# On random planar graphs, the number of planar graphs and their triangulations


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**Abstract**

Let $\mathcal{P}_n$ be the set of labelled planar graphs with $n$ vertices. Denise, Vasconcellos and Welsh proved that $|\mathcal{P}_n| \leq n!\,(75.8)^{n+o(n)}$ and Bender, Gao and Wormald proved that $|\mathcal{P}_n| \geq n!\,(26.1)^{n+o(n)}$. Gerke and McDiarmid proved that almost all graphs in $\mathcal{P}_n$ have at least $13/7n$ edges. In this paper, we show that $|\mathcal{P}_n| \leq n!\,(37.3)^{n+o(n)}$ and that almost all graphs in $\mathcal{P}_n$ have at most $2.56n$ edges. The proof relies on a result of Tutte on the number of plane triangulations, the above result of Bender, Gao and Wormald and the following result, which we also prove in this paper: every labelled planar graph $G$ with $n$ vertices and $m$ edges is contained in at least $\varepsilon 3^{(3n-m)/2}$ labelled triangulations on $n$ vertices, where $\varepsilon$ is an absolute constant. In other words, the number of triangulations of a planar graph is exponential in the number of edges which are needed to triangulate it. We also show that this bound on the number of triangulations is essentially best possible.


## 1 Introduction

Compared to the wealth of knowledge one has about random graphs in general, rather little is known about the likely properties of a random planar graph on $n$ vertices – not even the expected number of edges. Here we consider the uniform model: let $\mathcal{P}_n$ be the set of labelled planar graphs with vertex set $\{1, \ldots, n\}$. A random planar graph $P_n$ is chosen uniformly from $\mathcal{P}_n$. This should not be confused with a random planar map, since a planar map is defined as a connected graph which is embedded in the plane, whereas a planar graph may have several embeddings. Moreover, we consider only simple graphs whereas maps may usually have multiple edges.

Random planar graphs were first investigated by Denise, Vasconcellos and Welsh [2]. They showed that $n!\,6^{n+o(n)} \leq |\mathcal{P}_n| \leq n!\,(75.8)^{n+o(n)}$, that the limiting probability that $P_n$ is connected is greater than zero and that the expected number of edges of $P_n$ is at least $3n/2$. They also introduced a Markov chain whose stationary distribution is the uniform measure on $\mathcal{P}_n$. This Markov chain was investigated in much more detail by Gerke and McDiarmid [4], who showed that almost surely $P_n$ has at least $13n/7$ edges. Complementing the result of [2] on the connectivity of $P_n$, McDiarmid, Steger, and Welsh [6] proved (amongst other results) that the limiting probability that $P_n$ is connected is less than one. Using generating function techniques, Bender, Gao and Wormald [1] proved that the number of 2-connected graphs in $\mathcal{P}_n$ is in fact asymptotically $Cn!\alpha^n n^{-7/2}$, where $C$ is some positive constant and $\alpha \sim 26.1876$, which gives the best known lower bound on $|\mathcal{P}_n|$.

Concerning upper bounds on $|\mathcal{P}_n|$, in the final section we will prove upper bounds for the number of graphs in $\mathcal{P}_n$ with a given number of edges (Theorem 12), which will immediately imply the following result.

**Theorem 1.**

$$|\mathcal{P}_n| \leq n!\,(37.3)^{n+o(n)}.$$

Theorem 12 will turn out to be an immediate consequence of a result of Tutte [8] on the number of planar triangulations and the following result, which states that the number of triangulations of *every* planar graph is exponential in the number of edges which are needed in order to triangulate the graph. We will prove this result in Section 3, where we will also see (Proposition 11) that the bound given in Theorem 2 is essentially best possible for $m \geq 2n$.

**Theorem 2.** *Every labelled planar graph $G$ with $n$ vertices and $m$ edges is contained in at least $\varepsilon 3^{(3n-m)/2}$ labelled triangulations on $n$ vertices, where $\varepsilon$ is an absolute constant.*

Combining our upper bounds with the result in [1] mentioned earlier, we will deduce the following result in the final section. Given a class $\mathcal{A}$ of graphs and a property $\mathcal{Q}$, we say that *almost all* graphs in $\mathcal{A}$ have $\mathcal{Q}$ if the proportion of graphs in $\mathcal{A}$ on $n$ vertices which have $\mathcal{Q}$ tends to one as $n \to \infty$.

**Theorem 3.** *Almost all graphs in $\mathcal{P}_n$ have less than $2.56n$ edges.*

## 2 Definitions and basic facts

For convenience, for $n \in \mathbb{N}$ we let $[n] := \{1, \ldots, n\}$. A *plane* graph is a planar graph together with an embedding into the plane. A planar graph $G$ is called *rigid* if any two embeddings of $G$ are equivalent. By the theorem of Whitney, every 3-connected planar graph is rigid (see e.g. [3]). Given a face $f$, we denote its boundary by $b(f)$. Moreover, if $f$ is not the outer face, we say that the *bounding cycle* of $f$ is the shortest cycle containing $f$ in its interior and denote it by $bc(f)$. Note that $bc(f) \subset b(f)$ and that we have equality if $f$ is 2-connected. We say that a vertex $x$ *lies in* $f$ if $x \in b(f)$.

Given a face $f$, we treat the boundary of $f$ as an ordered sequence of not necessarily distinct vertices $v_1, \ldots, v_k$. If $i < j$ and $v_i \neq v_j$ are not adjacent, then the two sequences $v_{i+1}, \ldots, v_{j-1}$ and $v_{j+1}, \ldots, v_k, v_1, \ldots, v_{i-1}$ are non-empty and we call them the left and the right $v_i$-$v_j$-*side* of $f$, respectively.

For a set of *labelled* graphs $\mathcal{A}$, denote by $\mathcal{A}^u$ the set of distinct *unlabelled* graphs contained in $\mathcal{A}$, each representing an isomorphism class of $\mathcal{A}$. Denote by $\mathcal{A}_n$ the set of those graphs in $\mathcal{A}$ with vertex set $[n]$ and by $\mathcal{A}_{n,m}$ the set of those graphs in $\mathcal{A}_n$ with exactly $m$ edges.

Denote by $\mathcal{P}$ the set of all labelled planar graphs. We shall need a few classes of special planar graphs. Define

$$\mathcal{T} \;:=\; \{G \in \mathcal{P} : \; |E(G)| = 3|V(G)| - 6\}\,.$$

Thus $\mathcal{T}$ is the set of all maximal planar graphs. It is well known that in every embedding of a graph $G \in \mathcal{T}$ all faces are bounded by triangles. Tutte [8] proved that

$$|\mathcal{T}_n^u| = (1 + o(1))C\gamma^n n^{-5/2}, \qquad \text{where } \gamma = \frac{256}{27} = 9.48.. \tag{1}$$

and where $C$ is some positive constant. It is well known (see e.g. [5] for a proof) that the number of triangulations of the interior of an $\ell$-cycle is given by $\mathrm{Cat}(\ell)$, the $\ell$-th Catalan number:

$$\mathrm{Cat}(\ell) = \frac{1}{\ell - 1}\binom{2\ell - 4}{\ell - 2} = 4^{\ell + o(\ell)}. \tag{2}$$

# 3 Triangulating a planar graph

The aim of this section is to prove Theorem 2. The basic idea is as follows. Consider a plane graph $G \in \mathcal{P}_{n,m}$. We would like to generate as many triangulations containing $G$ as possible, and the easiest way would be to simply triangulate each (non-triangular) face independently. This may of course not always be possible, because two non-adjacent vertices which are connected in order to triangulate one face can then not be connected in any of the neighbouring faces.

It turns out (see Proposition 5) that this approach does work for 3-connected planar graphs. However, in general, in order to generate many triangulations, we have to make use of the different embeddings that a planar graph may have. For instance, the graph in Figure 3 has only one triangulation when viewed as a plane graph but superexponentially many when viewed as a labelled planar graph.

Before dealing with this, let us first consider the 3-connected case. We say that a planar graph has the 1-*face property* if it has an embedding so that the intersection of the boundaries of two faces consists of either an edge, a vertex or is empty. Equivalently, if $x$ and $y$ lie on the boundary of some face $f$ and are not adjacent, then they do not both also lie on the boundary of some other face $f'$.

**Proposition 4.** *A 3-connected plane graph $G$ has the 1-face property.*

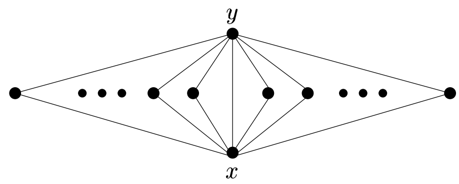

Figure 1: A plane graph with only one triangulation

This seems to be a folklore result (see e.g. [7]). As the proof is short, we include it here for completeness. We mention (but will not make use of this) that the converse is also true: a 2-connected graph which has the 1-face property must be 3-connected.

*Proof of Proposition 4.* Suppose that $G$ does not have the 1-face property. Then there exist vertices $x$ and $y$ which are contained in the boundaries of two faces $f_1$ and $f_2$ and where the edge $xy$ (which may or may not be present in $G$) does not lie on the boundary of both faces. But this implies that there are two faces of $G - xy$ into which we can insert the edge $xy$ and thus that $G + xy$ does not have a unique embedding. By Whitney's theorem, $G+xy$ cannot be 3-connected (and hence neither is $G$). $\square$

The above proposition implies that it is easy to prove Theorem 2 for 3-connected graphs.

**Proposition 5.** *Let $G$ be a 3-connected plane graph in $\mathcal{P}_{n,m}$. Then the number of triangulations on $n$ vertices which contain $G$ is at least $2^{3n-m-6}$.*

*Proof.* Using Proposition 4, it is easy to see that the number of triangulations of $G$ is equal to the product (over all faces) of the number of triangulations of each face. To calculate the product, first note that the boundary of each face of $G$ is a cycle, and so the number of triangulations of such a face $f$ is given by the Catalan number $\mathrm{Cat}(\ell)$, where $\ell$ is the length of the bounding cycle of $f$. Using (2), it is easy to show by induction that $\mathrm{Cat}(\ell) \geq 2^{\ell-3}$ for any $\ell \geq 3$. Now denote by $\ell_j$ the length of the boundary of the $j$-th face. Since every edge lies on the boundary of two faces, this gives us

$$\sum_j \ell_j = 2m, \tag{3}$$

where the sum is over all faces of $G$. Moreover, from Euler's formula we know that there are $2 - n + m$ faces. Combining all this, the number of triangulations on $n$ vertices of $G$ is

$$\prod_j \mathrm{Cat}(\ell_j) \geq \prod_j 2^{\ell_j - 3} = 2^{\sum_j (\ell_j - 3)} = 2^{2m-3(2-n+m)} = 2^{3n-m-6}.$$

$\square$

So we have proven that every 3-connected graph $G \in \mathcal{P}_{n,m}$ is contained in at least $2^{3n-6-m}$ triangulations. In other words, the deletion of an edge doubles the number of triangulations if the resulting graph is 3-connected. The above proof shows that this is tight if and only if all faces are triangles or quadrilaterals.

Before we move closer towards the proof of Theorem 2, we make some preliminary steps (Propositions 6 and 7). We say that two planar graphs $G_0, G_1 \in \mathcal{P}_n$ are *incomparable* if there is no planar graph $H \in \mathcal{P}_n$ containing both $G_0$ and $G_1$ as a subgraph. Obviously it is true that

$$\begin{aligned} & G_0 \subseteq H_0,\ G_1 \subseteq H_1,\ G_0 \text{ and } G_1 \text{ are incomparable} \\ & \quad \Rightarrow\ H_0 \text{ and } H_1 \text{ are incomparable.} \end{aligned} \tag{4}$$

(Here the $H_i$ are allowed to have more vertices than the $G_i$). For a *planar* graph $G$ denote by $\mathrm{Add}(G)$ the set of all edges $e \in E(\overline{G})$ so that $G + e$ is still planar. In contrast, for a *plane* graph $G$ denote by $\mathrm{Ins}(G)$ the set of those edges $e \in \mathrm{Add}(G)$, so that $e$ can be inserted into the current embedding of $G$. Suppose that $G$ and $H$ are planar graphs. Then

$$G \subseteq H \quad \Longrightarrow \quad E(\overline{G}) \cap E(H) \ \subseteq \ \mathrm{Add}(G). \tag{5}$$

Observe that for a rigid plane graph $G$ we have that $\mathrm{Add}(G) = \mathrm{Ins}(G)$. Moreover note that rigidity is not necessarily preserved when adding vertices and/or edges to a graph. On the other hand, every subdivision of a rigid graph is rigid, as this means nothing else than replacing edges by paths. In particular, every subdivision of a 3-connected graph is rigid.

The following actually rather obvious proposition will turn out to be useful. Suppose we have two plane graphs and the first one has an edge which the second one does not have and cannot have – given its present embedding. If the second one is rigid, then this means that it will never be able to get it, and therefore there is no third planar graph containing both of the two graphs.

**Proposition 6.** *Let $G_0$ and $G_1$ be two plane graphs with vertex set $[n]$ where $G_1$ is a subdivision of a 3-connected graph. If there exists an edge $e_0 \in E(G_0) \cap E(\overline{G_1})$ such that $e_0 \notin \mathrm{Ins}(G_1)$, then $G_0$ and $G_1$ are incomparable.*

*Proof.* Suppose to the contrary that there is a planar graph $H \in \mathcal{P}_n$ containing $G_0$ and $G_1$ as subgraphs. Then $e_0 \in E(H)$, and applying (5) to $G_1$ and $H$ shows that $e_0 \in \mathrm{Add}(G_1)$. On the other hand, $G_1$ is rigid because it is a subdivision of a 3-connected graph. Therefore $\mathrm{Add}(G_1) = \mathrm{Ins}(G_1)$. Hence $e_0 \in \mathrm{Ins}(G_1)$, contradicting the assumption. □

Stepping back from planarity for a moment, consider a graph $G$ containing a triangle $T$. Suppose that $\{x, y\}$ is a cut-set. Then there exists exactly one component $H$ of $G - x - y$ which contains all vertices in $T \setminus \{x, y\}$ and we call $H$ the *$T$-component* of $G - x - y$.

**Proposition 7.** *Let $G$ be a graph which is 2-connected but not 3-connected and let $T$ be a triangle in $G$. Let $\{x, y\}$ be a cut-set of $G$ which minimizes the* *cardinality of the $T$-component $H$ of $G - x - y$, and let $H^+$ be the subgraph of $G$ induced by $V(H) \cup \{x, y\}$, where we only include the edge $xy$ if $xy \in T$. Then $H^+$ is 2-connected.*

*Proof.* Suppose to the contrary that there exists a vertex $z$ in $H^+$ such that $H^+ - z$ has two components $H_1$ and $H_2$. We can assume without loss of generality that $x \in H_1$ and $y \in H_2$, because if one of the $H_i$ contains neither $x$ nor $y$, then $G - z$ would not be connected. As there are no edges between $H_1$ and $H_2$, this implies that $xy \notin H^+$, and hence by definition of $H^+$, $xy \notin T$. But then $T$ must lie either in $H_1 + z$ or in $H_2 + z$. Suppose (again without loss of generality) that $T \subseteq H_1 + z$. But then $H_1 - x$ is a component of $G - x - z$ which contains $T \setminus \{x, z\}$, and so it is a $T$-component which is smaller than $H$. □

Returning to planar graphs, consider a plane 2-connected graph $G$ whose outer face $f$ is bounded by a triangle $T$. In this case we will refer to the $T$-component as the *external component.* Two paths are called *internally disjoint* if their intersection contains at most their endvertices. For two vertices $u, w$, a path denoted by $P(u, w)$ is a path connecting $u$ and $w$.

Now reconsider our position with respect to proving Theorem 2. Recall that if $G$ is not 3-connected, then we cannot apply Proposition 5. Instead, we consider a cut-set $\{x, y\}$ of $G$ and consider the components $H_0, \ldots, H_k$ of $G - x - y$. In Lemma 8, which constitutes the core of the proof of Theorem 2, we fix the embedding of $H_0 + x + y$, and then embed all other components in several ways into that face of $H_0 + x + y$ which contains $x$ and $y$ . For every such embedding, we fix the positions of the $H_i$ by inserting additional edges, so that the resulting graph is rigid as far as the relative positions of the $H_i$ are concerned. Finally we make sure that the graphs obtained in this way are not only distinct but also incomparable – in other words, no matter how we later add more edges, the resulting graphs will continue to be distinct, so that in the end we really have the required number of distinct triangulations.

**Lemma 8.** *Let $G \in \mathcal{P}_{n,m}$ be a 2-connected plane graph whose outer face $f$ is a triangle. Let $\{x, y\}$ be a cut-set which minimizes the cardinality of the external component $H_0$. Suppose that $G - x - y$ has $k + 1 \geq 2$ components. Set $t := k$ if $xy \in G$ and $t := k + 1$ if $xy \notin G$.*

*Then there is a family of pairwise incomparable plane graphs $G_1, \ldots, G_s$ whose outer face is still $f$, such that for all $1 \leq i \leq s$, the embeddings of the $G_i$ and $G$ are the same when restricted to $V(H_0)$,*

$$G_i \in \mathcal{P}_{n,m+t}, \quad G \subset G_i, \quad G_i - x - y \text{ is connected}$$

*and $s \geq 3^{t/2}$. In the exceptional case where $xy \in f$, we only require that $s \geq 3^{(t-1)/2}$.*

*Proof.* Denote by $H_0, \ldots, H_k$ the plane subgraphs of $G$ which are induced by the connected components of $G - x - y$, where $H_0$ is the external component. Denote by $H_i^+$ the plane subgraph of $G$ induced by $V(H_i) \cup \{x, y\}$, where we

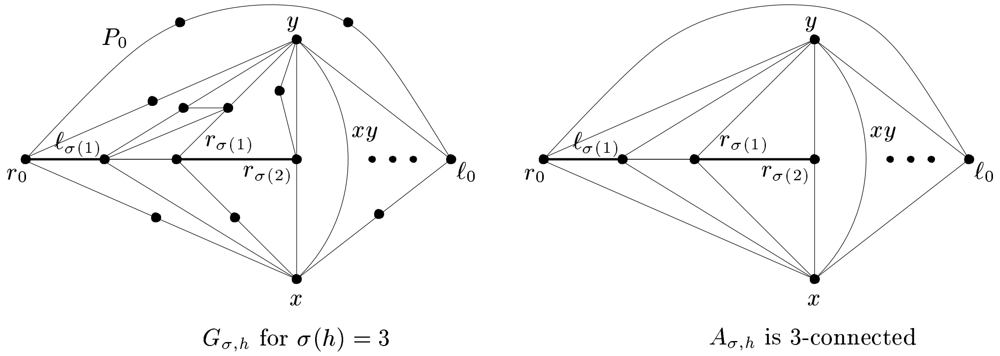


Figure 2: Illustrating the proof of Lemma 8. The thick edges are the edges which are added to $G$ to form $G_{\sigma,h}$.

include the edge $xy$ if and only if $i = 0$ and $xy \in f$. (By Proposition 7, this will imply that $H_0^+$ is 2-connected.) For each $i \geq 1$, it is clear that $x$ and $y$

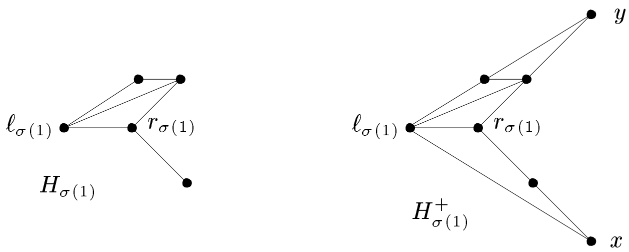


Figure 3: The graphs $H_{\sigma(1)}$ and $H^+_{\sigma(1)}$ corresponding to the example in the previous figure.

lie in the outer face of $H_i^+$. As $xy \notin H_i^+$ for $i \geq 1$, following the notation in the beginning of Section 2, we choose a shortest path $P_i$ in $H_i$ connecting the left and the right $x$-$y$-side of the outer face of $H_i^+$, and let $\ell_i$ be the first and $r_i$ be the last vertex of $P_i$. If $\ell_i \neq r_i$ then in $H_i^+$ there exist pairwise internally disjoint paths

$$P(\ell_i, x),\ P(r_i, x),\ P(\ell_i, y),\ P(r_i, y).$$

If $|P_i| = 1$ and $\ell_i = r_i$, then the respective paths above coincide.

*Case* 1*:* $xy \notin G$. Consider $H_0^+$. It is clear that there exists a face $f'$ of $H_0^+$ such that $x$ and $y$ now lie on $b(f')$. By Proposition 7, $H_0^+$ is 2-connected, so both $x$ and $y$ must lie in $bc(f') = b(f')$. As $xy$ is not contained in $G$ and thus neither in $H_0^+$, this immediately implies that $b(f')$ has at least four vertices, and in particular $f' \neq f$. Moreover, let $P_0$ be a shortest path in $H_0$ connecting the left and right $x$-$y$-side of $b(f')$. $P_0$ exists because $H_0$ is connected. Denote by $\ell_0 \neq r_0$ the two endvertices of $P_0$. Observe that $b(f') \cap P_0 = \{\ell_0, r_0\}$. Similarly to the paths in $H_i^+$, we divide $b(f')$ into four paths

$$P(\ell_0, x),\ P(r_0, x),\ P(\ell_0, y),\ P(r_0, y)$$

which are all pairwise internally disjoint.

Now choose a permutation $\sigma$ on $[k]$ and an integer $h$ from $\{0,\dots,k+1\}$. Given $\sigma$ and $h$, construct the graph $G_{\sigma,h}$ as follows. For convenience, set $\sigma(0) := 0$ and $\sigma(k+1) := 0$. Successively, for every $i \in [k]$, embed $H^+_{\sigma(i)}$ into $f'$ in such a way that the edge $r_{\sigma(i-1)}\ell_{\sigma(i)}$ can be added (and add it). Having done this for all $i$, add the edge $r_{\sigma(k)}\ell_0$. Finally, if $h > 0$, remove the edge $r_{\sigma(h-1)}\ell_{\sigma(h)}$ and insert the edge $xy$ instead. Note that $G_{\sigma,h}$ has $m+t$ edges, where $t = k+1$ as required. Moreover, we have constructed

$$s := k!(k+2) \geq 3^{\frac{k+1}{2}} = 3^{\frac{t}{2}}$$

graphs $G_{\sigma,h}$. It is clear from the construction that each $G_{\sigma,h}$ is a plane graph whose outer face is identical to the outer face $f$ of $G$ and whose embedding when restricted to $V(H_0)$ is the same as that of $G$. Obviously, each $G_{\sigma,h}$ also contains $G$. As all components $H_i$ of $G-x-y$ are now connected by new edges, we also know that $G_{\sigma,h} - x - y$ is connected.

It remains to prove that the $G_{\sigma,h}$ are pairwise incomparable. To this end, we define two auxiliary graphs which are obtained from $G_{\sigma,h}$ as follows. We first define $B_{\sigma,h} \subseteq G_{\sigma,h}$. For $h = 0$,

$$B_{\sigma,0} := \bigcup_{i=1}^{k+1} \left(r_{\sigma(i-1)}\ell_{\sigma(i)} \cup P_{\sigma(i)} \cup P(\ell_{\sigma(i)},x) \cup P(r_{\sigma(i)},x) \cup P(\ell_{\sigma(i)},y) \cup P(r_{\sigma(i)},y)\right)$$

and for $h > 0$, we obtain $B_{\sigma,h}$ from $B_{\sigma,0}$ by removing the edge $r_{\sigma(h-1)}\ell_{\sigma(h)}$ and inserting the edge $xy$ instead.

Observe that the vertices in $B_{\sigma,h}$ which have degree at least 3 are exactly $x, y, \ell_0, r_0, \dots, \ell_k, r_k$. Now obtain $A_{\sigma,h}$ by successively contracting all paths between these vertices into edges. Alternatively, $A_{\sigma,h}$ can also be obtained from a complete bipartite graph with classes $\{x,y\}$ and $\{\ell_0, r_0, \dots, \ell_k, r_k\}$ by adding either a Hamilton cycle to the second class (if $h = 0$) or Hamilton paths to both classes (if $h > 0$). It is straightforward to check that $A_{\sigma,h}$ is 3-connected, and hence $B_{\sigma,h}$ is a subdivision of a 3-connected graph.

We now show that the $B_{\sigma,h}$ are pairwise incomparable. By the construction of $B_{\sigma,h}$ (in particular, since $B_{\sigma,h}$ is rigid), it is clear that there is no edge $r_i\ell_j \in \mathrm{Ins}(B_{\sigma,h})$ with $i \neq j$. Thus, when considering $B = B_{\sigma,h}$ and $B' = B_{\sigma',h'}$, it suffices to find an edge $r_i\ell_j \in E(B) \cap E(\overline{B'})$ with $i \neq j$ in order to apply Proposition 6.

If $\sigma \neq \sigma'$, then $k \geq 2$ and, recalling that $\sigma(0) = \sigma'(0) = 0$ and $\sigma(k+1) = \sigma'(k+1) = 0$, let $i \geq 1$ be the smallest integer so that $\sigma(i) \neq \sigma'(i)$ and let $j \leq k$ be the largest integer so that $\sigma(j) \neq \sigma'(j)$. Note that $i < j$. Then $r_{\sigma(i-1)}\ell_{\sigma(i)}, r_{\sigma(j)}\ell_{\sigma(j+1)} \in E(B) \cap E(\overline{B'})$, unless they are removed from $B$ before adding $xy$. But as at most one of them will be removed, we are done. If $\sigma = \sigma'$, then $r_{\sigma(h'-1)}\ell_{\sigma(h')} \in E(B) \cap E(\overline{B'})$. Thus the $B_{\sigma,h}$ are pairwise incomparable, and hence by (4), so are the $G_{\sigma,h}$. This completes the proof of the lemma for Case 1.

*Case 2a: $xy \in G$, $xy \notin f$.* By definition, $xy \notin H_0^+$, so the only difference to Case 1 is that we need the edge $xy$ to be present in all our graphs $G_{\sigma,h}$; in other words, we require that $1 \le h \le k+1$, so that the total number of graphs is $s := k!(k+1)$ when adding $t := k$ edges to $G$ (one edge less as before, because $xy$ already exists in $G$). Checking that

$$s = (k+1)! \ge 3^{\frac{k}{2}} = 3^{\frac{t}{2}}$$

completes this case.

*Case 2b: $xy \in G$, $xy \in f$.* As $xy \in H_0^+$ and $H_0^+$ is 2-connected, there are two faces in $H_0^+$ whose boundaries contain $xy$. Note that one of them is $f$, and denote the other one by $f'$. Furthermore the face $f'$ into which we will be embedding the other $H_i$ is bounded by a cycle $b(f')$ containing $x$ and $y$. However, as $xy \subset b(f')$, we now cannot assume the existence of two vertices $r_0$ and $\ell_0$ on $b(f')$ to which we can connect the $H_i$ as before. So we still proceed as in Case 1, except that we fix $h := k+1$ and do not include $\ell_0$ and $P_0$ in $B_{\sigma,k+1}$. (Nevertheless, it is easy to check that the resulting graphs $A_{\sigma,k+1}$ are still 3-connected.) Hence the total number of graphs is $s := k!$ when adding $t := k$ edges to $G$. By the statement of the lemma, this time we only need to check that

$$s = k! \ge 3^{\frac{k-1}{2}} = 3^{\frac{t-1}{2}}.$$

As our construction is a special case of the general construction for Case 1, it is clear that the graphs constructed in this way fulfil all the requirements of the lemma. □

Observe that we can apply Lemma 8 iteratively, so that starting from a 2-connected graph $G$, we produce 3-connected graphs $G_i$ satisfying the above requirements (once there are no more vertex pairs $x$ and $y$ to which we can apply the lemma, this means we have arrived at a graph which is 3-connected). We then apply Proposition 5 to the 3-connected graphs. So the only remaining problem is that we need to get started if $G$ is not 2-connected. Roughly speaking, we solve this problem by simply embedding loose blocks into faces of 3-connected components and fixing them there.

Recall that a *block* of a graph $G$ is a maximal 2-connected component. The following definitions will be convenient. Given a rooted tree, we say that the *children* of a vertex $v$ are those vertices which are adjacent to $v$ and whose distance to the root is greater than that of $v$. Given a plane graph $G$, a *triangulation tree of $G$* is a rooted tree whose vertices correspond to plane graphs (with vertex set $[n]$) containing $G$; whose root corresponds to $G$; whose leaves correspond to 3-connected graphs which are pairwise incomparable; for any vertex $v$ of the tree, the graphs corresponding to the children of $v$ all have the same number of edges and, finally, if a child has $t$ more edges than its parent $v$, then $v$ has at least $3^{t/2}$ children unless it is an *exceptional vertex*, in which case it still has at least $3^{(t-2)/2}$ children. Observe that in our definition we require the leaves of the tree to be pairwise incomparable, however this is guaranteed via

(4) if the children of each vertex are pairwise incomparable. Our aim will be to construct a triangulation tree of $G$ with few exceptional vertices on any path from the root to a leaf.

**Lemma 9.** *Consider a plane graph $G \in \mathcal{P}_{n,m}$ consisting of a 3-connected plane graph $G'$ (whose outer face is a triangle) and a block $H$ which has exactly one vertex $x$ in common with $G'$. Then there is a triangulation tree of $G$ with no exceptional vertices, unless $x$ is contained in the boundary of the outer face of $G$ and has degree at most six in $G'$, in which case the root of the tree may be exceptional.*

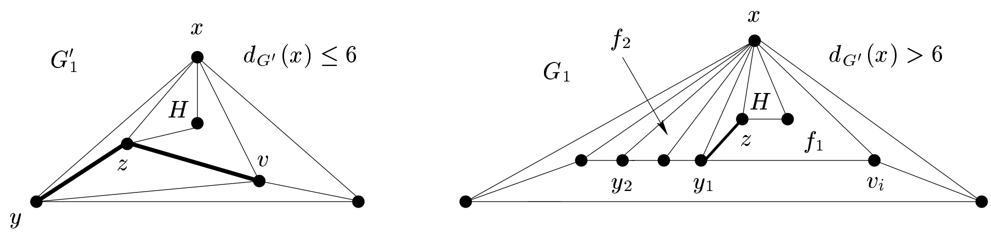


Figure 4: Illustrating the proof of Lemma 9 where the block $H$ is a triangle. The thick edges are those which were added to $G$.

*Proof.* Suppose first that $x$ lies on the boundary of the outer face and has degree at most six. Let $z$ be a vertex in $H$ which is adjacent to $x$. Let $y$ be a vertex on the boundary of the outer face which is adjacent to $x$ and let $v$ be a vertex which is not in the outer face but lies on the same face $f$ as $x$ and $y$ in $G'$ (such a $v$ exists since $x$ has degree at least three in $G'$). Let $G'_1$ be the graph obtained from $G$ by inserting the edges $zy$ and $zv$ (thus $H$ is embedded into $f$). Now apply Lemma 8 to $G'_1$. If the resulting graphs $G_1, \dots, G_s$ are not 3-connected, then we repeatedly apply Lemma 8 to those graphs and their children until we arrive at graphs which are 3-connected. We claim that these graphs then form the leaves of a triangulation tree of $G'_1$ containing no exceptional vertices. In other words, we can never encounter the exceptional case Lemma 8 where the two cut-vertices lie in the outer face. This follows since $G'$ is 3-connected and $V(H)\backslash\{x\}$ is connected to two distinct vertices of $G'$ in $G'_1$. Thus we cannot separate the graph into several components by a cut consisting of exactly two vertices on the outer face. Since the vertices on the outer face are always the same, this proves the claim. We then obtain a triangulation tree of $G$ by adding one more vertex (corresponding to $G$) and letting its only child be the vertex corresponding to $G'_1$. Since $G'_1$ had two more edges than $G$, the root of this tree is an exceptional vertex.

Now suppose that the degree of $x$ in $G'$ is more than six. Let $v_1, \dots, v_p$, where $p \geq 7$, be the neighbours of $x$ in $G'$, ordered clockwise when viewed from $x$. Then $x$ has at least four neighbours $y'_1, \dots, y'_4$ (also ordered clockwise when viewed from $x$) which are not contained in the outer face. Let $f_1$ be the face of $G'$ which contains $x$, $y'_1$ and the neighbour $v_i$ of $x$ which lies to the left of $y'_1$

when viewed from $x$. Then we may have $v_i = y'_4$ but we claim that $f_1$ cannot also contain $y'_2$ or $y'_3$. (Indeed, if $f_1$ contains $y'_2$ for instance, then one can easily verify that the removal of $x$ and $y'_2$ separates $y'_1$ from $v_i$, contradicting the fact that $G'$ is 3-connected.) Now let $f_2$ be the face of $G'$ which contains $x$, $y'_3$ and the neighbour $v_j$ of $x$ which lies to the left of $y'_3$ when viewed from $x$. Then as above, we may have $v_j = y'_2$ but $y'_1$ is not contained in $f'_2$. Similarly, it is easily checked that there is no face $f$ which contains $x$, $y'_1$ and $y'_3$. Now let $y_1 := y'_1$ and $y_2 := y'_3$. Let $z$ be a vertex in $H$ which is adjacent to $x$. For $i = 1, 2$, let $G_i$ be the graph obtained from $G'$ by inserting $H$ and the edges $xz$ and $zy_i$ into $f_i$. Obviously, $G_i$ contains $G$.

Hence an application of Lemma 8 to $G_i$ gives us graphs $G'_{i1}, \ldots, G'_{is'}$ and which contain $G$ and which are incomparable. By repeatedly applying the Lemma 8 to the graphs $G'_{ij}$ obtained and their children, we will eventually obtain graphs $G_{i1}, \ldots, G_{is}$ which have the additional property of being 3-connected and which form the leaves of a triangulation tree of $G_i$. As in the previous case, there are no exceptional vertices. (Consider two vertices $w_1$ and $w_2$ on the outer face of $G_i$ and let $W := \{w_1, w_2\}$. Since $f_i$ was not the outer face, we have $V(H) \cap W \subseteq \{x\}$. Since $H$ is 2-connected, it follows that $H - W$ is connected. Also, since the $y_i$ are not contained in the outer face, we have $y_i \notin W$ and thus $G_i - W$ contains the edge $zy_i$. Finally, since $G'$ is 3-connected, $G' - W$ is connected. Putting these three observations together, we see that $G_i - W$ is connected.)

We will now prove that for any $j$, $G_{1j}$ is incomparable with $G_2$ (and thus with $G_{2j'}$, for any $j'$). For this, by applying Proposition 6 with $e_0 = zy_2$, it suffices to show that $zy_2 \in E(G_2) \cap E(\overline{G_{1j}})$ and $zy_2 \notin \mathrm{Ins}(G_{1j})$. To see this, first note that as $G'$ is 3-connected and contains the outer face, in any of the above applications of Lemma 8, $G'$ will be contained in the external component and thus the embeddings of $G_{1j}$ when restricted to $G'$ will be the same as that of $G_2$ when restricted to $G'$. Now consider the embedding of $G_{1j}$ when restricted to $G' + z$. $G' + z$ contains both $zy_1$ and $zx$, so $z$ must be embedded into a face of $G'$ containing both $x$ and $y_1$. Since $G'$ contains no face which contains $x$ and both of the $y_i$, this means that $zy_2 \notin E(G_{1j})$ and $zy_2 \notin \mathrm{Ins}(G_{1j})$ and thus the conditions of the proposition are satisfied.

Since $xz \in G$, the $G_i$ have only one more edge than $G$ and we thus obtain a triangulation tree of $G$ (with no exceptional vertices) as follows. We form a single tree from the triangulation trees of the $G_i$ by adding a root vertex corresponding to $G$ and letting its two children correspond to the $G_i$. $\square$

We can now prove Theorem 2 for connected graphs.

**Theorem 10.** *Every connected planar graph $G \in \mathcal{P}_{n,m}$ is contained in at least $\delta 3^{(3n-m)/2}$ labelled triangulations, where $\delta$ is an absolute constant.*

*Proof.* By inserting at most two edges into $G$, we can obtain a graph $G'$ that has a block $B$ which contains a triangle and which contains at least four vertices. Our aim is to construct a triangulation tree whose root corresponds to $G'$ and whose vertices correspond to graphs obtained from repeated applications of Lemmas 8 and 9 (and so contain $G'$).

Fix an embedding of $B$ so that the outer face is a triangle. If $B$ is not 3-connected, we apply Lemma 8 to obtain a set of children of $B$. We then apply Lemma 8 to those children which are not 3-connected and continue in this way until we have obtained a triangulation tree of $B$ whose root corresponds to $B$, where the remaining vertices correspond to descendants of $B$ and whose leaves are $B_1, \dots, B_s$, say. Note that we do not assume that the graphs corresponding to the leaves all have the same number of edges. Also on any path from the vertex to the root of this tree, the number of exceptional vertices is at most three – they can only appear if we apply the lemma to a pair of vertices on the outer face and the vertices on the outer face are always the same.

If $B = G'$ (i.e. $G'$ was 2-connected), then we have a triangulation tree of $G'$. Now suppose that $G'$ was not 2-connected and let $H$ be a block of $G'$ which has a vertex $x$ in common with $B$ (there will be exactly one such vertex in $H$). Now apply Lemma 9 to the graphs $B_i + H$ to obtain triangulation trees of $B_i + H$. We merge these into a single triangulation tree of $B \cup H$ as follows: we start with the triangulation tree of $B$ except that a vertex of the tree which corresponded to a graph $F$ in the triangulation tree of $B$ now corresponds to $F + H$. We then identify the roots of the triangulation trees of $B_i + H$ with the leaves $B_i$ of the tree of $B$. If $B + H = G'$, then again we have a triangulation tree of $G'$. If this is not the case, we take a new block $H'$ and apply Lemma 9, until we have dealt with all the blocks of $G'$ and thus obtained a triangulation tree of $G'$.

It is now easy to verify by backward induction on the distance from the root in the triangulation tree that each graph corresponding to a vertex of the tree is contained in the required number of triangulations. By Proposition 5, this is certainly true (with $\delta = 2^{-6}$) for the leaves of the tree because they are 3-connected. For the induction step, consider a vertex $F \in \mathcal{P}_{n,m}$ and suppose that its children all have $m+t$ edges and are all contained in at least $q$ triangulations. Then Lemmas 8 and 9 imply that the number of triangulations of $F$ is at least $q\, 3^{t/2}$ (which is exactly what we are aiming for), unless we encountered the exceptional case of either of the lemmas, in which case we find at least $q\, 3^{(t-2)/2}$ triangulations. But it is easily seen that on any path from a leaf to the root of the tree we can encounter the exceptional case of Lemma 8 at most three times when we build a triangulation tree of $B$. When we apply Lemma 9 to incorporate the other blocks, we can encounter the exceptional case (where $x$ lies on the outer face and has degree at most six) at most 12 times altogether (since an application of the lemma increases the degree of the vertex $x$ in the statement of the lemma by at least one and there are three vertices on the outer face, which are always the same ones). Since $G'$ had at most two more edges than $G$, this proves the theorem (rather crudely) with $\delta = 3^{-(2/2+3+12)}\, 2^{-6}$.

□

Finally, we are in a position to deal with the case where $G$ is not necessarily connected.

*Proof of Theorem 2.* If $G$ has a component $K$ containing all but at most two vertices of $G$, then we are immediately done by applying Theorem 10 to $C$. So suppose that this is not the case. By inserting at most six edges into $G$, we can obtain a graph $G'$ which contains two triangles $t_a$ and $t_b$ which are both contained in distinct components. Let $v_a$ be a vertex on $t_a$ and $v_b$ one on $t_b$. Denote by $K_1, \ldots, K_k$ the components of $G'$ not containing $t_a$ or $t_b$. We now construct $2^k$ connected plane graphs $G_s$ which all contain $G'$ as a subgraph. Consider some $s$ with $0 \leq s < 2^k$ and consider the binary expansion of $s$. If the $\ell$th entry is equal to 0, we place the $K_\ell$ into $t_a$ and add an edge joining $K_\ell$ to $v_a$. If the $\ell$th entry is equal to 1, we do the same with $t_b$ instead. Carrying this out for all $\ell$ with $1 \leq \ell \leq k$, this gives us a plane graph $G_s$.

Thus for each $0 \leq i < 2^k$, the graph $G_i$ has two components $A_i$ and $B_i$ with $n_{i,a}$ and $n_{i,b}$ vertices and $m_{i,a}$ and $m_{i,b}$ edges respectively (where $n_{i,a} + n_{i,b} = n$ and $m_{i,a} + m_{i,b} \leq m + 6 + k$) and an embedding so that the outer face is bounded by $t_a$ and $t_b$. Now we apply Theorem 10 to all the $A_i$ and $B_i$. Clearly, we can turn any triangulation $T_{i,a}$ of $A_i$ and $T_{i,b}$ of $B_i$ into one of $G_i$ by considering an embedding of $T_{i,a}$ and $T_{i,b}$ where $t_a$ and $t_b$ are on the outer face and triangulating the outer face of the resulting plane graph in an arbitrary way (we call such triangulation of $G_i$ a *canonical* triangulation of $G_i$). Since in the last step we only added edges which have one endpoint in $t_a$ and the other in $t_b$, the number of canonical triangulations of $G_i$ is at least the product of the number of triangulations of $A_i$ and $B_i$.

Finally, we claim that for any $0 \leq i \neq j < 2^k$, any canonical triangulation $T$ of $G_i$ is incomparable with $G_j$. Indeed, since $i \neq j$, there must be an $\ell$ so that $K_\ell$ was inserted into $t_b$ (by adding an edge $e_b$ between $v_b$ and $K_\ell$) in $G_j$ but not in $G_i$. The claim now follows by applying Proposition 6 to $e_b$, $G_j$ and $T$. Theorem 10 now shows that the number of triangulations of $G'$ is at least

$$\sum_{i=0}^{2^k-1} \delta^2\, 3^{(3n_{i,a} - m_{i,a} + 3n_{i,b} - m_{i,b})/2} \geq 2^k\, \delta^2\, 3^{(3n-m-6-k)/2} = \frac{\delta^2}{3^3} 3^{(3n-m)/2} \left(\frac{2}{\sqrt{3}}\right)^k$$

as required. □

We conclude this section by showing that Theorem 2 is essentially best possible for $m \geq 2n$.

**Proposition 11.** *For any $c$ with $2 \leq c \leq 3$ and for all $n \in \mathbb{N}$, there is a graph $G$ in $\mathcal{P}_{n',m}$, where $n' = n + o(n)$ and $m = cn + o(n)$, so that the number of triangulations on $n'$ vertices which contain $G$ is $3^{(3n'-m+o(n))/2}$.*

*Proof.* We construct $G$ as follows. We begin with a square grid on $n/2 + o(n)$ vertices together with an arbitrary triangulation of the outer face of the grid (the number of edges needed to triangulate the outer face is $o(n)$) to obtain a plane graph $D$. Since we are considering labelled plane graphs, we can speak of the "top row" of the grid, etc. in what follows. We now "augment" each square inside the grid by adding a single new vertex into each face of the square grid and connecting it to both the bottom left and the top right vertex of the face (in other words, an augmented square is a four-cycle where two opposite vertices are connected by an additional path of length two). The resulting graph has $n + o(n)$ vertices, $2n + o(n)$ edges and $n/2 + o(n)$ augmented squares.

We obtain a graph $G$ with $m = cn + o(n)$ edges by selecting $(c - 2 + o(1))n/2$ squares of the original grid and connecting the corresponding new vertices inside these squares to the remaining two vertices on the outside of this square. We call an augmented square not triangulated in this way *open*.

It remains to verify the claim about the number of triangulations of $G$. Since $D$ is 3-connected, Proposition 4 implies that $D$ has the 1-face property (which can of course also be easily verified directly). Hence, as already noted at the beginning of the proof of Proposition 5, the number of triangulations of $D$ is the product of the number of triangulations of the squares in $D$. It is easily seen that this in turn implies that the number of triangulations of $G$ is the product of the number of triangulations of the open augmented squares. But, clearly, the number of triangulations of each open augmented square is exactly three and there are

$$\frac{n}{2} - \frac{m - 2n}{2} + o(n) = \frac{3n - m}{2} + o(n)$$

open squares. □

## 4 Upper bounds

In this section, we apply Theorem 2 to deduce upper bounds on the number of planar graphs (implying Theorem 1) and on their likely number of edges (Theorem 3). For $0 < x < 1$ let

$$H(x) := -x \log x - (1 - x) \log(1 - x) \tag{6}$$

be the *entropy function*, where log denotes the logarithm to base 2 and let $H(0) := H(1) := 0$ for convenience. It has the property that $\binom{n}{xn} = 2^{H(x)n+o(n)}$ for $0 \leq x \leq 1$.

For $G \in \mathcal{P}_n$ define

$$\beta(G) := |\{H \in \mathcal{T}_n : G \subseteq H\}|.$$

For $0 \leq c \leq 3$ we let

$$\beta(c, n) := \min_{G \in \mathcal{P}_{n,cn}} \beta(G).$$

By the result (1) of Tutte, the number $|\mathcal{T}_n|$ of labelled triangulations on $n$ vertices is at most $n!\,\gamma^{n+o(n)}$. This implies that

$$|\mathcal{P}_{n,cn}| \leq |\mathcal{T}_n| \frac{\binom{3n-6}{cn}}{\beta(c, n)} \leq n!\,\gamma^{n+o(n)} \frac{2^{3H(c/3)n+o(n)}}{\beta(c, n)}. \tag{7}$$

Theorem 2 implies that $\beta(c, n) \geq 3^{(3-c)n/2+o(n)}$ and thus has the following immediate consequence, which is a generalization of Theorem 1.

**Theorem 12.** *For $0 \le c \le 3$,*

$$|\mathcal{P}_{n,cn}| \le n!\,\gamma^{n+o(n)}\,2^{3H(c/3)n}\,3^{-(3-c)n/2}.$$

*In particular, $|\mathcal{P}_n| \le n!\,(37.3)^{n+o(n)}$ (the maximum is attained at $c \sim 1.902$).*

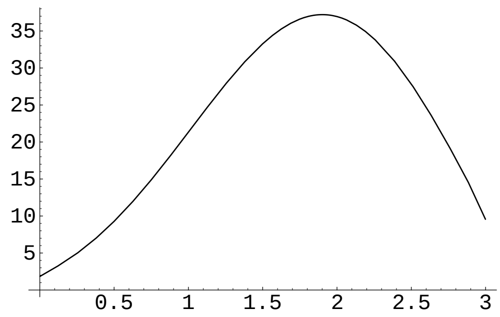


Figure 5: The upper bound on $|\mathcal{P}_{n,cn}|^{1/n}/n!$ in Theorem 12 versus $c$, where $0 \le c \le 3$.

Now we turn to the proof of Theorem 3, which stated that almost all graphs in $\mathcal{P}_n$ have at most $2.56n$ edges. Let $\mathcal{P}^2$ denote the set of 2-connected planar graphs. In the proof, we will employ the bound $|\mathcal{P}_n| \ge |\mathcal{P}_n^2| = n!\,\alpha^{n+o(n)}$, where $\alpha \sim 26.1876$, due to Bender, Gao and Wormald [1].

*Proof of Theorem 3.* Numerical calculation gives that $\gamma\,2^{3H(c/3)}\,3^{-(3-c)/2} < 26.18 - 0.1$ for $c = 2.56$. Note that this inequality also holds for $c \ge 2.56$ as $2^{3H(c/3)}\,3^{c/2}$ is decreasing in this range. Thus if we apply Theorem 12 with $c \ge 2.56$, we have

$$\bigcup_{2.56n \le m \le 3n-6} |\mathcal{P}_{n,m}| \le n!\,0.44n\ (26.18 - 0.1)^{n+o(n)} = o(|\mathcal{P}_n|),$$

which implies the result. □

Our upper bound in Theorem 12 applies only to labelled graphs. However, since any unlabelled graph on $n$ vertices corresponds to at most $n!$ labelled graphs the above result of [1] immediately implies that $|\mathcal{P}_n^u| \ge |\mathcal{P}_n^{2,u}| \ge (26.18)^{n+o(n)}$. Similarly to the proof of Theorem 3, one can compare this with the upper bound

$$|\mathcal{P}_{n,cn}^u| \le \gamma^{n+o(n)} \binom{3n-6}{cn} \tag{8}$$

(which follows from (1) again) to observe the following result.

**Theorem 13.** *Almost all graphs in $\mathcal{P}_n^u$ have at most $2.69n$ edges.*

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