Below, I have collected some open problems which I have come across in my research and which I believe are interesting and solvable. I'd be happy to hear about any updates on these problems: `osthus@maths.bham.ac.uk`

1. Cycles in oriented graphs

The famous Caccetta-Häggkvist conjecture asks for a cycle of length *at most* ℓ in an oriented graph. But what if we ask for a cycle of length exactly ℓ? The following conjecture asserts that the extremal example is always a blow-up of a cycle, e.g. if $\ell = 6$ then the extremal example is a blow-up of a 4-cycle.

Conjecture 1 (Kelly, Kühn and Osthus). *Let $\ell \geq 4$ be a positive integer and let k be the smallest integer that is greater than 2 and does not divide ℓ. Then there exists an integer $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum in- and outdegree at least $\lfloor n/k \rfloor + 1$ contains an ℓ-cycle.*

Some partial results are obtained in [3].

2. Cycles in directed graphs

Conjecture 2. *Let G be a digraph on $n \geq 3$ vertices with n even such that $d_i^+, d_i^- \geq i + 1$ for all $i < n/2$. Then G contains a cycle through any pair of given vertices.*

Actually, Nash-Williams conjectured that G should even contain a Hamilton cycle under the above conditions. (Approximate versions of this were proved in [1, 11].) This would provide a directed analogue of Posa's theorem. The above weakening was proposed by Bermond and Thomassen.

3. Short cycles in graphs

Conjecture 3 (Verstraëte). *For all integers $k < \ell$ there exists a positive $c = c(\ell)$ such that every $C_{2\ell}$-free graph G has a C_{2k}-free subgraph H with $e(H) \geq e(G)/c$.*

This conjecture was motivated by a result of Györi [2] who showed that every bipartite C_6-free graph G has a C_4-free subgraph which contains at least half of the edges of G. The case $k = 2$ was proven in [6].

A related (but probably harder) conjecture is the following:

Conjecture 4 (Thomassen [13]). *For all integers g and k there exists an integer d such that every graph G of average degree at least d contains a subgraph of average degree at least k and girth greater than g.*

Unlike what is expected in the case when $C_{2\ell} \not\subseteq G$ for some $\ell \geq g/2$, d clearly cannot depend linearly on k here (when g is fixed and even). For regular graphs the truth of Thomassen's conjecture was observed by Alon (see [4] for the argument) and the case $g \leq 4$ was proved in [4].

4. Partitions

Problem 5. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are r-uniform hypergraphs on the same vertex set V such that $\mathcal{H}_i$ has m_i hyperedges. Does there always exist a partition of V into r classes $V_1, \dots, V_r$ such that for both $i = 1, 2$ at least $r!m_i/r^r - o(m_i)$ hyperedges of $\mathcal{H}_i$ meet each of the classes $V_1, \dots, V_r$?*

The bound on the number of hyperedges is what one would expect for a random partition. For graphs, the question was answered in the affirmative in [9]. Keevash and Sudakov observed that the answer is negative if we consider many hypergraphs instead of just 2 (see [9] for the example).

5. Topological minors

Problem 6 (Shi). *Is there a function $h(r)$ such that every graph of minimum degree at least $r \geq 3$ and girth at least $h(r)$ contains a subdivision of K_{r+1} as an induced subgraph?*

The above problem has an affirmative answer if we assume that the minimum degree is sufficiently large compared to r [5] or if we omit the condition of being induced (see e.g. [7]).

6. Complexity of the H-factor problem

An H-factor in a graph G is a set of vertex-disjoint copies of H covering all vertices of G. (This is also called a perfect H-packing or a perfect H-matching). Let δ_H be the smallest number so that every graph G on n vertices and minimum degree at least $\delta_H n$ contains an H-factor. In [10], we determined $\delta_H n$ up to an additive constant.

Problem 7 (Kühn & D.O.). *Given $c < \delta_H$, is it NP-hard to determine whether a graph G on n vertices and minimum degree cn contains and H-factor?*

The answer is positive for cliques and a few other graphs [8].

7. Random partially ordered sets

Consider the random partially ordered set $\mathcal{P}(n,p)$ which is generated by first selecting each subset of $[n] = \{1, \dots, n\}$ with probability p and then ordering the selected subsets by inclusion. Sperner's theorem says that the largest antichain in the power set of $[n]$ is obtained by choosing the middle level, i.e. all sets of size $\lfloor n/2 \rfloor$. The following would yield a probabilistic analogue of Sperner's theorem

Conjecture 8 (D.O.). *When $pn \to \infty$, the size of the largest antichain in $\mathcal{P}(n,p)$ is asymptotically equal to the size of its 'middle level', i.e. $p\binom{n}{n/2}$.*

Note that if $pn \to 0$, then one can take more than one level as an antichain. Partial results appear in [12].

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