# ON A CONJECTURE OF ERDŐS ON LOCALLY SPARSE STEINER TRIPLE SYSTEMS

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ABSTRACT. A famous theorem of Kirkman says that there exists a Steiner triple system of order $n$ if and only if $n \equiv 1, 3 \mod 6$. In 1976, Erdős conjectured that one can find so-called 'sparse' Steiner triple systems. Roughly speaking, the aim is to have at most $j - 3$ triples on every set of $j$ points, which would be best possible. (Triple systems with this sparseness property are also referred to as having high girth.) We prove this conjecture asymptotically by analysing a natural generalization of the triangle removal process. Our result also solves a problem posed by Lefmann, Phelps and Rödl as well as Ellis and Linial in a strong form. Moreover, we pose a conjecture which would generalize the Erdős conjecture to Steiner systems with arbitrary parameters and provide some evidence for this.

## 1. Introduction

Given a set $X$ of size $n$, a set $\mathcal{S}$ of 3-subsets of $X$ is a *Steiner triple system of order* $n$ if every 2-subset of $X$ is contained in exactly one of the triples of $\mathcal{S}$ (if every 2-subset of $X$ lies in at most one of the triples of $\mathcal{S}$, we refer to $\mathcal{S}$ as a *partial Steiner triple system*). In 1847, Kirkman [19] proved that there exists a Steiner triple system of order $n$ if and only if $n \equiv 1, 3 \mod 6$. We shall call such $n$ *admissible*. In this paper, we investigate so-called 'sparse' Steiner triple systems, which do not contain certain 'forbidden configurations'. Erdős conjectured the existence of such sparse systems. A $(j, \ell)$*-configuration* is a set of $\ell$ triples on $j$ points every two of which intersect in at most one point. The 'forbidden configurations' are the $(j, j-2)$-configurations. For instance, the unique $(6, 4)$-configuration is called the *Pasch configuration* or *quadrilateral*. There are two $(7, 5)$-configurations, called *mitre* and *mia* (see Figure 1).

A Steiner triple system is called $k$*-sparse* if it does not contain any $(j+2, j)$-configuration for $2 \leq j \leq k$. Erdős conjectured that if $k$ is bounded, then all these configurations can be avoided.

**Conjecture 1.1** (Erdős [8])**.** *For every $k$, there exists an $n_k$ such that for all admissible $n > n_k$, there exists a $k$-sparse Steiner triple system of order $n$.*

We note that Conjecture 1.1 would be best possible in the following sense: it is easy to see that for all $n \geq j \geq 4$, every Steiner triple system of order $n$ contains a $(j, j-3)$-configuration. This is true in a very robust sense. For instance, the $(6, 3)$-theorem of Ruzsa and Szemerédi [28] implies that any partial Steiner triple system of order $n$ with no $(6, 3)$-configuration has only $o(n^2)$ triples.

The conjecture is trivial for $k \leq 3$ (in the sense that it follows directly from Kirkman's theorem). A Steiner triple system is 4-sparse if and only if it is Pasch-free. This case has received a lot of attention and has been settled in a series of papers [6, 15, 16, 23]. For $k = 5$, it was shown in [32] that 5-sparse Steiner triple systems exist for almost all admissible orders. 6-sparse Steiner triple systems for infinitely many orders have been constructed in [10]. However, not a single 7-sparse system is known (on at least 9 points). All of these and many other related results are usually based on algebraic techniques.

*Date*: 13th February 2018.

The research leading to these results was partially supported by the EPSRC, grant nos. EP/N019504/1 (D. Kühn) and EP/P002420/1 (A. Lo), by the Royal Society and the Wolfson Foundation (D. Kühn) as well as by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreement no. 306349 (S. Glock and D. Osthus).

Here, we prove Conjecture 1.1 approximately by analysing a natural random process. Roughly speaking, we show that when triples are randomly chosen one by one under the condition that the set of chosen triples remains sparse, then with high probability, this process runs almost to the end, i.e. almost as many triples are added as there are in a Steiner triple system of the same order (see Theorem 4.4). In particular, such a sparse 'approximate' Steiner triple system exists. The same result has been announced independently by Bohman and Warnke (personal communication).

**Theorem 1.2.** *For every fixed $k$ and $n$ tending to infinity, there exists a $k$-sparse partial Steiner triple system $\mathcal{S}$ on $n$ vertices with $|\mathcal{S}| = (1/6 - o(1))n^2$.*

This also solves a problem of Lefmann, Phelps and Rödl [22] in a strong form. They showed that for every $k$, there exists $c_k > 0$ such that for all $n$ there is a $k$-sparse partial Steiner triple $\mathcal{S}$ on $n$ vertices with $|\mathcal{S}| \geq c_k n^2$, where $c_k \to 0$ as $k \to \infty$. Lefmann, Phelps and Rödl asked whether $c_k$ could be bounded away from 0. The same question was also raised by Ellis and Linial [7]. Our Theorem 1.2 implies that $c_k \sim \frac{1}{6}$ for all $k$. Note that the property of being $k$-sparse is often referred to as having high girth. Thus our result can be interpreted as providing an asymptotically optimal density bound for the existence of triple systems of given girth.

It is not hard to check that, for Conjecture 1.1 to be true, we must have $k = \mathcal{O}(\sqrt{n_k})$. In fact, $k$ needs to be much smaller than that, as shown by the following result.

**Theorem 1.3** ([22])**.** *There exists $c > 0$ such that every Steiner triple system of order $n$ contains a $(j, j-2)$-configuration for some $4 \leq j < c \log n / \log\log n$.*

This raises the question whether it is possible to allow $k$ to grow with $n$ in Theorem 1.2, perhaps matching the upper bound given by Theorem 1.3, although it is not clear what the correct function should be.

We will view configurations and partial Steiner triple systems as (linear) 3-graphs. It will be convenient not to assume from the outset that the systems/configurations are linear, i.e. that every two triples meet in at most one point. Instead, we will force this condition by forbidding the so-called *diamond*, i.e. the 3-graph with 2 triples on 4 vertices. Thus, we define a *forbidden configuration* as a 3-graph $\mathcal{S}$ with $|V(\mathcal{S})| = j$ and $|\mathcal{S}| = j-2$ for some $j \geq 4$. An *Erdős-configuration* is a forbidden configuration which does not contain any forbidden configuration as a proper subgraph. Thus, the diamond is the smallest Erdős-configuration. There are no Erdős-configurations on 5 points. Pasch and mitre are Erdős-configurations, but the mia is not as it is not Pasch-free (cf. Figure 1). Clearly, a Steiner triple system is $k$-sparse if and only if it does not contain any Erdős-configuration on at most $k+2$ points. For instance, a Steiner triple system is 5-sparse if and only if it does not contain the Pasch or the mitre configuration. It is not too difficult to see that an Erdős-configuration exists for every order $j \geq 6$. For example, take vertices $e, o, x_1, \dots, x_{j-2}$ and all triples $ox_\ell x_{\ell+1}$ if $\ell \leq j-3$ is odd and all triples $ex_\ell x_{\ell+1}$ if $\ell \leq j-3$ is even. Moreover, if $j$ is even, then also take the triple $ex_{j-2}x_1$, and if $j$ is odd, then include the triple $x_{j-4}x_{j-2}x_1$ instead.

As indicated above, in order to prove Theorem 1.2, we will consider a natural random process, which can be seen as a generalization of the triangle removal process, or alternatively as an $\mathcal{H}$-free process for hypergraphs.

The triangle removal process starts with the complete graph $K_n$ and then repeatedly deletes the edges of a uniformly chosen triangle. This process terminates with a triangle-free graph, and along the way produces a partial Steiner triple system. The most natural question about this process is how long it typically runs for, or equivalently, how many edges are left when it terminates. With the motivation of determining the Ramsey number $R(3,t)$, Bollobás and Erdős conjectured in 1990 that with high probability the number of edges left is of order $n^{3/2}$. This problem attracted much attention (see e.g. [14, 27, 29]), culminating in a result of Bohman, Frieze and Lubetzky [3] where the exponent was finally approximately confirmed.

| $j$ | Name | Triples | |
|---|---|---|---|
| $4^*$ | diamond | $012, 013$ | Erdős |
| 6 | Pasch | $012, 034, 135, 245$ | Erdős |
| 7 | mitre | $012, 034, 135, 236, 456$ | Erdős |
| 7 | mia | $012, 034, 135, 245, 056$ | contains Pasch |
| 8 | 6-cycle | $012, 034, 135, 246, 257, 367$ | Erdős |
| 8 | crown | $012, 034, 135, 236, 147, 567$ | Erdős |
| 8 | | $012, 034, 135, 236, 146, 057$ | contains Pasch |
| 8 | | $012, 034, 135, 236, 146, 247$ | contains Pasch |
| 8 | | $012, 034, 135, 236, 147, 257$ | contains mitre |

FIGURE 1. The smallest forbidden configurations. There are more such configurations which are not linear (i.e. contain the diamond) and thus are omitted here. If we assume at the outset that all configurations are linear, then the Pasch configuration becomes the smallest forbidden configuration.

We adapt the triangle removal process so that it does not just produce a partial Steiner triple system, but a $k$-sparse one. Hence, in each step we delete the edges of a uniformly chosen triangle which does not produce an Erdős-configuration of order at most $k+2$ with some of the previously chosen triangles (cf. Algorithm 4.1). The process terminates if no such triangle is left. The question is of course again how long the process typically runs for. We prove that with high probability, the number of leftover edges is $o(n^2)$, implying Theorem 1.2. It would be interesting to find the correct order of magnitude of the number of leftover edges. It may be possible that this number is still of order $n^{3/2}$.

We actually formulate the above process as an $\mathcal{H}$-free process for hypergraphs. Let $\mathcal{H}$ be the set of Erdős-configurations up to order $k+2$. The *$\mathcal{H}$-free process* is the random process starting with an empty 3-graph on $n$ vertices where in each step a uniformly random hyperedge is added under the condition that no copy of a member of $\mathcal{H}$ is created. For a fixed (hyper-)graph $H$, the $H$-free process has been extensively studied, in particular if $H$ is 'strictly 2-balanced' (see e.g. [2, 4, 5, 9, 20, 25, 30, 31]). A particular challenge arising in the analysis of the current process is that each individual Erdős-configuration in $\mathcal{H}$ has a significant influence on the trajectory of the process.

An obvious question is whether our approximate result can be combined with the absorbing method in order to prove Conjecture 1.1, e.g. using approaches from [17, 18] or [12, 13]. One major difficulty here is that the absorbing method relies on the simple fact that, given two triangle packings which are edge-disjoint, their union also forms a triangle packing. On the contrary, the union of two edge-disjoint sparse triangle packings is not necessarily sparse.

Our paper is organised as follows. After introducing our basic terminology in Section 2, we will state Freedman's inequality in Section 3, which will be the main probabilistic tool to analyse our process. In Section 4, we define the process more formally, discuss the key random variables of the process and predict its behaviour heuristically using the differential equation method. Subsequently, in Section 5, we analyse the process. In particular, we establish trend hypotheses and boundedness hypotheses for the random variables which we track. In Section 6, we formulate a conjecture on the number of $k$-sparse Steiner triple systems. Finally, in Section 7, we propose a conjecture which would generalize Conjecture 1.1 to Steiner systems with arbitrary parameters and provide some evidence for our conjecture.

## 2. Notation

We let $[n]$ denote the set $\{1, \ldots, n\}$, where $[0] := \emptyset$. Moreover, we set $[n]_0 := [n] \cup \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given a set $X$ and $i \in \mathbb{N}_0$, we write $\binom{X}{i}$ for the collection of all $i$-subsets of $X$.

A *hypergraph* $H$ is a pair $(V, E)$, where $V = V(H)$ is the vertex set and the edge set $E$ is a set of subsets of $V$. We identify $H$ with $E$. In particular, we let $|H| := |E|$. We say that $H$ is an *$r$-graph* if every edge has size $r$. Given $U \subseteq V(H)$, we write $H[U]$ for the sub-hypergraph of $H$ induced by $U$. Given $S \subseteq V(H)$, we write $d_H(S)$ for the *degree of $S$ in $H$*, i.e. the number of hyperedges of $H$ containing $S$.

We say that an event holds *with high probability (whp)* if the probability that it holds tends to 1 as $n \to \infty$ (where $n$ usually denotes the number of vertices).

We write $a = b \pm c$ if $b - c \leq a \leq b + c$. Equations containing $\pm$ are always to be interpreted from left to right, e.g. $b_1 \pm c_1 = b_2 \pm c_2$ means that $b_1 - c_1 \geq b_2 - c_2$ and $b_1 + c_1 \leq b_2 + c_2$. Moreover, $a \wedge b$ denotes the minimum of $a$ and $b$.

We write $f = \mathcal{O}(g)$ if $|f| \leq C|g|$ for some constant $C$ (which by default may only depend on $k$). We write $\mathcal{O}_\gamma$ to indicate that the constant may also depend on $\gamma$. Similarly, we write $f = \Omega(g)$ if $f \geq c|g|$ for some constant $c > 0$ (which by default may depend only on $k$, and additional dependencies are indicated as indices). Note that if $x = \mathcal{O}(n^a)$ and $y = \Omega(n^b)$, then

$$\frac{x + \mathcal{O}(\varepsilon)n^a}{y + \mathcal{O}(\varepsilon)n^b} = \frac{x}{y} + \mathcal{O}(\varepsilon)n^{a-b}. \tag{2.1}$$

We write $x \ll y$ to mean that for any $y \in (0, 1]$ there exists an $x_0 \in (0, 1)$ such that for all $x \leq x_0$ the subsequent statement holds. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. We will always assume that the constants in our hierarchies are reals in $(0, 1]$. Moreover, if $1/x$ appears in a hierarchy, this implicitly means that $x$ is a natural number. More precisely, $1/x \ll y$ means that for any $y \in (0, 1]$ there exists an $x_0 \in \mathbb{N}$ such that for all $x \in \mathbb{N}$ with $x \geq x_0$ the subsequent statement holds.

## 3. Freedman's inequality

Let $X(0), X(1), \dots$ be a real-valued random process. We define

$$\Delta X(i) := X(i + 1) - X(i).$$

The process $X(0), X(1), \dots$ is a *supermartingale* (with respect to a filtration $\mathcal{F} = (\mathcal{F}(0), \mathcal{F}(1), \dots)$) if $\mathbb{E}\left(X(i + 1) \mid \mathcal{F}(i)\right) \leq X(i)$, or equivalently, $\mathbb{E}\left(\Delta X(i) \mid \mathcal{F}(i)\right) \leq 0$, for all $i \geq 0$.

The following tail probability is due to Freedman [11]. It was originally stated for martingales, but the proof for supermartingales is verbatim the same (cf. [21]).

**Lemma 3.1** (Freedman's inequality, cf. [11, 21])**.** *Let $X(0), X(1), \dots$ be a supermartingale with respect to a filtration $\mathcal{F} = (\mathcal{F}(0), \mathcal{F}(1), \dots)$. Suppose that $|\Delta X(i)| \leq K$ for all $i$, and let $V(i) := \sum_{j=0}^{i-1} \mathbb{E}\left((\Delta X(j))^2 \mid \mathcal{F}(j)\right)$. Then for any $t, v > 0$,*

$$\mathbb{P}\left(X(i) \geq X(0) + t \text{ and } V(i) \leq v \text{ for some } i\right) \leq \mathrm{e}^{-\frac{t^2}{2(v+Kt)}}.$$

We will apply Lemma 3.1 in the following scenario: There will be a parameter $n$ which measures the size of the probability space. There will be a (random) time $\tau_{freeze} = \mathcal{O}(n^2)$ such that $\Delta X(i) = 0$ for all $i \geq \tau_{freeze}$. Moreover, we will have $\Delta X(i) = \mathcal{O}(n^{\alpha_2})$ and $\mathbb{E}\left(|\Delta X(i)| \mid \mathcal{F}(i)\right) = \mathcal{O}(n^{\alpha_3})$ for all $i$, and $-X(0) = \Omega(n^{\alpha_1})$. Suppose that $\alpha_1 > \alpha_2$ and $\alpha_1 \geq \alpha_3 + 2$. Then we can conclude that

$$\mathbb{P}\left(\exists i \colon X(i) \geq 0\right) \leq \mathrm{e}^{-\Omega(n^{\alpha_1 - \alpha_2})}. \tag{3.1}$$

Indeed, we can apply Lemma 3.1 with $t := -X(0)$, $v = \mathcal{O}(n^{\alpha_2+\alpha_3+2})$ and $K = \mathcal{O}(n^{\alpha_2})$. For every $i$, we have $\mathbb{E}\left((\Delta X(i))^2 \mid \mathcal{F}(i)\right) \leq K \cdot \mathbb{E}\left(|\Delta X(i)| \mid \mathcal{F}(i)\right) = \mathcal{O}(n^{\alpha_2+\alpha_3})$ and thus $\sum_{i=0}^{\infty} \mathbb{E}\left((\Delta X(i))^2 \mid \mathcal{F}(i)\right) \leq v$. Hence, $V(i) \leq v$ for all $i$. Note that $t^2 = \Omega(n^{2\alpha_1})$ and $v + Kt = \mathcal{O}(n^{\alpha_1+\alpha_2})$.

The supermartingales we consider are obtained as follows: Let $X$ be a random variable of the process, e.g. the number of available triples containing a fixed edge. Using the differential equation method, we will have a rough idea of how $X$ should behave, i.e. we will find a (smooth) deterministic function $f_X$ and predict that $X \approx f_X$. We call $f_X$ the *trajectory* of $X$. In order to control the deviation of $X$ from $f_X$, we introduce an *error function* $\varepsilon_X$. We now define

$$X^+(i) := X(i) - f_X(i) - \varepsilon_X(i),$$
$$X^-(i) := -X(i) + f_X(i) - \varepsilon_X(i).$$

(In the actual proof we will actually 'freeze' these variables after a certain random time.) Note that if $X^\pm(i) \le 0$, then $|X(i) - f_X(i)| \le \varepsilon_X(i)$. (We write $X^\pm(i) \le 0$ to mean that both $X^+(i) \le 0$ and $X^-(i) \le 0$ hold.) Our aim is to show that the $X^\pm$ define two supermartingales and then to use (3.1) to show that $X^\pm \le 0$ throughout the process. In order to show that $X^\pm$ are supermartingales (with respect to a filtration $\mathcal{F} = (\mathcal{F}(0), \mathcal{F}(1), \dots))$, it is enough to show that $\mathbb{E}(\Delta X^\pm(i) \mid \mathcal{F}(i)) \le 0$ for all $i \ge 0$ (usually referred to as the 'trend hypothesis'). Observe that

$$\mathbb{E}\left(\Delta X^\pm(i) \mid \mathcal{F}(i)\right) = \pm\mathbb{E}\left(\Delta X(i) \mid \mathcal{F}(i)\right) \mp \Delta f_X(i) - \Delta\varepsilon_X(i).$$

In order to determine $\Delta f_X$ and $\Delta\varepsilon_X$, we use the following simple consequence of Taylor's theorem with remainder in Lagrange form: for a sufficiently smooth function $f$, we have

$$\Delta f(i) := f(i+1) - f(i) = f'(i) \pm \sup_{\xi\in[i,i+1]} f''(\xi). \tag{3.2}$$

The terms $\mathbb{E}(\Delta X(i) \mid \mathcal{F}(i))$ and $\Delta f_X(i)$ will almost cancel out, and the purpose of $\Delta\varepsilon_X(i)$ is to make the sum negative. For this to work, $\varepsilon_X$ has to have a large enough growth rate throughout the process. On the other hand, it must not grow too fast, otherwise we would lose control of $X$. A careful calibration is thus essential for the analysis to work.

Once we have established that $X^+$ is a supermartingale, it remains to give bounds on $|\Delta X^+(i)|$ ('boundedness hypothesis') and $\mathbb{E}(|\Delta X^+(i)| \mid \mathcal{F}(i))$. For this, we simply use $|\Delta X^+(i)| \le |\Delta X(i)| + |\Delta f_X(i)| + |\Delta\varepsilon_X(i)|$.

Let $\mathcal{X}(i)$ be a set (which contains all objects of a certain type at time $i$) and suppose that our random variable is defined as $X(i) := |\mathcal{X}(i)|$. Suppose we consider our process at time $i$. For every object $x \in \mathcal{X}(i)$, $x$ could potentially be removed from $\mathcal{X}(i)$, i.e. $x \notin \mathcal{X}(i+1)$. We denote the indicator function of this event by $\mathbb{1}_{-x}$. Moreover, there is a set $\mathcal{X}^{pot}(i)$ of potential new elements which might be added to $\mathcal{X}(i)$, i.e. for every $x \in \mathcal{X}^{pot}(i)$, we have $x \notin \mathcal{X}(i)$ but with non-zero probability we have $x \in \mathcal{X}(i+1)$. We denote the indicator function of this event by $\mathbb{1}_{+x}$. Thus, we have

$$\Delta X(i) = |\mathcal{X}(i+1)| - |\mathcal{X}(i)| = -\sum_{x\in\mathcal{X}(i)} \mathbb{1}_{-x} + \sum_{x\in\mathcal{X}^{pot}(i)} \mathbb{1}_{+x}. \tag{3.3}$$

## 4. The process

We now describe the process that we wish to analyse. Let $V$ be a set of $n$ vertices. Suppose that we want to construct a $k$-sparse triple system, with $k \ge 2$. Let

$$j_{max} := k + 2$$

and consider Algorithm 4.1.

The last step of the process is $\tau_{max} := \min\{i : \mathcal{A}(i) = \emptyset\}$. At time $i$, we say that $\mathcal{A}(i)$ is the set of *available triples* and $\mathcal{C}(i)$ is the set of *chosen triples*. Clearly, we have $\mathcal{A}(i+1) \subseteq \mathcal{A}(i)$, $\mathcal{C}(i+1) \supseteq \mathcal{C}(i)$ and $\mathcal{A}(i) \cap \mathcal{C}(i) = \emptyset$ for all $i$. We refer to $T^*(i)$ as the *selected triple* in step $i$. For a 3-set $T \subseteq V$, let $\tau_T := \min\{i : T \notin \mathcal{A}(i)\}$.

**Fact 4.2.** *$|\mathcal{C}(i)| = i$ and $\mathcal{C}(i)$ is $k$-sparse for all $i \le \tau_{max}$.*

**Algorithm 4.1**

$\mathcal{A}(0) := \binom{V}{3}$, $\mathcal{C}(0) := \emptyset$, $i := 0$
**while** $\mathcal{A}(i) \neq \emptyset$ **do**
  select $T^*(i) \in \mathcal{A}(i)$ uniformly at random
  let $\mathcal{A}'(i)$ consist of all $T \in \mathcal{A}(i)$ for which there is $\mathcal{C}' \subseteq \mathcal{C}(i)$ such that $\{T, T^*(i)\} \cup \mathcal{C}'$ is an Erdős-configuration on at most $j_{max}$ points
  $\mathcal{A}(i+1) := \mathcal{A}(i) \setminus (\mathcal{A}'(i) \cup \{T^*(i)\})$
  $\mathcal{C}(i+1) := \mathcal{C}(i) \cup \{T^*(i)\}$
  $i := i+1$
**end while**

In particular, $\mathcal{C}(i)$ is a linear 3-graph, i.e. $|T^*(i') \cap T^*(i'')| \leq 1$ for all distinct $i', i'' < i$.

A 2-set $e \subseteq V$ is called *covered* (at time $i$) if $e \subseteq T$ for some $T \in \mathcal{C}(i)$, otherwise it is *uncovered*. We often refer to 2-sets of $V$ as *edges*. Let $E(i)$ be the set of uncovered edges at time $i$. Since $\mathcal{C}(i)$ is linear, we have $|E(i)| = \binom{n}{2} - 3|\mathcal{C}(i)| = \binom{n}{2} - 3i$ for all $i \leq \tau_{max}$. For a 2-set $e$, we define the random time $\tau_e := \min\{i : e \notin E(i)\}$, where $\tau_e := \infty$ if $e \in E(\tau_{max})$.

**Fact 4.3.** *If $T \in \binom{V}{3}$ is available, then every edge contained in $T$ is uncovered.*

By Fact 4.2, the following result implies Theorem 1.2.

**Theorem 4.4.** *Suppose that $\gamma \in (0,1)$ and $k \in \mathbb{N}$. Then whp as $n \to \infty$, $\tau_{max} \geq (1-\gamma)n^2/6$.*

4.1. **Key variables and threats.** Define the densities

$$p(i) := |E(i)|/\binom{n}{2}, \quad p_{\mathcal{C}}(i) := |\mathcal{C}(i)|/\binom{n}{3}, \quad p_{\mathcal{A}}(i) := |\mathcal{A}(i)|/\binom{n}{3}. \tag{4.1}$$

The following equalities clearly hold throughout the process, i.e. for all $i \leq \tau_{max}$:

$$p(i) = 1 - \frac{3i}{\binom{n}{2}}, \tag{4.2}$$

$$p_{\mathcal{C}}(i) = \frac{i}{\binom{n}{3}}. \tag{4.3}$$

However, this gives no information as to how long the process continues. For this, we need to track the number $|\mathcal{A}(i)|$ of available triples.

For $T_1, T_2 \in \mathcal{A}(i)$, we say that $T_1$ and $T_2$ *exclude each other*, denoted by $T_1 \leftrightarrow T_2$, if there is $\mathcal{C}' \subseteq \mathcal{C}(i)$ such that $\{T_1, T_2\} \cup \mathcal{C}'$ is an Erdős-configuration on at most $j_{max}$ points.

For $T \in \mathcal{A}(i)$, let $\mathcal{T}_T(i) := \{T^* \in \mathcal{A}(i) : T \leftrightarrow T^*\}$. Hence, if $T^*(i) \in \mathcal{T}_T(i)$ then $T \in \mathcal{A}'(i)$. Note that $T \notin \mathcal{T}_T(i)$. Since $T^*(i)$ is selected uniformly at random, we have that the probability that $T$ is not in $\mathcal{A}(i+1)$ is $\frac{|\mathcal{T}_T(i)|+1}{|\mathcal{A}(i)|}$.

For a 2-set $e$, let $\mathcal{X}_e(i) := \{T \in \mathcal{A}(i) : T \supseteq e\}$ be the set of available triples containing $e$ at time $i$. Moreover, we set $X_e(i) := |\mathcal{X}_e(i)|$. Clearly, we have $X_e(0) = n-2$.

**Fact 4.5.** $|\mathcal{A}(i)| = \frac{1}{3}\sum_{e \in E(i)} X_e(i)$.

**Proof.** By Fact 4.3, every available triple contains 3 uncovered edges, and $X_e(i) = |\mathcal{X}_e(i)|$ for all $e \in E(i)$. $\square$

Let $\mathfrak{J}_j$ be the set of all unlabelled Erdős-configurations on $j$ vertices in $V$. For a triple $T$, we let $\mathfrak{J}_j(T) := \{\mathcal{S} \in \mathfrak{J}_j : T \in \mathcal{S}\}$. By symmetry, we have that $|\mathfrak{J}_j(T)| =: J_j$ is the same for all triples $T$. We will not compute the precise number, but only need that $J_j = \Theta(n^{j-3})$ for $j \geq 6$.

For a triple $T$, $j \in \{4, \dots, j_{max}\}$ and $c \in \{0, \dots, j-4\}$, we define

$$\mathcal{X}_{T,j,c}(i) := \{\mathcal{S} \in \mathfrak{J}_j(T) : |(\mathcal{S} - \{T\}) \cap \mathcal{C}(i)| = c, |(\mathcal{S} - \{T\}) \cap \mathcal{A}(i)| = j - 3 - c\}. \tag{4.4}$$

Note that if $\mathcal{S} \in \mathcal{X}_{T,j,c}(i)$, then every $T' \in \mathcal{S} - \{T\}$ is either chosen or available (at time $i$). We make no assumption on the status of $T$, however we will only be interested in $\mathcal{X}_{T,j,c}(i)$ as long as $T$ is available. Define $X_{T,j,c}(i) := |\mathcal{X}_{T,j,c}(i)|$. Note that $X_{T,j,0}(0) = J_j$ and $X_{T,j,c}(0) = 0$ if $c > 0$.

Note that since $\mathfrak{J}_5 = \emptyset$, we always have $\mathcal{X}_{T,5,c}(i) = \emptyset$. Moreover, note that $\mathcal{X}_{T,4,0}(i)$ corresponds to the set of all $T' \in \mathcal{A}(i)$ with $|T' \cap T| = 2$.

We call elements of $\mathcal{X}_{T,j,j-4}$ *dangerous configurations.*

**Fact 4.6.** *For $T \in \mathcal{A}(i)$, we have*

$$\mathcal{T}_T(i) = \{T^* \,:\, \exists \mathcal{S} \in \bigcup_{j=4}^{j_{max}} \mathcal{X}_{T,j,j-4}(i) \text{ such that } (\mathcal{S} - \{T\}) \cap \mathcal{A}(i) = \{T^*\}\}.$$

**Proof.** Suppose $T^* \in \mathcal{T}_T(i) \subseteq \mathcal{A}(i)$. Then there is $\mathcal{C}' \subseteq \mathcal{C}(i)$ such that $\{T, T^*\} \cup \mathcal{C}'$ forms an Erdős-configuration $\mathcal{S}$ on $j \leq j_{max}$ points. Then $\mathcal{S} \in \mathcal{X}_{T,j,j-4}(i)$ and $(\mathcal{S} - \{T\}) \cap \mathcal{A}(i) = \{T^*\}$. Conversely, if there is $\mathcal{S} \in \bigcup_{j=4}^{j_{max}} \mathcal{X}_{T,j,j-4}(i)$ with $(\mathcal{S} - \{T\}) \cap \mathcal{A}(i) = \{T^*\}$, then $\mathcal{C}' := \mathcal{S} - \{T, T^*\} \subseteq \mathcal{C}(i)$ is such that $\{T, T^*\} \cup \mathcal{C}'$ forms an Erdős-configuration on at most $j_{max}$ points. □

By showing that most $T^*$ are only contained in at most one $\mathcal{S} \in \bigcup_{j=4}^{j_{max}} \mathcal{X}_{T,j,j-4}(i)$, we will see (cf. Proposition 5.12) that

$$|\mathcal{T}_T(i)| \approx \sum_{j=4}^{j_{max}} X_{T,j,j-4}(i). \tag{4.5}$$

**Fact 4.7.** *For $T \in \mathcal{A}(i)$, $X_{T,4,0}(i) = \sum_{e \in \binom{T}{2}} X_e(i) - 3$.*

**Proof.** For $T \in \mathcal{A}(i)$, $X_{T,4,0}(i)$ counts the number of $T^* \in \mathcal{A}(i)$ with $|T \cap T^*| = 2$. If for such $T^*$, we have $T \cap T^* = e \in \binom{T}{2}$, then $T^* \in \mathcal{X}_e(i) \setminus \{T\}$. □

For $e \in E(i)$ and $T \in \mathcal{X}_e(i)$, we say that $T^* \in \mathcal{A}(i)$ *threatens* $T, e$ if $e \not\subseteq T^*$ and $T \leftrightarrow T^*$. This means that if $T^*$ is the selected triple $T^*(i)$, then $T \notin \mathcal{A}(i+1)$, but still $e \in E(i+1)$. Let $th_{T,e}(i)$ be the number of threats to $T, e$.

**Proposition 4.8.** *For $e \in E(i)$ and $T \in \mathcal{X}_e(i)$, we have $th_{T,e}(i) = |\mathcal{T}_T(i)| - X_e(i) + 1$.*

**Proof.** We have $th_{T,e}(i) = |\mathcal{T}_T(i)| - |\{T^* \in \mathcal{T}_T(i) \,:\, e \subseteq T^*\}|$. Since for $j \geq 5$ and $T^* \in \mathcal{T}_T(i)$ with $e \subseteq T^*$, there is no Erdős-configuration on $j$ points which contains $T$ and $T^*$, we have $\{T^* \in \mathcal{T}_T(i) \,:\, e \subseteq T^*\} = \{T^* \in \mathcal{A}(i) \setminus \{T\} \,:\, e \subseteq T^*\} = \mathcal{X}_e(i) \setminus \{T\}$. □

Together with (4.5) and Fact 4.7, we have that

$$th_{T,e}(i) \approx \sum_{e' \in \binom{T}{2} \setminus \{e\}} X_{e'}(i) + \sum_{j=6}^{j_{max}} X_{T,j,j-4}(i). \tag{4.6}$$

For $T \in \mathcal{A}(i)$ and $\mathcal{S} \in \mathcal{X}_{T,j,c}(i)$, we say that $T^* \in \mathcal{A}(i)$ *threatens* $\mathcal{S}, T$ if $T \not\leftrightarrow T^*$ and $T \neq T^*$ and there is $T' \in (\mathcal{S} - \{T\}) \cap \mathcal{A}(i)$ with $T' \leftrightarrow T^*$ or $T' = T^*$. (Note that if $c = j - 4$, then the case $T' = T^*$ cannot happen as this would imply $T \leftrightarrow T^*$.) This means that if $T^*$ is the selected triple $T^*(i)$, then $\mathcal{S} \notin \mathcal{X}_{T,j,c}(i+1)$, but still $T \in \mathcal{A}(i+1)$. Let $th_{\mathcal{S},T}(i)$ be the number of threats to $\mathcal{S}, T$.

By showing that, for fixed $\mathcal{S}, T$, most $T^*$ exclude only one $T' \in (\mathcal{S} - \{T\}) \cap \mathcal{A}(i)$, we will see (cf. Proposition 5.14) that

$$th_{\mathcal{S},T}(i) \approx \sum_{T' \in (\mathcal{S} - \{T\}) \cap \mathcal{A}(i)} |\mathcal{T}_{T'}(i)|. \tag{4.7}$$

4.2. **Heuristics.** We now use the differential equation method to predict the behaviour of the process. This is only a heuristic argument and not a part of the formal proof, yet should provide motivation for our choice of the trajectories $f_X$. As part of the exposition, we define some key functions which will play a crucial role in the remainder of the paper.

We make the assumptions that for all $e \in E(i)$, we have $X_e(i) \approx f_{edge}(i)$ for some function $f_{edge}$. Similarly, for all $T \in \mathcal{A}(i)$ and $j \in \{6, \dots, j_{max}\}$, we have $X_{T,j,j-4}(i) \approx f_{j,j-4}(i)$ for some function $f_{j,j-4}$. In order to use the differential equation method, we interpret the term $\frac{\mathrm{d}f_{edge}(i)}{\mathrm{d}i}$ as the expectation of $\Delta X_e(i)$. Note that $f_{edge}(i)$ approximates $X_e(i)$ only for uncovered edges $e$, whereas for all covered edges $e$ we have $X_e(i) = 0$. Thus, it is important to consider the conditional expectation of $\Delta X_e(i)$ under the event that $e$ remains uncovered. In this conditional probability space, $T^*(i)$ is chosen uniformly from $\mathcal{A}(i) \setminus \mathcal{X}_e(i)$, and for a fixed triple $T \in \mathcal{X}_e(i)$, we have $T \notin \mathcal{X}_e(i+1)$ if and only if $T^*(i)$ threatens $T, e$. Thus, the (conditional) probability that $T$ becomes unavailable is given by $\frac{th_{T,e}(i)}{|\mathcal{A}(i)\setminus\mathcal{X}_e(i)|}$. For brevity, define

$$A(i) = \frac{1}{3}|E(i)|f_{edge}(i) \overset{(4.1)}{=} \frac{1}{3}p(i)\binom{n}{2}f_{edge}(i), \tag{4.8}$$

$$F(i) = \sum_{j=6}^{j_{max}} f_{j,j-4}(i). \tag{4.9}$$

Fact 4.5 indicates that $|\mathcal{A}(i) \setminus \mathcal{X}_e(i)| \approx |\mathcal{A}(i)| \approx A(i)$. From (4.6), we deduce that $th_{T,e}(i) \approx 2f_{edge}(i) + F(i)$. Thus, we approximate the conditional expectation of $\Delta X_e(i)$ as

$$-\sum_{T\in\mathcal{X}_e(i)} \frac{th_{T,e}(i)}{|\mathcal{A}(i) \setminus \mathcal{X}_e(i)|} \approx -\frac{2f_{edge}(i) + F(i)}{A(i)} f_{edge}(i).$$

We obtain the following differential equation for $f_{edge}(i)$:

$$\frac{\mathrm{d}f_{edge}(i)}{\mathrm{d}i} = -\frac{2f_{edge}(i) + F(i)}{A(i)} f_{edge}(i). \tag{4.10}$$

In order to obtain an expression for $F(i)$, we make the additional assumption that $\mathcal{A}(i)$ and $\mathcal{C}(i)$ are random 3-graphs obtained by including every triple independently with probability $p_{\mathcal{A}}$ and $p_{\mathcal{C}}$, respectively, conditioned on $\mathcal{A}(i)\cap\mathcal{C}(i) = \emptyset$. Fix a triple $T \subseteq V$. Recall that $J_j = |\mathfrak{J}_j(T)|$ denotes the number of unlabelled Erdős-configurations on $j$ points in $V$ which contain $T$ as a triple. For each $\mathcal{S} \in \mathfrak{J}_j(T)$, $\mathcal{S}$ belongs to $\mathcal{X}_{T,j,j-4}$ if and only if one triple of $\mathcal{S} - \{T\}$ is available, and the other $j-4$ are chosen. Under the above assumption, the probability for this is $(j-3)p_{\mathcal{C}}(i)^{j-4}p_{\mathcal{A}}(i)$. We thus guess that

$$f_{j,j-4}(i) = (j-3)p_{\mathcal{C}}(i)^{j-4}p_{\mathcal{A}}(i)J_j. \tag{4.11}$$

By (4.1) and (4.9), we then have

$$F(i) = \sum_{j=6}^{j_{max}} (j-3)\left(\frac{i}{\binom{n}{3}}\right)^{j-4} \frac{|\mathcal{A}(i)|}{\binom{n}{3}} J_j = |\mathcal{A}(i)| \sum_{j=6}^{j_{max}} \frac{(j-3)J_j}{\binom{n}{3}^{j-3}} i^{j-4}.$$

This motivates the definition of the following function, which turns out to be a crucial parameter of the process.

$$\rho'(i) = \sum_{j=6}^{j_{max}} \frac{(j-3)J_j}{\binom{n}{3}^{j-3}} i^{j-4}.$$

We obtain

$$\frac{2f_{edge}(i) + F(i)}{A(i)} \approx \frac{2f_{edge}(i) + A(i)\rho'(i)}{A(i)} = \frac{6}{p(i)\binom{n}{2}} + \rho'(i).$$

Substituting this into (4.10) yields the linear differential equation

$$\frac{\mathrm{d}f_{edge}(i)}{\mathrm{d}i} = -f_{edge}(i)\left(\frac{6}{p(i)\binom{n}{2}} + \rho'(i)\right).$$

For this equation, we can find the solution (e.g. using separation of variables)

$$f_{edge}(i) = \mathrm{e}^{-\rho(i)}p(i)^2 f_{edge}(0) = \mathrm{e}^{-\rho(i)}p(i)^2(n-2),$$

where

$$\rho(i) = \sum_{j=6}^{j_{max}} \frac{J_j}{\binom{n}{3}^{j-3}} i^{j-3}$$

is the integral of $\rho'$ with $\rho(0) = 0$.

We briefly interpret this result. Note that since $J_j = \Theta(n^{j-3})$ and $i = \mathcal{O}(n^2)$, we have that $\rho(i) = \mathcal{O}(1)$. Also, as long as $i = o(n^2)$, we have $\rho(i) = o(1)$, i.e. the effect of the term $\mathrm{e}^{-\rho(i)}$ is negligible. This means that in the early stages of the process, we expect $X_e$ to behave as in the standard random triangle removal process. Once $i$ is quadratic in $n$, sufficiently many dangerous configurations have been created to affect $X_e(i)$ significantly. However, their influence is limited in the sense that they modify $X_e(i)$ only by a multiplicative constant.

4.3. **Trajectories.** As a result of the heuristic argument, we conclude that we wish to track the random variables $X_e$ and $X_{T,j,j-4}$. Clearly, in order to track $X_{T,j,j-4}$, we also need to track $X_{T,j,j-5}$, and so on. A guess for the trajectory of $X_{T,j,c}$ can be obtained similarly to (4.11). We now define the trajectories for these key variables formally. For clarity, we also define the other relevant functions from above again.

**Definition 4.9** (Trajectories)**.** For $0 \le i \le \binom{n}{2}/3$, define the functions

$$\rho(i) := \sum_{j=6}^{j_{max}} \frac{J_j}{\binom{n}{3}^{j-3}} i^{j-3}, \tag{4.12}$$

$$f_{edge}(i) := \mathrm{e}^{-\rho(i)}p(i)^2(n-2), \tag{4.13}$$

$$A(i) := \mathrm{e}^{-\rho(i)}p(i)^3\binom{n}{3} \quad \left(= \frac{1}{3}p(i)\binom{n}{2}f_{edge}(i)\right). \tag{4.14}$$

Moreover, for all $j \in \{6, \dots, j_{max}\}$, define

$$f_{j,c}(i) := \binom{j-3}{c}\mathrm{e}^{-(j-3-c)\rho(i)}p(i)^{3(j-3-c)}i^c\binom{n}{3}^{-c}J_j \tag{4.15}$$

for all $c \in \{1, \dots, j-4\}$ and $f_{j,0}(i) := \mathrm{e}^{-(j-3)\rho(i)}p(i)^{3(j-3)}J_j$. Finally, define

$$F(i) := \sum_{j=6}^{j_{max}} f_{j,j-4}(i). \tag{4.16}$$

We close this section by observing the following properties of the functions defined above.

**Proposition 4.10.** *For $0 \le i \le \binom{n}{2}/3$, the following hold:*

$$\rho(i) = \mathcal{O}(1),\ \rho'(i) = \mathcal{O}(n^{-2}),\ \rho''(i) = \mathcal{O}(n^{-4}); \tag{4.17}$$

$$f_{j,c}(i) = \mathcal{O}(n^{j-3-c}) \text{ for all } j, c; \tag{4.18}$$

$$f_{edge}(i) = \mathcal{O}(n),\ F(i) = \mathcal{O}(n). \tag{4.19}$$

**Proof.** For (4.17), recall that $J_j = \Theta(n^{j-3})$. (4.18) and (4.19) then follow. □

## 5. Analysis of the process

In this section, we prove Theorem 4.4. Choose constants

$$\varepsilon_0 \ll 1/C \ll \gamma \ll 1/k. \tag{5.1}$$

In all calculations, we assume that $n$ is sufficiently large once all other constants are fixed.

5.1. **Extension types.** As mentioned before, in order to track our key variables, we also need to track a number of auxiliary variables, e.g. to account for double counting when estimating the threat functions $th_{T,e}$ and $th_{\mathcal{S},T}$. We will also need such auxiliary variables to establish boundedness conditions for our key variables. Fortunately, it suffices to have (generous) upper bounds on these variables. This allows us to treat all the auxiliary variables we need using a unified framework.

An *extension type* is a pair $(H, U)$ where $H$ is a 3-graph and $U \subseteq V(H)$ is such that $|H[U]| = 0$. We can think of $U$ as a set of *root vertices*, whereas the vertices in $V(H) \setminus U$ are *free*. Given a 3-graph $G$ and a set $R \subseteq V(G)$ with $|R| = |U|$, an $(H, U)$*-extension at* $R$ *in* $G$ is an embedding $\phi\colon H \to G$ such that $\phi(U) = R$, i.e. an injective map $\phi\colon V(H) \to V(G)$ such that $\phi(U) = R$ and $\phi(e) \in G$ for all $e \in H$. Note that if $G$ is a random 3-graph on $n$ vertices, where edges appear independently with probability $1/n$, then the expected number of $(H, U)$-extensions at a fixed set $R$ is of order $n^{|V(H)\setminus U|-|H|}$.

Let

$$m := 2j_{max}.$$

For an extension type $(H, U)$ with $|V(H)| \le m$ and a set $R \subseteq V$ with $|R| = |U|$, we define the random variable $X_{R,(H,U)}(i)$ counting the number of $(H, U)$-extensions at $R$ in $\mathcal{C}(i)$. Note that $X_{R,(H,U)}(0) = 0$ if $H$ is non-empty.

**Definition 5.1.** Call an extension type $(H, U)$ $\kappa$*-balanced* if for all $U \subseteq U' \subseteq V(H)$, we have $|H - H[U']| \ge |V(H) \setminus U'| - \kappa$. Let $\kappa(H, U) := \min\{\kappa \ge 0 \,:\, (H, U) \text{ is } \kappa\text{-balanced}\}$.

For $\ell \in \{1, \dots, m\}$ and $\kappa \in \{0, \dots, \ell\}$, let $Ext(\kappa, \ell)$ denote the set of all extension types $(H, U)$ with $|V(H)| \le m$, $\kappa(H, U) = \kappa$ and $|V(H) \setminus U| = \ell$, and such that $H$ is not empty. We do not distinguish between isomorphic extension types here. In particular, $|Ext(\kappa, \ell)| = \mathcal{O}(1)$.

We gather a few easy facts about balanced extension types.

**Fact 5.2.** *Let $(H, U)$ be an extension type, $\kappa := \kappa(H, U)$ and $\ell := |V(H)\setminus U|$. Then the following assertions hold.*

(i) $|V(H) \setminus U| - |H| \le \kappa$.
(ii) $\kappa \le \ell$.
(iii) *If $H$ is empty, then $\kappa = \ell$.*
(iv) *If $R \subseteq V$ with $|R| = |U|$, then $X_{R,(H,U)}(i) \le |U|!n^{\ell}$.*

Recall that in our process, we have $p_{\mathcal{C}}(i) = i/\binom{n}{3}$, and since $i = \mathcal{O}(n^2)$, we have $p_{\mathcal{C}}(i) = \mathcal{O}(1/n)$. Fact 5.2(i) tells us that if the triples in $\mathcal{C}(i)$ appeared independently at random, then we would have $\mathbb{E}\left(X_{R,(H,U)}(i)\right) = \mathcal{O}(n^{\kappa(H,U)})$.

In order to track $X_{R,(H,U)}(i)$ during the process, the following observations are crucial. We will use (i) to establish a trend hypothesis and (ii) to establish a boundedness hypothesis for $X_{R,(H,U)}$.

**Fact 5.3.** *Let $(H, U)$ be a $\kappa$-balanced extension type and $e \in H$. Then the following hold:*

(i) *$(H - e, U)$ is a $(\kappa + 1)$-balanced extension type.*
(ii) *$(H - H[U \cup e], U \cup e)$ is a $\kappa$-balanced extension type.*

**Proof.** (i) is obvious. For (ii), let $H_1 := H - H[U \cup e]$. Note that $(H_1, U \cup e)$ clearly is an extension type. For any $U \cup e \subseteq U' \subseteq V(H_1) = V(H)$ we have $|H - H[U']| \ge |V(H) \setminus U'| - \kappa$ since $(H, U)$ is $\kappa$-bounded. As $H_1 - H_1[U'] = H - H[U']$, we conclude that $|H_1 - H_1[U']| \ge |V(H_1) \setminus U'| - \kappa$. □

We will now make some observations as to how Erdős-configurations (and combinations thereof) can be viewed as balanced extension types.

**Fact 5.4.** *Let $\mathcal{S}$ be an Erdős-configuration with $T' \in \mathcal{S}$ and let $U \subseteq V(\mathcal{S})$ with $|U| \geq 4$. Let $\mathcal{S}^- := \mathcal{S} - \{T'\}$. Then $|\mathcal{S}^- - \mathcal{S}^-[U]| \geq |V(\mathcal{S}) \setminus U|$.*

**Proof.** Suppose not. Then $|\mathcal{S}^-[U]| > |U| + |\mathcal{S}^-| - |V(\mathcal{S})| = |U| - 3$. But this means that $\mathcal{S}^-[U]$ contains a forbidden configuration, a contradiction. $\square$

The following result will be used to establish various boundedness conditions.

**Proposition 5.5.** *Let $\mathcal{S}$ be an Erdős-configuration on $j$ points and $\mathcal{S}' \subseteq \mathcal{S}$ with $V(\mathcal{S}') = V(\mathcal{S})$ and $|\mathcal{S}'| = c$. Let $U \subseteq V(\mathcal{S})$ with $|U| \geq 4$ and $|(\mathcal{S} - \mathcal{S}')[U]| \geq a$. Define $\kappa := j - 3 - c - a$. Then $(\mathcal{S}' - \mathcal{S}'[U], U)$ is $\max\{\kappa, 0\}$-balanced.*

**Proof.** Suppose not. Then there exists $U \subseteq U' \subseteq V(\mathcal{S})$ such that $|\mathcal{S}' - \mathcal{S}'[U']| < |V(\mathcal{S}) \setminus U'| - \max\{\kappa, 0\}$. Note that this implies $U' \neq V(\mathcal{S})$ and that $|\mathcal{S}'| - |\mathcal{S}'[U']| \leq |V(\mathcal{S})| - |U'| - \kappa - 1$. Thus $|\mathcal{S}'[U']| \geq |U'| - 2 - a$. We conclude that $|\mathcal{S}[U']| \geq |\mathcal{S}'[U']| + |(\mathcal{S} - \mathcal{S}')[U]| \geq |U'| - 2$. Since $|U'| \geq 4$, $\mathcal{S}[U']$ is a forbidden configuration, a contradiction to $\mathcal{S}$ being an Erdős-configuration. $\square$

We will also need to bound the number of pairs $\mathcal{S}, \mathcal{S}'$ of Erdős-configurations appearing in some specified constellation.

The following proposition yields a 'global' edge count of two overlapping Erdős-configurations. After specifying some root set, it can be used to compare the number of edges with the number of free vertices.

**Proposition 5.6.** *Let $\mathcal{S}_1, \mathcal{S}_2$ be distinct Erdős-configurations.*

(i) *If $|V(\mathcal{S}_1) \cap V(\mathcal{S}_2)| \geq 4$, then $|\mathcal{S}_1 \cup \mathcal{S}_2| \geq |V(\mathcal{S}_1) \cup V(\mathcal{S}_2)| - 1$.*
(ii) *If $|V(\mathcal{S}_1) \cap V(\mathcal{S}_2)| = 3$, then $|\mathcal{S}_1 \cup \mathcal{S}_2| \geq |V(\mathcal{S}_1) \cup V(\mathcal{S}_2)| - 2$.*

**Proof.** (i) View $\mathcal{S}_1 \cap \mathcal{S}_2$ as a set of triples on $V(\mathcal{S}_1) \cap V(\mathcal{S}_2)$. Since $\mathcal{S}_2$ is an Erdős-configuration, we must have $|\mathcal{S}_1 \cap \mathcal{S}_2| \leq |V(\mathcal{S}_1) \cap V(\mathcal{S}_2)| - 3$. This implies that

$$|\mathcal{S}_1 \cup \mathcal{S}_2| = |\mathcal{S}_1| + |\mathcal{S}_2| - |\mathcal{S}_1 \cap \mathcal{S}_2| = |V(\mathcal{S}_1)| + |V(\mathcal{S}_2)| - 4 - |\mathcal{S}_1 \cap \mathcal{S}_2| \geq |V(\mathcal{S}_1) \cup V(\mathcal{S}_2)| - 1.$$

(ii) follows similarly by using $|\mathcal{S}_1 \cap \mathcal{S}_2| \leq 1$. $\square$

**Proposition 5.7.** *Let $\mathcal{S}_1, \mathcal{S}_2$ be distinct Erdős-configurations. Suppose that $T' \in \mathcal{S}_1 \cap \mathcal{S}_2$, $T_1 \in \mathcal{S}_1$, $T_2 \in \mathcal{S}_2$. We require $T' \neq T_1, T_2$, but allow $T_1 = T_2$. Define $H := (\mathcal{S}_1 \cup \mathcal{S}_2) - \{T_1, T_2, T'\}$ and $U := T_1 \cup T_2$. Suppose $U \subseteq U' \subseteq V(H)$ and $U' \neq U$. Then $|H - H[U']| \geq |V(H) \setminus U'|$.*

**Proof.** Define $U'_\ell := U' \cap V(\mathcal{S}_\ell)$ for $\ell \in [2]$. Suppose, for a contradiction, that $|U'_1|, |U'_2| \leq 3$. This implies $U'_1 = T_1$ and $U'_2 = T_2$ and hence $U' = U'_1 \cup U'_2 = U$, a contradiction. Now, without loss of generality, assume that $|U'_1| \geq 4$. Define $U''_2 := (U' \cup V(\mathcal{S}_1)) \cap V(\mathcal{S}_2)$. Note that $|U''_2| \geq 4$ as well since $T_2 \cup T' \subseteq U''_2$. Let $\mathcal{S}^-_\ell := \mathcal{S}_\ell - \{T'\}$. By Fact 5.4, we have $|\mathcal{S}^-_1 - \mathcal{S}^-_1[U'_1]| \geq |V(\mathcal{S}_1) \setminus U'_1|$ and $|\mathcal{S}^-_2 - \mathcal{S}^-_2[U''_2]| \geq |V(\mathcal{S}_2) \setminus U''_2|$. Observe that $\mathcal{S}^-_1 - \mathcal{S}^-_1[U'_1], \mathcal{S}^-_2 - \mathcal{S}^-_2[U''_2] \subseteq H - H[U']$ and $(\mathcal{S}^-_1 - \mathcal{S}^-_1[U'_1]) \cap (\mathcal{S}^-_2 - \mathcal{S}^-_2[U''_2]) = \emptyset$. We conclude that

$$\begin{aligned} |H - H[U']| &\geq |\mathcal{S}^-_1 - \mathcal{S}^-_1[U'_1]| + |\mathcal{S}^-_2 - \mathcal{S}^-_2[U''_2]| \\ &\geq |V(\mathcal{S}_1) \setminus U'_1| + |V(\mathcal{S}_2) \setminus U''_2| = |V(H) \setminus U'|, \end{aligned}$$

as required. $\square$

The next two propositions will be used to establish trend and boundedness hypotheses for our key variables.

**Proposition 5.8.** *Let $T_1, T_2, T'$ be distinct triples and let $\mathcal{S}_1, \mathcal{S}_2$ be distinct Erdős-configurations where at least one is not the diamond. Suppose that $T' \in \mathcal{S}_1 \cap \mathcal{S}_2$, $T_1 \in \mathcal{S}_1$, $T_2 \in \mathcal{S}_2$. Define $H := (\mathcal{S}_1 \cup \mathcal{S}_2) - \{T_1, T_2, T'\}$ and $U := T_1 \cup T_2$. Suppose that $H[U]$ is empty. Then the extension type $(H, U)$ is 0-balanced.*

**Proof.** Note that $|H| = |\mathcal{S}_1 \cup \mathcal{S}_2| - 3$ and $V(H) = V(\mathcal{S}_1) \cup V(\mathcal{S}_2)$. By Proposition 5.6, we have $|\mathcal{S}_1 \cup \mathcal{S}_2| \geq |V(\mathcal{S}_1) \cup V(\mathcal{S}_2)| - 2$, implying $|H| \geq |V(H)| - 5$. Thus, if $|U| \geq 5$, we have $|H| \geq |V(H) \setminus U|$. Suppose $|U| = 4$, i.e. $|T_1 \cap T_2| = 2$. Since either $T_1$ and $T'$ or $T_2$ and $T'$ are edge-disjoint, we have $T_1 \cap T_2 \nsubseteq T'$, implying $|V(\mathcal{S}_1) \cap V(\mathcal{S}_2)| \geq 4$. Proposition 5.6(i) implies $|\mathcal{S}_1 \cup \mathcal{S}_2| \geq |V(\mathcal{S}_1) \cup V(\mathcal{S}_2)| - 1$ and thus $|H| \geq |V(H)| - 4 = |V(H) \setminus U|$.

Now, let $U \subseteq U' \subseteq V(H)$ with $U' \neq U$. Then we can apply Proposition 5.7 to conclude that $|H - H[U']| \geq |V(H) \setminus U'|$. □

**Proposition 5.9.** *Let $\mathcal{S}_1, \mathcal{S}_2$ be distinct Erdős-configurations with distinct $T, T' \in \mathcal{S}_1 \cap \mathcal{S}_2$. Define $H := (\mathcal{S}_1 \cup \mathcal{S}_2) - \{T, T'\}$. Then $(H, T)$ is 0-balanced.*

**Proof.** Clearly, $H[T]$ is empty. Let $T \subseteq U' \subseteq V(H)$. Note that $|H| = |\mathcal{S}_1 \cup \mathcal{S}_2| - 2$ and $V(H) = V(\mathcal{S}_1) \cup V(\mathcal{S}_2)$. Since $T, T' \in \mathcal{S}_1 \cap \mathcal{S}_2$, we have $|V(\mathcal{S}_1) \cap V(\mathcal{S}_2)| \geq 4$ and thus $|\mathcal{S}_1 \cup \mathcal{S}_2| \geq |V(\mathcal{S}_1) \cup V(\mathcal{S}_2)| - 1$ by Proposition 5.6. We conclude that $|H| \geq |V(H)| - 3 = |V(H) \setminus T|$. Thus, if $U' = T$, our condition is satisfied.

Now, suppose $U' \neq T$. Then we can apply Proposition 5.7 (with $T_1 = T_2 = T$) to conclude that $|H - H[U']| \geq |V(H) \setminus U'|$. □

5.2. **Stopping and freezing times.** In order to keep track of our key variables, we define the following error function:

$$\varepsilon(i) := \left(1 + \frac{C}{n^2}\right)^i \varepsilon_0. \tag{5.2}$$

Note that we have

$$\Delta\varepsilon(i) = C\varepsilon(i)n^{-2} \text{ and } \varepsilon(i) \leq \mathrm{e}^C \varepsilon_0. \tag{5.3}$$

To control the extension types, for all $\ell \in \{1, \ldots, m\}$, $\kappa \in \{0, \ldots, \ell\}$, define

$$\varepsilon_{\kappa,\ell}(i) := n^{\kappa + \frac{\ell}{m+\kappa}}(1 + i/n^2). \tag{5.4}$$

Let $\tau_{cut} := \lfloor (1-\gamma)n^2/6 \rfloor$. During the process (at least up to time $\tau_{cut}$), we aim to show that the following hold:

- for all $e \in E(i)$, we have

$$X_e(i) = f_{edge}(i) \pm \varepsilon(i)n, \tag{5.5}$$

- for all $T \in \mathcal{A}(i)$, $j \in \{6, \ldots, j_{max}\}$ and $c \in \{0, \ldots, j-4\}$, we have

$$X_{T,j,c}(i) = f_{j,c}(i) \pm \varepsilon(i)n^{j-3-c}, \tag{5.6}$$

- for all $\ell \in \{1, \ldots, m\}$, $\kappa \in \{0, \ldots, \ell\}$, $(H, U) \in Ext(\kappa, \ell)$ and all sets $R \subseteq V$ with $|R| = |U|$, we have

$$X_{R,(H,U)}(i) \leq \varepsilon_{\kappa,\ell}(i). \tag{5.7}$$

Let $\tau_{violated}$ be the smallest $i$ such that at least one of these conditions is violated ($\tau_{violated} := \infty$ if this never happens). Let

$$\tau_{stop} := \tau_{violated} \wedge \tau_{cut}. \tag{5.8}$$

Clearly, the (random) times $\tau_{cut}, \tau_{violated}, \tau_{stop}$ are stopping times of the process (so for example whether $\tau_{violated} = i$ can be decided upon observing the process until time $i$). We now define additional 'freezing times'.

Define $\tau_{freeze,e} := \tau_{stop} \wedge (\tau_e - 1)$ and $\tau_{freeze,T} := \tau_{stop} \wedge (\tau_T - 1)$. (Recall that $\tau_e$ and $\tau_T$ were defined at the beginning of Section 4.) We note that the random times $\tau_{freeze,e}$ and $\tau_{freeze,T}$ are not stopping times of the process.

For every 2-set $e$, define

$$X_e^{\pm}(i) := \begin{cases} \pm X_e(i) \mp f_{edge}(i) - \varepsilon(i)n & \text{if } i \le \tau_{freeze,e}, \\ X_e^{\pm}(i-1) & \text{if } i > \tau_{freeze,e}. \end{cases} \tag{5.9}$$

Alternatively, we can write

$$X_e^{\pm}(i) := \pm X_e(i \wedge \tau_{freeze,e}) \mp f_{edge}(i \wedge \tau_{freeze,e}) - \varepsilon(i \wedge \tau_{freeze,e})n.$$

For every triple $T$, $j \in \{6, \dots, j_{max}\}$ and $c \in \{0, \dots, j-4\}$, define

$$X_{T,j,c}^{\pm}(i) := \begin{cases} \pm X_{T,j,c}(i) \mp f_{j,c}(i) - \varepsilon(i)n^{j-3-c} & \text{if } i \le \tau_{freeze,T}, \\ X_{T,j,c}^{\pm}(i-1) & \text{if } i > \tau_{freeze,T}. \end{cases} \tag{5.10}$$

For all $\ell \in \{1, \dots, m\}$, $\kappa \in \{0, \dots, \ell\}$, $(H,U) \in Ext(\kappa, \ell)$ and all sets $R \subseteq V$ with $|R| = |U|$, define

$$X_{R,(H,U)}^{+}(i) := \begin{cases} X_{R,(H,U)}(i) - \varepsilon_{\kappa,\ell}(i) & \text{if } i \le \tau_{stop}, \\ X_{R,(H,U)}^{+}(i-1) & \text{if } i > \tau_{stop}. \end{cases} \tag{5.11}$$

Recall from Section 4 that $\tau_{max} := \min\{i : \mathcal{A}(i) = \emptyset\}$.

**Fact 5.10.** *Suppose that all the variables $X^{\pm}$ defined in (5.9), (5.10), (5.11) are non-positive for all $i$. Then $\tau_{max} \ge \tau_{cut}$.*

**Proof.** Clearly, if $E(\tau_{max}) = \emptyset$, then $\tau_{max} = \lfloor \binom{n}{2}/3 \rfloor$. Thus, we can assume that there exists $e^* \in E(\tau_{max})$.

We first claim that $\tau_{violated} \ge \tau_{cut}$. Suppose for a contradiction that this is not the case. Say, for example, that condition (5.5) for $e$ is violated at time $\tau_{violated}$. In particular, we have $\tau_{violated} \le \tau_e - 1$. We conclude that $\tau_{violated} = \tau_{freeze,e}$. However, $X_e^{+}(\tau_{violated}) > 0$ or $X_e^{-}(\tau_{violated}) > 0$ since (5.5) is violated for $e$ at time $\tau_{violated}$, a contradiction to our assumption. The argument for the case when (5.6) or (5.7) is violated is similar.

We deduce that $\tau_{stop} = \tau_{cut}$ and hence $\tau_{freeze,e^*} = \tau_{cut}$. From $X_{e^*}^{-}(\tau_{cut}) \le 0$, we infer that $|\mathcal{X}_{e^*}(\tau_{cut})| = X_{e^*}(\tau_{cut}) \ge f_{edge}(\tau_{cut}) - \varepsilon(\tau_{cut})n > 0$, where the last inequality follows from (4.1), (4.13), (4.17) and (5.3). In particular, $\mathcal{A}(\tau_{cut}) \ne \emptyset$ and hence $\tau_{max} \ge \tau_{cut}$. □

The following lemma thus implies Theorem 4.4.

**Lemma 5.11.** *Whp, all the variables $X^{\pm}$ defined in (5.9), (5.10), (5.11) are non-positive for all $i$.*

We first remark that all these variables are negative at the start. Since $\mathcal{A}(0) = \binom{V}{3}$ and $\mathcal{C}(0) = \emptyset$, we observe that for every 2-set $e$, we have $X_e(0) = n - 2 = f_{edge}(0)$, and for every triple $T$, we have $X_{T,j,c}(0) = 0 = f_{j,c}(0)$ if $c > 0$ and $X_{T,j,0}(0) = J_j = f_{j,0}(0)$. Moreover, for every extension type $(H,U)$ with $H$ not being empty, we have $X_{R,(H,U)}(0) = 0$. Hence, by (5.9)–(5.11), the following initial conditions hold for our variables $X^{\pm}$:

$$X_e^{\pm}(0) = -\varepsilon_0 n; \tag{5.12}$$

$$X_{T,j,c}^{\pm}(0) = -\varepsilon_0 n^{j-3-c}; \tag{5.13}$$

$$X_{R,(H,U)}^{+}(0) = -n^{\kappa+\frac{\ell}{m+\kappa}}. \tag{5.14}$$

Our strategy to prove Lemma 5.11 is as follows: In the next subsection, we show that each such variable $X^{\pm}$ induces a supermartingale. We then establish some additional boundedness conditions which we need to finally apply Freedman's inequality to prove Lemma 5.11.

5.3. **Counting double configurations.** For a triple $T$, we also define the variable $X_{T,double}(i)$ which counts the number of pairs $\mathcal{S},\mathcal{S}' \in \bigcup_{j=4}^{j_{max}} \mathcal{X}_{T,j,j-4}(i)$ such that $\mathcal{S} \neq \mathcal{S}'$ and $(\mathcal{S} - \{T\}) \cap \mathcal{A}(i) = (\mathcal{S}' - \{T\}) \cap \mathcal{A}(i)$. Recall that $\mathcal{T}_T(i)$ was defined in the beginning of Section 4.1.

**Proposition 5.12.** *For all $i$ and $T \in \mathcal{A}(i)$, $0 \leq \sum_{j=4}^{j_{max}} X_{T,j,j-4}(i) - |\mathcal{T}_T(i)| \leq 2X_{T,double}(i)$.*

**Proof.** For $T^* \in \mathcal{T}_T(i)$, let $z_{T^*}$ denote the number of $\mathcal{S} \in \bigcup_{j=4}^{j_{max}} \mathcal{X}_{T,j,j-4}(i)$ with $(\mathcal{S} - \{T\}) \cap \mathcal{A}(i) = \{T^*\}$. Thus, by Fact 4.6, we have $\sum_{j=4}^{j_{max}} X_{T,j,j-4}(i) = \sum_{T^* \in \mathcal{T}_T(i)} z_{T^*} \geq |\mathcal{T}_T(i)|$, which establishes the first inequality. Crucially, we have $\sum_{T^* \in \mathcal{T}_T(i)} \binom{z_{T^*}}{2} = X_{T,double}(i)$, implying that $\sum_{T^* \in \mathcal{T}_T(i)\colon z_{T^*}>1} z_{T^*} \leq 2X_{T,double}(i)$. Thus,

$$\begin{aligned}|\mathcal{T}_T(i)| \geq |\{T^* \in \mathcal{T}_T(i) \,:\, z_{T^*} = 1\}| &= \sum_{T^* \in \mathcal{T}_T(i)} z_{T^*} - \sum_{T^* \in \mathcal{T}_T(i)\colon z_{T^*}>1} z_{T^*} \\ &\geq \sum_{j=4}^{j_{max}} X_{T,j,j-4}(i) - 2X_{T,double}(i).\end{aligned}$$

□

The following is an immediate consequence of Proposition 5.9.

**Corollary 5.13.** *For all triples $T$ and all times $i < \tau_{stop}$, $X_{T,double}(i) = \mathcal{O}(n^{1-\frac{1}{m}})$.*

**Proof.** Suppose the pair $\mathcal{S},\mathcal{S}'$ is counted by $X_{T,double}(i)$. Let $T' \in \mathcal{A}(i)$ be the unique available triple in $\mathcal{S} - \{T\}$ and $\mathcal{S}' - \{T\}$. Let $H := (\mathcal{S} \cup \mathcal{S}') - \{T, T'\}$. Note that $H \subseteq \mathcal{C}(i)$. By Proposition 5.9, the extension type $(H,T)$ is 0-balanced. Thus, $(H,T) \in Ext(0,\ell)$ with $\ell := |V(H) \setminus T| \leq m-1$ (with room to spare).

Hence, we have

$$X_{T,double}(i) = \mathcal{O}\left(\sum_{\ell=0}^{m-1} \sum_{(H',T) \in Ext(0,\ell)} X_{T,(H',T)}(i)\right).$$

By (5.7) and (5.4), we conclude that $X_{T,double}(i) = \mathcal{O}(n^{\frac{m-1}{m}})$. □

For distinct triples $T_1, T_2$, we let $\mathcal{X}_{T_1,T_2}(i)$ be the set of all pairs $\mathcal{S}_1 \neq \mathcal{S}_2$, not both diamonds, such that for each $\ell \in [2]$, $\mathcal{S}_\ell \in \mathcal{X}_{T_\ell,j_\ell,j_\ell-4}(i)$ with $4 \leq j_\ell \leq j_{max}$, and such that $(\mathcal{S}_1 - \{T_1\}) \cap \mathcal{A}(i) = (\mathcal{S}_2 - \{T_2\}) \cap \mathcal{A}(i)$. We let $X_{T_1,T_2}(i) := |\mathcal{X}_{T_1,T_2}(i)|$.

Recall that $th_{\mathcal{S},T}(i)$ was defined after (4.6).

**Proposition 5.14.** *For all $i$, all $T \in \mathcal{A}(i)$, all $j \in \{6,\ldots,j_{max}\}$, all $c \in \{0,\ldots,j-4\}$ and all $\mathcal{S} \in \mathcal{X}_{T,j,c}(i)$,*

$$\left| th_{\mathcal{S},T}(i) - \sum_{T' \in (\mathcal{S}-\{T\}) \cap \mathcal{A}(i)} |\mathcal{T}_{T'}(i)| \right| \leq \mathcal{O}(1) + \sum_{T' \neq T'' \in \mathcal{S} \cap \mathcal{A}(i)} X_{T',T''}(i).$$

**Proof.** We have

$$th_{\mathcal{S},T}(i) = \left| \left( \bigcup_{T' \in (\mathcal{S}-\{T\}) \cap \mathcal{A}(i)} (\{T'\} \cup \mathcal{T}_{T'}(i)) \right) \setminus (\{T\} \cup \mathcal{T}_T(i)) \right|,$$

from which we immediately have $th_{\mathcal{S},T}(i) \le j + \sum_{T' \in (\mathcal{S}-\{T\}) \cap \mathcal{A}(i)} |\mathcal{T}_{T'}(i)|$. Moreover, we have

$$th_{\mathcal{S},T}(i) \ge \sum_{T' \in (\mathcal{S}-\{T\}) \cap \mathcal{A}(i)} |\mathcal{T}_{T'}(i)| - \sum_{T' \neq T'' \in \mathcal{S} \cap \mathcal{A}(i)} |\mathcal{T}_{T'}(i) \cap \mathcal{T}_{T''}(i)| - 1.$$

We claim that $|\mathcal{T}_{T'}(i) \cap \mathcal{T}_{T''}(i)| \le X_{T',T''}(i) + \mathcal{O}(1)$, which completes the proof. Let $T^* \in \mathcal{T}_{T'}(i) \cap \mathcal{T}_{T''}(i)$. By Fact 4.6, there are $\mathcal{S}' \in \mathcal{X}_{T',j',j'-4}(i)$ and $\mathcal{S}'' \in \mathcal{X}_{T'',j'',j''-4}(i)$ with $4 \le j', j'' \le j_{max}$ and such that $(\mathcal{S}' - \{T'\}) \cap \mathcal{A}(i) = \{T^*\} = (\mathcal{S}'' - \{T''\}) \cap \mathcal{A}(i)$. Clearly, we have $\mathcal{S}' \neq \mathcal{S}''$. Thus, unless $j' = j'' = 4$, this pair $\mathcal{S}', \mathcal{S}''$ is counted by $X_{T',T''}(i)$. Finally, if both $\mathcal{S}', \mathcal{S}''$ are diamonds, then since $T'$ and $T''$ are edge-disjoint, we must have $T^* \subseteq T' \cup T''$, for which there are only $\mathcal{O}(1)$ possibilities. □

The following fact will be useful in Section 5.5 to bound the negative change of $X_{T,j,j-4}(i)$.

**Fact 5.15.** *Let $j \in \{6, \ldots, j_{max}\}$ and $T, T^* \in \mathcal{A}(i)$ be distinct. Then*

$$|\{\mathcal{S} \in \mathcal{X}_{T,j,j-4}(i) \,:\, T^* \text{ threatens } \mathcal{S}, T\}| \le X_{T,T^*}(i).$$

**Proof.** Let $\mathcal{S} \in \mathcal{X}_{T,j,j-4}(i)$ and assume that $T^*$ threatens $\mathcal{S}, T$. Let $\{T'\} = (\mathcal{S} - \{T\}) \cap \mathcal{A}(i)$. We cannot have $T^* = T'$ as this would mean $T^* \leftrightarrow T$. Hence, by Fact 4.6, there is $\mathcal{S}' \in \mathcal{X}_{T^*,j',j'-4}(i)$ with $4 \le j' \le j_{max}$ such that $(\mathcal{S}' - \{T^*\}) \cap \mathcal{A}(i) = \{T'\}$. Since $T \neq T^*$, the pair $\mathcal{S}, \mathcal{S}'$ is counted by $X_{T,T^*}(i)$. □

**Corollary 5.16.** *For $i < \tau_{stop}$ and distinct $T_1, T_2 \in \mathcal{A}(i)$, we have $X_{T_1,T_2}(i) = \mathcal{O}(n^{1-\frac{1}{m}})$.*

**Proof.** Let $U := T_1 \cup T_2$. Suppose the pair $\mathcal{S}_1, \mathcal{S}_2$ is counted by $X_{T_1,T_2}(i)$. Let $T'$ be the unique available triple in $\mathcal{S}_1 - \{T_1\}$ and $\mathcal{S}_2 - \{T_2\}$. Let $H := (\mathcal{S}_1 \cup \mathcal{S}_2) - \{T_1, T_2, T'\}$. Note that $H \subseteq \mathcal{C}(i)$. In particular, since $T_1, T_2$ are still available, we have that $H[U]$ is empty. By Proposition 5.8, the extension type $(H, U)$ is 0-balanced. Thus, $(H, U) \in Ext(0, \ell)$ with $\ell := |V(H) \setminus U| \le m - 1$ (with room to spare).

Hence, we have

$$X_{T_1,T_2}(i) = \mathcal{O}\left( \sum_{\ell=0}^{m-1} \sum_{(H',U) \in Ext(0,\ell)} X_{U,(H',U)}(i) \right).$$

By (5.7) and (5.4), we conclude that $X_{T_1,T_2}(i) = \mathcal{O}(n^{\frac{m-1}{m}})$. □

We will also need the following consequence of Proposition 5.5.

**Corollary 5.17.** *Let $R \subseteq V$ with $|R| \ge 4$, $j \in \{6, \ldots, j_{max}\}$ and $c \in \{0, \ldots, j-4\}$. For $i < \tau_{stop}$, the number of Erdős-configurations $\mathcal{S}$ with $R \subseteq V(\mathcal{S})$, $|V(\mathcal{S})| = j$, $|\mathcal{S} \cap \mathcal{C}(i)| = c$ and $|(\mathcal{S} - \mathcal{C}(i))[R]| \ge a$, is $\mathcal{O}(n^{\max\{j-3-c-a,0\}+\frac{1}{2}})$.*

**Proof.** Let $\mathcal{S}$ be an Erdős-configuration with $R \subseteq V(\mathcal{S})$, $|V(\mathcal{S})| = j$, $|\mathcal{S} \cap \mathcal{C}(i)| = c$ and $|(\mathcal{S} - \mathcal{C}(i))[R]| \ge a$. Let $\mathcal{S}' := \mathcal{S} \cap \mathcal{C}(i)$ and $\kappa := \max\{j - 3 - c - a, 0\}$. By Proposition 5.5, we know that $(\mathcal{S}' - \mathcal{S}'[R], R)$ is $\kappa$-balanced. Thus, unless $\mathcal{S}' - \mathcal{S}'[R]$ is empty, we have $(\mathcal{S}' - \mathcal{S}'[R], R) \in Ext(\kappa', \ell)$ for some $0 \le \kappa' \le \kappa$ and $\ell := j - |R| \le j_{max} = m/2$. Therefore, the number of such $\mathcal{S}$ is $\mathcal{O}(n^{\kappa + \frac{j_{max}}{m}})$ by (5.7) and (5.4). In case $\mathcal{S}' - \mathcal{S}'[R]$ is empty, we have that the number of free vertices is $j - |R| \le \kappa$ by Fact 5.2(iii), and thus obtain the trivial upper bound $\mathcal{O}(n^\kappa)$. □

5.4. **Trend hypotheses.** Our goal is now to show that the variables $X^{\pm}$ form supermartingales. For $i \geq 0$, define the random variable

$$L(i) := (T^*(0), T^*(1), \dots, T^*((i-1) \wedge (\tau_{max} - 1)))$$

which lists the sequence of chosen triples until time $i$. Thus, $L(i)$ contains all the information about the process until time $i$. Let $\mathcal{L}(i)$ denote the set of all possible outcomes of $L(i)$. Moreover, let $\mathcal{L}^*(i)$ denote the set of all $\tilde{L} \in \mathcal{L}(i)$ for which $\tau_{stop} > i$.

Now, let $X$ be one of our variables $X^{\pm}$ defined in (5.9)–(5.11). We will show that $(X(0), X(1), \dots)$ is a supermartingale with respect to (the filtration induced by) $(L(0), L(1), \dots)$. Thus, we need to show that for all $i \geq 0$ and all $\tilde{L} \in \mathcal{L}(i)$, we have

$$\mathbb{E}\left(\Delta X(i) \mid L(i) = \tilde{L}\right) \leq 0.$$

Recall that $X$ comes with a (random) 'freezing' time $\tau$ (e.g. $\tau_{freeze,e}$, $\tau_{freeze,T}$, $\tau_{stop}$), of which we know that

$$\tau \leq \tau_{stop} \text{ and } \Delta X(i) = 0 \text{ for all } i \geq \tau. \tag{5.15}$$

Consider $i \geq 0$ and $\tilde{L} \in \mathcal{L}(i)$. We may transition to the probability space $\mathbb{P}_{\tilde{L}}$ obtained by conditioning on the event $L(i) = \tilde{L}$. Thus, we need to show that $\mathbb{E}_{\tilde{L}}(\Delta X(i)) \leq 0$. If $\mathbb{P}_{\tilde{L}}(i < \tau) = 0$, then trivially $\mathbb{E}_{\tilde{L}}(\Delta X(i)) = 0$ by (5.15). Note that if $\tilde{L} \in \mathcal{L}(i) \setminus \mathcal{L}^*(i)$, then we have $\mathbb{P}_{\tilde{L}}(i < \tau) = 0$ by (5.15). If $\tilde{L} \in \mathcal{L}^*(i)$ and $\mathbb{P}_{\tilde{L}}(i < \tau) > 0$, then by the law of total expectation, we obtain

$$\mathbb{E}_{\tilde{L}}(\Delta X(i)) \;=\; \mathbb{E}_{\tilde{L}}(\Delta X(i) \mid i < \tau)\, \mathbb{P}_{\tilde{L}}(i < \tau) + \mathbb{E}_{\tilde{L}}(\Delta X(i) \mid i \geq \tau)\, \mathbb{P}_{\tilde{L}}(i \geq \tau),$$

where the second summand trivially vanishes, again by (5.15).

To summarise, in order to show that $(X(0), X(1), \dots)$ is a supermartingale, it suffices to show that

$$\mathbb{E}_{\tilde{L}}(\Delta X(i) \mid i < \tau) \leq 0 \tag{5.16}$$

for all $i \geq 0$ and all $\tilde{L} \in \mathcal{L}^*(i)$ with $\mathbb{P}_{\tilde{L}}(i < \tau) > 0$.

Similarly, in order to show that $\mathbb{E}(|\Delta X(i)| \mid L(i)) \leq K$, it suffices to show that

$$\mathbb{E}_{\tilde{L}}(|\Delta X(i)| \mid i < \tau) \leq K \tag{5.17}$$

for all $i \geq 0$ and all $\tilde{L} \in \mathcal{L}^*(i)$ with $\mathbb{P}_{\tilde{L}}(i < \tau) > 0$.

**Lemma 5.18.** *The following hold:*

(i) *For every 2-set $e$, $(X_e^+(0), X_e^+(1), \dots)$ and $(X_e^-(0), X_e^-(1), \dots)$ are supermartingales with respect to $(L(0), L(1), \dots)$. Moreover,*

$$\mathbb{E}\left(|\Delta X_e^{\pm}(i)| \mid L(i)\right) = \mathcal{O}_\gamma(n^{-1})$$

*for all $i$.*

(ii) *For every triple $T$, all $j \in \{6, \dots, j_{max}\}$ and $c \in \{0, \dots, j-4\}$, $(X_{T,j,c}^+(0), X_{T,j,c}^+(1), \dots)$ and $(X_{T,j,c}^-(0), X_{T,j,c}^-(1), \dots)$ are supermartingales with respect to $(L(0), L(1), \dots)$. Moreover,*

$$\mathbb{E}\left(|\Delta X_{T,j,c}^{\pm}(i)| \mid L(i)\right) = \mathcal{O}_\gamma(n^{j-5-c})$$

*for all $i$.*

(iii) *For all $\ell \in \{1, \dots, m\}$, $\kappa \in \{0, \dots, \ell\}$, $(H, U) \in Ext(\kappa, \ell)$ and all sets $R \subseteq V$ with $|R| = |U|$, $(X_{R,(H,U)}^+(0), X_{R,(H,U)}^+(1), \dots)$ is a supermartingale with respect to $(L(0), L(1), \dots)$. Moreover,*

$$\mathbb{E}\left(|\Delta X_{R,(H,U)}^+(i)| \mid L(i)\right) \leq 2n^{\kappa - 2 + \frac{\ell}{m+\kappa}}$$

*for all $i$.*

We will prove this lemma at the end of this subsection. To continue, we need to gain control over $|\mathcal{A}(i)|$, $th_{T,e}(i)$ and $th_{\mathcal{S},T}(i)$. Recall from (4.13), (4.14) and (4.16) that

$$A(i) = \mathrm{e}^{-\rho(i)}p(i)^3\binom{n}{3} = \frac{1}{3}p(i)\binom{n}{2}f_{edge}(i),$$
$$F(i) = \sum_{j=6}^{j_{max}} f_{j,j-4}(i).$$

Note that if $i \leq \tau_{cut}$, then

$$p(i) = \Omega_\gamma(1),\ A(i) = \Omega_\gamma(n^3), \tag{5.18}$$

where we use (4.17) to deduce the latter from the first.

**Lemma 5.19.** *Let $i < \tau_{stop}$. Then the following hold:*

(i) $|\mathcal{A}(i)| = A(i) \pm \varepsilon(i)n^3$.
(ii) *For all $T \in \mathcal{A}(i)$, $|\mathcal{T}_T(i)| = \mathcal{O}(n)$.*
(iii) *For all $e \in E(i)$ and $T \in \mathcal{X}_e(i)$,*
$$th_{T,e}(i) = 2f_{edge}(i) + F(i) + \mathcal{O}(\varepsilon(i))n.$$
(iv) *For all $T \in \mathcal{A}(i)$, all $j \in \{6, \dots, j_{max}\}$, $c \in \{0, \dots, j-4\}$ and $\mathcal{S} \in \mathcal{X}_{T,j,c}(i)$,*
$$th_{\mathcal{S},T}(i) = (j-3-c)(3f_{edge}(i) + F(i)) + \mathcal{O}(\varepsilon(i))n.$$

**Proof.** By (5.8), $i < \tau_{violated}$, so we can make use of (5.5), (5.6) and (5.7). Using Fact 4.5, we can deduce that

$$|\mathcal{A}(i)| = \frac{1}{3}\sum_{e\in E(i)} X_e(i) \overset{(4.1),(5.5)}{=} \frac{1}{3}p(i)\binom{n}{2}(f_{edge}(i) \pm \varepsilon(i)n) = A(i) \pm \varepsilon(i)n^3,$$

i.e. (i) holds.

Moreover, from Fact 4.7, (5.5), (5.6), and (4.16), we obtain that for all $T \in \mathcal{A}(i)$ we have

$$\sum_{j=4}^{j_{max}} X_{T,j,j-4}(i) = 3f_{edge}(i) + F(i) + \mathcal{O}(\varepsilon(i))n.$$

From Corollary 5.13, we infer that $X_{T,double}(i) \leq \varepsilon(i)n$ for all triples $T$. With Proposition 5.12 and the above, we have

$$|\mathcal{T}_T(i)| = 3f_{edge}(i) + F(i) + \mathcal{O}(\varepsilon(i))n. \tag{5.19}$$

for all $T \in \mathcal{A}(i)$, so (ii) holds by (4.19).

By (5.19), Proposition 4.8 and (5.5), for all $e \in E(i)$ and $T \in \mathcal{X}_e(i)$ we have

$$th_{T,e}(i) = 2f_{edge}(i) + F(i) + \mathcal{O}(\varepsilon(i))n,$$

i.e. (iii) holds.

Finally, let $T \in \mathcal{A}(i)$, $j \in \{6, \dots, j_{max}\}$, $c \in \{0, \dots, j-4\}$ and $\mathcal{S} \in \mathcal{X}_{T,j,c}(i)$. From Proposition 5.14 and Corollary 5.16, we deduce that

$$th_{\mathcal{S},T}(i) = \sum_{T'\in(\mathcal{S}-\{T\})\cap\mathcal{A}(i)} |\mathcal{T}_{T'}(i)| + \mathcal{O}(n^{1-\frac{1}{m}}) \overset{(5.19)}{=} (j-3-c)(3f_{edge}(i) + F(i)) + \mathcal{O}(\varepsilon(i))n.$$

Thus (iv) holds too. $\square$

In order to compare the expectation of $\Delta X$ with its trajectory, we collect some important properties of the relevant trajectories in the following lemma.

**Lemma 5.20.** *The following hold for* $0 \le i \le \tau_{cut}$*:*

(i) $f'_{edge}(i) = -\frac{(2f_{edge}(i)+F(i))f_{edge}(i)}{A(i)}$.

(ii) *For all* $j \in \{6, \dots, j_{max}\}$ *and* $c \in \{1, \dots, j-4\}$,
$$f'_{j,c}(i) = \frac{-(j-3-c)(3f_{edge}(i)+F(i))f_{j,c}(i) + (j-2-c)f_{j,c-1}(i)}{A(i)}.$$

(iii) *For all* $j \in \{6, \dots, j_{max}\}$, $f'_{j,0}(i) = \frac{-(j-3)(3f_{edge}(i)+F(i))f_{j,0}(i)}{A(i)}$.

(iv) $f'_{edge}(i) = \mathcal{O}_\gamma(n^{-1})$ *and* $f''_{edge}(i) = \mathcal{O}_\gamma(n^{-3})$.

(v) *For all* $j \in \{6, \dots, j_{max}\}$ *and* $c \in \{0, \dots, j-4\}$, $f'_{j,c}(i) = \mathcal{O}_\gamma(n^{j-5-c})$ *and* $f''_{j,c}(i) = \mathcal{O}_\gamma(n^{j-7-c})$.

(vi) $\Delta f_{edge}(i) = f'_{edge}(i) + \mathcal{O}_\gamma(n^{-3}) = \mathcal{O}_\gamma(n^{-1})$.

(vii) *For all* $j \in \{6, \dots, j_{max}\}$ *and* $c \in \{0, \dots, j-4\}$, $\Delta f_{j,c}(i) = f'_{j,c}(i) + \mathcal{O}_\gamma(n^{j-7-c}) = \mathcal{O}_\gamma(n^{j-5-c})$.

**Proof.** First, we observe the following key identities:

$$\frac{F(i)}{A(i)} = \frac{\sum_{j=6}^{j_{max}} f_{j,j-4}(i)}{e^{-\rho(i)}p(i)^3\binom{n}{3}} \overset{(4.15)}{=} \sum_{j=6}^{j_{max}} \frac{(j-3)J_j}{\binom{n}{3}^{j-3}} i^{j-4} \overset{(4.12)}{=} \rho'(i), \tag{5.20}$$

$$\frac{f_{edge}(i)}{A(i)} \overset{(4.14)}{=} \frac{6}{p(i)n(n-1)} \overset{(4.2)}{=} -\frac{p'(i)}{p(i)}, \tag{5.21}$$

and for all $j \in \{6, \dots, j_{max}\}$ and $c \in \{1, \dots, j-4\}$,

$$\frac{f_{j,c-1}(i)}{f_{j,c}(i)} \overset{(4.15)}{=} \frac{\binom{j-3}{c-1}}{\binom{j-3}{c}} e^{-\rho(i)}p(i)^3 i^{-1}\binom{n}{3} = \frac{c}{(j-2-c)i}A(i). \tag{5.22}$$

Using the chain rule, we can now easily check that

$$\begin{aligned} f'_{edge}(i) &= -\rho'(i)f_{edge}(i) + 2\frac{p'(i)}{p(i)}f_{edge}(i) \overset{(5.20),(5.21)}{=} -\frac{(2f_{edge}(i)+F(i))f_{edge}(i)}{A(i)}, \\ f'_{j,c}(i) &= -(j-3-c)\rho'(i)f_{j,c}(i) + 3(j-3-c)\frac{p'(i)}{p(i)}f_{j,c}(i) + \frac{c}{i}f_{j,c}(i) \\ &\overset{(5.20),(5.21)}{=} \frac{-(j-3-c)(3f_{edge}(i)+F(i))f_{j,c}(i)}{A(i)} + \frac{c}{i}f_{j,c}(i), \end{aligned}$$

where the last summand vanishes if $c = 0$ and can be replaced with $\frac{(j-2-c)f_{j,c-1}(i)}{A(i)}$ otherwise by (5.22). Hence, (i), (ii) and (iii) hold.

We continue with computing the second derivatives. Note that $\left(\frac{p'(i)}{p(i)}\right)' = -\frac{p'(i)^2}{p(i)^2}$. Therefore,

$$\begin{aligned} f''_{edge}(i) &= -\rho''(i)f_{edge}(i) - \rho'(i)f'_{edge}(i) - 2\frac{p'(i)^2}{p(i)^2}f_{edge}(i) + 2\frac{p'(i)}{p(i)}f'_{edge}(i), \\ f''_{j,c}(i) &= -(j-3-c)\left(f'_{j,c}(i)(-3\frac{p'(i)}{p(i)} + \rho'(i)) + f_{j,c}(i)(3\frac{p'(i)^2}{p(i)^2} + \rho''(i))\right) \\ &\quad + (j-2-c)\frac{f'_{j,c-1}(i)A(i) - f_{j,c-1}A'(i)}{A(i)^2}, \end{aligned}$$

where the last summand is not present if $c = 0$. We clearly have $p'(i) = \mathcal{O}(n^{-2})$. Moreover, for the specified range of $i$, we have $\rho'(i) = \mathcal{O}(n^{-2})$ and $\rho''(i) = \mathcal{O}(n^{-4})$ by (4.17) and, crucially, $p(i) = \Omega_\gamma(1)$ and $A(i) = \Omega_\gamma(n^3)$ by (5.18). This also implies that $A'(i) = -\rho'(i)A(i) + 3\frac{p'(i)}{p(i)}A(i) = \mathcal{O}_\gamma(n)$.

Together with (4.19), we can infer that $f'_{edge}(i) = \mathcal{O}_\gamma(n^{-1})$ and $f'_{j,c}(i) = \mathcal{O}_\gamma(n^{j-5-c})$ and can conclude that $f''_{edge}(i) = \mathcal{O}_\gamma(n^{-3})$ and $f''_{j,c}(i) = \mathcal{O}_\gamma(n^{j-7-c})$. Thus, (iv) and (v) hold as well.

Finally, (vi) and (vii) follow from the previous and (3.2). □

We are now in a position to show that the variables $X^\pm$ indeed form supermartingales.

**Proof of Lemma 5.18.** Consider any $i \geq 0$ and any $\tilde{L} \in \mathcal{L}^*(i)$. We consider the probability space $\mathbb{P}_{\tilde{L}}$. The values of all (random) variables at time $i$ are now determined by $\tilde{L}$. Recall that by definition of $\mathcal{L}^*(i)$, we have $i < \tau_{stop}$. Hence, by (5.18), we have that $p(i) = \Omega_\gamma(1)$ and $A(i) = \Omega_\gamma(n^3)$.

*Step 1: The expected change of $X_e$*

Consider a 2-set $e$. By the observation at (5.16), we may assume that $\mathbb{P}_{\tilde{L}}(i < \tau_{freeze,e}) > 0$. In particular, we have $e \in E(i)$.

For every $T \in \mathcal{X}_e(i)$, the probability that $T \notin \mathcal{X}_e(i+1)$, conditioned on the event $e \in E(i+1)$, is $\frac{th_{T,e}(i)}{|\mathcal{A}(i) \setminus \mathcal{X}_e(i)|}$ (cf. Section 4.2). Thus,

$$\mathbb{E}_{\tilde{L}}\left(\Delta X_e(i) \mid i < \tau_{freeze,e}\right) = \mathbb{E}_{\tilde{L}}\left(\Delta X_e(i) \mid e \in E(i+1)\right) = -\sum_{T \in \mathcal{X}_e(i)} \frac{th_{T,e}(i)}{|\mathcal{A}(i) \setminus \mathcal{X}_e(i)|}.$$

By (5.5), we have $|\mathcal{X}_e(i)| = f_{edge}(i) \pm \varepsilon(i)n$. By Lemma 5.19(i), we have $|\mathcal{A}(i)| = A(i) \pm \varepsilon(i)n^3$ and thus $|\mathcal{A}(i) \setminus \mathcal{X}_e(i)| = A(i) \pm 2\varepsilon(i)n^3$. Moreover, by Lemma 5.19(iii), we have $th_{T,e}(i) = 2f_{edge}(i) + F(i) + \mathcal{O}(\varepsilon(i))n$ for all $T \in \mathcal{X}_e(i)$. We conclude that

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}\left(\Delta X_e(i) \mid i < \tau_{freeze,e}\right) &= -\left(f_{edge}(i) \pm \varepsilon(i)n\right)\frac{2f_{edge}(i) + F(i) + \mathcal{O}(\varepsilon(i))n}{A(i) \pm 2\varepsilon(i)n^3} \\
&\overset{(2.1),(4.19)}{=} -\frac{(2f_{edge}(i) + F(i))f_{edge}(i)}{A(i)} + \mathcal{O}_\gamma(\varepsilon(i)n^{-1}) \qquad (5.23)\\
&= f'_{edge}(i) + \mathcal{O}_\gamma(\varepsilon(i)n^{-1}) = \Delta f_{edge}(i) + \mathcal{O}_\gamma(\varepsilon(i)n^{-1}),
\end{aligned}$$

where the last two equalities are implied by Lemma 5.20(i) and (vi).

We conclude that

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}\left(\Delta X_e^\pm(i) \mid i < \tau_{freeze,e}\right) &\overset{(5.9)}{=} \pm\mathbb{E}_{\tilde{L}}\left(\Delta X_e(i) \mid i < \tau_{freeze,e}\right) \mp \Delta f_{edge}(i) \\
&\quad -\Delta\varepsilon(i)n \\
&\overset{(5.23),(5.3)}{=} \mathcal{O}_\gamma(\varepsilon(i))n^{-1} - C\varepsilon(i)n^{-1} \overset{(5.1)}{\leq} 0.
\end{aligned}$$

With the observation at (5.16), this completes the proof that $(X_e^\pm(0), X_e^\pm(1), \dots)$ is a supermartingale with respect to $(L(0), L(1), \dots)$.

Moreover, since $\Delta X_e(i) \leq 0$, we can also deduce that

$$\mathbb{E}_{\tilde{L}}\left(|\Delta X_e(i)| \mid i < \tau_{freeze,e}\right) = |\Delta f_{edge}(i)| + \mathcal{O}_\gamma(\varepsilon(i)n^{-1}) = \mathcal{O}_\gamma(n^{-1}) \qquad (5.24)$$

by Lemma 5.20(iv).

*Step 2: The expected change of $X_{T,j,c}$*

Now, consider a triple $T$, $j \in \{6, \dots, j_{max}\}$ and $c \in \{0, \dots, j-4\}$. By the observation at (5.16), we may assume that $\mathbb{P}_{\tilde{L}}(i < \tau_{freeze,T}) > 0$. In particular, we have $T \in \mathcal{A}(i)$. We split the expected change of $X_{T,j,c}(i)$ into an expected loss and an expected gain, i.e.

$$\mathbb{E}_{\tilde{L}}(\Delta X_{T,j,c}(i) \mid i < \tau_{freeze,T}) = \mathbb{E}_{\tilde{L}}(\Delta X_{T,j,c}(i) \mid T \in \mathcal{A}(i+1)) = -E^{loss} + E^{gain},$$

where $E^{loss}$ is the conditional expected size of $\mathcal{X}_{T,j,c}(i) \setminus \mathcal{X}_{T,j,c}(i+1)$ and $E^{gain}$ is the conditional expected size of $\mathcal{X}_{T,j,c}(i+1) \setminus \mathcal{X}_{T,j,c}(i)$. As we condition on the event that $i < \tau_{freeze,T}$, the chosen triple $T^*(i)$ is chosen uniformly from the available triples which do not render $T$ unavailable, i.e. $T^*(i) \notin \mathcal{T}_T(i) \cup \{T\}$.

We first consider $E^{loss}$. For every $\mathcal{S} \in \mathcal{X}_{T,j,c}(i)$, we have that the (conditional) probability that $\mathcal{S} \notin \mathcal{X}_{T,j,c}(i+1)$, is $\frac{th_{\mathcal{S},T}(i)}{|\mathcal{A}(i) \setminus (\mathcal{T}_T(i) \cup \{T\})|}$ by definition of $th_{\mathcal{S},T}(i)$. By (5.6), we have $|\mathcal{X}_{T,j,c}(i)| = f_{j,c}(i) \pm \varepsilon(i) n^{j-3-c}$. Thus, using Lemma 5.19(i),(ii) and (iv), we conclude that

$$\begin{aligned}
E^{loss} &= \sum_{\mathcal{S} \in \mathcal{X}_{T,j,c}(i)} \frac{th_{\mathcal{S},T}(i)}{|\mathcal{A}(i) \setminus (\mathcal{T}_T(i) \cup \{T\})|} \\
&= \left(f_{j,c}(i) \pm \varepsilon(i) n^{j-3-c}\right) \frac{(j-3-c)(3f_{edge}(i) + F(i)) + \mathcal{O}(\varepsilon(i))n}{A(i) \pm 2\varepsilon(i) n^3} \\
&\overset{(2.1),(4.18)}{=} \frac{(j-3-c)(3f_{edge}(i) + F(i))}{A(i)} f_{j,c}(i) + \mathcal{O}_\gamma(\varepsilon(i)) n^{j-5-c}.
\end{aligned}$$

We now consider $E^{gain}$. Observe that if $\mathcal{S} \in \mathcal{X}_{T,j,c}(i+1) \setminus \mathcal{X}_{T,j,c}(i)$, then we must have $\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i)$ and $T^*(i) \in \mathcal{S} - \{T\}$ (and in particular, $c > 0$). Hence, if $c = 0$, then $E^{gain} = 0$. Assume now that $c > 0$. Given $\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i)$, the (conditional) probability that $\mathcal{S} \in \mathcal{X}_{T,j,c}(i+1)$ is

$$\frac{|((\mathcal{S} - \{T\}) \cap \mathcal{A}(i)) \setminus \mathcal{T}_T(i)|}{|\mathcal{A}(i) \setminus (\mathcal{T}_T(i) \cup \{T\})|}.$$

We have $|(\mathcal{S} - \{T\}) \cap \mathcal{A}(i)| = j - 3 - (c-1)$ by definition of $\mathcal{X}_{T,j,c-1}(i)$ (cf. (4.4)). We claim that for most $\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i)$, we have $(\mathcal{S} - \{T\}) \cap \mathcal{A}(i) \cap \mathcal{T}_T(i) = \emptyset$. Indeed, for fixed $T' \in \mathcal{T}_T(i)$, Corollary 5.17 (applied with $T \cup T'$, 2, $c-1$ playing the roles of $R, a, c$) implies that the number of $\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i)$ with $T' \in \mathcal{S}$ is $\mathcal{O}(n^{j-3-(c-1)-2+\frac{1}{2}})$. Since $|\mathcal{T}_T(i)| = \mathcal{O}(n)$ by Lemma 5.19(ii), we conclude that the number of $\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i)$ with $(\mathcal{S} - \{T\}) \cap \mathcal{A}(i) \cap \mathcal{T}_T(i) \neq \emptyset$ is $\mathcal{O}(n^{j-3-c+\frac{1}{2}})$. From (5.6), we have that $|\mathcal{X}_{T,j,c-1}(i)| = f_{j,c-1}(i) \pm \varepsilon(i) n^{j-2-c}$. Using Lemma 5.19(i) and (ii), we conclude that

$$\begin{aligned}
E^{gain} &= \sum_{\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i)} \frac{|((\mathcal{S} - \{T\}) \cap \mathcal{A}(i)) \setminus \mathcal{T}_T(i)|}{|\mathcal{A}(i) \setminus (\mathcal{T}_T(i) \cup \{T\})|} \\
&= \left(|\mathcal{X}_{T,j,c-1}(i)| - \mathcal{O}(n^{j-3-c+\frac{1}{2}})\right) \frac{j-2-c}{|\mathcal{A}(i) \setminus (\mathcal{T}_T(i) \cup \{T\})|} \\
&= (f_{j,c-1}(i) \pm 2\varepsilon(i) n^{j-2-c}) \frac{j-2-c}{A(i) \pm 2\varepsilon(i) n^3} \\
&\overset{(2.1),(4.18)}{=} \frac{j-2-c}{A(i)} f_{j,c-1}(i) + \mathcal{O}_\gamma(\varepsilon(i)) n^{j-5-c}.
\end{aligned}$$

Thus, using Lemma 5.20(ii), (iii) and (vii), we obtain

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}(\Delta X_{T,j,c}(i) \mid i < \tau_{freeze,T}) = -E^{loss} + E^{gain} &= f'_{j,c}(i) + \mathcal{O}_\gamma(\varepsilon(i)) n^{j-5-c} \\
&= \Delta f_{j,c}(i) + \mathcal{O}_\gamma(\varepsilon(i)) n^{j-5-c}. \qquad (5.25)
\end{aligned}$$

We infer that

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}\left(\Delta X^{\pm}_{T,j,c}(i) \mid i < \tau_{freeze,T}\right) &\overset{(5.10)}{=} \pm\mathbb{E}_{\tilde{L}}\left(\Delta X_{T,j,c}(i) \mid i < \tau_{freeze,T}\right) \\
&\quad \mp\Delta f_{j,c}(i) - \Delta\varepsilon(i)n^{j-3-c} \\
&\overset{(5.25),(5.3)}{=} \mathcal{O}_{\gamma}(\varepsilon(i))n^{j-5-c} - C\varepsilon(i)n^{j-5-c} \overset{(5.1)}{\leq} 0.
\end{aligned}$$

With the observation at (5.16), this shows that $(X^{\pm}_{T,j,c}(0), X^{\pm}_{T,j,c}(1), \dots)$ is a supermartingale with respect to $(L(0), L(1), \dots)$.

Moreover, using (4.18), (4.19) and (5.18), we can also deduce that

$$\mathbb{E}_{\tilde{L}}\left(|\Delta X_{T,j,c}(i)| \mid i < \tau_{freeze,T}\right) \leq E^{loss} + E^{gain} = \mathcal{O}_{\gamma}(n^{j-5-c}). \tag{5.26}$$

*Step 3: The expected change of $X_{R,(H,U)}$*

Finally, consider $\ell \in \{1, \dots, m\}$, $\kappa \in \{0, \dots, \ell\}$, $(H, U) \in Ext(\kappa, \ell)$ and $R \subseteq V$ with $|R| = |U|$. By Fact 5.3(i), we have that $\kappa(H - e, U) \leq \kappa + 1$ for all $e \in H$. Thus, using (5.7), Lemma 5.19(i) and the fact that $i < \tau_{stop}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}\left(\Delta X_{R,(H,U)}(i)\right) &\leq \sum_{e\in H} \frac{X_{R,(H-e,U)}(i)}{|\mathcal{A}(i)|} \leq \frac{|H| 2n^{\kappa+1+\frac{\ell}{m+\kappa+1}}}{A(i) - \varepsilon(i)n^3} \\
&= \mathcal{O}_{\gamma}(n^{\kappa-2+\frac{\ell}{m+\kappa+1}}) \leq n^{\kappa-2+\frac{\ell}{m+\kappa}}.
\end{aligned} \tag{5.27}$$

(Here, we use Fact 5.2(iii) and (iv) instead of (5.7) if $H - e$ is empty.)

We continue to obtain

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}\left(\Delta X^{+}_{R,(H,U)}(i)\right) &\overset{(5.11)}{=} \mathbb{E}_{\tilde{L}}\left(\Delta X_{R,(H,U)}(i)\right) - \Delta\varepsilon_{\kappa,\ell}(i) \\
&\overset{(5.27),(5.4)}{\leq} n^{\kappa-2+\frac{\ell}{m+\kappa}} - n^{\kappa-2+\frac{\ell}{m+\kappa}} = 0.
\end{aligned}$$

By the observation at (5.16), $(X^{+}_{R,(H,U)}(0), X^{+}_{R,(H,U)}(1), \dots)$ is a supermartingale with respect to $(L(0), L(1), \dots)$.

Moreover, since $\Delta X_{R,(H,U)}(i) \geq 0$, we immediately have that

$$\mathbb{E}_{\tilde{L}}\left(|\Delta X_{R,(H,U)}(i)|\right) \leq n^{\kappa-2+\frac{\ell}{m+\kappa}}. \tag{5.28}$$

*Step 4: Expected absolute changes*

From (5.24), (5.26) and (5.28), it is now easy to deduce with the triangle inequality, Lemma 5.20(vi),(vii) and (5.3), (5.4) that

$$\begin{aligned}
\mathbb{E}_{\tilde{L}}\left(|\Delta X^{\pm}_{e}(i)| \mid i < \tau_{freeze,e}\right) &= \mathcal{O}_{\gamma}(n^{-1}), \\
\mathbb{E}_{\tilde{L}}\left(|\Delta X^{\pm}_{T,j,c}(i)| \mid i < \tau_{freeze,T}\right) &= \mathcal{O}_{\gamma}(n^{j-5-c}), \\
\mathbb{E}_{\tilde{L}}\left(|\Delta X^{+}_{R,(H,U)}(i)|\right) &\leq 2n^{\kappa-2+\frac{\ell}{m+\kappa}}.
\end{aligned}$$

With (5.17), this completes the proof. □

5.5. **Boundedness hypotheses.** We now establish boundedness hypotheses for the variables we track.

**Lemma 5.21.** *For every 2-set $e$, we have $\Delta X^{\pm}_{e}(i) = \mathcal{O}(n^{\frac{1}{2}})$ for all $i$.*

**Proof.** Fix a 2-set $e$. For $i \geq \tau_{freeze,e}$, we trivially have $\Delta X_e^{\pm}(i) = 0$. Suppose that $i < \tau_{freeze,e}$. In particular, $e \in E(i)$ and $e \in E(i+1)$. Note that $\mathcal{X}_e(i+1) \setminus \mathcal{X}_e(i) = \emptyset$ and $\mathcal{X}_e(i) \setminus \mathcal{X}_e(i+1) = \{T \in \mathcal{X}_e(i) : T \leftrightarrow T^*(i)\}$. Thus,

$$|\Delta X_e(i)| \leq \max_{T^* \in \mathcal{A}(i) \setminus \mathcal{X}_e(i)} |\{T \in \mathcal{X}_e(i) : T \leftrightarrow T^*\}|.$$

Fix any $T^* \in \mathcal{A}(i) \setminus \mathcal{X}_e(i)$. It follows that $|T^* \cup e| \geq 4$. The number of $T \in \mathcal{X}_e(i)$ with $T \leftrightarrow T^*$ is bounded from above by the number of Erdős-configurations $\mathcal{S}$ on $j \leq j_{max}$ points with $e \cup T^* \subseteq V(\mathcal{S})$, $|\mathcal{S} \cap \mathcal{C}(i)| = j-4$ and $|(\mathcal{S} - \mathcal{C}(i))[e \cup T^*]| \geq 1$, which by Corollary 5.17 is $\mathcal{O}(n^{\frac{1}{2}})$.

It follows that $\Delta X_e(i) = \mathcal{O}(n^{\frac{1}{2}})$, which, via (5.9), implies $\Delta X_e^{\pm}(i) = \mathcal{O}(n^{\frac{1}{2}})$ using Lemma 5.20(vi) and (5.3). □

**Lemma 5.22.** *For every 3-set $T$, all $j \in \{6, \ldots, j_{max}\}$ and $c \in \{0, \ldots, j-4\}$, we have $\Delta X_{T,j,c}^{\pm}(i) = \mathcal{O}(n^{j-3-c-\frac{1}{m}})$ for all $i$.*

**Proof.** For $i \geq \tau_{freeze,T}$, we trivially have $\Delta X_{T,j,c}^{\pm}(i) = 0$. Suppose that $i < \tau_{freeze,T}$. In particular, $T \in \mathcal{A}(i)$ and $T \in \mathcal{A}(i+1)$. We first examine the maximum positive change of $X_{T,j,c}(i)$. Note that if $c = 0$, we clearly have $\Delta X_{T,j,c}(i) \leq 0$. If $c > 0$, we have

$$\begin{aligned}
\Delta X_{T,j,c}(i) &\leq \max_{T^* \in \mathcal{A}(i) \setminus \{T\}} |\{\mathcal{S} \in \mathcal{X}_{T,j,c-1}(i) : T^* \in \mathcal{S}\}| \\
&\leq \max_{T^* \in \mathcal{A}(i) \setminus \{T\}} |\{\mathcal{S} \in \mathfrak{I}_j : T, T^* \in \mathcal{S}, |\mathcal{S} \cap \mathcal{C}(i)| = c-1\}| \\
&= \mathcal{O}(n^{j-3-(c-1)-2+\frac{1}{2}}) = \mathcal{O}(n^{j-4-c+\frac{1}{2}})
\end{aligned}$$

by Corollary 5.17 (with $T \cup T^*$, 2, $c-1$ playing the roles of $R, a, c$).

We next examine the maximum negative change of $X_{T,j,c}(i)$. Note that

$$-\Delta X_{T,j,c}(i) \leq \max_{T^* \in \mathcal{A}(i) \setminus \{T\}} |\{\mathcal{S} \in \mathcal{X}_{T,j,c}(i) : T^* \text{ threatens } \mathcal{S}, T\}|.$$

Fix $T^* \in \mathcal{A}(i) \setminus \{T\}$. Suppose first that $c < j-4$. Observe that $|\mathcal{T}_{T^*}(i)| = \mathcal{O}(n)$ by Lemma 5.19(ii). Thus, using Corollary 5.17, we obtain

$$\begin{aligned}
|\{\mathcal{S} \in \mathcal{X}_{T,j,c}(i) : T^* \text{ threatens } \mathcal{S}, T\}| &\leq \sum_{T' \in (\mathcal{T}_{T^*} \cup \{T^*\}) \setminus \{T\}} |\{\mathcal{S} \in \mathfrak{I}_j : T, T' \in \mathcal{S}, |\mathcal{S} \cap \mathcal{C}(i)| = c\}| \\
&= \mathcal{O}(n) \cdot \mathcal{O}(n^{\max\{j-3-c-2,0\}+\frac{1}{2}}) = \mathcal{O}(n^{j-4-c+\frac{1}{2}}),
\end{aligned}$$

as desired. If $c = j-4$, then we have $|\{\mathcal{S} \in \mathcal{X}_{T,j,j-4}(i) : T^* \text{ threatens } \mathcal{S}, T\}| \leq X_{T,T^*}(i) \leq \mathcal{O}(n^{1-\frac{1}{m}})$ by Fact 5.15 and Corollary 5.16.

We conclude that $\Delta X_{T,j,c}(i) = \mathcal{O}(n^{j-3-c-\frac{1}{m}})$, which, via (5.10), implies $\Delta X_{T,j,c}^{\pm}(i) = \mathcal{O}(n^{j-3-c-\frac{1}{m}})$ using Lemma 5.20(vii) and (5.3). □

**Lemma 5.23.** *Let $\ell \in \{1, \ldots, m\}$, $\kappa \in \{0, \ldots, \ell\}$, $(H, U) \in Ext(\kappa, \ell)$ and $R \subseteq V$ with $|R| = |U|$. Then $\Delta X_{R,(H,U)}^{+}(i) = \mathcal{O}(n^{\kappa + \frac{\ell-1}{m+\kappa}})$ for all $i$.*

**Proof.** For $i \geq \tau_{stop}$, we trivially have $\Delta X_{R,(H,U)}^{+}(i) = 0$. Suppose that $i < \tau_{stop}$. We bound the maximum change of $X_{R,(H,U)}(i)$.

By Fact 5.3(ii), we have that $\kappa(H - [U \cup e], U \cup e) \leq \kappa$ for all $e \in H$. Fix any $T^* \in \mathcal{A}(i)$. We need to give an upper bound on the number of $\phi \colon V(H) \to V$ which are $(H, U)$-extensions at $R$ in $\mathcal{C}(i) \cup \{T^*\}$, but not in $\mathcal{C}(i)$. Fix any such $\phi$. Then there must be $e \in H$ with $\phi(e) = T^*$, and $\phi(e') \in \mathcal{C}(i)$ for all $e' \in H - \{e\}$. Thus, we have that $\phi$ is an $(H - [U \cup e], U \cup e)$-extension at $R \cup T^*$ in $\mathcal{C}(i)$. The number $|V(H) \setminus (U \cup e)|$ of free vertices in the new extension type is at most $\ell - 1$ since $e \not\subseteq U$.

Hence, by (5.7), the number of all possible $\phi$ is at most

$$\sum_{e \in H\colon |U\cup e|=|R\cup T^*|} X_{R\cup T^*,(H-[U\cup e],U\cup e)}(i) = \mathcal{O}(n^{\kappa+\frac{\ell-1}{m+\kappa}}).$$

(Here, we use Fact 5.2(iii) and (iv) instead of (5.7) if $H-[U\cup e]$ is empty.) This implies $\Delta X^+_{R,(H,U)}(i) = \mathcal{O}(n^{\kappa+\frac{\ell-1}{m+\kappa}})$ since $\Delta\varepsilon_{\kappa,\ell}(i) = n^{\kappa-2+\frac{\ell}{m+\kappa}}$ by (5.4). □

5.6. **Proof of Lemma 5.11.** We now prove Lemma 5.11, which in turn implies Theorem 4.4 and hence Theorem 1.2.

**Proof of Lemma 5.11.** Fix a 2-set $e$. By Lemma 5.18, $X^\pm_e$ form supermartingales, and $\mathbb{E}\left(|\Delta X^\pm_e(i)| \mid L(i)\right) = \mathcal{O}_\gamma(n^{-1})$ for all $i$. By Lemma 5.21 we have $\Delta X^\pm_e(i) = \mathcal{O}(n^{\frac{1}{2}})$ for all $i$. By (5.12), we have $-X^\pm_e(0) = \Omega_{\varepsilon_0}(n)$. Thus, we can apply (3.1) with $(\alpha_1,\alpha_2,\alpha_3) = (1,\frac{1}{2},-1)$ to conclude that $\mathbb{P}\left(\exists i\colon X^\pm_e(i) \geq 0\right) \leq \mathrm{e}^{-\Omega_{\varepsilon_0}(n^{1/2})}$.

Fix a triple $T$, $j \in \{6,\dots,j_{max}\}$ and $c \in \{0,\dots,j-4\}$. By Lemma 5.18, $X^\pm_{T,j,c}$ form supermartingales, and $\mathbb{E}\left(|\Delta X^\pm_{T,j,c}(i)| \mid L(i)\right) = \mathcal{O}_\gamma(n^{j-5-c})$ for all $i$. By Lemma 5.22 we have $\Delta X^\pm_{T,j,c}(i) = \mathcal{O}(n^{j-3-c-\frac{1}{m}})$ for all $i$. By (5.13), we have $-X^\pm_{T,j,c}(0) = \Omega_{\varepsilon_0}(n^{j-3-c})$. Thus, we can apply (3.1) with $(\alpha_1,\alpha_2,\alpha_3) = (j-3-c, j-3-c-\frac{1}{m}, j-5-c)$ to conclude that $\mathbb{P}\left(\exists i\colon X^\pm_{T,j,c}(i) \geq 0\right) \leq \mathrm{e}^{-\Omega_{\varepsilon_0}(n^{1/m})}$.

Fix $\ell \in \{1,\dots,m\}$, $\kappa \in \{0,\dots,\ell\}$, $(H,U) \in Ext(\kappa,\ell)$ and $R \subseteq V$ with $|R| = |U|$. By Lemma 5.18, $X^+_{R,(H,U)}$ forms a supermartingale, and $\mathbb{E}\left(|\Delta X^+_{R,(H,U)}(i)| \mid L(i)\right) = \mathcal{O}(n^{\kappa-2+\frac{\ell}{m+\kappa}})$ for all $i$. By Lemma 5.23, we have $\Delta X^+_{R,(H,U)}(i) = \mathcal{O}(n^{\kappa+\frac{\ell-1}{m+\kappa}})$ for all $i$. Since $-X^+_{R,(H,U)}(0) = n^{\kappa+\frac{\ell}{m+\kappa}}$ by (5.14), we can apply (3.1) with $(\alpha_1,\alpha_2,\alpha_3) = (\kappa+\frac{\ell}{m+\kappa}, \kappa+\frac{\ell-1}{m+\kappa}, \kappa-2+\frac{\ell}{m+\kappa})$ to conclude that $\mathbb{P}\left(\exists i\colon X^+_{R,(H,U)} \geq 0\right) \leq \mathrm{e}^{-\Omega(n^{1/2m})}$.

Thus, a final union bound shows that whp, all the variables $X^\pm$ are non-positive. □

## 6. Counting sparse Steiner triple system

Wilson conjectured that the number $STS(n)$ of non-isomorphic Steiner triple systems on $n$ vertices (provided $n$ is admissible) is $(n/\mathrm{e}^2+o(n))^{n^2/6}$. This was recently proved by Keevash [18]. (The upper bound was previously established by Linial and Luria [24].) Letting $STS_k(n)$ denote the number of $k$-sparse Steiner triple systems on $n$ vertices, we expect from our heuristics in Section 4.2 that

$$\log STS_k(n) \approx \log STS(n) - \int_0^{n^2/6} \rho(i), \tag{6.1}$$

where $\rho(i) = \sum_{j=6}^{k+2} \frac{J_j}{\binom{n}{3}^{j-3}} i^{j-3}$. Let $erd_j$ denote the number of unlabelled Erdős-configurations on $[j]$ containing the triple 123. Thus, $J_j = erd_j\binom{n-3}{j-3}$.

We thus have

$$\int_0^{n^2/6} \rho(i) \approx \frac{n^2}{6}\sum_{j=6}^{k+2} \frac{erd_j}{(j-2)!}. \tag{6.2}$$

Hence, we conjecture that

$$STS_k(n) = \left(n\mathrm{e}^{-2-\sum_{j=6}^{k+2}\frac{erd_j}{(j-2)!}} + o(n)\right)^{\frac{n^2}{6}}. \tag{6.3}$$

In particular, since $erd_6 = 3$, we conjecture that the number of Pasch-free Steiner triple systems is $\left(ne^{-2-1/8} + o(n)\right)^{\frac{n^2}{6}}$. It would be interesting to find out whether the upper bound could be established using the entropy method as in [24].

## 7. General sparse designs

In this section, we discuss the possible existence of sparse Steiner systems with more general parameters. Given $n \geq q > r \geq 2$, a *partial* $(n, q, r)$*-Steiner system* is a set $\mathcal{S}$ of $q$-subsets of some $n$-set $V$ such that every $r$-subset of $V$ is contained in at most one $q$-set in $\mathcal{S}$. An $(n, q, r)$*-Steiner system* is a partial $(n, q, r)$-Steiner system $\mathcal{S}$ with $|\mathcal{S}| = \binom{n}{r}/\binom{q}{r}$, i.e. every $r$-set is covered. For fixed $q$ and $r$, we call $n$ *admissible* if $\binom{q-i}{r-i} \mid \binom{n-i}{r-i}$ for all $0 \leq i \leq r-1$. It is easy to see that this condition is necessary for the existence of an $(n, q, r)$-Steiner system. Recently, Keevash [17] was able to settle the so-called existence conjecture, stating that for sufficiently large $n$, there exists an $(n, q, r)$-Steiner system whenever $n$ is admissible (see [12] for an alternative proof).

### 7.1. A generalized Erdős-conjecture.

In order to formulate a generalized Erdős-conjecture, we first consider what the generalized Erdős-configurations might be. A $(j, \ell)_{q,r}$*-configuration* is a set of $\ell$ $q$-sets on $j$ points every two of which intersect in at most $r-1$ points.

The reason why $(j, j-3)$-configurations appear in every Steiner triple system $\mathcal{S}$ is that whenever we have a $(j, \ell)$-configuration $\mathcal{L}$ in $\mathcal{S}$ with an uncovered pair, then the triple in $\mathcal{S}$ which covers this pair determines only one new point, i.e. we can extend $\mathcal{L}$ to a $(j+1, \ell+1)$-configuration. Since there trivially are $(4, 1)$-configurations to start with, we can obtain $(j, j-3)$-configurations for all $j \geq 4$ in this way. For more general Steiner systems, the argument is similar. Having already found some configuration, we consider an uncovered $r$-set inside this configuration, and the Steiner system returns a $q$-set which covers this $r$-set. Hence, we add one $q$-set on the expense of maximally $q-r$ new points. This leads to the following family of functions:

$$\kappa_{q,r}(j) := \left\lfloor \frac{j-r-1}{q-r} \right\rfloor.$$

Note that $\kappa_{3,2}(j) = j-3$, and more generally, $\kappa_{r+1,r}(j) = j-r-1$.

**Proposition 7.1.** *For all $n \geq j > q > r \geq 2$, every $(n, q, r)$-Steiner system $\mathcal{S}$ contains a $(j, \kappa_{q,r}(j))_{q,r}$-configuration.*

**Proof.** We first prove by induction on $x \in \mathbb{N}_0$ that the statement holds for all $j$ of the form $j = x(q-r) + q + 1$ (for which we have $\kappa_{q,r}(j) = x+1$). For $x = 0$, we have $j = q+1$ and $\kappa_{q,r}(j) = 1$, and can thus just take any $q$-set of $\mathcal{S}$ together with an arbitrary additional point.

Suppose now that $x \geq 1$. Let $j' := (x-1)(q-r)+q+1$. Note that $\kappa_{q,r}(j') = x$. By induction, there is a $(j', x)_{q,r}$-configuration $\mathcal{L}$ in $\mathcal{S}$. Since

$$x\binom{q}{r} < \binom{(x-1)(q-r)+q+1}{r} = \binom{j'}{r},$$

there exists an $r$-set $e \subseteq V(\mathcal{L})$ with $e \notin \mathcal{L}$.

This is covered by a unique $q$-set $Q$ of $\mathcal{S}$. Thus, $\mathcal{L} \cup \{Q\}$ is a collection of $x+1$ $q$-sets of $\mathcal{S}$ on at most $j'+q-r$ points. By adding isolated vertices if necessary, we may assume that this yields a $(j'+q-r, x+1)_{q,r}$-configuration, i.e. a $(j, \kappa_{q,r}(j))_{q,r}$-configuration.

For general $j$, write $j = x(q-r)+q+1+y$ for $x \in \mathbb{N}_0$ and $0 \leq y < q-r$. Then $\kappa_{q,r}(j) = x+1 = \kappa_{q,r}(j-y)$. Thus, by the above, there exists a $(j-y, \kappa_{q,r}(j))_{q,r}$-configuration, and we may simply add isolated vertices to obtain a $(j, \kappa_{q,r}(j))_{q,r}$-configuration. $\square$

Proposition 7.1 tells us that we cannot forbid $(j, \kappa_{q,r}(j))_{q,r}$-configurations. Motivated by this, we say that an $(n, q, r)$-Steiner system $\mathcal{S}$ is *k-sparse* if there is no $(j, \kappa_{q,r}(j)+1)_{q,r}$-configuration in $\mathcal{S}$ with $2 \leq \kappa_{q,r}(j)+1 \leq k$. Note that this coincides with the definition of $k$-sparseness for triple systems. We propose the following generalization of Erdős's conjecture.

**Conjecture 7.2.** *For all $q > r \geq 2$ and every $k$, there exists an $n_k$ such that for all admissible $n > n_k$, there exists a $k$-sparse $(n, q, r)$-Steiner system.*

7.2. **Partial result.** It is not clear why the proof of Proposition 7.1 would yield the 'correct' function $\kappa_{q,r}$. We now provide some evidence that $\kappa_{q,r}$ is indeed the correct function. It would be interesting to see whether our process can be generalized to prove Conjecture 7.2 approximately. We take a much simpler route here and show that if we allow even one more $q$-set per $j$ vertices, then the conjecture approximately holds. We say that a (partial) $(n, q, r)$-Steiner system $\mathcal{S}$ is *weakly k-sparse* if there is no $(j, \kappa_{q,r}(j)+2)_{q,r}$-configuration in $\mathcal{S}$ with $\kappa_{q,r}(j)+2 \leq k$.

As tools, we use the Lovász local lemma and a result on almost perfect matchings in hypergraphs due to Pippenger. The idea is to first randomly sparsify the set of $q$-sets in such a way that no $j$-set contains too many $q$-sets, whilst preserving certain degree and codegree conditions. This allows to find an almost perfect matching in a suitable auxiliary hypergraph, producing an approximate Steiner system which is automatically sparse.

For events $B_1, \ldots, B_n$ in a common probability space, we say that the graph $\Gamma$ with $V(\Gamma) = [n]$ is a *dependency graph* if $B_i$ is mutually independent of all $B_j$ with $ij \notin \Gamma$.

**Lemma 7.3** (Lovász local lemma, cf. [1])**.** *Let $B_1, \ldots, B_n$ be events with dependency graph $\Gamma$. If there exist $x_1, \ldots, x_n \in [0, 1)$ such that for all $i \in [n]$, we have*

$$\mathbb{P}(B_i) \leq x_i \prod_{j \in N_\Gamma(i)} (1 - x_j), \tag{7.1}$$

*then*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{B_i}\right) \geq \prod_{i=1}^{n} (1 - x_i).$$

The following is a well-known result due to Pippenger, which has never been published, but several stronger versions have been proven since (see e.g. [26]).

**Theorem 7.4** (Pippenger)**.** *Suppose $1/D, \varepsilon \ll \gamma, 1/r$. Let $H$ be an $r$-graph on $n$ vertices and suppose that $d_H(x) = (1 \pm \varepsilon)D$ for all $x \in V(H)$ and $d_H(\{x, y\}) \leq \varepsilon D$ for all distinct $x, y \in V(H)$. Then there exists a matching in $H$ covering all but $\gamma n$ vertices.*

**Theorem 7.5.** *Let $1/n \ll \gamma, 1/k, 1/q$ and $2 \leq r < q$. There exists a weakly $k$-sparse partial $(n, q, r)$-Steiner system $\mathcal{S}$ on $n$ vertices with $|\mathcal{S}| \geq (1-\gamma)\binom{n}{r}/\binom{q}{r}$.*

**Proof.** Let $V$ be a set of size $n$. Choose new constants $\varepsilon, \theta$ such that $1/n \ll \varepsilon, \theta \ll \gamma, 1/k, 1/q$. Let $j_{max}$ be the maximal $j$ such that $\kappa_{q,r}(j)+2 \leq k$. Thus, we may assume that $\frac{j-r}{q-r}+1 \geq \frac{j-r+\theta}{q-r-\theta}$ for all $q+1 \leq j \leq j_{max}$. Note that $\kappa_{q,r}(j) \geq \frac{j-r}{q-r} - 1$ and thus we have

$$(q - r - \theta)(\kappa_{q,r}(j) + 2) \geq j - r + \theta \tag{7.2}$$

for all $q+1 \leq j \leq j_{max}$. Let $\mathcal{A}$ be the random $q$-graph on $V$ obtained by selecting every $Q \in \binom{V}{q}$ independently with probability $p := n^{-(q-r)+\theta}$.

For $q+1 \leq j \leq j_{max}$ and a set $S \in \binom{V}{j}$, we let $B_S$ denote the event that $|\mathcal{A}[S]| \geq \kappa_{q,r}(j)+2$. For an $r$-set $e \subseteq V$, let $B_e$ be the event that $d_{\mathcal{A}}(e) \neq (1 \pm \varepsilon)n^\theta/(q-r)!$. Finally, let $V_{codeg}$ be the set of all pairs $e, e'$ of distinct $r$-sets in $V$. For $ee' \in V_{codeg}$, let $B_{ee'}$ be the event that $d_{\mathcal{A}}(e \cup e') \geq n^{\theta/10}$.

We claim that with positive probability, none of the events $B_S$, $B_e$, $B_{ee'}$ occurs. (This will allow us to apply Theorem 7.4.) Define the graph $\Gamma$ with vertex set $V(\Gamma) = \bigcup_{j=q+1}^{j_{max}} \binom{V}{j} \cup \binom{V}{r} \cup$

$V_{codeg}$ and add the following edges: add a clique on $\binom{V}{r} \cup V_{codeg}$, and for $S, S' \in \bigcup_{j=q+1}^{j_{max}} \binom{V}{j}$, $e \in \binom{V}{r}$ and $e_1e_2 \in V_{codeg}$, add

$$\begin{aligned} SS' \in E(\Gamma) &\text{ if } |S \cap S'| \geq q, \\ eS \in E(\Gamma) &\text{ if } e \subseteq S, \\ \{e_1, e_2\}S \in E(\Gamma) &\text{ if } e_1 \cup e_2 \subseteq S. \end{aligned}$$

Clearly, $\Gamma$ is a dependency graph for the events $(B_v)_{v \in V(\Gamma)}$. We now aim to fulfill the conditions of the Lovász local lemma.

Clearly, for $q+1 \leq j \leq j_{max}$ and $S \in \binom{V}{j}$, we have

$$\mathbb{P}(B_S) \leq \mathcal{O}(1) p^{\kappa_{q,r}(j)+2}. \tag{7.3}$$

Now, consider $e \in \binom{V}{r}$. Note that $\mathbb{E}(d_{\mathcal{A}}(e)) = p\binom{n-r}{q-r}$. Using a standard Chernoff-Hoeffding bound, we have that

$$\mathbb{P}(B_e) \leq \mathrm{e}^{-\frac{\varepsilon^2}{4} n^{\theta}}. \tag{7.4}$$

Finally, consider $ee' \in V_{codeg}$. Note that $\mathbb{E}(d_{\mathcal{A}}(e \cup e')) \leq pn^{q-r-1} = o(1)$. Thus, using a standard Chernoff-Hoeffding bound, we have that

$$\mathbb{P}(B_{ee'}) \leq \mathrm{e}^{-n^{\theta/10}}. \tag{7.5}$$

For all $q+1 \leq j \leq j_{max}$ and $S \in \binom{V}{j}$, define $x_S := x_j := n^{-j+r}$. For all $e \in \binom{V}{r}$, define $x_e := x_{deg} := \mathrm{e}^{-\frac{\varepsilon^2}{4} n^{\theta/2}}$. For all $ee' \in V_{codeg}$, define $x_{ee'} := x_{codeg} := \mathrm{e}^{-n^{\theta/20}}$.

We now check condition (7.1). First consider $q+1 \leq j \leq j_{max}$ and $S \in \binom{V}{j}$. We have

$$\prod_{v \in N_{\Gamma}(S)} (1 - x_v) \geq \prod_{j'=q+1}^{j_{max}} (1 - x_{j'})^{\binom{j}{q} n^{j'-q}} \cdot (1 - x_{deg})^{\binom{j}{r}} \cdot (1 - x_{codeg})^{\binom{j}{r}^2} \geq 1/2.$$

Since

$$\mathbb{P}(B_S) \overset{(7.3)}{\leq} \mathcal{O}(1) p^{\kappa_{q,r}(j)+2} = \mathcal{O}(1) n^{-(q-r-\theta)(\kappa_{q,r}(j)+2)} \overset{(7.2)}{\leq} \frac{n^{-(j-r)}}{2},$$

we conclude that (7.1) is satisfied for $S$. Now consider $e \in \binom{V}{r}$. We have

$$\prod_{v \in N_{\Gamma}(e)} (1 - x_v) \geq \prod_{j=q+1}^{j_{max}} (1 - x_j)^{n^{j-r}} \cdot (1 - x_{deg})^{n^r} \cdot (1 - x_{codeg})^{n^{2r}} \geq \mathrm{e}^{-j_{max}}.$$

Since $\mathbb{P}(B_e) \leq \mathrm{e}^{-\frac{\varepsilon^2}{4} n^{\theta}} \leq \mathrm{e}^{-j_{max}} x_{deg}$ by (7.4), we deduce that (7.1) is satisfied for $e$. A similar calculation also shows that (7.1) is satisfied for $ee' \in V_{codeg}$.

Thus, with Lemma 7.3 we can infer that with positive probability none of the events $(B_v)_{v \in V(\Gamma)}$ occurs. Let $\mathcal{A}$ be such a $q$-graph. Define the auxiliary $\binom{q}{r}$-graph $H$ with $V(H) = \binom{V}{r}$ and $E(H) = \{\binom{Q}{r} : Q \in \mathcal{A}\}$. Since no event $B_e$ occurred, we have $d_H(e) = (1 \pm \varepsilon) n^{\theta}/(q-r)!$ for all $e \in V(H)$. Moreover, since no event $B_{ee'}$ occurred, we have $d_H(\{e, e'\}) \leq n^{\theta/10}$ for all distinct $e, e' \in V(H)$. Hence, by Theorem 7.4, there exists a matching in $H$ covering all but $\gamma\binom{n}{r}$ vertices of $H$. This correponds to a partial $(n, q, r)$-Steiner system $\mathcal{S}$ on $V$ covering all but $\gamma\binom{n}{r}$ $r$-sets. Thus, $|\mathcal{S}| \geq (1-\gamma)\binom{n}{r}/\binom{q}{r}$. Finally, every $q$-set of $\mathcal{S}$ is contained in $\mathcal{A}$, and since no event $B_S$ occurred, $\mathcal{S}$ is weakly $k$-sparse. □

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