A PROOF OF SUMNER'S UNIVERSAL TOURNAMENT CONJECTURE FOR LARGE TOURNAMENTS

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Abstract. Sumner's universal tournament conjecture states that any tournament on $2n-2$ vertices contains any directed tree on n vertices. In this paper we prove that this conjecture holds for all sufficiently large n. The proof makes extensive use of results and ideas from a recent paper by the same authors, in which an approximate version of the conjecture was proved.

1. Introduction

1.1. **Introduction.** A tournament is an orientation of a complete graph. Obviously one cannot guarantee any substructures which contain a cycle within an arbitrary tournament. On the other hand, Sumner's universal tournament conjecture states that one can find any directed tree T within an arbitrary tournament G, even if the order of T is rather large compared to that of G. More precisely, the conjecture states that any tournament on $2n-2$ vertices contains any directed tree on n vertices. Many partial results towards this conjecture (made in 1971) have been proved – some of them are described below. Here we prove this conjecture for all large n.

Theorem 1.1. *There exists n_0 such that the following holds. Let T be a directed tree on $n \geq n_0$ vertices, and G a tournament on $2n-2$ vertices. Then G contains a copy of T.*

To see that the bound is best possible, let T be a star with all edges directed inwards, and let G be a regular tournament on $2n-3$ vertices. Then every vertex of G has $n-2$ inneighbours and $n-2$ outneighbours, and so G does not contain a copy of T, whose central vertex has $n-1$ inneighbours. There are also 'near-extremal' examples which have a different structure to the one given above: let T be obtained from a directed path on $\ell \geq 1$ vertices by adding $y := (n-\ell)/2$ outneighbours to the terminal vertex of the path and y inneighbours to the initial vertex of the path. Let G consist of regular tournaments Y and Z, each on $2y-1$ vertices, together with an arbitrary tournament X on $\ell-1$ vertices so that all edges are oriented from Z to X, from X to Y and from Z to Y. Then $|G| = 2n-\ell-3$ as well as $|T| = n$, and it is easy to see that G does not contain T. These examples will play a significant role in the proof (see Section 1.2).

In [10], we used a randomised embedding algorithm to prove an approximate version of Sumner's universal tournament conjecture, and also a stronger result for directed trees of bounded degree. Both of these results will be important tools in this paper.

Theorem 1.2 ([10], Theorem 1.4)**.** *Let $\alpha > 0$. Then the following properties hold.*

(i) *There exists n_0 such that for any $n \geq n_0$, any tournament G on $2(1+\alpha)n$ vertices contains any directed tree T on n vertices.*
(ii) *Let Δ be any positive integer. Then there exists n_0 such that for any $n \geq n_0$, any tournament G on $(1+\alpha)n$ vertices contains any directed tree T on n vertices with $\Delta(T) \leq \Delta$.*

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Let $f(n)$ denote the smallest integer such that any tournament on $f(n)$ vertices contains any directed tree on n vertices. So Sumner's conjecture states that $f(n) = 2n-2$. Chung (see [15]) observed that $f(n) \leq n^{1+o(1)}$, and Wormald [15] improved this to $f(n) \leq O(n \log n)$. The first linear bound on $f(n)$ was established by Häggkvist and Thomason [4]. Havet [5] then showed that $f(n) \leq 38n/5$, and later Havet and Thomassé [7] used their notion of median orders to improve this to $f(n) \leq 7n/2$. Finally El Sahili used the same notion to prove the best known bound for general n, namely that $f(n) = 3n - 3$. We shall make extensive use of this result in this paper (actually, any linear bound would suffice for our purposes; the factor of 3 is not essential.)

Theorem 1.3 (El Sahili [3]). *Let T be a directed tree on n vertices, and let G be a tournament on $3n - 3$ vertices. Then G contains a copy of T.*

Sumner's conjecture is also known to hold for special classes of trees. In particular, Havet and Thomassé [7] proved it for 'outbranchings', again using median orders. Here an *outbranching* is a directed tree T in which we may choose a root vertex $t \in T$ so that for any vertex $t' \in T$, the path between t and t' in T is directed from t to t'. (Outbranchings are also known as arborescences.)

Theorem 1.4 (Havet and Thomassé [7]). *Let T be an outbranching on n vertices, and let G be a tournament on $2n - 2$ vertices. Then G contains a copy of T.*

For many types of trees, Sumner's conjecture holds with room to spare. A classical result of this type is Redei's theorem.

Theorem 1.5 (Redei [13]). *Any tournament contains a spanning directed path.*

This was generalized considerably by Havet and Thomassé [8] who showed that every tournament on $n \geq 8$ vertices contains every orientation of the path on n vertices (which proved a conjecture of Rosenfeld). They also proposed the following generalization of Sumner's conjecture (see [6]): Let T be a directed tree on n vertices with k leaves. Then every tournament on $n + k - 1$ vertices contains a copy of T. Some special cases are known (see e.g. [2]). It would be interesting to know whether our methods can be used to prove this conjecture.

As illustrated in the next section, our proof relies on all of the above theorems (i.e. Theorems 1.2–1.5), as well as a directed version of Szemerédi's regularity lemma and several structural results proved in [10].

1.2. **Outline of the proof.** In Section 2, we shall introduce some notation, before introducing some key ideas and lemmas. In particular we shall define the core tree T_Δ of a tree T. This is a subtree of T consisting of all the 'central' vertices of T, which has the important property that every component of $T - T_\Delta$ is small. This is useful for the problem of embedding T in a tournament G, as we may first embed T_Δ and then proceed to embed the components of $T - T_\Delta$ one by one, using the fact that each such component is small. We also introduce the notion of an 'almost-regular' tournament G, which is a tournament in which every vertex has in- and outdegree approximately equal to $|G|/2$. Section 2 also contains three auxiliary lemmas for embedding a directed tree T in a tournament G which are derived from Theorems 1.2 and 1.3 and which we shall use extensively in later sections:

- Lemma 2.5 is designed to embed a directed tree T which is similar to an outstar, in the sense that T contains a vertex t with no inneighbours such that every component of $T - t$ is small.

- In Lemma 2.6, we consider a subtree T_c of T with the property that every component of $T - T_c$ is small, showing that a suitable embedding of T_c in G can be extended to an embedding of T in G.
- In Lemma 2.7 we consider the case where the vertices of G can be partitioned into disjoint sets Y and Z such that almost all edges between Y and Z can be directed the same way. Here we show that if the vertices of T are partitioned appropriately between forests F^- and F^+, then to be able to embed T in G it is sufficient to embed the largest component of F^+ within Y.

We begin the proof of Theorem 1.1 in Section 3, by proving the case where $|T_\Delta| = 1$ (Lemma 3.1). Note that the extremal case when T is a star is covered by this case. To do this, we first embed the single vertex of T_Δ to a vertex of G with appropriate in- and outdegree. We then use Lemma 2.5, Lemma 2.6 and Theorem 1.4 to embed the components of $T - T_\Delta$ appropriately among the remaining vertices of G to obtain a copy of T in G.

Then in Section 4 we introduce the digraph regularity lemma, which yields a partition of the vertex set of G into clusters so that the edges between pairs of clusters of G form quasi-random bipartite subgraphs. We use the regularity lemma to prove

- Lemma 4.6, which states that Theorem 1.1 holds in the case where G is almost-regular and T_Δ is small enough to be embedded within a single cluster of G.

To prove this, we first select an appropriate cluster or pair of clusters of G in which to embed T_Δ, and then use Lemma 2.6 to extend this embedding of T_Δ to an embedding of T in G. We also prove that if we additionally assume that $|T_\Delta| \geq 2$ then the result holds with room to spare, i.e. we can allow G to be of order $(2-\alpha)n$, where α is small.

Next, in Section 5 we consider the case when the tournament G is a 'robust outexpander'. The latter implies that every set S of reasonable size has a large outneighbourhood. A key lemma in [10] showed that if G is a robust outexpander tournament on at least $(2+\alpha)n$ vertices with large minimum semidegree, then G contains any directed tree T on n vertices. However, the αn error term was only required in the case where T_Δ is small. In Section 5 we modify the argument from [10] to prove

- Lemma 5.3, which states that if T_Δ is large, then any robust outexpander tournament on at least $(2-\alpha)n$ vertices with large minimum semidegree contains a copy of T.

(The proof relies on further results from [10].) It is easy to see that any almost-regular tournament is a robust outexpander tournament. So we can combine Lemmas 4.6 and 5.3 to deduce

- Lemma 5.8, which states that Theorem 1.1 holds with a little room to spare if G is a large almost-regular tournament and $|T_\Delta| \geq 2$.

We also prepare the ground for the proof of Theorem 1.1 by modifying an algorithm from [10] to prove Lemma 5.2. This states that any tournament G may be split into disjoint subtournaments, each of which is either small or a robust outexpander with large minimum semidegree. This will allow us to apply our results on robust outexpander tournaments to (subtournaments of) general tournaments G.

In Section 6 we prove Lemma 6.1, which states that Theorem 1.1 holds for all directed trees T for which T_Δ is small. In particular, the 'near extremal' construction described in the introduction is dealt with in this part of the proof. Lemma 6.1 is proved in four steps. Firstly, in Lemma 6.2 we show that we may assume the tournament G contains two almost-regular subtournaments on vertex sets Y and Z which between them contain almost all of the vertices of G. Using this structural information, we show in Lemmas 6.3 and 6.4 that we may assume that T_Δ is a short directed path and that most of the remainder of T is attached to the endvertices of this path. (Lemma 5.8 is used as a tool here: we can apply it to embed a suitable subforest of T into Y or Z, and afterwards use Lemma 2.7 to embed the remainder of T.) We then consider the case $|T_\Delta| = 2$ separately, proving that Theorem 1.1 holds for such T. This allows us to assume for the proof of Lemma 6.1 that $|T_\Delta| \geq 3$. Since T_Δ is a directed path, we can use Redei's theorem to embed T_Δ within a set W of $|T_\Delta|$ vertices which have high in- and outdegree, and then apply Lemmas 2.5 and 2.6 to complete the embedding again.

Finally, in Section 7 we complete the proof of Theorem 1.1. By Lemma 6.1 we may assume for this that T_Δ is large. None of the extremal or near-extremal cases satisfy this condition, so we will always have a little room to spare in our calculations in this part of the proof. We proceed by using Lemma 5.2 to split the tournament G into disjoint robust outexpander subtournaments of large minimum semidegree. If there is just one such subtournament then this subtournament contains a copy of T by Lemma 5.3. By using Lemma 2.7 we prove Lemma 7.2, which shows that if there are two such subtournaments then these must also together contain a copy of T. We may therefore assume in the proof of Theorem 1.1 that there are at least three such subtournaments of G. In this case we use Lemma 5.3, Theorem 1.2 and Theorem 1.3 to embed T into these subtournaments.

2. Definitions and basic tools

2.1. **Notation.** For a graph G, we write $V(G)$ and $E(G)$ to denote the vertex set and edge set of G respectively. Then $|G| := |V(G)|$ denotes the number of vertices of G, and $e(G) := |E(G)|$ is the number of edges of G. We shall sometimes write $v \in G$ to mean $v \in V(G)$. A *tree* is a connected graph which does not contain any cycles, and we say that a vertex of a tree is a *leaf* if it has degree one.

A *directed graph* G, or digraph, consists of a vertex set $V(G)$ and an edge set $E(G)$, where each edge $e \in E$ is an ordered pair (u, v) of vertices of G. For vertices $u, v \in V(G)$ we write $u \to v$ or $v \leftarrow u$ to denote that $(u, v) \in E(G)$. If $u \to v$ then we say that v is an *outneighbour* of u, that u is an *inneighbour* of v, and that the edge (u, v) is *directed from u to v*. Sometimes we shall use the term *neighbour* of v to mean a vertex which is either an inneighbour or an outneighbour of v. For any vertex $v \in G$, we denote the set of all outneighbours of v by $N^+_G(v)$, or simply $N^+(v)$ when G is clear from the context. Similarly we write $N^-_G(v)$ or $N^-(v)$ to denote the set of all inneighbours of v. Then the *outdegree* of v, denoted $d^+_G(v)$, is defined by $d^+_G(v) := |N^+_G(v)|$. Similarly the *indegree* of v, denoted $d^-_G(v)$, is defined by $d^-_G(v) := |N^-_G(v)|$. Again we may write $d^+(v)$ or $d^-(v)$ when G is clear from the context. We define the minimum outdegree of G, denoted $\delta^+(G)$, to be the minimum of $d^+(v)$ taken over all vertices $v \in G$, and the minimum indegree, denoted $\delta^-(G)$, to be the minimum of $d^-(v)$ taken over all vertices $v \in G$. Then the minimum *semidegree* of G, denoted $\delta^0(G)$, is the minimum of $\delta^-(G)$ and $\delta^+(G)$. We write $G[U \to V]$ to denote the bipartite subgraph of G formed by edges directed from U to V.

We say that a directed graph G is an *oriented graph* if for any $u, v \in G$ at most one of $u \to v$ and $u \leftarrow v$ holds. So an oriented graph may be obtained by assigning a direction to each edge of an undirected graph. We call this undirected graph the *underlying graph*, and denote it by G_{under}. An oriented graph is a *tournament* if for any distinct $u, v \in V(G)$ precisely one of $u \to v$ and $u \leftarrow v$ holds. Equivalently, the underlying graph of a tournament is a complete graph. A *directed tree* is an oriented graph T for which the underlying graph T_{under} is a tree. The maximum degree of T, denoted $\Delta(T)$, is defined to be equal to $\Delta(T_{under})$. A tree or directed tree T may be *rooted* by identifying a specific vertex r as the *root* of T.

Let T be a directed tree, and let x be a vertex of T. Then for any edge $e \in E(T)$ incident to x, the *weight of* e *at* x, denoted $w_e(x)$, is the number of vertices y of T for which e (ignoring the orientation) is the first edge of the path in T_{under} from x to y. We say that a component of $T-x$ is an *incomponent of* x if the unique edge between x and this component is directed towards x, and an *outcomponent of* x if this edge is directed away from x. The *inweight of* x, denoted $w^-(x)$, is then the number of vertices in incomponents of x, and the *outweight of* x, denoted $w^+(x)$, is the number of vertices in outcomponents of x. Equivalently, the inweight of x is the sum of $w_e(x)$ taken over all edges e incident to x which are directed towards x, and the outweight can be defined similarly.

In the same way we define incomponents and outcomponents for a subtree T_c of T. Indeed, for any component T' of $T - T_c$ there is precisely one edge between T' and T_c. If this edge is directed towards a vertex of T' then we say that T' is an *outcomponent of* T_c, whereas if this edge is directed towards T_c we say that T' is an *incomponent of* T_c. As when T_c is a single vertex we define the *inweight of* T_c, denoted $w^-(T_c)$, to be the number of vertices in incomponents of T_c, and the *outweight of* T_c, denoted $w^+(T_c)$, to be the number of vertices in outcomponents of T_c. Again these inweights and outweights can equivalently be defined as the sum of the weights of the appropriate edges of T.

Throughout this paper we shall write $x \ll y$ to indicate that for any $y > 0$ there exists $x_0 > 0$ such that for any $0 < x \leq x_0$ the subsequent statements hold. Such statements with more variables are defined similarly.

2.2. The core tree.

Let T be a tree on n vertices, and let $\Delta \geq 2$ be fixed. Then we say that a vertex x of T is Δ-*core* if every edge e incident to x has $w_e(x) \leq (1 - 1/\Delta)n$. We call the subgraph of T induced by Δ-core vertices of T the *core tree of* T *with parameter* Δ, and denote it by T_Δ. With this definition, for any tree T, the core tree T_Δ is the same as the Δ-heart of T considered by Häggkvist and Thomason in [4]. The following proposition from [10] gives some important properties of the core tree.

Proposition 2.1 ([10], Proposition 4.2)**.** *Let T be a tree on n vertices and let $\Delta \geq 2$. Then:*

(i) *T_Δ is a tree containing at least one vertex.*
(ii) *$w_e(x) \geq n/\Delta$ if $e = xy$ is an edge of T_Δ.*
(iii) *$\Delta(T_\Delta) \leq \Delta$.*
(iv) *Every component subtree T' of $T - T_\Delta$ has $|T'| \leq n/\Delta$.*

Note that T_Δ is an undirected tree obtained from an undirected tree T. However we will frequently refer to the core tree of a directed tree T; this means the directed tree formed by taking the core tree T_Δ of the underlying graph T_{under} (an undirected tree) of T and directing each edge of T_Δ as it is directed in T.

The following proposition is needed in the proof of Lemma 2.3. Essentially the latter states that if trees T^1 and T^2 almost partition a tree T, then the core tree T_Δ is not much larger than $T^1_\Delta \cup T^2_\Delta$.

Proposition 2.2. *Let T be a tree on n vertices, let x be a leaf of T, and let $\Delta \geq 2$. Then $|(T-x)_\Delta| \geq |T_\Delta| - 1$.*

Proof. Let y be a vertex of $T_\Delta - (T-x)_\Delta$, and let z be an arbitrary vertex of $(T-x)_\Delta$. Then for some edge e incident to y we have $w_e(y) > (1-1/\Delta)(n-1)$ in $T-x$. Since by Proposition 2.1(iv) the component of $(T-x)-(T-x)_\Delta$ containing y contains at most $(n-1)/\Delta$ vertices, this edge must in fact be the first edge of the path in T from y to z. If e is also the first edge of the path in T from y to x then we have $w_e(y) > (1-1/\Delta)(n-1)+1 \geq (1-1/\Delta)n$ in T, and so $y \notin T_\Delta$, giving a contradiction. So y must lie on the path in T from x to z. Since $y \in T_\Delta$ we must have $w_e(y) \leq (1-1/\Delta)n$ in T, and so in T we have

$$(1-\frac{1}{\Delta})n - 1 \leq (1-\frac{1}{\Delta})(n-1) < w_e(y) \leq (1-\frac{1}{\Delta})n.$$

Clearly this can hold for at most one vertex y on the path from x to z. So $|T_\Delta - (T-x)_\Delta| \leq 1$, as desired. □

Lemma 2.3. *Let T be a tree on n vertices, let $\Delta \geq 2$ and let $\gamma, \alpha > 0$. Also let T^1 and T^2 be subtrees of T such that $|T^1 \cup T^2| \geq (1-\gamma)n$. Suppose also that $|T^1_\Delta|, |T^2_\Delta| \leq \alpha n$. Then $|T_\Delta| \leq \gamma n + 2\alpha n + 2n/\Delta$.*

Proof. Arbitrarily choose vertices $x_1 \in T^1_\Delta$ and $x_2 \in T^2_\Delta$, and let P be the path from x_1 to x_2 (so P is also a subtree of T). Then let $T^* := T^1 \cup P \cup T^2$, so $|T^*| \geq (1-\gamma)n$. Furthermore, T^* can be formed from T by repeated leaf-deletions. So by Proposition 2.2 we must have $|T| - |T^*| \geq |T_\Delta| - |T^*_\Delta|$, and so

$$|T_\Delta| \leq |T| - |T^*| + |T^*_\Delta| \leq \gamma n - |P - (T^1 \cup T^2)| + |T^*_\Delta|. \tag{1}$$

Let $T^*_c := T^1_\Delta \cup P \cup T^2_\Delta$. We claim that $T^*_\Delta \subseteq T^*_c$. Indeed, suppose for a contradiction that there exists a vertex $y \in T^*_\Delta - T^*_c$. Since T^*_c is a subtree of T, every vertex of T^*_c lies in the same component C of $T^* - y$. Note that $T^* - C$ is a tree. Now, T^1_Δ and T^2_Δ are subtrees of C, so by Proposition 2.1(iv) $T^* - C$ contains at most $|T^1|/\Delta$ vertices of T^1 and at most $|T^2|/\Delta$ vertices of T^2. Let e be the edge of T^* between y and C. Then since $y \in T^*_\Delta$, $w_e(y) \leq (1-1/\Delta)|T^*|$ in T^*. So at least $|T^*|/\Delta$ vertices of T^* lie in components of $T^* - y$ other than C. As every vertex of P lies in C, either at least $|T^1|/\Delta$ vertices of T^1 lie in components of $T^* - y$ other than C, or at least $|T^2|/\Delta$ vertices of T^2 lie in components of $T^* - y$ other than C. In the former case this implies that $T^* - C$ contains more that $|T^1|/\Delta$ vertices of T^1, and in the latter case this implies that $T^* - C$ contains more that $|T^2|/\Delta$ vertices of T^2. In either case this yields a contradiction.

Now, $|T^*_c| \leq 2\alpha n + |P - (T^1_\Delta \cup T^2_\Delta)|$. Since $(P \cap T^1) - T^1_\Delta$ is contained within a single component of $T^1 - T^1_\Delta$, $|(P \cap T^1) - T^1_\Delta| \leq |T^1|/\Delta$, by Proposition 2.1(iv). Similarly $|(P \cap T^2) - T^2_\Delta| \leq |T^2|/\Delta$. So

$$|T^*_\Delta| \leq |T^*_c| \leq 2\alpha n + (|T^1| + |T^2|)/\Delta + |P - (T^1 \cup T^2)|.$$

So by (1)

$$|T_\Delta| \leq \gamma n + (|T^1| + |T^2|)/\Delta + 2\alpha n \leq \gamma n + 2n/\Delta + 2\alpha n.$$

□

2.3. **Almost-regular tournaments.** In a regular directed graph G, every vertex v has $d^+(v) = d^-(v) = e(G)/|G|$. We say that a directed graph G is *γ-almost-regular* if every vertex $v \in G$ has $d^+(v), d^-(v) \geq (1-\gamma)e(G)/|G|$. In particular, if G is a tournament then G is γ-almost-regular if and only if every vertex $v \in G$ has $d^+(v), d^-(v) \geq (1-\gamma)(|G|-1)/2$. The next proposition shows that for a large tournament G only one of these two bounds is needed to ensure that G contains an almost-spanning almost-regular tournament.

Proposition 2.4. *Suppose that $1/n \ll \alpha \ll \gamma \ll 1$. Let G be a tournament on n vertices in which at least one of the following holds:*

(i) $d^+(v) \geq (1-\alpha)(n-1)/2$ *for every* $v \in G$,
(ii) $d^-(v) \geq (1-\alpha)(n-1)/2$ *for every* $v \in G$,
(iii) $d^+(v) \leq (1+\alpha)(n-1)/2$ *for every* $v \in G$,
(iv) $d^-(v) \leq (1+\alpha)(n-1)/2$ *for every* $v \in G$.

Then G contains a γ-almost-regular subtournament G' on at least $(1-\gamma)n$ vertices.

Proof. We shall prove (i); then (ii), (iii) and (iv) follow immediately. Suppose that G has at least $\sqrt{\alpha}n$ vertices with $d^+(v) > (1+\sqrt{\alpha})(n-1)/2$. Then

$$\binom{n}{2} = e(G) = \sum_{v \in G} d^+(v) > (1-\alpha)\binom{n}{2} + \sqrt{\alpha}n \cdot \sqrt{\alpha}(n-1)/2 = \binom{n}{2},$$

giving a contradiction. So there are at most $\sqrt{\alpha}n$ vertices of G with $d^+(v) > (1+\sqrt{\alpha})(n-1)/2$. Delete all of these vertices of G, and let G' be the obtained subtournament. Then $n - \sqrt{\alpha}n \leq |G'| \leq n$. Also, every vertex of G' has

$$d^+_{G'}(v) \geq \frac{(1-\alpha)(n-1)}{2} - \sqrt{\alpha}n \geq \frac{(1-\gamma)(|G'|-1)}{2}$$

and

$$d^-_{G'}(v) \geq n - 1 - \sqrt{\alpha}n - \frac{(1+\sqrt{\alpha})(n-1)}{2} \geq \frac{(1-\gamma)(|G'|-1)}{2}.$$

So G' is a γ-almost-regular tournament on at least $(1-\gamma)n$ vertices, as desired. □

2.4. **Some embedding results.** The following three lemmas will be the main tools we shall use to embed directed trees in tournaments. We use Theorem 1.3 in the proofs of all three lemmas, although the factor of 3 in Theorem 1.3 is not critical to our proof; any linear bound would suffice. For the proof of Lemma 2.7 we also require the use of Theorem 1.2.

Lemma 2.5. *Let T be a directed tree on n vertices, rooted at t, such that t has no inneighbours in T, and every component of $T-t$ contains at most d vertices. Let G be a tournament whose vertex set is partitioned into three sets, $\{v\}, N$ and X, where $|N| \geq n-1$, every vertex of N is an outneighbour of v, and at least $3d$ vertices of N each have at least $6d$ inneighbours in X and at least $6d$ outneighbours in X. Then T can be embedded in G in such a way that t is embedded to v and at most $4d$ vertices of X are occupied by this embedding.*

Proof. Let $N' \subseteq N$ consist of all vertices of N with at least $6d$ inneighbours in X and at least $6d$ outneighbours in X. Then $|N'| \geq 3d$. We begin by embedding t to the vertex v. Now let $T_1, \dots, T_r$ be the components of $T - t$, in order of decreasing order. For each i, let t_i be the single vertex of T_i which is an outneighbour of t. Then we shall embed $T_1, \dots, T_r$ in turn in $N \cup X$, with each t_i embedded in N and each T_i embedded in the vertices not occupied by the embeddings of $T_1, \dots, T_{i-1}$. This will give an embedding of T in G. So suppose that we have embedded $T_1, \dots, T_{i-1}$ in this manner, and we now wish to embed T_i. Then at most $n-1$ vertices of T have been embedded. At least one of these vertices (namely t) was not embedded in N, so at least one vertex of N must be unoccupied.

Suppose that N' contains at least one unoccupied vertex v_i, and also that fewer than $3d$ vertices of X have been occupied. Then v_i has at least $3d$ unoccupied inneighbours in X and at least $3d$ unoccupied outneighbours in X. Embed t_i to v_i. We then proceed through the outcomponents of t_i in T_i in turn. Suppose that when we come to embed an outcomponent of t_i we have previously embedded m vertices of T_i. Then the current outcomponent has order at most $d-m$. Also, v_i has at least $3d-m \geq 3(d-m)$ outneighbours in X which have not yet been occupied, so by Theorem 1.3 we may embed this outcomponent amongst the outneighbours of v_i in X. Similarly we may embed the incomponents of t_i in turn amongst the inneighbours of v_i in X, and so we obtain an embedding of T_i in the unoccupied vertices of G. Note that all vertices of T_i apart from t_i are embedded in X.

Now suppose instead that every vertex of N' has been occupied, but still that fewer than $3d$ vertices of X have been occupied. Then at least one of the T_j with $j < i$ must have had $|T_j| = 1$, and so T_i consists of one single vertex, namely t_i. We may therefore embed t_i to any unoccupied vertex of N (recall that there is at least one such vertex).

Finally, suppose that at least $3d$ vertices of X have been occupied. Then at least $3d+1$ vertices of T have been embedded outside N, and so N contains at least $n-1-(n-(3d+1)) = 3d$ unoccupied vertices. Since $|T_i| \leq d$, by Theorem 1.3 we may embed T_i among these unoccupied vertices.

By embedding each T_i in this fashion we obtain an embedding of T in G with t embedded to v. Furthermore, the only vertices embedded in X are those in some T_i such that when we came to embed T_i, N' contained at least one unoccupied vertex v_i, and fewer than $3d$ vertices of X had been occupied. The embedding of T_i occupied at most another d vertices of X, and so at most $4d$ vertices of X can have been occupied in total. □

Lemma 2.6. (a) *Let T be a directed tree, and let T_c be a subtree of T such that every component of $T - T_c$ contains at most d vertices. Let G be a tournament whose vertices are partitioned into two sets S and N such that for every vertex $v \in S$ we have*

(i) $|N^+(v) \cap N| \geq |T - T_c| + 2d$, *and*
(ii) $|N^-(v) \cap N| \geq |T - T_c| + 2d$.

Then any embedding of T_c in $G[S]$ can be extended to an embedding of T in G.

(b) *Suppose that in addition to the above assumptions we choose a set $N' \subseteq N$ and an integer $r \leq |T - T_c|$, so that every vertex $v \in S$ satisfies*

(iii) $|N^+(v) \cap N'| \geq r + 2d$, *and*
(iv) $|N^-(v) \cap N'| \geq r + 2d$.

Then any embedding of T_c in $G[S]$ can be extended to an embedding of T in G such that at least r vertices of T are embedded in N'.

(c) *Suppose that no edges of T are directed from T_c to $T - T_c$. Then conditions (i) and (iii) may be dropped without affecting the validity of the above result. Likewise if no edges of T are directed from $T - T_c$ to T_c, then the above results hold even without conditions (ii) and (iv).*

Proof. Let $n := |T|$. We shall prove (b) and (c); for (a), apply (b) with $r := |T - T_c|$ and $N' := N$. Let $T_1, \dots, T_q$ be the components of $T - T_c$, so $|T_i| \leq d$ for each i. Suppose now that we have successfully extended the embedding of T_c in $G[S]$ to an embedding of $T_c \cup T_1 \cup \dots \cup T_{s-1}$ in G. We shall demonstrate how to extend this embedding to an embedding of $T_c \cup T_1 \cup \dots \cup T_s$ in G. Indeed, there is precisely one edge between T_c and T_s. Let $t \in T_c$ and $t_s \in T_s$ be the endvertices of this edge, and let v be the vertex in S to which t is embedded.

Suppose that t_s is an outneighbour of t. By (i), v has at least $|T - T_c| + 2d$ outneighbours in N. At most $|T_1| + \dots + |T_{s-1}|$ of these outneighbours are occupied by the embedding of $T_c \cup T_1 \cup \dots \cup T_{s-1}$, and so v has at least $|T_s| + 2d \geq 3|T_s|$ outneighbours in N which are not occupied by this embedding. Now, by (iii), v has at least $r + 2d$ outneighbours in N'. If at most $r - |T_s|$ of these outneighbours are occupied by the embedding of $T_c \cup T_1 \cup \dots \cup T_{s-1}$, then by Theorem 1.3 we may embed T_s amongst the at least $2d + |T_s| \geq 3|T_s|$ unoccupied outneighbours of v in N'. If instead $r - k$ of these outneighbours are occupied, for some $1 \leq k \leq |T_s| - 1$, then by Theorem 1.3 we may embed T_s amongst the $2|T_s| + k$ unoccupied outneighbours in N' and some arbitrary $|T_s| - k$ outneighbours of v in $N \setminus N'$. Then at least k vertices of N' will be occupied by this embedding of T_s. Finally, if at least r outneighbours of v in N' have been occupied by this embedding, then we may embed T_s within the at least $3|T_s|$ unoccupied outneighbours of v in N.

If instead t_s is an inneighbour of t, then we may extend the embedding similarly, using (ii) and (iv) rather than (i) and (iii). So we may extend the embedding of T_c in $G[S]$ to an embedding of T in G by proceeding through each T_i in this manner. Also conditions (i) and (iii) will only be required if at least one edge of T is directed from T_c to $T - T_c$, and conditions (ii) and (iv) will only be required if at least one edge of T is directed from $T - T_c$ to T_c. Finally, note that after each T_s is embedded, either every vertex of $T_1 \cup \dots \cup T_s$ will have been embedded in N', or at least r vertices of $T_1 \cup \dots \cup T_s$ will have been embedded in N'. Since $|T_1 \cup T_2 \cup \dots \cup T_q| = |T - T_c| \geq r$, we can be sure that at least r vertices of N' will be occupied by the embedding of T, as desired. □

Lemma 2.7. *Suppose that $1/n \ll \gamma \ll \alpha \ll 1$. Let T be a directed tree on n vertices, and let forests F^- and F^+ be induced subgraphs of T such that $V(F^-)$ and $V(F^+)$ partition $V(T)$ and every edge between F^- and F^+ is directed from F^- to F^+. Let T_1^+ and T_2^+ be the largest and second largest components of F^+ respectively. Also, let Y and Z be disjoint sets such that*

$$|Y| \geq |F^+| + |T_2^+| + \alpha n \text{ and } |Z| \geq 2|F^-| + \alpha n.$$

Let G be a tournament on vertex set $Y \cup Z$ such that every vertex of Y has at most γn outneighbours in Z, and every vertex of Z has at most γn inneighbours in Y. Then any embedding of T_1^+ in $G[Y]$ can be extended to an embedding of T in G.

Proof. Let $T_1, \dots, T_r$ be the components of F^- and F^+, ordered so that $T_1 = T_1^+$ and so that for each $2 \leq i \leq r$ there is exactly one edge of T between T_i and $T_1 \cup \dots \cup T_{i-1}$. Then we have an embedding of T_1 in $G[Y]$. We shall proceed through the trees T_i in turn, embedding each T_i in $G[Y]$ if T_i is a component of F^+, or in $G[Z]$ if T_i is a component of F^-. Each T_i will be embedded so that the embeddings of $T_1, \dots, T_i$ form an embedding of the subtree of T induced by the vertices of $T_1, \dots, T_i$. Suppose that we have successfully embedded $T_1, \dots, T_{i-1}$ in this manner, and we wish to extend this embedding to include T_i. Note that there is precisely one edge e between T_i and $T_1 \cup \dots \cup T_{i-1}$. Let t be the endvertex of e in $T_1 \cup \dots \cup T_{i-1}$, and let v be the vertex to which t was embedded.

If T_i is a component of F^+, then $t \in F^-$, so $v \in Z$. In this case we will embed T_i within the unoccupied outneighbours of v in Y. Since $v \in Z$, $|N^+(v) \cap Y| \geq |Y| - \gamma n \geq$

$|F^+| + |T_2^+| + \alpha n/2$. At most $|F^+| - |T_i|$ of these vertices are occupied by the embeddings of $T_1, \dots, T_{i-1}$. Since $i \geq 2$, T_i is not the largest component of F^+, and so has order $|T_i| \leq |T_2^+|$. So at least $2|T_i| + \alpha n/2$ outneighbours of v in Y remain unoccupied. So if $|T_i| \geq \alpha n/2$ then by Theorem 1.2(i) we may embed T_i in these unoccupied vertices of $N^+(v) \cap Y$. On the other hand, if $|T_i| < \alpha n/2$ then by Theorem 1.3 we may embed T_i in these unoccupied vertices of $N^+(v) \cap Y$.

Now suppose instead that T_i is a component of F^-. Then $t \in F^+$, so $v \in Y$. Here we will embed T_i within the unoccupied inneighbours of v in Z. Since $v \in Y$, $|N^-(v) \cap Z| \geq |Z| - \gamma n \geq 2|F^-| + \alpha n/2$, and at most $|F^-| - |T_i|$ of these vertices are occupied by the embeddings of $T_1, \dots, T_{i-1}$. So at least $2|T_i| + \alpha n/2$ such vertices remain unoccupied. So as before, if $|T_i| \geq \alpha n/2$ then by Theorem 1.2(i) we may embed T_i in these unoccupied vertices of $N^-(v) \cap Z$, whereas if $|T_i| < \alpha n/2$ then by Theorem 1.3 we may embed T_i in these unoccupied vertices of $N^-(v) \cap Z$. By proceeding through all of the trees T_i in this manner we will obtain an embedding of T in G. □

Observe that if in the statement of Lemma 2.7 we let T_1^- and T_2^- be the largest and second-largest components of F^- respectively, and replaced the conditions on the sizes of Z and Y by the conditions that $|Y| \geq 2|F^+| + \alpha n$ and $|Z| \geq |F^-| + |T_2^-| + \alpha n$, then we could conclude that any embedding of T_1^- in $G[Z]$ can be extended to an embedding of T in G. To see this, either note that the proof will still be valid with appropriate changes (switching inneighbours and outneighbours and so forth) or observe that this is the effect of reversing the direction of every edge of T and every edge of G, in which case the embedding problem is the same. Sometimes when referring to Lemma 2.7 we will implicitly mean this 'dual' of Lemma 2.7 instead.

3. Embedding trees whose core tree is a single vertex

In this section we shall verify that Sumner's universal tournament conjecture holds for large directed trees T whose core tree T_Δ contains only one vertex, that is, trees which are 'star-shaped'. Such trees can be embedded by selecting an appropriate vertex to which to embed the single vertex of T_Δ, and then embedding the components of $T - T_\Delta$ one by one.

Lemma 3.1. *Suppose that $1/n \ll 1/\Delta \ll 1$. Let T be a directed tree on n vertices with $|T_\Delta| = 1$, and let G be a tournament on $2n - 2$ vertices. Then G contains a copy of T.*

Proof. Introduce constants α and γ with $1/\Delta \ll \alpha \ll \gamma \ll 1$. Let t be the single vertex of T_Δ, let y be the outweight of T_Δ, and let z be the inweight of T_Δ. Also, let T_1 be the subtree of T formed by t and all of its outcomponents, and let T_2 be the subtree of T formed by t and all of its incomponents. Then $y + z = n - 1$, $|T_1| = y + 1$ and $|T_2| = z + 1$. Now, suppose that G contains a vertex v such that

(i) either $d^+(v) \geq y + 2n/\Delta$ or $y = 0$, and
(ii) either $d^-(v) \geq z + 2n/\Delta$ or $z = 0$.

Then embed t to v. By Proposition 2.1 each component of $T - t$ contains at most n/Δ vertices. So by Lemma 2.6 we may extend the embedding of t in $\{v\}$ to an embedding of T_1 in $\{v\} \cup N^+(v)$ (since if $y = 0$ then T_1 consists of the single vertex t). Also by Lemma 2.6, we may extend the embedding of t in $\{v\}$ to an embedding of T_2 in $\{v\} \cup N^-(v)$ (since if $z = 0$ then t is the only vertex of T_2). These two embeddings only overlap in the vertex v, and so combining these two embeddings gives an embedding of T in G.

So we may assume that every vertex $v \in G$ has either $d^+(v) < y + 2n/\Delta$ or $d^-(v) < z + 2n/\Delta$. Let $Y := \{v \in G : d^+(v) < y + 2n/\Delta\}$ and let $Z := \{v \in G : d^-(v) < z + 2n/\Delta\}$. Then every vertex of G lies in precisely one of Y and Z, so $|Y| + |Z| = 2n - 2$. Thus we must have either $|Y| \geq 2y$ or $|Z| \geq 2z$. Furthermore, if $y = 0$ and $|Y| \geq 1$ then each $v \in Y$ has $d^+(v) < 2n/\Delta$ and therefore $d^-(v) \geq z + 2n/\Delta$, and so satisfies (ii). We may therefore assume that if $y = 0$ then $|Y| = 0$ and similarly that if $z = 0$ then $|Z| = 0$. So without loss of generality we may assume that $|Y| \geq 2y$ and $y > 0$ (otherwise reverse the direction of every edge of T and every edge of G; then we would have $|Y| \geq 2y$ and $y > 0$ at this stage, and the embedding problem is the same). Observe that by definition of Y we must also have $|Y| \leq 2y + 4n/\Delta + 1$.

Now suppose that $y \geq \alpha n$. Since $y \in \mathbb{N}$ and $|Y| \geq 2y$, Y must contain a vertex v which satisfies $|N^+(v) \cap Y| \geq y$. Choose a subset $N' \subseteq N^+(v) \cap Y$ of size y. For any vertex $u \in Y$,

$$d^+_{G[Y]}(u) = |N^+(u) \cap Y| \leq d^+_G(u) < y + 2n/\Delta \leq (1+\alpha)(|Y| - 1)/2.$$

So by Proposition 2.4 $G[Y]$ contains a γ-almost-regular tournament on at least $2(1-\gamma)y$ vertices. So at most $|Y| - 2(1-\gamma)y \leq 3\gamma y$ vertices of Y have fewer than $(1-2\gamma)y$ inneighbours in Y or fewer than $(1-2\gamma)y$ outneighbours in Y. Since $|N'| = y$, at most $6\gamma y + 1$ vertices of N' have more than $(1-3\gamma)y$ inneighbours in N', and at most $6\gamma y + 1$ vertices of N' have more than $(1-3\gamma)y$ inneighbours in N'. So at least $(1-16\gamma)y$ vertices of N' have at least γy inneighbours in $Y \setminus N'$ and at least γy outneighbours in $Y \setminus N'$. Certainly therefore at least $3n/\Delta$ vertices of N' have at least $6n/\Delta$ inneighbours in $Y \setminus (\{v\} \cup N')$ and at least $6n/\Delta$ outneighbours in $Y \setminus (\{v\} \cup N')$. So by Lemma 2.5 we may embed T_1 in Y, with t embedded to v, and at most $4n/\Delta$ vertices embedded outside $N' \cup \{v\}$. Let V' be the set of vertices of G not occupied by this embedding of T_1. Since v has at least $|G| - 1 - (y + 2n/\Delta) \geq z + 6n/\Delta$ inneighbours in G, all outside $N' \cup \{v\}$, v must have at least $z + 2n/\Delta$ unoccupied inneighbours in V'. So by Lemma 2.6 we may extend the embedding of t in $\{v\}$ to an embedding of T_2 in $\{v\} \cup V'$. These two embeddings only overlap in the vertex v, and so combine to give an embedding of T in G.

So we may assume that $1 \leq y < \alpha n$. Then every vertex $v \in Y$ has

$$d^-(v) \geq |G| - 1 - y - 2n/\Delta \geq n + 2n/\Delta. \tag{2}$$

Let T_3 be the subtree of T formed by every vertex $t' \in T$ for which T contains a directed path from from t to t'. Then $t \in T_3$, and (taking t as the root vertex) T_3 is an outbranching. Also $T_3 \subseteq T_1$, so $|T_3| \leq y + 1$, and so by Theorem 1.4, we may embed T_3 in $G[Y]$. Since $T_\Delta \subseteq T_3$, by Proposition 2.1(iv) each component of $T - T_3$ contains at most n/Δ vertices. So as every edge of T between $T - T_3$ and T_3 is directed from $T - T_3$ to T_3, and also since by (2) every vertex of Y has at least $|T - T_3| + 2n/\Delta$ inneighbours which were not occupied by the embedding of T_3, we may extend the embedding of T_3 in $G[Y]$ to an embedding of T in G by Lemma 2.6. □

4. The regularity lemma and its applications to embedding trees

In this section we shall present a degree form of the regularity lemma for directed graphs, and show how this may be used to embed trees. In particular, the regularity lemma is useful for embedding directed trees T for which T_Δ is substantially smaller than the size of a cluster obtained by applying the regularity lemma to a tournament G; our approach here is essentially to select an appropriate cluster in G in which to embed T_Δ so that we may then embed the components of $T - T_\Delta$ in the remaining clusters of G. By using this method we shall prove Lemma 4.6, which states that Theorem 1.1 holds in the case where G is a large and almost-regular tournament, and T is a directed tree such that T_Δ is small.

Let U and V be disjoint sets, and let G be a directed graph on vertex set $U \cup V$. Recall that $G[U \to V]$ denotes the bipartite subgraph of G formed by edges directed from U to V. The *density from U to V*, denoted $d(G[U \to V])$, is then defined by

$$d(G[U \to V]) := \frac{e(G[U \to V])}{|U||V|}.$$

We say that $G[U \to V]$ is *ε-regular* if for any $U' \subseteq U$ and $V' \subseteq V$ with $|U'| > \varepsilon|U|$ and $|V'| > \varepsilon|V|$ we have $d(G[U' \to V']) = d(G[U \to V]) \pm \varepsilon$.

The next lemma is the degree form of the regularity lemma which we shall use. A regularity lemma for digraphs was proven by Alon and Shapira [1]. The degree form follows from this in the same way as in the undirected case (see [11] for a sketch of the latter).

Lemma 4.1 (Regularity Lemma for directed graphs)**.** *Suppose that $1/n \ll 1/M \ll 1/M' \ll \varepsilon$. Let G be a directed graph on n vertices. Then there exists a partition of $V(G)$ into $V_0, \dots, V_k$ and a spanning subgraph G' of G such that*

(1) $M' \leq k \leq M$,
(2) $|V_0| \leq \varepsilon n$,
(3) $|V_1| = \dots = |V_k|$,
(4) $d^+_{G'}(x) > d^+_G(x) - \varepsilon n$ *for all vertices* $x \in V(G)$,
(5) $d^-_{G'}(x) > d^-_G(x) - \varepsilon n$ *for all vertices* $x \in V(G)$,
(6) *for all* $i \in [k]$ *the directed graph* $G'[V_i]$ *is empty,*
(7) *for all* $i, j \in [k]$ *with* $i \neq j$ *the directed graph* $G'[V_i \to V_j]$ *is* ε*-regular.*

We say that an oriented graph G on clusters $V_1, \dots, V_k$ of equal size is an *ε-regular cluster tournament* if for any $i, j \in [k]$ with $i \neq j$ the subdigraph $G[V_i \to V_j]$ is ε-regular and for any $i \in [k]$ the subdigraph $G[V_i]$ is a tournament. If G is a cluster tournament on clusters $V_1, \dots, V_k$ then we shall denote the density of $G[V_i \to V_j]$ by d_{ij} for any $i, j \in [k]$ (the tournament G will be clear from the context). The following corollary of the regularity lemma shows that any sufficiently large tournament G contains an almost-spanning ε-regular cluster tournament G^* such that vertices have similar in- and outdegrees in both G and G^*.

Corollary 4.2. *Suppose that $1/n \ll 1/M \ll 1/M' \ll \varepsilon$. Let G be a tournament on n vertices. Then there exist disjoint subsets $V_1, \dots, V_k \subseteq V(G)$ of equal size and a subgraph $G^* \subseteq G$ on vertex set $V_1 \cup \dots \cup V_k$ such that:*

(i) $M' \leq k \leq M$,
(ii) G^* *is an* ε*-regular cluster tournament,*
(iii) $\bigcup_{i \in [k]} V_i \geq (1 - \varepsilon)n$,
(iv) $d^+_{G^*}(x) > d^+_G(x) - 2\varepsilon n$ *for all vertices* $x \in V(G)$*, and*
(v) $d^-_{G^*}(x) > d^-_G(x) - 2\varepsilon n$ *for all vertices* $x \in V(G)$.

Proof. Apply Lemma 4.1 to obtain a partition $V_0, \dots, V_k$ of $V(G)$ and a subgraph $G' \subseteq G$ which satisfy the conditions of Lemma 4.1. In particular (i) and (iii) are satisfied. Now form G^* from $G'[V_1 \cup \dots \cup V_k]$ by adding every edge of G for which both endvertices lie in the same cluster V_i. So $G^* \subseteq G$, and by (7) of Lemma 4.1 and the fact that $G^*[V_i]$ is a tournament for each $i \in [k]$ we have (ii). Finally note that using (4) of Lemma 4.1 we have

$$d^+_{G^*}(x) \geq d^+_{G'}(x) - |V_0| \geq d^+_G(x) - 2\varepsilon n.$$

Similarly $d^-_{G^*}(x) \geq d^-_G(x) - 2\varepsilon n$ using (5) of Lemma 4.1. □

It follows immediately from the definition of regularity that if U and V are sets of size m, and $G[U \to V]$ is ε-regular with density d, then all but at most $2\varepsilon m$ vertices of U have $(d \pm \varepsilon)m$ outneighbours in V. The next lemma is a generalisation of this fact, considering the number of outneighbours of vertices in one cluster within a cluster tournament.

Lemma 4.3. *Suppose that $1/m \ll 1/k \ll \varepsilon \ll \varepsilon' \ll 1$. Let G be an ε-regular cluster tournament on clusters $V_1, \dots, V_k$, each of size m. Let $V'_j \subseteq V_j$ for each $j \in [k]$ be fixed. Then for any i, all but at most $\varepsilon' m$ vertices of V_i have $\sum_{j\in[k]\setminus\{i\}} d_{ij}|V'_j| \pm \varepsilon' km$ outneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$ and $\sum_{j\in[k]\setminus\{i\}} d_{ji}|V'_j| \pm \varepsilon' km$ inneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$.*

Proof. Fix some $i \in [k]$. Then let L be the set of all $j \in [k] \setminus \{i\}$ such that $|V'_j| \geq \varepsilon m$ and $d_{ij} \geq \sqrt{\varepsilon}$. For each $j \in L$, let A_j denote the set of vertices of V_i which have fewer than $(1-\sqrt{\varepsilon})d_{ij}|V'_j|$ outneighbours in V'_j. Then for each $j \in L$, the subdigraph of $G[V_i \to V_j]$ induced by A_j and V'_j has density less than $(1-\sqrt{\varepsilon})d_{ij} \leq d_{ij} - \varepsilon$. Since $G[V_i \to V_j]$ is ε-regular with density d_{ij}, and $|V'_j| \geq \varepsilon m$, we must have $|A_j| < \varepsilon m$.

Now, fix a vertex $v \in V_i$. Suppose that v appears in at most $\sqrt{\varepsilon}|L|$ of the sets A_j with $j \in L$. Then

$$\begin{aligned}
|N^+(v) \cap \bigcup_{j\in L} V'_j| &\geq \sum_{j\in L: v\notin A_j} (1-\sqrt{\varepsilon})d_{ij}|V'_j| \\
&\geq \sum_{j\in[k]\setminus\{i\}} (1-\sqrt{\varepsilon})d_{ij}|V'_j| - \sum_{j\in[k]\setminus(L\cup\{i\})} d_{ij}|V'_j| - \sum_{j\in L: v\in A_j} d_{ij}|V'_j| \\
&\geq \sum_{j\in[k]\setminus\{i\}} d_{ij}|V'_j| - \sqrt{\varepsilon}km - \sqrt{\varepsilon}km - \sqrt{\varepsilon}|L|m \\
&\geq \sum_{j\in[k]\setminus\{i\}} d_{ij}|V'_j| - 3\sqrt{\varepsilon}km.
\end{aligned}$$

Since at most $\sqrt{\varepsilon}m$ vertices $v \in V_i$ appear in more than $\sqrt{\varepsilon}|L|$ of the sets A_j with $j \in L$, we may conclude that there are at most $\sqrt{\varepsilon}m$ vertices $v \in V_i$ with fewer than $\sum_{j\in[k]\setminus\{i\}} d_{ij}|V'_j| - 3\sqrt{\varepsilon}km$ outneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$. A similar argument shows that there are at most $\sqrt{\varepsilon}m$ vertices $v \in V_i$ with more than $\sum_{j\in[k]\setminus\{i\}} d_{ij}|V'_j| + 3\sqrt{\varepsilon}km$ outneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$.

Now, let L' be the set of all $j \in [k]$ such that $|V'_j| \geq \varepsilon m$ and $d_{ji} \geq \sqrt{\varepsilon}$. Then the same argument applied to inneighbours rather than outneighbours shows that there are at most $\sqrt{\varepsilon}m$ vertices $v \in V_i$ with fewer than $\sum_{j\in[k]\setminus\{i\}} d_{ji}|V'_j| - 3\sqrt{\varepsilon}km$ inneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$ and at most $\sqrt{\varepsilon}m$ vertices $v \in V_i$ with more than $\sum_{j\in[k]\setminus\{i\}} d_{ji}|V'_j| + 3\sqrt{\varepsilon}km$ inneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$. Since $\varepsilon \ll \varepsilon'$, this completes the proof. □

The next two lemmas will be used in the proof of Lemma 4.6; we state them separately as we shall also refer to them in Section 6. Both of these consider an ε-regular cluster tournament G on k clusters with the property that for some cluster V_i the density of edges leaving V_i and the density of edges entering V_i are each roughly $1/2$. Lemma 4.4 considers the case where for many clusters V_j the density of edges between V_i and V_j is large in both directions, showing that in this case G contains a copy of a directed tree T of the type considered. Lemma 4.5 considers the alternative, namely that for almost all clusters V_j the density of edges between V_i and V_j is small in one direction, showing that in this case G contains a copy of T provided that T_Δ has large inweight and large outweight.

Lemma 4.4. *Suppose that $1/n \ll 1/\Delta', \beta \ll 1/k \ll \varepsilon \ll \gamma \ll \alpha \ll 1/\Delta \ll 1$. Let T be a directed tree on n vertices with $|T_{\Delta'}| \leq \beta n$ and $|T_\Delta| \geq 2$, and let G be an ε-regular cluster tournament on clusters $V_1, \dots, V_k$, each of size $m \geq 2(1-\gamma)n/k$. Suppose also that for some $i \in [k]$ we have*

$$\sum_{j \in [k] \setminus \{i\}} d_{ij} \geq \frac{(1-3\gamma)k}{2} \quad \text{and} \quad \sum_{j \in [k] \setminus \{i\}} d_{ji} \geq \frac{(1-3\gamma)k}{2},$$

and also that there are at least αk values of $j \in [k] \setminus \{i\}$ such that $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$. Then G contains a copy of T.

Proof. Fix such a value of i, and introduce a new constant ε' with $\varepsilon \ll \varepsilon' \ll \gamma$. Since $\Delta \leq \Delta'$, we must have $T_\Delta \subseteq T_{\Delta'}$. Also, since $|T_\Delta| \geq 2$, we may choose an edge $t^- \to t^+$ of T_Δ, which therefore is also an edge of $T_{\Delta'}$. Let T^+ and T^- be the two components formed when this edge is deleted from T, labelled so that $t^+ \in T^+$ and $t^- \in T^-$. Similarly, let $T^+_{\Delta'}$ and $T^-_{\Delta'}$ be the two components formed by the deletion of the edge $t^- \to t^+$ from $T_{\Delta'}$, labelled with $t^+ \in T^+_{\Delta'}$ and $t^- \in T^-_{\Delta'}$. Then T^+ and T^- partition the vertices of T, and there is precisely one edge of T between T^+ and T^-, which is directed towards T^+. Furthermore, since $t^- \to t^+$ was an edge of T_Δ, by Proposition 2.1(ii) we have $|T^+|, |T^-| \geq n/\Delta$.

Let $J \subseteq [k] \setminus \{i\}$ satisfy $|J| \geq \alpha k$ and also that for any $j \in J$ we have $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$. Then $\sum_{j \in J} d_{ij} \geq \alpha^2 k$ and $\sum_{j \in J} d_{ji} \geq \alpha^2 k$. By Lemma 4.3 (applied with $V'_j = \emptyset$ for each $j \notin J$) at most $\varepsilon' m$ vertices of V_i have fewer than

$$\sum_{j \in J} d_{ij} m - \varepsilon' km \geq \alpha^2 km - \varepsilon' km \geq \frac{\alpha^2 km}{2} \tag{3}$$

outneighbours in $\bigcup_{j \in J} V_j$ or fewer than $\sum_{j \in J} d_{ji} m - \varepsilon' km \geq \alpha^2 km/2$ inneighbours in $\bigcup_{j \in J} V_j$. Also by Lemma 4.3 at most $\varepsilon' m$ vertices of V_i have fewer than

$$\sum_{j \in [k] \setminus \{i\}} d_{ij} m - \varepsilon' km \geq \frac{(1-3\gamma-2\varepsilon')km}{2} \geq (1-5\gamma)n \tag{4}$$

outneighbours in $\bigcup_{j \in [k] \setminus \{i\}} V_j$ or fewer than $\sum_{j \in [k] \setminus \{i\}} d_{ji} m - \varepsilon' km \geq (1-5\gamma)n$ inneighbours in $\bigcup_{j \in [k] \setminus \{i\}} V_j$. Finally, at most $m/2 + 1$ vertices of V_i have fewer than $m/4$ inneighbours in V_i. So we may choose a set S^+ of $m/10$ vertices of V_i which do not fall into any of these categories. Since $|T^+_{\Delta'}| \leq |T_{\Delta'}| \leq \beta n \leq m/30$, by Theorem 1.3 we may embed $T^+_{\Delta'}$ in S^+. Let $S^+_{\Delta'}$ be the set of vertices of S^+ occupied by this embedding of $T^+_{\Delta'}$, and let v^+ be the vertex to which t^+ was embedded. Recall that $|T^-| \geq n/\Delta$, so

$$|T^+| = n - |T^-| \leq (1 - 1/\Delta)n.$$

Furthermore, every component of $T^+ - T^+_{\Delta'}$ is a component of $T - T_{\Delta'}$ and thus has order at most n/Δ' by Proposition 2.1. So by (3) and (4), and since $\gamma \ll 1/\Delta$, we may apply Lemma 2.6(b) to extend the embedding of $T^+_{\Delta'}$ in $S^+_{\Delta'}$ to an embedding of T^+ in $S^+_{\Delta'} \cup$

$\bigcup_{j\in[k]\setminus\{i\}} V_j$ so that at least $\alpha^2 n/3$ vertices of $\bigcup_{j\in J} V_j$ are occupied by this embedding of T^+.

Now, at least $m/4 - m/10 = 3m/20$ vertices of $V_i \setminus S^+_{\Delta'}$ are inneighbours of v^+. For each $j \in [k] \setminus \{i\}$, let o_j denote the number of vertices of V_j which are occupied by our embedding of T^+, and let $V'_j \subseteq V_j$ consist of those vertices of V_j which are not occupied by this embedding. So $|V'_j| = m - o_j$ for each j. Note that since $d_{ij} + d_{ji} \le 1$ we have $d_{ij} \le 1 - \alpha$ for each $j \in J$. Then by Lemma 4.3, at most $\varepsilon' m$ vertices of V_i have fewer than

$$\begin{aligned}
\sum_{j\in[k]\setminus\{i\}} d_{ij}(m-o_j) - \varepsilon' km &\ge \sum_{j\in[k]\setminus\{i\}} d_{ij}m - \varepsilon' km - \sum_{j\in J} d_{ij}o_j - \sum_{j\in[k]\setminus(\{i\}\cup J)} d_{ij}o_j \\
&\overset{(4)}{\ge} (1-5\gamma)n - (1-\alpha)\sum_{j\in J} o_j - \sum_{j\in[k]\setminus(\{i\}\cup J)} o_j \\
&\ge (1-5\gamma)n - \sum_{j\in[k]\setminus\{i\}} o_j + \alpha\sum_{j\in J} o_j \\
&\ge (1-5\gamma)n - \sum_{j\in[k]\setminus\{i\}} o_j + \alpha^3 n/3 \ge n - \sum_{j\in[k]\setminus\{i\}} o_j + 2n/\Delta' \qquad (5)
\end{aligned}$$

outneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$ or fewer than

$$\sum_{j\in[k]\setminus\{i\}} d_{ji}(m-o_j) - \varepsilon' km \ge n - \sum_{j\in[k]\setminus\{i\}} o_j + 2n/\Delta',$$

inneighbours in $\bigcup_{j\in[k]\setminus\{i\}} V'_j$. So we may choose a set S^- of $m/10$ vertices of $V_i \setminus S^+_{\Delta'}$, none of which fall into these two categories, and all of which are inneighbours of v^+. Since $|T^-_{\Delta'}| \le |T_{\Delta'}| \le \beta n \le m/30$, by Theorem 1.3 we may embed $T^-_{\Delta'}$ in S^-. Let $S^-_{\Delta'}$ be the set of vertices of S^- occupied by this embedding of $T^-_{\Delta'}$. Then since

$$|T^-| = n - |T^+| \le n - \sum_{j\in[k]\setminus\{i\}} o_j,$$

the right hand side of (5) is at least $|T^-| + 2n/\Delta'$. Also every component of $T^- - T^-_{\Delta'}$ is a component of $T - T_{\Delta'}$ (and so has order at most n/Δ' by Proposition 2.1(iv)). So by Lemma 2.6 we may extend the embedding of $T^-_{\Delta'}$ in $S^-_{\Delta'}$ to an embedding of T^- in $S^-_{\Delta'} \cup \bigcup_{j\in[k]\setminus\{i\}} V'_j$. Then the embeddings of T^+ and T^- do not overlap, and so together these embeddings form an embedding of T in G. □

Given an ε-regular cluster tournament G on clusters $V_1, \dots, V_k$, we define the *reduced digraph of G with parameter d*, denoted $R_G(d)$, to be the directed graph on vertex set $[k]$ in which $i \to j$ if and only if $d_{ij} \ge d$. Observe that since $d_{ij} + d_{ji} \le 1$ for any i and j, if $d > 1/2$ then $R_G(d)$ is an oriented graph.

Lemma 4.5. *Suppose that $1/n \ll 1/\Delta', \beta \ll 1/k \ll \varepsilon \ll \gamma \ll \alpha \ll 1$. Let T be a directed tree on n vertices with $|T_{\Delta'}| \le \beta n$, and let y and z be the outweight and inweight of $T_{\Delta'}$ respectively. Let G be an ε-regular cluster tournament on clusters $V_1, \dots, V_k$, each of size $m \ge 2(1-\gamma)n/k$. Suppose that for some $i \in [k]$ we have*

$$\sum_{j\in[k]\setminus\{i\}} d_{ij} \ge \frac{(1-3\gamma)k}{2} \quad \text{and} \quad \sum_{j\in[k]\setminus\{i\}} d_{ji} \ge \frac{(1-3\gamma)k}{2},$$

and also that there are at most αk values of $j \in [k] \setminus \{i\}$ such that $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$. Then:

(i) *There are at most $2\alpha k$ values of $j \in [k] \setminus \{i\}$ such that $d_{ij} < 1-2\alpha$ and $d_{ji} < 1-2\alpha$.*
(ii) *Let $R := R_G(1-2\alpha)$. Then $|N^+_R(i)|, |N^-_R(i)| \geq (1-10\alpha)k/2$.*
(iii) *If $y, z \geq 14\alpha n$, then G contains a copy of T.*

Proof. Fix such an i, and introduce a new constant ε' with $\varepsilon \ll \varepsilon' \ll \gamma$. For (i), note that since $d_{ij} + d_{ji} \leq 1$ for any $j \in [k] \setminus \{i\}$, and

$$\sum_{j \in [k] \setminus \{i\}} (d_{ij} + d_{ji}) \geq (1-3\gamma)k,$$

there are at most $3\sqrt{\gamma}k \leq \alpha k$ values of $j \in [k] \setminus \{i\}$ for which $d_{ij} + d_{ji} < 1 - \sqrt{\gamma}$. So there are at most $2\alpha k$ values of $j \in [k] \setminus \{i\}$ for which $d_{ij} < 1 - \alpha - \sqrt{\gamma}$ and $d_{ji} < 1 - \alpha - \sqrt{\gamma}$, so (i) holds.

For (ii), observe that by (i) we have

$$\begin{aligned}\frac{(1-3\gamma)k}{2} &\leq \sum_{j \in [k] \setminus \{i\}} d_{ij} \leq \sum_{\substack{j \in [k] \setminus \{i\} \\ d_{ij} \geq 1-2\alpha}} d_{ij} + \sum_{\substack{j \in [k] \setminus \{i\} \\ d_{ij}, d_{ji} < 1-2\alpha}} d_{ij} + \sum_{\substack{j \in [k] \setminus \{i\} \\ d_{ij} \leq 2\alpha}} d_{ij} \\ &\leq |N^+_R(i)| + 2\alpha k + 2\alpha k,\end{aligned}$$

so $|N^+_R(i)| \geq (1-10\alpha)k/2$. A similar calculation shows that $|N^-_R(i)| \geq (1-10\alpha)k/2$.

For (iii), let N^+ and N^- denote $N^+_R(i)$ and $N^-_R(i)$ respectively, and let $V^+ := \bigcup_{j \in N^+} V_j$ and $V^- := \bigcup_{j \in N^-} V_j$, so V^+ and V^- are disjoint. By Lemma 4.3, V_i contains at most $\varepsilon' m$ vertices with fewer than

$$\begin{aligned}\sum_{j \in N^+} d_{ij}m - \varepsilon' km &\geq |N^+_R(i)|(1-2\alpha)m - \varepsilon' km \geq (1-10\alpha)(1-2\alpha)km/2 - \varepsilon' km \\ &\geq (1 - 12\alpha - 2\varepsilon')km/2 \geq (1-13\alpha)n\end{aligned}$$

outneighbours in V^+ and at most $\varepsilon' m$ vertices with fewer than $\sum_{j \in N^-} d_{ji}m - \varepsilon' km \geq (1-13\alpha)n$ inneighbours in V^-. Choose a set S of $m/2$ vertices of V_i, not including any of these at most $2\varepsilon' m$ vertices. Since $|T_{\Delta'}| \leq \beta n \leq m/6$, by Theorem 1.3 we may embed $T_{\Delta'}$ in S. Let $S_{\Delta'}$ be the set of vertices of S occupied by this embedding of $T_{\Delta'}$. Also let T_1 be the tree formed by $T_{\Delta'}$ and all of its outcomponents, and let T_2 be the tree formed by $T_{\Delta'}$ and all of its incomponents. Note that all of these out- and incomponents have order at most $n/\Delta' \ll \alpha n$ by Proposition 2.1(iv). In addition $|T_1| = n - z \leq (1-14\alpha)n$ and $|T_2| = n - y \leq (1-14\alpha)n$. So by Lemma 2.6 we may extend the embedding of $T_{\Delta'}$ in $S_{\Delta'}$ to an embedding of T_1 in $S_{\Delta'} \cup V^+$. Similarly by Lemma 2.6 we may extend the embedding of $T_{\Delta'}$ in $S_{\Delta'}$ to an embedding of T_2 in $S_{\Delta'} \cup V^-$. Then these embeddings do not overlap outside $T_{\Delta'}$, so we may combine them to form an embedding of T in G. □

To finish this section we shall show how Lemma 4.1 can be used to show that Sumner's universal tournament conjecture holds for any large and almost-regular tournament with a small core tree. Actually we shall prove a slightly stronger result in this case, considering a tournament on fewer than $2n-2$ vertices. Later on we shall make use of the fact that we have a little room to spare in the order of the tournament. Much of the work for this lemma is done by the two previous lemmas.

Lemma 4.6. *Suppose that $1/n \ll 1/\Delta', \beta \ll \gamma \ll 1/\Delta \ll 1$. Let T be a directed tree on n vertices such that $|T_{\Delta'}| \leq \beta n$ and $|T_\Delta| \geq 2$. Let G be a γ-almost-regular tournament on at least $(2-\gamma)n$ vertices. Then G contains a copy of T.*

Proof. Introduce new constants $\varepsilon, \varepsilon', \alpha, M$, and M' with

$$1/n \ll 1/\Delta', \beta \ll 1/M \ll 1/M' \ll \varepsilon \ll \varepsilon' \ll \gamma \ll \alpha \ll 1/\Delta \ll 1.$$

If $|G| \geq (2+\gamma)n$, then G contains a copy of T by Theorem 1.2(i). So we may assume that $|G| = (2 \pm \gamma)n$. Observe that $d^+(v), d^-(v) \geq (1-\gamma)(|G|-1)/2 \geq (1-2\gamma)n$ for all $v \in G$.

Since $\Delta \leq \Delta'$, we must have $T_\Delta \subseteq T_{\Delta'}$. Also, since $|T_\Delta| \geq 2$, we may choose an edge $t^- \to t^+$ of T_Δ, which must also lie in $T_{\Delta'}$. Let T^+ and T^- be the two components formed when this edge is deleted from T, labelled so that $t^+ \in T^+$ and $t^- \in T^-$. Similarly, let $T^+_{\Delta'}$ and $T^-_{\Delta'}$ be the two components formed by the deletion of the edge $t^- \to t^+$ from $T_{\Delta'}$, labelled with $t^+ \in T^+_{\Delta'}$ and $t^- \in T^-_{\Delta'}$. Then T^+ and T^- partition the vertices of T, and there is precisely one edge of T between T^+ and T^-, which is directed towards T^+. Furthermore, $|T^+|, |T^-| \geq n/\Delta$.

Let disjoint subsets $V_1, \dots, V_k$ and a subgraph $G^* \subseteq G$ satisfy the conditions of Corollary 4.2. So $M' \leq k \leq M$, and G^* is an ε-regular cluster tournament on clusters $V_1, \dots, V_k$ of equal size m, where

$$\frac{2(1-\gamma)n}{k} \leq \frac{(2-\gamma)n - 3\varepsilon n}{k} \leq m \leq \frac{(2+\gamma)n}{k}. \tag{6}$$

Also, for each $v \in G^*$ we have $d^+_{G^*}(v) \geq d^+_G(v) - 2\varepsilon|G| \geq d^+_G(v) - 5\varepsilon n$ and $d^-_{G^*}(v) \geq d^-_G(v) - 5\varepsilon n$. So for each $i \in [k]$ we have

$$\begin{aligned} \sum_{j \in [k]\setminus\{i\}} d_{ij} &= \sum_{j \in [k]\setminus\{i\}} \frac{e_{G^*}(V_i \to V_j)}{m^2} \geq \sum_{v \in V_i} \frac{d^+_{G^*}(v) - m}{m^2} \\ &\geq \sum_{v \in V_i} \frac{d^+_G(v) - 5\varepsilon n - m}{m^2} \geq \frac{(1-2\gamma)n - 5\varepsilon n - m}{m} \overset{(6)}{\geq} \frac{(1-3\gamma)k}{2}, \end{aligned} \tag{7}$$

and similarly $\sum_{j \in [k]\setminus\{i\}} d_{ji} \geq (1-3\gamma)k/2$.

So if there exists some $i \in [k]$ for which there are at least αk values of $j \in [k] \setminus \{i\}$ such that $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$, then by Lemma 4.4 we may embed T in G^*, and therefore in G. So we may assume that for each $i \in [k]$ fewer than αk values of $j \in [k] \setminus \{i\}$ satisfy $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$. Then by Lemma 4.5 we may assume that $R := R_G(1-2\alpha)$ has

$$\delta^0(R) \geq (1-10\alpha)k/2. \tag{8}$$

Let y be the number of vertices in outcomponents of $T_{\Delta'}$, and let z be the number of vertices in incomponents of $T_{\Delta'}$, so $y + z + |T_{\Delta'}| = n$. So if $y, z \geq 14\alpha n$ then G^* (and therefore G) contains a copy of T by Lemma 4.5. We may therefore assume without loss of generality that $z < 14\alpha n$.

Now, since $|R| = k$ we may choose a vertex $i \in R$ with $d^+_R(i) \leq k/2$. Then we may choose a vertex $j \in N^+_R(i)$ with at most $d^+_R(i)/2$ outneighbours in $N^+_R(i)$. So $i \to j$ and

$|N_R^+(i) \cap N_R^+(j)| \leq k/4$. For this choice of i and j, let

$$\begin{aligned} A &:= N_R^+(i) \cap N_R^+(j), \\ B &:= N_R^+(i) \setminus N_R^+(j), \\ C &:= N_R^+(j) \setminus N_R^+(i). \end{aligned}$$

Then A, B and C are disjoint, and $|B|, |C| \geq k/2 - 5\alpha k - |A| \geq k/4 - 5\alpha k$ by (8). Now, choose a set S^+ of $m/2$ vertices of V_j such that each vertex $v \in S^+$ has

(i) at least $m/2$ inneighbours in V_i,
(ii) at least $\sum_{\ell \in A} d_{j\ell} m - \varepsilon' km \geq (1-2\alpha)m|A| - \varepsilon' km$ outneighbours in $\bigcup_{\ell \in A} V_\ell$, and
(iii) at least $\sum_{\ell \in C} d_{j\ell} m - \varepsilon' km \geq (1-2\alpha)m|C| - \varepsilon' km$ outneighbours in $\bigcup_{\ell \in C} V_\ell$.

We can be sure that such a choice is possible, as by Lemma 4.3 there are at most $2\varepsilon' m$ vertices of V_j which fail either of (ii) and (iii), and since $G^*[V_i \to V_j]$ is ε-regular with density $d_{ij} \geq 1 - 2\alpha$ there are at most εm vertices of V_j which fail (i). Then since $|T^+_{\Delta'}| \leq \beta n \leq m/6$, by Theorem 1.3 we can embed $T^+_{\Delta'}$ in S^+. Let v^+ be the vertex to which t^+ is embedded. Then v^+ has at least $m/2$ inneighbours in V_i. Choose a set S^- of $m/3$ of these inneighbours so that every vertex $v \in S^-$ has at least

$$\sum_{\ell \in A \cup B = N_R^+(i)} d_{i\ell} m - \varepsilon' km \geq (1-2\alpha)m|N_R^+(i)| - \varepsilon' km \overset{(8)}{\geq} (1-13\alpha)n \tag{9}$$

outneighbours in $\bigcup_{\ell \in A \cup B} V_\ell$. Again we can be sure that such a choice is possible, since by Lemma 4.3 at most $\varepsilon' m$ vertices of V_i fail this condition. Then since $|T^-_{\Delta'}| \leq \beta n \leq m/9$, by Theorem 1.3 we can embed $T^-_{\Delta'}$ in S^-. Let $S^+_{\Delta'}$ and $S^-_{\Delta'}$ be the sets of vertices of G occupied by $T^+_{\Delta'}$ and $T^-_{\Delta'}$ respectively.

Let T_3 be the tree formed by $T_{\Delta'}$ and all of its incomponents. Let T_4 be the tree formed by $T^+_{\Delta'}$ and all of its outcomponents, and let T_5 be the tree formed by $T^-_{\Delta'}$ and all of its outcomponents in T^- (i.e. all of its outcomponents except T^+). Note that $T_3 \cup T_4 \cup T_5 = T$. Then $|T_3| = |T_{\Delta'}| + z < 15\alpha n$, $|T_4| \leq |T^+| \leq n - |T^-| \leq (1 - 1/\Delta)n$, and similarly $|T_5| \leq (1 - 1/\Delta)n$. Every vertex of G has at least $(1-2\gamma)n$ inneighbours in G, so by Lemma 2.6(c) we may extend the embedding of $T_{\Delta'}$ in $S^+_{\Delta'} \cup S^-_{\Delta'}$ to an embedding of T_3 in G. For each $\ell \in [k] \setminus \{i\}$, let $V'_\ell \subseteq V_\ell$ consist of the vertices of V_ℓ which are not occupied by this embedding.

By (ii) and (iii), every vertex of $S^+_{\Delta'}$ then has at least $(1-2\alpha)(|A|+|C|)m - 2\varepsilon' km - |T_3| \geq (1-28\alpha)n$ outneighbours in $\bigcup_{\ell \in A \cup C} V'_\ell$ (here we also use the fact that $|A| + |C| = |N_R^+(j)| \geq (1-10\alpha)k/2$ by (8)). Since also $1/\Delta' \ll \alpha \ll 1/\Delta$ and every component of $T_4 - T^+_{\Delta'}$ has order at most n/Δ', by Lemma 2.6 we may extend the embedding of $T^+_{\Delta'}$ in $S^+_{\Delta'}$ to an embedding of T_4 in $S^+_{\Delta'} \cup \bigcup_{\ell \in A \cup C} V'_\ell$. Furthermore, since every vertex of $S^+_{\Delta'}$ has at least $(1-2\alpha)|C|m - \varepsilon' km - |T_3| \geq n/\Delta$ outneighbours in $\bigcup_{\ell \in C} V'_\ell$, and $|T_4 - T^+_{\Delta'}| = |T^+ - T_3| \geq n/2\Delta$, by Lemma 2.6(b) we can ensure that this embedding of T_4 occupies at least $n/2\Delta$ vertices of $\bigcup_{\ell \in C} V'_\ell$. So crucially at most $|T_4| - n/2\Delta$ vertices of T_4 are embedded in $\bigcup_{\ell \in A \cup B} V_\ell$. For each $\ell \in A \cup B$, let $V''_\ell \subseteq V_\ell$ consist of those vertices which are not occupied by the embedding of T_3 and T_4.

Finally, by (9), every vertex of $S^-_{\Delta'}$ has at least

$$(1-13\alpha)n - (|T_4| - n/2\Delta) - |T_3| \geq n - |T_4| + n/3\Delta$$

outneighbours in $\bigcup_{\ell\in A\cup B} V''_\ell$. Since $|T_5 - T^-_{\Delta'}| \leq n - |T_4|$, by Lemma 2.6(c) we can extend the embedding of $T^-_{\Delta'}$ in $S^-_{\Delta'}$ to an embedding of T_5 in $S^-_{\Delta'} \cup \bigcup_{\ell\in A\cup B} V''_\ell$. Then the embeddings of T_3, T_4 and T_5 do not overlap outside $S^+_{\Delta'} \cup S^-_{\Delta'}$, and so together form an embedding of T in G. □

5. Embedding trees in robust outexpander tournaments

Let G be a tournament on n vertices, and let $\mu \leq \nu$ be positive constants. Then the *robust outneighbourhood* $RN^+_\mu(S)$ of a set $S \subseteq V(G)$ is the set of vertices of G with at least μn inneighbours in S. We say that G is a *robust* (μ,ν)*-outexpander* if for any $S \subseteq V(G)$ with $\nu n \leq |S| \leq (1-\nu)n$ we have $|RN^+_\mu(S)| \geq |S| + \mu n$.

If a tournament G is not a robust outexpander, then the following lemma shows that G contains two subtournaments which partition the vertices of G and which have almost all edges between them directed the same way.

Lemma 5.1 ([10], Lemma 2.8). *Suppose that $1/n \ll \mu \ll \nu$, that G is a tournament on n vertices and that G is not a robust (μ,ν)-outexpander. Then we can partition $V(G)$ into sets S and S' such that $\nu n < |S|, |S'| < (1-\nu)n$ and $e(G[S \to S']) \leq 4\mu n^2$.*

By iterating this split, we obtain a decomposition of G into sets S_i which either induce robust expanders or are small, and where for all $i < j$, almost all edges are directed from S_i to S_j. (So if all the S_i are small, then G is close to being a transitive tournament.) We will use this decomposition in Section 7 to prove Theorem 1.1.

Lemma 5.2. *Suppose that $1/n \ll \mu \ll \nu \ll \eta \ll \gamma \ll 1$. Let G be a tournament on n vertices. Then we may choose disjoint subsets $S_1, \dots, S_r$ of $V(G)$ such that:*

(i) $|\bigcup_{i\in[r]} S_i| \geq (1-\gamma)n$,

(ii) *for each $i \in [r]$, any vertex $v \in S_i$ has at most γn inneighbours in $\bigcup_{j>i} S_j$ and at most γn outneighbours in $\bigcup_{j<i} S_j$, and*

(iii) *for each $i \in [r]$, either $G[S_i]$ is a robust (μ,ν)-outexpander with $\delta^0(G[S_i]) \geq \eta n$ or $|S_i| < \gamma n$.*

Proof. We shall use a modified version of an algorithm from [10], which keeps track of an ordered family $\mathcal{S}^\tau$ of disjoint subsets of $V(G)$, and a set B^τ of bad edges of G, at each time τ. The analysis of this algorithm is also similar to the analysis in [10]. Initially, let $\mathcal{S}^1 := (V(G))$, and let $B^1 := \emptyset$. Then at time $\tau \geq 1$, we have $\mathcal{S}^\tau = (S^\tau_1, \dots, S^\tau_\tau)$, and the algorithm proceeds as follows.

(1) Let S^τ_ℓ be the largest member of $\mathcal{S}^\tau$ which is not a robust (μ,ν)-outexpander with $\delta^0(G[S^\tau_\ell]) \geq \eta n$. If there is no such member of $\mathcal{S}^\tau$, or if $|S^\tau_\ell| < \gamma n$, then terminate. If there is more than one largest such member, then choose one of these arbitrarily.

(2) If some $v \in S^\tau_\ell$ has $d^+_{G[S^\tau_\ell]}(v) < \eta n$, then let

$$\mathcal{S}^{\tau+1} := (S^\tau_1, \dots, S^\tau_{\ell-1}, S^\tau_\ell \setminus \{v\}, \{v\}, S^\tau_{\ell+1}, \dots, S^\tau_\tau),$$

let $B^{\tau+1} := B^\tau \cup E(\{v\} \to S^\tau_\ell \setminus \{v\})$, and proceed to step (5).

(3) Similarly, if some $v \in S^\tau_\ell$ has $d^-_{G[S^\tau_\ell]}(v) < \eta n$, then let

$$\mathcal{S}^{\tau+1} := (S^\tau_1, \dots, S^\tau_{\ell-1}, \{v\}, S^\tau_\ell \setminus \{v\}, S^\tau_{\ell+1}, \dots, S^\tau_\tau),$$

let $B^{\tau+1} := B^\tau \cup E(S^\tau_\ell \setminus \{v\} \to \{v\})$, and proceed to step (5).

(4) If $G[S^\tau_\ell]$ is not a robust (μ,ν)-outexpander then apply Lemma 5.1 to partition the vertices of S^τ_ℓ into sets S' and S'' such that $\nu|S^\tau_\ell| \leq |S'|, |S''| \leq (1-\nu)|S^\tau_\ell|$ and at most $4\mu|S^\tau_\ell|^2$ edges of $G[S^\tau_\ell]$ are directed from S'' to S'. Then let
$$\mathcal{S}^{\tau+1} := (S^\tau_1, \dots, S^\tau_{\ell-1}, S', S'', S^\tau_{\ell+1}, \dots, S^\tau_\tau)$$
and let $B^{\tau+1} := B^\tau \cup E(S'' \to S')$.

(5) Finally, for each $i \in [\tau+1]$, delete from $S^{\tau+1}_i$ any vertex v which lies in more than $\sqrt{\eta}n$ edges of $B^{\tau+1}$.

At any time τ, if the algorithm does not terminate at step (1) then S^τ_ℓ will be split in precisely one of steps (2), (3) and (4). So at each time τ, either the algorithm terminates or $|\mathcal{S}^\tau|$ increases from τ to $\tau+1$ (in forming $\mathcal{S}^{\tau+1}$) by reducing the size of the largest piece. Therefore the algorithm must terminate at some time $\tau_{end} \leq n$. Take $r := \tau_{end}$, and $S_i := S^r_i$ for each i. Then since the algorithm terminated at step (1) of time r, (iii) must hold.

To see (i), observe that the split in step (4) will occur for at most $1/\gamma\nu$ times $\tau < \tau_{end}$. This is because any set obtained by a split in step (4) must have size at least $\gamma\nu n$ (since $|S^\tau_\ell| \geq \gamma n$, and the sets S', S'' obtained have $|S'|, |S''| \geq \nu|S^\tau_\ell|$). Also, at each time $\tau \leq \tau_{end}$, the number of edges added to form $B^{\tau+1}$ from B^τ is at most ηn if the algorithm carried out the split in step (2) or (3), and at most $4\mu n^2$ if the algorithm carried out the split in step (4). Since $\tau_{end} \leq n$, and the split in step (4) is carried out in at most $1/\gamma\nu$ steps, we must have
$$|B^{\tau_{end}}| \leq \eta n^2 + 4\mu n^2/\nu\gamma \leq 2\eta n^2.$$
Since $B^1 \subseteq \dots \subseteq B^{\tau_{end}}$, any vertex of G which was ever deleted in step (5) must lie in at least $\sqrt{\eta}n$ edges of $B^{\tau_{end}}$, and so at most $4\sqrt{\eta}n \leq \gamma n$ vertices of G can have been deleted in step (5) over the entire course of the algorithm. But any vertex which was not deleted lies in some S_i, and so (i) holds.

Finally, for (ii) fix any $i \in [r]$ and any $v \in S_i$. Observe that all edges directed from v to $\bigcup_{j<i} S_j$ and all edges directed from $\bigcup_{j>i} S_j$ to v are contained in B^r. This means that there are at most $\sqrt{\eta}n$ such edges, as otherwise v would have been deleted in step (5) at some point. Since i and v were arbitrary, (ii) must hold. □

We now consider the case when G is a robust outexpander. Lemma 4.1 of [10] stated that if T is a directed tree on n vertices, and G is a robust outexpander tournament on at least $(2+\alpha)n$ vertices with large minimum semidegree, then G contains a copy of T. However, in the proof of this lemma, the αn error term was only needed in the case when T_Δ is small. Indeed, in this section we modify this proof to show that Sumner's universal tournament conjecture holds for such G in the case when T_Δ is large. This is the following lemma.

Lemma 5.3. *Suppose that $1/n \ll 1/\Delta \ll \mu \ll \nu \ll \eta \ll \gamma \ll \beta \ll 1$. Let T be a directed tree on n vertices such that $|T_\Delta| \geq \beta n$, and let G be a robust (μ,ν)-outexpander tournament on at least $(2-\gamma)n$ vertices, with $\delta^0(G) \geq \eta|G|$. Then G contains a copy of T.*

Before we can present the proof of this lemma, we must give some definitions from [10]. Let $V_1, \dots, V_k$ be disjoint sets of equal size. A digraph G on vertex set $V_1, \dots, V_k$ is a *ε-regular d-dense cycle of cluster tournaments* if for each i, $G[V_i]$ is a tournament and $G[V_i \to V_{i+1}]$ is ε-regular with density at least d (where addition on the index of V_{i+1} is taken modulo k). The following lemma from [10] (an immediate consequence of two results from [12]) will help us to find such digraphs.

Lemma 5.4 ([10], Lemma 2.7). *Suppose that $1/n \ll 1/M \ll 1/M' \ll \varepsilon \ll d \ll \mu \ll \nu \ll \eta \ll 1$. Let G be a tournament on n vertices which is a robust (μ,ν)-outexpander with $\delta^0(G) \geq \eta n$. Then G contains an ε-regular d-dense cycle of cluster tournaments on clusters $V_1,\dots,V_k$, where $|\bigcup_{i=1}^k V_i| > (1-\varepsilon)n$, and $M' \leq k \leq M$.*

Let T be a directed tree. Then the *distance* between vertices $u,v \in T$, denoted $d(u,v)$, is the length of the shortest path connecting u and v in the underlying graph T_{under}. Similarly for a set X of vertices of T, the distance $d(u,X)$ is the minimum of $d(u,x)$ taken over all vertices $x \in X$. If T is a rooted tree with root r, then the *children* of a vertex $u \in T$ are those neighbours v of u for which $d(r,u) = d(r,v)+1$.

Let T be a tree on n vertices, rooted at t_1, and let $H \subseteq V(T)$. Also let k be a positive integer. For any vertex $x \in T$, there is a unique path in T from x to t_1; let P_x denote the set of the first k vertices of this path, starting from x. Let $H^1 := \bigcup_{x\in H} P_x$, and then for each $i \geq 1$ let H^{i+1} be formed from H^i by adding the vertices of P_x for any $x \in H^i$ with at least two children in H^i. After at most n steps we must have $H^i = H^{i+1}$, when we terminate the process. We refer to this final H^i as H *with leading paths included*, denoted $\mathcal{P}_k(H)$. So $H \subseteq \mathcal{P}_k(H) \subseteq V(T)$. Note that $\mathcal{P}_k(H)$ depends on both the value of k and the root t_1 of T.

We may now present the key lemma from [10] we shall use to prove Lemma 5.3. This says that a directed tree of bounded degree can be embedded in a robust outexpander tournament of large minimum semidegree such that the vertices in a small set H of vertices of T are embedded within a chosen set $U \subseteq V(G)$.

Lemma 5.5 ([10], Lemma 4.6). *Suppose that $1/n \ll 1/\Delta, 1/k \ll \varepsilon \ll d \ll \alpha,\lambda \leq 1/2$, that $m := n/k$, that $\lambda \leq \alpha/4$ and that $\delta := d\lambda/8k$. Let T be a directed tree on n vertices rooted at t_1 and with $\Delta(T) \leq \Delta$. Let $H \subseteq V(T)$ be such that $|H| \leq \delta n/7k$ and $|\{x \in T\colon 1 \leq d(x,\mathcal{P}_k(H)) \leq k^3\}| \leq \delta n$. Let G be an ε-regular d-dense cycle of cluster tournaments on clusters $V_1,\dots,V_k$, each of size $(1+\alpha)m$, and let $U \subseteq V_1 \cup \dots \cup V_k$ have size $|U| \geq \lambda n$. Then T can be embedded in G so that each vertex $t \in H$ is embedded to some $u \in U$.*

We will also use the following lemma, again from [10]. This shows that we can extend T_Δ to an 'extended tree' T_{ext}, with desired properties. We will apply Lemma 5.5 to T_{ext} and embed H within a set U of vertices of high in- and outdegree.

Proposition 5.6 ([10], Lemma 4.5). *Suppose that $1/n, 1/\Delta^* \ll 1/\Delta, 1/k, \omega \ll 1$. Let T be a directed tree on n vertices. Choose any vertex $t_1 \in T_\Delta$ as the root of T. Then there exists a subtree T_{ext} of T and a subset $H \subseteq V(T_{ext})$ which satisfy the following properties.*

(i) $T_\Delta \subseteq T_{ext}$.
(ii) $\Delta(T_{ext}) \leq \Delta^*$.
(iii) *For any edge e between $T - T_{ext}$ and T_{ext}, the endvertex of e in T_{ext} lies in H.*
(iv) *The number of vertices $v \in T_{ext}$ which satisfy $1 \leq d(v,\mathcal{P}_k(H)) \leq k^3$ is at most ωn.*
(v) $|H| \leq n/\Delta^{k^{1/\omega}}$.

The final lemma we shall need to prove Lemma 5.3 gives standard Chernoff-type bounds for the binomial and hypergeometric distributions. The binomial random variable X with parameters (n,p) is defined to be the number of successes in n independent trials, each of which has probability p of success. So $\mathbb{E}X = np$. The hypergeometric random variable Y with parameters (n,m,k) is defined as follows. Let N be a set of size n, and fix a set $S \subseteq N$ of size $|S| = m$. Now choose a set $T \subseteq N$ of size $|T| = k$ uniformly at random. Then $Y = |T \cap S|$. Note that $\mathbb{E}Y = km/n$.

Proposition 5.7 ([9], Corollary 2.3 and Theorem 2.10)**.** *Suppose X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

Proof of Lemma 5.3. We begin by introducing new constants $\Delta^*, M, M', \varepsilon, d$ and α which satisfy

$$1/n \ll 1/\Delta^* \ll 1/M \ll 1/M', 1/\Delta \ll \varepsilon \ll d \ll \mu \ll \nu \ll \eta \ll \gamma \ll \alpha \ll \beta \ll 1.$$

Now, if $|G| \geq (2+\gamma)n$, then by Theorem 1.2(i), G contains a copy of T. So we may assume that $|G| = (2 \pm \gamma)n$. Since G is a robust (μ, ν)-outexpander with $\delta^0(G) \geq \eta|G|$, Lemma 5.4 implies that G contains an ε-regular d-dense cycle of cluster tournaments on clusters $V_1, \dots, V_k$ each of equal size between $(1-\varepsilon)|G|/k \geq (1-\varepsilon)(2-\gamma)m \geq 2(1-\gamma)m$ and $|G|/k \leq (2+\gamma)m$, where $m := n/k$ and $M' \leq k \leq M$. So we may remove vertices from each V_i to obtain a 2ε-regular $(d/2)$-dense cycle of cluster tournaments G' on clusters $V'_1, \dots, V'_k$ each of size $2(1-\gamma)m$. So $|G'| = 2(1-\gamma)n$. Let

$$\delta := d\alpha\beta/160k.$$

Choose any vertex $t_1 \in T_\Delta$ as the root of T. Then let T_{ext} and H satisfy the properties of Proposition 5.6, with $\omega := \delta\beta$. Let T_1 denote the subtree of T formed by T_{ext} and all of its outcomponents, and let T_2 denote the subtree of T formed by T_{ext} and all of its incomponents. Since $T_\Delta \subseteq T_{ext}$ (this is (i) of Proposition 5.6), all of these incomponents and outcomponents have order at most n/Δ by Proposition 2.1. Let $x := |T_{ext}|, y := |T_1 - T_{ext}|, z := |T_2 - T_{ext}|$, so $x + y + z = n$. Since $T_\Delta \subseteq T_{ext}$, we have $x \geq \beta n$. Also, all but at most $2y + x - \alpha n/2$ vertices of G have at least $y + x/2 - \alpha n/4$ outneighbours, and all but at most $2z + x - \alpha n/2$ vertices of G have at least $z + x/2 - \alpha n/4$ inneighbours. So at least $(2-\gamma)n - 2y - 2z - 2x + \alpha n \geq \alpha n/2$ vertices of G satisfy both of these conditions. Let U_0 be the set of these vertices, so $|U_0| \geq \alpha n/2$, and each $v \in U_0$ has at least $y + x/2 - \alpha n/4$ outneighbours and at least $z + x/2 - \alpha n/4$ inneighbours.

From each cluster V'_i of G' choose a set X_i of $(1+\alpha)x/k$ vertices uniformly at random, and let $X := X_1 \cup \dots \cup X_k$. Then $|X| = (1+\alpha)x$. For any single vertex $u \in G'$, the probability that u is included in X is $(1+\alpha)x/|G'| \geq x/2n$, so by Proposition 5.7, with probability at least $2/3$ the set $U := X \cap U_0$ satisfies $|U| \geq \alpha x/5 \geq \alpha\beta n/5$. Also, for any vertex $v \in U$, the expected number of outneighbours of v outside X is at least

$$\begin{aligned}\left(y + \frac{x}{2} - \frac{\alpha n}{4}\right)\left(1 - \frac{(1+\alpha)x}{|G'|}\right) &\geq y - \frac{\alpha n}{4} + \frac{x}{2} - \frac{(1+\alpha)xy}{2(1-\gamma)n} - \frac{(1+\alpha)x^2}{4(1-\gamma)n} \\ &\geq y - \frac{\alpha n}{4} + \frac{2xn - 2xy - x^2 - 2\gamma xn - 2\alpha xy - \alpha x^2}{4(1-\gamma)n} \\ &\geq y + \frac{x^2}{4n} - 2\alpha n \geq y + \frac{\beta^2 n}{4} - 2\alpha n \geq y + 2\alpha n,\end{aligned}$$

where in the first inequality of the third line we used the fact that $2n - 2y - x \geq x$. A similar calculation shows that for each $v \in U$, the expected number of inneighbours of v outside X is at least $z + 2\alpha n$. So by Proposition 5.7 we find that with probability at least $2/3$, every vertex $v \in U$ has at least $y + \alpha n$ outneighbours outside X and at least $z + \alpha n$ inneighbours outside X. Fix a choice of X such that both these events of probability at least $2/3$ occur.

Since every vertex of U has either at least $(|G| - |X|)/2 \geq y + z + \alpha n$ inneighbours outside X or at least $y + z + \alpha n$ outneighbours outside X, we may choose a set $U' \subseteq U$ of size $|U'| \geq |U|/2 \geq \alpha\beta n/10$ such that either

(α_1) every $v \in U'$ has at least $y + \alpha n$ outneighbours outside X and at least $y + z + \alpha n$ inneighbours outside X, or

(α_2) every $v \in U'$ has at least $y + z + \alpha n$ outneighbours outside X and at least $z + \alpha n$ inneighbours outside X.

So $G'[X]$ is a $(2\varepsilon/\beta)$-regular $(d/2)$-dense cycle of cluster tournaments on clusters $X_1, \dots, X_k$ of size $(1+\alpha)x/k$, and $U' \subseteq X_1 \cup \dots \cup X_k$ has size $|U'| \geq \alpha\beta x/10$. Also T_{ext} is a directed tree on x vertices rooted at t_1 and with $\Delta(T_{ext}) \leq \Delta^*$, and $H \subseteq V(T_{ext})$ has $|H| \leq n/\Delta^{k^{1/\beta\delta}} \leq \delta x/7k$ and $|\{t \in T_{ext} : 1 \leq d(t, \mathcal{P}_k(H)) \leq k^3\}| \leq \delta\beta n \leq \delta x$. So by Lemma 5.5 (with $\alpha\beta/10$, Δ^* and $d/2$ in place of λ, Δ and d respectively), $G'[X]$ contains a copy of T_{ext} in which every vertex of H is embedded to a vertex of U'.

So every vertex $t \in H$ has been embedded to some vertex $v(t) \in U'$. Suppose that (α_1) holds. Then for every $t \in H$, $v(t)$ has at least $y + 2n/\Delta$ outneighbours outside X (and so unoccupied by vertices of T_{ext}). Since the only vertices of T_{ext} which may have neighbours in $T_1 - T_{ext}$ are the vertices of H, we may use Theorem 1.3 to extend the embedding of T_{ext} in $G[X]$ to an embedding of T_1 in G in the same way as in the proof of Lemma 2.6 (we cannot just apply Lemma 2.6 as vertices of G to which we embedded $T_{ext} - H$ may not have sufficiently many outneighbours, but since vertices of $T_{ext} - H$ do not have any outneighbours outside T_{ext} this does not cause any problems). Then for every $t \in H$, $v(t)$ has at least $z + 2n/\Delta$ inneighbours outside X which are not occupied by this embedding of T_1. So in the same way we may extend the embedding of T_{ext} in $G[X]$ to an embedding of T_2 in the vertices of G not occupied by $T_1 - T_{ext}$. So the embeddings of T_1 and T_2 only overlap in T_{ext}, and so together form an embedding of T in G. If instead (α_2) holds we may embed T in G similarly by first embedding T_2 then T_1. □

We can now deduce that if G is a large almost-regular tournament and if $|T_\Delta| > 1$, then Sumner's conjecture holds with a little room to spare (we shall need this extra room in the proof of Lemmas 6.2 and 6.3). Indeed, we shall see that a large almost-regular tournament G is also a robust outexpander, and so if T_Δ is large, then we can embed T in G by Lemma 5.3. On the other hand, if T_Δ is small but has more than one vertex, then we may embed T in G by Lemma 4.6.

In particular, together with Lemma 3.1 (which deals with the case $|T_\Delta| = 1$), this means that at this stage, we have proved that Sumner's conjecture holds for all large almost-regular tournaments.

Lemma 5.8. *Suppose that $1/n \ll \gamma \ll 1/\Delta \ll 1$. Let T be a directed tree on n vertices with $|T_\Delta| > 1$. Then every γ-almost-regular tournament G on at least $(2-\gamma)n$ vertices contains a copy of T.*

Proof. Introduce constants $\mu, \nu, \eta, \Delta', \beta, \gamma'$ such that

$$1/n \ll 1/\Delta' \ll \mu \ll \nu \ll \eta \ll \gamma \ll \beta \ll \gamma' \ll 1/\Delta \ll 1.$$

Let G be a γ-almost-regular tournament on at least $(2-\gamma)n$ vertices. Then we shall show that G is a robust (μ, ν)-outexpander. Indeed, let $S \subseteq V(G)$ satisfy $\nu|G| \leq |S| \leq 2|G|/3$. Then at least $(1-\gamma)|S|(|G|-1)/2$ edges originate in S. At most $\binom{|S|}{2}$ of these have both endvertices in S, so at least $(1-\gamma)|S|(|G|-1)/2 - \binom{|S|}{2} \geq |S|((1-\gamma)(|G|-1) - |S|)/2 \geq \nu|G|^2/10$ edges leave S. So at least $\nu|G|/20 \geq 3\mu|G|$ vertices outside S have at least $\nu|G|/20 \geq 3\mu|G|$ inneighbours in S. At most $2\mu|G|$ vertices of S have fewer than $\mu|G|$ inneighbours in S, and so $|RN^+_\mu(S)| \geq |S| + \mu|G|$, as desired. On the other hand, if $S \subseteq V(G)$ satisfies $2|G|/3 < |S| \leq (1-\nu)|G|$, every vertex of G has at least $|G|/7 \geq \mu|G|$ inneighbours in S. So $|RN^+_\mu(S)| = |G| \geq |S| + \mu|G|$, as desired.

So G is indeed a robust (μ, ν)-outexpander. Clearly $\delta^0(G) \geq \eta|G|$. So if $|T_{\Delta'}| \geq \beta n$, then by Lemma 5.3, G contains a copy of T. So we may assume that $|T_{\Delta'}| \leq \beta n$. But G is also a γ'-almost-regular tournament on at least $(2-\gamma')n$ vertices, and so by Lemma 4.6, G contains a copy of T. □

6. Embedding trees whose core tree is small

We now turn our attention to the general case of the problem. As when considering almost-regular tournaments, we consider the problem of embedding directed trees whose core trees are small separately from the case when the core trees are large. In this section we shall consider directed trees with small core trees, proving the following lemma.

Lemma 6.1. *Suppose $1/n \ll \beta, 1/\Delta' \ll 1$. Let T be a directed tree on n vertices with $|T_{\Delta'}| \leq \beta n$, and let G be a tournament on $2n-2$ vertices. Then G contains a copy of T.*

We begin by showing that we may assume that the tournament G consists of two large disjoint almost-regular tournaments, with almost all of the edges between them directed the same way.

Lemma 6.2. *Suppose that $1/n \ll \beta, 1/\Delta \ll \gamma \ll \eta \ll 1$. Let T be a directed tree on n vertices with $|T_\Delta| \leq \beta n$, and let G be a tournament on $2n-2$ vertices. Let y be the outweight of T_Δ, and let z be the inweight of T_Δ. Then the following properties hold.*

(i) *If $z < \eta n$ or $y < \eta n$ then G contains a copy of T.*

(ii) *Either G contains a copy of T, or we can find disjoint sets $Y, Z \subseteq V(G)$ such that $|Y| \geq (2-\gamma)y$ and $|Z| \geq (2-\gamma)z$, $G[Y]$ and $G[Z]$ are γ-almost-regular, any vertex of Y has at most $3\gamma n$ outneighbours in Z and any vertex of Z has at most $3\gamma n$ inneighbours in Y.*

Proof. Introduce new constants $M, M', \varepsilon, \varepsilon', \alpha, \gamma^*$ and Δ^* such that

$$1/n \ll \beta, 1/\Delta \ll 1/M \ll 1/M' \ll \varepsilon \ll \varepsilon' \ll \gamma \ll \alpha \ll \eta \ll \gamma^* \ll 1/\Delta^* \ll 1.$$

Partition the vertex set of G into sets A, B, C, D, E such that:

$$\begin{aligned}
A &\subseteq \{v \in G : d^+(v) \leq y + \varepsilon n\},\\
B &\subseteq \{v \in G : y + \varepsilon n < d^+(v) < n - \varepsilon n\},\\
C &\subseteq \{v \in G : d^+(v), d^-(v) \geq n - \varepsilon n\},\\
D &\subseteq \{v \in G : z + \varepsilon n < d^-(v) < n - \varepsilon n\},\\
E &\subseteq \{v \in G : d^-(v) \leq z + \varepsilon n\}.
\end{aligned}$$

These subset relations may not all be equality, for example in the case where z is very small, when we have $y + \varepsilon n \geq n - \varepsilon n$. However, it is clear that each vertex $v \in G$ lies in at least one of these five sets, so we may choose such a partition of $V(G)$. Let $x := |T_\Delta|$, so $x + y + z = n$ and $x \leq \beta n$.

Suppose that $|B| \geq 3x$. Then by Theorem 1.3 we may embed T_Δ in $G[B]$. Let $S_\Delta \subseteq B$ be the set of vertices occupied by this embedding of T_Δ. Then every vertex of S_Δ has at least

$y+\varepsilon n-x \geq y+2n/\Delta$ outneighbours outside S_Δ and at least $|G|-x-(n-\varepsilon n) \geq y+z+2n/\Delta$ inneighbours outside S_Δ. Let T_1 be the subtree of T formed by T_Δ and all outcomponents of T_Δ, and let T_2 be the subtree of T formed by T_Δ and all incomponents of T_Δ. Then $|T_1| = x+y$ and $|T_2| = x+z$. By Proposition 2.1(iv), all incomponents and outcomponents of T_Δ contain at most n/Δ vertices, so by Lemma 2.6(c) we may extend our embedding of T_Δ in S_Δ to an embedding of T_1 in G. Then each vertex of S_Δ still has at least $z+2n/\Delta$ inneighbours outside S_Δ which are not occupied by this embedding of T_1, so by Lemma 2.6(c) we may also extend our embedding of T_Δ in S_Δ to an embedding of T_2 in G which avoids vertices occupied by the embedding of $T_1 - T_\Delta$. Then these embeddings of T_1 and T_2 do not overlap outside T_Δ, and so together form an embedding of T in G. We may therefore assume that $|B| < 3x \leq 3\beta n$. By the same argument (embedding first T_2 and then T_1 in G) we may assume that $|D| < 3x \leq 3\beta n$.

If $|T_{\Delta^*}| = 1$, then G contains a copy of T by Lemma 3.1. So we may assume that $|T_{\Delta^*}| \geq 2$. Now, if $z < \eta n$, then every $v \in E$ satisfies $d^-(v) < (\eta+\varepsilon)n < 2\eta n$, so $|E| \leq 4\eta n + 1$, and so $|B \cup D \cup E| \leq 4\eta n + 1 + 6\beta n \leq 5\eta n$. Let $G' := G[A \cup C]$. Then $|G'| \geq 2n - 2 - 5\eta n$, and every vertex $v \in G'$ has $d^+(v) \leq n + \varepsilon n$. So by Proposition 2.4, G' contains a γ^*-almost-regular subtournament G' on at least $(2-\gamma^*)n$ vertices. Since $|T_{\Delta^*}| \geq 2$, by Lemma 5.8 G' contains a copy of T, so G contains a copy of T also. If instead we have $y < \eta n$, then we may similarly embed T in $G[C \cup E]$. So if $z < \eta n$ or $y < \eta n$ then G contains a copy of T, completing the proof of (i). So for (ii), we may assume that $y, z \geq \eta n$.

Suppose now that $|C| \geq 5\varepsilon' n$. Let disjoint subsets $V_1, \dots, V_k$ and a subgraph $G^* \subseteq G$ satisfy the conditions of Corollary 4.2. So $M' \leq k \leq M$, and G^* is an ε-regular cluster tournament on clusters $V_1, \dots, V_k$ of equal size m, where

$$\frac{(1-\varepsilon)|G|}{k} \leq m \leq \frac{|G|}{k}.$$

We shall show that G^* has the property that for some $i \in [k]$ we have

$$\sum_{j \in ([k]\setminus\{i\})} d_{ij} \geq \frac{(1-3\varepsilon')k}{2} \qquad \text{and} \sum_{j \in ([k]\setminus\{i\})} d_{ji} \geq \frac{(1-3\varepsilon')k}{2}. \tag{10}$$

Indeed, if for some $i \in [k]$ we have $\sum_{j \in ([k]\setminus\{i\})} d_{ij} < (1-3\varepsilon')k/2$, then by Lemma 4.3 all but at most $\varepsilon' m$ vertices of V_i have at most

$$\sum_{j \in ([k]\setminus\{i\})} d_{ij} m + \varepsilon' km < \frac{(1-\varepsilon')km}{2} < n - 8\varepsilon n$$

outneighbours in $\bigcup_{j \in ([k]\setminus\{i\})} V_j$ (in the graph G^*), and hence at most $n - 8\varepsilon n + (|G| - |G^*|) + |V_i| + 2\varepsilon|G| < n - \varepsilon n$ outneighbours in G. So at most $\varepsilon' m$ vertices of V_i lie in C. Similarly if for some $i \in [k]$ we have $\sum_{j \in ([k]\setminus\{i\})} d_{ji} < (1-3\varepsilon')k/2$ then again at most $\varepsilon' m$ vertices of V_i lie in C. Since $|C| \geq 5\varepsilon' n > 2\varepsilon' mk + (|G| - |G^*|)$, there must be some $i \in [k]$ which satisfies (10). Fix such an i. Then if at least αk values of $j \in [k] \setminus \{i\}$ have $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$ then G^* contains a copy of T by Lemma 4.4 (applied with ε' in the place of γ). Alternatively, if at most αk values of $j \in [k] \setminus \{i\}$ have $d_{ij} \geq \alpha$ and $d_{ji} \geq \alpha$ then since $y, z \geq \eta n$, G^* contains a copy of T by Lemma 4.5(iii) (again applied with ε' in the place of γ). So in either case G contains a copy of T, and so we may assume that $|C| < 5\varepsilon' n$.

So to prove (ii), observe that we must therefore have $|B \cup C \cup D| \leq 5\varepsilon' n + 6\beta n \leq 6\varepsilon' n$. Trivially $|A| \leq 2y + 2\varepsilon n + 1$ and $|E| \leq 2z + 2\varepsilon n + 1$, and so we must have

$$|A| \geq 2n - 2 - 6\varepsilon' n - 2z - 2\varepsilon n - 1 \geq 2y - 7\varepsilon' n, \text{ and}$$
$$|E| \geq 2n - 2 - 6\varepsilon' n - 2y - 2\varepsilon n - 1 \geq 2z - 7\varepsilon' n.$$

So by Proposition 2.4, $G[A]$ contains a γ-almost-regular subtournament on at least $(2-\gamma)y$ vertices, and $G[E]$ contains a γ-almost-regular subtournament on at least $(2-\gamma)z$ vertices. Let Y and Z be the vertex sets of these subtournaments respectively. Then any vertex of Y has at least $(1-2\gamma)y$ outneighbours in Y, and so has at most $y + \varepsilon n - (1-2\gamma)y \leq 3\gamma n$ outneighbours in Z. Similarly any vertex of Z has at least $(1-2\gamma)z$ inneighbours in Z, and so has at most $3\gamma n$ inneighbours in Y. So Y and Z are as required for (ii). □

The next lemma builds on the previous lemma and will in turn be used in the proof of Lemma 6.4.

Lemma 6.3. *Suppose that $1/n \ll \beta, 1/\Delta' \ll \alpha \ll 1/\Delta \ll 1$. Let T be a directed tree on n vertices with $|T_{\Delta'}| \leq \beta n$. Let y and z be the outweight and inweight of $T_{\Delta'}$ respectively. Suppose that forests F^- and F^+ are induced subgraphs of T which partition the vertices of T, such that $|F^+| \leq y + 2\alpha n$, $|F^-| \leq z - \alpha n$, and every edge of T between F^- and F^+ is directed from F^- to F^+. Suppose also that either*

(i) *no component of F^+ has order greater than $y - \alpha n$, or*
(ii) *the largest component T_1 of F^+ has $|(T_1)_\Delta| \geq 2$.*

Then any tournament G on $2n-2$ vertices contains a copy of T.

Proof. Let G be a tournament on $2n-2$ vertices, and let T_1 and T_2 be the largest and second largest components of F^+ respectively. Introduce new constants γ and η with

$$1/n \ll \beta, 1/\Delta' \ll \gamma \ll \alpha \ll 1/\Delta \ll \eta \ll 1.$$

Then by Lemma 6.2 we may assume that $y, z \geq \eta n$. Also by Lemma 6.2 we may find subsets $Y, Z \subseteq V(G)$ such that $|Y| \geq (2-\gamma)y$, $|Z| \geq (2-\gamma)z$, $G[Y]$ is γ-almost-regular, each vertex of Y has at most $3\gamma n$ outneighbours in Z, and each vertex of Z has at most $3\gamma n$ inneighbours in Y. Then $|Y| \geq 3|F^+|/2 + \alpha n \geq |F^+| + |T_2| + \alpha n$, and $|Z| \geq 2|F^-| + \alpha n$, and so by Lemma 2.7 any embedding of T_1 in $G[Y]$ may be extended to an embedding of T in G.

It therefore suffices to embed T_1 in $G[Y]$. If $|T_1| < y/2$ then we may do this by Theorem 1.3. If instead $|T_1| \geq y/2 \geq \eta n/2$ and we also have (i), then $|T_1| \leq y - \alpha n$. Since $|Y| \geq (2-\gamma)y \geq 2|T_1| + \alpha n$ we may embed T_1 in $G[Y]$ by Theorem 1.2(i). Finally, if $|T_1| \geq \eta n/2$ and we also have (ii), then $|T_1| \leq |F^+| \leq y + 2\alpha n$ and $|(T_1)_\Delta| \geq 2$. Since $\gamma \leq 9\alpha/\eta$, $G[Y]$ is a $9\alpha/\eta$-almost-regular tournament on at least $(2-\gamma)y \geq (2 - 9\alpha/\eta)|T_1|$ vertices, and so we may embed T_1 in $G[Y]$ by Lemma 5.8. So in any case we may embed T_1 in $G[Y]$, completing the proof. □

Observe that as with Lemma 2.7 a 'dual' form of Lemma 6.3 can be proved similarly. For this we instead require that $|F^+| \leq y - \alpha n$ and $|F^-| \leq z + 2\alpha n$, and also either that no component of F^- has order greater than $z - \alpha n$ or that the largest component T_1 of F^- has $|(T_1)_\Delta| \geq 2$. If these conditions are met then we may conclude that G contains a copy of T. As with Lemma 2.7, we shall sometimes implicitly refer to this 'dual' when referring to Lemma 6.3.

In the next lemma we show that Lemma 6.1 holds for any directed tree T whose core tree T_Δ is not a directed path in which most of the outweight and inweight of T_Δ lies at the endvertices of T_Δ. We say that a vertex t of a directed tree T is an *outleaf* if t has one inneighbour and no outneighbours, or an *inleaf* if t has one outneighbour and no inneighbours.

Lemma 6.4. *Suppose that $1/n \ll \beta, 1/\Delta' \ll 1/\Delta \ll \sigma \ll 1$. Let T be a directed tree on n vertices with $|T_{\Delta'}| \leq \beta n$, and let y and z be the outweight and inweight of $T_{\Delta'}$ respectively. Let G be a tournament on $2n-2$ vertices. Then either G contains a copy of T, or T_Δ is a directed path whose outleaf has outweight at least $y - \sigma n$ and whose inleaf has inweight at least $z - \sigma n$.*

Proof. Introduce new constants α and η with

$$1/n \ll \beta, 1/\Delta' \ll \alpha \ll 1/\Delta \ll \sigma \ll \eta \ll 1.$$

Then by Lemma 6.2 we may assume that $y, z \geq \eta n$. Also, if $|T_\Delta| = 1$ then G contains a copy of T by Lemma 3.1, so we may assume that $|T_\Delta| \geq 2$.

Suppose that some vertex $t \in T$ has the property that $w^-(t) \leq z - \alpha n - 1$, and also that every outcomponent of t contains at most $w^+(t) - 3\alpha n = |V^+| - 3\alpha n$ vertices. Then let the set V^- consist of t and every vertex in an incomponent of t, and let $V^+ := V(T) \setminus V^-$. Then $|V^-| \leq w^-(t) + 1 \leq z - \alpha n$, and every edge of T between V^- and V^+ is directed from V^- to V^+. Also, each component of $T[V^+]$ contains at most $w^+(t) - 3\alpha n$ vertices. Now, select a source vertex from the largest component of $T[V^+]$, delete this vertex from V^+, and add it to V^-. Repeat this step until we have $|V^+| \leq y + 2\alpha n$ and $|V^-| \leq z - \alpha n$. For these final V^+ and V^-, let $F^+ := T[V^+]$ and let $F^- := T[V^-]$. Then F^- and F^+ are forests which partition the vertices of T, with $|F^+| \leq y + 2\alpha n$ and $|F^-| \leq z - \alpha n$. Also, every edge of T between F^- and F^+ is directed from F^- to F^+. Finally, since we always deleted a vertex from the largest component of $T[V^+]$, no component of F^+ contains more than $|F^+| - 3\alpha n \leq y - \alpha n$ vertices. So by Lemma 6.3(i) G contains a copy of T. So we may assume that

there is no vertex $t \in T$ such that $w^-(t) \leq z - \alpha n - 1$ and every outcomponent of t contains at most $w^+(t) - 3\alpha n$ vertices. In particular, this implies that for every inleaf t of T_Δ, at least $n/2\Delta$ vertices of T lie in incomponents of t. (†)

Indeed, if T_Δ contains some inleaf t such that fewer than $n/2\Delta \leq z - \alpha n - 1$ vertices of T lie in incomponents of t, then by the definition of T_Δ at least $n/2\Delta - 1$ vertices of T lie in outcomponents of t other than the outcomponent containing the remaining vertices of T_Δ. Moreover, the definition of T_Δ also implies that at least n/Δ vertices of T lie in the one component of $T - t$ containing $T_\Delta - t$. Altogether this shows that every outcomponent of t contains at most $w^+(t) - n/2\Delta + 1 \leq w^+(t) - 3\alpha n$ vertices, a contradiction. By the same argument with the roles of incomponents and outcomponents switched, we may assume that

there is no vertex $t \in T$ such that $w^+(t) \leq y - \alpha n - 1$ and every incomponent of t contains at most $w^-(t) - 3\alpha n$ vertices. It follows from this that for every outleaf t of T_Δ, at least $n/2\Delta$ vertices of T lie in outcomponents of t. (††)

Claim. *If T_Δ has at least two inleaves or at least two outleaves, then G contains a copy of T.*

To prove the claim, suppose that T_Δ has two outleaves t and t' (the proof for inleaves is similar). Then we shall form a set V^+ of size between $n-z+\alpha n$ and $y+2\alpha n$ such that any edge of T between V^+ and $V^- := V(G)\setminus V^+$ is directed from V^- to V^+. We may do this by repeatedly selecting a sink vertex of T, adding it to V^+ and removing it from T. Now, by (††) at least $n/2\Delta$ vertices lie in outcomponents of t, and at least $n/2\Delta$ vertices lie in outcomponents of t'. Furthermore, if T' is an outcomponent of t, then any sink vertex in T' is a sink vertex in T, and the same is true if T' is instead an outcomponent of t'. So we may form V^+ and V^- as described above so that additionally V^+ contains at least $n/2\Delta$ vertices from outcomponents of t and at least $n/2\Delta$ vertices from outcomponents of t'. Fix such a choice of V^+ and V^-, and let $F^+ := T[V^+]$ and $F^- := T[V^-]$ be the induced forests. Then $|F^+| \leq y+2\alpha n$ and $|F^-| = n-|F^+| \leq z-\alpha n$, and every edge of T between F^- and F^+ is directed from F^- to F^+. So if every component of F^+ contains at most $y-\alpha n$ vertices, then G contains a copy of T by Lemma 6.3(i). We may therefore assume that the largest component T^+ of F^+ contains more than $y-\alpha n \geq |F^+|-n/4\Delta$ vertices. Since F^+ includes at least $n/2\Delta$ vertices from outcomponents of t and at least $n/2\Delta$ vertices from outcomponents of t', it follows that T^+ contains at least $n/4\Delta$ vertices from outcomponents of t and at least $n/4\Delta$ vertices from outcomponents of t'. As a consequence T^+ must contain t and t'. Furthermore, we must have $t, t' \in (T^+)_{4\Delta}$, and so $|(T^+)_{4\Delta}| \geq 2$. So G contains a copy of T by Lemma 6.3(ii), which proves the claim.

We may therefore assume that T_Δ has at most one outleaf and at most one inleaf. So T_Δ is a path with one inleaf and one outleaf. Let $t_1, \dots, t_x$ be the vertices of this path, labelled so that t_1 is the inleaf of T_Δ (so $t_1 \to t_2$), t_x is the outleaf of T_Δ (so $t_{x-1} \to t_x$), and for each $i \in [x-1]$ there is an edge of T_Δ between t_i and t_{i+1}.

Now suppose that the inweight of T_Δ is less than $z-2\alpha n$. Let the set V^- consist of all vertices of T which lie in T_Δ or in incomponents of T_Δ. Then $|V^-| \leq z-2\alpha n+|T_\Delta| \leq z-\alpha n$ (since $|T_\Delta| \leq |T_{\Delta'}| \leq \beta n$). Also, every edge of T between V^- and $V^+ := V(T)\setminus V^-$ is directed from V^- to V^+. Choose a source vertex of $T[V^+]$, delete it from V^+, and add it to V^-, and repeat this step until we have $|V^-| \leq z-\alpha n$ and $|V^+| \leq y+2\alpha n$. For these final V^- and V^+, let $F^+ := T[V^+]$ and $F^- := T[V^-]$ be the induced forests. Then $|F^-| \leq z-\alpha n$, $|F^+| \leq y+2\alpha n$, and every edge of T between F^- and F^+ is directed from F^- to F^+. Also, every component of F^+ is contained within a component of $T-T_\Delta$, and so has order at most $n/\Delta \leq y-\alpha n$ by Proposition 2.1. So G contains a copy of T by Lemma 6.3(i). We may therefore assume that the inweight of T_Δ is at least $z-2\alpha n$, and by a similar argument we may also assume that the outweight of T_Δ is at least $y-2\alpha n$. It follows that the outweight of T_Δ is at most $n-(z-2\alpha n) \leq y+3\alpha n$ and that the inweight of T_Δ is at most $n-(y-2\alpha n) \leq z+3\alpha n$.

We now suppose that fewer than $y-\sigma n$ vertices of T lie in outcomponents of t_x. Let T_1 be the subtree of T formed by T_Δ and all of its outcomponents. Initially let the set $V^+ := V(T_1)$, so $|V^+| \leq y+4\alpha n$, and every edge of T between V^+ and $V^- := V(G)\setminus V^+$ is directed from V^- to V^+. Choose a sink vertex of $T[V^-]$, delete it from V^- and add it to V^+, and repeat this step until we have $|V^+| \leq y+4\alpha n$ and $|V^-| \leq z-2\alpha n$. Fix these final V^+ and V^- and let $F^- := T[V^-]$ and $F^+ := T[V^+]$ be the induced forests. So $|F^+| \leq y+4\alpha n$, $|F^-| \leq z-2\alpha n$, and every edge of T between F^- and F^+ is directed from

F^- to F^+. Also $T_1 \subseteq F^+$, so T_1 is contained within a single component T^+ of F^+. Since at least $y - 2\alpha n$ vertices of T lie in outcomponents of T_Δ, at least $\sigma n/2$ vertices of T lie in outcomponents of T_Δ other than the outcomponents of t_x. Moreover, since t_x is an outleaf of T_Δ, by (††) at least $n/2\Delta$ vertices lie in outcomponents of t_x. So $t_{x-1} \in (T^+)_{2\Delta}$ and $t_x \in (T^+)_{2\Delta}$. So $|(T^+)_{2\Delta}| \geq 2$. But since the outweight of T_Δ is at least $y - 2\alpha n$ we have $|T^+| \geq |T_1| \geq y - 2\alpha n$, and so T^+ must be the largest component of F^+. So G contains a copy of T by Lemma 6.3(ii).

So we may assume that at least $y - \sigma n$ vertices of T lie in outcomponents of t_x, as desired. If fewer than $z - \sigma n$ vertices of T lie in incomponents of t_1, then we may similarly embed T in G, so we may also assume that at least $z - \sigma n$ vertices of T lie in incomponents of t_1. So at most $3\sigma n$ vertices of T do not lie in incomponents of t_1 or outcomponents of t_x. It remains only to show that T_Δ is a directed path. So suppose for a contradiction that T_Δ is not a directed path. Then there is some $i \in [x-1]$ such that $t_i \leftarrow t_{i+1}$. Choose the minimal such i (note $i > 1$ as t_1 is an inleaf of T_Δ). Then t_i has two inneighbours and no outneighbours in T_Δ. So at least two incomponents of t_i contain at least n/Δ vertices, and so no incomponent of t_i contains more than $w^-(t_i) - n/\Delta \leq w^-(t_i) - 3\alpha n$ vertices. Also, at most $3\sigma n \leq y - \alpha n - 1$ vertices of T lie in outcomponents of t_i, contradicting (††). □

We can now prove that Sumner's universal tournament conjecture holds for any large directed tree T whose core tree T_Δ contains precisely two vertices.

Lemma 6.5. *Suppose that $1/n \ll 1/\Delta' \ll 1$. Let T be a directed tree on n vertices with $|T_{\Delta'}| = 2$, and let G be a tournament on $2n-2$ vertices. Then G contains a copy of T.*

Proof. Introduce new constants $\Delta, \varepsilon, \gamma$ and η with

$$1/n \ll \beta, 1/\Delta' \ll 1/\Delta \ll \varepsilon \ll \gamma \ll \eta \ll 1.$$

Then $|T_{\Delta'}| = 2 \leq \beta n$. Also, since $\Delta \leq \Delta'$ we have $T_\Delta \subseteq T_{\Delta'}$. If $|T_\Delta| = 1$, then by Lemma 3.1 G contains a copy of T. So we may assume that $T_\Delta = T_{\Delta'}$. Let t_2 and t_1 be the vertices of T_Δ, labelled so that $t_2 \to t_1$. Let y be the outweight of T_Δ, and let z be the inweight of T_Δ, so $y + z = n - 2$. Then by Lemma 6.4 (with ε in the place of σ), we may assume that t_2 has inweight at least $z - \varepsilon n$, and also that t_1 has outweight at least $y - \varepsilon n$. Let T_1 be the subtree of T consisting of all vertices which lie in T_Δ or in outcomponents of T_Δ, and let T_2 be the subtree of T consisting of all vertices which lie in T_Δ or in incomponents of T_Δ. So $|T_1| = y + 2$ and $|T_2| = z + 2$. By Lemma 6.2(i) we may assume that $y, z \geq \eta n$.

As in the proof of Lemma 6.2, we partition the vertices of G into sets A, B, C, D and E, where:

$$\begin{aligned}
A &:= \{v \in G : d^+(v) \leq y + \varepsilon n\},\\
B &:= \{v \in G : y + \varepsilon n < d^+(v) < n - \varepsilon n\},\\
C &:= \{v \in G : d^+(v), d^-(v) \geq n - \varepsilon n\},\\
D &:= \{v \in G : z + \varepsilon n < d^-(v) < n - \varepsilon n\},\\
E &:= \{v \in G : d^-(v) \leq z + \varepsilon n\}.
\end{aligned}$$

Since $y, z \geq \eta n$ and $\varepsilon \ll \eta$ this is indeed a partition. Suppose first that $|B| \geq 2$. Then we may embed T_Δ in $G[B]$. Let $S_\Delta \subseteq B$ be the set of vertices occupied by T_Δ. Then every vertex of S_Δ has at least $y + \varepsilon n - 1 \geq y + 2n/\Delta$ outneighbours outside S_Δ and at least $|G| - 2 - (n - \varepsilon n) \geq y + z + 2n/\Delta$ inneighbours outside S_Δ. So by Lemma 2.6(c) we may extend the embedding of T_Δ in S_Δ to an embedding of T_1 in G. This embedding of T_1 occupies at most y vertices of G outside S_Δ, and so we may apply Lemma 2.6(c) again to extend the embedding of T_Δ in S_Δ to an embedding of T_2 in G so that the embeddings of T_1 and T_2 do not overlap outside T_Δ. Then together the embeddings of T_1 and T_2 form an embedding of T in G. So we may assume that $|B| \leq 1$. If $|D| \geq 2$ we may embed T in G in the same way by embedding T_Δ in D and then extending this embedding to embeddings of first T_2 and then T_1 in G which do not overlap outside T_Δ. So we may also assume that $|D| \leq 1$.

Now suppose that $|C| \geq 3$. Then we may choose vertices $v_2, v_1 \in C$ with $v_2 \to v_1$ and $|N^+(v_1) \cap N^+(v_2)| \geq \eta n \geq \eta n/2 + 2n/\Delta$. Embed t_1 to v_1 and t_2 to v_2. Then since $|N^+(v_1)|, |N^+(v_2)| \geq n - \varepsilon n \geq y + 2n/\Delta$, by Lemma 2.6(b) and (c) we may extend the embedding of T_Δ in $\{v_1, v_2\}$ to an embedding of T_1 in G so that at least $\eta n/2$ vertices of T_1 are embedded in $N^+(v_1) \cap N^+(v_2)$. Then at most $y + 2 - \eta n/2$ vertices of $N^-(v_1) \cup N^-(v_2)$ are occupied by this embedding, and so in each of $N^-(v_1)$ and $N^-(v_2)$ at least $n - \varepsilon n - (y + 2 - \eta n/2) \geq z + 2n/\Delta$ vertices remain unoccupied. So by Lemma 2.6(a) and (c) we may extend the embedding of T_Δ in $\{v_1, v_2\}$ to an embedding of T_2 in G which does not overlap with the embedding of T_1 outside T_Δ. Then together these embeddings form an embedding of T in G. So we may assume that $|C| \leq 2$, and hence that $|A \cup E| \geq 2n - 6$.

Claim. *Either some vertex of A has at least y outneighbours in $A \cup B \cup D$ or some vertex of E has at least z inneighbours in $B \cup D \cup E$.*

Indeed, suppose for a contradiction that both of these statements are false. Then certainly every vertex of A has fewer than y outneighbours in A and every vertex of E has fewer than z inneighbours in E. So $|A| \leq 2y - 1$ and $|E| \leq 2z - 1$. Since $y + z = n - 2$ and $|A \cup E| \geq 2n - 6$, we must have $|A| = 2y - 1$ and $|E| = 2z - 1$, and also $|B| = 1, |D| = 1$ and $|C| = 2$. Then every vertex of A must have $y - 1$ outneighbours in A, and so no vertex of A can have an outneighbour in B or in D. Likewise, every vertex of E must have $z - 1$ inneighbours in E, and so no vertex of E can have an inneighbour in B or in D. But then if we let b be the vertex in B and d be the vertex in D we have $d^+(b) = d^+(d) \pm 3$, contradicting the definition of B and D. So either some vertex of A has at least y outneigbours in $A \cup B \cup D$ or some vertex of E has at least z inneighbours in $B \cup D \cup E$. This completes the proof of the claim.

If some $v \in A$ has at least y outneighbours in $A \cup B \cup D$, then we shall embed T_1 in $G[A]$ so that we may then embed the incomponents of t_2 and t_1 in the unoccupied vertices of E and A respectively. For this, note that $|E| \leq 2(z + \varepsilon n) + 1$, so $|A| \geq 2n - 2z - 2\varepsilon n - 7 \geq 2y - 3\varepsilon n$ (and similarly we have $|E| \geq 2z - 3\varepsilon n$). Since every $a \in A$ has at most $y + \varepsilon n$ outneighbours in A, by Proposition 2.4 $G[A]$ contains a γ-almost-regular subtournament on at least $(2 - \gamma)y$ vertices. Let Y be the vertex set of this subtournament. Now,

$$|(A \cup B \cup D) \setminus Y| \leq 2 + (2y + 2\varepsilon n + 1) - (2 - \gamma)y \leq 2\gamma y,$$

so v must have at least $(1 - 2\gamma)y$ outneighbours in Y. Also, since $v \in A$ we have

$$(1 - 2\gamma)y \leq |N^+(v) \cap Y| \leq y + \varepsilon n \leq (1 + 2\gamma)y.$$

So at most $10\gamma y$ vertices of $N^+(v) \cap Y$ have more than $(1 - 3\gamma)y$ outneighbours in $N^+(v) \cap Y$, and at most $10\gamma y$ vertices of $N^+(v) \cap Y$ have more than $(1 - 3\gamma)y$ inneighbours in $N^+(v) \cap Y$. Since every vertex of Y has at least $(1 - 2\gamma)y$ inneighbours in Y and at least $(1 - 2\gamma)y$ outneighbours in Y, this means that at least $|N^+(v) \cap Y| - 20\gamma y \geq 3n/\Delta$ vertices of $N^+(v) \cap Y$

have at least $\gamma y \geq 6n/\Delta$ outneighbours in $Y \setminus N^+(v)$ and at least $6n/\Delta$ inneighbours in $Y \setminus N^+(v)$. Let T^+ be the tree formed by t_1 and its outcomponents, so $|T^+| \leq y+1$. Then every component of $T^+ - t_1$ is a component of $T - T_\Delta$ and so has order at most n/Δ by Proposition 2.1. So by Lemma 2.5 (applied with $N := N^+(v) \cap (A \cup B \cup D)$ and $X := Y \setminus N^+(v)$), we may embed T^+ in $G[A \cup B \cup D]$ so that t_1 is embedded to v and at most $4n/\Delta$ vertices are embedded outside $N^+(v)$.

Since $v \in A$ we have $d^+(v) \leq y + \varepsilon n$, and so v has at least

$$|Y| - 1 - (y + \varepsilon n) - 4n/\Delta \geq 7\varepsilon n \tag{11}$$

inneighbours in Y which are not occupied by the embedding of T^+. Let T^* be the tree formed by all vertices of T which do not lie in outcomponents of t_1 or incomponents of t_2. Then every edge incident to t_1 in T^* is directed towards t_1. Also, $|T^*| \leq n - (y - \varepsilon n) - (z - \varepsilon n) = 2\varepsilon n + 2$, so certainly every component of $T^* - t_1$ has order at most $2\varepsilon n + 1$. Together with (11) and Theorem 1.3 this shows that we may extend the embedding of t_1 in $\{v\}$ to an embedding of T^* in $\{v\} \cup (N^-(v) \cap Y)$ so that the embeddings of T^+ and T^* only overlap in the vertex t_1. Then in particular t_2 is embedded to some vertex $v_2 \in Y$.

To complete the embedding, observe that every vertex of Y has at least $(1-2\gamma)y$ outneighbours in Y, and therefore at most $3\gamma y$ outneighbours outside Y. So v_2 has at least $|E| - 3\gamma y \geq z + 2n/\Delta$ inneighbours in E, none of which have been occupied by the embeddings of T^+ and T^*. Let T^- be the subtree of T consisting of t_2 and all of its incomponents. Then $|T^-| \leq z+1$, and each component of $T^- - t_2$ is a component of $T - T_\Delta$ and so has order at most n/Δ by Proposition 2.1. So by Lemma 2.6(c) we may extend the embedding of t_2 in $\{v_2\}$ to an embedding of T^- in $\{v_2\} \cup E$. These embeddings together form an embedding of T in G.

If instead some $v \in E$ has at least z inneighbours in $B \cup D \cup E$ then we may similarly embed T in G by choosing Z to be the vertex set of a γ-almost-regular subtournament of $G[E]$ on at least $(2-\gamma)z$ vertices and embedding T^- in $G[B \cup D \cup E]$, then embedding $T^* - t_2$ in the unoccupied vertices of Z, before finally embedding $T^+ - t_1$ in $G[A]$. □

We can now give the proof of Lemma 6.1. It was necessary to prove Lemma 6.5 separately from this as the method of proof does not hold for $|T_\Delta| = 2$ (we cannot obtain the partition of $V(G)$ into Y^* and Z^* in this case).

Proof of Lemma 6.1. Introduce new constants γ, α, Δ and η with

$$1/n \ll \beta, 1/\Delta' \ll 1/\Delta \ll \gamma \ll \alpha \ll \eta \ll 1.$$

Let y' be the outweight of $T_{\Delta'}$ and let z' be the inweight of $T_{\Delta'}$. Then by Lemma 6.2 we may assume that $y', z' \geq \eta n$. Similarly let y and z be the outweight and inweight of T_Δ respectively. If $|T_\Delta| = 1$, then G contains a copy of T by Lemma 3.1. If instead $|T_\Delta| = 2$ then G contains a copy of T by Lemma 6.5. So we may assume that $\ell := |T_\Delta| \geq 3$, and by Lemma 6.4 we may assume that T_Δ is a directed path. Let $t_1, \dots, t_\ell$ be the vertices of T_Δ, labelled so that $t_i \to t_{i+1}$ for each $i \in [\ell - 1]$. Then by Lemma 6.4 we may also assume that the inweight of t_1 is at least $z' - \gamma n$ and that the outweight of t_ℓ is at least $y' - \gamma n$. This implies that $z \geq z' - \gamma n$ and $y \geq y' - \gamma n$. Since $y' + z' + |T_{\Delta'}| = y + z + |T_\Delta| = n$ it follows that we must have

$$y = y' \pm 2\gamma n \text{ and } z = z' \pm 2\gamma n. \tag{12}$$

Finally, by Lemma 6.2 we may assume that there are disjoint sets $Y, Z \subseteq V(G)$ such that:

(a) $|Y| \geq (2-\gamma)y'$ and $|Z| \geq (2-\gamma)z'$,
(b) $G[Y]$ and $G[Z]$ are γ-almost-regular, and
(c) any vertex of Y has at most $3\gamma n$ outneighbours in Z and any vertex of Z has at most $3\gamma n$ inneighbours in Y.

Let $X := V(G)\setminus(Y\cup Z)$, so $|X| \leq 2\gamma n$. Let T^* be the subtree of T formed by deleting from T all vertices in outcomponents of t_ℓ or incomponents of t_1. So $|T^*| \leq n-(z'-\gamma n)-(y'-\gamma n) \leq 3\gamma n$. Let T^+ be the subtree of T formed by t_ℓ and its outcomponents, and let T^- be the subtree of T formed by t_1 and its incomponents. So $|T^+| \leq y+1$ and $|T^-| \leq z+1$. Also, each component of $T^+ - t_\ell$ and each component of $T^- - t_1$ is a component of $T - T_\Delta$ and so has order at most n/Δ by Proposition 2.1.

Suppose that some vertex $v \in X$ has at least αn inneighbours in Y and at least αn outneighbours in Z. Since $\ell \geq 3$, we may choose i with $1 < i < \ell$. Embed t_i to v. Let T_a be the subtree of T^* consisting of t_i and all of its outcomponents, and let T_b be the subtree of T^* consisting of t_i and all of its incomponents. Then $|T_a|, |T_b| \leq |T^*| \leq 3\gamma n$. So by Lemma 2.6 we may extend the embedding of t_i in $\{v\}$ to an embedding of T_a in $Z\cup\{v\}$, and similarly we may extend the embedding of t_i in $\{v\}$ to an embedding of T_b in $Y\cup\{v\}$. Then in particular t_1 is embedded to some $v_1 \in Y$ and t_ℓ is embedded to some $v_\ell \in Z$. So v_1 has at least $|Z| - 3\gamma n \geq z + 3\gamma n + 2n/\Delta$ inneighbours in Z, at most $3\gamma n$ of which are occupied by the embedding of T_a. Similarly v_ℓ has at least $|Y|-3\gamma n \geq y+3\gamma n+2n/\Delta$ outneighbours in Y, at most $3\gamma n$ of which are occupied by the embedding of T_b. So by Lemma 2.6 we may extend the embedding of t_1 in $\{v_1\}$ to an embedding of T^- in $\{v_1\}\cup Z$ and also extend the embedding of t_ℓ in $\{v_\ell\}$ to an embedding of T^+ in $\{v_\ell\}\cup Y$ so that these embeddings together form a copy of T in G.

So we may assume that no vertex of X has at least αn inneighbours in Y and at least αn outneighbours in Z. Let $X^+ \subseteq X$ consist of all vertices of X with fewer than αn inneighbours in Y, and let $X^- \subseteq X \setminus X^+$ consist of all vertices of $X \setminus X^+$ with fewer than αn outneighbours in Z. Let $Y^* := Y \cup X^-$ and let $Z^* := Z \cup X^+$, so Y^* and Z^* partition the vertices of G. Then any vertex of Y^* has at most αn outneighbours in Z, and thus at least $z+\alpha n$ inneighbours in Z^* (by (a), (12) and the fact that $z' \geq \eta n$). Similarly any vertex of Z^* has at most αn inneighbours in Y, and therefore at least $y+\alpha n$ outneighbours in Y^*. Let $W \subseteq V(G)$ consist of all vertices in Y^* with at least $y+\alpha n$ outneighbours in Y^* and all vertices in Z^* with at least $z+\alpha n$ inneighbours in Z^*.

Now suppose that $|W| \geq |T_\Delta|$. Since T_Δ is a directed path, by Theorem 1.5 we may embed T_Δ in $G[W]$. Let $S_\Delta \subseteq W$ be the set of vertices occupied by this embedding. Then $|S_\Delta| = |T_\Delta| \leq |T_{\Delta'}| \leq \beta n$. So every vertex of S_Δ has at least $y + \alpha n/2 \geq y + 2n/\Delta$ outneighbours in $Y^* \setminus S_\Delta$ and at least $z+\alpha n/2 \geq z+2n/\Delta$ inneighbours in $Z^* \setminus S_\Delta$. Let T_1 be the subtree of T consisting of T_Δ and all of its outcomponents, and let T_2 be the subtree of T consisting of T_Δ and all of its incomponents. So $|T_1| = \ell + y$ and $|T_2| = \ell + z$. Also, each component of $T_1 - T_\Delta$ and each component of $T_2 - T_\Delta$ is a component of $T - T_\Delta$, and so has order at most n/Δ by Proposition 2.1. So by Lemma 2.6 we may extend the embedding of T_Δ in S_Δ to an embedding of T_1 in $Y^* \cup S_\Delta$. Similarly by Lemma 2.6 we may extend the embedding of T_Δ in S_Δ to an embedding of T_2 in $Z^* \cup S_\Delta$. These embeddings of T_1 and T_2 do not overlap outside T_Δ, and so together form an embedding of T in G.

We may therefore assume that $|W| < |T_\Delta|$, and hence that $|G - W| \geq 2n - 1 - \ell$. Since $y+z = n-\ell$, we must have either $|Y^*\setminus W| \geq 2y$ or $|Z^*\setminus W| \geq 2z$. Suppose that $|Y^*\setminus W| \geq 2y$. Then $Y^* \setminus W$ contains a vertex v_ℓ with at least y outneighbours in Y^*. So we may choose a set $N \subseteq N^+(v_\ell) \cap Y^*$ with $|N| = y$. Then $|N \cap Y| \geq y - (|Y^*| - |Y|) \geq y - 2\gamma n$. Now, by (a), (b) and (12) every vertex of Y has at least $(1-2\sqrt{\gamma})y$ inneighbours in Y and at least $(1-2\sqrt{\gamma})y$ outneighbours in Y. Since $|N| = y$, at most $6\sqrt{\gamma}y$ vertices of $N \cap Y$ have more than $(1-3\sqrt{\gamma})y$ inneighbours in $N \cap Y$, and at most $6\sqrt{\gamma}y$ vertices of $N \cap Y$ have more than $(1-3\sqrt{\gamma})y$ outneighbours in $N \cap Y$. So at least $|N \cap Y| - 12\sqrt{\gamma}n \geq 3n/\Delta$ vertices of N have at least $6n/\Delta$ inneighbours in $Y^* \setminus (N \cup \{v_\ell\})$ and at least $6n/\Delta$ outneighbours in $Y^* \setminus (N \cup \{v_\ell\})$. This means that by Lemma 2.5 (applied with $Y^* \setminus (N \cup \{v_\ell\})$ playing the role of X) we may embed T^+ in Y^* with t_ℓ embedded to v_ℓ, and at most $4n/\Delta$ vertices of T^+ embedded outside N. Since $v_\ell \notin W$, v_ℓ has at most $y + \alpha n$ outneighbours in Y^*, and so v_ℓ has at least $|Y| - 1 - (y + \alpha n) - 4n/\Delta \geq 9\gamma n$ inneighbours in Y which are not occupied by the embedding of T^+. Since $|T^*| \leq 3\gamma n$, by Lemma 2.6 we may extend the embedding of t_ℓ in v_ℓ to an embedding of T^* in Y which only overlaps the embedding of T^+ in t_ℓ. The vertex t_1 of T will therefore be embedded to some vertex $v_1 \in Y$. By (3), v_1 then has at least $|Z| - 3\gamma n \geq z + 2n/\Delta$ inneighbours in Z, none of which will have been occupied by the embeddings of T^* and T^+ so far. So by Lemma 2.6 we may extend the embedding of t_1 in $\{v_1\}$ to an embedding of T^- in $Z \cup \{v_1\}$. Then the embeddings of T^+, T^- and T^* combine to form an embedding of T in G. If instead we have $|Z^* \setminus W| \geq 2z$, then we may embed T in G similarly, first embedding T^- in Z^*, then embedding T^* in the unoccupied vertices of Z, and finally embedding T^+ in Y. So in either case G contains a copy of T, completing the proof. □

7. Proof of Theorem 1.1

Having proved that Sumner's conjecture holds for directed trees of small core, we now show that the same is true for directed trees of large core, which will complete the proof of Theorem 1.1. We begin with an embedding result similar to Lemma 6.3.

Lemma 7.1. *Suppose that $1/n \ll 1/\Delta \ll \mu \ll \nu \ll \eta \ll \gamma \ll \alpha \ll \beta \ll 1$. Let T be a directed tree on n vertices, and let forests F^- and F^+ be induced subgraphs of T which partition the vertices of T such that $|F^+| \geq 6\alpha n$. Suppose also that every edge of T between F^- and F^+ is directed from F^- to F^+. Let Y and Z be disjoint sets with $|Y| \geq 2|F^+| - 2\alpha n$ and $|Z| \geq 2|F^-| + \alpha n$, and let G be a tournament on vertex set $Y \cup Z$ such that every vertex of Y has at most $\gamma|G|$ outneighbours in Z and every vertex of Z has at most $\gamma|G|$ inneighbours in Y. Finally, let T_1^+ be the largest component of F^+, and suppose that either*

(i) $|T_1^+| \leq |F^+| - 3\alpha n$,
(ii) *$G[Y]$ is a robust (μ, ν)-outexpander with $\delta^0(G[Y]) \geq \eta|Y|$ and $|(T_1^+)_\Delta| \geq \beta n$, or*
(iii) $\Delta(T_1^+) \leq \Delta$.

Then G contains a copy of T.

Proof. First observe that if $|G| \geq 3n$, then G contains a copy of T by Theorem 1.3. So we may assume that $|G| < 3n$, and hence that every vertex of Y has at most $3\gamma n$ outneighbours in Z and every vertex of Z has at most $3\gamma n$ inneighbours in Y. Let T_2^+ be the second largest component of F^+. Then $|F^+| - |T_2^+| \geq |F^+|/2 \geq 3\alpha n$, so $|Y| \geq |F^+| + |T_2^+| + \alpha n$. Since $|Z| \geq 2|F^-| + \alpha n$, by Lemma 2.7 any embedding of T_1^+ in $G[Y]$ may be extended to an embedding of T in G. So it is sufficient to embed T_1^+ in $G[Y]$.

Note that $|Y| \geq 10\alpha n$, so if $|T_1^+| < \alpha n$, then $G[Y]$ contains a copy of T_1^+ by Theorem 1.3. Alternatively, suppose that $|T_1^+| \geq \alpha n$. If (i) holds, then $|T_1^+| \leq |Y|/2 - 2\alpha n$, and so $|Y| \geq (2+\alpha)|T_1^+|$. So $G[Y]$ contains a copy of T_1^+ by Theorem 1.2(i). If instead (ii) holds then G contains a copy of T_1^+ by Lemma 5.3. Finally, if (iii) holds then G contains a copy of T_1^+ by Theorem 1.2(ii), completing the proof. □

Observe that as with Lemma 2.7 and Lemma 6.3, a 'dual' form of Lemma 7.1 can be proved similarly. For this we instead require that that $|F^-| \geq 6\alpha n$, $|Y| \geq 2|F^+| + \alpha n$ and $|Z| \geq 2|F^-| - 2\alpha n$, and also either that the largest component $(T_1^-)_\Delta$ of F^- contains at most $|F^-| - 3\alpha n$ vertices, or that $G[Z]$ is a robust (μ, ν)-outexpander with $\delta^0(G[Z]) \geq \eta|Z|$ and $|(T_1^-)_\Delta| \geq \beta n$, or that $\Delta(T_1^-) \leq \Delta$. If these conditions are met we may conclude that G contains a copy of T. As with Lemma 2.7, we shall sometimes implicitly refer to this 'dual' when referring to Lemma 7.1.

The next lemma is our final result we need to proof Theorem 1.1. It states that if we can find disjoint subsets $Y, Z \subseteq V(G)$ containing almost all of the vertices of G, so that $G[Y]$ and $G[Z]$ are robust outexpanders of large minimum semidegree with almost all edges between Y and Z directed the same way, then G contains a copy of T.

Lemma 7.2. *Suppose that $1/n \ll 1/\Delta \ll \mu \ll \nu \ll \eta \ll \gamma \ll \alpha \ll \beta \ll 1$. Let T be a directed tree on n vertices with $|T_\Delta| \geq \beta n$. Let Y and Z be disjoint sets with $|Y \cup Z| \geq (2-\alpha)n$, and let G be a tournament on vertex set $Y \cup Z$ such that*

(i) *$G[Y]$ is a robust (μ, ν)-outexpander with $\delta^0(G[Y]) \geq \eta|Y|$,*
(ii) *$G[Z]$ is a robust (μ, ν)-outexpander with $\delta^0(G[Z]) \geq \eta|Z|$, and*
(iii) *every vertex of Y has at most $\gamma|G|$ outneighbours in Z, and every vertex of Z has at most $\gamma|G|$ inneighbours in Y.*

Then G contains a copy of T.

Proof. If $|Y \cup Z| \geq (2+\alpha)n$, then G contains a copy of T by Theorem 1.2(i). So we may assume that $|Y \cup Z| = (2 \pm \alpha)n$. Suppose first that $|Z| < 64\alpha n$. Then $|Y| \geq (2 - 65\alpha)n$, and hence $G[Y]$ contains a copy of T by (i) and Lemma 5.3. Similarly if $|Y| < 64\alpha n$, then by (ii) and Lemma 5.3 $G[Z]$ contains a copy of T. So we may assume that $|Y| \geq 64\alpha n$ and $|Z| \geq 64\alpha n$.

So we may form a forest F_1^+ of order between $|Y|/2 + 4\alpha n$ and $|Y|/2 + 5\alpha n$ by repeatedly choosing a sink vertex of T, deleting it from T and adding it to F_1^+. Let $F_1^- := T - F_1^+$, so that

$$\frac{|Z|}{2} - 6\alpha n \leq n - \frac{|Y|}{2} - 5\alpha n \leq |F_1^-| \leq n - \frac{|Y|}{2} - 4\alpha n \leq \frac{|Z|}{2} - 3\alpha n. \tag{13}$$

We therefore have $|Y| \geq 2|F_1^+| - 10\alpha n$ and $|Z| \geq 2|F_1^-| + 6\alpha n$. Note also that $|F_1^+| \geq 36\alpha n$. Let T' be the largest component of F_1^+. If $|T'| \leq |F_1^+| - 18\alpha n$ or $|T'_\Delta| \geq \beta n/3$ then G contains a copy of T by (i), (iii) and Lemma 7.1. So we may assume that $|T'| > |F_1^+| - 18\alpha n$, and that $|T'_\Delta| < \beta n/3$.

Next we form a forest F_2^- which is a subgraph of T and which contains F_1^-. To do this, take F_2^- initially to be F_1^-. Then select a source vertex of F_1^+, delete it from F_1^+ and add it to F_2^-, and repeat this step until $|Z|/2 + 4\alpha n \leq |F_2^-| \leq |Z|/2 + 5\alpha n$, and let $F_2^+ := T - F_2^-$. Then by (13) we have $|F_1^+ \cap F_2^-| = |F_2^-| - |F_1^-| \leq 11\alpha n$. Also $|F_2^+| \leq |Y|/2 - 3\alpha n$, and so we have both $|Z| \geq 2|F_2^-| - 10\alpha n$ and $|Y| \geq 2|F_2^+| + 6\alpha n$. Observe also that $|F_2^-| \geq 36\alpha n$.

Let T'' be the largest component of F_2^-. Then if $|T''| \leq |F_2^-| - 18\alpha n$ then G contains a copy of T by (ii), (iii) and Lemma 7.1. So we may assume that $|T''| > |F_2^-| - 18\alpha n$. Clearly $|T' \cap T''| \leq |F_1^+ \cap F_2^-| \leq 11\alpha n$, and so $|T' \cup T''| \geq |T'| + |T''| - |T' \cap T''| > (1 - 47\alpha)n$. This implies that $|T''_\Delta| \geq \beta n/3$, as otherwise by Lemma 2.3 we would have $|T_\Delta| < \beta n$, a contradiction. Thus G contains a copy of T by (ii), (iii) and Lemma 7.1, as desired. □

Proof of Theorem 1.1. Introduce new constants with

$$1/n \ll 1/\Delta \ll \mu \ll \nu \ll \eta \ll \gamma \ll \alpha \ll \alpha' \ll \beta \ll 1.$$

If $|T_\Delta| < \beta n$ then G contains a copy of T by Lemma 6.1. So we may assume that $|T_\Delta| \geq \beta n$. Let $x := |T_\Delta|$, let y be the outweight of T_Δ, and let z be the inweight of T_Δ, so $x+y+z = n$. Also let T_1 be the subtree of T formed by T_Δ and all outcomponents of T_Δ, and let T_2 be the subtree of T formed by T_Δ and all incomponents of T_Δ, so $|T_1| = x+y$, and $|T_2| = x+z$.

By Lemma 5.2 we may choose disjoint subsets $S_1, \dots, S_r$ of $V(G)$ such that

- (i) $|\bigcup_{i\in[r]} S_i| \geq (1-\gamma)|G|$,
- (ii) for each $i \in [r]$, any vertex $v \in S_i$ has at most $\gamma|G|$ inneighbours in $\bigcup_{j>i} S_j$ and at most $\gamma|G|$ outneighbours in $\bigcup_{j<i} S_j$, and
- (iii) for each $i \in [r]$, either $G[S_i]$ is a robust (μ,ν)-outexpander with $\delta^0(G[S_i]) \geq \eta|G|$ or $|S_i| < \gamma|G|$.

Let i be maximal such that $|S_1 \cup \dots \cup S_{i-1}| < \max\{2(z-\alpha n), 4\alpha n\}$, and let j be minimal such that $|S_{j+1} \cup \dots \cup S_r| < \max\{2(y-\alpha n), 4\alpha n\}$. Since $y+z \leq n - \beta n$, by (i) we have $i \leq j$ (though equality is possible here). Let $Z := S_1 \cup \dots \cup S_i$, let $Y := S_j \cup \dots \cup S_r$ and let $X := S_{i+1} \cup \dots \cup S_{j-1}$. Then we have

$$|Z \setminus S_i| < \max\{2(z-\alpha n), 4\alpha n\} \text{ and } |Y \setminus S_j| < \max\{2(y-\alpha n), 4\alpha n\}. \tag{14}$$

Also, by the maximality of i and the minimality of j we have

$$|Z| \geq z + \alpha n \text{ and } |Y| \geq y + \alpha n. \tag{15}$$

Claim. *If $|Z \setminus S_i| \geq 11\alpha n$ or $|Y \setminus S_j| \geq 11\alpha n$ then G contains a copy of T.*

To prove the claim, suppose first that $|Y \setminus S_j| \geq 11\alpha n$. Let $X^- := Z \cup X \cup S_j$ and $X^+ := Y \setminus S_j$. By (14) we have $|X^+| < 2y - 2\alpha n$. Also, by (ii) every vertex in X^- has at most $\gamma|G|$ inneighbours in X^+ and every vertex in X^+ has at most $\gamma|G|$ outneighbours in X^-. Now, $T_1 - T_\Delta$ is a forest on $y > |X^+|/2 + \alpha n$ vertices in which each component has order at most n/Δ by Proposition 2.1(iv). So by repeatedly deleting a source vertex of $T_1 - T_\Delta$, we may obtain a subforest F^+ on between $|X^+|/2 + 2\alpha n/3$ and $|X^+|/2 + \alpha n$ vertices. So $|F^+| \geq 6\alpha n$, and each component of F^+ has order at most $n/\Delta \leq |F^+| - 3\alpha n$. Let $F^+ := T - F^-$, so every edge of T between F^- and F^+ is directed from F^- to F^+. Since $|X^+| + |X^-| \geq (1-\gamma)|G|$ by (i), we have

$$|F^-| = n - |F^+| \leq n - \frac{|X^+|}{2} - \frac{2\alpha n}{3} \leq \frac{|X^-|}{2} - \frac{\alpha n}{2}.$$

So $|X^-| \geq 2|F^-| + \alpha n$, and $|X^+| \geq 2|F^+| - 2\alpha n$, and so G contains a copy of T by Lemma 7.1(i). If instead $|Z \setminus S_i| \geq 11\alpha n$ then G contains a copy of T similarly. This proves the claim.

We may therefore assume that $|Z \setminus S_i| < 11\alpha n$ and $|Y \setminus S_j| < 11\alpha n$. Suppose first that $i = j$. Then $|S_i| \geq (1-\gamma)|G| - 22\alpha n \geq (2-\alpha')n$, so by (iii) $G[S_i]$ is a robust (μ,ν)-outexpander with $\delta^0(G[S_i]) \geq \eta|G| \geq \eta|S_i|$. Thus G contains a copy of T by Lemma 5.3. Now suppose instead that $i \neq j$, and also that $|X| < 12\alpha' n$. Then $|S_i \cup S_j| \geq (1-\gamma)|G| - |X| - 22\alpha n \geq (2-13\alpha')n$. Now if $|S_i| < \gamma|G|$, then we must have $|S_j| \geq (2-14\alpha')n$. Then by (iii) $G[S_j]$ must be a robust (μ,ν)-outexpander with $\delta^0(G[S_j]) \geq \eta|G| \geq \eta|S_j|$, so $G[S_j]$ contains a copy of T by Lemma 5.3. Alternatively, if $|S_j| < \gamma|G|$ then $G[S_i]$ contains a copy of T similarly. Finally, if $|S_i|, |S_j| \geq \gamma|G|$, then by (iii) $G[S_i]$ and $G[S_j]$ must both be robust (μ,ν)-outexpanders with $\delta^0(G[S_i]) \geq \eta|G| \geq \eta|S_i|$ and $\delta^0(G[S_j]) \geq \eta|S_j|$. Also, by (ii) every vertex of S_i has at most $\gamma|G|$ inneighbours in S_j, and every vertex of S_j has at most $\gamma|G|$ outneighbours in S_i. So $G[S_i \cup S_j]$ contains a copy of T by Lemma 7.2.

So we may assume that $i \neq j$, and also that $|X| \geq 12\alpha' n$. We next consider two cases for the size of X, in each case showing that T may be embedded in G.

Case 1: $|X| \geq (1+\alpha)x$**.**

Since by Proposition 2.1(iii) we have $\Delta(T_\Delta) \leq \Delta$, by Theorem 1.2(ii) we may embed T_Δ in $G[X]$. Let $X' \subseteq X$ consist of the vertices occupied by this embedding. Now, by (ii) every vertex of X' has at most $\gamma|G|$ inneighbours in Y, and hence by (15) at least $y + \alpha n/2$ outneighbours in Y. Since by Proposition 2.1(iv) every component of $T_1 - T_\Delta$ has order at most n/Δ, by Lemma 2.6 we may extend the embedding of T_Δ in $G[X']$ to an embedding of T_1 in $G[X' \cup Y]$. Similarly by (ii) every vertex of X' has at most $\gamma|G|$ outneighbours in Z, and hence by (15) at least $z + \alpha n/2$ inneighbours in Z. Since by Proposition 2.1(iv) every component of $T_2 - T_\Delta$ has order at most n/Δ, by Lemma 2.6 we may extend the embedding of T_Δ in $G[X']$ to an embedding of T_2 in $G[X' \cup Z]$. Since these embeddings of T_1 and T_2 only overlap in T_Δ, they together form an embedding of T in G.

Case 2: $|X| < (1+\alpha)x$**.**

Observe that if $|Z| \leq 2z + \alpha n$ and $|Y| \leq 2y + \alpha n$, then by (i) and the fact that $x = |T_\Delta| \geq \beta n$ we have

$$|X| \geq (1-\gamma)|G| - |Z| - |Y| \geq 2n - 2z - 2y - 3\alpha n \geq 2x - 3\alpha n \geq (1+\alpha)x,$$

contradicting our assumption on X. So at least one of $|Z| > 2z + \alpha n$ and $|Y| > 2y + \alpha n$ must hold. This gives us three further cases, which we consider separately.

Case 2(a): $|Z| > 2z + \alpha n$**,** $|Y| \leq 2y + \alpha n$**.**

In this case it is sufficient to embed T_2 in $G[X \cup Z]$. Indeed, by (ii) every vertex of $X \cup Z$ has at most $\gamma|G|$ inneighbours in Y, and therefore by (15) at least $y + \alpha n/2$ outneighbours in Y. Since by Proposition 2.1(iv) every component of $T - T_2$ has order at most n/Δ, any embedding of T_2 in $G[X \cup Z]$ can be extended to an embedding of T in G by Lemma 2.6.

Now, if $|X \cup Z| \geq 2|T_2| + 2\alpha n$, then we may embed T_2 in $G[X \cup Z]$ by Theorem 1.2(i). So we may assume that $|X \cup Z| < 2|T_2| + 2\alpha n$. Also, by (i) we have

$$|X \cup Z| \geq (1-\gamma)|G| - |Y| \geq 2n - 2y - 2\alpha n = 2x + 2z - 2\alpha n = 2|T_2| - 2\alpha n.$$

So $|X \cup Z| = 2|T_2| \pm 2\alpha n$. In particular, since $|T_2| \geq |T_\Delta| \geq \beta n$, we have $|X \cup Z| \geq \beta n$. By repeatedly deleting a source vertex of T_Δ, we may form a forest F which is an induced subgraph of T_Δ (consisting of the undeleted vertices of T_Δ) so that every edge between $T_\Delta - F$ and F is directed from T_Δ to F, and also so that

$$\frac{|X|}{2} + \frac{2\alpha'|T_2|}{3} \leq |F| \leq \frac{|X|}{2} + \alpha'|T_2|.$$

Let $F^- := T_2 - F$. Then

$$|F^-| = |T_2| - |F| \leq |T_2| - \frac{|X|}{2} - \frac{2\alpha'|T_2|}{3} \leq \frac{|Z|}{2} - \frac{\alpha'|T_2|}{2}.$$

So $|X| \geq 2|F| - 2\alpha'|T_2|$ and $|Z| \geq 2|F^-| + \alpha'|T_2|$. Also, $|F| \geq |X|/2 \geq 6\alpha'|T_2|$, and since F is a subtree of T_Δ, by Proposition 2.1(iii) each component C of F has $\Delta(C) \leq \Delta$. Since by (ii) every vertex of X has at most $\gamma|G| \leq 2\gamma|X \cup Z|/\beta$ outneighbours in Z and every vertex of Z has at most $\gamma|G| \leq 2\gamma|X \cup Z|/\beta$ inneighbours in X, $G[X \cup Z]$ contains a copy of T_2 by Lemma 7.1, as required.

Case 2(b): $|Z| \leq 2z + \alpha n$, $|Y| > 2y + \alpha n$.

In this case T may be embedded in G by the same method as in the previous case, with the roles of inneighbours and outneighbours switched. So we begin by embedding T_1 in $G[X \cup Y]$, and then use Lemma 2.6 to extend this embedding to an embedding of T in G.

Case 2(c): $|Z| > 2z + \alpha n$, $|Y| > 2y + \alpha n$.

In this case, we shall partition T into three forests as follows. Initially take F^- to be the forest formed by all incomponents of T_Δ, and F^+ to be the forest formed by all outcomponents of T_Δ. Then select a source vertex of T_Δ, delete it from T_Δ and add it to F^-. Repeat this step until $2|F^-| + \alpha n \leq |Z| \leq 2|F^-| + 2\alpha n$. Next, select a sink vertex of T_Δ, delete it from T_Δ and add it to F^+. Repeat this step until $2|F^+| + \alpha n \leq |Y| \leq 2|F^+| + 2\alpha n$. Then let F consist of all vertices remaining in T_Δ. So F is a subgraph of T_Δ. Also, by (i)

$$|F| = n - |F^-| - |F^+| \leq n - |Y|/2 - |Z|/2 + 2\alpha n \leq |X|/2 + 3\alpha n,$$

so (since $|X| \geq \alpha' n$) $|X| \geq |F| + \alpha n$. We shall embed the components of F^-, F and F^+ in turn amongst the vertices of Z, X and Y respectively. Indeed, the proof is similar to the proof of Lemma 2.7, but with three forests instead of two.

Let $C_1, \dots, C_s$ be the components of F^-, F and F^+, ordered so that C_1 is a component of F, and for each $i \in [s-1]$, C_{i+1} has precisely one neighbour in $C_1 \cup \dots \cup C_i$. We shall embed the C_i in turn, so that each component of F^- is embedded in $G[Z]$, each component of F is embedded in $G[X]$, and each component of F^+ is embedded in $G[Y]$. We also require that after each C_i is embedded, the embeddings of $C_1, \dots, C_i$ together form an embedding in G of the subtree of T induced by the vertices of $C_1, \dots, C_i$. So suppose that we have successfully embedded $C_1, \dots, C_{i-1}$ in this manner, and we now wish to extend this embedding to include C_i. Then if $i \geq 2$, there is precisely one edge of T between C_i and $C_1 \cup \dots \cup C_{i-1}$. Let t be the endvertex of this edge in $C_1 \cup \dots \cup C_{i-1}$, and let v be the vertex to which t was embedded. If C_i is a component of F^-, then $i \geq 2$, the edge between t and C_i is directed towards t and $v \in X \cup Y$. So we may let S consist of the inneighbours of v in Z. Then by (ii) we have $|S| \geq |Z| - \gamma|G|$. Let $S' \subseteq S$ consist of the unoccupied vertices of S. Since at most $|F^-| - |C_i|$ vertices of S are occupied by the embeddings of $C_1, \dots, C_{i-1}$,

$$|S'| \geq |Z| - \gamma|G| - |F^-| + |C_i| \geq 2|C_i| + \alpha n/2.$$

So if $|C_i| < \alpha n/2$ then $G[S']$ contains a copy of T by Theorem 1.3, and if $|C_i| \geq \alpha n/2$ then $G[S']$ contains a copy of T by Theorem 1.2(i). Alternatively, if C_i is a component of F^+, then $i \geq 2$, the edge between t and C_i is directed towards C_i and $v \in X \cup Z$. So we may let S consist of the outneighbours of v in Y, and let $S' \subseteq S$ consist of the unoccupied vertices of S. Then we may embed C_i in S' by the same argument as used when C_i is a component of F^-. Finally, suppose that C_i is a component of F. Then if $i \geq 2$ and $t \in F^+$, let S consist of the inneighbours of v in X. If instead $i \geq 2$ and $t \in F^-$, let S

consist of the outneighbours of v in X. If $i = 1$ then let $S = X$. Then by (ii) we have $|S| \geq |X| - \gamma|G|$. Again let $S' \subseteq S$ consist of the unoccupied vertices of S. Then it suffices to embed C_i in $G[S']$. Since at most $|F| - |C_i|$ vertices have been embedded in S, we have $|S'| \geq |X| - \gamma|G| - |F| + |C_i| \geq |C_i| + \alpha n/2$. Now, C_i is a subtree of T_Δ, so $\Delta(C_i) \leq \Delta$ by Proposition 2.1(iii). So if $|C_i| \geq \alpha n/4$, then $G[S']$ contains a copy of C_i by Theorem 1.2(ii). On the other hand, if $|C_i| < \alpha n/4$, then $G[S']$ contains a copy of C_i by Theorem 1.3. So in any case we may embed C_i as desired, completing the proof. □

References

[1] N. Alon and A. Shapira, Testing subgraphs in directed graphs, *Journal of Computer and System Sciences* **69** (2004), 354–382.

[2] S. Céroi and F. Havet, Trees with three leaves are $(n+1)$-unavoidable, *Discrete Applied Mathematics* **141** (2004), 19–39.

[3] A. El Sahili, Trees in tournaments, *Journal of Combinatorial Theory, Series B* **92** (2004), 183–187.

[4] R. Häggkvist and A.G. Thomason, Trees in tournaments, *Combinatorica* **11** (1991), 123–130.

[5] F. Havet, Trees in tournaments, *Discrete Mathematics* **243** (2002), 121–134.

[6] F. Havet, On unavoidability of trees with k leaves, *Graphs and Combinatorics* **19** (2003), 101–110.

[7] F. Havet and S. Thomassé, Median orders of tournaments: a tool for the second neighbourhood problem and Sumner's conjecture, *Journal of Graph Theory* **35** (2000), 244–256.

[8] F. Havet and S. Thomassé, Oriented Hamiltonian paths in tournaments: a proof of Rosenfeld's conjecture, *Journal of Combinatorial Theory, Series B* **78** (2000), 243–273.

[9] S. Janson, T. Łuczak and A. Ruciński, Random graphs, Wiley-Interscience, 2000.

[10] D. Kühn, R. Mycroft and D. Osthus, An approximate version of Sumner's universal tournament conjecture, preprint.

[11] D. Kühn and D. Osthus, Embedding large subgraphs into dense graphs, in Surveys in Combinatorics 2009, eds. S. Huczynska, J.D. Mitchell and C.M. Roney-Dougal, *London Math. Soc. Lecture Notes* **365**, Cambridge University Press, 2009, 137–168.

[12] D. Kühn, D. Osthus and A. Treglown, Hamiltonian degree sequences in digraphs, *Journal of Combinatorial Theory, Series B*, to appear.

[13] L. Redei, Ein kombinatorischer Satz, *Acta Lit. Szeged* **7** (1934), 39–43.

[14] K.B. Reid and N.C. Wormald, Embedding oriented n-trees in tournaments, *Studia Scientiarum Mathematicarum Hungarica* **18** (1983) 377–387.

[15] N.C. Wormald, Subtrees of large tournaments, *Combinatorial Mathematics X, Springer Lecture Notes in Mathematics* **1036** (1983) 417–419.

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