# EULER TOURS IN HYPERGRAPHS

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Abstract. We show that a quasirandom $k$-uniform hypergraph $G$ has a tight Euler tour subject to the necessary condition that $k$ divides all vertex degrees. The case when $G$ is complete confirms a conjecture of Chung, Diaconis and Graham from 1989 on the existence of universal cycles for the $k$-subsets of an $n$-set.

## 1. Introduction

Finding an *Euler tour* in a graph is a problem as old as graph theory itself: Euler's negative resolution of the Seven Bridges of Königsberg problem in 1736 is widely considered the first theorem in graph theory. Euler observed that if a (multi-)graph contains a closed walk which traverses every edge exactly once, then all the vertex degrees are even. He also stated that every connected graph with only even vertex degrees contains such a walk, which was later proved by Hierholzer and Wiener.

There are several ways of generalising the concept of paths/cycles, and similarly Euler trails/tours, to hypergraphs. Not least due to its connection to universal cycles, we focus in this paper on the so-called 'tight' regime. We will discuss related notions in Section 1.3.

### 1.1. Universal cycles.

Let $[n]$ denote the set $\{1, \dots, n\}$ and $\binom{[n]}{k}$ the set of all $k$-subsets of $[n]$. A *universal cycle for* $\binom{[n]}{k}$ is a cyclic $n$-ary sequence in which every $k$ consecutive elements are distinct and every element of $\binom{[n]}{k}$ appears exactly once consecutively (but in an arbitrary order). For example, 1234524135 is a universal cycle for $\binom{[5]}{2}$. The study of these objects was initiated by Chung, Diaconis and Graham [5] in a paper where they define universal cycles for various combinatorial structures (see Section 1.3).

It is easy to see that if a universal cycle for $\binom{[n]}{k}$ exists, then $k$ divides $\binom{n-1}{k-1}$, or equivalently, $n$ divides $\binom{n}{k}$. In 1989, Chung, Diaconis and Graham conjectured that the converse should also be true, at least if $n$ is sufficiently large, and offered \$100 for the resolution of this problem.

**Conjecture 1** (Chung, Diaconis, Graham [4, 5]). For every $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exists a universal cycle for $\binom{[n]}{k}$ whenever $k$ divides $\binom{n-1}{k-1}$.

It is easy to see that Conjecture 1 is true for $k = 2$. Numerous partial results have been obtained. In particular, Jackson proved the conjecture for $k = 3$ [15] and for $k \in \{4, 5\}$ (unpublished), and Hurlbert [14] confirmed the cases $k \in \{3, 4, 6\}$ if $n$ and $k$ are coprime. Approximate versions of Conjecture 1 have been obtained in [6, 7]. We prove Conjecture 1 in a strong form by showing the existence of tight Euler tours in 'typical' $k$-graphs.

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1.2. **Tight Euler tours in typical hypergraphs.** Given a $k$-graph $G$ (i.e. a $k$-uniform hypergraph $G$), a sequence of vertices $\mathcal{W} = x_1x_2\dots x_\ell$ is a *(tight self-avoiding) walk in $G$* if $\{x_i, x_{i+1}, \dots, x_{i+k-1}\} \in E(G)$ for all $i \in [\ell-k+1]$, and no edge of $G$ appears more than once in this way. Similarly, we say that $\mathcal{W}$ is a *closed walk* if $\{x_i, x_{i+1}, \dots, x_{i+k-1}\} \in E(G)$ for all $i \in [\ell]$, with indices modulo $\ell$, and no edge of $G$ appears more than once in this way. We let $E(\mathcal{W})$ denote the set of edges appearing in $\mathcal{W}$. An *Euler tour of $G$* is a closed walk $\mathcal{W}$ in $G$ with $E(\mathcal{W}) = E(G)$, and an *Euler trail of $G$* is a walk $\mathcal{W}$ in $G$ with $E(\mathcal{W}) = E(G)$. Clearly, a universal cycle for $\binom{[n]}{k}$ is equivalent to an Euler tour of the complete $n$-vertex $k$-graph $K_n^k$.

The problem of deciding whether a given 3-graph has an Euler tour has been shown to be NP-complete [20]. Thus, when $k > 2$, there is probably no simple characterisation of $k$-graphs having an Euler tour. However, we show that for 'typical' $k$-graphs, the existence of an Euler tour hinges only on a simple divisibility condition.

A $k$-graph $G$ on $n$ vertices is called *$(c, h, p)$-typical* if for any set $A$ of $(k-1)$-subsets of $V(G)$ with $|A| \leq h$, we have $|\bigcap_{S\in A} N_G(S)| = (1 \pm c)p^{|A|}n$, where $N_G(S)$ denotes the *neighbourhood of $S$*, i.e. the set of all vertices which together with $S$ form an edge. Note that this is what one would expect in a random $n$-vertex $k$-graph in which every edge appears independently with probability $p$. It is easy to see that the complete $k$-graph $K_n^k$ is $(hk/n, h, 1)$-typical. Thus, the following more general result implies Conjecture 1.

**Theorem 2.** *For all $k \in \mathbb{N}$ and $p \in (0, 1]$, there exist $c > 0$ and $h, n_0 \in \mathbb{N}$ such that the following holds: Let $G$ be a $(c, h, p)$-typical $k$-graph on at least $n_0$ vertices with all vertex degrees divisible by $k$. Then $G$ has a tight Euler tour.*

Clearly, the condition that all vertex degrees are divisible by $k$ is necessary for the existence of a tight Euler tour. Instead of an Euler tour, we can also easily obtain a tight Euler trail (see end of Section 2).

Our proof goes as follows. In the first step, we find a 'spanning' walk $\mathcal{W}$ in $G$, where spanning means that every ordered $(k-1)$-set of vertices appears at least once consecutively in the vertex sequence of $\mathcal{W}$. For this, we show that a self-avoiding random walk yields such a walk $\mathcal{W}$ (after an appropriate number of steps) with high probability. This step will use only a small fraction of the edges of $G$. We then extend $\mathcal{W}$ to a closed walk $\mathcal{W}'$. Subsequently, we remove $E(\mathcal{W}')$ from $G$ and decompose the remaining $k$-graph into tight cycles using results on the existence of $F$-designs from [11]. Each such cycle can be incorporated into $\mathcal{W}'$, which finally yields a tight Euler tour.

1.3. **Related research and open questions.** The most prominent example of universal cycles are de Bruijn cycles. A *de Bruijn cycle of order $k$* is a binary cyclic sequence in which every binary sequence of length $k$ appears as a subsequence (of consecutive terms) exactly once. Chung, Diaconis and Graham [5] extended this notion to various other combinatorial objects, for instance permutations (see [16]) and partitions of an $n$-set. The general idea is that a universal cycle for a set $S$ of combinatorial objects is a cyclic sequence which contains a 'representation' of every element of $S$ exactly once as a subsequence of consecutive terms. Due to their rich symmetry, such structures have found many applications, for instance in cryptography, computer graphics, database theory, digital fault testing and neural decoding. A common and natural approach to find universal cycles is via *transition graphs*. Suppose that every element of $S$ has a unique representation as a sequence of length $k$. One can then define a directed graph $G_S$ with vertex set $S$ where there is an arc from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ if and only if $y_i = x_{i+1}$ for all $i \in [k-1]$. With this terminology, a universal cycle for $S$ corresponds to a directed Hamilton cycle in $G_S$. One obstacle to finding universal cycles for $\binom{[n]}{k}$, which was noted in [5], is that it is not even possible to define such a transition graph (since each $k$-set is represented by several sequences).

Rather than seeking an Euler tour in a $k$-graph $G$, an alternative way to cover all edges of $G$ is to ask for a Hamilton decomposition, i.e. to ask for a collection of edge-disjoint Hamilton cycles in $G$ such that every edge is contained in exactly one such Hamilton cycle. In 1892, Walecki showed that the complete (2-)graph $K_n$ has a Hamilton decomposition whenever $n$ is odd. As mentioned before, there are several natural definitions of paths/cycles in hypergraphs. One of the earliest such concepts was introduced by Berge. A *Berge cycle* consists of a cyclic alternating sequence $v_1e_1v_2e_2 \dots v_\ell e_\ell$ of distinct vertices and edges such that $v_i, v_{i+1} \in e_i$ for all $i \in [\ell]$. (Here $v_{\ell+1} := v_1$ and the edges $e_i$ are also allowed to contain vertices outside $\{v_1, \dots, v_\ell\}$.) For $n \geq 100$, it is shown in [18] that $K_n^k$ has a decomposition into Berge Hamilton cycles if and only if $n \mid \binom{n}{k}$. Bailey and Stevens [1] conjectured that $K_n^k$ has a decomposition into tight Hamilton cycles if and only if $n \mid \binom{n}{k}$. This conjecture is generalised in [18] to include other notions of cycles such as *loose cycles*. A related conjecture on wreath decompositions was independently brought forward by Baranyai [3] and Katona in the 1970s. (If $k$ and $n$ are coprime, then a tight Hamilton cycle coincides with the notion of a *wreath*.) Approximate results in the sense of packing many edge-disjoint Hamilton cycles into $K_n^k$ have been obtained in [2, 9, 10].

Some results on Euler tours in hypergraphs have been obtained using the Berge notion (such Euler tours are defined analogously, except that vertices may be repeated). In [19], it is shown that the problem of deciding whether a $k$-graph has a Berge Euler tour is NP-complete for all $k > 2$. On the other hand, a characterization is obtained for so-called 'strongly connected' $k$-graphs: such a $k$-graph $G$ has a Berge Euler tour if and only if the number of odd degree vertices of $G$ is at most $(k-2)|E(G)|$. The existence of Berge Euler tours has also been investigated with the host hypergraphs being designs [12, 13, 22].

It is also natural to seek the above structures within $k$-graphs of large minimum degree. To formalize this, for a set $S \subseteq V(G)$ with $0 \leq |S| \leq k$, we let $d_G(S)$ denote the *degree of $S$ in $G$*, that is, the number of edges which contain $S$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum $(k-1)$-degree of a $k$-graph $G$, respectively, that is, the minimum/maximum value of $d_G(S)$ over all $S \subseteq V(G)$ of size $k-1$. Rödl, Ruciński and Szemerédi [21] showed that a $k$-graph $G$ on $n$ vertices with $\delta(G) \geq (1/2 + o(1))n$ contains a tight Hamilton cycle. Many related results have been obtained, see e.g. [24] for a recent survey. We pose the following question, which would show that the degree threshold for a tight Euler tour and that for a tight Hamilton cycle coincide asymptotically.

**Conjecture 3.** For all $k > 2$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every $k$-graph $G$ on $n \geq n_0$ vertices with $\delta(G) \geq (1/2 + \varepsilon)n$ has a tight Euler tour if all vertex degrees are divisible by $k$.

It follows from our Theorem 2 that this holds with $1/2 + \varepsilon$ being replaced by $1 - \varepsilon$ for some small $\varepsilon$. The following adaptation of a well-known construction shows that the conjecture would be asymptotically best possible for infinitely many $n$: Consider the $k$-graph $G$ with vertex set $V(G) = A \cup B$, where $|A| = |B|$ is divisible by $k$, and all possible edges except those which intersect $B$ in precisely one vertex. By removing up to $k-1$ perfect matchings from $G[A]$ and from $G[B]$, we can ensure that all vertex degrees of the resulting $k$-graph $G'$ are divisible by $k$. Moreover, $G'$ does not have a tight Euler tour.

## 2. Proof

For a $k$-graph $G$, we let $|G|$ and $e(G)$ denote the number of vertices and edges of $G$, respectively. Given a (closed) walk $\mathcal{W}$ in $G$, we will often view $\mathcal{W}$ as the subgraph $(V(G), E(\mathcal{W}))$ of $G$ and accordingly use terminology such as $e(\mathcal{W})$ and $\Delta(\mathcal{W})$.

2.1. **Spanning walk.** We call a walk $\mathcal{W}$ in a $k$-graph $G$ *spanning* if every ordered $(k-1)$-set of vertices appears consecutively in $\mathcal{W}$ at least once. One important ingredient of our approach for the proof of Theorem 2 is to find a sparse spanning walk in a given $k$-graph $G$.

This spanning walk will form a 'backbone' structure to which we will subsequently add smaller closed walks until every edge of $G$ is used exactly once.

We show that such a spanning walk can be obtained randomly, by following a self-avoiding random walk for a suitable number of steps. More precisely, let $G$ be a $k$-graph. We define a simple random process $X = (X_t)_{t\in\mathbb{N}}$ as follows: Arbitrarily choose distinct starting vertices $x_1, \dots, x_{k-1} \in V(G)$ and set $X_t := x_t$ for all $t \in [k-1]$. Moreover, let $G_{k-1} := G$. For all $t \geq k$, proceed as follows. Among all edges in $G_{t-1}$ that contain the $(k-1)$-set $\{X_{t-k+1}, \dots, X_{t-1}\}$ choose one edge $e$ uniformly at random and let $X_t$ be the vertex in $e \setminus \{X_{t-k+1}, \dots, X_{t-1}\}$ and set $G_t := G_{t-1} - e$. If no such edge is available, then terminate the process and set $X_{t'} := \emptyset$ and $G_{t'} := G_{t-1}$ for all $t' \geq t$.

Clearly, this yields a walk in $G$ as long as the process does not terminate. We write $\mathcal{W}_t$ for the walk $X_1X_2 \dots X_{t'}$ in $G$, where $t' \leq t$ is maximal such that $X_{t'} \neq \emptyset$. Note that $E(\mathcal{W}_t) = E(G) \setminus E(G_t)$.

We will only perform a very crude analysis of this process here, which is sufficient for our purposes. Clearly, for every ordered $(k-1)$-set of vertices to appear in the walk, we need a walk of length $\Omega(|G|^{k-1})$. We show that if we follow the random walk for a slightly larger number of steps, then with high probability, every ordered $(k-1)$-set of vertices will indeed appear at least once, and the walk will still be sparse in the sense that no $(k-1)$-set is contained in too many edges of the walk.

For the analysis of the process, we will use the following Chernoff bound for moderately dependent Bernoulli variables (see e.g. [8]).

**Lemma 4.** *Suppose $X_1, \dots, X_n$ are Bernoulli random variables, and let $X := \sum_{i=1}^n X_i$. Suppose $0 \leq \varepsilon \leq 3/2$. If for all $i \in [n]$, we have $\mathbb{P}[X_i = 1 \mid X_1, \dots, X_{i-1}] \leq p^+$, then*

$$\mathbb{P}\left[X \geq (1+\varepsilon)np^+\right] \leq \mathrm{e}^{-\varepsilon^2 np^+/3}.$$

*Similarly, if for all $i \in [n]$, we have $\mathbb{P}[X_i = 1 \mid X_1, \dots, X_{i-1}] \geq p^-$, then*

$$\mathbb{P}\left[X \leq (1-\varepsilon)np^-\right] \leq \mathrm{e}^{-\varepsilon^2 np^-/3}.$$

The following definition turns out to be a suitable assumption on $G$ which enables a convenient analysis of the process. We call a $k$-graph $G$ *$\alpha$-connected* if for all distinct $v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_{2k-1} \in V(G)$, there exist at least $\alpha|G|$ vertices $v_k$ such that $v_i v_{i+1} \dots v_{i+k-1} \in E(G)$ for all $i \in [k]$. This property is present in natural classes of $k$-graphs. For instance, if $G$ is $(c, k, p)$-typical, then $G$ is $(1-c)p^k$-connected. Similarly, if $\delta(G) \geq (1 - \frac{1}{k} + \alpha)|G|$, then $G$ is $k\alpha$-connected.

**Lemma 5.** *Let $k \geq 2$ and $\alpha > 0$. Suppose $n$ is sufficiently large in terms of $k$ and $\alpha$. Suppose that $G$ is an $\alpha$-connected $k$-graph on $n$ vertices and $X = (X_t)_{t\in\mathbb{N}}$ is the process defined above. Let $T := \lfloor n^{k-1} \log^2 n \rfloor$. Then with probability at least $1 - 1/n$, $\mathcal{W}_T$ is a spanning walk in $G$ and $\Delta(\mathcal{W}_T) \leq \log^3 n$.*

**Proof.** We denote by $\mathcal{E}_t$ the event that $\Delta(\mathcal{W}_t) \leq \sqrt{n}$. We say that distinct $v_1, \dots, v_{k-1} \in V(G)$ (or simply an ordered $(k-1)$-set) are *met* by $X$ at step $t$ if $X_{t-k+1+i} = v_i$ for all $i \in [k-1]$. A $(k-1)$-set $S$ is *covered* by $X$ at step $t$ if $S \subseteq \{X_{t-k+1}, \dots, X_t\}$ and $X_t \neq \emptyset$.

In the following, fix distinct $v_1, \dots, v_{k-1} \in V(G)$. The key idea is to show that for any given time $t$, the probability that $X$ meets/covers $v_1, \dots, v_{k-1}$ at step $t + 2k$ is $\Theta_\alpha(n^{-k+1})$, respectively.

For $t \in \mathbb{N}$, let $I_t$ be the indicator random variable of the event that $X$ meets $v_1, \dots, v_{k-1}$ at step $t$, and let $C_t$ be the indicator random variable of the event that $X$ covers $\{v_1, \dots, v_{k-1}\}$ at step $t$. Note that $d_{\mathcal{W}_t}(\{v_1, \dots, v_{k-1}\}) = \sum_{t'=k}^{t} C_{t'}$.

Consider $t \geq k-1$ and suppose we know the outcome of the process up to and including step $t$. Clearly, in every step of the process, there are at most $n$ choices for the next vertex. Moreover, if $\mathcal{E}_t$ holds, then $G_{t-1+i}$ will be $2\alpha/3$-connected and $\delta(G_{t-1+i}) \geq 2\alpha n/3$ for all

$i \in [2k]$. In particular, in each of the following $2k$ steps, the process has at least $\alpha n/2$ choices for the next vertex $X_{t+i}$ (even if we require that $X_{t+i} \notin \{v_1, \ldots, v_{k-1}\}$). Thus, the process will not terminate within the next $2k$ steps.

We claim that

$$\mathbb{P}[I_{t+2k} = 1 \mid (X_{t'})_{t' \in [t]}, \mathcal{E}_t] \geq \frac{(\alpha n/2)^{k+1}}{n^{2k}} = \frac{\alpha^{k+1}}{2^{k+1} n^{k-1}} =: p^-. \tag{1}$$

Clearly, there are at most $n^{2k}$ choices for the vertices $X_{t+1}, \ldots, X_{t+2k}$. Moreover, for at least $(\alpha n/2)^{k+1}$ choices of $X_{t+1}, \ldots, X_{t+2k}$, the vertices $v_1, \ldots, v_{k-1}$ are met at step $t + 2k$. This is because for $X_{t+1}, \ldots, X_{t+k}$ there are at least $(\alpha n/2)^k$ choices that avoid $v_1, \ldots, v_{k-1}$, and then there are at least $\alpha n/2$ choices for $X_{t+k+1}$ such that $X_{t+k+1+i} = v_i$ for all $i \in [k-1]$ is a valid choice for the process. (Here, the step of choosing $X_{t+k+1}$ is the part where the definition of $\alpha$-connectedness is crucial.)

We also claim that

$$\mathbb{P}[C_{t+2k} = 1 \mid (X_{t'})_{t' \in [t]}, \mathcal{E}_t] \leq \frac{k! n^{k+1}}{(\alpha n/2)^{2k}} = \frac{2^{2k} k!}{\alpha^{2k} n^{k-1}} =: p^+. \tag{2}$$

To prove this claim, we make three observations. Firstly, recall that in each of the next $2k$ steps, the process has at least $\alpha n/2$ choices for the next vertex. Secondly, note that if $C_{t+2k} = 1$, then $\{X_{t+k+1}, \ldots, X_{t+2k}\} \supseteq \{v_1, \ldots, v_{k-1}\}$. There are at most $k!$ ways of assigning $v_1, \ldots, v_{k-1}$ to $X_{t+k+1}, \ldots, X_{t+2k}$. Thirdly, there are at most $n^{k+1}$ choices for the remaining $k + 1$ vertices.

Note that the probability estimates (1) and (2) rely on the assumption that $\mathcal{E}_t$ holds. To account for the complementary case, we define auxiliary $0/1$ random variables $Y_t^-$ and $Y_t^+$ for $t \geq 4k + 1$ as follows. Let $Y_t^- := I_t$ and $Y_t^+ := C_t$ if $\mathcal{E}_{t-2k}$ holds and otherwise let $Y_t^- := 1$ with probability $p^-$ and $Y_t^+ := 1$ with probability $p^+$ independently of all other random choices.

Since the bounds (1) and (2) only hold if we condition on the process until $2k$ steps earlier, we consider $2k$ disjoint subsequences and analyse each subsequence individually. Define $T' := \lfloor T/2k \rfloor$ and for all $i \in [2k]$, define $Z_i^{\pm} := \sum_{t'=1}^{T'} Y_{i+2k(t'+1)}^{\pm}$. Observe that for each $i \in [2k]$ and all $t' \in \mathbb{N}$, we have

$$\mathbb{P}[Y_{i+2k(t'+1)}^- = 1 \mid Y_{i+4k}^-, Y_{i+6k}^-, \ldots, Y_{i+2kt'}^-] \geq p^-,$$
$$\mathbb{P}[Y_{i+2k(t'+1)}^+ = 1 \mid Y_{i+4k}^+, Y_{i+6k}^+, \ldots, Y_{i+2kt'}^+] \leq p^+.$$

This follows from (1) and (2) if $\mathcal{E}_{i+2kt'}$ holds and from the definition of $Y_{i+2k(t'+1)}^{\pm}$ otherwise. Note that $T' \cdot p^- \geq \log^{3/2} n$ and $\log^2 n \leq T' \cdot p^+ \leq \log^{5/2} n$. Hence, by Lemma 4 we conclude that

$$\mathbb{P}[Z_i^- \leq (\log^{3/2} n)/2] \leq \mathrm{e}^{-(\log^{3/2} n)/12} \text{ and } \mathbb{P}[Z_i^+ \geq 2\log^{5/2} n] \leq \mathrm{e}^{-(\log^2 n)/3}.$$

Therefore, with probability at least $1 - 1/n$ say, we have $Z_i^- \geq (\log^{3/2} n)/2$ and $Z_i^+ \leq 2\log^{5/2} n$ for all $i \in [2k]$ and all choices of $v_1, \ldots, v_{k-1}$ simultaneously.

Finally, suppose that the above hold. From the upper bound on $Z_i^+$, we obtain that $\mathcal{E}_T$ holds. This implies that $Y_t^- = I_t$ and $Y_t^+ = C_t$ for all $t \leq T$. Consequently, all ordered $(k-1)$-sets are met at least once until step $T$ and $\Delta(\mathcal{W}_T) \leq \log^3 n$. □

2.2. **$F$-decompositions.** In 1976, Wilson [23] proved the fundamental result that given any graph $F$, for sufficiently large $n$, the complete graph $K_n$ has an $F$-decomposition whenever it satisfies some necessary divisibility conditions (see below). This was generalised to hypergraphs in [11]. In order to formally state the required result, we define the following. Let $G$ and $F$ be $k$-graphs, where $F$ is non-empty. An *$F$-decomposition of $G$* is a collection of copies of $F$ in $G$ such that every edge of $G$ is contained in exactly one of these copies. It is easy to see that the existence of an $F$-decomposition necessitates certain divisibility conditions. For instance, we surely need $e(F) \mid e(G)$. More generally, define $d_F(i) := \gcd\{d_F(S) \ : \ S \in \binom{V(F)}{i}\}$ for all $0 \leq i \leq k-1$. Note that $d_F(0) = e(F)$. Now, $G$ is called *$F$-divisible* if $d_F(i) \mid d_G(S)$ for all $0 \leq i \leq k-1$ and all $S \in \binom{V(G)}{i}$. It is easy to see that $G$ must be $F$-divisible in order to admit an $F$-decomposition. The converse implication is in general not true. However, if $G$ is a large typical $k$-graph, then divisibility guarantees the existence of a decomposition. For $G = K_n^k$, this generalises Wilson's theorem to hypergraphs.

**Theorem 6** ([11]). *For all $k \in \mathbb{N}$, $p \in [0,1]$ and any $k$-graph $F$, there exist $c > 0$ and $h, n_0 \in \mathbb{N}$ such that the following holds. Suppose that $G$ is a $(c,h,p)$-typical $k$-graph on at least $n_0$ vertices. Then $G$ has an $F$-decomposition whenever it is $F$-divisible.*

We remark that explicit bounds for $c$ and $h$ were obtained in [11]. Using these one can also obtain such explicit bounds in Theorem 2. For an alternative proof of Theorem 6 see [17].

Let $C_\ell^k$ denote the tight $k$-uniform cycle of length $\ell$, that is, the vertices of $C_\ell^k$ are $v_1, \dots, v_\ell$, and the edges are all the $k$-tuples of the form $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$, with indices modulo $\ell$.

Here, we will apply Theorem 6 with $F = C_{2k}^k$. Clearly, we have $d_{C_{2k}^k}(0) = e(C_{2k}^k) = 2k$ and $d_{C_{2k}^k}(1) = k$. Moreover, for every $i \in \{2, \dots, k-1\}$, we have $d_{C_{2k}^k}(\{v_1, \dots, v_{i-1}, v_k\}) = 1$ and hence $d_{C_{2k}^k}(i) = 1$. Conveniently, a $k$-graph $G$ is thus $C_{2k}^k$-divisible whenever $2k \mid e(G)$ and $k \mid d_G(v)$ for all $v \in V(G)$.

## 2.3. **Proof of Theorem 2.** We can now prove our main theorem.

**Proof of Theorem 2.** Given $k$ and $p$, choose $c > 0$ sufficiently small and $h, n_0$ sufficiently large. In particular, we assume that $h \geq k$ and that we can apply Theorem 6 with $k$, $p$, $2c$, $h$, $n_0$, $C_{2k}^k$ playing the roles of $k, p, c, h, n_0, F$. Suppose that $G$ is a $(c,h,p)$-typical $k$-graph on $n \geq n_0$ vertices with all vertex degrees divisible by $k$. Since $G$ is $(1-c)p^k$-connected, by Lemma 5, there exists a spanning walk $\mathcal{W} = v_1 v_2 \dots v_\ell$ in $G$ such that $\Delta(\mathcal{W}) \leq \log^3 n$. Next, we extend $\mathcal{W}$ to a closed walk $\mathcal{W}'$. Choose $k \leq \ell' \leq 3k-1$ such that $\ell' \equiv e(G) - \ell \mod 2k$. Now, find distinct vertices $v_{\ell+1}, \dots, v_{\ell+\ell'} \in V(G) \setminus \{v_1, \dots, v_{k-1}, v_{\ell-k+2}, \dots, v_\ell\}$ such that $v_i v_{i+1} \dots v_{i+k-1} \in E(G) \setminus E(\mathcal{W})$ for all $i$ with $\ell - k + 1 < i \leq \ell + \ell'$, with indices modulo $\ell + \ell'$. We can find such vertices one-by-one using the typicality of $G$. Let $\mathcal{W}' := v_1 \dots v_{\ell+\ell'}$. Clearly, $\mathcal{W}'$ is a spanning closed walk in $G$ and $\Delta(\mathcal{W}') \leq 2\log^3 n$.

Now, let $G' := G - E(\mathcal{W}')$. We have $e(G') = e(G) - (\ell + \ell') \equiv 0 \mod 2k$. Moreover, since $\mathcal{W}'$ is a closed walk, we have $k \mid d_{\mathcal{W}'}(v)$ for all $v \in V(G)$. Combining this with the initial divisibility condition of $G$, we have that $k \mid d_{G'}(v)$ for all $v \in V(G')$. Thus, $G'$ is $C_{2k}^k$-divisible. Moreover, since $\Delta(\mathcal{W}') \leq 2\log^3 n$, we have that $G'$ is still $(2c, h, p)$-typical. Invoking Theorem 6, we conclude that $G'$ has a $C_{2k}^k$-decomposition $\mathcal{C}$. We can now simply incorporate each cycle of $\mathcal{C}$ one-by-one into the spanning closed walk $\mathcal{W}'$. For this, suppose that $\mathcal{W}''$ is the current spanning closed walk and let $C \in \mathcal{C}$ be a copy of $C_{2k}^k$ with vertices $v_1, \dots, v_{2k}$ appearing in this order on $C$. Since $\mathcal{W}''$ is spanning, $v_1, \dots, v_{k-1}$ appear consecutively in $\mathcal{W}''$, say $\mathcal{W}'' = \mathcal{W}_1'' v_1 \dots v_{k-1} \mathcal{W}_2''$. We can then simply replace $\mathcal{W}''$ with $\mathcal{W}_1'' v_1 \dots v_{2k} v_1 \dots v_{k-1} \mathcal{W}_2''$ to obtain a new spanning closed walk $\mathcal{W}'''$ with $E(\mathcal{W}''') = E(\mathcal{W}'') \cup E(C)$. Adding all cycles of $\mathcal{C}$ in this way yields the desired Euler tour. $\square$

As pointed out after Theorem 2, our proof can also be easily adapted to obtain a tight Euler trail. If a (2-)graph has an Euler trail, then there are precisely two vertices of odd degree. If an Euler trail in a $k$-graph $G$ starts with the sequence $v_1 \dots v_{k-1}$ and ends with the sequence $w_{k-1} \dots w_1$, then (assuming that $v_1, \dots, v_{k-1}, w_1, \dots, w_{k-1}$ are distinct) we must have $d_G(v_i), d_G(w_i) \equiv i \mod k$ for all $i \in [k-1]$, and all other vertex degrees are divisible by $k$. On the other hand, if these conditions hold and $G$ is typical, then $G$ has an Euler trail. For this, in the above proof of Theorem 2, instead of extending the spanning walk $\mathcal{W}$ to a closed walk, one simply extends both ends to the designated start and end $(k-1)$-tuples.

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