Large topological cliques in graphs without a 4-cycle

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Abstract

Mader asked whether every C_4-free graph G contains a subdivision of a complete graph whose order is at least linear in the average degree of G. We show that there is a subdivision of a complete graph whose order is almost linear. More generally, we prove that every $K_{s,t}$-free graph of average degree r contains a subdivision of a complete graph of order $r^{\frac{1}{2}+\frac{1}{2(s-1)}-o(1)}$.

1 Introduction

Bollobás and Thomason [4] as well as Komlós and Szemerédi [10] independently proved the following result, which improved an earlier bound of Mader.

Theorem 1 [4, 10] *There exists a positive constant c such that every graph of average degree r contains a subdivision of a complete graph of order at least $c\sqrt{r}$.*

It is easy to see (and was first observed by Jung [7]) that the complete bipartite graph $K_{r,r}$ contains no subdivision of a complete graph K_ℓ with $\ell \geq \sqrt{8r}$. So in general Theorem 1 is best possible up to the value of the constant c. However, it turns out that dense bipartite graphs are the only counterexamples in the sense that we can improve Theorem 1 if we forbid a fixed complete bipartite subgraph $K_{s,t}$:

Theorem 2 *For all integers $t \geq s \geq 2$ there exists an $r_0 = r_0(s,t)$ such that every $K_{s,t}$-free graph G of average degree $r \geq r_0$ contains a subdivision of a complete graph of order at least*

$$\frac{r^{\frac{1}{2}+\frac{1}{2(s-1)}}}{(\log r)^{12}}. \tag{1}$$

By Jung's observation, clearly we cannot hope for a similar result if we forbid a non-bipartite graph H instead of a $K_{s,t}$ since then $K_{r,r}$ would be H-free.

In the C_4-free case $s = t = 2$ the bound (1) is 'almost linear' and thus best possible up to the logarithmic term. For arbitrary $t \geq s \geq 2$ a classical conjecture on the existence of dense $K_{s,t}$-free graphs (see e.g. Bollobás [2, p. 362]) would also imply that the bound (1) is best possible up to the logarithmic term. We will give the details in Section 4.

Up to the logarithmic term, the special case $s = t = 2$ of Theorem 2 gives an affirmative answer to a question of Mader [16], who asked whether every graph G of girth at least 5 (and hence also every C_4-free graph) contains a subdivision of a complete graph whose order is at least linear in the average degree of G. This is true if the girth is at least 15: based on earlier results of Mader [15] we showed that every graph of minimum degree $r \geq 500$ and girth at least 15 contains a subdivision of a complete graph of order $r+1$ [11]. An analogue of Theorem 2 for ordinary minors was proved in [14]: every $K_{s,t}$-free graph of average degree r contains a complete graph of order $r^{1+\frac{1}{2(s-1)}-o(1)}$ as minor. (This implies Hadwiger's conjecture for $K_{s,t}$-free graphs whose chromatic number is large compared with s and t.) Further results about (topological) minors in graphs of large girth can be found in [17, 12, 13].

The proof of Theorem 2 uses results of Komlós and Szemerédi [9, 10]. In fact, Theorems 2.1 and 2.2 of [10] together with Theorem 9 below already imply the weaker bound $r^{\frac{1}{2}+\frac{1}{6(s-1)}-o(1)}$ instead of (1) in Theorem 2.

This paper is organized as follows. In Section 2 we state several results which we will need later on. We prove Theorem 2 in Section 3. In the final section we derive the upper bounds mentioned above.

2 Notation and tools

All logarithms in this paper are base e, where e denotes the Euler number. We write $e(G)$ for the number of edges of a graph G and $|G|$ for its order. We denote the maximum degree of a graph G by $\Delta(G)$, its minimum degree by $\delta(G)$ and its average degree by $d(G) := 2e(G)/|G|$. We write $d_G(x)$ for the degree of a vertex $x \in G$. Given $X \subseteq V(G)$, we denote by $N_G(X)$ the set of all those neighbours of vertices in X that lie outside X. Given disjoint $A, B \subseteq V(G)$, we write $(A,B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all the edges of G between A and B. We denote by $e_G(A,B)$ the number of those edges. More generally, we write (A,B) for a bipartite graph with vertex classes A and B.

A *subdivision* of a graph G is a graph TG obtained from G by replacing the edges of G with internally disjoint paths. The *branch vertices* of TG are all those vertices that correspond to vertices of G. We say that G is a *topological minor* of a graph H if H contains a subdivision of G as a subgraph.

We will now collect some results which we need in our proof of Theorem 2. The following lemma [14, Lemma 12] allows us to assume that in the proof of Theorem 2 our given graph G is 'almost regular' in the sense that its maximum degree is not much larger than its minimum degree.

Lemma 3 *For all integers $t \geq 2$ and all $r \geq 10^9 t^4$ every $K_{t,t}$-free graph G of average degree at least r either contains a subdivision of some graph of average degree at least r^3 or a bipartite subgraph H such that $\delta(H) \geq \frac{r}{400t\log r}$ and $\Delta(H) \leq r$.*

Throughout this paper, we fix a constant κ such that

$$1 < \kappa < 6/5 \quad \text{and} \quad \kappa^2 + 3\kappa + 3 < 8. \tag{2}$$

Given positive constants d and ε_0, let

$$\varepsilon(x) := \begin{cases} 0 & \text{if } x < d/4 \\ \varepsilon_0/(\log(8x/d))^\kappa & \text{if } x \geq d/4. \end{cases} \tag{3}$$

Note that $\varepsilon(x)x$ is monotone increasing for all $x \geq d/2$. We call a graph H a $(\kappa, d, \varepsilon_0)$*-expander for sets of size at least* x_0 if every $X \subseteq V(H)$ with $x_0 \leq |X| \leq |H|/2$ satisfies $|N_H(X)| \geq \varepsilon(|X|)|X|$, where ε is the function defined in (3). H is a $(\kappa, d, \varepsilon_0)$*-expander* if we can take $x_0 = 0$.

The following result of Komlós and Szemerédi [9, Thm. 2.2] shows that every graph G contains an expander whose average degree is not much smaller than that of G.

Theorem 4 *Let $d, \varepsilon_0 > 0$ and suppose that the function ε defined in* (3) *satisfies $\sum_{x=1}^{\infty} \varepsilon(x)/x \leq 1/6$ (which holds if ε_0 is sufficiently small compared with κ). Then every graph G has a subgraph H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$ which is a $(\kappa, d, \varepsilon_0)$-expander for sets of size at least $3d/4$.*

Corollary 5 *There is a positive $\varepsilon_0 = \varepsilon_0(\kappa) < 1$ such that every graph G has a subgraph H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$ which is a $(\kappa, d(H), \varepsilon_0)$-expander.*

Proof. Let G' be a subgraph of G which maximizes $d(G')$. Put $d' := d(G')/6$. If ε_0 is sufficiently small, we may apply Theorem 4 to G' to obtain a graph H which is a $(\kappa, d', 8\varepsilon_0)$-expander for sets of size at least $3d'/4$. Using that $d(H) \leq 6d'$, it is easy to check that for $x \geq d(H)/4$ we have

$$\frac{8\varepsilon_0}{(\log(8x/d'))^{\kappa}} \geq \frac{\varepsilon_0}{(\log(8x/d(H)))^{\kappa}}.$$

Since $d(H)/4 \geq 3d'/4$ this shows that H is a $(\kappa, d(H), \varepsilon_0)$-expander. $\square$

The following simple consequence of expansion is implicit in [9]. It shows that expanders have 'robustly small diameter'. A proof is included in [10, Lemma 2.1].

Lemma 6 *Let $d > 0$, $1 > \varepsilon_0 > 0$ and let G be a $(\kappa, d, \varepsilon_0)$-expander. Let ε be as defined in* (3) *and suppose that $X, Y, Z \subseteq V(G)$ such that $|X|, |Y| \geq x \geq d$, $|Z| \leq \varepsilon(x)x/4$ and $(X \cup Y) \cap Z = \emptyset$. Then the distance between X and Y in $G - Z$ is at most*

$$\frac{2\log(|G|/x)}{\log(1 + \varepsilon(|G|)/2)} \leq \frac{8(\log(8|G|/d))^{1+\kappa}}{\varepsilon_0}.$$

In the proof of Theorem 2 we will first replace our given graph G with an 'almost regular' subgraph obtained by Lemma 3. Then we apply Corollary 5 to this subgraph to obtain an expander H which is still 'almost regular'. The following result of Komlós and Szemerédi [9, Thm. 3.1] implies that we are already done if the order of H is sufficiently large compared with the average degree of H.

Theorem 7 *Let $\varepsilon_0 > 0$ and let $\alpha > \kappa^2 + 3\kappa + 3 > 7$. Then there exists a positive constant c such that every graph G which is a $(\kappa, d(G), \varepsilon_0)$-expander satisfying $d(G)/2 \leq \delta(G) \leq \Delta(G) \leq 72(d(G))^2$ and $\log|G| \geq (\log d(G))^{\alpha}$ contains a subdivision of a complete graph of order at least $cd(G)$.*

In the remainder of this section we collect some other results which we will use in our proof of Theorem 2. A proof of the following lemma can be found in [3, Ch. IV, Lemma 7].

Lemma 8 *Let (A, B) be a bipartite graph that does not contain a $K_{s,t}$ with t vertices in A and s vertices in B. Suppose that on average each vertex in A has d neighbours in B. Then*

$$|A|\binom{d}{s} \leq t\binom{|B|}{s}.$$

Lemma 8 can be used to prove the following lower bound on the order of $K_{s,t}$-free graphs (see e.g. [2, Ch. VI, Thm. 2.3]).

Theorem 9 *Let $t \geq s \geq 2$ be integers. Then every $K_{s,t}$-free graph G has at most $t|G|^{2-1/s}$ edges and thus satisfies*

$$|G| \geq \left(\frac{\delta(G)}{2t}\right)^{1+\frac{1}{s-1}}. \tag{4}$$

Moreover, we will use the following Chernoff bound (see e.g. [6, Cor. 2.3 and 2.4]).

Lemma 10 *Let $X_1, \ldots, X_n$ be independent* 0-1 *random variables with $\mathbb{P}(X_i = 1) = p$ and let $X := \sum_{i=1}^{n} X_i$. Then*

$$\mathbb{P}(X \leq \mathbb{E}X/2) \leq 2\mathrm{e}^{-\mathbb{E}X/12}, \tag{5}$$

$$\mathbb{P}(X \geq x) \leq \mathrm{e}^{-x} \quad \text{for all } x \geq 7\mathbb{E}(X). \tag{6}$$

The next result is an easy consequence of Hall's matching theorem (see e.g. [3, Ch. III, Thm. 7] or [5, Thm. 2.1.2]).

Corollary 11 *Let $G = (A, B)$ be a bipartite graph such that $d_G(a) \geq d_A$ for all $a \in A$ and $d_G(B) \leq d_B$ for all $b \in B$. Then G contains $|A|$ disjoint stars with centres in A and such that each of them has $\lfloor d_A/d_B \rfloor$ leaves.*

Proof. Form a new bipartite graph $G' = (A', B)$ by replacing every vertex $a \in A$ with $\tau := \lfloor d_A/d_B \rfloor$ new vertices and joining each such vertex to all the neighbours of a. For every $A^* \subseteq A'$ we have

$$|A^*|d_A \leq e_{G'}(A^*, N_{G'}(A^*)) \leq |N_{G'}(A^*)|\tau d_B$$

and thus $|N_{G'}(A^*)| \geq |A^*|$. So by Hall's theorem there exists a matching of A' in G'. But this corresponds to the required disjoint stars in G. $\square$

3 Proof of Theorem 2

As indicated in Section 2, in the proof of Theorem 2 we may assume the we are given a graph H which is an 'almost regular' expander such that $\log|H| \le (\log d(H))^\alpha$. But then by Lemma 6, the distance in H between any two sufficiently large sets is small in terms of $d(H)$ and this remains true if we delete a few vertices of H. Roughly, we shall use this property as follows. Let ℓ be the value of (1) in Theorem 2. So we are seeking a subdivision TK_ℓ of K_ℓ in H. Lemma 12 below implies that we can find ℓ disjoint stars in H such that the neighbourhood in H of each star is large even if we delete a small but arbitrary subset of the leaves. The centres of these stars will form the branch vertices of our TK_ℓ. To find the subdivided edges, we will apply Lemma 6 to obtain for every pair of stars a short path joining the neighbourhoods of the stars. All these paths will be disjoint, will avoid the stars themselves and they can be extended to subdivided edges of the TK_ℓ.

Given a star S, we denote by $L(S)$ the set of its leaves.

Lemma 12 *For all integers $t \ge s \ge 2$ there exists an $r_0 = r_0(s,t)$ such that for each $r \ge r_0$ every $K_{s,t}$-free graph G with $\delta(G) \ge r/1600t\log r$ and $\Delta(G) \le r$ contains at least*

$$k := \left\lfloor \frac{r^{\frac{1}{2}+\frac{1}{2(s-1)}}}{t(1600\log r)^2} \right\rfloor \tag{7}$$

disjoint stars where each such star S satisfies the following two conditions.

(i) $|L(S)| = k$.

(ii) *For every $v \in L(S)$ there is a set N_v of k neighbours of v outside $V(S)$ such that $N_v \cap N_w = \emptyset$ for distinct $v, w \in L(S)$.*

As described in Section 4, it is believed that for $t \ge s \ge 2$ there are $K_{s,t}$-free graphs of average degree r and order at most $c_s r^{1+1/(s-1)}$. Note that for such graphs G the union of the stars in Lemma 12 (and thus the subdivision of the complete graph which we will construct in our proof of Theorem 2) would cover a significant portion of $V(G)$.

Proof of Lemma 12. Throughout the proof of the lemma we will assume that r is sufficiently large compared with s and t. Put $n := |G|$, $\delta := \delta(G)$ and $f := 2(\log r)r^{\frac{1}{2}-\frac{1}{2(s-1)}}$. Consider a random subset X_p of $V(G)$ which is obtained by including each vertex of G with probability $p := f/\mathrm{e}^2 r$ in X_p, independently of all other vertices of G. Call a vertex $v \in G$ *good* if it has at most f neighbours in X_p. Then Stirling's inequality (see e.g. [3, p. 124]) implies that

$$\mathbb{P}(v \text{ is not good}) \le \binom{d_G(v)}{f} p^f \le \left(\frac{\mathrm{e}r}{f}\cdot p\right)^f = \mathrm{e}^{-f} \le r^{-2}. \tag{8}$$

Let n_p denote the number of vertices in G which are not good or have a neighbour that is not good. Then (8) implies that $\mathbb{E}(n_p) \le (r+1)n/r^2$. So writing

$m_p := |X_p| - n_p$, we have

$$\mathbb{E}(m_p) \geq pn - \frac{(r+1)n}{r^2} \geq \frac{pn}{2} \overset{(4)}{\geq} \frac{f}{2\mathrm{e}^2 r} \cdot \left(\frac{\delta}{2t}\right)^{1+\frac{1}{s-1}} \geq \frac{r^{\frac{1}{2}+\frac{1}{2(s-1)}}}{(2\mathrm{e}1600t^2)^2 \log r} \geq 2k.$$

Hence there is an outcome X_p which contains least $2k$ vertices that are good and have only good neighbours in G. Let X denote the set of all these vertices.

We remark that for the case $t = s = 2$ the lemma now follows easily. Indeed, since every vertex $x \in X$ is good, it has at least $\delta - f \geq \delta/2$ neighbours outside X and, since each such neighbour is good, it sends at most f edges to X. As

$$\frac{\delta}{2f} \geq 100k \tag{9}$$

we can apply Corollary 11 to $(X, N_G(X))_G$ to obtain $|X|$ disjoint stars whose centres are the vertices in X and where each such star has k leaves. Then these stars S are as required in the lemma. (Given $v \in L(S)$, we can take for N_v any set of k neighbours of v outside S. As $t = s = 2$ these sets are disjoint for distinct $v \in L(S)$.) The argument easily extends to the case $t \geq s = 2$ but not to the general case. However, we will show that a random assignment of leaves (these will be the vertices in $N_G(X)$) to star centres (which will be the vertices in X) works for all $t \geq s \geq 2$.

Given a vertex $v \in N_G(X)$, with probability $|N_G(v) \cap X|/f$ choose one of the vertices $x \in N_G(v) \cap X$. Here each of these vertices is equally likely to be chosen and so the corresponding probability is $1/f$. Choose no vertex at all with the remaining probability $1 - |N_G(v) \cap X|/f$. (Recall that $|N_G(v) \cap X| \leq f$ since $N_G(X) \ni v$ consists of good vertices. So the probability is well defined.) Do this independently for all vertices $v \in N_G(X)$. Let S_x denote the star in G whose centre is x and whose leaves are the vertices in $N_G(X)$ that have chosen x. Thus the S_x are disjoint for distinct x. We will now show that with positive probability at least half of the stars S_x $(x \in X)$ contain a substar which satisfies (i) and (ii). So call S_x *useful* if there is a set $L_x \subseteq L(S_x)$ satisfying the following two conditions.

(a) $|L_x| = k$.

(b) For every $v \in L_x$ there is a set N_v of k neighbours of v outside $L_x \cup \{x\}$ such that $N_v \cap N_w = \emptyset$ for distinct $v, w \in L_x$.

Call S_x *useless* if it is not useful. Fix a set A_x of $\lfloor \delta/2 \rfloor$ neighbours of x in G that lie outside X. For each $v \in A_x$ fix a set V_v of $\lceil \delta/2 \rceil$ neighbours of v in G outside $A_x \cup \{x\}$. Let G_x denote the bipartite subgraph of G whose vertex classes are A_x and $B_x := \bigcup_{v \in A_x} V_v$ and in which each vertex $v \in A_x$ is joined to precisely the vertices in V_v. So $e(G_x) = \lceil \delta/2 \rceil |A_x|$. Denote by B_x^1 the set of all vertices in B_x whose degree in G_x is at most f^2 and let $B_x^2 := B_x \setminus B_x^1$.

We now claim that $e_{G_x}(A_x, B_x^2) \leq e(G_x)/2$. Suppose not. Then on average each vertex in A_x has at least $\delta/4$ neighbours in B_x^2. Since $(A_x, B_x^2)_{G_x}$ does not contain a $K_{s-1,t}$ with t vertices in A_x and $s-1$ vertices in B_x^2 (such a $K_{s-1,t}$

would yield a $K_{s,t}$ in G together with x), Lemma 8 implies that

$$|A_x|\binom{\delta/4}{s-1} \leq t\binom{|B_x^2|}{s-1}.$$

As $|A_x| \geq \lfloor \delta/2 \rfloor \geq \delta/4$ it follows that

$$|B_x^2| \geq \frac{\delta|A_x|^{\frac{1}{s-1}}}{8t} \geq \frac{\delta^{1+\frac{1}{s-1}}}{32t}. \tag{10}$$

On the other hand, we have $f^2|B_x^2| \leq e_{G_x}(A_x, B_x^2) \leq \delta^2$, and thus

$$|B_x^2| \leq \frac{\delta^2}{f^2} \leq \frac{\delta^{1+\frac{1}{s-1}}}{(2\log r)^2},$$

contradicting (10). So $e_{G_x}(A_x, B_x^1) \geq e(G_x)/2$. Let A'_x be the set of all those vertices in A_x which have at least $\delta/8$ neighbours in G_x that lie inside B_x^1. Then

$$|A'_x|\delta + |A_x|\delta/8 \geq e_{G_x}(A_x, B_x^1) \geq e(G_x)/2 \geq |A_x|\delta/4$$

and thus $|A'_x| \geq |A_x|/8$. We claim that S_x is useful if $|A'_x \cap L(S_x)| \geq k$ and if in G_x each vertex in B_x^1 has at most $7f$ neighbours lying inside $L(S_x)$. To see this, apply Corollary 11 with $A := A'_x \cap L(S_x)$, $B := B_x^1$, $d_A := \delta/8$ and $d_B := 7f$ to the graph $(A,B)_{G_x}$ to obtain $|A'_x \cap L(S_x)| \geq k$ disjoint stars with centres in $A'_x \cap L(S_x)$ and such that each star has at least $\lfloor \delta/56f \rfloor$ leaves. Since $\lfloor \delta/56f \rfloor \geq k$ by (9), we can take for the set L_x in the definition of a useful star S_x any set of k centres of these stars. Hence S_x is useful.

So it remains to estimate the probability that $|A'_x \cap L(S_x)| \leq k$ or that B_x^1 contains a vertex with more than $7f$ neighbours in $L(S_x)$. As each vertex in $A_x \supseteq A'_x$ chooses x with probability $1/f$, we have

$$\mathbb{E}(|A'_x \cap L(S_x)|) = |A'_x|/f \geq |A_x|/8f \geq \delta/32f \overset{(9)}{\geq} 2k.$$

Together with (5) this implies

$$\mathbb{P}(|A'_x \cap L(S_x)| \leq k) \leq 2\mathrm{e}^{-2k/12} \leq 1/4. \tag{11}$$

Furthermore, the definition of B_x^1 implies that for every vertex $b \in B_x^1$

$$\mathbb{E}(|N_{G_x}(b) \cap L(S_x)|) \leq f,$$

and thus from (6) it follows that

$$\mathbb{P}(|N_{G_x}(b) \cap L(S_x)| \geq 7f) \leq \mathrm{e}^{-7f}.$$

Hence

$$\mathbb{P}(\exists\, b \in B_x^1 \text{ with } |N_{G_x}(b) \cap L(S_x)| \geq 7f) \leq \mathrm{e}^{-7f}|B_x^1| \leq \mathrm{e}^{-7f}\delta^2 \leq 1/4.$$

Together with (11) this implies that with probability at most $1/2$ the star S_x is useless. Hence the expected number of useless stars S_x is at most $|X|/2$, and therefore for some outcome at least $|X|/2 \geq k$ of the stars S_x ($x \in X$) are useful. For each such S_x let $S'_x \subseteq S_x$ be the star whose centre is x and whose leaves are the vertices in a set L_x satisfying (a) and (b). Then the S'_x are stars as required in the lemma. □

Proof of Theorem 2. Throughout the proof we assume that r is sufficiently large compared with s and t. Let k be as defined in (7) and put

$$\ell := \left\lfloor \frac{k}{(\log r)^{4+5\kappa}} \right\rfloor \overset{(2)}{\geq} \frac{r^{\frac{1}{2}+\frac{1}{2(s-1)}}}{(\log r)^{12}}.$$

We will show that G contains a subdivision of K_ℓ. First we apply Lemma 3 to G. Since by Theorem 1 every graph of average degree r^3 contains a subdivision of a complete graph of order $r \geq \ell$, we may assume that the lemma returns a subgraph G' with $\delta(G') \geq \frac{r}{400t\log r}$ and $\Delta(G') \leq r$. Apply Corollary 5 to G' to obtain a positive constant $\varepsilon_0 = \varepsilon_0(\kappa) < 1$ and a subgraph H which is a $(\kappa, d(H), \varepsilon_0)$-expander and satisfies $d := d(H) \geq d(G')/2$ and $\delta(H) \geq d(H)/2$. Since $72d^2 \geq r \geq \Delta(H)$, Theorem 7 with $\alpha := 8$ shows that H contains a subdivision of a complete graph of order at least $cd \geq \ell$, provided that $\log|H| \geq (\log d)^8$. Thus, setting $n := |H|$, we may assume that

$$\log n < (\log d)^8 \leq (\log r)^8. \tag{12}$$

Apply Lemma 12 to H to obtain k disjoint stars as described there. Pick ℓ of these stars, $S_1, \dots, S_\ell$ say. For all leaves v of S_i fix a set N_v^i satisfying condition (ii) of Lemma 12 and let $A_i := \bigcup_{v \in L(S_i)} N_v^i$. So $|A_i| = k^2$. The branch vertices of our subdivision of K_ℓ in G will be the centres of the S_i and each edge ij of K_ℓ will correspond to a path joining a leaf of S_i to a leaf of S_j. We will find disjoint such paths as follows. For each edge $ij \in K_\ell$ in turn we use Lemma 6 to find a short A_i–A_j path in the graph obtained from H by deleting $S_1, \dots, S_\ell$ as well as all previously constructed paths. We have to take care that for every vertex $i \in K_\ell$ the paths that correspond to the edges of K_ℓ which are incident with i start in distinct sets $N_v^i \subseteq A_i$ and thus can be joined by independent edges to distinct leaves of S_i. Thus when defining the A_i–A_j path corresponding to the edge $ij \in K_\ell$, we will also delete all those sets N_v^i from H which contain the starting point of a previously constructed A_i–$A_{j'}$ path; and similarly for j.

More formally, we proceed as follows. Fix an enumeration $i_1 j_1, \dots, i_{\binom{\ell}{2}} j_{\binom{\ell}{2}}$ of the edges of K_ℓ. We will show that for all $b \leq \binom{\ell}{2}$ there is a path P_b whose length is at most $2 + 8(\log n)^{1+\kappa}/\varepsilon_0 =:$ diam, which joins a leaf of S_{i_b} to a leaf of S_{j_b}, has no inner vertices in $\bigcup_{i=1}^{\ell} S_i$ and such that the P_b are disjoint for distinct $b \leq \binom{\ell}{2}$. Suppose inductively that for some $a \geq 1$ we have already defined P_b for all $b < a$.

To find P_a, let N_{i_a} be the union of all those $N_v^{i_a}$ for which the leaf v of S_{i_a} is an endpoint of a path P_b constructed previously. (In other words, these paths P_b are precisely the previously constructed paths that correspond to an edge of K_ℓ incident with i_a, i.e. for which $i_a \in \{i_b, j_b\}$.) Define N_{j_a} similarly. Let Z be the set consisting of the vertices in $N_{i_a} \cup N_{j_a}$ together with all vertices lying in some S_i ($i \leq \ell$) and all vertices on the paths P_b already constructed. So

$$\begin{aligned} |Z| &\leq 2\ell k + \ell(k+1) + (\mathrm{diam}+1) \cdot \binom{\ell}{2} \leq 4\ell k + \frac{5\ell^2(\log n)^{1+\kappa}}{\varepsilon_0} \\ &\overset{(12)}{\leq} 4\ell k + \frac{5\ell^2(\log r)^{8+8\kappa}}{\varepsilon_0} \leq \frac{6k^2}{\varepsilon_0(\log r)^{2\kappa}}. \end{aligned} \tag{13}$$

Let $A'_{i_a} := A_{i_a} \setminus Z$ and define A'_{j_a} similarly. Then

$$|A'_{i_a}| \geq |A_{i_a}| - |Z| \overset{(13)}{\geq} k^2 - \frac{6k^2}{\varepsilon_0(\log r)^{2\kappa}} \geq \frac{k^2}{2}. \tag{14}$$

Let ε be as defined in (3). Using that $\varepsilon(x)x$ is monotone increasing for all $x \geq d/2$ and thus for $x \geq k^2/2$, it is easy to check that (13) and (14) imply $|Z| \leq |A'_{i_a}|\varepsilon(|A'_{i_a}|)/4$. Similarly it follows that $|Z| \leq |A'_{j_a}|\varepsilon(|A'_{j_a}|)/4$. Thus we may apply Lemma 6 to obtain an A'_{i_a}–A'_{j_a} path P in $H - Z$ of length at most

$$\frac{8(\log(8n/d))^{1+\kappa}}{\varepsilon_0} \leq \mathrm{diam} - 2.$$

The definition of A'_{i_a} implies that the endpoint of P in A'_{i_a} can be joined by an edge to some leaf of S_{i_a} which is not already an endpoint of a path P_b constructed previously. The same is true for j_a. Altogether this shows that the S_{i_a}–S_{j_a} path P_a obtained from P in this way has the required properties. □

4 Upper bounds

The following proposition shows that the existence of sufficiently dense $K_{s,t}$-free graphs would imply that the bound (1) in Theorem 2 is best possible up to the logarithmic factor.

Proposition 13 *For every $c > 0$ and all $t \geq s \geq 2$ there is a constant $C = C(c, s, t)$ such that no $K_{s,t}$-free graph G with $e(G) \geq c|G|^{2-1/s}$ contains a subdivision of a complete graph of order at least $Cd(G)^{\frac{1}{2}+\frac{1}{2(s-1)}}$.*

Proof. We will show that $C := (16t)^s/c^{\frac{1}{2}+\frac{1}{2(s-1)}}$ works. Let $n := |G|$ and $r := d(G)$. Clearly, we may assume that G contains a subdivision TK_ℓ of K_ℓ for some $\ell \geq (16t)^s$. Recall that by Theorem 9, every subgraph H of G has at most $t|H|^{2-1/s}$ edges. In particular, the subgraph of G induced by the branch vertices of TK_ℓ contains at most $t\ell^{2-1/s} \leq \ell^2/16 \leq e(K_\ell)/4$ edges. So at least $3/4$ of the edges of K_ℓ correspond to paths in TK_ℓ of length at least two. Thus $n \geq 3e(K_\ell)/4 \geq \ell^2/4$. On the other hand, our assumption on G implies that $n \leq (r/c)^{1+1/(s-1)}$. Hence

$$\ell \leq 2\left(\frac{r}{c}\right)^{\frac{1}{2}+\frac{1}{2(s-1)}},$$

as required. □

It is widely believed that $K_{s,t}$-free graphs as in the statement of Proposition 13 do exist, i.e. that for all $t \geq s \geq 2$ there exists a positive constant $c = c(s,t)$ such that there are arbitrarily large $K_{s,t}$-free graphs G with $e(G) \geq c|G|^{2-1/s}$. Note that this would mean that the upper bound of Theorem 9 gives the correct order of magnitude. The case $s = t$ of the conjecture (which of course would already imply the general case) as well as a proof of the conjecture for the case $t \geq s$ with $s = 2, 3$ can be found in Bollobás [2]. Based on a construction in [8], Alon, Rónyai and Szabó [1] proved that the conjecture is also true for all $t \geq s \geq 2$ with $t > (s-1)!$.

For $s \geq 4$ the best known lower bound on the maximum number of edges of a $K_{s,s}$-free graph G is $c|G|^{2-2/(s+1)}$ (see e.g. [2, Ch. VI, Thm. 2.10]). Using this bound, the proof of Proposition 13 still yields an upper bound of $C' r^{\frac{1}{2}+\frac{1}{s-1}}$ for the order of the complete topological minor in Theorem 2.

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