Randomness to the rescue: if in doubt, flip a coin

Deryk Osthus

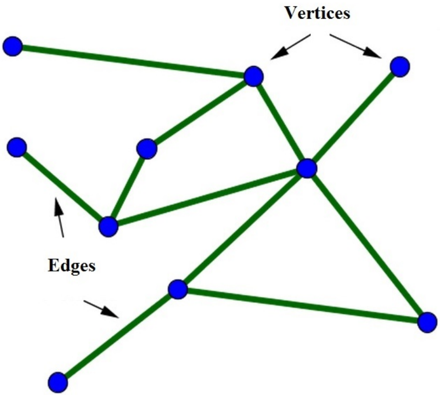
March 26, 2014

If in doubt, flip a coin: randomness helps

But Combinatorialists do.
Why, and what do we gain?

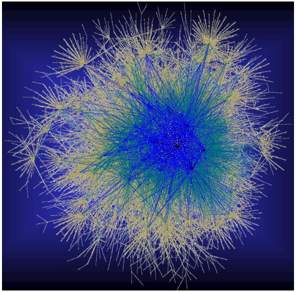
What is a graph?

Graphs:



Modelling large 'real world' graphs

Internet graph:



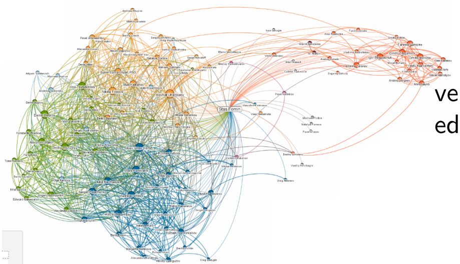
vertices $=$ webpages
edges $=$ links

or

vertices $=$ computers
edges $=$ physical links

Modelling large ‘real world’ graphs

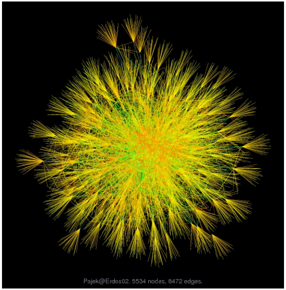
Social networks
(e.g. linkedin, facebook):



vertices = people
edges = friends/ colleagues

Modelling large 'real world' graphs

Erdős collaboration graph:

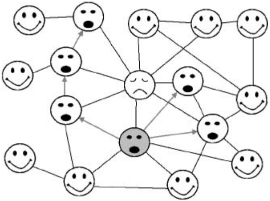


vertices = mathematicians
edges = have collaborated on a paper

Modelling large 'real world' graphs

Questions

- How do these networks grow?
- Do they have common features?
- Can one model/ predict rumor or virus spread?

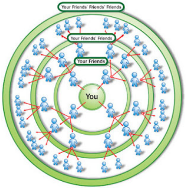


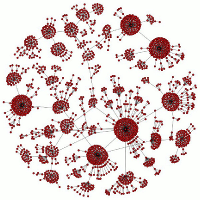
Obtain answers by modelling these networks as large **random** graphs. We can do simulations or prove theorems about these models to make predictions.

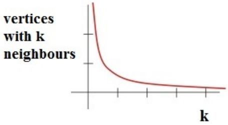
Modelling large 'real world' graphs

Observed features of networks

- Small distances ('small world phenomenon')
- Local clustering (friends of friends are likely to be friends)
- Scale free degree distribution: the proportion of vertices with k neighbours is $\sim k^{-\gamma}$ ($\gamma = 2.1$ for the internet graph)



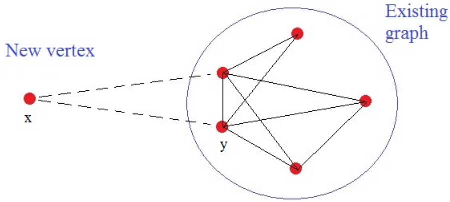




Modelling large 'real world' graphs

Random model: Rich get richer (Barabási-Albert model)

Graph grows by adding vertices which are more likely to attach to vertices of already high degree:



Prob(x connects to y) is proportional to:
the number of neighbours of y + initial attractivity γ

Bollobás & Riordan: small distances
Buckley & Osthus: scale free degree distribution, $k^{-\gamma}$

Modelling large ‘real world’ graphs

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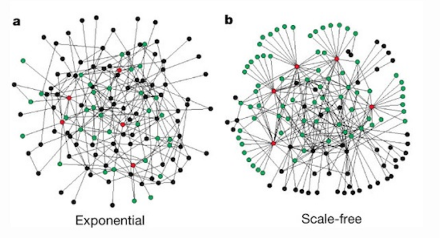
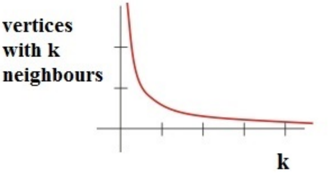
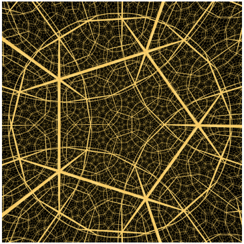
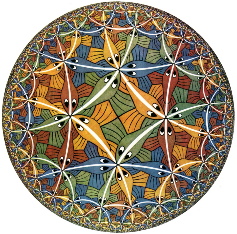




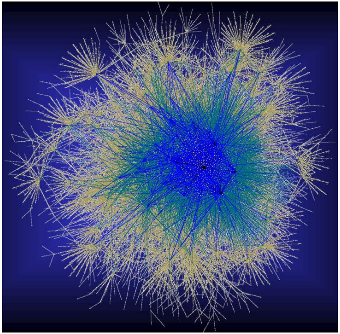
What about local clustering?
Need more complicated models which involve the ‘geometry’ of the network.
Recent approach: embed network into ‘hyperbolic space’.

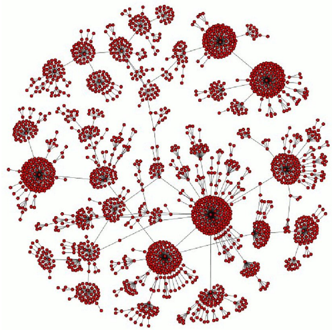




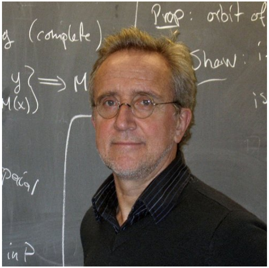
Current work by several groups, including Fountoulakis (Birmingham)

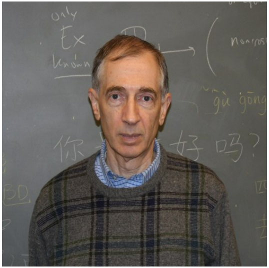
Have seen:
Use probability ("coin flipping") to model 'real world' networks.





Next:
Use probability to prove the existence of good solutions to hard problems.

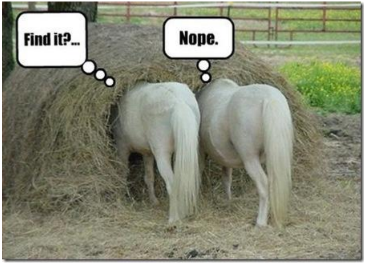




Anders Björner and **Richard P. Stanley**

"Combinatorics is somewhat of a Cinderella story. It used to be looked down on by "mainstream" mathematicians as being somehow less respectable than other areas. Then along came the prince of computer science with its many mathematical problems and needs – and it was combinatorics that best fitted the glass slipper held out."

Suppose we are given a difficult problem (which amounts to finding a needle in a haystack)...



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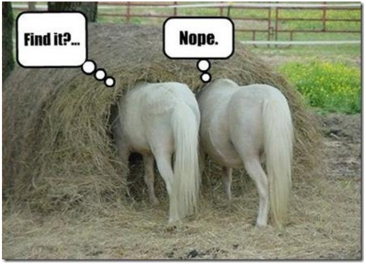

Idea: Use coin flips to decide what a solution should look like.

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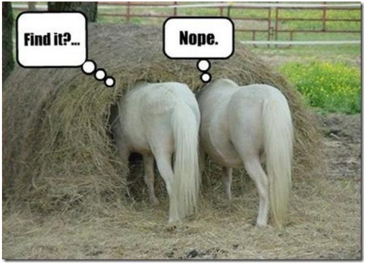


Idea: Use coin flips to decide what a solution should look like.

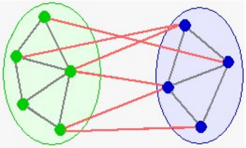
More precisely, we obtain a good solution if we can show

$$\mathbb{P}(\text{good solution exists}) > 0.$$

Hence one such solution exists!

MaxCut problem in graphs

Split vertices so that most edges go across split.
Applications e.g. in computer chip design.



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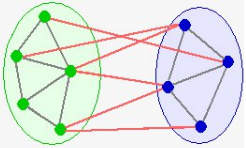

'Extremal' solution to this problem:

Theorem (Edwards, University of Birmingham 1973)

If a graph has m edges, we can always guarantee

$$\frac{m}{2} + \sqrt{\frac{m}{8}}$$

edges across.

bound is best possible

Multicoloured MaxCut

Have edges in several colours/types (red blue ...)

Aim: Find cut with many edges of each colour going across.

Question (Bollobás and Scott)

Can one ensure that at least half of red and half of blue go across?

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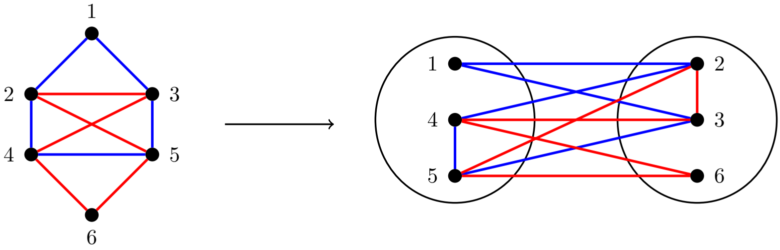

4 blue across
4 red across

(Multicoloured) MaxCut

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(Multicoloured) MaxCut

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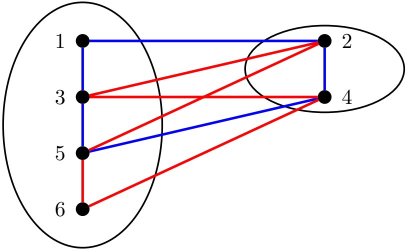
Answer (Kühn and Osthus)

Yes! Flip a coin for each vertex.



2 blue across and 4 red across

(Multicoloured) MaxCut

Question (Bollobás and Scott)

Can one ensure that at least half of red and half of blue go across?

Answer (Kühn and Osthus)

Yes! Flip a coin for each vertex.

Can show:

$$\mathbb{P}(\text{random solution is `good'}) > 0.$$

So: there is a good solution.

Question (Bollobás and Scott)

Can one ensure that at least half of red and half of blue go across?

Answer (Kühn and Osthus)

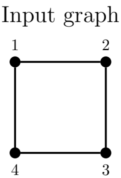
Yes! Flip a coin for each vertex.

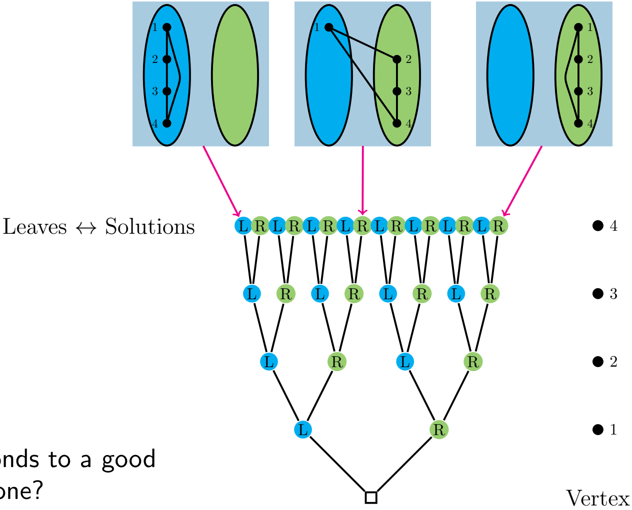
So there exists a good solution.

What if we want to find it?
For simplicity, forget about colours.

Trying out possibilities is like finding a needle in a haystack.

Solution: Assign vertices to Left (L) and Right (R) one by one and look at decision tree.



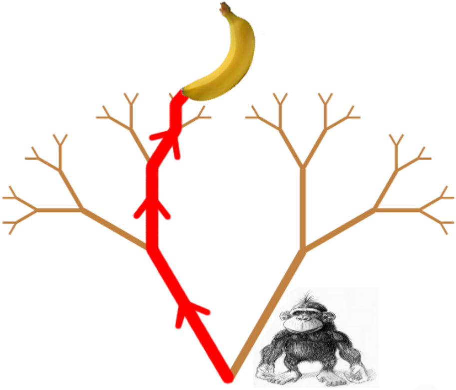


One leaf corresponds to a good solution. Which one?

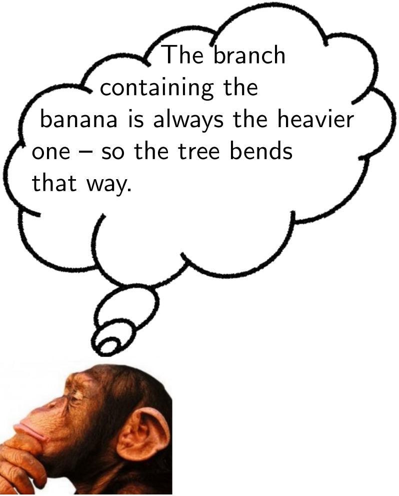
(Multicoloured) MaxCut

Ask an ape!!

(how he finds a banana in a binary tree without looking.)



(Multicoloured) MaxCut



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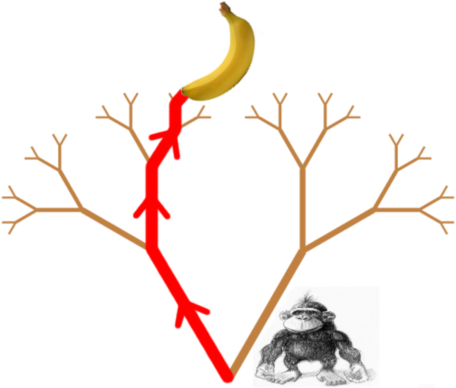
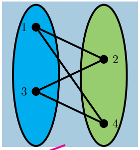
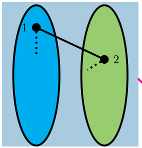


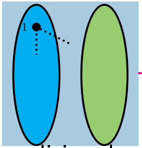
(Multicoloured) MaxCut

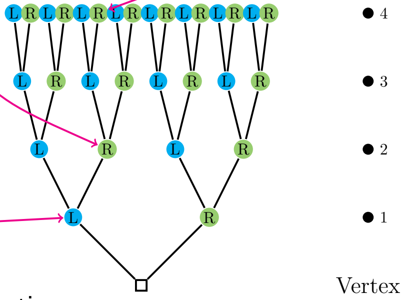
To solve the ‘real’ decision tree problem, give each partial solution a weight according to its (predicted) quality.



Follow the path in the decision tree which predicts the best quality solution (largest weight).







Do this via computing conditional expectations.

Summary of approach

Probabilistic reasoning

$\Downarrow$

$\mathbb{P}(\text{good solution exists}) > 0$

$\Downarrow$

There is a good solution

Consider random cut

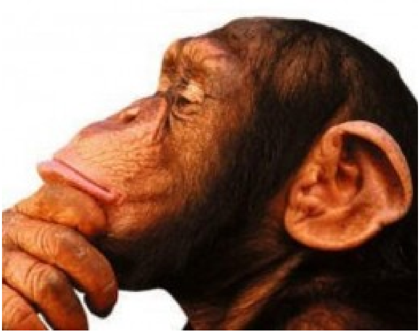
$\Downarrow$

$\mathbb{P}(\text{large cut exists}) > 0$

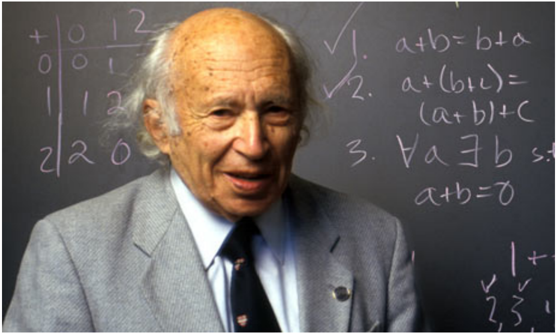
$\Downarrow$

There is a large cut

Sometimes can use 'ape in a tree' argument to <u>find</u> a good solution efficiently.

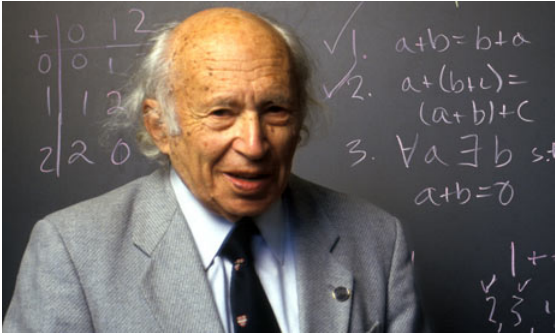


What is combinatorics?



Once, Israil Gelfand said that mathematics has three parts:

What is combinatorics?



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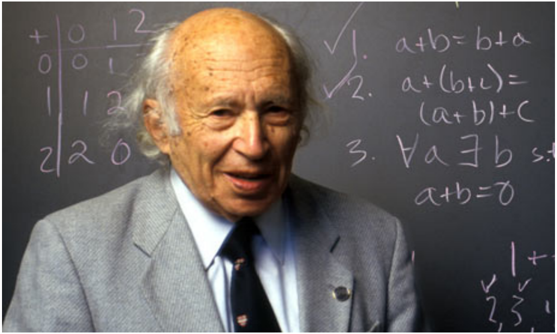
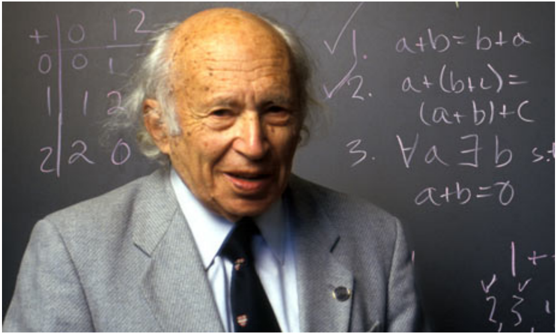


Once, Israil Gelfand said that mathematics has three parts: xxx, yyy, and Combinatorics.
"What is combinatorics?" the listeners asked.

What is combinatorics?



Once, Israil Gelfand said that mathematics has three parts:
xxx, yyy, and Combinatorics.
"What is combinatorics?" the listeners asked. The answer was:

"This is a science not yet created ..."

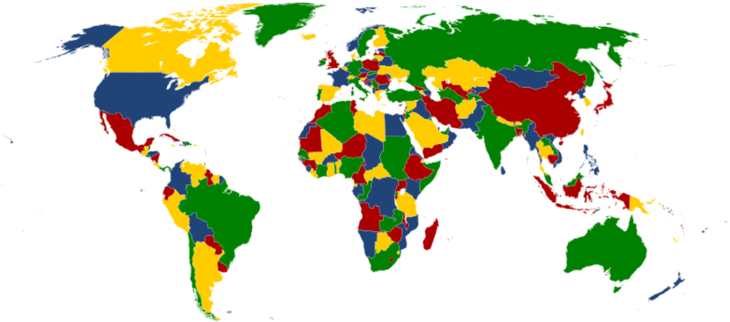
— Israil M. Gelfand (1980s)

Colouring maps

Can we colour the countries of any given map with at most 4 colours so that neighbouring countries get different colours?

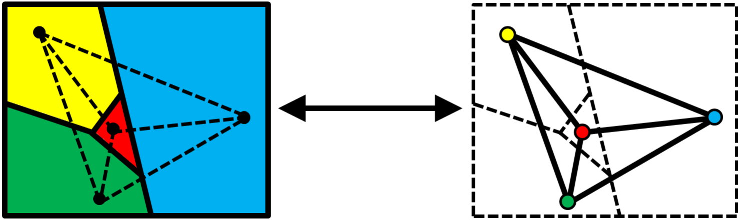
A 4-colouring of the world's countries

Can we colour the countries of any given map with at most 4 colours so that neighbouring countries get different colours?

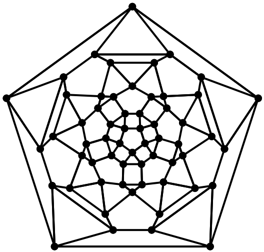


From maps to graphs/networks

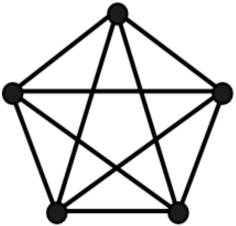
- For each country, introduce a vertex.
- Connect vertices by an edge if the countries are neighbours.



Note that the graphs we obtain are always planar, which means they can be drawn without crossing edges.



Planar



Non-planar

Question (4-colour problem for graphs (asked in 1852))

Can one colour the vertices of every planar graph with at most 4 colours so that neighbouring vertices get different colours?

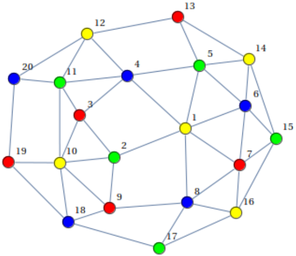
Appel & Haken, 1976: YES

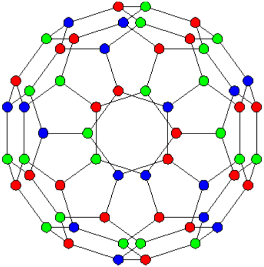
Vertex colourings

Definition

Proper colouring = *a vertex colouring where adjacent vertices receive different colours.*

Chromatic number: $\chi(G)$ = *smallest number of colours in a proper colouring of* G.

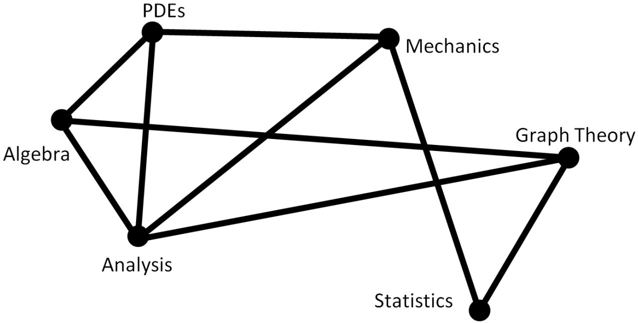




Application: Timetabling and scheduling

Exam scheduling conflict graph

- vertices $\leftrightarrow$ exams
- edge between two vertices if someone is taking both exams
- $\chi(G)=$ number of time slots needed



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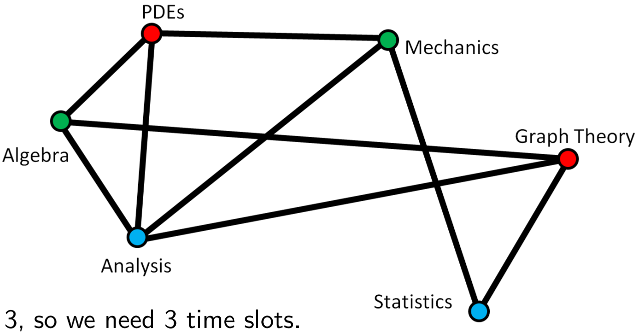
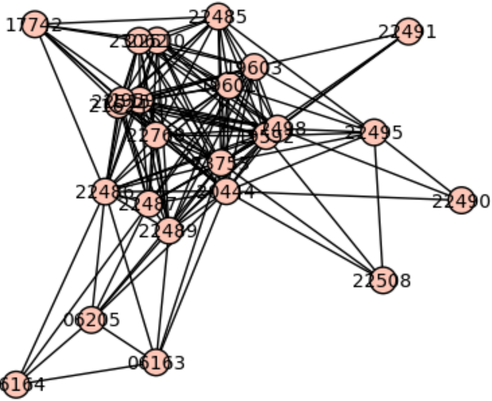


$\chi(G) = 3$, so we need 3 time slots.

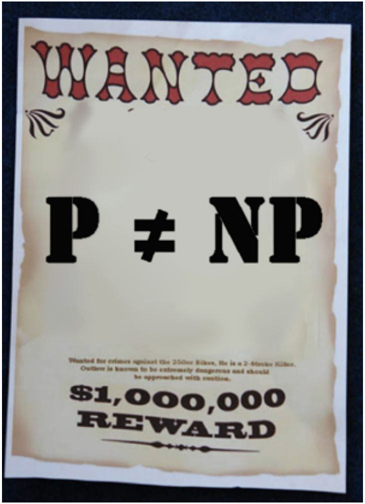
Things rapidly get complicated...

Conflict graph for some modules in the School of Mathematics



$\chi(G) = 11$

Graph colouring is hard



Bad news: Determining $\chi(G)$ is NP-complete

Clay Institute Millennium Problem

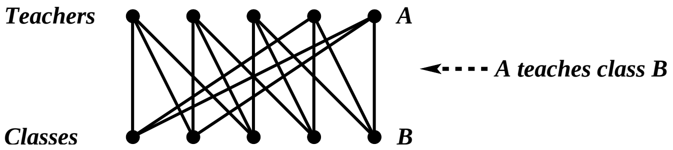
Decide whether P=NP.

(one of 7 millennium problems, with a \$1,000,000 reward!)

Efficient algorithm for determining $\chi(G)$ would imply P=NP

Edge colourings

Now we colour **edges** instead, so that adjacent edges get different colours.



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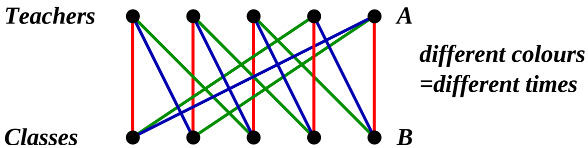


- **edge chromatic number:** $\chi_{\text{edge}}(G) =$ smallest number of colours needed in a proper edge-colouring of G
- Determining $\chi_{\text{edge}}(G)$ in general is NP-complete

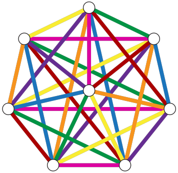
Edge colourings

A graph is **D-regular** if every vertex has exactly D neighbours.

Conjecture (1-factorization conjecture, Dirac 1950's)

Let G be a D-regular graph on n vertices, where n is even and $D \geq n/2$. Then $\chi_{edge}(G) = D$.

A 7-edge-colouring of the complete graph K_8 on 8 vertices.



Bounds in Conjecture would be best possible:

- trivially, $\chi_{\text{edge}}(G) \geq D$.
- cannot replace $n/2$ by $n/2 - 1$

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- True for $D = n - 1$, i.e. complete graphs.
- Chetwynd and Hilton (1989), and independently Niessen and Volkmann (1990), for $D \geq (\sqrt{7} - 1)n/2 \approx 0.82n$.
- Perkovic and Reed (1997) for $D \geq (1/2 + \varepsilon)n$ with $\varepsilon > 0$.
- Vaughan (2013) : an approximate multigraph version.

Edge colourings

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Conjecture (1-factorization conjecture, Dirac 1950's)

Let G be a D-regular graph on n vertices, where n is even and $D \geq n/2$. Then $\chi_{edge}(G) = D$.

Proof.

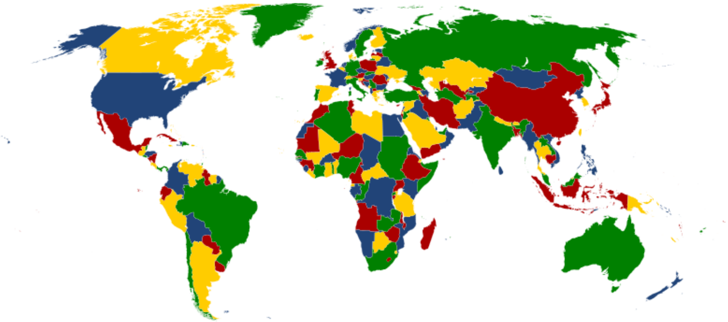
Proved by Csaba, Kühn, Lo, Osthus & Treglown, 2013.

- Proof is 200 pages and uses probabilistic arguments.
- Proof argument gives 2 more conjectures on Hamilton cycles from 1970's.

□

Theorem (Appel & Haken, 1976)

The vertices of every planar graph can be coloured with at most 4 colours, i.e. every planar graph G has $\chi(G) \leq 4$.

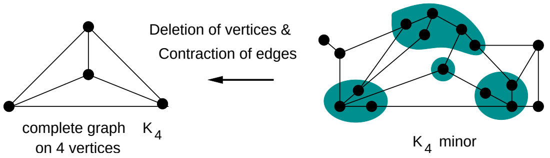


Proof of the 4-colour theorem is long and needs computer help.

4-colour theorem vs Hadwiger's conjecture

Proof of the 4-colour theorem is long and needs computer help.

Complete graphs K_r as minors of other graphs:



Conjecture (Hadwiger's conjecture, 1943)

If r colours are needed to colour graph G, then G contains a complete graph K_r on r vertices as a minor.

4-colour theorem vs Hadwiger's conjecture

Conjecture (Hadwiger's conjecture, 1943)

If r colours are needed to colour graph G, then G contains a complete graph K_r on r vertices as a minor.

Remark

Hadwiger's conjecture for $r = 5$ implies the 4-colour theorem, i.e. that every planar graph G has $\chi(G) \leq 4$.

4-colour theorem vs Hadwiger's conjecture

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Remark

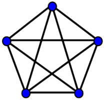
Hadwiger's conjecture for $r = 5$ implies the 4-colour theorem, i.e. that every planar graph G has $\chi(G) \leq 4$.

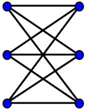
To prove this remark, we use:

Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

K_5





$K_{3,3}$

4-colour theorem vs Hadwiger's conjecture

Conjecture (Hadwiger's conjecture, 1943)

If r colours are needed to colour graph G, then G contains a complete graph K_r on r vertices as a minor.

Remark

Hadwiger's conjecture for $r = 5$ implies the 4-colour theorem, i.e. that every planar graph G has $\chi(G) \leq 4$.

Proof (of Remark).

Suppose that $\chi(G) > 4$.
Hadwiger's conjecture implies that G has a K_5 minor.
Then Kuratowski's theorem implies that G is not planar. $\square$

4-colour theorem vs Hadwiger's conjecture

Conjecture (Hadwiger's conjecture, 1943)

If r colours are needed to colour graph G, then G contains a complete graph K_r on r vertices as a minor.

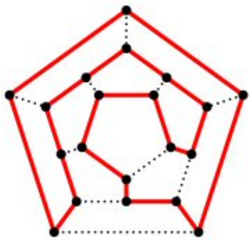
Remark

Hadwiger's conjecture for $r = 5$ implies the 4-colour theorem, i.e. that every planar graph G has $\chi(G) \leq 4$.

- Robertson, Seymour & Thomas 1993: true for $r = 6$
- Kühn & Osthus 2003: true for 'locally sparse' graphs
- ...

Hamilton cycles in graphs

Hamilton cycle contains every vertex exactly once



Question

Can you decide if a graph contains a Hamilton cycle? Is this a difficult problem?

- no characterization of Hamiltonian graphs known
- decision problem NP-complete

Hamilton cycles in graphs

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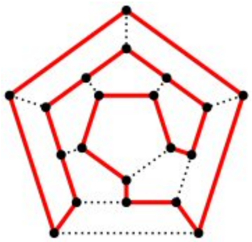

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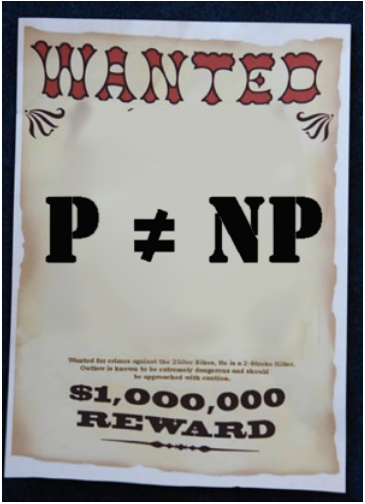


Clay Institute Millennium Problem

Decide whether P=NP.

efficient algorithm for checking Hamiltonicity would imply P=NP

Aim

Find simple sufficient conditions which guarantee a Hamilton cycle

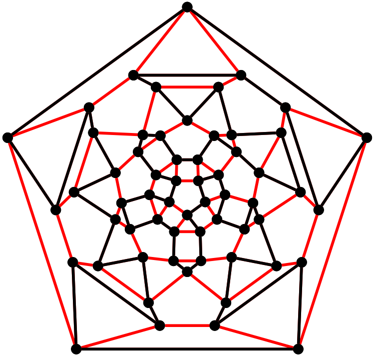
Aim

Simple sufficient conditions which guarantee a Hamilton cycle

Proved longstanding conjectures by:

- Thomassen 1979 (Keevash, Kühn, Osthus)
- Thomassen 1982 (Kühn, Lapinskas, Osthus, Patel)
- Bollobás & Häggkvist 1970's (Kühn, Lo, Osthus, Staden)
- Nash-Williams 1970 (Csaba, Kühn, Lapinskas, Lo, Osthus, Treglown)
- Kelly 1968 (Kühn, Osthus)
- Frieze & Krivelevich, 2005 (Knox, Kühn & Osthus)
- ...

Hamilton decomposition of G
$=$ set of edge-disjoint Hamilton cycles covering all edges of G

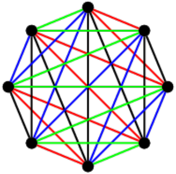


Which graphs/digraphs have Hamilton decompositions?
Very few general conditions known

Theorem (Walecki, 1892)

Complete graph K_n has a Hamilton decomposition $\Leftrightarrow$ n odd

Construction: find Hamilton path decomposition for K_{n-1}



then add extra vertex and close paths into Hamilton cycles

Hamilton decomposition conjecture

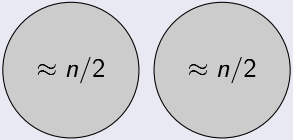
A graph is **D-regular** if every vertex has exactly D neighbours.

Hamilton decomposition conjecture (Nash-Williams 1970)

Every D-regular graph on n vertices with D even and $D \geq \lfloor n/2 \rfloor$ has a decomposition into Hamilton cycles.

Extremal examples

No disconnected graph contains a Hamilton cycle.



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- Nash-Williams (1969), $D \geq \lfloor n/2 \rfloor$ guarantees a Hamilton cycle.
- Jackson (1979), $D/2 - n/6$ edge-disjoint Hamilton cycles
- Christofides, Kühn and Osthus (2012) $D \geq n/2 + \varepsilon n$ guarantees an almost Hamilton decomposition.
- Kühn and Osthus (2014) $D \geq n/2 + \varepsilon n$ guarantees Hamilton decomposition.

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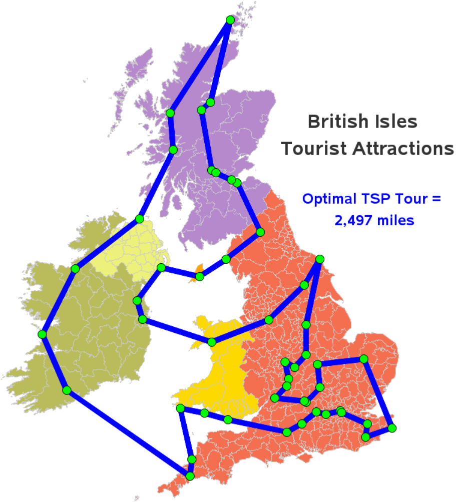
Theorem (Csaba, Kühn, Lo, Osthus, Treglown 2014^{+})

Hamilton decomposition conjecture holds for sufficiently large n.



Travelling salesman problem

Given a graph and weights on the edges, find a shortest tour which visits all vertices.

- Weighted version of the Hamilton cycle problem.
- Problem is NP-complete.

Travelling salesman tour domination

Popular approach:
Find 'approximate' solutions which are close to optimal

Alternative approach:

Domination ratio of an algorithm A

A has domination ratio d $\Leftrightarrow$ the proportion of solutions which are worse than those produced by A is at least d.

So want an algorithm with d close to 1.

Travelling salesman tour domination

Domination ratio of an algorithm A

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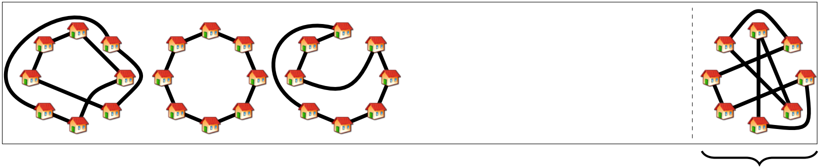
Tours ranked by cost
Number of cities $= n$



Best known result: guaranteed not to lie in bottom $\frac{1}{n}$ fraction.

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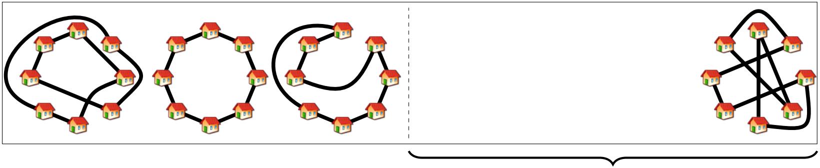

Kühn and Osthus: guaranteed not to lie in bottom $\frac{1}{2}$.

Travelling salesman problem

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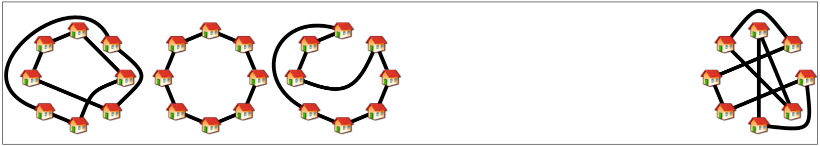
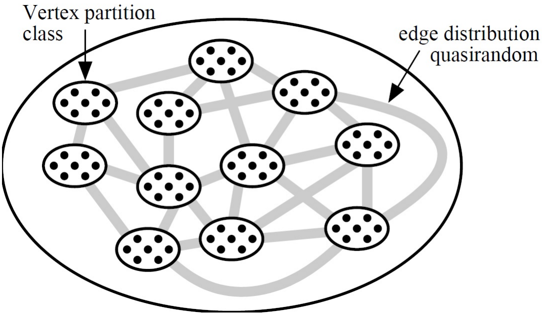


Kühn, Osthus and Patel 2014: guaranteed to lie in top $1/n$ fraction for 1–2 weights.

Proofs use more sophisticated versions of coin flipping; for example:

Szemerédi's Regularity Lemma

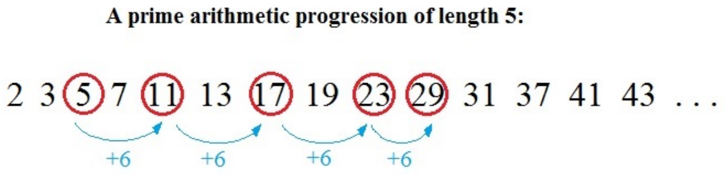
Every dense graph can be approximated by a small number of random graphs.



These tools and ideas have applications outside of Graph Theory:

Theorem (Green, Tao, 2004)

The primes contain arithmetic progressions of arbitrary length.



The proof uses the fact that primes are 'randomly distributed' on any long interval of the integers.

The development of Combinatorics

Nati Linial:

Combinatorics must always have been fun. But when and how did it become a serious subject? I see several main steps in this development:

- *The asymptotic perspective.*
- *Extremal combinatorics (in particular extremal graph theory).*
- *The emergence of the probabilistic method.*
- *The computational perspective.'*

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Many thanks to: Katherine Staden, Amelia Taylor, Tim Townsend