not an endpoint of any non-trivial path in L.

If D and $\mathcal{F}$ are clear from the context, then we omit "with respect to $(D, \mathcal{F})$".

Fact 5.1. *Let D be a digraph on a vertex set V. Let $V = W \cup V'$ be a partition of V. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| \leq \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ and $\mathcal{F} \subseteq E(D)$. Let $(L_1, F_1) \dots (L_\ell, F_\ell)$ be W-exceptional layouts. For each $i \in [\ell]$, let $\mathcal{H}_i$ be a W-exceptional spanning configuration of shape (L_i, F_i). Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ are pairwise edge-disjoint. For each $i \in [\ell]$, denote by $\mathcal{P}_i$ a path decomposition of $\mathcal{H}_i$ induced by (L_i, F_i). Define the multiset L by $L \coloneqq \bigcup_{i\in[\ell]} L_i$. Let $F \coloneqq \bigcup_{i\in[\ell]} F_i$, $\mathcal{H} \coloneqq \bigcup_{i\in[\ell]} \mathcal{H}_i$, and $\mathcal{P} \coloneqq \bigcup_{i\in[\ell]} \mathcal{P}_i$. Then, for all $v \in V$,*

$$d^\pm_{\mathcal{H}}(v) = d^\pm_L(v) + |\{i \in [\ell] \mid v \in V' \setminus V(L_i)\}|.$$

Moreover, if $(L_1, F_1) \dots (L_\ell, F_\ell)$ are U^*-path consistent with respect to $(D \setminus \mathcal{F}, \mathcal{F})$, then $\mathcal{P}$ is a partial path decomposition of D such that $|\mathcal{P}|$ is equal to the number of non-trivial paths in L, $E_W(\mathcal{P}) \subseteq F \subseteq \mathcal{F}$, and, for each $v \in U^0(D)$, if v is an endpoint of a (non-trivial) path in $\mathcal{P}$, then $v \in U^*$.

In the following lemma, we construct W-exceptional layouts which will then be turned into W-exceptional spanning configurations in the approximate decomposition step.

Lemma 5.2. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$ and $d \in \mathbb{N}$. Let D be a oriented graph on a vertex set V of size n such that the following hold.*

(a) *Let $W_1 \cup W_2 \cup V'$ be a partition of V. Denote $W \coloneqq W_1 \cup W_2$. Suppose $|W| \leq \varepsilon n$ and $E(D[W]) = \emptyset$.*
(b) *Let $U^* \subseteq U^0(D) \setminus W$ be such that $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Moreover, each $v \in U^*$ satisfies $d^+_D(v) = d^-_D(v) \leq \widetilde{\mathrm{ex}}(D) - 1$.*
(c) *Let A^+ and A^- be absorbing sets of (W_1, V')-starting and (V', W_1)-ending edges for D, respectively, and denote $A \coloneqq A^+ \cup A^-$. Suppose $X^\pm \subseteq (U^\pm(D) \cup U^*) \setminus W$ are such that $|A^\pm| + |X^\pm| = \lceil \eta n \rceil$. Define $\phi^\pm : V \longrightarrow \{0, 1\}$ by*

$$\phi^\pm(v) \coloneqq \begin{cases} 1 & \text{if } v \in X^\pm, \\ 0 & \text{otherwise.} \end{cases}$$

(d) *$d \geq \eta n$.*
(e) *$\widetilde{\mathrm{ex}}(D) \geq d + \lceil \eta n \rceil$.*
(f) *For all $v \in W_1$, $10\varepsilon n \leq d_{D\setminus A}(v) \leq 2d - \lceil \eta n \rceil$. Moreover, if $\widetilde{\mathrm{ex}}(D) < 2d + \lceil \eta n \rceil$, then, for each $v \in W_1$, one of the following holds:*
    - *$|\mathrm{ex}_D(v)| = d_D(v)$; or*
    - *$d^{\min}_D(v) \geq \eta n$ and $|\mathrm{ex}_{D\setminus A}(v)| \leq \lceil \eta n \rceil$; or*
    - *$d^{\min}_D(v) \geq \eta n$ and $d_A(v) = \lceil \eta n \rceil$.*
(g) *For all $v \in W_2$, $d^{\min}_D(v) \geq \lceil \eta n \rceil$ and $d_D(v) \leq 2d + 2\lceil \eta n \rceil$.*
(h) *For all $v \in V'$, $2d + 2\lceil \eta n \rceil - \varepsilon n \leq d_D(v) \leq 2d + 2\lceil \eta n \rceil$ and $|\mathrm{ex}_D(v)| \leq \varepsilon n$.*

Let $\mathcal{F} := E_W(D) \setminus A$ and $D' := D \setminus \mathcal{F}$. Then, there exist $\ell \in \mathbb{N}$ and W-exceptional layouts $(L_1, F_1), \dots, (L_\ell, F_\ell)$ which are U^*-path consistent with respect to $(D', \mathcal{F})$ and satisfy the following, where L is the multiset defined by $L := L_1 \cup \dots \cup L_\ell$.

(i) $d \leq \ell \leq d + \sqrt{\varepsilon} n$.
(ii) *L contains exactly $\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$ non-trivial paths.*
(iii) *For all $v \in W_1$, $d_L^\pm(v) = d_{\mathcal{F}}^\pm(v) = d_{D\setminus A}^\pm(v)$.*
(iv) *For all $v \in W_2$, $d_L^\pm(v) = d_{\mathcal{F}}^\pm(v) - \lceil \eta n \rceil = d_D^\pm(v) - \lceil \eta n \rceil$.*
(v) *For all $v \in V'$, $d_L^\pm(v) = d_D^\pm(v) - |\{i \in [\ell] \mid v \notin V(L_i)\}| - \lceil \eta n \rceil + \phi^\mp(v)$.*
(vi) *For all $i \in [\ell]$, $|V(L_i)|, |E(L_i)| \leq 3\varepsilon^{\frac{1}{3}} n$.*
(vii) *For each $v \in V'$, $d_L(v) \leq 8\varepsilon n$ and there exist at most $3\sqrt{\varepsilon} n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.*

Note that W_1 will consist of the vertices of very high excess and of the vertices adjacent to absorbing edges. W_2 will consist of the other vertices with excess greater than εn. The cleaning procedure (see Lemma 7.1) will ensure that (a)–(h) are satisfied. In the approximate decomposition step, we will construct W-exceptional spanning configurations of shapes (L_1, F_1), $\dots$, (L_ℓ, F_ℓ). We will use Fact 5.1 and (iii) to show that, after the approximate decomposition, the only remaining edges at W_1 will be the absorbing edges. Similarly, (iv) and (v) will imply that, after the approximate decomposition, each $v \in W_2$ will satisfy $d^\pm(v) = \lceil \eta n \rceil$ and each $v \in V'$ will satisfy $d^\pm(v) \in \{\lceil \eta n \rceil - 1, \lceil \eta n \rceil\}$ (depending on whether $v \in X^\mp$ or not). This is exactly the structure desired for the final step of the decomposition (recall (†) in Section 2 and Corollary 4.30).

Altogether we will obtain a path decomposition of D of size $\widetilde{\mathrm{ex}}(D)$ where each vertex in U^* will be the starting point of exactly one path and the ending point of precisely one path. This implies that each $v \in U^*$ needs to satisfy $d_D(v) \leq 2\,\widetilde{\mathrm{ex}}(D) - 2$, which explains the "moreover" condition of (b).

We will prove Lemma 5.2 as follows. In Step 1, we choose a set $\widetilde{E}$ of auxiliary edges which "neutralise" the excess of the vertices in D. In Step 2, we then subdivide these edges into paths which form a layout $(\widetilde{L}, \widetilde{F})$. In Step 3, we subdivide the paths in $\widetilde{L}$ further to obtain layouts $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ which cover the edges of $D \setminus A$ at W in such a way that (iii) and (iv) are satisfied. Finally, in Step 4, we adjust the degrees of the vertices in V' as follows. For those vertices $v \in V'$ where the current layouts would result in a degree which is too small after the approximate decomposition, we add v as an isolated vertex to some of the layouts. For vertices $v \in V'$ whose degree would be too large, we subdivide two edges from a suitable layout and include v into both of the resulting paths.

Proof of Lemma 5.2. Let $W^\pm := W \cap U^\pm(D)$. Denote $\phi := \phi^+ + \phi^-$. For each $v \in V$, define $\widehat{\mathrm{ex}}^\pm(v) := \widetilde{\mathrm{ex}}^\pm_{D,U^*}(v) - \phi^\pm(v)$. Let $\widehat{U}^\pm := \{v \in V \mid \widehat{\mathrm{ex}}^\pm(v) > 0\}$. Note that (4.3) implies that $\widehat{\mathrm{ex}}(D) := \sum_{v\in V} \widehat{\mathrm{ex}}^+(v) = \sum_{v \in V} \widehat{\mathrm{ex}}^-(v) = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$. If $(\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil) - d < \sqrt{\varepsilon} n$, let $\ell := \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$; otherwise, let $\ell := d$. Note that, by (e), $\ell \geq d$. Thus, (i) holds, as desired. Observe that either $(\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil) - \ell \geq \sqrt{\varepsilon} n$ or $(\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil) - \ell = 0$.

We claim that each $v \in V'$ satisfies

$$d_D^\pm(v) \leq \widetilde{\mathrm{ex}}(D) - \widetilde{\mathrm{ex}}^\mp_{D,U^*}(v). \tag{5.1}$$

Indeed, if $v \in U^*$, then $\widetilde{\mathrm{ex}}^\pm_{D,U^*}(v) \leq 1$ and so (5.1) holds by (b). We may therefore assume that $v \notin U^*$. Suppose without loss of generality that $d_D^+(v) \geq d_D^-(v)$. Then, $\widetilde{\mathrm{ex}}^-_{D,U^*}(v) = 0$ and so $\widetilde{\mathrm{ex}}(D) - \widetilde{\mathrm{ex}}^-_{D,U^*}(v) \geq \Delta^0(D) \geq d_D^+(v)$. Finally, $\widetilde{\mathrm{ex}}^+_{D,U^*}(v) = \mathrm{ex}_D^+(v)$ and so $\widetilde{\mathrm{ex}}(D) - \widetilde{\mathrm{ex}}^+_{D,U^*}(v) \geq d_D^+(v) - \mathrm{ex}_D^+(v) = d_D^-(v)$, as desired.

Note that throughout this proof, given a multiset L' of paths, the corresponding edge set F' in the layout (L', F') we construct will always satisfy $F' = E(L') \cap \mathcal{F} = E(L') \cap (E_W(D) \setminus A)$.

Step 1: Choosing suitable endpoints. In this step, we will select suitable endpoints for the (non-trivial) paths in L.

Let $s := \widehat{\text{ex}}(D) = \widetilde{\text{ex}}(D) - \lceil \eta n \rceil$. For each $\diamond \in \{+,-\}$, let $v_1^\diamond, \dots, v_s^\diamond \in \widehat{U}^\diamond$ be such that, for each $v \in \widehat{U}^\diamond$, there exist exactly $\widehat{\text{ex}}^\diamond(v)$ indices $i \in [s]$ for which $v = v_i^\diamond$. Since $s \geq d > 1$ and each $v \in \widehat{U}^+ \cap \widehat{U}^-$ satisfies $\widehat{\text{ex}}^+(v) = \widehat{\text{ex}}^-(v) = 1$, we may assume without loss of generality that, for each $i \in [s]$, $v_i^+ \neq v_i^-$. Let $\widetilde{E} := \{v_j^+ v_j^- \mid j \in [s]\}$. Note that,

$$|\widetilde{E}| = \widehat{\text{ex}}(D) = \widetilde{\text{ex}}(D) - \lceil \eta n \rceil \tag{5.2}$$

and, for each $v \in V$,

$$d_{\widetilde{E}}^{\pm}(v) = \widehat{\text{ex}}^{\pm}(v). \tag{5.3}$$

Step 2: Constructing layouts. Recall that $\mathcal{F} = E_W(D) \setminus A$ and $D' = D \setminus \mathcal{F}$. In this step, we will use $\widetilde{E}$ to construct a layout $(\widetilde{L}, \widetilde{F})$ such that the following hold.

(α) $(\widetilde{L}, \widetilde{F})$ is a W-exceptional layout such that $\widetilde{F} \subseteq \mathcal{F}$ and $\widetilde{L}$ contains no isolated vertex.
(β) $\widetilde{L}$ has shape $\widetilde{E}$ (and thus, by (5.3), $(\widetilde{L}, \widetilde{F})$ is U^*-path consistent with respect to $(D', \mathcal{F})$).
(γ) For each $P \in \widetilde{L}$ and $v \in V(P) \cap V'$, v is an endpoint of P or has (in P) a neighbour in W.
(δ) For each $P \in \widetilde{L}$, $V^0(P) \subseteq V'$.
(ε) Each $P \in \widetilde{L}$ has at most 4 vertices and contains an edge which lies entirely in V'

Initially, let $\widetilde{L}^0 := \widetilde{E}$ and $\widetilde{F}^0 := \emptyset$. Let $w_1, \dots, w_k$ be an enumeration of $W \cap V(\widetilde{L}^0)$. Note that, by (5.3), for each $i \in [k]$, $\text{ex}_{D \setminus A}(w_i) \neq 0$. Assume inductively that for some $0 \leq m \leq k$, we have constructed, for each $i \in [m]$, a multiset of paths $\widetilde{L}^i$ and a set of edges $\widetilde{F}^i$ such that the following are satisfied.

(I) Let $i \in [m]$. Let $S_i := \{e \in E(\widetilde{L}^{i-1}) \mid w_i \in V(e)\}$. Then, $\widetilde{L}^i$ is the multiset of paths obtained from $\widetilde{L}^{i-1}$ by subdividing each edge $e \in S_i$ with some vertex $z_e \in N_{D \setminus A}^{\max}(w_i) \cap V'$, where the vertices z_e are distinct for different edges $e \in S_i$.
(II) For each $i \in [m]$, $\widetilde{F}^i = \widetilde{F}^{i-1} \cup E_{\{w_i\}}(\widetilde{L}^i)$.

Note that (I) and (II) imply that, for each $i \in [m]$, $\widetilde{F}^i$ is a set of edges obtained from $\widetilde{F}^{i-1}$ by adding all the edges of the form $w_i z_e$ or $z_e w_i$ from (I). In particular, $\widetilde{F}^i \subseteq E_{\{w_j \mid j \in [i]\}}(D) \setminus A \subseteq \mathcal{F}$ is satisfied.

If $m = k$, then let $\widetilde{L} := \widetilde{L}^k$ and $\widetilde{F} := \widetilde{F}^k$. Observe that ($\alpha$)–($\varepsilon$) hold.

We may therefore assume that $m < k$. Note that, by (5.3), $w_{m+1} \notin U^0(D)$. Thus, we may assume without loss of generality that $w_{m+1} \in W^+$. This implies that $\text{ex}_D(w_{m+1}) = |\text{ex}_D(w_{m+1})| = \text{ex}_D^+(w_{m+1})$ and $d_D^{\min}(w_{m+1}) = d_D^-(w_{m+1})$. Moreover, by Definition 4.28, (4.2), and since, by (c), $w_{m+1} \notin X^+$, we have $\widehat{\text{ex}}^+(w_{m+1}) = \widetilde{\text{ex}}_{D,U^*}^+(w_{m+1}) = \text{ex}_D^+(w_{m+1}) - d_A^+(w_{m+1}) = \text{ex}_{D \setminus A}^+(w_{m+1})$. Construct $(\widetilde{L}^{m+1}, \widetilde{F}^{m+1})$ as follows. Define the multiset X by $X := \{v \mid w_{m+1}v \in E(\widetilde{L}^m)\}$. Construct an auxiliary bipartite graph G on vertex classes X and $Y := N_{D \setminus A}^+(w_{m+1}) \subseteq V'$ by joining $v \in X$ and $u \in Y$ if and only if $u \neq v$. Observe that, by (5.3) and (I), $|X| = \widehat{\text{ex}}^+(w_{m+1}) \leq |N_{D \setminus A}^+(w_{m+1})| = |Y|$ and, by (f) and (g), $|Y| \geq 5\varepsilon n$. Clearly, for each $v \in X$, $d_G(v) \geq |Y| - 1$. Note that (I) implies that if $v \in V'$ is contained in a path P in $\widetilde{L}^m$, then v is an endpoint of P or has (in P) a neighbour in W. Hence, together with (5.3) and (h), we have, for each $v \in Y \subseteq V'$,

$$d_G(v) \geq |X| - \widehat{\text{ex}}^-(v) - |W| \geq |X| - 2\varepsilon n.$$

Thus, applying Proposition 4.18 with X and Y playing the roles of A and B gives a matching M of G covering X.

Let $\widetilde{L}^{m+1}$ be obtained from $\widetilde{L}^m$ by subdividing, for each $vu \in M$ (with $v \in X$ and $u \in Y$), the edge $w_{m+1}v \in E(\widetilde{L}^m)$ into the path $w_{m+1}uv$. Note that this is a valid subdivision since (I) implies that the path $P \in \widetilde{L}^m$ containing $w_{m+1}v$ satisfies $V' \cap V(P) \subseteq \{v\}$. Let $\widetilde{F}^{m+1} := \widetilde{F}^m \cup E_{w_{m+1}}(\widetilde{L}^{m+1})$. Clearly, (I) and (II) are satisfied with $m+1$ playing the role of m, as desired. This completes Step 2.

Step 3: Covering additional edges incident to W. We now proceed similarly as above to ensure (iii) and (iv) are satisfied. More precisely, we construct $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ such that the following hold, where $\widehat{L} := \bigcup_{i\in[\ell]} \widehat{L}_i$ and $\widehat{F} := \bigcup_{i\in[\ell]} \widehat{F}_i$.

- (α') $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are W-exceptional layouts such that $\widehat{F} \subseteq \mathcal{F}$ and $\widehat{L}$ contains no isolated vertex.
- (β') $\widehat{L}$ is a subdivision of $\widetilde{L}$ (and thus, by (β), $\widehat{L}$ has shape $\widetilde{E}$ and $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are U^*-path consistent with respect to $(D', \mathcal{F})$).
- (γ') For each $v \in V'$, $d_{\widehat{L}}(v) \leq |\mathrm{ex}_D(v)| - \phi(v) + 2 + 2|W|$.
- (δ') For each $i \in [\ell]$, $|V(\widehat{L}_i)| \leq 5\sqrt{\varepsilon}n$ and $|E(\widehat{L}_i)| \leq 4\sqrt{\varepsilon}n$. Moreover, each path $P \in \widehat{L}$ contains an edge which lies entirely in V'.
- (ε') Either $\ell = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$ or there exist at least $\sqrt{\varepsilon}n$ indices $i \in [\ell]$ such that $\widehat{L}_i$ contains at least 2 paths. Moreover, if $\ell = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$, then for all $i \in [\ell]$, $|\widehat{L}_i| = 1$.
- (ζ') For each $v \in W_1$, $d^{\pm}_{\widehat{L}}(v) = d^{\pm}_{D\setminus A}(v)$.
- (η') For each $v \in W_2$, $d^{\pm}_{\widehat{L}}(v) = d^{\pm}_{D}(v) - \lceil \eta n \rceil$.

Let $\widehat{L}^0 := \widetilde{L}$ and $\widehat{F}^0 := \widetilde{F}$. Let $w_1, \dots, w_k$ be an enumeration of W. Let $\widehat{Q}^0$ be a set of paths in $\widehat{L}^0$ of size $\min\{2\ell, |\widehat{L}^0|\}$. Assume inductively that, for some $0 \leq m \leq k$, we have constructed, for each $i \in [m]$, two multisets of paths $\widehat{L}^i$ and $\widehat{Q}^i$, and a set of edges $\widehat{F}^i$ such that the following hold.

- (I$'$) Let $i \in [m]$. Then, for each $P \in \widehat{Q}^i$, either $P \in \widehat{Q}^{i-1}$ or there exist $P' \in \widehat{Q}^{i-1}$, an edge $e = u_e v_e \in E(P') \setminus \widehat{F}^{i-1}$ with $u_e, v_e \in V'$, and distinct $u'_e, v'_e \in V' \setminus V(P')$ such that P is obtained from P' by subdividing the edge $e = u_e v_e$ into the path $u_e u'_e w_i v'_e v_e$, where $u'_e w_i, w_i v'_e \in E(D \setminus A) \setminus \widetilde{F}$ and $\{u'_e, v'_e\} \cap \{u'_{e'}, v'_{e'}\} = \emptyset$ whenever $e, e' \in E(\widehat{Q}^{i-1})$ are distinct edges to be subdivided in order to form $\widehat{Q}^i$. Moreover, $\widehat{L}^i = (\widehat{L}^{i-1} \setminus \widehat{Q}^{i-1}) \cup \widehat{Q}^i$.
- (II$'$) For each $i \in [m]$, $\widehat{F}^i = \widehat{F}^{i-1} \cup E_{\{w_i\}}(\widehat{L}^i)$.
- (III$'$) Let $i \in [m]$. If $w_i \in W_1$, then $N^{\pm}_{\widehat{L}^m}(w_i) = N^{\pm}_{D\setminus A}(w_i)$ and, if $w_i \in W_2$, then $N^{\pm}_{\widehat{L}^m}(w_i) \subseteq N^{\pm}_{D}(w_i)$ and $d^{\pm}_{\widehat{L}^m}(w_i) = d^{\pm}_{D}(w_i) - \lceil \eta n \rceil$.

Note that (ε) and (I$'$) imply the following.

- (IV$'$) For each $i \in [m]$, each $P \in \widehat{L}^i$ contains an edge which lies entirely in V'.

Also note that, by (I$'$) and (II$'$), for each $i \in [m]$, $\widehat{F}^i$ is a set of edges (rather than a multiset) and is obtained from $\widehat{F}^{i-1}$ by adding all the edges of the form $u'_e w_i$ and $w_i v'_e$ in (I$'$). In particular,

$$\widehat{F}^{i-1} \subseteq \widehat{F}^i = E_W(\widehat{L}^i) \subseteq E_W(D) \setminus A = \mathcal{F}. \tag{5.4}$$

If $m = k$, then note that, by (5.2), we have $|\widehat{L}^k| = |\widetilde{L}| = |\widetilde{E}| = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$. We partition $\widehat{L}^k$ into $\widehat{L}_1, \dots, \widehat{L}_\ell$ such that, for each $i, j \in [\ell]$, $||\widehat{L}_i| - |\widehat{L}_j|| \leq 1$ and $|\widehat{L}_i \cap \widehat{Q}^k| \leq 2$. Note that, for each $i \in [\ell]$,

$$\begin{aligned}
|\widehat{L}_i| &\leq \left\lceil \frac{\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil}{\ell} \right\rceil \stackrel{\text{(d),(i)}}{\leq} \frac{\max\{\Delta^0(D), \mathrm{ex}(D)\} - \lceil \eta n \rceil}{\eta n} + 1 \\
&\stackrel{\text{(h)}}{\leq} \frac{\max\{n, n|W| + \varepsilon n|V'|\} - \lceil \eta n \rceil}{\eta n} + 1 \stackrel{|W| \leq \varepsilon n}{\leq} \frac{\max\{n, 2\varepsilon n^2\} - \lceil \eta n \rceil}{\eta n} + 1 \leq \sqrt{\varepsilon} n.
\end{aligned}$$

For each $i \in [\ell]$, define $\widehat{F}_i := E(\widehat{L}_i) \cap \mathcal{F} = E(\widehat{L}_i) \cap \widehat{F}^k$. Then, ($\alpha'$) holds by ($\alpha$), (I′), and (II′), while (β') follows from (I′). For (γ'), note that, by (β), (γ), (I′), and (II′), each $v \in V'$ satisfies

$$\begin{aligned} d_{\widehat{L}}(v) &\leq \widehat{\mathrm{ex}}^+(v) + \widehat{\mathrm{ex}}^-(v) + 2|N_D(v) \cap W| \\ &= \widetilde{\mathrm{ex}}^+_{D,U^*}(v) - \phi^+(v) + \widetilde{\mathrm{ex}}^-_{D,U^*}(v) - \phi^-(v) + 2|N_D(v) \cap W| \\ &\leq |\operatorname{ex}_D(v)| - \phi(v) + 2 + 2|W|. \end{aligned}$$

For (δ'), note that, by (I′) and (ε), for each $Q \in \widehat{Q}^k$, $|V(Q)| \leq 4 + 3|W|$. Since $|\widehat{L}_i \cap \widehat{Q}^k| \leq 2$, this implies that $|V(\widehat{L}_i)| \leq 4|\widehat{L}_i| + 6|W| \leq 4\sqrt{\varepsilon} n + 6\varepsilon n \leq 5\sqrt{\varepsilon} n$. Similarly, for each $Q \in \widehat{Q}^k$, $|E(Q)| \leq 3 + 3|W|$. Thus, $|E(\widehat{L}_i)| \leq 3|\widehat{L}_i| + 6|W| \leq 3\sqrt{\varepsilon} n + 6\varepsilon n \leq 4\sqrt{\varepsilon} n$. The rest of ($\delta'$) also holds by (I′) and (ε). For (ε'), note that, if $\ell \neq \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$, then, by (5.2) and choice of ℓ, $|\widehat{L}^k| = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil \geq \ell + \sqrt{\varepsilon} n$ and so, since $||\widehat{L}_i| - |\widehat{L}_j|| \leq 1$ for each $i, j \in [\ell]$, there exist at least $\sqrt{\varepsilon} n$ indices $i \in [\ell]$ such that $|\widehat{L}_i| \geq 2$. Finally, (ζ') and (η') follow from (III′).

If $m < k$, then assume without loss of generality that $w_{m+1} \notin W^-$. Thus, $w_{m+1} \in U^+(D) \cup U^0(D)$ and so $\operatorname{ex}_D(w_{m+1}) = |\operatorname{ex}_D(w_{m+1})| = \operatorname{ex}^+_D(w_{m+1})$ and $d^{\min}_D(w_{m+1}) = d^-_D(w_{m+1})$. Moreover, by Definition 4.28, $d^-_{D\setminus A}(w_{m+1}) = d^-_D(w_{m+1})$. Finally, by assumptions (b) and (c), $\phi^{\pm}(w_{m+1}) = 0$ and $U^* \cap W = \emptyset$. Thus,

$$\widehat{\mathrm{ex}}^+(w_{m+1}) = \widetilde{\mathrm{ex}}^+_{D,U^*}(w_{m+1}) = \operatorname{ex}^+_D(w_{m+1}) - d^+_A(w_{m+1}) = \operatorname{ex}^+_{D\setminus A}(w_{m+1}). \tag{5.5}$$

Proceed as follows.

First, note that, by (5.3), (β), (δ), and (I′), we have $d^-_{\widehat{L}^m}(w_{m+1}) = 0$. Fix a bijection $\sigma : N^-_{D\setminus A}(w_{m+1}) \longrightarrow N^+_{D\setminus(A\cup \widehat{F}^m)}(w_{m+1})$. Note that this is possible since

$$d^+_{\widehat{F}^m}(w_{m+1}) \overset{(5.4)}{=} d^+_{\widehat{L}^m}(w_{m+1}) \overset{(\mathrm{I}')}{=} d^+_{\widehat{L}}(w_{m+1}) \overset{(\beta),(\delta)}{=} d^+_{\widetilde{E}}(w_{m+1}) \overset{(5.3)}{=} \widehat{\mathrm{ex}}^+(w_{m+1}) \overset{(5.5)}{=} \operatorname{ex}^+_{D\setminus A}(w_{m+1}).$$

Thus, $d^+_{D\setminus(A\cup\widehat{F}^m)}(w_{m+1}) \overset{(5.4)}{=} d^+_{D\setminus A}(w_{m+1}) - d^+_{\widehat{F}^m}(w_{m+1}) = d^+_{D\setminus A}(w_{m+1}) - \operatorname{ex}^+_{D\setminus A}(w_{m+1}) = d^-_{D\setminus A}(w_{m+1})$, as desired.

Let $X := \{(u, \sigma(u)) \mid u \in N^-_{D\setminus A}(w_{m+1})\}$. Let $Y \subseteq \widehat{Q}^m$ be obtained from $\widehat{Q}^m$ by deleting all the paths that contain w_{m+1}. Define an auxiliary bipartite graph G with vertex classes X and Y by joining $(u, v) \in X$ and $P \in Y$ if and only if both $u, v \notin V(P)$.

Claim 1. *If $w_{m+1} \in W_1$, then G contains a matching M covering X. If $w_{m+1} \in W_2$, then G contains a matching M of size $|X| - \lceil \eta n \rceil$.*

Let M be as in Claim 1. We obtain $\widehat{Q}^{m+1}$ from $\widehat{Q}^m$ by subdividing, for each $(u', v')P \in M$, an edge $uv \in P$ that lies entirely in V' (which exists by (IV′)) into the path $uu'w_{m+1}v'v$. Let $\widehat{L}^{m+1} := (\widehat{L}^m \setminus \widehat{Q}^m) \cup \widehat{Q}^{m+1}$ and $\widehat{F}^{m+1} := \widehat{F}^m \cup E_{\{w_{m+1}\}}(\widehat{L}^{m+1})$. One can easily verify that (I′)–(III′) are satisfied with $m + 1$ playing the role of m. There only remains to show Claim 1.

Proof of Claim 1. Clearly, we may assume that $X \neq \emptyset$. Moreover, by (g), if $w_{m+1} \in W_2$, then $|X| = d^-_{D\setminus A}(w_{m+1}) = d^{\min}_D(w_{m+1}) \geq \lceil \eta n \rceil$. Note that ($\gamma$) and (I′) and imply that if $v \in V'$ is contained in a path $P \in \widetilde{L}^m$, then v is an endpoint of P or has (in P) a neighbour in W. Moreover, by (a), we have $u, v \in V'$ for each $(u, v) \in X$. Hence, together with (5.3) and (h), we have, for each $(u, v) \in X$,

$$\begin{aligned} d_G((u,v)) \geq\ & |Y| - (\widehat{\mathrm{ex}}^+(u) + \widehat{\mathrm{ex}}^-(u)) - 2|N_{D\setminus A}(u) \cap W| \\ & \qquad - (\widehat{\mathrm{ex}}^+(v) + \widehat{\mathrm{ex}}^-(v)) - 2|N_{D\setminus A}(v) \cap W| \\ \geq\ & |Y| - (\widetilde{\mathrm{ex}}^+_{D,U^*}(u) + \widetilde{\mathrm{ex}}^-_{D,U^*}(u) - \phi(u)) - (\widetilde{\mathrm{ex}}^+_{D,U^*}(v) + \widetilde{\mathrm{ex}}^-_{D,U^*}(v) - \phi(v)) - 4|W| \\ \geq\ & |Y| - (|\operatorname{ex}_D(u)| + 2 - \phi(u)) - (|\operatorname{ex}_D(v)| + 2 - \phi(v)) - 4|W| \\ \geq\ & |Y| - |\operatorname{ex}_D(u)| - |\operatorname{ex}_D(v)| - 4 - 4|W| \geq |Y| - 7\varepsilon n. \end{aligned}$$

By (ε) and (I′), we have, for each $P \in Y$,

$$d_G(P) \geq |X| - |V(P) \cap V'| \geq |X| - (4 + 2|W|) \geq |X| - 3\varepsilon n.$$

Suppose that

$$|Y| \geq \begin{cases} \max\{|X|, \eta n\} & \text{if } w_{m+1} \in W_1, \\ |X| - \lceil \eta n \rceil & \text{if } w_{m+1} \in W_2. \end{cases} \tag{5.6}$$

Then, we are done by Proposition 4.18, applied with $\{X, Y\}$ playing the roles of $\{A, B\}$ with $|A| \leq |B|$. To complete the proof of the claim, it suffices to prove (5.6).

Note that, by (δ) and (I′), w_{m+1} is not an internal vertex of any path in $\widehat{Q}^m$. Thus, by (β) and (5.3),

$$|Y| \geq |\widehat{Q}^m| - \widehat{\text{ex}}^+(w_{m+1}) \stackrel{(5.5)}{=} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}). \tag{5.7}$$

Assume first that $|\widehat{Q}^m| \geq 2d$ and $w_{m+1} \in W_1$. Then,

$$\begin{aligned} |Y| &\stackrel{(5.7)}{\geq} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}) \geq 2d - \text{ex}^+_{D\setminus A}(w_{m+1}) \\ &\stackrel{\text{(f)}}{\geq} d_{D\setminus A}(w_{m+1}) + \eta n - \text{ex}^+_{D\setminus A}(w_{m+1}) = 2d^-_{D\setminus A}(w_{m+1}) + \eta n \geq |X| + \eta n. \end{aligned}$$

Similarly, if $|\widehat{Q}^m| \geq 2d$ and $w_{m+1} \in W_2$, then

$$\begin{aligned} |Y| &\stackrel{(5.7)}{\geq} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}) \stackrel{w_{m+1}\notin V(A)}{=} |\widehat{Q}^m| - \text{ex}^+_{D}(w_{m+1}) \\ &\geq 2d - \text{ex}^+_D(w_{m+1}) \stackrel{\text{(g)}}{\geq} d_D(w_{m+1}) - 2\lceil \eta n \rceil - \text{ex}^+_D(w_{m+1}) \\ &= 2d^-_D(w_{m+1}) - 2\lceil \eta n \rceil = |X| - \lceil \eta n \rceil + d^-_D(w_{m+1}) - \lceil \eta n \rceil \stackrel{\text{(g)}}{\geq} |X| - \lceil \eta n \rceil, \end{aligned}$$

as desired.

Next, assume that $|\widehat{Q}^m| < 2d$. Since, by (i), $d \leq \ell$, we have $|\widehat{L}^m| = |\widehat{L}^0| = |\widehat{Q}^0| = |\widehat{Q}^m|$ and so

$$\widetilde{\text{ex}}(D) - \lceil \eta n \rceil \stackrel{(5.2)}{=} |\widetilde{E}| \stackrel{(\beta),(\text{I}')}{=} |\widehat{L}^m| = |\widehat{Q}^m| < 2d. \tag{5.8}$$

Thus,

$$\begin{aligned} |Y| &\stackrel{(5.7)}{\geq} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}) \\ &\stackrel{(5.8)}{=} \widetilde{\text{ex}}(D) - \lceil \eta n \rceil - \text{ex}^+_{D\setminus A}(w_{m+1}) \\ &\geq d^+_D(w_{m+1}) - \lceil \eta n \rceil - \text{ex}^+_D(w_{m+1}) = d^-_D(w_{m+1}) - \lceil \eta n \rceil \stackrel{w_{m+1}\notin V(A^-)}{=} |X| - \lceil \eta n \rceil. \end{aligned} \tag{5.9}$$

We may therefore assume that $w_{m+1} \in W_1$ and $|\widehat{Q}^m| < 2d$. We need to show that $|Y| \geq \max\{|X|, \eta n\}$. Recall that $d^-_D(w_{m+1}) = d^-_{D\setminus A}(w_{m+1}) = |X| > 0$. Then, $d^+_D(w_{m+1}) > \text{ex}^+_D(w_{m+1})$. Thus, by (f) and (5.8), we have $|X| \geq \eta n$ and one of the following holds: $\text{ex}^+_{D\setminus A}(w_{m+1}) \leq \lceil \eta n \rceil$ or $d^+_A(w_{m+1}) = d_A(w_{m+1}) = \lceil \eta n \rceil$. Thus, it suffices to show that $|Y| \geq |X|$. If $\text{ex}^+_{D\setminus A}(w_{m+1}) \leq \lceil \eta n \rceil$, then, by (5.9) and (e),

$$\begin{aligned} |Y| &\geq d - \text{ex}^+_{D\setminus A}(w_{m+1}) \stackrel{\text{(f)}}{\geq} \frac{d_{D\setminus A}(w_{m+1}) + \lceil \eta n \rceil}{2} - \text{ex}^+_{D\setminus A}(w_{m+1}) \\ &\stackrel{\text{Fact 4.21(b)}}{=} d^-_{D\setminus A}(w_{m+1}) + \frac{\lceil \eta n \rceil - \text{ex}^+_{D\setminus A}(w_{m+1})}{2} \geq |X|, \end{aligned}$$

as desired. If $d^+_A(w_{m+1}) = \lceil \eta n \rceil$, then (5.9) implies that

$$|Y| \geq d^+_D(w_{m+1}) - \lceil \eta n \rceil - \text{ex}^+_{D\setminus A}(w_{m+1}) = d^+_{D\setminus A}(w_{m+1}) - \text{ex}^+_{D\setminus A}(w_{m+1}) = d^-_{D\setminus A}(w_{m+1}) = |X|,$$

as desired. ◇

Step 4: Adjusting the degree of the vertices in V'. Finally, we add isolated vertices and subdivide paths to ensure (v) is satisfied. Let $v_1, \ldots, v_k$ be an enumeration of V' and, for each $i \in [k]$, define

$$n_i := d^+_{\widehat{L}}(v_i) + |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| + \lceil \eta n \rceil - \phi^-(v_i) - d^+_D(v_i). \tag{5.10}$$

Claim 2. *For each $i \in [k]$,*

$$d^\pm_D(v_i) = d^\pm_{\widehat{L}}(v_i) + |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| - n_i + \lceil \eta n \rceil - \phi^\mp(v).$$

Proof of Claim. Let $i \in [k]$. The equality for $+$ holds immediately by definition of n_i. One can easily verify that, in order to show that the equality for $-$ holds, it is enough to prove that

$$d^+_{\widehat{L}}(v_i) - \phi^-(v_i) - d^+_D(v_i) = d^-_{\widehat{L}}(v_i) - \phi^+(v_i) - d^-_D(v_i). \tag{5.11}$$

We now show that (5.11) is satisfied. First, note that, by (5.3) and (β'),

$$d^+_{\widehat{L}}(v_i) = (d^-_{\widehat{L}}(v_i) - \widehat{\mathrm{ex}}^-(v_i)) + \widehat{\mathrm{ex}}^+(v_i). \tag{5.12}$$

Assume without loss of generality that $d^+_D(v_i) \geq d^-_D(v_i)$. Suppose first that $v_i \notin U^*$. Then, $\widehat{\mathrm{ex}}^\pm(v_i) = \mathrm{ex}^\pm_D(v_i) - \phi^\pm(v_i)$. Moreover, $\mathrm{ex}^-_D(v_i) = 0$ and so $\widehat{\mathrm{ex}}^-(v_i) = -\phi^-(v_i)$. Thus, by (5.12),

$$\begin{aligned} d^+_{\widehat{L}}(v_i) &= (d^-_{\widehat{L}}(v_i) + \phi^-(v_i)) + (\mathrm{ex}^+_D(v_i) - \phi^+(v_i)) \\ &= d^-_{\widehat{L}}(v_i) + \phi^-(v_i) + (d^+_D(v_i) - d^-_D(v_i)) - \phi^+(v_i), \end{aligned}$$

so (5.11) holds, as desired. Now suppose that $v_i \in U^*$. Then, $\widehat{\mathrm{ex}}^\pm(v_i) = 1 - \phi^\pm(v_i)$ and $d^+_D(v_i) = d^-_D(v_i)$. Thus, by (5.12),

$$d^+_{\widehat{L}}(v_i) = (d^-_{\widehat{L}}(v_i) - 1 + \phi^-(v_i)) + (1 - \phi^+(v_i)) = d^-_{\widehat{L}}(v_i) + \phi^-(v_i) - \phi^+(v_i) + (d^+_D(v_i) - d^-_D(v_i)),$$

so (5.11) holds, as desired. $\diamond$

Claim 3. *For each $i \in [k]$,*

$$-2\varepsilon n \leq n_i \leq 2\sqrt{\varepsilon} n.$$

Proof of Claim. Let $i \in [k]$. We have

$$2n_i \overset{\text{Claim 2}}{=} d_{\widehat{L}}(v_i) + 2|\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| + 2\lceil \eta n \rceil - \phi(v_i) - d_D(v_i).$$

Thus,

$$\begin{aligned} 2n_i &\geq d_{\widehat{L}}(v_i) + 2(\ell - d_{\widehat{L}}(v_i)) + 2\lceil \eta n \rceil - \phi(v_i) - d_D(v_i) \\ &= 2\ell - d_{\widehat{L}}(v_i) + 2\lceil \eta n \rceil - \phi(v_i) - d_D(v_i) \\ &\overset{(\gamma')}{\geq} 2\ell - (|\mathrm{ex}_D(v_i)| - \phi(v_i) + 2 + 2|W|) + 2\lceil \eta n \rceil - \phi(v_i) - (2d + 2\lceil \eta n \rceil) \\ &= 2(\ell - d) - |\mathrm{ex}_D(v_i)| - 2 - 2|W| \overset{\text{(a),(h),(i)}}{\geq} -4\varepsilon n. \end{aligned}$$

Similarly,

$$\begin{aligned} 2n_i &\overset{(\gamma')}{\leq} (|\mathrm{ex}_D(v_i)| - \phi(v_i) + 2 + 2|W|) + 2\ell + 2\lceil \eta n \rceil - \phi(v_i) - (2d + 2\lceil \eta n \rceil - \varepsilon n) \\ &\leq 2(\ell - d) + |\mathrm{ex}_D(v_i)| + 2 + 2|W| + \varepsilon n \overset{\text{(a),(h),(i)}}{\leq} 4\sqrt{\varepsilon} n, \end{aligned}$$

which proves the claim. $\diamond$

If $n_i > 0$, then, in order to satisfy (v), it is enough to add v_i as an isolated vertex to exactly n_i of the sets of paths $\widehat{L}_1, \dots, \widehat{L}_\ell$ that do not contain v_i. If $n_i < 0$, then it is enough to find $-n_i$ indices $j \in [\ell]$ such that $v_i \notin V(\widehat{L}_j)$ and $|\widehat{L}_j| \geq 2$, and add v_i as an internal vertex in exactly two paths in $\widehat{L}_j$. We do so inductively as follows.

Assume without loss of generality, $(n_i)_{i \in [k]}$ is an increasing sequence and so, for any $i, j \in [k]$, if $n_i < 0$ but $n_j \geq 0$, then $i < j$. For each $i \in [\ell]$, let $L_i^0 := \widehat{L}_i$. Assume inductively that, for some $0 \leq m \leq k$, we have constructed, for each $i \in [\ell]$ and $j \in [m]$, a multiset L_i^j of paths and isolated vertices such that the following are satisfied, where $L^j := \bigcup_{i \in [\ell]} L_i^j$ for each $j \in [m]$.

(I″) For each $j \in [m]$, if $n_j < 0$, then there exists $N_j \subseteq [\ell]$ such that $|N_j| = -n_j$ and the following hold. For each $i \in N_j$, $v_j \notin V(L_i^{j-1})$ and there exist two paths $P_1, P_2 \in L_i^{j-1}$ such that L_i^j is obtained from L_i^{j-1} by subdividing, for each $s \in [2]$, an edge $uw \in E(P_s) \setminus E_W(P_s)$ into the path uv_jw. For each $i \in [\ell] \setminus N_j$, $L_i^j = L_i^{j-1}$.

(II″) For each $j \in [m]$, if $n_j \geq 0$, then there exists $N_j \subseteq [\ell]$ such that $|N_j| = n_j$ and the following hold. For each $i \in N_j$, $v_j \notin V(L_i^{j-1})$ and L_i^j is obtained from L_i^{j-1} by adding v_j as an isolated vertex. For each $i \in [\ell] \setminus N_j$, $L_i^j = L_i^{j-1}$.

(III″) For each $i \in [\ell]$ and $j \in [m]$, $|V(L_i^j) \setminus V(\widehat{L}_i)| \leq \varepsilon^{\frac{1}{3}} n$.

Note that (I″) and (II″) imply that the following hold.

(IV″) For each $i \in [\ell]$ and $j \in [m]$, $|E(L_i^j) \setminus E(\widehat{L}_i)| \leq 2|V(L_i^j) \setminus V(\widehat{L}_i)|$.

(V″) For each $j \in [m]$, $\sum_{i \in [\ell]} |V(L_i^j)| = \sum_{i \in [\ell]} |V(\widehat{L}_i)| + \sum_{j' \in [j]} |n_{j'}|$.

If $m = k$, then let $L_i := L_i^k$ and $F_i := \widehat{F}_i$ for each $i \in [\ell]$. Then, note that, by (α'), (β'), (I″), and (II″), $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are W-exceptional U^*-path consistent layouts with respect to $(D', \mathcal{F})$. Moreover, (i)–(vii) hold. Indeed, we have shown before Step 1 that (i) holds. (ii) holds by (5.2), (β'), (I″), and (II″). (iii) and (iv) hold by (ζ'), (η'), (I″), and (II″). (v) holds by Claim 2, (I″), and (II″). (vi) follows from (δ'), (III″), and (IV″). For (vii), note that, for each $i \in [k]$, by (a), (h), (γ'), Claim 3, (I″), and (II″), $d_L(v_i) = d_{\widehat{L}}(v_i) + 2\max\{-n_i, 0\} \leq |\mathrm{ex}_D(v_i)| + 2 + 2|W| + 2\max\{-n_i, 0\} \leq \varepsilon n + 2 + 2\varepsilon n + 4\varepsilon n \leq 8\varepsilon n$. Moreover, by (a), (h), (α'), (γ'), Claim 3, (I″), and (II″), for each $i \in [k]$, there exist at most $d_{\widehat{L}}(v_i) + |n_i| \leq |\mathrm{ex}_D(v_i)| + 2 + 2|W| + |n_i| \leq \varepsilon n + 2 + 2\varepsilon n + 2\sqrt{\varepsilon} n \leq 3\sqrt{\varepsilon} n$ indices $j \in [\ell]$ such that $v_i \in V(L_j)$.

Assume $m < k$. By Claim 2, $n_i = d_{\widehat{L}}^{\pm}(v_i) + |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| + \lceil \eta n \rceil - \phi^{\mp}(v_i) - d_D^{\pm}(v_i)$ and so we may suppose without loss of generality that $v_{m+1} \notin U^-(D)$. Let X be the set of indices $i \in [\ell]$ such that $|V(L_i^m) \setminus V(\widehat{L}_i)| = \lfloor \varepsilon^{\frac{1}{3}} n \rfloor$. Let Z be the set of indices $i \in [\ell]$ such that $v_{m+1} \in V(L_i^m)$. By (a), (h), (α'), (γ'), (I″), and (II″), $|Z| \leq d_{L^m}(v_{m+1}) = d_{\widehat{L}}(v_{m+1}) \leq |\mathrm{ex}_D(v_{m+1})| - \phi(v) + 2 + 2|W| \leq 4\varepsilon n$.

If $n_{m+1} < 0$, then proceed as follows. Let Y be the set of indices $i \in [\ell] \setminus (X \cup Z)$ such that $|L_i^m| \geq 2$. We claim that $|Y| \geq -n_{m+1}$. By our choice of ordering $v_1, \dots, v_k$ of V', Claim 3, and (V″), we have $|X| \leq \frac{n \cdot 2\varepsilon n}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \leq 3\varepsilon^{\frac{2}{3}} n$. Assume for a contradiction that $\ell = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$. Then, by (ε'), for each $i \in [\ell]$, $|\widehat{L}_i| = 1$. Thus, by (5.3) and (β'), $d_{\widehat{L}}^+(v_{m+1}) + |\{i \in [\ell] \mid v_{m+1} \notin V(\widehat{L}_i)\}| = \ell - \widehat{\mathrm{ex}}^-(v_{m+1})$. Therefore,

$$\begin{aligned} n_{m+1} &\overset{(5.10)}{=} (\ell + \lceil \eta n \rceil) - (\widehat{\mathrm{ex}}^-(v_{m+1}) + \phi^-(v_{m+1})) - d_D^+(v_{m+1}) \\ &= \widetilde{\mathrm{ex}}(D) - \widetilde{\mathrm{ex}}_{D,U^*}^-(v_{m+1}) - d_D^+(v_{m+1}) \overset{(5.1)}{\geq} 0, \end{aligned}$$

a contradiction. Therefore, by (ε') and Claim 3, $|Y| \geq \sqrt{\varepsilon} n - 3\varepsilon^{\frac{2}{3}} n - 4\varepsilon n \geq -n_{m+1}$, as desired.

Let $N_{m+1} \subseteq Y$ be such that $|N_{m+1}| = -n_{m+1}$ and, for each $i \in N_{m+1}$, fix two paths $P_{i,1}, P_{i,2} \in \widehat{L}_i^m$. For each $i \in N_{m+1}$ and $j \in [2]$, let $u_{i,j}w_{i,j} \in E(P_{i,j}) \setminus E_W(P_{i,j})$, which exists by (δ') and (I″). For each $i \in [\ell] \setminus N_{m+1}$, let $L_i^{m+1} := L_i^m$. For each $i \in N_{m+1}$, let L_i^{m+1} be obtained from L_i^m by subdividing, for each $j \in [2]$, the edge $u_{i,j}w_{i,j}$ in $P_{i,j}$ into the path $u_{i,j}v_{m+1}w_{i,j}$. Then, (I″)–(III″) are satisfied with $m+1$ playing the role of m.

If $n_{m+1} \geq 0$, then proceed as follows. Let $Y := [\ell] \setminus (X \cup Z)$. We claim that $|Y| \geq n_{m+1}$. By (V″) and Claim 3, $|X| \leq \frac{2\sqrt{\varepsilon} n \cdot n}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \leq 3\varepsilon^{\frac{1}{6}} n$. Recall that $|Z| \leq 4\varepsilon n$. Thus, by (d), (i), and Claim 3, $|Y| \geq \eta n - 3\varepsilon^{\frac{1}{6}} n - 4\varepsilon n \geq n_{m+1}$.

Let $N_{m+1} \subseteq Y$ satisfy $|N_{m+1}| = n_{m+1}$. For each $i \in [\ell] \setminus N_{m+1}$, let $L_i^{m+1} := L_i^m$ and, for each $i \in N_{m+1}$, let L_i^{m+1} be obtained from L_i^m by adding v_{m+1} as an isolated vertex. Clearly, (I″)–(III″) hold with $m+1$ playing the role of m, as desired. □

6. Approximate decomposition

We are now ready for the approximate decomposition. In this step, we transform the layouts constructed in Section 5 into spanning configurations. Recall that we constructed W-exceptional layouts and, as mentioned in Section 5, we aim to construct W-exceptional spanning configurations so that, by Fact 5.1, the leftover will have the desired degrees.

It turns out to be convenient to first transform a W-exceptional layout on V into a layout on $V \setminus W$. This allows us to ignore W when turning layouts into spanning configurations.

Definition 6.1. *Let V be a vertex set and $W \subseteq V$. Suppose (L, F) is a W-exceptional layout on V. We denote by $(L^{\upharpoonright W}, F^{\upharpoonright W})$ the layout on $V \setminus W$ obtained from (L, F) as follows.*

Let $\mathcal{P}$ be the multiset of maximal paths P such that $P \subseteq P'$ for some $P' \in L$, $V^0(P) \subseteq W$, and $V(P) \cap W \neq \emptyset$ (in particular, each isolated vertex $v \in V(L) \cap W$ is a path in $\mathcal{P}$ but no isolated vertex $v \in V(L) \setminus W$ is a path in $\mathcal{P}$). Note that, since (L, F) is W-exceptional, each $P \in \mathcal{P}$ satisfies $E(P) \subseteq F$. Let $P_1, \dots, P_k$ be an enumeration of $\mathcal{P}$ and, for each $i \in [k]$, let x_i and y_i denote the starting and ending points of P_i, respectively. Then, let $L^{\upharpoonright W}$ be obtained from L as follows. For each $i \in [k]$, if both $x_i, y_i \in V \setminus W$, then contract the subpath P_i into an edge $x_i y_i$ and, otherwise, delete $E(P_i)$ as well as $V(P_i) \cap W$. Note that $V(L^{\upharpoonright W}) = V(L) \setminus W \subseteq V \setminus W$. Define $F^{\upharpoonright W} := \{x_i y_i \mid i \in [k], x_i, y_i \in V \setminus W\} \cup (F \setminus E_W(F)) = \{x_i y_i \mid i \in [k], x_i, y_i \in V \setminus W\} \cup (F \setminus E_W(L))$.

The following fact states that a spanning configuration of shape $(L^{\upharpoonright W}, F^{\upharpoonright W})$ in $D[V \setminus W]$ can easily be transformed into a W-exceptional spanning configuration of shape (L, F) in D. In other words, it allows us to reverse the process described in Definition 6.1.

Fact 6.2. *Let D be a digraph on a vertex set V. Let $W \subseteq V$ and denote $V' := V \setminus W$. Let (L, F) be a W-exceptional layout on V. Let $(L^{\upharpoonright W}, F^{\upharpoonright W})$ be as in Definition 6.1. Suppose $\mathcal{H}^{\upharpoonright W} \subseteq D[V'] \cup F^{\upharpoonright W}$ is a spanning configuration of shape $(L^{\upharpoonright W}, F^{\upharpoonright W})$. Let $\mathcal{H}$ be the multidigraph with $V(\mathcal{H}) := V$ and $E(\mathcal{H}) := (E(\mathcal{H}^{\upharpoonright W}) \setminus F^{\upharpoonright W}) \cup F$. Then, $\mathcal{H} \subseteq D[V'] \cup F$ and $\mathcal{H}$ is a W-exceptional spanning configuration of shape (L, F).*

This has the advantage that it suffices to find spanning configurations in an almost regular robust outexpander, which corresponds to the setting of Lemma 6.3 below. More precisely, if we let $V' := V \setminus W$ be the set of "non-exceptional vertices" described in Section 2.3 and let D' be the remainder of the tournament $T[V']$ after the cleaning step, then D' is almost complete and almost regular, and hence a robust outexpander. Then, we can split D' into D and Γ as required for Lemma 6.3.

Lemma 6.3. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \eta, \delta \leq 1$. Suppose $\ell \in \mathbb{N}$ satisfies $\ell \leq (\delta - \eta)n$. If $\ell \leq \varepsilon^2 n$, then let $p \leq n^{-1}$; otherwise, let $p \leq n^{-2}$. Let D and Γ be edge-disjoint digraphs on a common vertex set V of size n. Suppose D is (δ, ε)-almost regular and Γ is (γ, ε)-almost regular. Suppose further that Γ is an (ε, p)-robust (ν, τ)-outexpander. Let $\mathcal{F}$ be a multiset of directed edges on V. Any edge in $\mathcal{F}$ is considered to be distinct from the edges of $D \cup \Gamma$, even if the starting and ending points are the same (recall Section 3). Let $F_1, \dots, F_\ell$ be a partition of $\mathcal{F}$. Assume $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are layouts such that $V(L_i) \subseteq V$ for each $i \in [\ell]$ and the following hold, where $L := \bigcup_{i \in [\ell]} L_i$.*

(a) *For each $i \in [\ell]$, $|V(L_i)| \leq \varepsilon^2 n$ and $|E(L_i)| \leq \varepsilon^4 n$.*
(b) *Moreover, for each $v \in V$, $d_L(v) \leq \varepsilon^3 n$ and there exist at most $\varepsilon^2 n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.*

Then, there exist edge-disjoint $\mathcal{H}_1, \dots, \mathcal{H}_\ell \subseteq D \cup \Gamma \cup \mathcal{F}$ such that, for each $i \in [\ell]$, $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) and the following hold, where $\mathcal{H} := \bigcup_{i \in [\ell]} \mathcal{H}_i$, $D' := D \setminus \mathcal{H}$, and $\Gamma' := \Gamma \setminus \mathcal{H}$.

(i) *If $\ell \leq \varepsilon^2 n$, then Γ' is obtained from Γ by removing at most $3\varepsilon^3 \nu^{-4} n$ edges incident to each vertex.*
(ii) *If $\ell \leq \nu^5 n$, then D' is $(\delta - \frac{\ell}{n}, 2\varepsilon)$-almost regular and Γ' is $(\gamma, 2\varepsilon)$-almost regular. Moreover, Γ' is a $(\sqrt{\varepsilon}, p)$-robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander.*

(iii) $D' \cup \Gamma'$ *is a robust* $(\frac{\nu}{2}, \tau)$*-outexpander.*

The approximate decomposition is constructed in stages. The core of the approximate decomposition occurs in Lemma 6.3(i), where a small set of layouts is converted into spanning configurations one by one. Repeated applications of Lemma 6.3(i) will then enable us to transform larger sets of layouts into spanning configurations (Lemma 6.3(ii)). Then, one can obtain the final approximate decomposition (Lemma 6.3(iii)) by repeatedly applying Lemma 6.3(ii), adjusting the parameters in each iteration. This can be seen as a semirandom "nibble" process, where the applications of Lemma 6.3(i) are the "nibbles" (which are chosen via a probabilistic argument) and the applications of Lemma 6.3(ii) correspond to "bites" consisting of several "nibbles". We prove (ii), (iii), and (i) in this order.

Proof of Lemma 6.3(ii). Let $\ell' := \lfloor \varepsilon^2 n \rfloor$ and $k := \lceil \frac{\ell}{\ell'} \rceil$. Note that $k \leq 2\nu^5\varepsilon^{-2}$. We now group $(L_1, F_1), \dots, (L_\ell, F_\ell)$ into k batches, each of size at most ℓ'. We aim to apply Lemma 6.3(i) to each batch in turn.

Assume that, for some $0 \leq m \leq k$, we have constructed edge-disjoint $\mathcal{H}_1, \dots, \mathcal{H}_{\min\{m\ell', \ell\}} \subseteq D \cup \Gamma \cup \mathcal{F}$ such that, for each $i \in [\min\{m\ell', \ell\}]$, $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) satisfying $E(\mathcal{H}_i) \cap E(\mathcal{F}) = E(F_i)$ and the following holds. Let $\Gamma_m := \Gamma \setminus \bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i$. Then, for each $v \in V$,

$$|N_{\Gamma \cap \bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i}(v)| = |N_{\Gamma \setminus \Gamma_m}(v)| \leq 25\varepsilon^3\nu^{-4}mn \leq 50\varepsilon\nu n \leq \frac{\varepsilon n}{2}. \tag{6.1}$$

Let $D_m := D \setminus \bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i$. Observe that, by Fact 5.1 and (b), $\bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i$ is $(\frac{\min\{m\ell', \ell\}}{n}, \varepsilon^2 + \varepsilon^3)$-almost regular. Together with (6.1), this implies that D_m is $(\delta - \frac{\min\{m\ell', \ell\}}{n}, 2\varepsilon)$-almost regular and Γ_m is $(\gamma, 2\varepsilon)$-almost regular. Moreover, by Lemma 4.15, Γ_m is a $(\sqrt{\varepsilon}, p)$-robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander. Thus, if $m = k$, we are done.

Suppose $m < k$. We show that Γ_m is a $(2\varepsilon, n^{-1})$-robust $(\nu - \varepsilon, \tau)$-outexpander. If $m = 0$, then $\Gamma_m = \Gamma$ and we are done. We may therefore assume that $m \geq 1$. Then, note that $k \geq 2$ so $\ell > \ell' = \lfloor \varepsilon^2 n \rfloor$ and, thus, $p \leq n^{-2}$. Fix an integer $k' \geq 2\varepsilon n$. Suppose $S \subseteq V$ is a random subset of size k'. We show that $\Gamma_m[S]$ is a robust $(\nu - \varepsilon, \tau)$-robust outexpander with probability at least $1 - n^{-1}$. Let $v \in V$. If $|N_{\Gamma \setminus \Gamma_m}(v)| \leq \varepsilon^2 n$, then $|N_{\Gamma \setminus \Gamma_m}(v) \cap S| \leq \varepsilon^2 n \leq \varepsilon k'$. Suppose $|N_{\Gamma \setminus \Gamma_m}(v)| \geq \varepsilon^2 n$. Then, by (6.1), $\mathbb{E}[|N_{\Gamma \setminus \Gamma_m}(v) \cap S|] = \frac{k'}{n}|N_{\Gamma \setminus \Gamma_m}(v)| \leq \frac{\varepsilon k'}{2}$. Thus, Lemma 4.11 implies that

$$\mathbb{P}\left[|N_{\Gamma \setminus \Gamma_m}(v) \cap S| > \varepsilon k'\right] \leq \mathbb{P}\left[|N_{\Gamma \setminus \Gamma_m}(v) \cap S| > 2\mathbb{E}[|N_{\Gamma \setminus \Gamma_m}(v) \cap S|]\right] \leq \exp\left(-\frac{2\varepsilon^3 n}{3}\right).$$

Therefore, by a union bound, with probability at least $1 - n\exp\left(-\frac{2\varepsilon^3 n}{3}\right)$, the digraph $\Gamma_m[S]$ is obtained from $\Gamma[S]$ by removing at most $\varepsilon k'$ edges incident to each vertex. Our assumption on Γ implies that $\Gamma[S]$ is a robust (ν, τ)-outexpander with probability at least $1 - p \geq 1 - n^{-2}$. Therefore, by Lemma 4.2, we conclude that $\Gamma_m[S]$ is a robust $(\nu - \varepsilon, \tau)$-outexpander with probability at least $1 - p - n\exp\left(-\frac{2\varepsilon^3 n}{3}\right) \geq 1 - n^{-1}$. Thus, Γ_m is a $(2\varepsilon, n^{-1})$-robust $(\nu - \varepsilon, \tau)$-outexpander.

Let $\ell'' := \min\{\ell - m\ell', \ell'\}$ and $\mathcal{F}' := \bigcup_{i \in [\ell'']} \mathcal{F}_{m\ell'+i}$. Apply Lemma 6.3(i) with D_m, Γ_m, $\mathcal{F}'$, n^{-1}, $\delta - \frac{m\ell'}{n}$, $\nu - \varepsilon$, 2ε, ℓ'', $L_{m\ell'+1}, \dots, L_{m\ell'+\ell''}$, and $F_{m\ell'+1}, \dots, F_{m\ell'+\ell''}$ playing the roles of D, Γ, $\mathcal{F}$, p, δ, ν, ε, ℓ, $L_1, \dots, L_\ell$, and $F_1, \dots, F_\ell$ to obtain edge-disjoint $\mathcal{H}_{m\ell'+1}, \dots, \mathcal{H}_{m\ell'+\ell''} \subseteq D_m \cup \Gamma_m \cup \mathcal{F}'$ such that, for each $i \in [\ell'']$, $\mathcal{H}_{m\ell'+i}$ is a spanning configuration of shape $(L_{m\ell'+i}, F_{m\ell'+i})$ and, for each $v \in V$, $|N_{\Gamma_m \setminus \Gamma_{m+1}}(v)| \leq 3(2\varepsilon)^3(\nu - \varepsilon)^{-4}n \leq 25\varepsilon^3\nu^{-4}n$, where $\Gamma_{m+1} := \Gamma_m \setminus \bigcup_{i \in [\ell'']} \mathcal{H}_{m\ell'+i}$. In particular, (6.1) holds. This completes the proof. □

Proof of Lemma 6.3(iii). Let $\ell' := \lfloor \nu^5 n \rfloor$ and $k := \lceil \frac{\ell}{\ell'} \rceil$. Note that $k \leq \nu^{-5}$. For each $i \in \mathbb{N}$, denote $\varepsilon_i := 2^i \varepsilon^{\frac{1}{2^i}}$. Assume inductively that, for some $0 \leq m \leq k$, we have constructed edge-disjoint $\mathcal{H}_1, \dots, \mathcal{H}_{\min\{m\ell', \ell\}} \subseteq D \cup \Gamma \cup \mathcal{F}$ such that

- for each $i \in [\min\{m\ell', \ell\}]$, $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) satisfying $E(\mathcal{H}_i) \cap E(\mathcal{F}) = E(F_i)$;

– for each $i \in [m]$, $D_i \coloneqq D \setminus \bigcup_{j \in [\min\{i\ell', \ell\}]} \mathcal{H}_j$ is $(\delta - \frac{\min\{i\ell',\ell\}}{n}, \varepsilon_i)$-almost regular; and
– for each $i \in [m]$, $\Gamma_i \coloneqq \Gamma \setminus \bigcup_{j \in [\min\{i\ell', \ell\}]} \mathcal{H}_j$ is a (γ, ε_i)-almost regular (ε_i, p)-robust $(\nu - \varepsilon_i, \tau)$-outexpander.

If $m = k$, then, since $k \leq \nu^{-5}$ and $\varepsilon \ll \nu$, Γ_m is a robust $(\frac{\nu}{2}, \tau)$-outexpander and so is $D_m \cup \Gamma_m$, as desired. Assume $m < k$. Let $\ell'' \coloneqq \min\{\ell - m\ell', \ell'\}$ and $\mathcal{F}' \coloneqq \bigcup_{i \in [\ell'']} \mathcal{F}_{m\ell'+i}$. Then, apply Lemma 6.3(ii) with $D_m, \Gamma_m, \mathcal{F}', \delta - \frac{m\ell'}{n}, \nu - \varepsilon_m, \varepsilon_m, \ell''$, $L_{m\ell'+1}, \dots, L_{m\ell'+\ell''}$, and $F_{m\ell'+1}, \dots, F_{m\ell'+\ell''}$ playing the roles of $D, \Gamma, \mathcal{F}, \delta, \nu, \varepsilon, \ell, L_1, \dots, L_\ell$, and $F_1, \dots, F_\ell$ to obtain edge-disjoint $\mathcal{H}_{m\ell'+1}, \dots, \mathcal{H}_{m\ell'+\ell''} \subseteq D_m \cup \Gamma_m \cup \mathcal{F}'$ such that the following hold. For each $i \in [\ell'']$, $\mathcal{H}_{m\ell'+i}$ is a spanning configuration of shape $(L_{m\ell'+i}, F_{m\ell'+i})$. Moreover, $D_{m+1} \coloneqq D_m \setminus \bigcup_{i \in [\ell'']} \mathcal{H}_{m\ell'+i}$ is $(\delta - \frac{\min\{(m+1)\ell', \ell\}}{n}, \varepsilon_{m+1})$-almost regular and $\Gamma_{m+1} \coloneqq \Gamma_m \setminus \bigcup_{i \in [\ell'']} \mathcal{H}_{m\ell'+i}$ is a $(\gamma, \varepsilon_{m+1})$-almost regular (ε_{m+1}, p)-robust $(\nu - \varepsilon_{m+1}, \tau)$-outexpander, as desired. □

The key idea in the proof of Lemma 6.3(i) is how to use the robust outexpander Γ efficiently, i.e. to find the required number of spanning configurations $\mathcal{H}_i$ without using too many edges of Γ. We achieve this by considering a random partition $A_1, \dots, A_a$ of V. To build $\mathcal{H}_i$, we find an almost cover of V in D with few long paths (which exists since D is almost regular) and tie them together into a single spanning path using only $\Gamma[A_j]$ for a suitable $j \in [a]$. The remainder of $\mathcal{H}_i$ is comparatively small and its construction does not affect Γ significantly.

Proof of Lemma 6.3(i). Let $a \coloneqq \lceil \varepsilon^{-1}\nu^4 \rceil$. By Lemma 4.17 (successively applied to D and Γ) and since Γ is an (ε, p)-robust (ν, τ)-outexpander, we can fix a partition $A_1, \dots, A_a$ of V such that, for each $i \in [a]$, the following hold.

(α) $|A_i| = \frac{n}{a} \pm 1 = \varepsilon(\nu^{-4} \pm 1)n$.
(β) $\Gamma[A_i]$ is a robust (ν, τ)-outexpander.
(γ) For each $v \in V$, $|N_\Gamma^\pm(v) \cap A_i| = (\gamma \pm 2\varepsilon)\frac{n}{a}$.
(δ) For each $v \in V$, $|N_D^\pm(v) \cap A_i| = (\delta \pm 2\varepsilon)\frac{n}{a}$.

For each $i \in [\ell]$, let $j \in [a]$ be such that $i \equiv j \mod a$ and define $A_i' \coloneqq A_j \setminus V(L_i)$. Using Lemma 4.2 and (a), it is easy to check that, for each $i \in [\ell]$, the following hold.

(α') $|A_i'| = \varepsilon(\nu^{-4} \pm 2)n$.
(β') $\Gamma[A_i']$ and $\Gamma - A_i'$ are both robust $(\frac{\nu}{2}, 2\tau)$-outexpanders.
(γ') $\Gamma[A_i']$ and $\Gamma - A_i'$ are both $(\gamma, 3\varepsilon)$-almost regular.
(δ') $D - A_i'$ is $(\delta, 3\varepsilon)$-almost regular.
(ε') For each $v \in V \setminus A_i'$, $|N_D^\pm(v) \cap A_i'| \geq \frac{\varepsilon\delta n}{2\nu^4}$.

For each $i \in [\ell]$, fix $e_i \in E(L_i) \setminus F_i$ (this is possible by (L3)). Assume inductively that for some $0 \leq m \leq \ell$ we have constructed, for each $i \in [m]$, a set of paths $\mathcal{P}_i = \{P_e^i \mid e \in E(L_i) \setminus F_i\}$ in $D \cup \Gamma$ such that $\mathcal{P}_1, \dots, \mathcal{P}_m$ are edge-disjoint and the following hold.

(A) Let $i \in [m]$. For each $e \in E(L_i) \setminus F_i$, P_e^i is a path of shape e. Moreover, the paths in $\mathcal{P}_i$ are internally vertex-disjoint and $V^0(\mathcal{P}_i) = V \setminus V(L_i)$. In particular, $\mathcal{P}_i \cup F_i$ is a spanning configuration of shape (L_i, F_i).
(B) For each $i \in [m]$ and $e \in E(L_i) \setminus (F_i \cup \{e_i\})$, $P_e^i \subseteq \Gamma - A_i'$ and $e(P_e^i) \leq 8\nu^{-1}$. Moreover, for each $v \in V$, there exist at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V^0(\mathcal{P}_i \setminus \{P_{e_i}^i\})$.
(C) For each $i \in [m]$, $E(P_{e_i}^i) \cap E(\Gamma) \subseteq E(\Gamma[A_i'])$.

Denote $D_m \coloneqq D \setminus \bigcup_{i \in [m]} E(\mathcal{P}_i)$ and $\Gamma_m \coloneqq \Gamma \setminus \bigcup_{i \in [m]} E(\mathcal{P}_i)$. For each $i \in [m]$, define $\mathcal{H}_i \coloneqq \mathcal{P}_i \cup F_i$. Denote $\mathcal{H}^m \coloneqq \bigcup_{i \in [m]} \mathcal{H}_i$. Then, note that, for each $v \in V$, since $d_L(v) \leq \varepsilon^3 n$, there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V^+(\mathcal{P}_i \setminus \{P_{e_i}^i\}) \cup V^-(\mathcal{P}_i \setminus \{P_{e_i}^i\})$ and, by (B), there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V^0(\mathcal{P}_i \setminus \{P_{e_i}^i\})$. Moreover, by (C) and construction of the A_i', there are, for each $v \in V$, at most $\lceil \frac{\ell}{a} \rceil$ indices $i \in [m]$ such that $v \in V(E(P_{e_i}^i) \cap E(\Gamma))$.

Hence, each $v \in V$ satisfies

$$(6.2) \qquad |N_{\mathcal{H}^m \cap \Gamma}(v)| \leq \varepsilon^3 n + 2\varepsilon^3 n + 2\left\lceil \frac{\ell}{a} \right\rceil \leq 3\varepsilon^3 n + \frac{2\varepsilon^2 n}{\varepsilon^{-1}\nu^4} + 2 \leq 3\varepsilon^3 \nu^{-4} n.$$

Assume $m = \ell$. Then, by (A), $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) for each $i \in [\ell]$. Moreover, (i) holds by (6.2) and we are done.

Assume $m < \ell$. Using (α')–(ε'), (6.2), and (b), it is easy to check that the following hold.

(I) $\Gamma_m[A'_{m+1}]$ and $\Gamma_m - A'_{m+1}$ are robust $(\frac{\nu}{4}, 2\tau)$-outexpanders.

(II) $\Gamma_m[A'_{m+1}]$ and $\Gamma_m - A'_{m+1}$ are both $(\gamma, 4\varepsilon)$-almost regular.

(III) $D_m - A'_{m+1}$ is $(\delta - \frac{m}{n}, 4\varepsilon)$-almost regular.

(IV) For each $v \in V \setminus A'_{m+1}$, $|N^{\pm}_{D_m}(v) \cap A'_{m+1}| \geq \frac{\varepsilon\delta n}{3\nu^4}$.

We first construct P^{m+1}_e for each $e \in E(L_{m+1}) \setminus (F_{m+1} \cup \{e_{m+1}\})$ in the following way. Let S be the set of vertices $v \in V$ for which there exist $\lfloor \varepsilon^3 n \rfloor$ indices $i \in [m]$ such that $v \in V^0(\mathcal{P}_i \setminus \{P^i_{e_i}\})$. Observe that, by (a) and (B), $|S| \leq \frac{8\nu^{-1}\cdot\ell\cdot\varepsilon^4 n}{\lfloor \varepsilon^3 n \rfloor} \leq \varepsilon|V \setminus A'_{m+1}|$. Denote $E(L_{m+1}) \setminus (F_{m+1} \cup \{e_{m+1}\}) =: \{x_1x'_1, \dots, x_kx'_k\}$ and apply Corollary 4.6 with $\Gamma_m - A'_{m+1}, \frac{\nu}{4}, 2\tau, \gamma - 4\varepsilon$, and $S \cup V(L_{m+1})$ playing the roles of D, ν, τ, δ, and S to obtain internally vertex-disjoint paths $P^{m+1}_{x_1x'_1}, \dots, P^{m+1}_{x_kx'_k} \subseteq \Gamma_m - A'_{m+1}$ such that, for each $i \in [k]$, $P^{m+1}_{x_ix'_i}$ is an (x_i, x'_i)-path of length at most $8\nu^{-1}$ with $V^0(P^{m+1}_{x_ix'_i}) \subseteq V \setminus (A'_{m+1} \cup S \cup V(L_{m+1}))$. Let $\mathcal{P}'_{m+1} := \{P^{m+1}_{x_ix'_i} \mid i \in [k]\}$.

Let $z \notin V$ be a new vertex. Let H be the digraph on vertex set $V(H) := V \setminus (V(L_{m+1}) \cup V(\mathcal{P}'_{m+1})) \cup \{z\}$ defined as follows. Denote $v^+v^- := e_{m+1}$ and recall that, by construction, $v^{\pm} \notin A'_{m+1}$. Then, let $N^{\pm}_H(z) := N^{\pm}_{D_m}(v^{\pm}) \cap V(H)$, $H[A'_{m+1}] := \Gamma_m[A'_{m+1}]$, and, for each $v \in V(H) \setminus (A'_{m+1} \cup \{z\})$, $N^{\pm}_{H-\{z\}}(v) := N^{\pm}_{D_m}(v) \cap V(H)$. Note that, by (I)–(IV), the following hold.

(I′) $H[A'_{m+1}]$ is a robust $(\frac{\nu}{4}, 2\tau)$-outexpander.

(II′) $H[A'_{m+1}]$ is $(\gamma, 4\varepsilon)$-almost regular.

(III′) $H - A'_{m+1}$ is $(\delta - \frac{m}{n}, 5\varepsilon)$-almost regular.

(IV′) For each $v \in V(H) \setminus A'_{m+1}$, $|N^{\pm}_H(v) \cap A'_{m+1}| \geq \frac{\varepsilon\delta n}{3\nu^4}$.

Indeed, to check (III′), note that, by (a), $H - A'_{m+1}$ is obtained from $D_m - A'_{m+1}$ by adding z and deleting $|V(L_{m+1}) \cup V(\mathcal{P}'_{m+1})| \leq \varepsilon^2 n + \varepsilon^4 n \cdot 8\nu^{-1} \leq 2\varepsilon^2 n$ vertices.

Our aim is to find a Hamilton cycle of H which contains few edges of $\Gamma[A'_{m+1}]$. First, we cover $V(H) \setminus A'_{m+1}$ with a small number of paths as follows. Let $k' := \left\lfloor \frac{|V(H) \setminus A'_{m+1}|}{\varepsilon n} \right\rfloor$. Apply Lemma 4.17 with $H - A'_{m+1}, |V(H) \setminus A'_{m+1}|, \delta - \frac{m}{n}$, and 5ε playing the roles of D, n, δ, and ε to obtain a partition $V_1, \dots, V_{k'}$ of $V(H) \setminus A'_{m+1}$ such that, for each $i \in [k']$, $|V_i| = (1 \pm 2\varepsilon)\varepsilon n$ and, for each $i \in [k']$ and $v \in V_i$, $|N^-_H(v) \cap V_{i-1}| = (\delta - \frac{m}{n} \pm 10\varepsilon)\varepsilon n$ if $i > 1$ and $|N^+_H(v) \cap V_{i+1}| = (\delta - \frac{m}{n} \pm 10\varepsilon)\varepsilon n$ if $i < k'$.

Then, for each $i \in [k'-1]$, apply Proposition 4.19 with $H[V_i, V_{i+1}]$, V_i, V_{i+1}, εn, $\delta - \frac{m}{n}$, and 10ε playing the roles of G, A, B, n, δ, and ε to obtain a matching M_i of $H[V_i, V_{i+1}]$ of size at least $\left(1 - \frac{31\varepsilon}{\delta}\right)\varepsilon n$. For each $i \in [k'-1]$, denote by $\overrightarrow{M}_i$ the directed matching obtained from M_i by directing all edges from V_i to V_{i+1}. Note that, by construction, $\overrightarrow{M}_i \subseteq H$. Define $F \subseteq H$ by letting $V(F) := V(H) \setminus A'_{m+1}$ and $E(F) := \bigcup_{i \in [k'-1]} \overrightarrow{M}_i$. Observe that F is a linear forest which spans $V(H) \setminus A'_{m+1}$ and has $f \leq \frac{33\varepsilon n}{\delta}$ components. Indeed, one can count the number of paths in F by counting the number of ending points as follows. (An isolated vertex is considered as the ending point of a trivial path of length 0.) Note that, for each $i \in [k'-1]$, $v \in V_i$ is the ending point of path in F if and only if $v \notin V(M_i)$, while every $v \in V_{k'}$ is the ending point of a path in F. Moreover, for each $i \in [k'-1]$, we have $|V_i \setminus V(M_i)| \leq |V_i| - |M_i| \leq \varepsilon n + 2\varepsilon^2 n - \left(1 - \frac{31\varepsilon}{\delta}\right)\varepsilon n \leq \frac{32\varepsilon^2 n}{\delta}$. Thus, since $k' - 1 \leq \varepsilon^{-1} - 1$, we have $f \leq \frac{32\varepsilon^2 n}{\delta}(\varepsilon^{-1} - 1) + |V_k| \leq \frac{33\varepsilon n}{\delta}$, as desired.

Denote the components of F by $P_1, \dots, P_f$. We now join $P_1, \dots, P_f$ into a Hamilton cycle as follows. Note that, by (α'), $f \le (\frac{\nu}{4})^3 |A'_{m+1}|$. For each $i \in [f]$, denote by v_i^+ and v_i^- the starting and ending points of P_i. By (IV′), for each $i \in [f]$, we have $|N_H^{\mp}(v_i^{\pm}) \cap A'_{m+1}| \ge 2f$. Apply Corollary 4.8(c) with $H, A'_{m+1}, \emptyset, f, \frac{\nu}{4}, 2\tau$, and $\gamma - \nu$ playing the roles of D, V', S, k, ν, τ, and δ to obtain a Hamilton cycle C of H such that $F \subseteq C$. Denote by $u^{\pm}$ the (unique) vertices such that $u^{\pm} \in N_C^{\pm}(z)$, respectively. Let $P_{e_{m+1}}^{m+1} := (C - \{z\}) \cup \{v^+u^+, u^-v^-\}$. By construction, $P_{e_{m+1}}^{m+1}$ is a path of shape e_{m+1} such that $P_{e_{m+1}}^{m+1} \subseteq (D_m \cup \Gamma_m) - V(\mathcal{P}'_{m+1})$ and $V^0(P_{e_{m+1}}^{m+1}) = V \setminus (V(\mathcal{P}'_{m+1}) \cup V(L_{m+1}))$. Moreover, $P_{e_{m+1}}^{m+1}[A'_{m+1}] \subseteq \Gamma_m$ and $P_{e_{m+1}}^{m+1} \setminus P_{e_{m+1}}^{m+1}[A'_{m+1}] \subseteq D_m$. Let $\mathcal{P}_{m+1} := \mathcal{P}'_{m+1} \cup \{P_{e_{m+1}}^{m+1}\}$. Thus, (A)–(C) hold. This completes the induction step. $\square$

7. Proof of Theorems 1.3 and 1.8 and Corollary 1.9

As discussed in the proof overview, vertices of very high excess and vertices adjacent to absorbing edges will be treated as exceptional vertices throughout the proof of Theorem 1.8. To be able to cover the edges at these vertices in the approximate decomposition step, we need to start with a cleaning procedure, which will ensure that no two exceptional vertices have an edge between them and that the degree of the exceptional vertices is not too large. This needs be done efficiently so that, after the cleaning step, the non-exceptional vertices still form an almost regular oriented graph of very large degree.

Lemma 7.1. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let $T \notin \mathcal{T}_{\rm excep}$ be a tournament on a vertex set V of size n satisfying the following properties.*

- (a) *Let $W_* \cup W_0 \cup V'$ be a partition of V such that, for each $w_* \in W_*$, $|\mathrm{ex}_T(w_*)| > (1-20\eta)n$; for each $w_0 \in W_0$, $|\mathrm{ex}_T(w_0)| \le (1-20\eta)n$; and, for each $v' \in V'$, $|\mathrm{ex}_T(v')| \le \varepsilon n$. Let $W := W_* \cup W_0$ and suppose $|W| \le \varepsilon n$.*
- (b) *Let $A^+, A^- \subseteq E(T)$ be absorbing sets of (W, V')-starting/(V', W)-ending edges for T of size at most $\lceil \eta n \rceil$. Denote $A := A^+ \cup A^-$. Let $W_A^{\pm} := V(A^{\pm}) \cap W$ and $W_A := V(A) \cap W$. Suppose that, for each $\diamond \in \{+, -\}$, if $|W_A^\diamond| \ge 2$, then $\mathrm{ex}_T^\diamond(v) < \lceil \eta n \rceil$ for each $v \in V$ and, if $|W_A^\diamond| = 1$, then, for each $v \in V$ and $w \in W_A^\diamond$, $\mathrm{ex}_T^\diamond(v) \le \mathrm{ex}_T^\diamond(w)$. Moreover, suppose that if $W_A \neq \emptyset$, then $\widetilde{\mathrm{ex}}(T) \ge \frac{n}{2} + 10\eta n$.*
- (c) *Let $U^* \subseteq U^0(T)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$.*

Then, there exist $d \in \mathbb{N}$ and a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ of T such that the following hold, where $D := T \setminus \mathcal{P}$.

- (i) $\lceil \frac{n}{2} \rceil - 10\eta n \le d \le \lceil \frac{n}{2} \rceil - \eta n$.
- (ii) *Each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ satisfies $d_D^+(v) = d_D^-(v) \le \widetilde{\mathrm{ex}}(D) - 1$.*
- (iii) $E(D[W]) = \emptyset$.
- (iv) $N^{\pm}(T) - N^{\pm}(D) \le 89\eta n$.
- (v) $\widetilde{\mathrm{ex}}(D) \ge d + \lceil \eta n \rceil$.
- (vi) *If $\widetilde{\mathrm{ex}}(D) < 2d + \lceil \eta n \rceil$, then each $w \in W_*$ satisfies $|\mathrm{ex}_D(w)| = d_D(w)$.*
- (vii) *For each $v \in W_* \cup W_A$, $2d - 3\sqrt{\eta} n \le d_D(v) \le 2d - \lceil \eta n \rceil$.*
- (viii) *For each $v \in W_0$, $2d + 2\lceil \eta n \rceil - 4\sqrt{\eta} n \le d_D(v) \le 2d + 2\lceil \eta n \rceil$ and $d_D^{\min}(v) \ge \lceil \eta n \rceil$.*
- (ix) *For each $v \in V'$, $2d + 2\lceil \eta n \rceil - 9\sqrt{\varepsilon} n \le d_D(v) \le 2d + 2\lceil \eta n \rceil$.*

Lemma 7.1 will be proved in Section 8. As mentioned above, the cleaning step has two aims: cleaning the edges inside W (property (iii)) and bounding the degree of vertices in $W_* \cup W_A$ from above (compare the upper bound in (vii) to those in (viii) and (ix)). As discussed in the second paragraph after the statement of Lemma 5.2, property (ii) is needed to ensure that our final decomposition contains the desired number of paths. (iv) ensures that there are few vertices whose excess has been completely used (this is necessary for the last step of the decomposition). (i) and (vii)–(ix) ensure that the leftover oriented graph D is still almost complete. (v) ensures that the leftover excess is sufficiently large compared to the number of edges left to cover. (vi) is a technical condition that will enable us to cover all edges at W_* when the total excess is not too large.

We are now ready to prove Theorem 1.8 using Lemma 7.1. We will proceed as follows. In Step 1, we select absorbing edges (if they are required). In Step 2, we clean up T by removing a small number of paths using Lemma 7.1. In Step 3, we obtain an approximate decomposition of T by Lemmas 5.2 and 6.3. Finally, in Step 4, we apply Corollary 4.30 to decompose the leftover.

Proof of Theorem 1.8. Assume without loss of generality that $\beta \leq 1$. Fix additional constants such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \alpha_1 \ll \alpha_2 \ll \eta \ll \beta \leq 1$. Let $T \notin \mathcal{T}_{\rm excep}$ be a tournament on $n \geq n_0$ vertices satisfying (a) or (b). By Theorem 1.2(b), we may assume that $\mathrm{ex}(T) \leq \varepsilon^2 n^2$. Denote $V := V(T)$. If both $N^\pm(T) \geq \alpha_1 n$, then redefine $\eta := \alpha_1^2$. Suppose not. If both $N^\pm(T) \leq \alpha_2 n$, then redefine $\varepsilon := \alpha_2$. Otherwise, there exists $\diamond \in \{+,-\}$ such that $N^\diamond(T) \leq \alpha_1 n$ and $\circ \in \{+,-\} \setminus \{\diamond\}$ satisfies $N^\circ(T) \geq \alpha_2 n$ and we redefine $\varepsilon := \alpha_1$ and $\eta := \alpha_2^2$. Thus, we have defined constants such that

$$0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \beta \leq 1,$$

$\mathrm{ex}(T) \leq \varepsilon^2 n^2$, and, for each $\diamond \in \{+,-\}$, either $N^\diamond(T) \geq \sqrt{\eta} n$ or $N^\diamond(T) \leq \varepsilon n$. Define additional constants such that

$$0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \eta \ll \beta \leq 1.$$

Let $r := \lceil \eta n \rceil$.

Step 1: Choosing absorbing edges. Let $\diamond \in \{+,-\}$ and define $W_A^\diamond$ as follows. If $N^\diamond(T) \geq \sqrt{\eta} n$, then let $W_A^\diamond = \emptyset$. Otherwise, by construction, $N^\diamond(T) \leq \varepsilon n$. Then, by Fact 4.20, $\mathrm{ex}(T) \geq \widetilde{\mathrm{ex}}(T) - N^\diamond(T) \geq \frac{n}{2} - \varepsilon n \geq r$. Let $W_A^\diamond \subseteq U^\diamond(T)$ be a smallest set such that $\mathrm{ex}_T^\diamond(W_A^\diamond) \geq r$. We further assume that, subject to this, $\mathrm{ex}_T^\diamond(W_A^\diamond)$ is maximum. Observe that, if $|W_A^\diamond| \geq 2$, then, for each $v \in V$, $\mathrm{ex}_T^\diamond(v) < r$, and, if $|W_A^\diamond| = 1$, then, for each $v \in V$ and $w \in W_A^\diamond$, $\widetilde{\mathrm{ex}}_T^\diamond(v) \leq \widetilde{\mathrm{ex}}_T^\diamond(w)$, as desired for Lemma 7.1(b). Moreover, note that $|U^\diamond(T)| \leq N^\diamond(T) \leq \varepsilon n$. Thus, $|W_A^\diamond| \leq \varepsilon n$.

Let $W_A := W_A^+ \cup W_A^-$. Let

$$W_* := \{v \in V \mid |\mathrm{ex}_T(v)| > (1-20\eta)n\} \quad \text{and} \quad W_0 := \{v \in V \setminus W_* \mid |\mathrm{ex}_T(v)| > \varepsilon n\} \cup (W_A \setminus W_*).$$

For each $\diamond \in \{*, 0\}$, let $W_\diamond^\pm := W_\diamond \cap U^\pm(T)$. Let $W^\pm := W_*^\pm \cup W_0^\pm$, $W := W^+ \cup W^-$, and $V' := V \setminus W$. Note that, by construction, $|W| \leq 4\varepsilon n$. (Here we also use that $\mathrm{ex}(T) \leq \varepsilon^2 n^2$.)

If $N^+(T) \geq \sqrt{\eta} n$, then let $A^+ := \emptyset$; otherwise, let $A^+ \subseteq E(T)$ be an absorbing set of r (W_A^+, V')-starting edges for T (A^+ exists since, by construction, $\mathrm{ex}_T^+(W_A^+) \geq r$ and $d_T^+(v) \geq \frac{n}{2}$ for each $v \in W_A^+$). Similarly, let $A^- \subseteq E(T)$ be an absorbing set of (V', W_A^-)-ending edges for T, of size 0 if $N^-(T) \geq \sqrt{\eta} n$ and r otherwise. Let $A := A^+ \cup A^-$. Note that, by construction, $V(A^\pm) \cap W = W_A^\pm$. In particular, each $v \in W_A^\pm$ satisfies

$$d_A^\pm(v) \geq 1. \tag{7.1}$$

Step 2: Cleaning. Let $U_1^* \subseteq U^0(T)$ be such that $|U_1^*| = \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ (this is possible by Proposition 4.24). Note that $W \cap U_1^* = \emptyset$ since $W \cap U^0(T) = \emptyset$. Moreover, if $W_A \neq \emptyset$, then, by Step 1, (b) does not hold and so $\widetilde{\mathrm{ex}}(T) \geq \frac{n}{2} + \beta n \geq \frac{n}{2} + 10\eta n$. Thus, the "moreover part" of Lemma 7.1(b) is satisfied. Apply Lemma 7.1 with U_1^* and 4ε playing the roles of U^* and ε to obtain $d \in \mathbb{N}$ and $\mathcal{P}_1 \subseteq T$ such that the following are satisfied, where $D_1 := T \setminus \mathcal{P}_1$.

(i) $\mathcal{P}_1$ is a good (U_1^*, W, A)-partial path decomposition of T. In particular, $\mathcal{P}_1$ is consistent with A^+ and A^- and so, by Fact 4.29, A^+ and A^- are absorbing sets of (W_A^+, V')-starting and (V', W_A^-)-ending edges for D_1.

(ii) $\lceil \frac{n}{2} \rceil - 10\eta n \leq d \leq \lceil \frac{n}{2} \rceil - \eta n$.

(iii) For each $v \in U_1^* \setminus (V^+(\mathcal{P}_1) \cup V^-(\mathcal{P}_1))$ satisfies $d_{D_1}^+(v) = d_{D_1}^-(v) \leq \widetilde{\mathrm{ex}}(D_1) - 1$.

(iv) $E(D_1[W]) = \emptyset$.

(v) $N^\pm(T) - N^\pm(D_1) \leq 89\eta n$.

(vi) $\widetilde{\mathrm{ex}}(D_1) \geq d + \lceil \eta n \rceil$.

(vii) If $\widetilde{\text{ex}}(D_1) < 2d + \lceil \eta n \rceil$, then each $w \in W_*$ satisfies $|\text{ex}_{D_1}(w)| = d_{D_1}(w)$.

(viii) For each $v \in W_* \cup W_A$, $2d - 3\sqrt{\eta}n \leq d_{D_1}(v) \leq 2d - 2\lceil \eta n \rceil$.

(ix) For each $v \in W_0$, $2d + 2\lceil \eta n \rceil - 4\sqrt{\eta}n \leq d_{D_1}(v) \leq 2d + 2\lceil \eta n \rceil$ and $d_{D_1}^{\min}(v) \geq \lceil \eta n \rceil$.

(x) For each $v \in V'$, $2d + 2\lceil \eta n \rceil - 18\sqrt{\varepsilon}n \leq d_{D_1}(v) \leq 2d + 2\lceil \eta n \rceil$ and $|\text{ex}_{D_1}(v)| \leq \varepsilon n$.

(The final part of (x) follows from the fact that $|\text{ex}_T(v)| \leq \varepsilon n$ for each $v \in V'$, and since $\mathcal{P}_1$ is a (U_1^*, W, A)-partial path decomposition of T.)

Step 3: Approximate decomposition. First, we will apply Lemma 5.2 to construct layouts for the approximate decomposition. Let $U_2^* := U_1^* \setminus (V^+(\mathcal{P}_1) \cup V^-(\mathcal{P}_1))$ and observe that, by Proposition 4.33 and (i), $|U_2^*| = \widetilde{\text{ex}}(D_1) - \text{ex}(D_1)$. Let $X^\pm \subseteq (U^\pm(D_1) \cup U_2^*) \setminus W$ be such that $|X^\pm| = r - |A^\pm|$. This is possible since, for each $\diamond \in \{+,-\}$, by construction, if $N^\diamond(T) \leq \varepsilon n$, then $|A^\diamond| = r$; otherwise, $N^\diamond(T) \geq \sqrt{\eta}n$ and so, by (v), $|U^\diamond(D_1) \cup U_2^*| = N^\diamond(D_1) \geq \sqrt{\eta}n - 89\eta n \geq r + |W|$. Define $\phi^\pm : V \longrightarrow \{0,1\}$ by

$$\phi^\pm(v) := \begin{cases} 1 & \text{if } v \in X^\pm, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $\mathcal{F} := E_W(D_1) \setminus A$ and $D_1' := D_1 \setminus \mathcal{F}$. Let $W_1 := W_* \cup W_A$ and $W_2 := W_0 \setminus W_A$.

We now verify that Lemma 5.2(a)–(h) hold with D_1, U_2^*, and $18\sqrt{\varepsilon}$ playing the roles of D, U^*, and ε. Lemma 5.2(a) holds by Step 1 and (iv). Lemma 5.2(b) holds by (iii) and the fact that $U_2^* \subseteq U_1^* \subseteq V'$. Lemma 5.2(c) holds by construction and (i). Lemma 5.2(d) follows from (ii) and Lemma 5.2(e) holds by (vi). Lemma 5.2(g) and (h) as well as the first part of Lemma 5.2(f) follow from (viii)–(x). Finally, we show that the "moreover" part of Lemma 5.2(f) holds. By (vii), if $\widetilde{\text{ex}}(D_1) < 2d + \lceil \eta n \rceil$, then each $v \in W_*$ satisfies $|\text{ex}_{D_1}(v)| = d_{D_1}(v)$. Moreover, by (ix), each $v \in W_A \setminus W_* \subseteq W_0$ satisfies $d_{D_1}^{\min}(v) \geq \lceil \eta n \rceil$. Finally, by Step 1 and (i), if $|W_A^\pm| \geq 2$, then each $v \in W_A^\pm$ satisfies $\text{ex}_{D_1 \setminus A}^\pm(v) \leq \text{ex}_{D_1}^\pm(v) \leq \text{ex}_T^\pm(v) \leq \lceil \eta n \rceil$ and, if $|W_A^\pm| = 1$, then $d_A^\pm(v) = \lceil \eta n \rceil$ for the unique $v \in W_A^\pm$.

Apply Lemma 5.2 with D_1, U_2^*, and $18\sqrt{\varepsilon}$ playing the roles of D, U^*, and ε to obtain $\ell \in \mathbb{N}$ and W-exceptional layouts $(L_1, F_1), \dots, (L_\ell, F_\ell)$ which are U_2^*-path consistent with respect to $(D_1', \mathcal{F})$ and such that the following hold. Let $L := \bigcup_{i \in [\ell]} L_i$.

(A) $d \leq \ell \leq d + \varepsilon^{\frac{1}{5}} n$.

(B) L contains exactly $\widetilde{\text{ex}}(D_1) - r$ non-trivial paths.

(C) For each $v \in W_1$, $d_L^\pm(v) = d_{D_1 \setminus A}^\pm(v)$.

(D) For each $v \in W_2$, $d_L^\pm(v) = d_{D_1}^\pm(v) - r$.

(E) For each $v \in V'$, $d_{D_1}^\pm(v) = d_L^\pm(v) + |\{i \in [\ell] \mid v \notin V(L_i)\}| + r - \phi^\mp(v)$.

(F) For each $i \in [\ell]$, $|V(L_i)|, |E(L_i)| \leq \varepsilon^{\frac{1}{7}} n$.

(G) For each $v \in V'$, $d_L(v) \leq \varepsilon^{\frac{1}{3}} n$ and there exist at most $\varepsilon^{\frac{1}{5}} n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.

For each $i \in [\ell]$, let $(L_i^{\restriction W}, F_i^{\restriction W})$ be as in Definition 6.1. Let $L^{\restriction W} := \bigcup_{i \in [\ell]} L_i^{\restriction W}$ and $\mathcal{F}^{\restriction W} := \bigcup_{i \in [\ell]} F_i^{\restriction W}$. Then, by construction, $(L_1^{\restriction W}, F_1^{\restriction W}), \dots, (L_\ell^{\restriction W}, F_\ell^{\restriction W})$ are layouts on V' such that, for each $i \in [\ell]$, $|V(L_i^{\restriction W})|, |E(L_i^{\restriction W})| \leq \varepsilon^{\frac{1}{7}} n \leq 2\varepsilon^{\frac{1}{7}}|V'|$ and, for each $v \in V'$, $d_{L^{\restriction W}}(v) \leq \varepsilon^{\frac{1}{3}} n \leq 2\varepsilon^{\frac{1}{3}}|V'|$ and there exist at most $\varepsilon^{\frac{1}{5}} n \leq 2\varepsilon^{\frac{1}{5}}|V'|$ indices $i \in [\ell]$ such that $v \in V(L_i^{\restriction W})$.

We now reserve a random slice $\Gamma \subseteq D_1[V']$ for Lemma 6.3 as follows. Let $\delta := \frac{d+r}{n}$. Note that, by (x), $D_1[V']$ is $(\delta, 10\sqrt{\varepsilon})$-almost regular. By Lemma 4.4 and (ii), $D_1[V']$ is a robust (ν, τ)-outexpander. Apply Lemma 4.16 with $D_1[V'], |V'|$, and $10\sqrt{\varepsilon}$ playing the roles of D, n, and ε to obtain $\Gamma \subseteq D_1[V']$ such that Γ is a $(\gamma, 10\sqrt{\varepsilon})$-almost regular $(10\sqrt{\varepsilon}, |V'|^{-2})$-robust (ν, τ)-outexpander and $D_1'' := D_1[V'] \setminus \Gamma$ is $(\delta - \gamma, 10\sqrt{\varepsilon})$-almost regular.

Observe that, by (A), $\ell \leq (\delta - \frac{\eta}{2})|V'|$. Apply Lemma 6.3 with $D_1'', \mathcal{F}^{\restriction W}, |V'|, |V'|^{-2}, \frac{\eta}{2}, \varepsilon^{\frac{1}{29}}$, and $(L_1^{\restriction W}, F_1^{\restriction W}), \dots, (L_\ell^{\restriction W}, F_\ell^{\restriction W})$ playing the roles of $D, \mathcal{F}, n, p, \eta, \varepsilon$, and $(L_1, F_1), \dots, (L_\ell, F_\ell)$ to obtain edge-disjoint $\mathcal{H}_1^{\restriction W}, \dots, \mathcal{H}_\ell^{\restriction W} \subseteq D_1'' \cup \Gamma \cup \mathcal{F}^{\restriction W} = D_1[V'] \cup \mathcal{F}^{\restriction W}$ such that, for each $i \in [\ell]$, $\mathcal{H}_i^{\restriction W}$ is a spanning configuration of shape $(L_i^{\restriction W}, F_i^{\restriction W})$ and the following holds. Let $\mathcal{H}^{\restriction W} \coloneqq \bigcup_{i \in [\ell]} \mathcal{H}_i^{\restriction W}$ and $D_2' \coloneqq D_1[V'] \setminus \mathcal{H}^{\restriction W}$. Then,

(I) D_2' is a robust $(\frac{\nu}{2}, \tau)$-outexpander.

Now construct W-exceptional spanning configurations as follows. For each $i \in [\ell]$, let $\mathcal{H}_i$ be the digraph with $V(\mathcal{H}_i) \coloneqq V$ and $E(\mathcal{H}_i) = (E(\mathcal{H}_i^{\restriction W}) \setminus F_i^{\restriction W}) \cup F_i$. Denote $\mathcal{H} \coloneqq \bigcup_{i \in [\ell]} \mathcal{H}_i$. Then, by Fact 6.2, for each $i \in [\ell]$, $\mathcal{H}_i \subseteq D_1[V'] \cup F_i$ and $\mathcal{H}_i$ is a W-exceptional spanning configuration of shape (L_i, F_i). Moreover, for each $i \in [\ell]$, since $F_i \subseteq E_W(D_1) \setminus A$, we have $\mathcal{H}_i \subseteq D_1 \setminus A$ and so $E(\mathcal{H}) \cap A = \emptyset$. Furthermore, by definition of U_2^*-path consistency with respect to $(D_1', \mathcal{F})$, the sets $F_1, \dots, F_\ell$ are edge-disjoint. Thus, since $\mathcal{H}_1^{\restriction W}, \dots, \mathcal{H}_\ell^{\restriction W}$ are edge-disjoint, $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ are edge-disjoint.

For each $i \in [\ell]$, let $\mathcal{P}_{2,i}$ be a path decomposition of $\mathcal{H}_i$ induced by (L_i, F_i). Let $\mathcal{P}_2 \coloneqq \bigcup_{i \in [\ell]} \mathcal{P}_{2,i}$ and $D_2 \coloneqq D_1 \setminus \mathcal{P}_2$. We claim that the following holds.

Claim 1. *$\mathcal{P}_2$ is a (U_2^*, W, A)-partial path decomposition of D_1.*

Proof of Claim. By Fact 5.1, since (L_1, F_1), $\dots$, (L_ℓ, F_ℓ) are U_2^*-path consistent with respect to $(D_1', \mathcal{F})$, $\mathcal{P}_2$ is a partial path decomposition of D_1 such that $U^0(D_1) \cap (V^+(\mathcal{P}_2) \cup V^-(\mathcal{P}_2)) \subseteq U_2^*$. Moreover, $E(\mathcal{P}_2) \cap A = \emptyset$. Thus, it suffices to show that each $v \in W_A$ is the starting point of exactly $\widetilde{\mathrm{ex}}^+_{D_1, U_2^*}(v)$ paths in $\mathcal{P}_2$ and the ending point of exactly $\widetilde{\mathrm{ex}}^-_{D_1, U_2^*}(v)$ paths in $\mathcal{P}_2$.

Let $v \in W_A^+$ (similar argument hold if $v \in W_A^-$) and recall that $W_A \subseteq W_1$. Note that, since $U_2^* \cap W_A = \emptyset$, we have $\widetilde{\mathrm{ex}}^-_{D_1, U_2^*}(v) = \mathrm{ex}^-_{D_1}(v) = 0$ and $\widetilde{\mathrm{ex}}^+_{D_1, U_2^*}(v) = \mathrm{ex}^+_{D_1}(v) - d_A^+(v)$. Since (L_1, F_1), $\dots$, (L_ℓ, F_ℓ) are U_2^*-path consistent with respect to $(D_1', \mathcal{F})$, v is not the ending point of any path in $\mathcal{P}_2$, as desired. In particular, v is an internal vertex of precisely $d^-_{\mathcal{P}_2}(v)$ paths in $\mathcal{P}_2$. Therefore, the number of paths in $\mathcal{P}_2$ whose starting point is v equals

$$d^+_{\mathcal{P}_2}(v) - d^-_{\mathcal{P}_2}(v) \overset{\text{Fact 5.1,(C)}}{=} d^+_{D_1 \setminus A}(v) - d^-_{D_1}(v) \overset{\text{(i)}}{=} \mathrm{ex}^+_{D_1}(v) - d^+_A(v) = \widetilde{\mathrm{ex}}^+_{D_1, U_2^*}(v),$$

as desired. $\diamond$

Then, by Fact 4.29 and since $W_A \subseteq W_1$ and $\mathcal{P}_2$ is a (U_2^*, W, A)-partial path decomposition of D_1,

(II) A^+ and A^- are absorbing sets of (W_1, V')-starting and (V', W_1)-ending edges for D_2.

Note that $D_2[V'] = D_2'$ and recall that $|W| \leq 4\varepsilon n$. Thus, by Lemma 4.2 and (I),

(I′) $D_2 - W_1$ is a robust $(\frac{\nu}{4}, 2\tau)$-outexpander.

Recall that by (i), $\mathcal{P}_1$ avoids all the edges in A. Thus, by Fact 5.1 and (C)–(E), the following hold.

(III) For each $v \in W_1$, $N^{\pm}_{D_2}(v) = N^{\pm}_A(v)$.

(IV) For each $v \in V \setminus (W_1 \cup X^-)$, $d^+_{D_2}(v) = r$.

(V) For each $v \in V \setminus (W_1 \cup X^+)$, $d^-_{D_2}(v) = r$.

(VI) For each $v \in X^{\pm}$, $d^{\mp}_{D_2}(v) = r - 1$.

Note that, since $|X^{\pm} \cup A^{\pm}| = r$ and $X^+ \cap X^- \subsetneq V \setminus W_1$, (III)–(VI) imply that

$$\widetilde{\mathrm{ex}}(D_2) = r. \tag{7.2}$$

Indeed, each $v \in V$ satisfies $d^{\max}_{D_2}(v) \leq r$ and each $v \notin (X^+ \cap X^-) \cup W_1$ satisfies $d^{\max}_{D_2}(v) = r$. Thus, $X^+ \cap X^- \subsetneq V \setminus W_1$ implies $\Delta^0(D_2) = r$. Moreover, $\mathrm{ex}(D_2) = |(X^{\pm} \setminus X^{\mp}) \cup A^{\pm}| \leq |X^{\pm} \cup A^{\pm}| = r$.

Thus, (B) implies that

(VII) $\mathcal{P}_2$ is a good partial path decomposition of D_1.

Step 4: Completing the path decomposition. Let $Y^{\pm} := X^{\pm} \setminus X^{\mp}$, $Y^* := X^+ \cap X^-$, and $Y^0 := V \setminus (Y^+ \cup Y^- \cup Y^*)$. Note that, if $A^{\pm} \neq \emptyset$, then, by Step 1, $|A^{\pm}| = r$ and so, by Step 3, $X^{\pm} = \emptyset$. Therefore, $V(A^{\pm}) \cap (V \setminus W_1) = V(A^{\pm}) \cap V' \subseteq Y^{\mp} \cup Y^0$. Apply Corollary 4.30 with $D_2, n - |W_1|, V \setminus W_1, W_1, Y^+, Y^-, Y^*, Y^0, \frac{\eta}{2}, \frac{\nu}{4}$, and 2τ playing the roles of $D, n, V', W, X^+, X^-, X^*, X^0, \delta, \nu$, and τ to obtain a path decomposition $\mathcal{P}_3$ of D_2 of size r. Note that, by (7.2), $\mathcal{P}_3$ a perfect path decomposition of D_2. Recall that by (i) and (VII), $\mathcal{P}_1$ and $\mathcal{P}_2$ are good. Then, by Fact 4.26(b), $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ is a perfect path decomposition of T. That is, $|\mathcal{P}| = \widetilde{\text{ex}}(T)$. This completes the proof. □

We now derive Theorem 1.3 from Theorem 1.8.

Proof of Theorem 1.3. Let $0 < \frac{1}{n_0} \ll \beta \ll 1$. Let $n \geq n_0$ be even and T be a tournament on n vertices. It is easy to see that $T \notin \mathcal{T}_{\text{excep}}$. We show that one of Theorem 1.8(a) and (b) holds. Suppose that Theorem 1.8(b) does not hold. Without loss of generality, we may assume that $N^+(T) \leq \beta n$. Thus, $|U^+(T)| \leq \beta n$. Since n is even, each $v \in V(T)$ satisfies $\text{ex}_T(v) \neq 0$. Thus, $\widetilde{\text{ex}}(T) \geq \text{ex}(T) \geq |U^-(T)| = n - |U^+(T)| \geq n - \beta n \geq \frac{n}{2} + \beta n$ and so Theorem 1.8(a) holds. Therefore, by Theorem 1.8 and Proposition 4.22, $\text{pn}(T) = \widetilde{\text{ex}}(T) = \text{ex}(T)$. □

Finally, we derive Corollary 1.9 from Theorem 1.8. The idea is that if none of Theorems 1.3, 1.8, and 4.35 apply to T then we can transform T into a tournament T' which satisfies the conditions of Theorem 1.8 by flipping a small number of edges, and so that $\text{pn}(T) \sim \text{pn}(T')$ and $\widetilde{\text{ex}}(T) \sim \widetilde{\text{ex}}(T')$.

Proof of Corollary 1.9. We may assume without loss of generality that $\beta \ll 1$. Let $0 < \frac{1}{n_0} \ll \beta \ll 1$. Let T be a tournament on $n \geq n_0$ vertices. By Theorems 1.3 and 4.35, we may assume that $T \notin \mathcal{T}_{\text{excep}}$ and that n is odd. If $\Delta^0(T) \geq \frac{n}{2} + \frac{\beta n}{5}$, then, by Theorem 1.8 applied with $\frac{\beta}{5}$ playing the role of β, $\text{pn}(T) = \widetilde{\text{ex}}(T)$ and we are done. We may therefore assume that $\Delta^0(T) < \frac{n}{2} + \frac{\beta n}{5}$. Let $v \in V$. Since T is not regular, we may assume without loss of generality that $v \in U^+(T)$. Then, note that $d_T^+(v) \geq \frac{n+1}{2}$. Let $S \subseteq N_T^-(v)$ satisfy $|S| = \lceil \frac{n}{2} + \frac{\beta n}{5} \rceil - d_T^+(v)$ (this is possible since $d_T^-(v) = (n-1) - d_T^+(v)$). Note that $|S| \leq \lceil \frac{n}{2} + \frac{\beta n}{5} \rceil - \frac{n+1}{2} \leq \frac{\beta n}{4}$.

Let T' be obtained from T by flipping the direction of all edges between v and S. Then, observe that $\widetilde{\text{ex}}(T') \geq \Delta^0(T') \geq d_{T'}^+(v) = \lceil \frac{n}{2} + \frac{\beta n}{5} \rceil$ and, in particular, $T' \notin \mathcal{T}_{\text{excep}}$. Moreover, we claim that $\widetilde{\text{ex}}(T') \leq \widetilde{\text{ex}}(T) + 2|S|$. Since $\Delta^0(T') \leq \Delta^0(T) + |S|$, it suffices to show that $\text{ex}(T') \leq \text{ex}(T) + 2|S|$. Note that, by Fact 4.21(c), $\text{ex}_{T'}^+(v) - \text{ex}_T^+(v) = 2(d_{T'}^+(v) - d_T^+(v)) = 2|S|$. For each $\diamond \in \{+, -, 0\}$, denote $S^\diamond := S \cup U^\diamond(T)$. Then, by Fact 4.21(c), for each $u \in S^+$, $\text{ex}_{T'}^+(u) - \text{ex}_T^+(u) = -2$ and, for each $u \in S^- \cup S^0$, $\text{ex}_{T'}^+(u) = 0 = \text{ex}_T^+(v)$. Thus, $\text{ex}(T') - \text{ex}(T) = 2|S| - 2|S^+| \leq 2|S|$, as desired.

By Theorem 1.8, $\text{pn}(T') = \widetilde{\text{ex}}(T') \leq \widetilde{\text{ex}}(T) + 2|S|$ and thus, since $|S| \leq \frac{\beta n}{4}$, it suffices to show that $\text{pn}(T) \leq \text{pn}(T') + 2|S|$. Let $\mathcal{P}'$ be a path decomposition of T' of size $\text{pn}(T')$. Let $\mathcal{P}_1'$ consist of all the paths $P \in \mathcal{P}'$ such that $E(P) \subseteq E(T)$. Let $\mathcal{P}_2' := \mathcal{P}' \setminus \mathcal{P}_1'$. Let $\mathcal{P}_2$ be set of paths obtained from $\mathcal{P}_2'$ by deleting all the edges in $E(T') \setminus E(T)$. Then, $\mathcal{P} := \mathcal{P}_1' \cup \mathcal{P}_2 \cup (E(T) \setminus E(T'))$ is a path decomposition of T. Moreover, by construction, all the edges in $E(T') \setminus E(T)$ are incident to v. Thus, each path in $\mathcal{P}_2'$ contains exactly one edge of $E(T') \setminus E(T)$ and so $|\mathcal{P}_2 \cup (E(T) \setminus E(T'))| \leq 3|\mathcal{P}_2'| = |\mathcal{P}_2'| + 2|E(T') \setminus E(T)| = |\mathcal{P}_2'| + 2|S|$. Therefore, $\text{pn}(T) \leq |\mathcal{P}| \leq |\mathcal{P}_1'| + |\mathcal{P}_2'| + 2|S| = \text{pn}(T') + 2|S|$. This completes the proof. □

8. Cleaning

We now prove Lemma 7.1 using Lemmas 8.1, 8.2, and 8.4 below. In Lemma 8.1, we cover all edges of $T[W_0]$. The edges of $T[W]$ which are incident to W_* will be covered in Lemma 8.2. Since the excess of T is proportional to $|W_*|$, the edges of $T[W]$ which are incident to W_* can be covered one by one with short paths. However, vertices in W_0 may have small excess and so $T[W_0]$ needs to be covered more efficiently. The idea will be to decompose $T[W_0]$ into matchings and then tie them into paths.

In Lemma 8.2, we also decrease the degree of the vertices in $W_* \cup W_A$ when $W_* \neq \emptyset$. This is achieved by covering edges at W_* one by one with short paths until the desired degree is attained.

(The endpoints of these paths are chosen via Lemma 8.3.) Finally, we will use Lemma 8.4 to decrease the degree at W_A when $W_* = \emptyset$. There, we will use long paths to decrease the degree at all vertices in W_A at the same time. This is necessary because the total excess may be relatively small, so we do not have room to cover the degree at each vertex in W_A one by one.

Lemma 8.1. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let $T \notin \mathcal{T}_{\rm excep}$ be a tournament on a vertex set V of size n satisfying the following properties.*

(a) *Let $W_* \cup W_0 \cup V'$ be a partition of V such that, for each $w_* \in W_*$, $|\mathrm{ex}_T(w_*)| > (1-20\eta)n$; for each $w_0 \in W_0$, $|\mathrm{ex}_T(w_0)| \leq (1-20\eta)n$; and, for each $v' \in V'$, $|\mathrm{ex}_T(v')| \leq \varepsilon n$. Let $W := W_* \cup W_0$ and suppose $|W| \leq \varepsilon n$.*

(b) *Let $A^+, A^- \subseteq E(T)$ be absorbing sets of (W, V')-starting/(V', W)-ending edges for T of size at most $\lceil \eta n \rceil$. Denote $A := A^+ \cup A^-$.*

(c) *Let $U^* \subseteq U^0(T)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$.*

Then, there exists a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ of T such that the following hold, where $D := T \setminus \mathcal{P}$.

(i) *$|\mathcal{P}| \leq 2\varepsilon n$.*

(ii) *$E(D[W_0]) = \emptyset$.*

(iii) *If $|U^+(D)| = |U^-(D)| = 1$, then $e(U^-(D), U^+(D)) = 0$ or $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D) \geq 2$.*

(iv) *Each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ satisfies $d_D^+(v) = d_D^-(v) \leq \widetilde{\mathrm{ex}}(D) - 1$.*

One can use the (iii) to ensure that the leftover oriented graph D does not have all its positive and negative excess concentrated on two vertices v^+ and v^-, respectively, with an edge v^-v^+ between them. Otherwise, we would encounter a similar problem as with the tournaments in the class $\mathcal{T}_{\rm apex}$ (recall Proposition 4.37 and Fact 4.38).

Proof of Lemma 8.1. If $W_0 = \emptyset$, then we can set $\mathcal{P} := \emptyset$ and, by Facts 4.20 and 4.38, we are done. Thus, we may assume that $W_0 \neq \emptyset$. Fix additional constants such that $\varepsilon \ll \nu \ll \tau \ll \eta$. Let $W^\pm := W \cap U^\pm(T)$ and, for each $\diamond \in \{*, 0\}$, denote $W_\diamond^\pm := W_\diamond \cap U^\pm(T)$.

Fix an optimal matching decomposition $M_1, \dots, M_m$ of $T[W_0]$ (where $m \leq \varepsilon n$ by Vizing's theorem). Assume inductively that for some $0 \leq k \leq m$, we have constructed edge-disjoint paths $P_{1,1}, P_{1,2}, P_{2,1} \dots, P_{k,2} \subseteq T$ such that $\mathcal{P}_k := \{P_{i,j} \mid i \in [k], j \in [2]\}$ is a (U^*, W, A)-partial path decomposition of T and the following hold.

(α) For each $i \in [k]$, $E(P_{i,1} \cup P_{i,2}) \cap E(T[W]) = M_i$.

(β) For each $i \in [k]$ and $j \in [2]$, $V \setminus W_* \subseteq V(P_{i,j})$.

(γ) For each $\diamond \in \{+, -\}$, if $W_*^\diamond \neq \emptyset$, then $V^\diamond(\mathcal{P}_k) \subseteq W_*^\diamond$.

Denote $D_k := T \setminus \mathcal{P}_k$. Then, following holds.

Claim 1. *We have $\widetilde{\mathrm{ex}}(D_k) = \widetilde{\mathrm{ex}}(T) - 2k \geq 2\eta n$. In particular, $\mathcal{P}_k$ is a good (U^*, W, A)-partial path decomposition of T.*

Proof of Claim. First, note that $\widetilde{\mathrm{ex}}(T) - 2k \geq 2\eta n$ holds by Fact 4.20 and since $k \leq \varepsilon n$. To show $\widetilde{\mathrm{ex}}(D_k) = \widetilde{\mathrm{ex}}(T) - 2k$, note that, by Proposition 4.25 and since $\mathcal{P}_k$ is a partial path decomposition of T, we have $\widetilde{\mathrm{ex}}(D_k) \geq \widetilde{\mathrm{ex}}(T) - 2k$ and $\mathrm{ex}(D_k) \leq \widetilde{\mathrm{ex}}(T) - 2k$. Thus, it is enough to show that $\Delta^0(D_k) \leq \widetilde{\mathrm{ex}}(T) - 2k$.

If there exists $\diamond \in \{+, -\}$ such that $|W_*^\diamond| \geq 2$, then $\mathrm{ex}(D_k) \geq 2(1-20\eta)n - 2k \geq n$ and so $\Delta^0(D_k) \leq \mathrm{ex}(D_k) \leq \widetilde{\mathrm{ex}}(T) - 2k$, as desired. Suppose both $|W_*^\pm| \leq 1$. If $v \in U^\pm(T)$, then, by (β), (γ), and since $\mathcal{P}_k$ is a (U^*, W, A)-partial path decomposition of T, $v \in V^\pm(P) \cup V^0(P)$ for each $P \in \mathcal{P}_k$ and so, $d_{D_k}^{\max}(v) = d_T^{\max}(v) - 2k \leq \widetilde{\mathrm{ex}}(T) - 2k$. Similarly, if $v \in U^0(T) \setminus U^*$, then $v \in V^0(P)$ for each $P \in \mathcal{P}_k$ and so, $d_{D_k}^{\max}(v) = d_T^{\max}(v) - 2k \leq \widetilde{\mathrm{ex}}(T) - 2k$. Thus, if $U^* = \emptyset$, then $\Delta^0(D_k) \leq \widetilde{\mathrm{ex}}(T) - 2k$, as desired. Suppose there exists $v \in U^* \subseteq U^0(T)$. Then, note that, by definition of U^*, $\widetilde{\mathrm{ex}}(T) > \mathrm{ex}(T)$ and, thus, by Proposition 4.22, n is odd. By (β), Fact 4.20, and since $\mathcal{P}_k$ is a (U^*, W, A)-partial path decomposition of T, $v \in V^+(P) \cup V^0(P)$ for all but at most one path $P \in \mathcal{P}_k$ and so, $d^+_{D_k}(v) \leq d^+_T(v) - (2k-1) = \frac{n+1}{2} - 2k \leq \widetilde{\text{ex}}(T) - 2k$. Similarly, $d^-_{D_k}(v) \leq \widetilde{\text{ex}}(T) - 2k$. Thus, $\Delta^0(D_k) \leq \widetilde{\text{ex}}(T) - 2k$, as desired. $\diamond$

If $k = m$, then let $\mathcal{P} := \mathcal{P}_m$. Then, $\mathcal{P}$ is good by Claim 1. Moreover, by construction, $|\mathcal{P}| = 2m \leq 2\varepsilon n$ and $D := T \setminus \mathcal{P}$ satisfies $E(D[W_0]) = \emptyset$ by (α). Thus, (i) and (ii) hold. For (iii), suppose both $|U^\pm(D)| = 1$, say $U^\pm(D) = \{u_\pm\}$, and assume that $u_-u_+ \in E(D)$. If there exists $\diamond \in \{+,-\}$ such that $u_\diamond \notin W_*$, then, as $|\mathcal{P}| \leq 2\varepsilon n$, $d^{\min}_D(u_\diamond) \geq 9\eta n$ and so, by Fact 4.21(d), $\widetilde{\text{ex}}(D) - \text{ex}(D) \geq 2$. We may therefore assume that both $u_\pm \in W_*$. Then, by (γ), all paths in $\mathcal{P}$ start in $W^+_* \subseteq U^+(T)$ and end in $W^-_* \subseteq U^-(T)$. Thus, as $\mathcal{P}$ is a partial path decomposition of T, we have $\text{ex}(D) = \text{ex}(T) - |\mathcal{P}|$ and, since $k \leq \varepsilon n$ and each $v \in W_*$ satisfies $|\text{ex}_T(v)| \geq (1-20\eta)n$, we have $U^\pm(D) = U^\pm(T)$. Therefore, by Claim 1 and since $T \notin \mathcal{T}_{\text{excep}}$, Fact 4.38 implies that $\widetilde{\text{ex}}(D) - \text{ex}(D) = \widetilde{\text{ex}}(T) - \text{ex}(T) \geq 2$ and so (iii) holds. Finally, for (iv), suppose that $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Then, note that $v \notin W_*$ and, by Proposition 4.22, n is odd. Thus, by Fact 4.20, (β), and Claim 1, $d^+_D(v) = d^-_D(v) = d^-_T(v) - |\mathcal{P}| = \frac{n-1}{2} - |\mathcal{P}| \leq \widetilde{\text{ex}}(T) - 1 - |\mathcal{P}| \leq \widetilde{\text{ex}}(D) - 1$ and so (iv) holds.

If $k < m$, then construct $P_{k+1,1}$ and $P_{k+1,2}$ as follows. Denote $M_{k+1} := \{x^+_1x^-_1, \dots, x^+_\ell x^-_\ell\}$. For each $\diamond \in \{+,-\}$, let $X^\diamond := \{x^\diamond_i \mid i \in [\ell]\}$. Let $U^*_k := U^* \setminus (V^+(\mathcal{P}_k) \cup V^-(\mathcal{P}_k))$. Note that, $U^*_k \subseteq U^0(D_k)$. Moreover, since $|U^*| = \widetilde{\text{ex}}(T) - \text{ex}(T)$, Claim 1 and Proposition 4.33 imply that $|U^*_k| = \widetilde{\text{ex}}(D_k) - \text{ex}(D_k)$.

We claim that there exist suitable endpoints $v^\pm_1$ and $v^\pm_2$ for $P_{k+1,1}$ and $P_{k+1,2}$. More precisely, we want to find $v^+_1, v^+_2, v^-_1, v^-_2 \in V$ such that the following hold.

(A) For each $i \in [2]$, $v^+_i \neq v^-_i$ and $v^\pm_i \in \widetilde{U}^\pm_{U^*_k}(D_k)$. Moreover, for each $\diamond \in \{+,-\}$, if $v^\diamond_1 = v^\diamond_2$, then $\widetilde{\text{ex}}^\diamond_{D_k,U^*_k}(v^\diamond_1) \geq 2$.

(B) For each $\diamond \in \{+,-\}$, if $W^\diamond_* \neq \emptyset$, then, for each $i \in [2]$, $v^\diamond_i \in W^\diamond_*$.

(C) There exists a partition $M_{k+1} = M_{k+1,1} \cup M_{k+1,2}$ such that, for each $i \in [2]$ and $x^+x^- \in M_{k+1,i}$, $x^\mp \neq v^\pm_i$ and $x^+x^- \neq v^+_iv^-_i$.

(A) will ensure that $\mathcal{P}_{k+1}$ is a (U^*, W, A)-partial path decomposition of T and (B) will ensure that (γ) holds. Finally, (C) will ensure that all edges of M_{k+1} can be covered by $P_{k+1,1} \cup P_{k+1,2}$.

To find $v^+_1, v^+_2, v^-_1, v^-_2 \in V$ satisfying (A)–(C), we will need the following claim.

Claim 2. *There exist $v^+_1, v^+_2, v^-_1, v^-_2 \in V$ such that, for each $\diamond \in \{+,-\}$ and $i, j \in [2]$, the following hold.*

(I) *If $W^\diamond_* \neq \emptyset$, then $v^\diamond_1 = v^\diamond_2 \in W^\diamond_*$ and $\widetilde{\text{ex}}^\diamond_{D_k,U^*_k}(v^\diamond_1) \geq 2$; otherwise, $v^\diamond_1, v^\diamond_2 \in \widetilde{U}^\diamond_{U^*_k}(D_k)$ are distinct.*

(II) *Both $v^-_1v^+_2, v^-_2v^+_1 \notin M_{k+1}$.*

(III) *$v^+_i \neq v^-_j$.*

Proof of Claim. If $W^+_* \neq \emptyset$, then pick $v^+_1 \in W^+_*$ and let $v^+_2 := v^+_1$ and note that, since $k \leq \varepsilon n$, $\widetilde{\text{ex}}^+_{D_k,U^*_k}(v^+_1) \geq \text{ex}^+_T(v^+_1) - d_{A^+}(v^+_1) - 2k \geq (1-20\eta)n - \lceil \eta n \rceil - 2\varepsilon n \geq 2$, as desired. Assume that $W^+_* = \emptyset$. We claim that $|\widetilde{U}^+_{U^*_k}(D_k)| \geq 2$. Assume not. Then, since $|A^+| \leq \lceil \eta n \rceil$, by Claim 1 and (4.3), $\widetilde{\text{ex}}^+_{U^*_k}(D_k) = \widetilde{\text{ex}}(D_k) - |A^+| \geq 2\eta n - \lceil \eta n \rceil > 1$ and so there exists $v \in V$ such that $\widetilde{U}^+_{U^*_k}(D_k) = \{v\}$ and $\widetilde{\text{ex}}^+_{D_k,U^*_k}(v) \geq 2$. Then, note that $v \notin U^0(D_k)$. As $v \notin W^+_*$ and $k \leq \varepsilon n$, $d^-_{D_k}(v) \geq 9\eta n$. Thus, by Fact 4.21(d), $\widetilde{\text{ex}}(D_k) - \text{ex}^+_{D_k}(v) \geq d^-_{D_k}(v) \geq 9\eta n$ and so, since $v \notin U^0(D_k)$, we have

$$
\begin{aligned}
0 &= \widetilde{\text{ex}}^+_{D_k,U^*_k}(V \setminus \{v\}) = \widetilde{\text{ex}}^+_{U^*_k}(D_k) - \widetilde{\text{ex}}^+_{D_k,U^*_k}(v) \\
&\overset{(4.2),(4.3)}{=} \widetilde{\text{ex}}(D_k) - |A^+| - (\text{ex}^+_{D_k}(v) - d_{A^+}(v)) \\
&\geq (\widetilde{\text{ex}}(D_k) - \text{ex}^+_{D_k}(v)) - |A^+| \geq 9\eta n - \lceil \eta n \rceil \geq 7\eta n,
\end{aligned}
$$

a contradiction. Thus, $|\widetilde{U}^+_{U^*_k}(D_k)| \geq 2$ and so we can pick distinct $v^+_1, v^+_2 \in \widetilde{U}^+_{U^*_k}(D_k)$.

Then, proceed similarly as above to pick $v_1^-, v_2^- \in \widetilde{U}^-_{U_k^*}(D_k) \setminus \{v_1^+, v_2^+\}$. Note that this is possible since, for each $i \in [2]$, $\widetilde{\mathrm{ex}}^-_{D_k,U_k^*}(v_i^+) \leq 1$. By relabelling v_1^- and v_2^- if necessary, we can ensure (II) holds. This completes the proof. $\diamond$

Fix $v_1^+, v_2^+, v_1^-, v_2^- \in V$ satisfying properties (I)–(III) of Claim 2. We claim that (A)–(C) hold. Indeed, (A) and (B) follow immediately from (I). Recall the notation $M_{k+1} = \{x_i^+x_i^- \mid i \in [\ell]\}$. For (C), let $M_{k+1,2} \coloneqq \{x_i^+x_i^- \in M_{k+1} \mid v_1^+ = x_i^- \text{ or } v_1^- = x_i^+ \text{ or } v_1^+v_1^- = x_i^+x_i^-\}$ and $M_{k+1,1} \coloneqq M_{k+1} \setminus M_{k+1,2}$. We claim that the partition $M_{k+1} = M_{k+1,1} \cup M_{k+1,2}$ witnesses that (C) holds. By definition, $M_{k+1,1}$ clearly satisfies the desired properties, so it is enough to show that $M_{k+1,2} \subseteq M_{k+1} \setminus \{x_i^+x_i^- \in M_{k+1} \mid v_2^+ = x_i^- \text{ or } v_2^- = x_i^+ \text{ or } v_2^+v_2^- = x_i^+x_i^-\}$. If $v_1^+v_1^- = x_i^+x_i^-$ for some $i \in [\ell]$, then, by (I), (III), and the fact that $V(M_{k+1}) \subseteq W_0$, we have $v_2^+, v_2^- \notin \{x_i^+, x_i^-\}$. Moreover, if $v_1^+ = x_i^-$ for some $i \in [\ell]$, then, by (I), $v_2^+ \neq x_i^-$, by (III), $v_2^- \neq v_1^+$ and so $v_2^+v_2^- \neq x_i^+x_i^-$, and, by (II), $v_2^- \neq x_i^+$. Similarly, if $v_1^- = x_i^+$ for some $i \in [\ell]$, then $v_2^- \neq x_i^+$, $v_2^+ \neq x_i^-$, and $v_2^+v_2^- \neq x_i^+x_i^-$. Therefore, (C) holds, as desired.

We will now construct, for each $i \in [2]$, a (v_i^+, v_i^-)-path $P_{k+1,i}$ covering $M_{k+1,i}$. The idea is to join together the edges in $M_{k+1,i}$ via V'. In order to satisfy (β), we also incorporate the vertices in $W_0 \setminus V(M_{k+1,i})$ in a similar fashion. This will be done using Corollary 4.8 as follows.

Denote

$$k' \coloneqq 2 + |M_{k+1,1} \setminus \{e \in M_{k+1,1} \mid V^+(e) = \{v_1^+\} \text{ or } V^-(e) = \{v_1^-\}\}| + |W_0 \setminus V(M_{k+1,1})|$$

(k' will play the role of k in Corollary 4.8). Now construct the k' paths for Corollary 4.8 as follows. If $v_1^+ \notin V(M_{k+1,1})$, let $Q_1 \coloneqq v_1^+$; otherwise, let Q_1 be the (unique) edge $e \in M_{k+1,1}$ such that $V^+(e) = \{v_1^+\}$. Similarly, if $v_1^- \notin V(M_{k+1,1})$, let $Q_{k'} \coloneqq v_1^-$; otherwise, let $Q_{k'}$ be the (unique) edge $e \in M_{k+1,1}$ such that $V^-(e) = \{v_1^-\}$. Let $Q_2, \dots, Q_{k'-1}$ be an enumeration of $(M_{k+1,1} \setminus \{Q_1, Q_{k'}\}) \cup (W_0 \setminus V(M_{k+1,1}))$. Recall that $V(M_{k+1,1}) \subseteq W_0$. Thus, $V' \cap \left(\bigcup_{i\in[k']} V(Q_i)\right) \subseteq \{v_1^+, v_1^-\}$ and so, since $k \leq \varepsilon n$, Lemma 4.4 implies that $D_k[V' \setminus \bigcup_{i\in[k']} V(Q_i)]$ is a robust (ν,τ)-outexpander. Then, apply Corollary 4.8(b) with $D_k \setminus A, V \setminus \bigcup_{i\in[k']} V(Q_i), k', \frac{3}{8}, W_* \setminus \{v_1^+, v_1^-\}$, and $Q_1, \dots, Q_{k'}$ playing the roles of D, V', k, δ, S, and $P_1, \dots, P_k$ to obtain a (v_1^+, v_1^-)-path $P_{k+1,1}$ covering the edges in $M_{k+1,1}$ and all vertices in $V' \cup W_0$. Note that all the conditions of Corollary 4.8 are satisfied since, by construction, the ending points of $Q_1, \dots, Q_{k'-1}$ lie in $V \setminus W_*$ and so, by Fact 4.21(b), (a), and (b), they have, in $D_k \setminus A$, at least $\frac{20\eta n}{2} - 2k - \lceil \eta n \rceil - |W| - 2k' \geq 8\eta n \geq 2k'$ outneighbours in $V' \setminus \bigcup_{i\in[k']} V(Q_i)$. Similarly, the starting points of $Q_2, \dots, Q_{k'}$ have, in $D_k \setminus A$, at least $2k'$ inneighbours in $V' \setminus \bigcup_{i\in[k']} V(Q_i)$, as desired. Construct $P_{k+1,2}$ similarly, but deleting the edges in $P_{k+1,1}$ before applying Corollary 4.8 (this will ensure that $P_{k+1,1}$ and $P_{k+1,2}$ are edge-disjoint). Thus, (α)–(γ) hold with k replaced by $k+1$. This completes the proof. $\square$

Since the vertices in W_* have almost all their edges in the same direction, one could not have covered the remaining edges in W with a similar approach as in Lemma 8.1. However, since the vertices in W_* have very large excess, $W_* \neq \emptyset$ implies that the excess of the tournament is very large and so we have room to cover each remaining edge in W one by one. One can also decrease the degree at $W_* \cup W_A$ with a similar approach. This is achieved in the next lemma.

Lemma 8.2. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let D be an oriented graph on a vertex set V of size n such that $\delta(D) \geq (1-\varepsilon)n$ and the following properties are satisfied.*

(a) *Let $W_* \cup W_0 \cup V'$ be a partition of V such that, for each $w_* \in W_*$, $|\mathrm{ex}_D(w_*)| > (1-21\eta)n$; for each $w_0 \in W_0$, $|\mathrm{ex}_D(w_0)| \leq (1-20\eta)n$; and, for each $v' \in V'$, $|\mathrm{ex}_D(v')| \leq \varepsilon n$. Let $W \coloneqq W_* \cup W_0$ and suppose $|W| \leq \varepsilon n$ and $W_* \neq \emptyset$.*

(b) *Let $A^+, A^- \subseteq E(D)$ be absorbing sets of (W, V')-starting/(V', W)-ending edges for D of size at most $\lceil \eta n \rceil$. Let $A \coloneqq A^+ \cup A^-$. Let $W_A^\pm \coloneqq V(A^\pm) \cap W$ and $W_A \coloneqq V(A) \cap W$. Suppose that, for each $\diamond \in \{+,-\}$, if $|W_A^\diamond| \geq 2$, then, for each $v \in V$, $\mathrm{ex}_D^\diamond(v) < \eta n$ and, if $|W_A^\diamond| = 1$, then, for each $v \in V$ and $w_A \in W_A^\diamond$, $\mathrm{ex}_D^\diamond(v) \leq \mathrm{ex}_D^\diamond(w_A) + \varepsilon n$.*

(c) *Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.*

(d) *Suppose* $E(D[W_0]) = \emptyset$ *and, if* $|U^+(D)| = |U^-(D)| = 1$*, then* $e(U^-(D), U^+(D)) = 0$ *or* $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D) \geq 2$.

Then, there exists a good (U^*, W, A)*-partial path decomposition* $\mathcal{P}$ *of* D *such that* $D' := D \setminus \mathcal{P}$ *satisfies the following.*

(i) $E(D'[W]) = \emptyset$.
(ii) $N^{\pm}(D) - N^{\pm}(D') \leq 88\eta n$.
(iii) *For each* $v \in W_* \cup W_A$, $(1 - 3\sqrt{\eta})n \leq d_{D'}(v) \leq (1 - 4\eta)n$.
(iv) *For each* $v \in W_0$, $d_{D'}(v) \geq (1 - 3\sqrt{\eta})n$ *and* $d_{D'}^{\min}(v) \geq 5\eta n$.
(v) *For each* $v \in V'$, $d_{D'}(v) \geq (1 - 3\sqrt{\varepsilon})n$.
(vi) *If* $|W_*^+|, |W_*^-| \leq 1$*, then each* $v \in W_*$ *satisfies* $|\mathrm{ex}_{D'}(v)| = d_{D'}(v)$.
(vii) *Each* $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ *satisfies* $d_{D'}^+(v) = d_{D'}^-(v) \leq \widetilde{\mathrm{ex}}(D') - 1$.

(vi) will enable us to satisfy Lemma 7.1(vi). As mentioned above, the strategy in the proof of Lemma 8.2 is to cover the remaining edges of $D[W]$ one by one. To decrease the degree at W_*, we further fix some additional edges that will be covered with short paths in the same way. The degree at $W_A \setminus W_*$ will be decreased by incorporating these vertices in some of these paths.

Similarly as in the proof of Lemma 8.1, given an edge that needs to be covered, we need to find suitable endpoints, that is, endpoints of the correct excess and which are "compatible" with the edge that needs to be covered. The next lemma enables us to find, given a set H of edges to be covered, pairs of suitable endpoints to cover each of these edges with a path.

Lemma 8.3. *Let* $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$*. Let* D *be an oriented graph on a vertex set* V *of size* n *such that Lemma 8.2(a)–(d) are satisfied. Let* $H \subseteq D$ *satisfy* $\Delta(H) \leq 11\eta n$ *and* $k := |E(H)| \leq 11\eta n|W_*|$*. Let* $w_1^+w_1^-, \dots, w_k^+w_k^-$ *be an enumeration of* $E(H)$.

Then, there exist pairs of vertices $(v_1^+, v_1^-), \dots, (v_k^+, v_k^-)$ *such that the following hold.*

(i) *For each* $v \in V$ *and* $\diamond \in \{+, -\}$*, there exist at most* $\min\{2\sqrt{\eta}n, \widetilde{\mathrm{ex}}_{D,U^*}^{\diamond}(v)\}$ *indices* $i \in [k]$ *such that* $v = v_i^{\diamond}$.
(ii) *For all* $i \in [k]$*, if* $w_i^{\pm} \in W_*^{\pm}$*, then* $v_i^{\pm} = w_i^{\pm}$.
(iii) *For all* $i \in [k]$*, if there exists* $\diamond \in \{+, -\}$ *such that* $w_i^{\diamond} \in V'$*, then* $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.
(iv) *For each* $i \in [k]$, $\{v_i^+, w_i^+\} \cap \{v_i^-, w_i^-\} = \emptyset$.
(v) *For each* $\diamond \in \{+, -\}$*, there exist at most* $88\eta n$ *vertices* $v \in V$ *such that there exist exactly* $\widetilde{\mathrm{ex}}_{D,U^*}^{\diamond}(v)$ *indices* $i \in [k]$ *such that* $v_i^{\diamond} = v$.
(vi) *If* $v \in V$ *satisfies* $d_D^+(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$*, then* $v \in \{w_i^+, w_i^-, v_i^+\} \setminus \{v_i^-\}$ *for all* $i \in [k]$*. Similarly, if* $v \in V$ *satisfies* $d_D^-(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$*, then* $v \in \{w_i^+, w_i^-, v_i^-\} \setminus \{v_i^+\}$ *for all* $i \in [k]$.

Since vertices in W_* have most of their edges in the same direction, (ii) will ensure that we will be able to tie up edges to the designated endpoints of the path. (iii) will ensure that some of the paths will have length more than one, which will enable us to cover a significant number of edges at $W_A \setminus W_*$. (iv) implies that the chosen endpoints are distinct, the chosen starting point for the path is not the ending point of the edge we want to cover and, similarly, that the chosen ending point is not the starting point of the edge we want to cover. Together with Proposition 4.27, property (vi) will ensure that the partial path decomposition constructed with this set of endpoints will be good.

Proof of Lemma 8.2. Recall that, by assumption, $W_* \neq \emptyset$. Fix additional constants such that $\varepsilon \ll \nu \ll \tau \ll \eta$. Let $W^{\pm} := W \cap U^{\pm}(D)$ and, for each $\diamond \in \{*, 0\}$, denote $W_{\diamond}^{\pm} := W_{\diamond} \cap U^{\pm}(D)$.

We now define a subdigraph $H \subseteq D$, whose edges will be covered by $\mathcal{P}$. If $\max\{|W_*^+|, |W_*^-|\} \geq 2$, then let $H \subseteq D \setminus A$ be obtained from $D[W]$ by adding, for each $v \in W_*$, $\lceil 4\eta n \rceil$ edges of $D \setminus A$ between v and V' (of either direction). Otherwise, let $H \subseteq D \setminus A$ be obtained from $D[W]$ by adding, for each $v \in W_*^{\pm}$, $\max\{d_D^{\mp}(v), \lceil 4\eta n \rceil\}$ edges of $D \setminus A$ between v and V' (of either direction) such that $d_H^{\mp}(v) = d_{D\setminus A}^{\mp}(v) = d_D^{\mp}(v)$. Note that $\Delta(H) \leq |W| + \max\{\max_{v \in W^*} d_D^{\min}(v), \lceil 4\eta n \rceil\} \leq$

$\varepsilon n + \frac{21\eta n}{2} \leq 11\eta n$, each $v \in V \setminus W_*$ satisfies $d_H(v) \leq |W_*| \leq \varepsilon n$, and

$$\lceil 4\eta n \rceil \leq k := |E(H)| \leq 11\eta n|W_*|. \tag{8.1}$$

Let $w_1^+w_1^-, \dots, w_k^+w_k^-$ be an enumeration of $E(H)$. Recall that $W_* \neq \emptyset$ and, for each $w \in W^*$, $|N_H(w) \cap V'| \geq \lceil 4\eta n \rceil$. Thus, we may assume without loss generality that, for each $i \in [\lceil 4\eta n \rceil]$, $w_i^+w_i^- \notin E(D[W])$. Apply Lemma 8.3 to obtain pairs of vertices $(v_1^+, v_1^-), \dots, (v_k^+, v_k^-)$ such that the following hold.

(α) For each $v \in V$ and $\diamond \in \{+,-\}$, there exist at most $\min\{2\sqrt{\eta}n, \widetilde{\mathrm{ex}}^\diamond_{D,U^*}(v)\}$ indices $i \in [k]$ such that $v = v_i^\diamond$.

(β) For all $i \in [k]$, if $w_i^\pm \in W_*^\pm$, then $v_i^\pm = w_i^\pm$.

(γ) For all $i \in [k]$, if there exists $\diamond \in \{+,-\}$ such that $w_i^\diamond \in V'$, then $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.

(δ) For each $i \in [k]$, $\{v_i^+, w_i^+\} \cap \{v_i^-, w_i^-\} = \emptyset$.

(ε) For each $\diamond \in \{+,-\}$, there exist at most $88\eta n$ vertices $v \in V$ such that there exist exactly $\widetilde{\mathrm{ex}}^\diamond_{D,U^*}(v)$ indices $i \in [k]$ such that $v_i^\diamond = v$.

(ζ) If $v \in V$ satisfies $d_D^+(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $v \in \{w_i^+, w_i^-, v_i^+\} \setminus \{v_i^-\}$ for all $i \in [k]$. Similarly, if $v \in V$ satisfies $d_D^-(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $v \in \{w_i^+, w_i^-, v_i^-\} \setminus \{v_i^+\}$ for all $i \in [k]$.

By assumption on our ordering of $E(H)$, (γ) implies that the following holds.

(γ') For all $i \in [\lceil 4\eta n \rceil]$, $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.

We now cover each edge $w_i^+w_i^-$ with a short (v_i^+, v_i^-)-path inductively as follows. Suppose that for some $0 \leq \ell \leq k$ we have constructed edge-disjoint paths $P_1, \dots, P_\ell \subseteq D \setminus A$. For each $0 \leq i \leq \ell$, let $D_i := D \setminus \bigcup_{j \in [i]} P_i$ and S_i be the set of vertices $w \in W_A \setminus W_*$ such that $d_{D_i}(w) > (1 - 4\eta)n$. (Note that S_ℓ corresponds to the set of vertices in $W_A \setminus W_*$ whose degree is currently too high.) Suppose furthermore that the following hold for each $i \in [\ell]$.

(I) P_i is a (v_i^+, v_i^-)-path.

(II) $w_i^+w_i^- \in E(P_i)$.

(III) $S_{i-1} \subseteq V(P_i)$.

(IV) For each $v \in V'$, there exist at most $\sqrt{\varepsilon}n$ indices $j \in [\ell]$ such that $v \in V^0(P_j) \setminus \{w_j^+, w_j^-\} = V(P_j) \setminus \{v_j^+, w_j^+, w_j^-, v_j^-\}$.

(V) For each $v \in V(P_i) \cap W$, $v \in \{v_i^+, v_i^-, w_i^+, w_i^-\} \cup S_{i-1}$.

(VI) $e(P_i) \leq 7\nu^{-1}(|S_{i-1}| + 1)$.

If $\ell = k$, then let $\mathcal{P} := \bigcup_{i \in [\ell]} P_i$ and $D' := D_\ell$. Note that $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D by (α) and (I). To show $\mathcal{P}$ is good, suppose first that $\max\{|W_*^+|, |W_*^-|\} \geq 2$. Then, $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$ and so, since $\mathcal{P}$ is a partial path decomposition of D, by Proposition 4.25(b),

$$\mathrm{ex}(D') = \mathrm{ex}(D) - |\mathcal{P}| \overset{(8.1)}{\geq} \max\{|W_*^+|, |W_*^-|\}(1 - 21\eta)n - 11\eta n|W_*| \geq n \geq \Delta^0(D').$$

Thus, if $\max\{|W_*^+|, |W_*^-|\} \geq 2$, then $\widetilde{\mathrm{ex}}(D') = \mathrm{ex}(D') = \mathrm{ex}(D) - |\mathcal{P}| = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$ and, so, $\mathcal{P}$ is good. We may therefore assume that both $|W_*^\pm| \leq 1$. Then, note that $k \leq 22\eta n$ and so, by Proposition 4.27(b), (ζ), (I), and (II), $\mathcal{P}$ is good, as desired.

We now check that (i)–(vii) are satisfied. First, note that (i) and (vi) hold by (II) and definition of H. Note that, since $k \geq \lceil 4\eta n \rceil$, $S_\ell = \emptyset$. Thus, (iii) holds by construction of H, (α), (I)–(III), and (V). Note that each $v \in W_0$ satisfies $d_D^{\min}(v) \geq (10\eta - \frac{\varepsilon}{2})n$ and $d_H(v) \leq |W| \leq \varepsilon n$. Thus, (iv) follows from (α), (I), and (V). Recall that each $v \in V'$ satisfies $|\mathrm{ex}_D(v)| \leq \varepsilon n$ and $E(H) \cap E(D[V']) = \emptyset$. Thus, (v) follows from (α), (I), and (IV). For (vii), note that, by (a), (α), and (I), each $v \in W^*$ satisfies $|\mathrm{ex}_{D'}(v)| \geq (1 - 21\eta)n - 2\sqrt{\eta}n \geq \frac{n}{2}$ and so $\widetilde{\mathrm{ex}}(D') \geq \mathrm{ex}(D') \geq \frac{n}{2}$. Thus, each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P})) =: U^{**}$ satisfies $d_{D'}^+(v) = d_{D'}^-(v) \leq \frac{n-1}{2} < \widetilde{\mathrm{ex}}(D')$ and so (vii) holds. Finally, for (ii), denote $S_\pm := \widetilde{U}^\pm_{U^*}(D) \cup \{v \in W_A^\pm \mid \mathrm{ex}_D^\pm(v) = d_A^\pm(v)\}$ and $S'_\pm := \widetilde{U}^\pm_{U^{**}}(D') \cup \{v \in W_A^\pm \mid \mathrm{ex}_{D'}^\pm(v) = d_A^\pm(v)\}$. Then, by (4.2) and since $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$,

$N^{\pm}(D) = |S_{\pm}|$. Similarly, since $\mathcal{P}$ is good, by Proposition 4.33, $N^{\pm}(D') = |S'_{\pm}|$. Since $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D, we have $S'_{\pm} \subseteq S_{\pm}$ and, for each $\diamond \in \{+,-\}$, $S_\diamond \setminus S'_\diamond$ is precisely the set of vertices $v \in V \setminus W_A^\diamond$ for which there exist exactly $\widetilde{\text{ex}}_{D,U^*}^\diamond(v)$ paths $P \in \mathcal{P}$ such that $v \in V^\diamond(\mathcal{P})$. Thus, by ($\varepsilon$) and (I), $N^{\pm}(D) - N^{\pm}(D') = |S_{\pm} \setminus S'_{\pm}| \leq 88\eta n$ and (ii) holds, as desired.

Suppose $\ell < k$. Note that, by (III), if $\lceil 4\eta n \rceil < i \leq \ell$, then $S_i = \emptyset$. We construct $P_{\ell+1}$ as follows. If $(v_{\ell+1}^+, v_{\ell+1}^-) = (w_{\ell+1}^+, w_{\ell+1}^-)$, then let $P_{\ell+1} := w_{\ell+1}^+ w_{\ell+1}^-$. Note that, in this case, by (γ'), $S_\ell = \emptyset$. Thus, (I)–(VI) hold with $\ell+1$ playing the role of ℓ and we are done. We may therefore assume that $(v_{\ell+1}^+, v_{\ell+1}^-) \neq (w_{\ell+1}^+, w_{\ell+1}^-)$. We construct $P_{\ell+1}$ using Corollary 4.8 as follows. Let X be the set of vertices $v \in V' \setminus \{v_{\ell+1}^+, v_{\ell+1}^-, w_{\ell+1}^+, w_{\ell+1}^-\}$ such that there exist $\lfloor \sqrt{\varepsilon} n \rfloor$ indices $i \in [\ell]$ such that $v \in V^0(P_i) \setminus \{w_i^+, w_i^-\}$. Note that, by (VI),

$$|X| \leq \frac{7\nu^{-1}(\sum_{i \in [\ell]} |S_i| + 2\ell)}{\lfloor \sqrt{\varepsilon} n \rfloor} \leq \frac{8\nu^{-1}(\lceil 4\eta n \rceil \cdot \varepsilon n + 22\eta n \cdot \varepsilon n)}{\sqrt{\varepsilon} n} \leq \varepsilon^{\frac{1}{3}} n.$$

Recall that each $v \in V'$ satisfies $|\text{ex}_D(v)| \leq \varepsilon n$. Thus, by Lemma 4.4, ($\alpha$), (I), and (IV), $D_\ell[V' \setminus (X \cup \{v_{\ell+1}^+, w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^-\})]$ is a robust (ν, τ)-outexpander. Let $u_1, \dots, u_s$ be an enumeration of $S_\ell \setminus \{v_{\ell+1}^+, w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^-\}$. If both $v_{\ell+1}^{\pm} \neq w_{\ell+1}^{\pm}$, then let $m := s+3$, $Q_1 := v_{\ell+1}^+$, $Q_2 := w_{\ell+1}^+ w_{\ell+1}^-$, $Q_m := v_{\ell+1}^-$, and, for each $i \in [s]$, let $Q_{i+2} := u_i$. If $v_{\ell+1}^+ \neq w_{\ell+1}^+$ and $v_{\ell+1}^- = w_{\ell+1}^-$, then let $m := s+2$, $Q_1 := v_{\ell+1}^+$, $Q_m := w_{\ell+1}^+ w_{\ell+1}^-$, and, for each $i \in [s]$, let $Q_{i+1} := u_i$. Similarly, if $v_{\ell+1}^+ = w_{\ell+1}^+$ and $v_{\ell+1}^- \neq w_{\ell+1}^-$, then let $m := s+2$, $Q_1 := w_{\ell+1}^+ w_{\ell+1}^-$, $Q_m := v_{\ell+1}^-$, and, for each $i \in [s]$, let $Q_{i+1} := u_i$. Note that, by (δ), this covers all possible cases. Moreover, we always have $m \leq |S_\ell| + 3$. Apply Corollary 4.8(a) with $D_\ell \setminus A, V \setminus (S_\ell \cup \{v_{\ell+1}^+, w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^-\}), m, \frac{3}{8}, X \cup (W \setminus (S_\ell \cup \{v_{\ell+1}^+, w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^-\}))$, and $Q_1, \dots, Q_m$ playing the roles of D, V', k, δ, S, and $P_1, \dots, P_k$ to obtain a $(v_{\ell+1}^+, v_{\ell+1}^-)$-path $P_{\ell+1}$ of length at most $2\nu^{-1}m + 1 \leq 2\nu^{-1}(|S_\ell| + 3) + 1 \leq 7\nu^{-1}(|S_\ell| + 1)$ which covers $w_{\ell+1}^+ w_{\ell+1}^-$ and the vertices in S_ℓ and avoids the vertices in $X \cup (W \setminus (S_\ell \cup \{v_{\ell+1}^+, w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^-\}))$. Note that all the conditions of Corollary 4.8 are satisfied. Indeed, observe that, by (α), (β), and construction, the ending points of $Q_1, \dots, Q_{m-1}$ lie in $V \setminus W_*^-$. We now verify that each $v \in V \setminus W_*^-$ satisfies $|N_{D_\ell \setminus A}^+(v) \cap (V' \setminus (\bigcup_{i \in [m]} V(Q_m) \cup X))| \geq 2m$. If $v \in W_*^+$, then, by Fact 4.21(c), (b), (α), (I), (V), and since $\Delta^+(H) \leq 11\eta n$, we have $|N_{D_\ell \setminus A}^+(v) \cap (V' \setminus (\bigcup_{i \in [m]} V(Q_m) \cup X))| \geq (1 - 11\eta)n - 2\sqrt{\eta} n - 11\eta n - \lceil \eta n \rceil - |W| - 2m - \varepsilon^{\frac{1}{3}} n \geq \eta n \geq 2m$. If $v \in V \setminus W_*$, then, recall that $d_H(v) \leq |W_*| \leq \varepsilon n$ and so, by Fact 4.21(b), Definition 4.28, (α), (I), (IV), (V), and definition of the S_i, we have $|N_{D_\ell \setminus A}^+(v) \cap (V' \setminus (\bigcup_{i \in [m]} V(Q_m) \cup X))| \geq \frac{(20\eta - \varepsilon)n}{2} - 1 - 1 - 4\eta n - \varepsilon n - |W| - 2m - \varepsilon^{\frac{1}{3}} n \geq \eta n \geq 2m$. Therefore, the ending points of $Q_1, \dots, Q_{m-1}$ have, in $D_\ell \setminus A$, at least $2m$ outneighbours in $V' \setminus (\bigcup_{i \in [m]} V(Q_m) \cup X)$ and, similarly, the starting points of $Q_2, \dots, Q_m$ have, in $D_\ell \setminus A$, at least $2m$ inneighbours in $V' \setminus (\bigcup_{i \in [m]} V(Q_m) \cup X)$, as desired. One can easily verify that (I)–(VI) hold with $\ell + 1$ playing the role of ℓ. $\square$

Proof of Lemma 8.3. Let $W^{\pm} := W \cap U^{\pm}(D)$ and, for each $\diamond \in \{*, 0\}$, denote $W_\diamond^{\pm} := W_\diamond \cap U^{\pm}(D)$. Observe that the following holds.

(††) *There are no distinct $v_+, v_-, v_0 \in V$ such that $v_+ v_- \in E(D)$ and both $\widetilde{U}_{U^*}^{\pm}(D) = \{v_\mp, v_0\}$.*

Indeed, suppose for a contradiction that $v_+, v_-, v_0 \in V$ are distinct and such that $v_+ v_- \in E(D)$ and both $\widetilde{U}_{U^*}^{\pm}(D) = \{v_\mp, v_0\}$. We now show that $U^{\pm}(D) = \{v_\mp\}$, which implies that $|U^{\pm}(D)| = 1$, $e(U^-(D), U^+(D)) \neq 0$ and $\widetilde{\text{ex}}(D) - \text{ex}(D) = |U^*| = |\widetilde{U}_{U^*}^+(D) \cap \widetilde{U}_{U^*}^-(D)| = 1 < 2$, a contradiction to (d). Note that, since $W_* \neq \emptyset$, $\text{ex}(D) \geq (1 - 21\eta)n$. Moreover, since $v_\mp \in \widetilde{U}_{U^*}^{\pm}(D) \setminus \widetilde{U}_{U^*}^{\mp}(D)$, we have $v_\mp \in U^{\pm}(D)$. Thus, by Fact 4.31 and (4.3), $\text{ex}_D^{\pm}(v_\mp) \geq \widetilde{\text{ex}}_{D,U^*}^{\pm}(v_\mp) = \widetilde{\text{ex}}_{U^*}^{\pm}(D) - \widetilde{\text{ex}}_{D,U^*}^{\pm}(v_0) \geq (\text{ex}(D) - \lceil \eta n \rceil) - 1 \geq 2\eta n$. Thus, by (b), $|W_A^{\pm}| \leq 1$. Moreover, $W_A^{\pm} \subseteq \{v_\mp\}$ since, if $v \in W_A^{\pm} \setminus \{v_\mp\}$, then $v \notin \widetilde{U}_{U^*}^{\pm}(D)$ and so, by (b), $\text{ex}_D^{\pm}(v_\mp) \leq \text{ex}_D^{\pm}(v) + \varepsilon n \leq \lceil \eta n \rceil + \varepsilon n < 2\eta n$, a contradiction. Thus, $U^{\pm}(D) = \widetilde{U}_{U^*}^{\pm}(D) \setminus U^* = \{v_\mp\}$, as desired.

Let $\widetilde{W}_*^\pm \coloneqq \{v \in V \mid \mathrm{ex}_D^\pm(v) \geq (1-86\eta)n\}$. For technical reasons, we will ensure that, for any $i \in [k]$ and $\diamond \in \{+,-\}$, if $w_i^\diamond \in \widetilde{W}_*^\diamond$, then $v_i^\diamond = w_i^\diamond$. Note that this will imply (ii), as $W_*^\pm \subseteq \widetilde{W}_*^\pm$. Without loss of generality, we may assume that, if $|\widetilde{W}_*^+| = |\widetilde{W}_*^-| = 1$ and there exists $i \in [k]$ such that $w_i^+ \in \widetilde{W}_*^-$ and $w_i^- \in \widetilde{W}_*^+$, then $i = 1$.

Suppose that, for some $0 \leq \ell \leq k$, we have already constructed pairs $(v_1^+, v_1^-), \dots, (v_\ell^+, v_\ell^-)$ such that the following hold. For each $v \in V$, define $\widehat{\mathrm{ex}}_\ell^\pm(v) \coloneqq \widetilde{\mathrm{ex}}_{D,U^*}^\pm(v) - |\{i \in [\ell] \mid v_i^\pm = v\}|$. Denote $\widehat{U}_\ell^\pm(D) \coloneqq \{v \in V \mid \widehat{\mathrm{ex}}_\ell^\pm(v) > 0\}$.

(α) For each $v \in V$, $\widehat{\mathrm{ex}}_\ell^\pm(v) \geq 0$.

(β) For each $v \in V$ and $\diamond \in \{+,-\}$, there exist at most $\sqrt{\eta}n$ indices $i \in [\ell]$ such that $v_i^\diamond = v \neq w_i^\diamond$.

(γ) For all $i \in [\ell]$ and $\diamond \in \{+,-\}$, if $w_i^\diamond \in \widetilde{W}_*^\diamond$, then $v_i^\diamond = w_i^\diamond$.

(δ) For all $i \in [\ell]$, if there exists $\diamond \in \{+,-\}$ such that $w_i^\diamond \in V'$, then $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.

(ε) For each $i \in [\ell]$, $\{v_i^+, w_i^+\} \cap \{v_i^-, w_i^-\} = \emptyset$.

(ζ) For each $\diamond \in \{+,-\}$, if $\widetilde{U}_{U^*}^\diamond(D) \setminus \widehat{U}_\ell^\diamond(D) \neq \emptyset$, then $|W_*^+| \leq 4$ and $|W_*^-| \leq 4$.

(η) If $v \in V$ satisfies $d_D^+(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $v \in \{w_i^+, w_i^-, v_i^+\} \setminus \{v_i^-\}$ for all $i \in [\ell]$. Similarly, if $v \in V$ satisfies $d_D^-(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $v \in \{w_i^+, w_i^-, v_i^-\} \setminus \{v_i^+\}$ for all $i \in [\ell]$.

Assume $\ell = k$. Then, since, by assumption, $\Delta(H) \leq 11\eta n$, (i) follows from (α) and (β). (ii)–(iv) and (vi) hold by (γ)–(ε) and (η), respectively. By (ζ), (v) holds if $\max\{|W_*^+|, |W_*^-|\} \geq 5$. Otherwise, $|\widetilde{U}_{U^*}^\pm(D) \setminus \widehat{U}_\ell^\pm(D)| \leq k \leq 88\eta n$ and (v) is also satisfied.

Suppose $\ell < k$. First, observe that, by definition of $\widehat{\mathrm{ex}}_\ell^\pm(v)$, the following hold.

$$\begin{aligned}
\widehat{\mathrm{ex}}_\ell^\pm(D) &\coloneqq \sum_{v \in V} \widehat{\mathrm{ex}}_\ell^\pm(v) = \widetilde{\mathrm{ex}}_{U^*}^\pm(D) - \ell \\
&\overset{(4.3)}{=} \widetilde{\mathrm{ex}}(D) - |A^\pm| - \ell \qquad (8.2)\\
&\geq \mathrm{ex}(D) - |A^\pm| - \ell \geq \max\{|W_*^+|, |W_*^-|\}(1-21\eta)n - \lceil \eta n \rceil - 11\eta n |W_*| \\
&\geq \max\{|W_*^+|, |W_*^-|\}(1-46\eta)n. \qquad (8.3)
\end{aligned}$$

Let $X^\pm$ be the set of vertices $v \in V \setminus \{w_{\ell+1}^\pm\}$ such that $\widetilde{\mathrm{ex}}_{D,U^*}^\pm(v) - \widehat{\mathrm{ex}}_\ell^\pm(v) = \lfloor \sqrt{\eta} n \rfloor$. Also observe that, by (β) and the fact that $\Delta(H) \leq 11\eta n$, the following hold.

(‡) *Each $v \in \widetilde{W}_*^\pm$ satisfies $\widehat{\mathrm{ex}}_\ell^\pm(v) \geq \mathrm{ex}_D^\pm(v) - |A^\pm| - \sqrt{\eta}n - \Delta(H) \geq 2$.*

One can easily verify that it is enough to find distinct $v_{\ell+1}^\pm \in V$ such that the following hold.

(I) $v_{\ell+1}^\pm \in \widehat{U}_\ell^\pm(D) \setminus (X^\pm \cup \{w_{\ell+1}^\mp\})$.

(II) If $w_{\ell+1}^\pm \in \widetilde{W}_*^\pm$, then $v_{\ell+1}^\pm = w_{\ell+1}^\pm$.

(III) If $\max\{|W_*^+|, |W_*^-|\} \geq 5$, then both $\widehat{\mathrm{ex}}_\ell^\pm(v_{\ell+1}^\pm) \geq 2$.

(IV) If $w_{\ell+1}^+ \in \widehat{U}_\ell^+(D)$, $w_{\ell+1}^- \in \widehat{U}_\ell^-(D)$, then for each $\diamond \in \{+,-\}$ such that $w_{\ell+1}^\diamond \in V'$, we have $v_{\ell+1}^\diamond \neq w_{\ell+1}^\diamond$.

(V) For each $\diamond \in \{+,-\}$, if $v \in V \setminus \{w_{\ell+1}^+, w_{\ell+1}^-\}$ satisfies $d_D^\diamond(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $v_{\ell+1}^\diamond = v$.

Indeed, in order to verify (δ), note that, if both $w_{\ell+1}^+, w_{\ell+1}^- \in W$, then ($\delta$) holds vacuously with $\ell+1$ playing the role of ℓ and, if there exists $\diamond \in \{+,-\}$ such that $w_{\ell+1}^\diamond \notin \widehat{U}_\ell^\diamond(D)$, then ($\delta$) follows from ($\alpha$). In the remaining cases, (δ) holds by (IV). In order to verify (η), first note that each $v \notin \{w_{\ell+1}^+, w_{\ell+1}^-\}$ satisfies (η) by (V). To check that the vertices $w_{\ell+1}^\pm$ satisfy (η), first note that, by Proposition 4.27(a.ii), if $d^\mp(w_{\ell+1}^\pm) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then $\mathrm{ex}_D^\mp(w_{\ell+1}^\pm) \geq 2$ and so $\widehat{\mathrm{ex}}_\ell^\pm(w_{\ell+1}^\pm) \leq \widetilde{\mathrm{ex}}_{D,U^*}^\pm(w_{\ell+1}^\pm) = 0$. Thus, ($\eta$) for the vertices $w_{\ell+1}^\pm$ follows from (I).

The following observations will enable us to ensure that (I)–(V) are satisfied simultaneously. Let $\diamond \in \{+,-\}$ and suppose $v \in V$ satisfies $d_D^\diamond(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$. By Proposition 4.27(a.i), $\widetilde{\mathrm{ex}}(D) \leq (1+22\eta)n$ and so $|\widetilde{W}_*^\pm| \leq 1$. In particular, $|W_*^\pm| \leq 1$ and so, $k \leq 22\eta n$. Thus, both $X^\pm = \emptyset$. By Proposition 4.27(a.ii), $v \in \widetilde{W}_*^\diamond$. Thus, $\widetilde{W}_*^\diamond = \{v\}$ and, by (‡), $\widehat{\mathrm{ex}}_\ell^\diamond(v) \geq 2$. By Proposition 4.27(a.ii), each $u \in V \setminus \{v\}$ satisfies $d_D^\diamond(u) < \widetilde{\mathrm{ex}}(D) - 22\eta n$ (otherwise $\mathrm{ex}_D^\diamond(u) \geq (1-86\eta)n$ and thus $|\widetilde{W}_*^\diamond| > 1$, a contradiction). Therefore, the following holds.

(‡‡) *Let $v^\pm \in V$ satisfy $d_D^\pm(v^\pm) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$. Then $\widehat{\mathrm{ex}}_\ell^\pm(v^\pm) \geq 2$ and, if $v^\pm \neq w_{\ell+1}^\mp$, then $v_{\ell+1}^\pm := v^\pm$ satisfies (the $\pm$ part of) (I)–(V). In particular, if $w_{\ell+1}^\pm \in \widetilde{W}_*^\pm$ and $v^\pm \neq w_{\ell+1}^\mp$, then $v^\pm = w_{\ell+1}^\pm$.*

To find $v_{\ell+1}^\pm$ when each $v^\pm \in V \setminus \{w_{\ell+1}^\mp\}$ satisfies $d_D^\pm(v^\pm) < \widetilde{\mathrm{ex}}(D) - 22\eta n$, we will use the following claim. For each $S \subseteq V$, denote $\widehat{\mathrm{ex}}_\ell^\pm(S) := \sum_{v \in S} \widehat{\mathrm{ex}}_\ell^\pm(v)$.

Claim 1. *The following hold.*

(A) *If $\max\{|W_*^+|, |W_*^-|\} \geq 5$, then there exists $v \in \widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^+, w_{\ell+1}^-\})$ satisfying $\widehat{\mathrm{ex}}_\ell^+(v) \geq 2$.*

(B) *Suppose $|W_*^+|, |W_*^-| \leq 4$. Then, $X^+ = \emptyset$ and $\widehat{U}_\ell^+(D) \setminus \{w_{\ell+1}^-\} \neq \emptyset$. Moreover, if $w_{\ell+1}^- \in \widehat{U}_\ell^-(D)$ and $w_{\ell+1}^+ \in V'$, then $\widehat{\mathrm{ex}}_\ell^+(V \setminus \{w_{\ell+1}^+, w_{\ell+1}^-\}) \geq 2$ (and thus, in particular, $\widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^+, w_{\ell+1}^-\}) \neq \emptyset$).*

Both statements also hold if + and − are swapped.

Proof of Claim. For (A), suppose that $\max\{|W_*^+|, |W_*^-|\} \geq 5$. Assume for a contradiction that each $v \in \widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^+, w_{\ell+1}^-\})$ satisfies $\widehat{\mathrm{ex}}_\ell^+(v) = 1$. Note that $\lfloor \sqrt{\eta} n \rfloor |X^+| \leq \ell < k \leq 11\eta n |W_*|$ and so $|X^+| \leq 23\sqrt{\eta} \max\{|W_*^+|, |W_*^-|\}$. Thus,

$$\begin{aligned}\widehat{\mathrm{ex}}_\ell^+(D) &\leq |X^+ \cup \{w_{\ell+1}^+, w_{\ell+1}^-\}| n + |\widehat{U}_\ell^+(D)| \leq \left(23\sqrt{\eta} \max\{|W_*^+|, |W_*^-|\} + 2\right) n + n \\ &\leq \max\{|W_*^+|, |W_*^-|\} 23\sqrt{\eta} n + 3n.\end{aligned}$$

But, by (8.3), $\widehat{\mathrm{ex}}_\ell^+(D) \geq \max\{|W_*^+|, |W_*^-|\}(1-46\eta)n$, so $\max\{|W_*^+|, |W_*^-|\} \leq \frac{3}{1-24\sqrt{\eta}} \leq 4$, a contradiction.

For (B), assume that $|W_*^+|, |W_*^-| \leq 4$. Then, $\ell \leq k \leq 88\eta n$ and so $X^+ = \emptyset$. If $w_{\ell+1}^- \in \widehat{U}_\ell^-(D) \subseteq \widetilde{U}_{U^*}^-(D)$ and $w_{\ell+1}^+ \in V'$, then $\widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^-) \leq \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) \leq 1$ and $\widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^+) \leq \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^+) \leq \max\{\mathrm{ex}_D^+(w_{\ell+1}^+), 1\} \leq \varepsilon n$. Hence, by (8.3), $\widehat{\mathrm{ex}}_\ell^+(V \setminus \{w_{\ell+1}^+, w_{\ell+1}^-\}) = \widehat{\mathrm{ex}}_\ell^+(D) - \widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^-) - \widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^+) \geq (1-46\eta)n - 1 - \varepsilon n \geq 2$, as desired. It only remains to show that $\widehat{U}_\ell^+(D) \setminus \{w_{\ell+1}^-\} \neq \emptyset$. By (8.3), $\widehat{\mathrm{ex}}_\ell^+(D) > 0$ and so $\widehat{U}_\ell^+(D) \neq \emptyset$. Suppose for a contradiction that $\widehat{U}_\ell^+(D) = \{w_{\ell+1}^-\}$. Note that, by (8.3), $\widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) \geq \widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^-) = \widehat{\mathrm{ex}}_\ell^+(D) \geq (1-46\eta)n$. Thus, $w_{\ell+1}^- \notin U^0(D)$ and so $\mathrm{ex}_D^+(w_{\ell+1}^-) \geq \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) \geq (1-46\eta)n$. Thus, $w_{\ell+1}^- \in \widetilde{W}_*^+$ and, by (b), $|W_A^+| \leq 1$. Moreover, each $v \in V \setminus \{w_{\ell+1}^-\}$ satisfies $\mathrm{ex}_D^+(v) \leq \widetilde{\mathrm{ex}}_{D,U^*}(v) + d_{A^+}^+(v) \leq (\widehat{\mathrm{ex}}_\ell^+(v) + \ell) + |A^+| \leq 0 + 88\eta n + \lceil \eta n \rceil < \mathrm{ex}_D^+(w_{\ell+1}^-) - \varepsilon n$. Thus, by (b), we have $W_A^+ \subseteq \{w_{\ell+1}^-\}$. Therefore, $d_{A^+}^+(w_{\ell+1}^-) = |A^+|$ and so

$$\widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) \overset{w_{\ell+1}^- \notin U^0(D)}{=} \mathrm{ex}_D^+(w_{\ell+1}^-) - d_{A^+}^+(w_{\ell+1}^-) = \mathrm{ex}_D^+(w_{\ell+1}^-) - |A^+|. \tag{8.4}$$

Then,

$$\widehat{\mathrm{ex}}_\ell^+(D) = \widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^-) \leq \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) \overset{(8.4)}{=} \mathrm{ex}_D^+(w_{\ell+1}^-) - |A^+| \leq n, \tag{8.5}$$

and so

$$\widetilde{\mathrm{ex}}(D) \overset{(8.2)}{=} \widehat{\mathrm{ex}}_\ell^+(D) + |A^+| + \ell \overset{(8.5)}{\leq} \mathrm{ex}_D^+(w_{\ell+1}^-) + \ell. \tag{8.6}$$

Suppose first that $d_D^+(w_{\ell+1}^-) < \widetilde{\mathrm{ex}}(D) - 22\eta n$. By (8.3) and (8.5), both $|W_*^\pm| \leq 1$. Thus, $\ell \leq 22\eta n$ and so, by (8.6), $d_D^+(w_{\ell+1}^-) < \widetilde{\mathrm{ex}}(D) - 22\eta n \leq \widetilde{\mathrm{ex}}(D) - \ell \leq \mathrm{ex}_D^+(w_{\ell+1}^-)$, a contradiction.

Therefore, $d_D^+(w_{\ell+1}^-) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$. Then, by ($\gamma$), ($\varepsilon$), ($\eta$), and since $w_{\ell+1}^- \in \widetilde{W}_*^+$, we have $\{i \in [\ell] \mid v_i^+ \neq w_{\ell+1}^-\} = \{i \in [\ell] \mid w_i^- = w_{\ell+1}^-\}$. Thus,

$$\begin{aligned}\widetilde{\mathrm{ex}}_{U^*}^+(D) &= \widehat{\mathrm{ex}}_\ell^+(D) + \ell = \widehat{\mathrm{ex}}_\ell^+(w_{\ell+1}^-) + \ell = \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) - |\{i \in [\ell] \mid v_i^+ = w_{\ell+1}^-\}| + \ell\\ &= \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) + |\{i \in [\ell] \mid w_i^- = w_{\ell+1}^-\}| < \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) + d_H^-(w_{\ell+1}^-)\\ &\leq \widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) + d_D^-(w_{\ell+1}^-). \end{aligned} \tag{8.7}$$

Therefore, we have

$$\begin{aligned}\widetilde{\mathrm{ex}}(D) &\overset{(4.3)}{=} \widetilde{\mathrm{ex}}_{U^*}^+(D) + |A^+| \overset{(8.7)}{<} (\widetilde{\mathrm{ex}}_{D,U^*}^+(w_{\ell+1}^-) + d_D^-(w_{\ell+1}^-)) + |A^+|\\ &\overset{(8.4)}{=} \mathrm{ex}_D^+(w_{\ell+1}^-) + d_D^-(w_{\ell+1}^-) = d_D^+(w_{\ell+1}^-) \leq \Delta^0(D),\end{aligned}$$

a contradiction.

The same arguments hold with $+$ and $-$ swapped. This concludes the proof of Claim 1. $\Diamond$

We are now ready to choose distinct $v_{\ell+1}^\pm \in V$ such that (I)–(V) are satisfied. Without loss of generality, suppose that $|\widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^-\})| \leq |\widehat{U}_\ell^-(D) \setminus (X^- \cup \{w_{\ell+1}^+\})|$. We start by picking $v_{\ell+1}^+$ as follows. If $w_{\ell+1}^+ \in \widetilde{W}_*^+$, then let $v_{\ell+1}^+ := w_{\ell+1}^+$ and if there exists $v \in V \setminus \{w_{\ell+1}^-\}$ such that $d_D^+(v) \geq \widetilde{\mathrm{ex}}(D) - 22\eta n$, then let $v_{\ell+1}^+ := v$. (Note that $v_{\ell+1}^+$ is well defined by the "in particular" part of (‡‡).) Then, by (‡) and (‡‡), properties (I)–(V) hold for $v_{\ell+1}^+$. Otherwise, by using Claim 1 and distinguishing the cases when $\max\{|W_*^+|, |W_*^-|\} \geq 5$ and when $|W_*^+|, |W_*^-| \leq 4$, it is easy to check that we can choose $v_{\ell+1}^+ \in \widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^-\})$ satisfying (I)–(V).

Note that, since $v_{\ell+1}^+$ satisfies (I), $\widehat{\mathrm{ex}}_\ell^-(v_{\ell+1}^+) \leq 1$. Therefore, by (‡), $v_{\ell+1}^+ \notin \widetilde{W}_*^-$ and, by (‡‡), $d_D^-(v_{\ell+1}^+) < \widetilde{\mathrm{ex}}(D) - 22\eta n$ (otherwise $\widehat{\mathrm{ex}}_\ell^-(v_{\ell+1}^+) \geq 2$, a contradiction). Thus, if $\widehat{U}_\ell^-(D) \setminus (X^- \cup \{w_{\ell+1}^+, v_{\ell+1}^+\}) \neq \emptyset$, then, we can proceed similarly as for $v_{\ell+1}^+$ to obtain $v_{\ell+1}^-$ satisfying (I)–(V). (To see that this can be done in the case when $|W_*^+|, |W_*^-| \leq 4$, note that, by Claim 1(B), if $|W_*^+|, |W_*^-| \leq 4$, $w_{\ell+1}^+ \in \widehat{U}_\ell^+(D)$, and $w_{\ell+1}^- \in V'$, then $\widehat{\mathrm{ex}}_\ell^-(V \setminus \{w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^+\}) \geq 2 - \widehat{\mathrm{ex}}_\ell^-(v_{\ell+1}^+) \geq 1$ and so $\widehat{U}_\ell^-(D) \setminus \{w_{\ell+1}^+, w_{\ell+1}^-, v_{\ell+1}^+\} \neq \emptyset$.) We may therefore assume that $\widehat{U}_\ell^-(D) \setminus (X^- \cup \{w_{\ell+1}^+, v_{\ell+1}^+\}) = \emptyset$. But, by Claim 1, $\widehat{U}_\ell^-(D) \setminus (X^- \cup \{w_{\ell+1}^+\}) \neq \emptyset$ and so $\widehat{U}_\ell^-(D) \setminus (X^- \cup \{w_{\ell+1}^+\}) = \{v_{\ell+1}^+\}$. Thus, by assumption, $|\widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^-\})| \leq |\widehat{U}_\ell^-(D) \setminus (X^- \cup \{w_{\ell+1}^+\})| = |\{v_{\ell+1}^+\}| = 1$. Then, since $v_{\ell+1}^+$ satisfies (I), $\widehat{U}_\ell^+(D) \setminus (X^+ \cup \{w_{\ell+1}^-\}) = \{v_{\ell+1}^+\}$. We will find a contradiction.

Note that, $v_{\ell+1}^+ \in \widehat{U}_\ell^+(D) \cap \widehat{U}_\ell^-(D) \subseteq U^0(D)$ and so $\widehat{\mathrm{ex}}_\ell^\pm(v_{\ell+1}^+) = 1$. Then, since $v_{\ell+1}^+$ satisfies (III), we have $|W_*^+|, |W_*^-| \leq 4$. Thus, by Claim 1(B) (and its analogue with $+$ and $-$ swapped), $X^\pm = \emptyset$. Therefore, we have $\widehat{U}_\ell^\pm(D) \setminus \{w_{\ell+1}^\mp\} = \{v_{\ell+1}^+\}$. Moreover, by (8.3), $\widehat{\mathrm{ex}}_\ell^\pm(D) \geq (1-46\eta)n$ and so, as $\widehat{\mathrm{ex}}_\ell^\pm(v_{\ell+1}^+) = 1$, we must have $\widehat{U}_\ell^\pm(D) = \{w_{\ell+1}^\mp, v_{\ell+1}^+\}$. Then, $\widetilde{\mathrm{ex}}_{D,U^*}^\pm(w_{\ell+1}^\mp) \geq \widehat{\mathrm{ex}}_\ell^\pm(w_{\ell+1}^\mp) = \widehat{\mathrm{ex}}_\ell^\pm(D) - 1 \geq (1-47\eta)n$. Thus, $w_{\ell+1}^\mp \notin U^*$ and so $\mathrm{ex}_D^\pm(w_{\ell+1}^\mp) \geq \widetilde{\mathrm{ex}}_{D,U^*}^\pm(w_{\ell+1}^\mp) \geq (1-47\eta)n$. Therefore, $w_{\ell+1}^\mp \in \widetilde{W}_*^\pm$. But, by (‡), $\widetilde{W}_*^\pm \subseteq \widehat{U}_\ell^\pm(D)$ and so $\widetilde{W}_*^\pm = \{w_{\ell+1}^\mp\}$. Thus, by assumption on our ordering of $E(H)$, it follows that $\ell = 0$. Therefore, $\widetilde{U}_{U^*}^\pm(D) = \widehat{U}_\ell^\pm(D) = \{w_{\ell+1}^\mp, v_{\ell+1}^+\}$, contradicting (††). $\square$

Note that, if $W_* = \emptyset$, then the excess of our tournament may be relatively small and so we do not have room to proceed similarly as in Lemma 8.2 to decrease the degree of the vertices in W_A. The strategy is to find a partial path decomposition $\mathcal{P}$ such that each vertex in W_A is covered by each of the paths in $\mathcal{P}$ and such that each vertex in V' is covered by half of the paths in $\mathcal{P}$. In this way, the degree at W_A is decreased faster than the degree at V'. Decreasing the degree at V' will ensure that the leftover excess is not too small compared to degree of the leftover oriented graph (recall Lemma 7.1(v)).

Lemma 8.4. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let D be an oriented graph on a vertex set V of size n satisfying $\delta(D) \geq (1-\varepsilon)n$, $\widetilde{\mathrm{ex}}(D) \geq \frac{n}{2} + 9\eta n$, and the following properties.*

(a) *Let $W \cup V'$ be a partition of V such that, for each $v \in V'$, $|\mathrm{ex}_D(v)| \leq \varepsilon n$ and, for each $v \in W$, $|\mathrm{ex}_D(v)| \leq (1-20\eta)n$. Suppose $E(D[W]) = \emptyset$ and $|W| \leq \varepsilon n$.*

(b) *Let $A^+, A^- \subseteq E(T)$ be absorbing sets of (W,V')-starting/(V',W)-ending edges for D of size at most $\lceil \eta n \rceil$. Let $A := A^+ \cup A^-$, $W_A^{\pm} := V(A^{\pm}) \cap W$, and $W_A := V(A) \cap W$. Assume $A \neq \emptyset$, i.e. $W_A \neq \emptyset$.*

(c) *Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.*

Then, there exists a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ of D such that $|\mathcal{P}| = 8\lceil \eta n \rceil$ and $D' := D \setminus \mathcal{P}$ satisfies the following.

(i) *For each $v \in W$, $d_{D'}(v) \leq d_D(v) - 12\lceil \eta n \rceil$.*

(ii) *For each $v \in V'$, $d_D(v) - 8\lceil \eta n \rceil \leq d_{D'}(v) \leq d_D(v) - 8\lceil \eta n \rceil + 1$.*

(iii) *Each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ satisfies $d^+_{D'}(v) = d^-_{D'}(v) \leq \widetilde{\mathrm{ex}}(D') - 1$.*

Proof. Fix additional constants such that $\varepsilon \ll \nu \ll \tau \ll \eta$. Let $k := 4\lceil \eta n \rceil$. Assume inductively that, for some $0 \leq \ell \leq k$, we have constructed edge-disjoint paths $P_{1,1}, P_{1,2}, P_{2,1}, \dots, P_{\ell,2} \subseteq D$ such that $\mathcal{P}_\ell := \{P_{i,j} \mid i \in [\ell], j \in [2]\}$ is a (U^*, W, A)-partial path decomposition of D such that the following hold, where $D_\ell := D \setminus \mathcal{P}_\ell$.

(α) For each $i \in [\ell]$ and $v \in W$, $v \in V(P_{i,1}) \cap V(P_{i,2})$.

(β) For each $i \in [\ell]$ and $v \in W$, v is the endpoint of at most one of $P_{i,1}$ and $P_{i,2}$.

(γ) For each $i \in [\ell]$ and $v \in V'$, either $v \in V(P_{i,1}) \triangle V(P_{i,2})$ or v is an endpoint of both $P_{i,1}$ and $P_{i,2}$.

(δ) For each $v \in V'$, there is at most one $i \in [\ell]$ such that v is an endpoint of exactly one of $P_{i,1}$ and $P_{i,2}$. Moreover, for each $v \in V'$, if there exists $i \in [\ell]$ such that v is an endpoint of exactly one of $P_{i,1}$ and $P_{i,2}$, then $\mathrm{ex}_{D_\ell}(v) = 0$.

If $\ell = k$, then let $\mathcal{P} := \mathcal{P}_k$ and $D' := D \setminus \mathcal{P}$.

Claim 1. *$\mathcal{P}$ is a good partial path decomposition of D.*

Note that if Claim 1 holds, then we are done. Indeed, (i) holds by (α) and (β) while (ii) holds by (γ) and (δ). Finally, (iii) follows from Claim 1. Indeed, for each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$, we have $d^+_{D'}(v) = d^-_{D'}(v) \leq \frac{n-1}{2} < \frac{n}{2} + 9\eta n - 2k \leq \widetilde{\mathrm{ex}}(D) - 2k = \widetilde{\mathrm{ex}}(D')$, as desired. Thus, it suffices to prove Claim 1.

Proof of Claim 1. Observe that, as $\mathcal{P}$ is a partial path decomposition of D, by Proposition 4.25, $\widetilde{\mathrm{ex}}(D') \geq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$ and $\mathrm{ex}(D') \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. Thus, it is enough to show that $\Delta^0(D') \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$.

Let $v \in V$. By (α), (γ), and since $\mathcal{P}$ is a partial path decomposition of D, the following hold.

- If $v \in U^{\pm}(D) \cap W$, then $v \in V^{\pm}(P) \cup V^0(P)$ for each $P \in \mathcal{P}$.
- If $v \in U^0(D) \cap W$, then for each $\diamond \in \{+,-\}$, $v \in V^{\diamond}(P) \cup V^0(P)$ for all but at most one $P \in \mathcal{P}$.
- If $v \in U^{\pm}(D) \cap V'$, then $v \in V^{\pm}(P) \cup V^0(P)$ for at least k paths $P \in \mathcal{P}$.
- If $v \in U^0(D) \cap V'$, then, for each $\diamond \in \{+,-\}$, $v \in V^{\diamond}(P) \cup V^0(P)$ for at least $k-1$ paths $P \in \mathcal{P}$.

Thus, since $\mathcal{P}$ is a partial path decomposition of D, we have

$$d^{\max}_{D'}(v) \leq \begin{cases} d^{\max}_D(v) - |\mathcal{P}| & \text{if } v \in W \setminus U^0(D), \\ d^{\max}_D(v) - |\mathcal{P}| + k + 1 & \text{if } v \in V' \cup U^0(D). \end{cases} \tag{8.8}$$

By Fact 4.21(c), for each $v \in V' \cup U^0(D)$, we have $d^{\max}_D(v) = \frac{d_D(v) + |\mathrm{ex}_D(v)|}{2} \leq \frac{n-1+\varepsilon n}{2} \leq \frac{n}{2} + 9\eta n - 4\lceil \eta n \rceil - 1 \leq \widetilde{\mathrm{ex}}(D) - k - 1$ and so, by (8.8), $\Delta^0(D') \leq \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. Thus, $\mathcal{P}$ is a good partial path decomposition of D, as desired. $\diamond$

If $\ell < k$, then let $D_\ell := D \setminus \mathcal{P}_\ell$ and $U^*_\ell := U^* \setminus (V^+(\mathcal{P}_\ell) \cup V^-(\mathcal{P}_\ell))$. We claim that there exist suitable endpoints $v_1^+, v_1^-, v_2^+, v_2^- \in V$ for $P_{\ell+1,1}$ and $P_{\ell+1,2}$.

Claim 2. *There exist $v_1^+, v_1^-, v_2^+, v_2^- \in V$ such that the following hold.*

(I) *For each $i \in [2]$, $v_i^+ \neq v_i^-$ and $v_i^\pm \in \widetilde{U}^\pm_{U^*_\ell}(D_\ell)$. Moreover, for each $\diamond \in \{+,-\}$, if $v_1^\diamond = v_2^\diamond$, then $\widetilde{\mathrm{ex}}^\diamond_{D_\ell, U^*_\ell}(v_1^\diamond) \geq 2$.*

(II) *For each $v \in W$, there exists at most one pair $(i, \diamond) \in [2] \times \{+,-\}$ such that $v_i^\diamond = v$.*

(III) *For each $v \in V'$, if there exists exactly one pair $(i, \diamond) \in [2] \times \{+,-\}$ such that $v_i^\diamond = v$, then $\mathrm{ex}^\diamond_{D_\ell}(v) = 1$.*

Before proving Claim 2, let us first apply it to construct $P_{\ell+1,1}$ and $P_{\ell+1,2}$. Let $v_1^+, v_1^-, v_2^+, v_2^- \in V$ be as in Claim 2. We construct a (v_1^+, v_1^-)-path $P_{\ell+1,1}$ and a (v_2^+, v_2^-)-path $P_{\ell+1,2}$ using Corollary 4.8 as follows. Observe that, by Lemma 4.4, $D_\ell[V' \setminus \{v_1^+, v_1^-, v_2^+, v_2^-\}]$ is a robust (ν, τ)-outexpander. Apply Corollary 4.8(a), with $D_\ell \setminus A, V' \setminus \{v_1^+, v_1^-\}, |W \cup \{v_1^+, v_1^-\}|, \frac{3}{8}, \{v_2^+, v_2^-\} \setminus (W \cup \{v_1^+, v_1^-\}), v_1^+, W \setminus \{v_1^+, v_1^-\}$, and v_1^- playing the roles of $D, V', k, \delta, S, P_1, \{P_2, \dots, P_{k-1}\}$, and P_k to obtain a (v_1^+, v_1^-)-path $P_{\ell+1,1}$ of length at most $\sqrt{\varepsilon} n$ which covers W and avoids $\{v_2^+, v_2^-\} \setminus (W \cup \{v_1^+, v_1^-\})$. Let $D'_\ell := D_\ell \setminus P_{\ell+1,1}$ and observe that, by Lemma 4.4, $D'_\ell[V' \setminus (V(P_{\ell+1,1}) \cup \{v_2^+, v_2^-\})]$ is still a robust (ν, τ)-outexpander. Then, apply Corollary 4.8(b) with $D'_\ell, V' \setminus \{v_2^+, v_2^-\}, |W \cup \{v_2^+, v_2^-\}|, \frac{3}{8}, V(P_{\ell+1,1}) \setminus (W \cup \{v_2^+, v_2^-\}), v_2^+, W \setminus \{v_2^+, v_2^-\}$, and v_2^- playing the roles of D, V', k, δ, S, P_1, $\{P_2, \dots, P_{k-1}\}$, and P_k to obtain a (v_2^+, v_2^-)-path $P_{\ell+1,2}$ satisfying $V \setminus V(P_{\ell+1,2}) = V(P_{\ell+1,1}) \setminus (W \cup \{v_2^+, v_2^-\})$. Then, note that, by (I), $\mathcal{P}_{\ell+1}$ is a (U^*, W, A)-partial path decomposition of D and, by (II) and (III), (β) and (δ) are satisfied with $\ell + 1$ playing the role of ℓ, respectively. Finally, by construction of $P_{\ell+1,1}$ and $P_{\ell+1,2}$, (α) and (γ) are satisfied.

It remains to prove Claim 2.

Proof of Claim 2. Since $\mathcal{P}_\ell$ is a (U^*, W, A)-partial path decomposition of D, $|A^\pm| \leq \lceil \eta n \rceil$, and $2\ell \leq 2k \leq 8\lceil \eta n \rceil$,

$$\widetilde{\mathrm{ex}}^\pm_{U^*_\ell}(D_\ell) \overset{\text{Fact 4.32}}{=} \widetilde{\mathrm{ex}}^\pm_{U^*}(D) - 2\ell$$

$$\overset{(4.3)}{=} \widetilde{\mathrm{ex}}(D) - |A^\pm| - 2\ell \tag{8.9}$$

$$\geq \frac{n}{2} - \eta n. \tag{8.10}$$

Thus, we can choose endpoints $v_1^+, v_1^-, v_2^+, v_2^- \in V$ satisfying (I)–(III) as follows.

If $|U^*_\ell \cap V'| \geq 2$, then pick distinct $u_1, u_2 \in U^*_\ell \cap V'$ and let $v_1^+ := u_1, v_2^+ := u_2, v_1^- := u_2$, and $v_2^- := u_1$. Then, (I)–(III) are satisfied, as desired.

We may therefore assume that $|U^*_\ell \cap V'| \leq 1$. We first pick $v_1^+, v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \setminus (U^*_\ell \cap V')$ as follows. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) \geq 2$, then pick $v_1^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap (V' \setminus U^*_\ell)$ such that $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v_1^+)$ is maximum. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v_1^+) \geq 2$, then let $v_2^+ := v_1^+$; otherwise, let $v_2^+ \in (\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap (V' \setminus U^*_\ell)) \setminus \{v_1^+\}$. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) = 1$, then, note that by Fact 4.31 and (8.10), $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(W) \geq \widetilde{\mathrm{ex}}^+_{U^*_\ell}(D_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(U^*_\ell \cap V') \geq \frac{n}{2} - \eta n - 1 - 1 \geq 1$. Thus, we can let $v_1^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap (V' \setminus U^*_\ell)$ and $v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W$. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) = 0$, then it is enough to show that $|\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W| \geq 2$ (so that we can take distinct $v_1^+, v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W$, as desired for (II)).

Note that, by Fact 4.31 and (8.10), $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(W) \geq \widetilde{\mathrm{ex}}^+_{U^*_\ell}(D_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(U^*_\ell \cap V') \geq \frac{n}{2} - \eta n - 0 - 1 \geq 2$ and so, in particular, $\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W \neq \emptyset$. Assume for a contradiction that $\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W = \{v\}$ for some $v \in W$. Note that since $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v) = \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(W) \geq 2$, $v \notin U^0(D_\ell)$ and so, $U^*_\ell \subseteq V'$ and $\mathrm{ex}^+_D(v) \geq \mathrm{ex}^+_{D_\ell}(v) \geq \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v)$. Thus, since $|U^*_\ell| = |U^*_\ell \cap V'| \leq 1$, $|A^+| \leq \lceil \eta n \rceil$, and $2\ell \leq 2k \leq 8\lceil \eta n \rceil$,

$$d^-_D(v) + \mathrm{ex}^+_D(v) \overset{\text{Fact 4.21(d)}}{\leq} \widetilde{\mathrm{ex}}(D) \overset{(8.9)}{=} \widetilde{\mathrm{ex}}^+_{U^*_\ell}(D_\ell) + 2\ell + |A^+|$$

$$\overset{\text{Fact 4.31}}{=} \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v) + |U^*_\ell| + 2\ell + |A^+| \leq \mathrm{ex}^+_D(v) + 1 + 9\lceil \eta n \rceil.$$

But, by (a) and Fact 4.21(b), $d_D^-(v) = \frac{d_D(v) - \mathrm{ex}_D^+(v)}{2} \geq \frac{(20\eta - \varepsilon)n}{2} > 9\lceil \eta n \rceil + 1$, a contradiction. Thus, $|\widetilde{U}_{U_\ell^*}^+(D_\ell) \cap W| \geq 2$ and we can let $v_1^+, v_2^+ \in \widetilde{U}_{U_\ell^*}^+(D_\ell) \cap W$ be distinct.

Now proceed analogously to pick $v_1^-, v_2^- \in \widetilde{U}_{U_\ell^*}^-(D_\ell) \setminus ((U_\ell^* \cap V') \cup \{v_1^+, v_2^+\})$ (this is possible since, for each $i \in [2]$, $\widetilde{\mathrm{ex}}_{D,U_\ell^*}^-(v_i^+) \leq 1$). One can easily verify that (I)–(III) are satisfied. $\diamond$

This completes the proof. $\square$

Proof of Lemma 7.1. Successively apply Lemmas 8.1, 8.2, and 8.4 as follows.

Step 1: Covering $T[W_0]$. First, apply Lemma 8.1 to obtain a good (U^*, W, A)-partial path decomposition $\mathcal{P}_1$ of T such that the following hold. Denote $D_1 := T \setminus \mathcal{P}_1$ and $U_1^* := U^* \setminus (V^+(\mathcal{P}_1) \cup V^-(\mathcal{P}_1))$.

(α) $\widetilde{\mathrm{ex}}(D_1) = \widetilde{\mathrm{ex}}(T) - |\mathcal{P}_1|$.
(β) $|\mathcal{P}_1| \leq 2\varepsilon n$.
(γ) $E(D_1[W_0]) = \emptyset$.
(δ) If $|U^+(D_1)| = |U^-(D_1)| = 1$, then $e(U^-(D_1), U^+(D_1)) = 0$ or $\widetilde{\mathrm{ex}}(D_1) - \mathrm{ex}(D_1) \geq 2$.
(ε) Each $v \in U_1^*$ satisfies $d_{D_1}^+(v) = d_{D_1}^-(v) \leq \widetilde{\mathrm{ex}}(D_1) - 1$.

In particular, observe that, by Fact 4.20, Lemma 7.1(b), and (β), the following hold.

(ζ) $\widetilde{\mathrm{ex}}(D_1) \geq \frac{n}{2} - 2\varepsilon n$ and, if $W_A \neq \emptyset$, then $\widetilde{\mathrm{ex}}(D_1) \geq \frac{n}{2} + 9\eta n$.
(η) For each $v \in V$, $d_{D_1}(v) \geq (1 - 5\varepsilon)n$.
(θ) For each $v \in W_*$, $|\mathrm{ex}_{D_1}(v)| > (1 - 21\eta)n$.
(ι) For each $\diamond \in \{+, -\}$, if $|W_A^\diamond| \geq 2$, then $\mathrm{ex}_{D_1}^\diamond(v) < \eta n$ for each $v \in V$ and, if $|W_A^\diamond| = 1$, then, for each $v \in V$ and $w \in W_A^\diamond$, $\mathrm{ex}_{D_1}^\diamond(v) \leq \mathrm{ex}_{D_1}^\diamond(w) + 5\varepsilon n$.

Step 2: Covering the remaining edges of $T[W]$ and decreasing the degree of the vertices in $W_* \cup W_A$ when $W_* \neq \emptyset$. If $W_* = \emptyset$, then let $\mathcal{P}_2 := \emptyset$. Otherwise, note that by Proposition 4.33, $|U_1^*| = \widetilde{\mathrm{ex}}(D_1) - \mathrm{ex}(D_1)$ and let $\mathcal{P}_2$ be the good (U_1^*, W, A)-partial path decomposition of D_1 obtained by applying Lemma 8.2 with D_1, U_1^*, and 5ε playing the roles of D, U^*, and ε. Denote $D_2 := D_1 \setminus \mathcal{P}_2$ and $U_2^* := U_1^* \setminus (V^+(\mathcal{P}_2) \cup V^-(\mathcal{P}_2))$. Then, note that, if $W_* \neq 0$, then the following hold.

(I) $\widetilde{\mathrm{ex}}(D_2) = \widetilde{\mathrm{ex}}(D_1) - |\mathcal{P}_2|$.
(II) $E(D_2[W]) = \emptyset$.
(III) $N^\pm(D_1) - N^\pm(D_2) \leq 88\eta n$.
(IV) For each $v \in W_* \cup W_A$, $(1 - 3\sqrt{\eta})n \leq d_{D_2}(v) \leq (1 - 4\eta)n$.
(V) For each $v \in W_0$, $d_{D_2}(v) \geq (1 - 3\sqrt{\eta})n$ and $d_{D_2}^{\min}(v) \geq 5\eta n$.
(VI) For each $v \in V'$, $d_{D_2}(v) \geq (1 - 8\sqrt{\varepsilon})n$.
(VII) If $|W_*^+|, |W_*^-| \leq 1$, then each $v \in W_*$ satisfies $|\mathrm{ex}_{D_2}(v)| = d_{D_2}(v)$.
(VIII) Each $v \in U_2^*$ satisfies $d_{D_2}^+(v) = d_{D_2}^-(v) \leq \widetilde{\mathrm{ex}}(D_2) - 1$.

Note that, by (IV), the following holds.

(IX) each $v \in W_*$ satisfies $|\mathrm{ex}_{D_2}(v)| \geq |\mathrm{ex}_T(v)| - 3\sqrt{\eta}n \geq (1 - 4\sqrt{\eta})n$.

Thus, (VII) implies the following.

(X) If $\widetilde{\mathrm{ex}}(D_2) \leq 2\lceil \frac{n}{2} \rceil - \lceil \eta n \rceil$, then $|\mathrm{ex}_{D_2}(v)| = d_{D_2}(v)$ for each $v \in W_*$.

Step 3: Decreasing the degree of the vertices in W_A when $W_* = \emptyset$. If $W_* \neq \emptyset$ or $W_A = \emptyset$, then let $\mathcal{P}_3 := \emptyset$. Assume $W_* = \emptyset$ and $W_A \neq \emptyset$. Recall that, by construction, $D_2 = D_1$ and $U_2^* = U_1^*$. In particular, (γ), (ζ), and (η) are satisfied and $|U_2^*| = \widetilde{\mathrm{ex}}(D_2) - \mathrm{ex}(D_2)$. Let $\mathcal{P}_3$ be the good (U_2^*, W, A)-partial path decomposition of D_2 obtained by applying Lemma 8.4 with

D_2, U_2^*, and 5ε playing the roles of D, U^*, and ε. Denote $D_3 := D_2 \setminus \mathcal{P}_3$ and note that, if $W_* = \emptyset$ and $W_A \neq \emptyset$, then the following hold.

(A) $\widetilde{\mathrm{ex}}(D_3) = \widetilde{\mathrm{ex}}(D_2) - |\mathcal{P}_3|$.
(B) Each $v \in U_2^* \setminus (V^+(\mathcal{P}_3) \cup V^-(\mathcal{P}_3))$ satisfies $d_{D_3}^+(v) = d_{D_3}^-(v) \leq \widetilde{\mathrm{ex}}(D_3) - 1$.
(C) $|\mathcal{P}_3| = 8\lceil \eta n \rceil$.
(D) For each $v \in W$, $d_{D_3}(v) \leq d_{D_2}(v) - 12\lceil \eta n \rceil$.
(E) For each $v \in V'$, $d_{D_2}(v) - 8\lceil \eta n \rceil \leq d_{D_3}(v) \leq d_{D_2}(v) - 8\lceil \eta n \rceil + 1$.

Step 4: Checking the assertions of Lemma 7.1. Let $\mathcal{P} := \bigcup_{i \in [3]} \mathcal{P}_i$ and $D := T \setminus \mathcal{P} = D_3$. If $W_* \neq \emptyset$ or $W_A = \emptyset$, then let $d := \min\{\lceil \frac{n}{2} \rceil - \lceil \eta n \rceil, \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil\}$. If $W_* = \emptyset$ and $W_A \neq \emptyset$, then let $d := \min\{\lceil \frac{n}{2} \rceil - 5\lceil \eta n \rceil, \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil\}$. In both cases, $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of T by Fact 4.34. Moreover, one can easily verify that (i)–(vi) are satisfied and, if $d \neq \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$, (vii)–(ix) hold too. Assume that $d = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$. By construction, $\widetilde{\mathrm{ex}}(D) \leq \lceil \frac{n}{2} \rceil$ and so, by (IX), $W_* = \emptyset$. Moreover, $2d + 2\lceil \eta n \rceil = 2\,\widetilde{\mathrm{ex}}(D) \geq d_D(v)$ for each $v \in V$ and so the upper bounds of (viii) and (ix) hold. If $W_A = \emptyset$, then (vii) holds vacuously and, by (η), each $v \in V$ satisfies $d_D^{\min}(v) \geq d_T^{\min}(v) - 5\varepsilon n \geq \lceil \eta n \rceil + 1$ and $d_D(v) \geq (1 - 5\varepsilon)n \geq 2\left(\lceil \frac{n}{2} \rceil - \lceil \eta n \rceil\right) + 2\lceil \eta n \rceil - 6\varepsilon n \geq 2d + 2\lceil \eta n \rceil - 6\varepsilon n$, so the lower bounds in (viii) and (ix) hold. Suppose $W_A \neq \emptyset$. Then, for each $w \in W$ and $v \in V'$,

$$\begin{aligned} d_D(w) &\overset{\text{(D)}}{\leq} d_{D_2}(w) - 12\lceil \eta n \rceil \leq (n-1) - 12\lceil \eta n \rceil \overset{(\beta)}{\leq} d_{D_2}(v) + 5\varepsilon n - 12\lceil \eta n \rceil \\ &\overset{\text{(E)}}{\leq} d_D(v) - 3\lceil \eta n \rceil \leq 2\Delta^0(D) - 3\lceil \eta n \rceil \leq 2\,\widetilde{\mathrm{ex}}(D) - 3\lceil \eta n \rceil = 2d - \lceil \eta n \rceil, \end{aligned}$$

so the upper bound of (vii) holds. Moreover, by (β) and (C), for each $v \in W$, $d_D(v) \geq n - 1 - 2|\mathcal{P}| \geq n - 1 - 2(2\varepsilon n + 8\lceil \eta n \rceil) \geq 2\left(\lceil \frac{n}{2} \rceil - 5\lceil \eta n \rceil\right) - 8\lceil \eta n \rceil \geq 2d + 2\lceil \eta n \rceil - 4\sqrt{\eta} n$ and, by Fact 4.21(b) and (a), $d_D^{\min}(v) \geq d_T^{\min}(v) - |\mathcal{P}| \geq \frac{10\eta n - 1}{2} - (2\varepsilon n + 8\lceil \eta n \rceil) \geq \lceil \eta n \rceil$. Thus, the lower bounds in (vii) and (viii) are satisfied. Finally, by (β) and (E), for each $v \in V'$, $d_D(v) \geq n - 1 - 4\varepsilon n - 8\lceil \eta n \rceil \geq 2\left(\lceil \frac{n}{2} \rceil - 5\lceil \eta n \rceil\right) + 2\lceil \eta n \rceil - 5\varepsilon n \geq 2d + 2\lceil \eta n \rceil - 9\sqrt{\varepsilon} n$, as desired for the lower bound in (ix). Thus, (vii)–(ix) hold in all cases and this completes the proof. □

9. Concluding remarks

9.1. **Approximate Hamilton decompositions of robust outexpanders.** In [17], Osthus and Staden showed that any regular robust outexpander of linear semidegree can be approximately decomposed into Hamilton cycles. This was used as a tool in [10] to prove that such graphs actually have a Hamilton decomposition.

Theorem 9.1 ([17])**.** *Let $0 < \frac{1}{n} \ll \tau \ll \alpha \leq 1$ and $0 \leq \frac{1}{n} \ll \varepsilon \ll \nu, \eta \leq 1$. If D is an (α, ε)-almost regular robust (ν, τ)-outexpander on n vertices, then D contains at least $(\alpha - \eta)n$ edge-disjoint Hamilton cycles.*

Lemma 6.3 can also be used to construct approximate Hamilton decompositions of (almost) regular robust outexpanders. In fact, our tools also enable us to assign some specific edges to each element of our approximate decomposition and so can be used to find approximate decompositions with prescribed edges.

Theorem 9.2. *Let $0 < \frac{1}{n} \ll \tau \ll \alpha \leq 1$ and $0 < \frac{1}{n} \ll \varepsilon \ll \eta, \nu \leq 1$. Let $\ell \leq (\alpha - \eta)n$. Suppose D is an (α, ε)-almost regular (ε, n^{-2})-robust (ν, τ)-outexpander on n vertices. Suppose that, for each $i \in [\ell]$, F_i is a linear forest on $V(D)$ satisfying $e(F_i) \leq \varepsilon n$ and such that, for each $v \in V(D)$, there exist at most εn indices $i \in [\ell]$ such that $v \in V(F_i)$. Define a multiset $\mathcal{F}$ by $\mathcal{F} := \bigcup_{i \in [\ell]} F_i$. Then, there exist edge-disjoint Hamilton cycles $C_1, \dots, C_\ell \subseteq D \cup \mathcal{F}$ such that, for each $i \in [\ell]$, $F_i \subseteq C_i$.*

Proof. By Lemma 4.3, we may assume without loss of generality that

$$0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \eta \ll \alpha \leq 1.$$

Define an additional constant γ such that $\tau \ll \gamma \ll \eta$. For each $i \in [\ell]$, let $v_{i,1}, v_{i,2} \in V(D) \setminus V(F_i)$ be distinct and such that, for any $v \in V$, there exists at most two $(i,j) \in [\ell] \times [2]$ such that $v = v_{i,j}$. For each $i \in [\ell]$, denote by $P_{i,1}, \dots, P_{i,f_i}$ the (non-trivial) components of F_i and, for $j \in [f_i]$, denote by $u_{i,j}$ and $w_{i,j}$ the starting and ending points of $P_{i,j}$. For each $i \in [\ell]$, let $L_i := \{v_{i,1}u_{i,1}P_{i,1}w_{i,1}u_{i,2}P_{i,2}w_{i,2}u_{i,3} \dots w_{i,f_i}v_{i,2}, v_{i,2}v_{i,1}\}$. Denote $L := \bigcup_{i\in[\ell]} L_i$. Note that $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are layouts such that, for each $i \in [\ell]$, $V(L_i) \subseteq V$, $|V(L_i)| \leq 3\varepsilon n$ and $|E(L_i)| \leq 3\varepsilon n$. Moreover, for each $v \in V(D)$, $d_L(v) \leq 3\varepsilon n$ and there exist at most $2\varepsilon n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.

By similar arguments as in Lemma 4.16, there exists a spanning subdigraph $\Gamma \subseteq D$ such that Γ is a (γ, ε)-almost regular (ε, n^{-2})-robust $(\frac{\nu\gamma}{2}, \tau)$-outexpander and $D' := D \setminus \Gamma$ is $(\alpha - \gamma, \varepsilon)$-almost regular.

Apply Lemma 6.3 with $D', \alpha - \gamma, \frac{\nu\gamma}{2}, \varepsilon^{\frac{1}{5}}$, and $\frac{\eta}{2}$ playing the roles of $D, \delta, \nu, \varepsilon$, and η to obtain edge-disjoint $C_1, \cdots, C_\ell \subseteq D \cup \mathcal{F}$ such that, for each $i \in [\ell]$, C_i is a spanning configuration of shape (L_i, F_i). Then, by construction, for each $i \in [\ell]$, C_i is a Hamilton cycle of $D \cup \mathcal{F}$ such that $F_i \subseteq E(C_i)$. □

Recall that, by Lemma 6.3(iii), the leftover from Theorem 9.2 is actually still a robust $(\frac{\nu\gamma}{4}, \tau)$-outexpander of linear minimum semidegree at least $\frac{\eta n}{2}$. Thus, if $D \cup \mathcal{F}$ is regular, we can actually obtain a Hamilton decomposition of $D \cup \mathcal{F}$ so that for all $i \in [\ell]$, the edges of F_i are contained in C_i (indeed, it suffices to apply to Theorem 4.9 to the leftover from Theorem 9.2).

Note that Theorem 9.2 requires D to be an (ε, n^{-2})-robust outexpander. One can show that this condition is in fact redundant and can be omitted. Indeed, Kühn, Osthus, and Treglown [12] showed that the "reduced digraph" of a robust outexpander inherits the robust outexpansion properties of the host graph (see [12, Lemma 14]). Thus, using Lemma 4.11 and basic properties of "ε-regular pairs", one can easily show that the following lemma holds. We omit the details.

Lemma 9.3. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu' \ll \alpha, \nu, \tau \ll 1$. Suppose D is a robust (ν, τ)-outexpander on n vertices satisfying $\delta^0(D) \geq \alpha n$. Then, D is an (ε, n^{-2})-robust $(\nu', 4\tau)$-outexpander.*

Thus, Theorem 9.2 and Lemma 9.3 imply Theorem 9.1. As the proof of Theorem 9.2 only relies on Lemma 6.3 (which in turn makes use of Corollary 4.8 as the main tool), this gives a much shorter proof than the original one.

9.2. **A remark about Conjecture 1.7.** Conjecture 1.7 and Theorem 4.35 state that any (large) tournament T can be decomposed into at most $\widetilde{\mathrm{ex}}(T) + 1$ paths. This cannot be generalised to digraphs or even oriented graphs. Indeed, it is easy to see that if D is a disconnected oriented graph then more than $\widetilde{\mathrm{ex}}(D) + 1$ paths may be required to decompose D. In fact, Conjecture 1.7 and Theorem 4.35 cannot even be generalised to strongly connected oriented graphs.

Proposition 9.4. *For any $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists a strongly connected oriented graph D on $n \geq n_0$ vertices such that $\mathrm{pn}(D) \geq \widetilde{\mathrm{ex}}(D) + \frac{(1-\varepsilon)n}{2}$.*

Proof. Fix additional integers m and k satisfying $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon$ and $m \geq n_0$. Let $V_1, \dots, V_k$ be disjoint sets of $2m+1$ vertices each. For each $i \in [k]$, let T_i be a regular tournament on V_i and $x_iy_i \in E(T_i)$. Let D be obtained from $\bigcup_{i\in[k]} T_i$ by deleting, for each $i \in [k]$, the edge x_iy_i and adding, for each $i \in [k]$, the edge x_iy_{i+1}, where $y_{k+1} := y_1$. Observe that D is a strongly connected m-regular oriented graph on $n := k(2m+1)$ vertices. Therefore, $\widetilde{\mathrm{ex}}(D) = \Delta^0(D) = m$. Moreover, note that, for each $i \in [k]$, $\mathrm{pn}(D[V_i]) \geq \widetilde{\mathrm{ex}}(D[V_i]) = m$.

Let $\mathcal{P}$ be a path decomposition of D of size $\mathrm{pn}(D)$. For each $i \in [k]$, let $\mathcal{P}_i$ be the set of paths $P \in \mathcal{P}$ such that $V(P) \subseteq V_i$. Then, by construction, $|\mathcal{P}_i| \geq \mathrm{pn}(D[V_i]) - 2 \geq m - 2$. Thus, $\mathrm{pn}(D) = |\mathcal{P}| \geq k(m-2) = \widetilde{\mathrm{ex}}(D) + (k-1)m - 2k \geq \widetilde{\mathrm{ex}}(D) + \frac{(1-\varepsilon)n}{2}$. □

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