Four-cycles in graphs without a given even cycle

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Abstract

We prove that every bipartite $C_{2\ell}$-free graph G contains a C_4-free subgraph H with $e(H) \geq e(G)/(\ell - 1)$. The factor $1/(\ell - 1)$ is best possible. This implies that $\mathrm{ex}(n, C_{2\ell}) \leq 2(\ell - 1)\mathrm{ex}(n, \{C_4, C_{2\ell}\})$, which settles a special case of a conjecture of Erdős and Simonovits.

1 Introduction

Given a finite family $\mathcal{F}$ of graphs and an integer n, the *Turán number* $\mathrm{ex}(n, \mathcal{F})$ *of* $\mathcal{F}$ is the largest integer m such that there exists a graph on n vertices with m edges which contains no member of $\mathcal{F}$ as a subgraph. Clearly $\mathrm{ex}(n, F) \geq \mathrm{ex}(n, \mathcal{F})$ for every $F \in \mathcal{F}$. Erdős and Simonovits [4] (see also [3]) asked whether there always exists an $F \in \mathcal{F}$ for which up to constants the converse also holds:

Conjecture A *For every finite family $\mathcal{F}$ of graphs there exists an $F \in \mathcal{F}$ such that* $\mathrm{ex}(n, F) = O(\mathrm{ex}(n, \mathcal{F}))$.

For the case when $\mathcal{F}$ consists of even cycles, this would mean that (up to constants) the Turán number of $\mathcal{F}$ is given by that of the longest cycle in $\mathcal{F}$. Verstraëte (personal communication) conjectured something stronger:

Conjecture B *For all integers $k < \ell$ there exists a positive $c = c(\ell)$ such that every $C_{2\ell}$-free graph G has a C_{2k}-free subgraph H with $e(H) \geq e(G)/c$.*

This conjecture was motivated by a result of Györi [5] who showed that every bipartite C_6-free graph G has a C_4-free subgraph which contains at least half of the edges of G. Our main result generalizes that of Györi:

Theorem 1 *Let $\ell \geq 3$ be an integer and G a $C_{2\ell}$-free bipartite graph. Then G contains a C_4-free subgraph H with $e(H) \geq e(G)/(\ell - 1)$.*

The complete bipartite graphs $K_{\ell-1,m}$ show that we cannot replace $1/(\ell-1)$ by anything larger (see Proposition 5). Moreover, as every graph G has a bipartite subgraph containing at least half of the edges of G, Theorem 1 carries over to arbitrary graphs with an extra factor 1/2. It would be interesting to know whether this factor is really necessary.

Theorem 1 immediately confirms Conjecture A in the following special case:

Corollary 2 *Let $\ell \geq 3$ be an integer. Then* $\mathrm{ex}(n, \{C_4, C_{2\ell}\}) \geq \frac{\mathrm{ex}(n, C_{2\ell})}{2(\ell-1)}$. □

Obviously Corollary 2 remains true if we replace the C_4 by any graph containing a C_4 as subgraph.

Conjecture B would follow if every $C_{2\ell}$-free graph G which can be obtained by pasting together C_{2k}'s has bounded average degree. (A graph G is *pasted together from* C_{2k}'s if G can be obtained from a C_{2k} by successively adding new C_{2k}'s which have at least one edge in common with the previous ones.) Indeed, we may assume that every edge of our given $C_{2\ell}$-free graph G lies in a C_{2k}. But every C_{2k} is contained in a maximal subgraph $D \subseteq G$ pasted together from C_{2k}'s and distinct such subgraphs D are edge-disjoint. If all these graphs D have bounded average degree, then the union of a spanning tree of each of them is a C_{2k}-free subgraph of G as desired in Conjecture B.

Problem C *Given integers $k < \ell$, does there always exist a number $d = d(\ell)$ such that every $C_{2\ell}$-free graph which is pasted together from C_{2k}'s has average degree at most d?*

Note that to settle Conjecture B, it would be sufficient to obtain an affirmative answer to the question in Problem C for bipartite graphs G. The next result deals with the simplest case of Problem C for bipartite graphs:

Theorem 3 *Let $\ell \geq 3$ be an integer and suppose that G is a bipartite $C_{2\ell}$-free graph which is obtained by pasting together C_4's. Then the average degree of G is at most 16ℓ.*

Clearly, Conjecture B would imply that every $C_{2\ell}$-free graph can be made into a graph of girth larger than 2ℓ by deleting at most a constant fraction of its edges. The following simple result shows that at least we can achieve any desired girth in this way by forbidding a longer even cycle of suitable length:

Theorem 4 *For every even integer $g \geq 4$ there are infinitely many integers ℓ such that every $C_{2\ell}$-free graph G contains a subgraph H whose girth is greater than g and such that $e(H) \geq e(G)/2(4\ell)^{(g-2)/2}$.*

A related conjecture of Thomassen [7] states that for all integers g and k there exists an integer d such that every graph G of average degree at least d contains a subgraph of average degree at least k and girth greater than g. Unlike what is expected in the case when $C_{2\ell} \not\subseteq G$ for some $\ell \geq g/2$, d clearly cannot depend linearly on k here (when g is fixed and even). For regular graphs the truth of Thomassen's conjecture was observed by Alon (see [6] for the argument) and the case $g \leq 4$ was proved in [6].

This paper is organized as follows. In Section 2 we prove Theorem 1 together with tighter bounds for graphs with few edges compared to ℓ (Lemma 6) and many edges (Corollary 7). Theorem 3 is then proved in Section 3 and Theorem 4 in the last section.

2 Eliminating 4-cycles in graphs without a given even cycle

Let us first introduce some notation. We denote the number of edges of a graph G by $e(G)$ and its number of vertices by $|G|$. We write $d(G) := 2e(G)/|G|$ for the *average degree* of G. The *girth* of G is the length of its shortest cycle. A *k-cycle* is a cycle of length k, which we abbreviate by C_k. The *successor* of a vertex x on a directed cycle $\vec{C}$ is the neighbour y of x on C for which the edge xy is directed from x to y. The *predecessor* of x is the neighbour y of x on C for which xy is directed from y to x. Given distinct vertices x and y on $\vec{C}$, we write $x\vec{C}y$ for the directed subpath of $\vec{C}$ joining x to y. The set of neighbours of a vertex $x \in G$ is denoted by $N_G(x)$. Given graphs F and G, we say that G is *F-free* if it does not contain F as a subgraph. We often denote a bipartite graph with vertex classes A and B by (A, B).

The following proposition implies that in Theorem 1 we cannot replace the factor $1/(\ell - 1)$ by anything larger.

Proposition 5 *Given integers $\ell \geq 3$ and m, there exists a $C_{2\ell}$-free bipartite graph G with at least m edges for which the maximum number of edges in a C_4-free subgraph is exactly $e(G)/(\ell-1) + \binom{\ell-1}{2}$.*

Proof. Consider the complete bipartite graph $G = (A, B)$ with $|A| = \ell - 1$ and $|B| = m$ and let H be any C_4-free subgraph of G. Let X be a maximal set of edges of H whose endvertices in B are distinct. (So $|X| \leq |B|$.) Given an edge $e = ab \in H - X$ with $a \in A$ and $b \in B$, let A_e be the (unordered) pair of vertices which consists of a and the endvertex in A of the unique edge in X incident with b. As H is C_4-free, the pairs A_e are distinct for different edges $e \in H - X$. Thus

$$e(H) \leq |X| + \binom{\ell-1}{2} \leq |B| + \binom{\ell-1}{2} = \frac{e(G)}{\ell-1} + \binom{\ell-1}{2}.$$

To see that this bound can be attained, suppose that $m \geq \binom{\ell-1}{2}$ and consider the C_4-free subgraph H of G which is obtained as follows. For each pair $a \neq a'$ of vertices in A choose a vertex $b \in B$ and join b to both a and a' but to none of the other vertices in A. Join every remaining vertex in B to exactly one (arbitrary) vertex in A. $\square$

Given a graph G and a cycle C of length at least 4 in G, we say that C is *augmented* if there is an edge xy on C such that the two neighbours of x, y in $C - xy$ are adjacent in G. Every such edge xy is called a *special edge* of C. So a special edge lies on a 4-cycle in G. Moreover, every 4-cycle C of a graph G is augmented and each edge on C is special.

The strategy of the proof of Theorem 1 is to successively mark edges of our given $C_{2\ell}$-free graph G and to ensure that no marked edge lies in a C_4 by deleting a bounded number of edges (at most $\ell - 2$) for each marked one. This strategy will also be used in the proof of Theorem 4, but the argument there will be much simpler.

Proof of Theorem 1. We will need the following claim.

Suppose that G is a $C_{2\ell}$-free bipartite graph and D is an even augmented cycle in G whose length $2k$ is maximal under the condition that $k \leq \ell$. Let xy be a special edge of D. Then there exists a set X of at most $\ell - 2$ edges of $G - xy$ such that, firstly, every edge in X lies on a 4-cycle of G and, secondly, every 4-cycle of G containing the edge xy has an edge in X. $(*)$

Before we prove $(*)$, let us show how it implies the theorem. Given our graph G, choose an augmented even cycle D_0 in G as in $(*)$. (As a 4-cycle is augmented, we may assume that such a D_0 exists.) Let x_0y_0 be a special edge of D_0 and apply $(*)$ to obtain a set $X_0 \subseteq E(G - x_0y_0)$. Mark x_0y_0 and delete all edges in X_0 from G to obtain a new graph G_1. If G_1 is not C_4-free, choose a maximal augmented even cycle D_1 as above and a special edge x_1y_1 of D_1. Apply $(*)$ to obtain a set $X_1 \subseteq E(G_1 - x_1y_1)$. As x_0y_0 does not lie in a 4-cycle of G_1 but x_1y_1 as well as every edge in X_1 does, the edges x_0y_0 and x_1y_1 are distinct and $x_0y_0 \notin X_1$. Mark x_1y_1 and delete all the edges in X_1 to obtain a new graph G_2. We continue in this fashion until we obtain a C_4-free graph H. As in the process no edge was marked more than once and no marked edges were deleted, it follows that $e(H) \geq e(G)/(\ell - 1)$, as desired.

To prove $(*)$, first note that $k < \ell$ since G is $C_{2\ell}$-free. Let $\mathcal{C}$ be the set of all 4-cycles in G which contain xy. Then each $C \in \mathcal{C}$ has a vertex in $D - \{x, y\}$ (otherwise $(D \cup C) - xy$ would be an augmented cycle of length $2k + 2 \leq 2\ell$, contradicting the choice of D). Denote by A_x the set of all those neighbours a of x in $V(D - y)$ for which xa lies in a 4-cycle (which may contain y). Let B_x be the set of all those neighbours b of y in $V(D - x)$ for which there exists a vertex outside D which is joined to both x and b. Then A_x and B_x are disjoint since G is bipartite, and all edges joining x to A_x or y to B_x lie on a 4-cycle. Since every $C \in \mathcal{C}$ contains an edge joining x to A_x or an edge joining y to B_x, we are done if $|A_x| + |B_x| \leq \ell - 2$, as in that case we can set $X := \{xa : a \in A_x\} \cup \{yb : b \in B_x\}$. Thus, if we define sets $A_y \subseteq N_G(y)$ and $B_y \subseteq N_G(x)$ similarly, we may assume that both $|A_x| + |B_x| \geq \ell - 1$ and $|A_y| + |B_y| \geq \ell - 1$.

Let V_x^{even} be the set of all vertices of $D - x$ whose distance from x on D is even and let V_x^{odd} be the set of all vertices of $D - y$ with odd distance from x. Thus $A_x, B_y \subseteq V_x^{\text{odd}}$ and $B_x, A_y \subseteq V_x^{\text{even}}$ since G is bipartite. Fix the orientation of D in which x is the successor of y. As $|V_x^{\text{even}}| = |V_x^{\text{odd}}| = k - 1$, but by our assumption $|A_x| + |B_x| \geq \ell - 1 \geq k$, there are vertices $a_x \in A_x$ and $b_x \in B_x$ such that b_x is the successor of a_x on $x\vec{D}y$. Indeed, if there are no such vertices then the successors of vertices in A_x cannot lie in B_x and thus $|B_x| \leq V_x^{\text{even}} - |A_x| = k - 1 - |A_x|$, a contradiction. Similarly, there are vertices $a_y \in A_y$ and $b_y \in B_y$ such that a_y is the successor of b_y on $x\vec{D}y$. Furthermore, if $k < \ell - 1$, then we can additionally require that $a_y \neq b_x$.

By definition of $B_x \ni b_x$, there is a vertex v outside D which is joined to both x and b_x. Similarly there is a vertex w outside D which is joined to both y and b_y. Then v and w are distinct since G is bipartite. We will now distinguish two cases.

Case 1. $a_y = b_x$

By construction, in this case we must have $k = \ell - 1$. But then the graph D' obtained from D by deleting both xy and $a_x b_x = b_y a_y$ and adding the two paths xvb_x and ywb_y is a cycle of length $2k + 2 = 2\ell$ (Fig. 1), a contradiction.

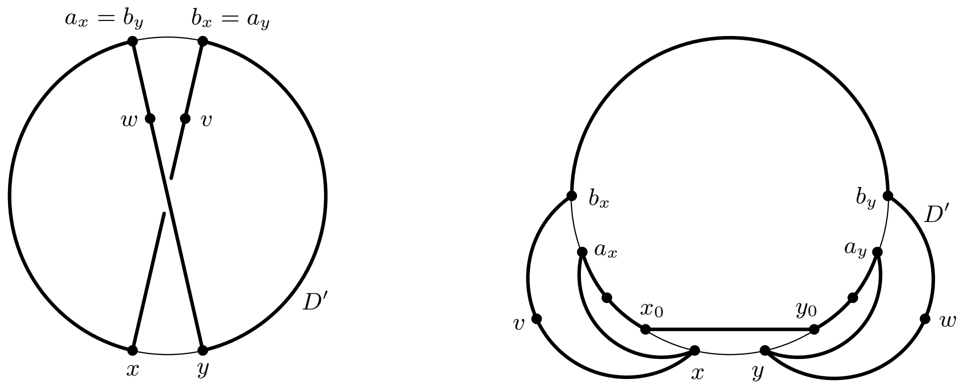

Figure 1: Illustrating Case 1 and Case 2 in the proof of Theorem 1

Case 2. $a_y \neq b_x$

Let x_0 be the successor of x on $\vec{D}$ and y_0 the predecessor of y. Thus $x_0 y_0 \in G$ since xy was a special edge of D. Denote by D' the cycle obtained from D by removing all edges of the path $x_0 x y y_0$ as well as the edges $a_x b_x$ and $a_y b_y$ and by adding the edge $x_0 y_0$ as well as the paths $a_x x v b_x$ and $a_y y w b_y$ (Fig. 1). Then D' is an augmented cycle of length $2k + 2 \leq 2\ell$, contradicting the choice of D. (Indeed, for example xv is a special edge of D'.) $\square$

The following simple probabilistic argument improves the bound in Theorem 1 in the case when $e(G) < (3(\ell - 1)/4)^3$.

Lemma 6 *Every bipartite graph G has a C_4-free subgraph H with $e(H) \geq \frac{3}{4} e(G)^{2/3}$.*

Proof. Set $m := e(G)$. Consider the random subgraph G' of G which is obtained by including each edge of G with probability $p := m^{-1/3}$ independently of all other edges of G. Denote the number of edges of G' by X and the number of its 4-cycles by Y. Since G is bipartite, every 4-cycle in G is determined by two of its independent edges (actually, we only need that G is K_4-free for this). As there are two possibilities for the choice of these edges in each 4-cycle, the number of 4-cycles in G is at most $m^2/4$. Thus

$$\mathbb{E}(X - Y) \geq pm - p^4 m^2/4 = 3m^{2/3}/4.$$

So there exists an outcome G' for which $X - Y \geq 3m^{2/3}/4$. But this means that the (C_4-free) graph H obtained from G' by deleting one edge on every C_4 of G' has at least $3m^{2/3}/4$ edges, as desired. $\square$

The complete bipartite graph $K_{r,\binom{r}{2}}$ shows that the bound in Lemma 6 is best possible up to the value of the constant. Indeed, as we saw in the proof of Proposition 5, the maximum number of edges of a C_4-free subgraph of $K_{r,\binom{r}{2}}$ is precisely $2\binom{r}{2}$. On the other hand, it is well known that the complete graph K_r has a C_4-free subgraph with $c_1 r^{3/2} = c_2 e(K_r)^{3/4}$ edges (see e.g. [1, Ch. VI, Thm. 2.8]). Thus at first sight one might think that perhaps every graph G should have a C_4-free subgraph with at least $c_3 e(G)^{3/4}$ edges, but the above remark shows that this is not the case.

Combining the deletion process of the proof of Theorem 1 with the probabilistic argument above we can come closer to the upper bound in Proposition 5: we obtain the correct order of magnitude for the additive term.

Corollary 7 *Let $\ell \geq 3$ be an integer and let G be a $C_{2\ell}$-free bipartite graph with $e(G) \geq (3(\ell-1)/8)^3$. Then G contains a C_4-free subgraph H with*

$$e(H) \geq \frac{e(G)}{\ell-1} + \frac{3^3(\ell-1)^2}{8^3} - 1.$$

Proof. Let $x_0 := (3(\ell-1)/8)^3$. Run the deletion process described after $(*)$ in the proof of Theorem 1 until it returns a graph G_i whose number of unmarked edges lies in the interval $[x_0, x_0+\ell-1]$. This can be done since in each step of the process precisely one edge will be marked and at most $\ell-2$ edges will be deleted. (The edge e_i is not counted as a marked edge of G_i.) Let x be the number of unmarked edges of G_i and let G' be the graph consisting of these edges. Apply Lemma 6 to obtain a C_4-free subgraph H' of G' with $e(H') \geq 3x^{2/3}/4$. As no marked edge of G_i lies in a C_4 of G_i, the graph H consisting of H' together with the set M of all marked edges of G_i is C_4-free. Moreover,

$$\begin{aligned} e(H) \geq |M| + e(H') &\geq \frac{e(G)-x}{\ell-1} + \frac{3x^{2/3}}{4} \\ &\geq \frac{e(G)}{\ell-1} - \frac{x_0+\ell-1}{\ell-1} + \frac{3x_0^{2/3}}{4} = \frac{e(G)}{\ell-1} + \frac{3^3(\ell-1)^2}{8^3} - 1, \end{aligned}$$

as required. □

We now outline how the proof of Corollary 7 can be modified to find a C_4-free subgraph H with $e(H) \geq e(G)/(\ell-1)+\ell-2$ in every $C_{2\ell}$-free bipartite graph G with $e(G) > (\ell-1)(\ell-2)$. This improves the bound in Corollary 7 when ℓ is small; and for $\ell = 3$ it matches the bound in Proposition 5 and is already due to Györi [5]. This time we stop the deletion process if the number x of unmarked edges of G_i lies in the interval $[(\ell-1)(\ell-2)+1, (\ell-1)^2]$. We then choose a spanning forest F of the graph consisting of the unmarked edges of G_i. As is easily seen (e.g. by induction on the number of edges), such a forest must have at least $(\ell-1)+(\ell-2)$ edges. So the C_4-free subgraph of G consisting of all the marked edges of G_i together with the edges in the chosen spanning forest F has at least

$$\frac{e(G)-x}{\ell-1} + (\ell-1) + (\ell-2) \geq \frac{e(G)}{\ell-1} + \ell - 2$$

edges.

3 Graphs pasted together from 4-cycles

Before proving Theorem 3, let us show that it is best possible in the sense that we cannot replace the condition that the graph G is pasted together from 4-cycles by the weaker assumption that every edge of G lies in many 4-cycles:

Proposition 8 *For all integers $\ell \geq 3$ and $k \geq 1$ there exists a $C_{2\ell}$-free bipartite graph G with minimum degree at least k such that each edge lies in at least $(\ell-2)(k-1)$ many 4-cycles.*

Proof. Let G' be a bipartite graph of minimum degree at least k and girth greater than 2ℓ. Denote the vertex classes of G' by A and B. Let G be the bipartite graph obtained from G' by replacing every vertex $b \in B$ by a set X_b of $\ell-1$ new vertices, each of which is joined to precisely the vertices in $N_{G'}(b)$. Then the minimum degree of G is at least k and each edge of G lies in at least $(\ell-2)(k-1)$ many 4-cycles. Moreover, it is easily checked that for every cycle C of length at most 2ℓ in G there exists a vertex $b \in B$ such that C lies within the complete bipartite subgraph of G whose vertex classes are X_b and $N_{G'}(b)$. Thus G is $C_{2\ell}$-free. □

For the proof of Theorem 3 we need the following result which was shown by Bondy and Simonovits in their proof of [2, Lemma 2].

Lemma 9 *Let $\ell \geq 2$ be an integer. Then every $C_{2\ell}$-free bipartite graph of radius at most ℓ has average degree at most 16ℓ.*

Proof of Theorem 3. By Lemma 9 it suffices to show that the radius of G is at most $\ell-1$. Suppose that G is obtained by successively pasting together the 4-cycles $C_1, C_2, \dots, C_m$ in that order. Fix a vertex $x \in C_1$. We will show that every other vertex $y \in G$ has distance at most $\ell-1$ from x. Since $x \in C_1$, there exists a sequence $C_1 =: D_1, D_2, \dots, D_k$ of distinct 4-cycles in G such that $y \in D_k$ and such that D_i and D_{i+1} share at least one edge for all $1 \leq i < k$. Let e_i be an edge of D_i that lies on D_{i+1} and let e_k be any edge of D_k incident with y. We now inductively define even cycles $D'_1, \dots, D'_k$ such that $x, e_i \in D'_i$ and $|D'_i| \leq |D'_{i-1}| + 2$. Set $D'_1 := D_1$ and suppose that we have already defined $D'_1, \dots, D'_i$ for some $i \geq 1$. Using that G is bipartite, it is easily seen that $D'_i \cup D_{i+1}$ contains a cycle D'_{i+1} with the required properties. Note that, since G is $C_{2\ell}$-free and $|D'_i| \leq |D'_{i-1}| + 2$, we must have $|D'_i| \leq 2\ell - 2$ for all $i \leq k$. But D'_k contains both x, y and thus the distance between x and y is at most $\ell - 1$, as desired. □

4 Longer even cycles

Theorem 4 follows immediately from repeated applications of the following lemma. The basic argument in the proof of this lemma was already used to prove Theorem 1. It relies on the fact that if ℓ satisfies the condition in the statement, then every graph obtained from (sufficiently many) C_g's by pasting them together in a chain-like fashion (where consecutive C_g's share exactly one edge) must contain a $C_{2\ell}$.

Lemma 10 *Let $g \geq 4$ be an even integer. For every integer $\ell > g/2$ with $2\ell \equiv 2 \mod (g-2)$, every $C_{2\ell}$-free graph G of girth at least g contains a subgraph H of girth greater than g and such that $e(H) \geq e(G)/4(\ell-1)$.*

Proof. Let $k \in \mathbb{N}$ be such that $2\ell = g + k(g-2)$. Let D_0 be an even cycle in G satisfying the following two properties:

- D_0 contains an edge $e_0 = xy$ that lies on a g-cycle in G,
- the length r_0 of D_0 is maximal under the condition that $r_0 \leq 2\ell$ and $r_0 \equiv 2 \mod (g-2)$.

(As every g-cycle is a candidate for D_0, we may assume that D_0 exists.) Then $r_0 < 2\ell$, as G is $C_{2\ell}$-free. Let $\mathcal{C}$ be a maximal family of g-cycles in G which contain e_0 but share no other edge incident to x or y with each other. We will show that $|\mathcal{C}| \leq r_0 - 2$. Let $C, C' \in \mathcal{C}$. By the choice of D_0, both C and C' meet $V(D_0) \setminus \{x, y\}$. (If for example $V(C) \cap (V(D_0) \setminus \{x, y\}) = \emptyset$ then we would have chosen the cycle $(C \cup D_0) - xy$ instead of D_0.) Moreover, since G has girth g, it follows that $V(C) \cap V(C') = \{x, y\}$. So C and C' meet $V(D_0) \setminus \{x, y\}$ in distinct vertices. Thus $|\mathcal{C}| \leq r_0 - 2$.

Now we mark e_0 and delete all those edges incident with x or y which lie on a cycle in $\mathcal{C}$ except for e_0. We say that each of the deleted edges is *assigned to* e_0. Let G_1 denote the resulting graph. Then e_0 does not lie on a g-cycle of G_1. If G_1 is not C_g-free, consider again an even cycle D_1 in G_1 which contains an edge e_1 lying on a g-cycle of G_1 and whose length r_1 is maximal under the condition that $r_1 \leq 2\ell$ and $r_1 \equiv 2 \mod (g-2)$. In a similar fashion as before, we mark e_1 and delete some edges from G_1 which we assign to e_1. Denote the graph thus obtained by G_2. As e_0 does not lie in a g-cycle of G_1 but e_1 as well as every edge in $E(G_1) \setminus E(G_2)$ does, we have $e_0 \in G_2$ and $e_0 \neq e_1$. We continue in this fashion until we obtain a C_g-free subgraph H of G. H contains all the marked edges and no edge was marked twice. Moreover, to each marked edge e_i we assigned at most $2(r_i - 2) < 4(\ell - 1)$ deleted edges. Thus $e(H) \geq e(G)/4(\ell-1)$, as required. □

Proof of Theorem 4. We will prove the following assertion.

> *For every even integer $g \geq 4$ there exists an integer k such that for each $\ell > k$ with $\ell \equiv 1 \mod k$ every $C_{2\ell}$-free graph G contains a subgraph H whose girth is greater than g and such that $e(H) \geq e(G)/2(4\ell)^{(g-2)/2}$.*

So suppose that we are given a $C_{2\ell}$-free graph G. First choose a bipartite subgraph G' of G with $e(G') \geq e(G)/2$. Note that now we may successively apply Lemma 10 to G' to obtain a subgraph H of G with the required properties provided that $2\ell > g$ and $2\ell \equiv 2 \mod (i-2)$ for all even integers $4 \leq i \leq g$.

The latter conditions are satisfied if $2\ell > g$ and $\ell \equiv 1 \mod j$ for all integers $1 < j < g/2$. But if we set $k := \prod_{j=1}^{g/2} j$ then this holds for every $\ell > k$ with $\ell \equiv 1 \mod k$. □

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