HAMILTONICITY OF RANDOM SUBGRAPHS OF THE HYPERCUBE

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Abstract. We study Hamiltonicity in random subgraphs of the hypercube $\mathcal{Q}^n$. Our first main theorem is an optimal hitting time result. Consider the random process which includes the edges of $\mathcal{Q}^n$ according to a uniformly chosen random ordering. Then, with high probability, as soon as the graph produced by this process has minimum degree $2k$, it contains k edge-disjoint Hamilton cycles, for any fixed $k \in \mathbb{N}$. Secondly, we obtain a perturbation result: if $H \subseteq \mathcal{Q}^n$ satisfies $\delta(H) \geq \alpha n$ with $\alpha > 0$ fixed and we consider a random binomial subgraph $\mathcal{Q}^n_p$ of $\mathcal{Q}^n$ with $p \in (0, 1]$ fixed, then with high probability $H \cup \mathcal{Q}^n_p$ contains k edge-disjoint Hamilton cycles, for any fixed $k \in \mathbb{N}$. In particular, both results resolve a long standing conjecture, posed e.g. by Bollobás, that the threshold probability for Hamiltonicity in the random binomial subgraph of the hypercube equals $1/2$. Our techniques also show that, with high probability, for all fixed $p \in (0, 1]$ the graph $\mathcal{Q}^n_p$ contains an almost spanning cycle. Our methods involve branching processes, the Rödl nibble, and absorption.

1. Introduction

The n-dimensional *hypercube* $\mathcal{Q}^n$ is the graph whose vertex set consists of all n-bit 01-strings, where two vertices are joined by an edge whenever their corresponding strings differ by a single bit. The hypercube and its subgraphs have attracted much attention in graph theory and computer science, e.g. as a sparse network model with strong connectivity properties. It is well known that hypercubes contain spanning paths (also called *Gray codes* or *Hamilton paths*) and, for all $n \geq 2$, they contain spanning cycles (also referred to as *cyclic Gray codes* or *Hamilton cycles*). Classical applications of Gray codes in computer science are described in the surveys of Savage [51] and Knuth [39]. Applications of hypercubes to parallel computing are discussed in the monograph of Leighton [48].

1.1. Spanning subgraphs in hypercubes.

The systematic study of spanning paths, trees and cycles in hypercubes was initiated in the 1970's. There is by now an extensive literature about subtrees of the hypercube; see, for instance, results of Bhatt, Chung, Leighton and Rosenberg [8] about embedding subdivided trees (instigated by processor allocation in distributed computing systems).

As a generalization of Hamilton paths, Caha and Koubek [21] considered the problem of finding a collection of spanning vertex-disjoint paths, given a prescribed set of endpoints. After several improvements [23, 32], this problem was recently resolved by Dvořák, Gregor and Koubek [25].

The applications of hypercubes as networks in computer science inspired questions about the reliability of its properties. This led to considering 'faulty' hypercubes in which some edges or vertices are missing. For instance, Chan and Lee [22] showed that, if $\mathcal{Q}^n$ has at most $2n-5$ faulty edges and every vertex has (non-faulty) degree at least 2, then there is a Hamilton cycle in $\mathcal{Q}^n$ which avoids all faulty edges (and this condition is best possible). They also showed

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that the general problem of determining the Hamiltonicity of $\mathcal{Q}^n$ with a larger number of faulty edges is NP-complete. More generally, Dvořák and Gregor [24] studied the existence of spanning collections of vertex-disjoint paths with prescribed endpoints in faulty hypercubes. (We will apply these results in our proofs, see Section 8.3 for details.) These can be seen as extremal results about the *robustness* of the hypercube with respect to containing spanning collections of paths and cycles.

1.2. **Hamilton cycles in binomial random graphs.** One of the most studied random graph models is the binomial random graph $G_{n,p}$. Here we have a (labelled) set of n vertices and we include each edge with probability p independently of all other edges.

Given some monotone increasing graph property $\mathcal{P}$, a function $p^* = p^*(n)$ is said to be a (coarse) *threshold* for $\mathcal{P}$ if $\mathbb{P}[G_{n,p} \in \mathcal{P}] \to 1$ whenever $p/p^* \to \infty$ and $\mathbb{P}[G_{n,p} \in \mathcal{P}] \to 0$ whenever $p/p^* \to 0$. One can define the stronger notion of a sharp threshold similarly: $p^* = p^*(n)$ is said to be a *sharp threshold* for $\mathcal{P}$ if, for all $\varepsilon > 0$, we have that $\mathbb{P}[G_{n,p} \in \mathcal{P}] \to 1$ whenever $p \geq (1+\varepsilon)p^*$ and $\mathbb{P}[G_{n,p} \in \mathcal{P}] \to 0$ whenever $p \leq (1-\varepsilon)p^*$. The problem of finding the threshold for the containment of a Hamilton cycle was solved independently by Pósa [50] and Koršunov [42]. Furthermore, Koršunov [42] determined the sharp threshold for Hamiltonicity to be $p^* = \log n/n$. These results were later made even more precise by Komlós and Szemerédi [41]. It is worth noting that $p^* = \log n/n$ is also the sharp threshold for the property of having minimum degree at least 2. In this sense, the results about Hamilton cycles in $G_{n,p}$ can be interpreted as saying that the natural obstruction of having sufficiently high minimum degree is also an 'almost sufficient' condition.

A property that generalises Hamiltonicity is that of containing k edge-disjoint Hamilton cycles, for some $k \in \mathbb{N}$. We will present more results in this direction in Section 1.4; for now, let us simply note that the sharp threshold for the containment of k edge-disjoint Hamilton cycles in $G_{n,p}$, for some $k \in \mathbb{N}$ independent of n, is $p^* = \log n/n$, i.e. the same as the threshold for Hamiltonicity.

The study of *robustness* of graph properties has also attracted much attention recently. For instance, given a graph G which is known to satisfy some property $\mathcal{P}$, consider a random subgraph G_p obtained by deleting each edge of G with probability $1-p$, independently of all other edges. The problem then is to determine the range of p for which G_p satisfies $\mathcal{P}$ with high probability. In this setting, a result of Krivelevich, Lee and Sudakov [44] asserts that, for any n-vertex graph G with minimum degree at least $n/2$, the graph G_p is asymptotically almost surely Hamiltonian whenever $p \gg \log n/n$. This can be viewed as a robust version of Dirac's theorem on Hamilton cycles.

1.3. **Hamilton cycles in binomial random subgraphs of the hypercube.** Throughout this paper, we will consider random subgraphs of the hypercube and show that the hypercube is robustly Hamiltonian in the above sense. We will denote by $\mathcal{Q}^n_p$ the random subgraph of the hypercube obtained by removing each edge of $\mathcal{Q}^n$ with probability $1-p$ independently of every other edge.

The random graph $\mathcal{Q}^n_p$ was first studied by Burtin [20], who proved that the sharp threshold for connectivity is $1/2$. This result was later made more precise by Erdős and Spencer [27] and Bollobás [10]. As a related problem, Dyer, Frieze and Foulds [26] determined the sharp threshold for connectivity in subgraphs of $\mathcal{Q}^n$ obtained by removing both vertices and edges uniformly at random. Later, Bollobás [12] proved that $1/2$ is also the sharp threshold for the containment of a perfect matching in $\mathcal{Q}^n_p$. As with the $G_{n,p}$ model, this also coincides with the threshold for having minimum degree at least 1.

The main goal of this paper is to study the analogous problem for Hamiltonicity in random subgraphs of the hypercube. There is a folklore conjecture that the sharp threshold for Hamiltonicity in $\mathcal{Q}^n_p$ should be $1/2$, i.e. the same as the threshold for having minimum degree at least 2. This question was explicitly asked by Bollobás [13] at several conferences in the 1980's, in the ICM surveys of Frieze [30] and Kühn and Osthus [46], as well as the recent survey of Frieze [31]. A special case of our first result resolves this problem.

Theorem 1.1. *For any $k \in \mathbb{N}$, the sharp threshold for the property of containing k edge-disjoint Hamilton cycles in $\mathcal{Q}^n_p$ is $p^* = 1/2$.*

For $k = 1$, this can be seen as a probabilistic version of the result on faulty hypercubes [22], and also as a statement about the robustness of Hamiltonicity in the hypercube.

While, for $p < 1/2$, with high probability $\mathcal{Q}^n_p$ will not contain a Hamilton cycle, it turns out that the reason for this is mostly due to local obstructions (e.g., vertices with degree zero or one). More precisely, we prove that, for any constant $p \in (0, 1/2)$, a.a.s. the random graph $\mathcal{Q}^n_p$ contains an almost spanning cycle.

Theorem 1.2. *For any $\delta, p \in (0, 1]$, a.a.s. the graph $\mathcal{Q}^n_p$ contains a cycle of length at least $(1 - \delta)2^n$.*

We believe that the probability bound is far from optimal, in the sense that random subgraphs of the hypercube where edges are picked with vanishing probability should also satisfy this property.

Conjecture 1.3. *Suppose that $p = p(n)$ satisfies that $pn \to \infty$. Then, a.a.s. $\mathcal{Q}^n_p$ contains a cycle of length $(1 - o(1))2^n$.*

Similarly, it would be interesting to determine which (long) paths and (almost spanning) trees can be found in $\mathcal{Q}^n_p$. Moreover, our methods might also be useful to embed other large subgraphs, such as F-factors.

Conjecture 1.4. *Suppose $\varepsilon > 0$ and an integer $\ell \geq 2$ are fixed and $p \geq 1/2 + \varepsilon$. Then, a.a.s. $\mathcal{Q}^n_p$ contains a C_{2^ℓ}-factor, that is, a set of vertex-disjoint cycles of length 2^ℓ which together contain all vertices of $\mathcal{Q}^n$.*

1.4. **Hitting time results.** Remarkably, the above intuition that having the necessary minimum degree is an 'almost sufficient' condition for the containment of edge-disjoint perfect matchings and Hamilton cycles can be strengthened greatly via so-called hitting time results. These are expressed in terms of random graph processes. The general setting is as follows. Let G be an n-vertex graph with $m = m(n)$ edges, and consider an arbitrary labelling $E(G) = \{e_1, \ldots, e_m\}$. The *$G$-process* is defined as a random sequence of nested graphs $\tilde{G}(\sigma) = (G_t(\sigma))_{t=0}^m$, where σ is a permutation of $[m]$ chosen uniformly at random and, for each $i \in [m]_0$, we set $G_i(\sigma) = (V(G), E_i)$, where $E_i \coloneqq \{e_{\sigma(j)} : j \in [i]\}$. Given any monotone increasing graph property $\mathcal{P}$ such that $G \in \mathcal{P}$, the *hitting time* for $\mathcal{P}$ in the above G-process is the random variable $\tau_{\mathcal{P}}(\tilde{G}(\sigma)) \coloneqq \min\{t \in [m]_0 : G_t(\sigma) \in \mathcal{P}\}$.

Let us denote the properties of containing a perfect matching by $\mathcal{PM}$, Hamiltonicity by $\mathcal{HAM}$, and connectivity by $\mathcal{CON}$, respectively. For any $k \in \mathbb{N}$, let $\boldsymbol{\delta} k$ denote the property of having minimum degree at least k, and let $\mathcal{HM}k$ denote the property of containing $\lfloor k/2 \rfloor$ edge-disjoint Hamilton cycles and, if k is odd, one matching of size $\lfloor n/2 \rfloor$ which is edge-disjoint from these Hamilton cycles. With this notion of hitting times, many of the results about thresholds presented in Sections 1.2 and 1.3 can be strengthened significantly. For instance, Bollobás and Thomason [16] showed that a.a.s. $\tau_{\mathcal{CON}}(\tilde{K_n}(\sigma)) = \tau_{\boldsymbol{\delta}1}(\tilde{K_n}(\sigma))$ and, if n is even, then a.a.s. $\tau_{\mathcal{PM}}(\tilde{K_n}(\sigma)) = \tau_{\boldsymbol{\delta}1}(\tilde{K_n}(\sigma))$. Ajtai, Komlós and Szemerédi [1] and Bollobás [11] independently proved that a.a.s. $\tau_{\mathcal{HAM}}(\tilde{K_n}(\sigma)) = \tau_{\boldsymbol{\delta}2}(\tilde{K_n}(\sigma))$. This was later generalised by Bollobás and Frieze [14], who proved that, given any $k \in \mathbb{N}$, for n even a.a.s. $\tau_{\mathcal{HM}k}(\tilde{K_n}(\sigma)) = \tau_{\boldsymbol{\delta}k}(\tilde{K_n}(\sigma))$.

A hitting time result for the property of having k edge-disjoint Hamilton cycles when k is allowed to grow with n is still not known, even in K_n-processes. As a slightly weaker notion, consider property $\mathcal{H}$, where we say that a graph G satisfies property $\mathcal{H}$ if it contains $\lfloor \delta(G)/2 \rfloor$ edge-disjoint Hamilton cycles, together with an additional edge-disjoint matching of size $\lfloor n/2 \rfloor$ if $\delta(G)$ is odd. Knox, Kühn and Osthus [38], Krivelevich and Samotij [45] as well as Kühn and Osthus [47] proved results for different ranges of p which, together, show that $G_{n,p}$ a.a.s. satisfies property $\mathcal{H}$.

For graphs other than the complete graph, Johansson [37] recently obtained a robustness version of the hitting time results for Hamiltonicity. In particular, for any n-vertex graph G with $\delta(G) \geq (1/2+\varepsilon)n$, he proved that a.a.s. $\tau_{\mathcal{HAM}}(\tilde{G}(\sigma)) = \tau_{\delta 2}(\tilde{G}(\sigma))$. This was later extended to a larger class of graphs G and to hitting times for $\mathcal{HM}2k$, for all $k \in \mathbb{N}$ independent of n, by Alon and Krivelevich [4].

In the setting of random subgraphs of the hypercube, Bollobás [12] determined the hitting time for perfect matchings by showing that a.a.s. $\tau_{\mathcal{PM}}(\tilde{\mathcal{Q}}^n(\sigma)) = \tau_{\mathcal{CON}}(\tilde{\mathcal{Q}}^n(\sigma)) = \tau_{\delta 1}(\tilde{\mathcal{Q}}^n(\sigma))$. One of our main results (which implies Theorem 1.1) is a hitting time result for Hamiltonicity (and, more generally, property $\mathcal{HM}k$) in $\mathcal{Q}^n$-processes. Again, this question was raised by Bollobás [13] at several conferences.

Theorem 1.5. *For all $k \in \mathbb{N}$, a.a.s. $\tau_{\mathcal{HM}k}(\tilde{\mathcal{Q}}^n(\sigma)) = \tau_{\delta k}(\tilde{\mathcal{Q}}^n(\sigma))$, that is, the hitting time for the containment of a collection of $\lfloor k/2 \rfloor$ Hamilton cycles and $k - 2\lfloor k/2 \rfloor$ perfect matchings, all pairwise edge-disjoint, in $\mathcal{Q}^n$-processes is a.a.s. equal to the hitting time for the property of having minimum degree at least k.*

We also wonder whether this is true if k is allowed to grow with n, and propose the following conjecture which, if true, would be an approximate version of the results of [38, 45, 47] in the hypercube.

Conjecture 1.6. *For all $p \in (1/2, 1]$ and $\eta > 0$, a.a.s. $\mathcal{Q}^n_p$ contains $(1/2-\eta)\delta(\mathcal{Q}^n_p)$ edge-disjoint Hamilton cycles.*

1.5. **Randomly perturbed graphs.** A relatively recent area at the interface of extremal combinatorics and random graph theory is the study of *randomly perturbed graphs*. Generally speaking, the idea is to consider a deterministic dense n-vertex graph H (usually satisfying some minimum degree condition) and a random graph $G_{n,p}$ on the same vertex set as H. The question is whether H is close to satisfying some given property $\mathcal{P}$ in the sense that a.a.s. $H \cup G_{n,p} \in \mathcal{P}$ for some small p. This line of research was sparked off by Bohman, Frieze and Martin [9], who showed that, if H is an n-vertex graph with $\delta(H) \geq \alpha n$, for any constant $\alpha > 0$, then a.a.s. $H \cup G_{n,p}$ is Hamiltonian for all $p \geq C(\alpha)/n$. Other properties that have been studied in this context are e.g. the existence of powers of Hamilton cycles and general bounded degree spanning graphs [19], F-factors [6] or spanning bounded degree trees [18, 43]. One common phenomenon in this model is that, by considering the union with a dense graph H (i.e. a graph H with linear degrees), the probability threshold of different properties is significantly lower than that in the classical $G_{n,p}$ model. The results for Hamiltonicity [9] were very recently generalised by Hahn-Klimroth, Maesaka, Mogge, Mohr and Parczyk [33] to allow α to tend to 0 with n (that is, to allow graphs H which are not dense).

We consider randomly perturbed graphs in the setting of subgraphs of the hypercube. To be precise, we take an arbitrary spanning subgraph H of the hypercube, with linear minimum degree, and a random subgraph $\mathcal{Q}^n_\varepsilon$, and consider $H \cup \mathcal{Q}^n_\varepsilon$. (Note here that $\mathcal{Q}^n_\varepsilon$ is a 'dense' subgraph of $\mathcal{Q}^n$, but for $\varepsilon < 1/2$ it will contain both isolated vertices and vertices of very low degrees.) In this setting, we show the following result.

Theorem 1.7. *For all $\varepsilon, \alpha \in (0, 1]$ and $k \in \mathbb{N}$, the following holds. Let H be a spanning subgraph of $\mathcal{Q}^n$ such that $\delta(H) \geq \alpha n$. Then, a.a.s. $H \cup \mathcal{Q}^n_\varepsilon$ contains k edge-disjoint Hamilton cycles.*

We can also allow H to have much smaller degrees, at the cost of requiring a larger probability to find the Hamilton cycles.

Theorem 1.8. *For every integer $k \geq 2$, there exists $\varepsilon > 0$ such that a.a.s., for every spanning subgraph H of $\mathcal{Q}^n$ with $\delta(H) \geq k$, the graph $H \cup \mathcal{Q}^n_{1/2-\varepsilon}$ contains a collection of $\lfloor k/2 \rfloor$ Hamilton cycles and $k - 2\lfloor k/2 \rfloor$ perfect matchings, all pairwise edge-disjoint.*

Note that Theorem 1.8 can be viewed as a 'universality' result for H, meaning that it holds for all choices of H simultaneously. It would be interesting to know whether such a result can also be obtained for the lower edge probability assumed in Theorem 1.7, i.e., is it the case that, for all

$\varepsilon, \alpha \in (0,1]$, a.a.s. $G \sim \mathcal{Q}^n_\varepsilon$ has the property that, for every spanning $H \subseteq \mathcal{Q}^n$ with $\delta(H) \geq \alpha n$, $G \cup H$ is Hamiltonian?

As we will prove, Theorem 1.1 follows straightforwardly from Theorem 1.7, and it follows trivially from Theorem 1.5. In turn, Theorem 1.5 follows from Theorem 1.8. On the other hand, Theorems 1.2, 1.7 and 1.8, while being proved with similar ideas, are incomparable.

1.6. **Percolation on the hypercube.** To build Hamilton cycles in random subgraphs of the hypercube, we will consider a random process which can be viewed as a branching process or percolation process on the hypercube. With high probability, for constant $p > 0$, this process results in a bounded degree tree in $\mathcal{Q}^n_p$ which covers most of the neighbourhood of every vertex in $\mathcal{Q}^n$, and thus spans almost all vertices of $\mathcal{Q}^n$. The version stated below is a special case of Theorem 7.1.

Theorem 1.9. *For any fixed $\varepsilon, p \in (0,1]$, there exists $D = D(\varepsilon)$ such that a.a.s. $\mathcal{Q}^n_p$ contains a tree T with $\Delta(T) \leq D$ and such that $|V(T) \cap N_{\mathcal{Q}^n}(x)| \geq (1-\varepsilon)n$ for every $x \in V(\mathcal{Q}^n)$.*

Further results concerning the local geometry of the giant component in $\mathcal{Q}^n_p$ for constant $p \in (0, 1/2)$ were proved recently by McDiarmid, Scott and Withers [49].

The random process we consider in the proof of Theorem 1.9 can be viewed as a branching random walk (with a bounded number of branchings at each step). Simpler versions of such processes (with infinite branchings allowed) have been studied by Fill and Pemantle [29] and Kohayakawa, Kreuter and Osthus [40], and we will base our analysis on these. Motivated by our approach, we raise the following question, which seems interesting in its own right.

Question 1.10. *Does a non-returning random walk on $\mathcal{Q}^n$ a.a.s. visit almost all vertices of $\mathcal{Q}^n$?*

More generally, there are many results and applications concerning random walks on the hypercube (but allowing for returns). For example, motivated by a processor allocation problem, Bhatt and Cai [7] studied a walk algorithm to embed large (subdivided) trees into the hypercube. Moreover, the analysis of (branching) random walks is a critical ingredient in the study of percolation thresholds for the existence of a giant component in $\mathcal{Q}^n_p$. These have been investigated e.g. by Bollobás, Kohayakawa and Łuczak [15], Borgs, Chayes, van der Hofstad, Slade and Spencer [17] and van der Hofstad and Nachmias [35].

1.7. **Organisation of the paper.** In Section 2 we provide an overview of our ideas and proof methods. In Section 3 we introduce the notation we will use throughout the paper. In Section 4 we state the different probabilistic tools, as well as some other well-known results, that we will call on, and in Section 5 we collect various results on matchings and random subgraphs of the hypercube. In Section 6 we prove Theorem 6.6, our main cube tiling result, and in Section 7 we prove Theorem 7.1, our main near-spanning tree result (see Section 2 for more details on each of these). Then, in Section 8 we prove Theorem 1.7 in the case $k = 1$ (see Theorem 8.1). In Section 8.5 we use this to deduce the general statement of Theorem 1.7, and also deduce Theorem 1.1 and explain how to obtain Theorem 1.2. Finally, in Section 9 we show how to modify the proof of Theorem 1.7 to obtain Theorem 1.8, and thus our hitting time result (Theorem 1.5).

2. Outline of the main proofs

2.1. **Overall outline.** We now sketch the key ideas for the proof of Theorem 1.7. We will first prove the case $k = 1$, and later use this to deduce the case when $k > 1$. Recall we are given $H \subseteq \mathcal{Q}^n$ with $\delta(H) \geq \alpha n$, and $G \sim \mathcal{Q}^n_\varepsilon$, with $\alpha, \varepsilon \in (0,1]$. Our aim is to show that a.a.s. $H \cup G$ is Hamiltonian.

Our approach for finding a Hamilton cycle is to first obtain a spanning tree. By passing along all the edges of a spanning tree T (with a vertex ordering prescribed by a depth first search), one can create a closed spanning walk W which visits every edge of T twice. The idea is then to modify such a walk into a Hamilton cycle. (This approach is inspired by the approximation algorithm for the Travelling Salesman Problem which returns a tour of at most twice the optimal length.) More precisely, our approach will be to obtain a near-spanning tree of $\mathcal{Q}^{n-s}$, for some suitable constant s, and to blow up vertices of this tree into s-dimensional cubes. These cubes can then be used to move along the tree without revisiting vertices, which will result in a near-Hamilton cycle $\mathfrak{H}$. All remaining vertices which are not included in $\mathfrak{H}$ will be absorbed into $\mathfrak{H}$ via absorbing structures that we carefully put in place beforehand.

In Sections 2.2 to 2.4 we outline in more detail how we find a long cycle in G (Theorem 1.2). Note that in Theorem 1.2 we have $G \sim \mathcal{Q}^n_\varepsilon$, so a.a.s. G will have isolated vertices which prevent any Hamilton cycle occurring as a subgraph. In Section 2.5 we outline how we build on this approach to obtain the case $k = 1$ of Theorem 1.7. In Section 2.6 we sketch how we obtain Theorem 1.5.

2.2. **Building block I: trees via branching processes.** We view each vertex in $\mathcal{Q}^n$ as an n-dimensional 01-coordinate vector. By fixing the first s coordinates, we fix one of 2^s *layers* $L_1, \dots, L_{2^s}$ of the hypercube, where $s \in \mathbb{N}$ will be constant. Thus, $L \cong \mathcal{Q}^{n-s}$ for each layer L. By considering a Hamilton cycle in $\mathcal{Q}^s$, we may assume that consecutive layers differ only by a single coordinate on the unique elements of $\mathcal{Q}^s$ which define them. Let $G \sim \mathcal{Q}^n_\varepsilon$. For each layer L, we let $L(G) \coloneqq G[V(L)]$ and define the *intersection graph* $I(G) \coloneqq \bigcap_{i=1}^{2^s} L_i(G)$. Hence, $I(G) \sim \mathcal{Q}^{n-s}_{\varepsilon^{2^s}}$. We view $I(G)$ as a subgraph of $\mathcal{Q}^{n-s}$. We first show that $I(G)$ contains a near-spanning tree T (Theorem 7.1). Thus, a copy of T is present in each of $L_1(G), \dots, L_{2^s}(G)$ simultaneously.

Since the walk W mentioned in Section 2.1 passes through each vertex x of T a total of $d_T(x)$ times, it will be important later for T to have bounded degree. In order to guarantee this, we run bounded degree branching processes (see Definition 7.3) from several far apart 'corners' of the hypercube. Roughly speaking, T will be formed by taking a union of these processes and removing cycles. Crucially, the model we introduce for these processes has a joint distribution with $\mathcal{Q}^{n-s}_{\varepsilon^{2^s}}$, so that T will in fact appear as a subgraph of $I(G)$. In applying Theorem 7.1, we obtain a bounded degree tree $T \subseteq I(G)$ which contains almost all of the neighbours of every vertex of $I(G)$. We also obtain a 'small' *reservoir* set $R \subseteq V(I(G))$, which T avoids and which will play a key role later in the absorption of vertices which do not belong to our initial long cycle. At this point, both T and R are now present in every layer of the hypercube simultaneously.

2.3. **Building block II: cube tilings via the nibble.** Let $\ell < s$ and $0 < \delta \ll 1$ be fixed. In order to gain more local flexibility when traversing the near-spanning tree T, we augment T by locally adding a near-spanning ℓ-cube factor of $I(G)$. One can use classical results on matchings in almost regular uniform hypergraphs of small codegree to show that $I(G)$ contains such a collection of $\mathcal{Q}^\ell$ spanning almost all vertices of $I(G)$. However, we require the following stronger properties, namely that there exists a collection $\mathcal{C}$ of vertex disjoint copies of $\mathcal{Q}^\ell$ in $I(G)$ so that, for each $x \in V(I(G))$,

(i) $\mathcal{C}$ covers almost all vertices in $N_{\mathcal{Q}^n}(x)$;
(ii) the directions spanned by the cubes intersecting $N_{\mathcal{Q}^n}(x)$ do not correlate too strongly with any given set of directions.

The precise statement is given in Theorem 6.6. Neither (i) nor (ii) follow from existing results on hypergraph matchings and the proofs strongly rely on geometric properties intrinsic to the hypercube.

To prove Theorem 6.6, we build on the so-called Rödl nibble. More precisely, we consider the hypergraph $\mathcal{H}$, with $V(\mathcal{H}) = V(\mathcal{Q}^{n-s})$, where the edge set is given by the copies of $\mathcal{Q}^\ell$ in $I(G)$. We run a random iterative process where at each stage we add a 'small' number of edges from $\mathcal{H}$ to $\mathcal{C}$, before removing all those remaining edges of $\mathcal{H}$ which 'clash' with our selection. A careful analysis and an application of the Lovász local lemma yield the existence of an instance of this process which terminates in the near-spanning ℓ-cube factor with the properties required for Theorem 6.6.

2.4. **Constructing a long cycle.** Roughly speaking, we will use T as a backbone to provide 'global' connectivity, and will use the near-spanning ℓ-cube factor $\mathcal{C}$ and the layer structure to gain high 'local' connectivity and flexibility. Let $T \cup \bigcup_{C \in \mathcal{C}} C =: \Gamma' \subseteq I(G)$ and let $\Gamma \subseteq \Gamma'$ be formed by removing all leaves and isolated cubes in Γ'. It follows by our tree and nibble results that almost all vertices of $I(G)$ are contained in Γ. Note that, for each $v \in V(\mathcal{Q}^{n-s}) = V(I(G))$, there is a unique vertex in each of the 2^s layers which corresponds to v. We refer to these 2^s vertices as *clones* of v and to the collection of these 2^s clones as a *vertex molecule*. Similarly, each ℓ-cube $C \in \mathcal{C}$ contained in Γ gives rise to a *cube molecule*. We construct a cycle in G which covers all of the cube molecules (and, therefore, almost all vertices in $\mathcal{Q}^n$).

Let Γ^* be the graph obtained from Γ by contracting each ℓ-cube $C \subseteq (\bigcup_{C\in\mathcal{C}} C) \cap \Gamma$ into a single vertex. We refer to such vertices in Γ^* as *atomic vertices*, and to all other vertices as *inner tree vertices*. We run a depth-first search on Γ^* to give an order to the vertices. Next, we construct a *skeleton* which will be the backbone for our long cycle. The skeleton is an ordered sequence of vertices in $\mathcal{Q}^n$ which contains the vertices via which our cycle will enter and exit each molecule. That is, given an *exit vertex* v for some molecule in the skeleton, the vertex u which succeeds v in the skeleton will be an *entry vertex* for another molecule, and such that $uv \in E(G)$. Here, a vertex in the skeleton belonging to an inner tree vertex molecule is referred to as both an entry and exit vertex. (Actually, we will first construct an 'external skeleton', which encodes this information. The skeleton then also prescribes some edges within molecules which go between different layers.) We use the ordering of the vertices of Γ^* to construct the skeleton in a recursive way starting from the lowest ordered vertex. It is crucial that our tree T has bounded degree (much smaller than 2^s), so that no molecule is overused in the skeleton.

Once the skeleton is constructed, we apply our 'connecting lemmas' (Lemmas 8.8 and 8.9). These connecting lemmas, applied to a cube molecule with a bounded number of pairs of entry and exit vertices as input (given by the skeleton), provide us with a sequence of vertex-disjoint paths which cover this molecule, where each path has start and end vertices consisting of an input pair. The union of all of these paths combined with all edges in G between the successive exit and entry vertices of the skeleton will then form a cycle $\mathfrak{H} \subseteq G$ which covers all vertices lying in the cube molecules (thus proving Theorem 1.2).

2.5. **Constructing a Hamilton cycle.** In order to construct a Hamilton cycle in $H \cup G$, we will absorb the vertices of $V(\mathcal{Q}^n) \setminus V(\mathfrak{H})$ into $\mathfrak{H}$. We achieve this via absorbing structures that we identify for each vertex (see Definition 8.2). To construct these absorbing structures, we will need to use some edges of H. Roughly speaking, to each vertex v we associate a left ℓ-cube $C_v^l \subseteq \mathcal{Q}^n$ and a right ℓ-cube $C_v^r \subseteq \mathcal{Q}^n$, where C_v^l, C_v^r are both clones of some ℓ-cubes $C^l, C^r \in \mathcal{C}$ contained in Γ. We choose these cubes so that v will have a neighbour $u \in V(C_v^l)$ and a neighbour $u' \in V(C_v^r)$, to which we refer as *tips* of the absorbing structure. Furthermore, u will have a neighbour $w \in V(C_v^r)$, which is also a neighbour of u'. Our near-Hamilton cycle $\mathfrak{H}$ will satisfy the following properties:

(a) $\mathfrak{H}$ covers all vertices in $C_v^l \cup C_v^r$ except for u, and
(b) $wu' \in E(\mathfrak{H})$.

These additional properties will be guaranteed by our connecting lemmas discussed in Section 2.4. We can then alter $\mathfrak{H}$ to include the segment $wuvu'$ instead of the edge wu', thus absorbing the vertices u and v into $\mathfrak{H}$. The following types of vertices will require absorption.

(i) Every vertex that is not covered by a clone of either some inner tree vertex or of some cube $C \in \mathcal{C}$ which is contained in Γ.
(ii) The cycle $\mathfrak{H}$ does not cover all the clones of inner tree vertices and, thus, the uncovered vertices of this type will also have to be absorbed.

However, we will not know precisely which of the vertices described in (i) and (ii) will be covered by $\mathfrak{H}$ and which of these vertices will need to be absorbed until after we have constructed the (external) skeleton. Moreover, many potential absorbing structures are later ruled out as candidates (for example, if they themselves contain vertices that will need to be absorbed). Therefore, it is important that we identify a 'robust' collection of many potential absorbing structures for every vertex in $\mathcal{Q}^n$ at a preliminary stage of the proof. The precise absorbing structure eventually assigned to each vertex will be chosen via an application of our rainbow matching lemma (Lemma 5.5) at a late stage in the proof.

We will now highlight the purpose of the reservoir R. Suppose $v \in V(\mathcal{Q}^n)$ is a vertex which needs to be absorbed via an absorbing structure with left ℓ-cube C_v^l and left tip $u \in V(C_v^l)$.

Recall that both u and C_v^l are clones of some $u^* \in V(\Gamma)$ and $C^l \in \mathcal{C}$, where $u^* \in V(C^l)$. If u^* has a neighbour w^* in $T - V(C^l)$, then it is possible that the skeleton will assign an edge from u to w for the cycle $\mathfrak{H}$ (where w is the clone of w^* in the same layer as u). Given that u is now incident to a vertex outside of C_v^l, we can no longer use the absorbing structure with u as a (left) tip (otherwise, we might disconnect T). To avoid this problem, we show that most vertices have many potential absorbing structures whose tips lie in the reservoir R (which T avoids). Here we make use of vertex degrees of H. A small number of *scant vertices* will not have high enough degree into R. For these vertices we fix an absorbing structure whose tips do not lie in R, and then alter T slightly so that these tips are deleted from T and reassigned to R. The fact that scant vertices are few and well spread out from each other will be crucial in being able to achieve this (see Lemma 7.20).

Let us now discuss two problems arising in the construction of the skeleton. Firstly, let $\mathcal{M}_C \subseteq \mathcal{Q}^n$ with $C \in \mathcal{C}$ be a cube molecule which is to be covered by $\mathfrak{H}$. Furthermore, suppose one of the clones C_v^l of C belongs to an absorbing structure for some vertex v. Let u be the tip of C_v^l and suppose that u has even parity. We would like to apply the connecting lemmas to cover $\mathcal{M}_C - \{u\}$ by paths which avoid u. But this would now involve covering one fewer vertex of even parity than of odd parity. This, in turn, has the effect of making the construction of the skeleton considerably more complicated (this construction is simplest when successive entry and exit vertices have opposite parities). To avoid this, we assign absorbing structures in pairs, so that, for each $C \in \mathcal{C}$, either two or no clones of C will be used in absorbing structures. In the case where two clones are used, we enforce that the tips of these clones will have opposite parities, and therefore each molecule $\mathcal{M}_C$ will have the same number of even and odd parity vertices to be covered by $\mathfrak{H}$. We use our robust matching lemma (see Lemma 5.2) to pair up the clones of absorbing structures in this way. To connect up different layers of a cube molecule, we will of course need to have suitable edges between these. Molecules which do not satisfy this requirement are called 'bondless' and are removed from Γ before the absorption process (so that their vertices are absorbed).

Secondly, another issue related to vertex parities arises from inner tree vertex molecules. Depending on the degree of an inner tree vertex $v \in V(T)$, the skeleton could contain an odd number of vertices from the molecule $\mathcal{M}_v$ consisting of all clones of v. All vertices in $\mathcal{M}_v$ outside the skeleton will need to be absorbed. But since the number of these vertices is odd, it would be impossible to pair up (in the way described above) the absorbing structures assigned to these vertices. To fix this issue, we effectively impose that $\mathfrak{H}$ will 'go around T twice'. That is, the skeleton will trace through every molecule beginning and finishing at the lowest ordered vertex in Γ^*. It will then retrace its steps through these molecules in an almost identical way, effectively doubling the size of the skeleton. This ensures that the skeleton contains an even number of vertices from each molecule, half of them of each parity.

Finally, once we have obtained an appropriate skeleton, we can construct a long cycle $\mathfrak{H}$ as described in Section 2.4. For every vertex in $\mathcal{Q}^n$ which is not covered by $\mathfrak{H}$ we have put in place an absorbing structure, which is covered by $\mathfrak{H}$ as described in (a) and (b). Thus, as discussed before, we can now use these structures to absorb all remaining vertices into $\mathfrak{H}$ to obtain a Hamilton cycle $\mathfrak{H}' \subseteq H \cup G$, thus proving the case $k = 1$ of Theorem 1.7.

2.6. **Hitting time for the appearance of a Hamilton cycle.** In order to prove Theorem 1.5, we consider $G \sim \mathcal{Q}^n_{1/2-\varepsilon}$. We show that a.a.s., for any graph H with $\delta(H) \geq 2$, the graph $G \cup H$ is Hamiltonian. The main additional difficulty faced here is that $G \cup H$ may contain vertices having degree as low as 2. For the set $\mathcal{U}$ of these vertices we cannot hope to use the previous absorption strategy: the neighbours of $v \in \mathcal{U}$ may not lie in cubes from $\mathcal{C}$. (In fact, v may not even have a neighbour within its own layer in $G \cup H$.) To handle such small degree vertices, we first prove that they will be few and well spread out (see Lemma 9.4). In Section 9.1 we define three types of new 'special absorbing structures'. The type of the special absorbing structure $SA(v)$ for v will depend on whether the neighbours a, b of v in H lie in the same layer as v. In each case, $SA(v)$ will consist of a short path P_1 containing the edges av and bv, and several other short paths designed to 'balance out' P_1 in a suitable way. (This is further discussed in

Section 10.1, see Figure 1). These paths will be incorporated into the long cycle $\mathfrak{H}$ described in Section 2.4. In particular, this allows us to 'absorb' the vertices of $\mathcal{U}$ into $\mathfrak{H}$. To incorporate the paths P_i forming $SA(v)$, we will proceed as follows.

Firstly, we make use of the fact that Theorem 7.1 allows us to choose our near-spanning tree T in such a way that it avoids a small ball around each $v \in \mathcal{U}$. Thus, (all clones of) T will avoid $SA(v)$, which has the advantage there will be no interference between T and the special absorbing structures. To link up each $SA(v)$ with the long cycle $\mathfrak{H}$, for each endpoint w of a path in $SA(v)$, we will choose an ℓ-cube in $I(G)$ which suitably intersects T and which contains w (or more precisely, the vertex in $I(G)$ corresponding to w). Altogether, these ℓ-cubes allow us to find paths between $SA(v)$ and vertices of $\mathfrak{H}$ which are clones of vertices in T. The remaining vertices in molecules consisting of clones of these ℓ-cubes will be covered in a similar way as in Section 2.4. All vertices in these balls around $\mathcal{U}$ which are not part of the special absorbing structures will be absorbed into $\mathfrak{H}$ via the same absorbing structures used in the proof of Theorem 1.7 to once again obtain a Hamilton cycle $\mathfrak{H}'$.

2.7. **Edge-disjoint Hamilton cycles.** The results on k edge-disjoint Hamilton cycles can be deduced from suitable versions (Theorems 8.1 and 9.6) of the case $k = 1$. Those versions are carefully formulated to allow us to repeatedly remove a Hamilton cycle from the original graph. We deduce Theorem 1.1 from Theorem 8.1 in Section 8.5, and deduce Theorem 1.5 from Theorem 9.6 in Section 9.4.

3. Notation

For $n \in \mathbb{Z}$, we denote $[n] \coloneqq \{k \in \mathbb{Z} : 1 \leq k \leq n\}$ and $[n]_0 \coloneqq \{k \in \mathbb{Z} : 0 \leq k \leq n\}$. Whenever we write a hierarchy of parameters, these are chosen from right to left. That is, whenever we claim that a result holds for $0 < a \ll b \leq 1$, we mean that there exists a non-decreasing function $f\colon [0,1) \to [0,1)$ such that the result holds for all $a > 0$ and all $b \leq 1$ with $a \leq f(b)$. We will not compute these functions explicitly. Hierarchies with more constants are defined in a similar way.

A *hypergraph* H is an ordered pair $H = (V(H), E(H))$ where $V(H)$ is called the vertex set and $E(H) \subseteq 2^{V(H)}$, the edge set, is a set of subsets of $V(H)$. If $E(H)$ is a multiset, we refer to H as a *multihypergraph*. We say that a (multi)hypergraph H is r*-uniform* if for every $e \in E(H)$ we have $|e| = r$. In particular, 2-uniform hypergraphs are simply called *graphs*. Given any set of vertices $V' \subseteq V(H)$, we denote the subhypergraph of H *induced* by V' as $H[V'] \coloneqq (V', E')$, where $E' \coloneqq \{e \in E(H) : e \subseteq V'\}$. We write $H - V' \coloneqq H[V \setminus V']$. Given any set $\hat{E} \subseteq E(H)$, we will sometimes write $V(\hat{E}) \coloneqq \{v \in V : \text{there exists } e \in \hat{E} \text{ such that } v \in e\}$.

Given any (multi)hypergraph H and any vertex $v \in V(H)$, let $E(H, v) \coloneqq \{e \in E(H) : v \in e\}$. We define the *neighbourhood* of v as $N_H(v) \coloneqq \bigcup_{e \in E(H,v)} e \setminus \{v\}$, and we define the *degree* of v by $d_H(v) \coloneqq |E(H, v)|$. We denote the minimum and maximum degrees of (the vertices in) H by $\delta(H)$ and $\Delta(H)$, respectively. Given any pair of vertices $u, v \in V(H)$, we define $E(H, u, v) \coloneqq \{e \in E(H) : \{u, v\} \subseteq e\}$. The *codegree* of u and v in H is given by $d_H(u, v) \coloneqq |E(H, u, v)|$. Given any set of vertices $W \subseteq V(H)$, we define $N_H(W) \coloneqq \bigcup_{w \in W} N_H(w)$. We denote $E(H, v, W) \coloneqq \{e \in E(H) : v \in e, e \setminus \{v\} \subseteq W\}$, $N_H(v, W) \coloneqq \bigcup_{e \in E(H,v,W)} e \setminus \{v\}$ and $d_H(v, W) \coloneqq |E(H, v, W)|$; we refer to the latter two as the neighbourhood and degree of v into W, respectively. Given $A, B \subseteq V(H)$ we denote $E_H(A, B) \coloneqq \{e \in E(H) : e \subseteq A \cup B, e \cap A \neq \varnothing, e \cap B \neq \varnothing\}$ and $e_H(A, B) \coloneqq |E_H(A, B)|$. Whenever $A = \{v\}$ is a singleton, we abuse notation and write $E_H(v, B)$ and $e_H(v, B)$. Thus, $e_H(v, B)$ and $d_H(v, B)$ may be used interchangeably.

Given any graph G and two vertices $u, v \in V(G)$, the *distance* $\mathrm{dist}_G(u, v)$ between u and v in G is defined as the length of the shortest path connecting u and v (and it is said to be infinite if there is no such path). Similarly, given any sets $A, B \subseteq V(G)$, the *distance* between A and B is given by $\mathrm{dist}_G(A, B) \coloneqq \min_{u \in A, v \in B} \mathrm{dist}_G(u, v)$. For any $r \in \mathbb{N}$, we denote $B^r_G(u) \coloneqq \{v \in V(G) : \mathrm{dist}_G(u, v) \leq r\}$ and $B^r_G(A) \coloneqq \{v \in V(G) : \mathrm{dist}_G(A, v) \leq r\}$; we refer to these sets as the *balls* of radius r around u and A, respectively.

A *directed graph* (or *digraph*) is a pair $D = (V(D), E(D))$, where $E(D)$ is a set of ordered pairs of elements of $V(D)$. If no pair of the form (v, v) with $v \in V(D)$ belongs to $E(D)$, we say that D is *loopless*. Given any $v \in V(D)$, we define its *inneighbourhood* as $N_D^-(v) \coloneqq \{u \in V(D) : (u,v) \in E(D)\}$, and its *outneighbourhood* as $N_D^+(v) \coloneqq \{u \in V(D) : (v,u) \in E(D)\}$. The *indegree* and *outdegree* of v are defined as $d_D^-(x) \coloneqq |N_D^-(x)|$ and $d_D^+(x) \coloneqq |N_D^+(x)|$, respectively. The minimum in- and outdegrees of (the vertices in) D are denoted by $\delta^-(D)$ and $\delta^+(D)$, respectively.

Given any multihypergraph or directed graph (V,E), a set $M \subseteq E$ is called a *matching* if its elements are pairwise disjoint. If the edges of M cover all of V, then it is said to be a *perfect matching*. Given an edge-colouring c of H, we say that a matching of H is *rainbow* if each of its edges has a different colour in c.

We often refer to the n-dimensional hypercube $\mathcal{Q}^n$ as an *n-cube* (the n is dropped whenever clear from the context). Given two vertices $v_1, v_2 \in V(\mathcal{Q}^n) = \{0,1\}^n$, we write $\mathrm{dist}(v_1,v_2)$ for the Hamming distance between v_1 and v_2. Thus, $\{v_1,v_2\} \in E(\mathcal{Q}^n)$ if and only if $\mathrm{dist}(v_1,v_2)=1$. Whenever the dimension n is clear from the context, we will use $\mathbf{0}$ to denote the vertex $\{0\}^n$. Given any $v \in \{0,1\}^n$, we will say that its *parity* is *even* if $\mathrm{dist}(v,\mathbf{0}) \equiv 0 \pmod 2$, and we will say that it is *odd* otherwise. This gives a natural partition of $V(\mathcal{Q}^n)$ into the sets of vertices with even and odd parities. Given any two vertices $v_1, v_2 \in \{0,1\}^n$, we will write $v_1 =_{\mathrm{p}} v_2$ if they have the same parity, and $v_1 \neq_{\mathrm{p}} v_2$ otherwise.

We will often consider the natural embedding of $V(\mathcal{Q}^n)$ into $\mathbb{F}_2^n$, which will allow us to use operations on the vertex set: whenever we write $v+u$, for some $u,v \in \{0,1\}^n$, we refer to their sum in $\mathbb{F}_2^n$. Given a vertex $v \in \{0,1\}^n$ and an edge $e = \{x,y\} \in E(\mathcal{Q}^n)$, we define $v+e$ to be the edge with endvertices $v+x$ and $v+y$. Given any two sets $A, B \subseteq \{0,1\}^n$, we will use the sumset notation $A+B \coloneqq \{a+b : a \in A, b \in B\}$, and we will abbreviate the k-fold sumset $A + \ldots + A$ by kA. Similarly, given any sets $A \subseteq \{0,1\}^n$ and $E \subseteq E(\mathcal{Q}^n)$, we write $A+E \coloneqq \{a+e : a \in A, e \in E\}$. Given a graph $G \subseteq \mathcal{Q}^n$ and a set of vertices $A \subseteq \{0,1\}^n$, $A+G$ will denote the graph with vertex set $A+V(G)$ and edge set $A+E(G)$. Note that this should never be confused with the notation $G-A$, which will be used exclusively to consider induced subgraphs of G. We will call the unitary vectors in $\mathbb{F}_2^n$ the *directions* of the hypercube. The set of directions will be denoted by $\mathcal{D}(\mathcal{Q}^n)$. Thus, $\mathcal{D}(\mathcal{Q}^n) = \{\hat{e} \in \{0,1\}^n : \mathrm{dist}(\hat{e},\mathbf{0}) = 1\}$. Note that two vertices $v_1, v_2 \in \{0,1\}^n$ are adjacent in $\mathcal{Q}^n$ if and only if there exists $\hat{e} \in \mathcal{D}(\mathcal{Q}^n)$ such that $v_1 = v_2 + \hat{e}$. Given any vertex $v \in \{0,1\}^n$ and any set $\mathcal{D} \subseteq \mathcal{D}(\mathcal{Q}^n)$, we will denote by $\mathcal{Q}^n(v,\mathcal{D}) \coloneqq \mathcal{Q}^n[v + n(\mathcal{D} \cup \{\mathbf{0}\})]$ the subcube of $\mathcal{Q}^n$ which contains v and all vertices in $\{0,1\}^n$ which can be reached from v by only adding directions in $\mathcal{D}$. Given any subcube $Q \subseteq \mathcal{Q}^n$, we will write $\mathcal{D}(Q)$ to denote the subset of $\mathcal{D}(\mathcal{Q}^n)$ such that, for any $v \in V(Q)$, we have $Q = \mathcal{Q}^n(v, \mathcal{D}(Q))$. Given any direction $\hat{e} \in \mathcal{D}(Q)$, we will sometimes informally say that Q *uses* $\hat{e}$. Given two vertices $v_1, v_2 \in \{0,1\}^n$, their *differing directions* are all directions in $\mathcal{D}(v_1,v_2) \coloneqq \{\hat{e} \in \mathcal{D}(\mathcal{Q}^n) : \mathrm{dist}(v_1+\hat{e}, v_2) < \mathrm{dist}(v_1,v_2)\}$. Observe that, if $\mathrm{dist}(v_1,v_2) = d$, then $|\mathcal{D}(v_1,v_2)| = d$ and $\mathcal{Q}^n(v_1, \mathcal{D}(v_1,v_2))$ is the smallest subcube of $\mathcal{Q}^n$ which contains both v_1 and v_2.

When considering random experiments for a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with $|V(G_n)|$ tending to infinity with n, we say that an event $\mathcal{E}$ holds *asymptotically almost surely* (*a.a.s.*) for G_n if $\mathbb{P}[\mathcal{E}] = 1 - o(1)$. When considering asymptotic statements, we will ignore rounding whenever this does not affect the argument.

4. Probabilistic tools

Here we list some probabilistic tools that we will use throughout the paper. The following can be proved easily with the Cauchy-Schwarz inequality.

Proposition 4.1. *Given a non-negative random variable X with finite support, we have that*

$$\mathbb{P}[X=0] \leq 1 - \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

Throughout the paper, we will be interested in proving concentration results for different random variables. We will often need the following Chernoff bound (see e.g. [36, Corollary 2.3]).

Lemma 4.2. *Let X be the sum of n mutually independent Bernoulli random variables and let $\mu := \mathbb{E}[X]$. Then, for all $0 < \delta < 1$ we have that $\mathbb{P}[X \geq (1+\delta)\mu] \leq e^{-\delta^2\mu/3}$ and $\mathbb{P}[X \leq (1-\delta)\mu] \leq e^{-\delta^2\mu/2}$. In particular, $\mathbb{P}[|X-\mu| \geq \delta\mu] \leq 2e^{-\delta^2\mu/3}$.*

Similar bounds hold for hypergeometric distributions (see e.g. [36, Theorem 2.10]). For $m, n, N \in \mathbb{N}$ with $m, n < N$, a random variable X is said to follow the hypergeometric distribution with parameters N, n and m if it can be defined as $X := |S \cap [m]|$, where S is a uniformly chosen random subset of $[N]$ of size n.

Lemma 4.3. *Suppose Y has a hypergeometric distribution with parameters N, n and m. Then, $\mathbb{P}[|Y - \mathbb{E}[Y]| \geq t] \leq 2e^{-t^2/(3n)}$.*

The following bound will also be used repeatedly (see e.g. [3, Theorem A.1.12]).

Lemma 4.4. *Let X be the sum of n mutually independent Bernoulli random variables. Let $\mu := \mathbb{E}[X]$, and let $\beta > 1$. Then, $\mathbb{P}[X \geq \beta\mu] \leq (e/\beta)^{\beta\mu}$. In particular, we have $\mathbb{P}[X \geq 7\mu] \leq e^{-\mu}$.*

Given any sequence of random variables $X = (X_1, \ldots, X_n)$ taking values in a set Ω and a function $f\colon \Omega^n \to \mathbb{R}$, for each $i \in [n]_0$ define $Y_i := \mathbb{E}[f(X) \mid X_1, \ldots, X_i]$. The sequence $Y_0, \ldots, Y_n$ is called the *Doob martingale* for f and X. All the martingales that appear in this paper will be of this form. To deal with them, we will need the following version of the well-known Azuma-Hoeffding inequality.

Lemma 4.5 (Azuma's inequality [5, 34])**.** *Let $X_0, X_1, \ldots$ be a martingale and suppose that $|X_i - X_{i-1}| \leq c_i$ for all $i \in \mathbb{N}$. Then, for any $n, t \in \mathbb{N}$,*

$$\mathbb{P}[|X_n - X_0| \geq t] \leq 2\exp\left(\frac{-t^2}{2\sum_{i=1}^n c_i^2}\right).$$

The following lemma, which concerns further bounds for martingales, is due to Alon, Kim and Spencer [2] (see also [3, Theorem 7.4.3]). Here, we describe a version which is tailored to our purposes. Let $r \in \mathbb{N}$ and let $\mathcal{H}$ be an r-uniform hypergraph. Let $\mathcal{H}' \subseteq \mathcal{H}$ be a random subgraph chosen according to any distribution for which the inclusion of edges are mutually independent. Let X be a random variable whose value is determined by the presence or absence of the edges of some collection $E' = \{e_1, \ldots, e_k\} \subseteq E(\mathcal{H})$ in $\mathcal{H}'$. Let p_i be the probability that e_i is present in $\mathcal{H}'$. Let c_i be the maximum value X could change, for some given choice of $\mathcal{H}'$, by changing the presence or absence of e_i. Let $C := \max_{i\in[k]} c_i$ and $\sigma^2 := \sum_{i\in[k]} p_i(1-p_i)c_i^2$.

Lemma 4.6 (Alon, Kim and Spencer [2])**.** *For all $\alpha > 0$ with $\alpha C < 2\sigma$ we have that*

$$\mathbb{P}[|X - \mathbb{E}[X]| > \alpha\sigma] \leq 2e^{-\alpha^2/4}.$$

We will also need the following special case of Talagrand's inequality (see e.g. [3, Theorem 7.7.1]). Let $\Omega := \prod_{i=1}^n \Omega_i$, where each Ω_i is a probability space. We say that $f\colon \Omega \to \mathbb{R}$ is *K-Lipschitz*, for some $K \in \mathbb{R}$, if for every $x, y \in \Omega$ which differ only on one coordinate we have $|f(x) - f(y)| \leq K$. We say that f is *h-certifiable*, for some $h\colon \mathbb{N} \to \mathbb{N}$, if, for every $x \in \Omega$ and $s \in \mathbb{R}$, whenever $f(x) \geq s$, there exists $I \subseteq [n]$ with $|I| \leq h(s)$ such that every $y \in \Omega$ that agrees with x on the coordinates in I satisfies $f(y) \geq s$.

Lemma 4.7 (Talagrand's inequality)**.** *Let $\Omega := \prod_{i=1}^n \Omega_i$, where each Ω_i is a probability space. Let $X\colon \Omega \to \mathbb{N}$ be K-Lipschitz and h-certifiable, for some $K \in \mathbb{N}$ and $h\colon \mathbb{N} \to \mathbb{N}$. Then, for all $b, t \in \mathbb{R}$,*

$$\mathbb{P}\left[X \leq b - tK\sqrt{h(b)}\right]\mathbb{P}[X \geq b] \leq \exp\left(\frac{-t^2}{4}\right).$$

Finally, the Lovász local lemma will come in useful. Let $\mathfrak{E} := \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$ be a collection of events. A *dependency graph* for $\mathfrak{E}$ is a graph H on vertex set $[m]$ such that, for all $i \in [m]$, $\mathcal{E}_i$ is mutually independent of $\{\mathcal{E}_j : j \neq i, j \notin N_H(i)\}$, that is, if $\mathbb{P}[\mathcal{E}_i] = \mathbb{P}[\mathcal{E}_i \mid \bigwedge_{j\in J} \mathcal{E}_j]$ for all $J \subseteq [m] \setminus (N_H(i) \cup \{i\})$. We will use the following version of the local lemma (it follows e.g. from [3, Lemma 5.1.1]).

Lemma 4.8 (Lovász local lemma)**.** *Let $\mathfrak{E} := \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$ be a collection of events and let H be a dependency graph for $\mathfrak{E}$. Suppose that $\Delta(H) \leq d$ and $\mathbb{P}[\mathcal{E}_i] \leq p$ for all $i \in [m]$. If $ep(d+1) \leq 1$, then*

$$\mathbb{P}\left[\bigwedge_{i=1}^{m} \overline{\mathcal{E}_i}\right] \geq (1-ep)^m.$$

5. Auxiliary results

5.1. **Results about matchings.** We will need three auxiliary results to help us find suitable absorbing cube pairs for different vertices. We will need to preserve the alternating parities of vertices that are absorbed by each molecule. The first lemma (Lemma 5.2) presented in this section, as well as its corollary, will help us to show that all vertices can be paired up in such a way that these parities can be preserved. The second lemma (Lemma 5.4) will be used to show that, for each such pair of vertices, there are many possible pairs of absorption cubes. Finally, the third lemma (Lemma 5.5) will allow us to assign one of those pairs of absorption cubes to each pair of vertices we need to absorb in such a way that these cube pairs are pairwise vertex disjoint.

To prove Lemma 5.2, as well as Lemma 7.16 and Theorem 7.19, the following consequence of Hall's theorem will be useful.

Lemma 5.1. *Let G be a bipartite graph with vertex partition $A \dot\cup B$. Assume that there is some integer $\ell \geq 0$ such that, for all $S \subseteq A$, we have $|N(S)| \geq |S| - \ell$. Then, G contains a matching which covers all but at most ℓ vertices in A.*

Given any graph G and a bipartition $(\mathfrak{A}, \mathfrak{B})$ of $V(G)$, we say that $(\mathfrak{A}, \mathfrak{B})$ is an *r-balanced bipartition* if $||\mathfrak{A}| - |\mathfrak{B}|| \leq r$. Let G be a graph on n vertices, and let $r, d \in \mathbb{N}$ with $r \leq d$. We say that G is *d-robust-parity-matchable* with respect to an r-balanced bipartition $(\mathfrak{A}, \mathfrak{B})$ if, for every $S \subseteq V(G)$ such that $|S| \leq d$ and $|\mathfrak{A} \setminus S| = |\mathfrak{B} \setminus S|$, the graph $G - S$ contains a perfect matching M with the property that every edge $e \in M$ has one endpoint in $\mathfrak{A} \setminus S$ and one endpoint in $\mathfrak{B} \setminus S$.

Given two disjoint sets of vertices A and B, the binomial random bipartite graph $G(A, B, p)$ is obtained by adding each possible edge with one endpoint in A and the other in B with probability p independently of every other edge. Given any two bipartite graphs on the same vertex set, $G_1 = (A, B, E_1)$ and $G_2 = (A, B, E_2)$, and any $\alpha \in \mathbb{R}$, we define $\Gamma^{\alpha}_{G_1,G_2}(A)$ as the graph with vertex set A where any two vertices $x, y \in A$ are joined by an edge whenever $|N_{G_1}(x) \cap N_{G_2}(y)| \geq \alpha|B|$ or $|N_{G_1}(y) \cap N_{G_2}(x)| \geq \alpha|B|$.

Lemma 5.2. *Let $d, k, r \in \mathbb{N}$ and $\alpha, \varepsilon, \beta > 0$ be such that $r \leq d$, $1/k \ll 1/d, \varepsilon, \alpha$ and $\beta \ll \varepsilon, \alpha$. Then, any bipartite graph $G = G(A, B, E)$ with $|B| = n \geq |A| \geq k$ such that $d_G(x) \geq \alpha n$ for every $x \in A$ satisfies the following with probability at least $1 - 2^{-10n}$: for any r-balanced bipartition of A into $(\mathfrak{A}, \mathfrak{B})$, the graph $\Gamma^{\beta}_{G,G(A,B,\varepsilon)}(A)$ is d-robust-parity-matchable with respect to $(\mathfrak{A}, \mathfrak{B})$.*

Proof. Let $\Gamma := \Gamma^{\beta}_{G,G(A,B,\varepsilon)}(A)$. Let Γ' be the auxiliary digraph with vertex set A where, for any pair of vertices $x, y \in A$, there is a directed edge from x to y if $|N_G(x) \cap N_{G(A,B,\varepsilon)}(y)| \geq \beta n$. Observe that the graph obtained from Γ' by ignoring the directions of its edges and identifying the possible multiple edges is exactly Γ, which means that $\delta(\Gamma) \geq \delta^+(\Gamma')$.

Given any two vertices $x, y \in A$, by Lemma 4.2 we have that

$$\mathbb{P}[(x, y) \notin E(\Gamma')] = \mathbb{P}[|N_G(x) \cap N_{G(A,B,\varepsilon)}(y)| < \beta n] \leq e^{-\varepsilon\alpha n/3}.$$

Furthermore, for a fixed $x \in A$, observe that the events that $(x, y) \notin E(\Gamma')$, for all $y \in A \setminus \{x\}$, are mutually independent. Therefore, $d^+_{\Gamma'}(x)$ is a sum of independent Bernoulli random variables. Let $m := |A|$. If $d^+_{\Gamma'}(x) < 4m/5$, that means that there is a set of $m/5$ vertices $Y \subseteq A \setminus \{x\}$ such that $(x, y) \notin E(\Gamma')$ for all $y \in Y$. We then conclude that

$$\mathbb{P}[d^+_{\Gamma'}(x) < 4m/5] \leq \sum_{Y \in \binom{A\setminus\{x\}}{m/5}} \mathbb{P}[(x, y) \notin E(\Gamma') \text{ for all } y \in Y] \leq \binom{m}{m/5} e^{-\varepsilon\alpha n m/15} \leq 2^{-20n}.$$

By a union bound over the choice of x, we conclude that

$$\mathbb{P}[\delta(\Gamma) < 4m/5] \leq \mathbb{P}[\delta^+(\Gamma') < 4m/5] \leq m2^{-20n} \leq 2^{-10n}.$$

Now, condition on the event that the previous holds. Fix any r-balanced bipartition $(\mathfrak{A}, \mathfrak{B})$ of A and let $\Gamma_{(\mathfrak{A},\mathfrak{B})}$ be the bipartite subgraph of Γ induced by this bipartition. Fix any set $S \subseteq A$ with $|S| \leq d$ and $|\mathfrak{A} \setminus S| = |\mathfrak{B} \setminus S|$. We have that $\delta(\Gamma_{(\mathfrak{A},\mathfrak{B})} - S) \geq 4m/5 - m/2 - d - r > m/4$. Therefore, by Lemma 5.1, $\Gamma_{(\mathfrak{A},\mathfrak{B})} - S$ contains a perfect matching. □

While Lemma 5.2 will be used in the proof of Theorem 8.1 in Section 8, we will instead need to use the following Corollary 5.3 in the proof of Theorem 9.6 in Section 9. Let G be a graph on $2n$ vertices. Let $(\mathfrak{A}, \mathfrak{B})$ be a balanced bipartition of $V(G)$ and let $(V_1, \ldots, V_k)$, for some $k \in \mathbb{N}$, be a partition of $V(G)$. Given any $d \in \mathbb{N}$, we say that G is d-robust-parity-matchable with respect to $(\mathfrak{A}, \mathfrak{B})$ *clustered in* $(V_1, \ldots, V_k)$ if, for every $S \subseteq V(G)$ with $|S| \leq d$ and $|S \cap \mathfrak{A}| = |S \cap \mathfrak{B}|$, the graph $G - S$ contains a perfect matching M such that every edge $e \in M$ has one endpoint in $\mathfrak{A} \setminus S$ and one endpoint in $\mathfrak{B} \setminus S$ and, for every $e = \{x, y\} \in M$, if $x \in V_j$ then $y \in V_{j-1} \cup V_j \cup V_{j+1}$ (where we take indices cyclically).

Corollary 5.3. *Let $d, k, t \in \mathbb{N}$ and $\alpha, \varepsilon, \beta > 0$ be such that $1/k \ll 1/d, \varepsilon, \alpha$ and $\beta \ll \varepsilon, \alpha$. Let $G = G(A, B, E)$ be a bipartite graph and $(A_1, \ldots, A_t)$ be a partition of A such that*

- *$|B| = n \geq |A|$,*
- *for every $i \in [t]$, we have that $|A_i| \geq k$ is even,*
- *for every $x \in A$, we have $d_G(x) \geq \alpha n$.*

Then, the following holds with probability at least $1 - 2^{-9n}$: for each $i \in [t]$ and for any balanced bipartition of A_i into $(\mathfrak{A}_i, \mathfrak{B}_i)$, the graph $\Gamma^{\beta}_{G,G(A,B,\varepsilon)}(A)$ is d-robust-parity-matchable with respect to $(\bigcup_{i=1}^t \mathfrak{A}_i, \bigcup_{i=1}^t \mathfrak{B}_i)$ clustered in $(A_1, \ldots, A_t)$.

Proof. Given any set $C \subseteq A$, for each $i \in [t]$, let $C_i := C \cap A_i$. Given any bipartition $(\mathfrak{A}, \mathfrak{B})$ of A, we write $C^{\mathfrak{A}} := C \cap \mathfrak{A}$ and $C^{\mathfrak{B}} := C \cap \mathfrak{B}$. Throughout this proof, we consider the indices in $[t]$ to be taken cyclically.

For each set $D \subseteq A$ with $|D| \leq d$, and for each $i \in [t]$, we apply Lemma 5.2 to the graph $G[D_i \cup A_{i-1}, B]$, with $2d$, k, d, α, ε and β playing the roles of d, k, r, α, ε and β, respectively. Then, by a union bound over all choices of D and all choices of $i \in [t]$, the following holds with probability at least $1 - 2^{-9n}$. For each $i \in [t]$, consider any balanced bipartition $(\mathfrak{A}_i, \mathfrak{B}_i)$ of A_i. Consider any $D \subseteq A$ with $|D_i| \leq d$ for each $i \in [t]$. Then, for each $i \in [t]$, the graph $\Gamma^{\beta}_{G[D_i \cup A_{i-1}, B], G(D_i \cup A_{i-1}, B, \varepsilon)}(D_i \cup A_{i-1})$ is $2d$-robust-parity-matchable with respect to $((\mathfrak{A}_i \cap D) \cup \mathfrak{A}_{i-1}, (\mathfrak{B}_i \cap D) \cup \mathfrak{B}_{i-1})$. Condition on the event that the above holds.

Now, for each $i \in [t]$, fix a balanced bipartition $(\mathfrak{A}_i, \mathfrak{B}_i)$ of A_i. Let $\mathfrak{A} := \bigcup_{i=1}^t \mathfrak{A}_i$ and $\mathfrak{B} := \bigcup_{i=1}^t \mathfrak{B}_i$. Let $S \subseteq A$ be a subset of size $|S| \leq d$ such that $|S^{\mathfrak{A}}| = |S^{\mathfrak{B}}|$. We want to show that $\Gamma^{\beta}_{G,G(A,B,\varepsilon)}(A) - S$ contains a perfect matching M such that every edge $e \in M$ has one endpoint in $\mathfrak{A} \setminus S$ and one endpoint in $\mathfrak{B} \setminus S$ and, for every $e = \{x, y\} \in M$, if $x \in A_j$ then $y \in A_{j-1} \cup A_j \cup A_{j+1}$. We begin by proving the following claim.

Claim 5.1. *There exists a set $D \subseteq A \setminus S$ satisfying the following properties:*

(RM1) *for every $i \in [t]$ we have $|D_i| \leq d$, and*
(RM2) *for every $i \in [t]$ we have $|D^{\mathfrak{A}}_{i+1} \cup D^{\mathfrak{B}}_i \cup S^{\mathfrak{B}}_i| = |D^{\mathfrak{B}}_{i+1} \cup D^{\mathfrak{A}}_i \cup S^{\mathfrak{A}}_i|$.*

Proof. We will construct one such set D by constructing the sets $D_i \subseteq A_i$ inductively. We will argue by induction on $i \in [t]$ in decreasing order. Let $D_t := \varnothing$. Now, suppose that, for some

$i \in [t-1]$, we have already constructed the sets $D_j \subseteq A_j$ for all $j \in [t] \setminus [i]$. Then, let $D_i \subseteq A_i \setminus S_i$ be a smallest set such that

$$|D^{\mathfrak{A}}_{i+1} \cup D^{\mathfrak{B}}_i \cup S^{\mathfrak{B}}_i| = |D^{\mathfrak{B}}_{i+1} \cup D^{\mathfrak{A}}_i \cup S^{\mathfrak{A}}_i|. \tag{5.1}$$

Observe that either $D_i = D^{\mathfrak{A}}_i$ or $D_i = D^{\mathfrak{B}}_i$. Furthermore, observe that $||D^{\mathfrak{A}}_{i+1} \cup S^{\mathfrak{B}}_i| - |D^{\mathfrak{B}}_{i+1} \cup S^{\mathfrak{A}}_i|| \leq |D_{i+1} \cup S_i|$. Therefore, there exists a set $D_i \subseteq A_i$ as required with $|D_i| \leq |D_{i+1} \cup S_i|$.

In order to prove that this results in a set D which satisfies the required properties, consider the following. First, by following the induction above, we have that $|D_t| = 0$ and $|D_i| \leq |D_{i+1}| + |S_i|$, hence $|D_i| \leq \sum_{j=1}^{t} |S_j| = |S| \leq d$ for all $i \in [t]$, thus (RM1) holds. On the other hand, (RM2) holds by (5.1) for all $i \in [t-1]$, so we must prove that it also holds for $i = t$. But this follows by summing (5.1) over all $i \in [t-1]$, and using the fact that $|S^{\mathfrak{A}}| = |S^{\mathfrak{B}}|$. ◀

Let D be the set given by Claim 5.1. Now, for each $i \in [t]$, let $J_i := D_i \cup S_i$. By Claim 5.1 (RM1) we have that $|J_i| \leq 2d$. Furthermore, by Claim 5.1 (RM2) it follows that $|(\mathfrak{A}_i \cup D^{\mathfrak{A}}_{i+1}) \setminus J_i| = |(\mathfrak{B}_i \cup D^{\mathfrak{B}}_{i+1}) \setminus J_i|$. By the conditioning above, this means that $\Gamma^{\beta}_{G[D_{i+1} \cup A_i, B], G(D_{i+1} \cup A_i, B, \varepsilon)}(D_{i+1} \cup A_i) - J_i$ contains a perfect matching M_i such that every edge of M_i has one endpoint in $(\mathfrak{A}_i \cup D^{\mathfrak{A}}_{i+1}) \setminus J_i$ and one endpoint in $(\mathfrak{B}_i \cup D^{\mathfrak{B}}_{i+1}) \setminus J_i$. Finally, let $M := \bigcup_{i=1}^{t} M_i$. It is clear that M satisfies the required conditions. The statement follows. □

The second lemma will be stated in terms of directed graphs.

Lemma 5.4. *Let $c, C > 0$ and let $\alpha \in (0, 1/(1 + c/C))$. Let D be a loopless n-vertex digraph such that*

(i) *for every $A \subseteq V(D)$ with $|A| \geq \alpha n$ we have $\sum_{v \in A} d^-(v) \geq c\alpha n$, and*
(ii) *for every $B \subseteq V(D)$ with $|B| \leq c\alpha n/C$ we have $\sum_{v \in B} d^+(v) \leq c\alpha n$.*

Then, D contains a matching M with $|M| > c\alpha n/(2C)$.

Proof. Assume for a contradiction that the largest matching M in D has size $|M| \leq c\alpha n/(2C)$. Since $\alpha < 1/(1 + c/C)$, there exists a set $A \subseteq V(D) \setminus V(M)$ with $|A| \geq \alpha n$, and thus, by (i), $\sum_{v \in A} d^-(v) \geq c\alpha n$. Since M is the largest matching, all edges that enter A must come from vertices of M (otherwise, we could add one such edge to M, finding a larger matching). However, by (ii), the number of edges going out of $V(M)$ is less than $c\alpha n$, a contradiction. □

For convenience, we state the third lemma in terms of rainbow matchings in hypergraphs.

Lemma 5.5. *Let $n, r \in \mathbb{N}$ and let $\mathcal{H}$ be an n-edge-coloured r-uniform multihypergraph. Then, for any $m \geq 10$, the following holds. Suppose $\mathcal{H}$ satisfies the following two properties:*

(i) *For every $i \in [n]$, there are at least m edges of colour i.*
(ii) *$\Delta(\mathcal{H}) \leq m/(6r)$.*

Then, there exists a rainbow matching of size n.

Proof. The idea is to pick a random edge from each colour class and prove that with non-zero probability this results in a rainbow matching. First, for each $i \in [n]$, let M_i be a set of m edges of colour i. We choose an edge from each M_i uniformly at random, independently of the other choices. For any $i, j \in [n]$ with $i \neq j$ and for any two edges $e \in M_i$ and $e' \in M_j$ for which $e \cap e' \neq \varnothing$, we denote by $A_{e,e'}$ the event that both e and e' are picked. We observe that

$$\mathbb{P}[A_{e,e'}] = \left(\frac{1}{m}\right)^2.$$

Moreover, note that every event $A_{e,e'}$ is independent of all other events $A_{f,f'}$ but at most $2m \cdot r \cdot \Delta(\mathcal{H}) \leq m_2/3$. Indeed, this holds because $A_{e,e'}$ can only depend on those events which involve at least one edge from either colour i or colour j. Applying now Lemma 4.8, we deduce that with non-zero probability no event $A_{e,e'}$ occurs, as required. □

5.2. **Properties of random subgraphs of the hypercube.** In this section we state and prove some basic properties of random subgraphs of the hypercube. The first one guarantees that the degrees of all vertices are linear in the dimension.

Lemma 5.6. *Let $0 < \delta \ll \varepsilon \leq 1/2$. Then, we a.a.s. have that $\delta(\mathcal{Q}^n_{1/2+\varepsilon}) \geq \delta n$.*

Proof. Let $p := 1/2 + \varepsilon$. Fix any $v \in \{0,1\}^n$. Throughout this proof, we write $d(v)$ to refer to the degree of v in $\mathcal{Q}^n_p$.

Note that $d(v)$ follows a binomial distribution with parameters n and p. Since $\delta < 1/2$, it follows that

$$\mathbb{P}[d(v) \leq \delta n] \leq \delta n \binom{n}{\delta n} p^{\delta n}(1-p)^{(1-\delta)n}.$$

Using the Stirling formula, we conclude that

$$\mathbb{P}[d(v) \leq \delta n] \leq (1 + \mathcal{O}(n^{-1}))\sqrt{\frac{\delta n}{2\pi(1-\delta)}}\left(\left(\frac{p}{\delta}\right)^{\delta}\left(\frac{1-p}{1-\delta}\right)^{1-\delta}\right)^{n}.$$

By the union bound, it now suffices to show that

$$\left(\frac{p}{\delta}\right)^{\delta}\left(\frac{1-p}{1-\delta}\right)^{1-\delta} = \left(\frac{1+2\varepsilon}{2\delta}\right)^{\delta}\left(\frac{1-2\varepsilon}{2(1-\delta)}\right)^{1-\delta} < \frac{1}{2},$$

but this follows since $\delta \ll \varepsilon$. $\square$

Furthermore, it will be important to show that the number of vertices whose degree deviates from the expected degree is small.

Lemma 5.7. *Let $\varepsilon \in (0,1)$ and $a \in (1/2, 1)$. Let X be the number of vertices $v \in \{0,1\}^n$ for which $d_{\mathcal{Q}^n_\varepsilon}(v) \neq \varepsilon n \pm n^a$. Then,*

$$\mathbb{P}[X \geq e^{-n^{2a-1}/(6\varepsilon)}2^n] \leq 2e^{-n^{2a-1}/(6\varepsilon)}.$$

Proof. Throughout this proof, we use $d(v)$ for $d_{\mathcal{Q}^n_\varepsilon}(v)$. Note that $d(v)$ follows a binomial distribution with parameters n and ε, so $\mathbb{E}[d(v)] = \varepsilon n$ and, by Lemma 4.2,

$$\mathbb{P}[d(v) \neq \varepsilon n \pm n^a] \leq 2e^{-n^{2a-1}/(3\varepsilon)}.$$

We then have that $\mathbb{E}[X] \leq 2^{n+1}e^{-n^{2a-1}/(3\varepsilon)}$, and the statement follows by Markov's inequality. $\square$

Remark 5.8. *In particular, for any $a \in (1/2, 1)$ we have that a.a.s. the number of vertices whose degree deviates from the expectation by more than n^a is at most $e^{-n^{2a-1}/(6\varepsilon)}2^n$.*

It will be important to show that a.a.s. there are not too many of the above vertex type 'close' to any given vertex.

Lemma 5.9. *Let $\varepsilon \in (0,1)$, $a \in (2/3, 1)$ and $\ell \in \mathbb{N}$. Then, for any $b > 2 - 2a$, a.a.s. there are no vertices $v \in \{0,1\}^n$ for which $|\{u \in B^\ell(v) : d_{\mathcal{Q}^n_\varepsilon}(u) \neq \varepsilon n \pm n^a\}| \geq n^b$.*

Proof. First, note that we may assume that $b < a$ (otherwise, choose a value $b' \in (2-2a, a)$ and prove the statement for this value, which in turn implies the result for b). Throughout this proof, we write $d(v)$ for $d_{\mathcal{Q}^n_\varepsilon}(v)$. Fix any vertex $v \in \{0,1\}^n$. Let $X(v) := |\{u \in B^\ell(v) : d(u) \neq \varepsilon n \pm n^a\}|$.

If $X(v) \geq n^b$, there exists a set $A \subseteq B^\ell(v)$ of size $|A| = n^b$ such that $d(u) \neq \varepsilon n \pm n^a$ for all $u \in A$. We call such a set A *bad.* We now bound the probability that such a bad set exists. Given any set $A \in \binom{B^\ell(v)}{n^b}$, for each $u \in A$ let $d^A(u) := |N_{\mathcal{Q}^n_\varepsilon}(u) \setminus A|$. Observe that $d^A(u) = d(u) \pm n^b$ and, since $b < a$, for any $u \in A$ we have that, if $d(u) \neq \varepsilon n \pm n^a$, then $d^A(u) \neq \varepsilon n \pm n^a/2$.

Fix a set $A \in \binom{B^\ell(v)}{n^b}$. Observe that $\mathbb{E}[d^A(u)] \in [\varepsilon n(1 - n^{b-1}), \varepsilon n]$ for all $u \in A$. Furthermore, the variables $\{d^A(u) : u \in A\}$ are mutually independent, and each of them follows a binomial distribution. By Lemma 4.2, for each $u \in A$ we have that

$$\mathbb{P}[d(u) \neq \varepsilon n \pm n^a] \leq \mathbb{P}[d^A(u) \neq \varepsilon n \pm n^a/2] \leq 2e^{-n^{2a-1}/(40\varepsilon)}.$$

Since the variables $d^A(u)$ are mutually independent, it follows that

$$\mathbb{P}[A \text{ is bad}] \leq \left(2e^{-n^{2a-1}/(40\varepsilon)}\right)^{n^b}.$$

Now consider a union bound over all possible choices of A and all choices of v. It suffices to prove that

$$\binom{\ell n^\ell}{n^b}\left(2e^{-n^{2a-1}/(40\varepsilon)}\right)^{n^b} < 2.1^{-n}.$$

Since $\binom{\ell n^\ell}{n^b} \leq (e\ell n^{\ell-b})^{n^b}$, it suffices to show that

$$n^b(1+\ln 2+\ln \ell+(\ell-b)\ln n - n^{2a-1}/(40\varepsilon)) < -n\ln 2.1.$$

But this follows for n sufficiently large, from the fact that $b > 2-2a$. □

Next we show that, in any ball of radius ℓ, the number of vertices whose degree is far from the expected is much smaller (at most a constant) if we allow larger deviations for the degrees. Even more, we can prove a similar statement if we restrict the degrees to some linear subsets of the total neighbourhood in $\mathcal{Q}^n$. Recall that, for any vertex $v \in \{0,1\}^n$, any graph $G \subseteq \mathcal{Q}^n$ and a set $S \subseteq N_{\mathcal{Q}^n}(v)$, we denote $d_G(v,S) = |N_G(v) \cap S|$.

Lemma 5.10. *Let $\varepsilon, \delta, \gamma \in (0,1)$ and $\ell \in \mathbb{N}$. For each $v \in \{0,1\}^n$, let $S(v) \subseteq N_{\mathcal{Q}^n}(v)$ satisfy $|S(v)| \geq \gamma n$. Let $\mathcal{E}$ be the event that there are no vertices $v \in \{0,1\}^n$ for which $|\{u \in B^\ell(v) : d_{\mathcal{Q}^n_\varepsilon}(u, S(u)) \neq (1\pm\delta)\varepsilon|S(u)|\}| \geq 100/(\delta^2\varepsilon\gamma)$. Then, for n sufficiently large, $\mathbb{P}[\mathcal{E}] \geq 1-e^{-4n}$.*

Proof. Throughout this proof, we write $d(v)$ for $d_{\mathcal{Q}^n_\varepsilon}(v)$ and $d(v,S)$ for $d_{\mathcal{Q}^n_\varepsilon}(v,S)$, for any set S.

Let $C := \lceil 100/(\delta^2\varepsilon\gamma)\rceil$. Fix any vertex $v \in \{0,1\}^n$ and $A \in \binom{B^\ell(v)}{C}$. Observe that for any $u \in A$, if $d(u,S(u)) \neq (1\pm\delta)\varepsilon|S(u)|$, then $d(u, S(u)\setminus A) \neq (1\pm\delta/2)\varepsilon|S(u)|$. Observe that $\mathbb{E}[d(u,S(u)\setminus A)] \in [\varepsilon(|S(u)|-C), \varepsilon|S(u)|]$ for all $u \in A$. Furthermore, the variables $\{d(u,S(u)\setminus A) : u \in A\}$ are mutually independent, and each of them follows a binomial distribution. By Lemma 4.2, for each $u \in A$ we have that, for n sufficiently large,

$$\mathbb{P}[d(u) \neq (1\pm\delta)\varepsilon|S(u)|] \leq \mathbb{P}[d(u,S(u)\setminus A) \neq (1\pm\delta/2)\varepsilon|S(u)|] \leq 2e^{-\delta^2\varepsilon\gamma n/19} \leq e^{-\delta^2\varepsilon\gamma n/20}.$$

We say that A is bad if $d(u,S(u)) \neq (1\pm\delta)\varepsilon|S(u)|$ for all $u \in A$. Since the variables $d(u,S(u)\setminus A)$ are mutually independent, it follows that

$$\mathbb{P}[A \text{ is bad}] \leq \left(e^{-\delta^2\varepsilon\gamma n/20}\right)^C \leq e^{-5n}.$$

Observe that $\mathcal{E}$ holds if there are no bad sets A. By a union bound over all choices of v and all choices of A, it follows that

$$\mathbb{P}[\overline{\mathcal{E}}] \leq 2^n\binom{\ell n^\ell}{C}e^{-5n} \leq e^{-4n}.$$ □

In more generality than Lemmas 5.7 and 5.10, we will need to use the fact that in $\mathcal{Q}^n_\varepsilon$ the directions in which the neighbours of a vertex lie are not correlated too much between vertices. Given a graph $G \subseteq \mathcal{Q}^n$, for any set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ and any vertex $x \in \{0,1\}^n$, we denote $N_{G,S}(x) := \{x+\hat{e} : \hat{e} \in S, \{x, x+\hat{e}\} \in E(G)\}$ and $d_{G,S}(x) := |N_{G,S}(x)|$. Similarly, for any $y \in \{0,1\}^n$ such that $\mathrm{dist}(x,y) = 1$, we denote $N_{G,S,x}(y) := N_{G,S}(y) \setminus \{x\}$ and $d_{G,S,x}(y) := |N_{G,S,x}(y)|$.

Lemma 5.11. *For every $\varepsilon, \delta \in (0,1)$, a.a.s. the following holds for every $x \in \{0,1\}^n$: for any set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$, all but at most $100/(\varepsilon\delta)$ vertices $y \in N_{\mathcal{Q}^n}(x)$ satisfy $d_{\mathcal{Q}^n_\varepsilon,S,x}(y) \geq 2\varepsilon|S|/3$.*

Proof. Fix a vertex $x \in \{0,1\}^n$ and a set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$. Choose any vertex $y \in N_{\mathcal{Q}^n}(x)$, and consider the variable $X(y) := d_{\mathcal{Q}^n_\varepsilon,S}(y)$. It suffices to prove that a.a.s. $X(y) > 2\varepsilon|S|/3$ for all but $100/(\varepsilon\delta)$ vertices $y \in N_{\mathcal{Q}^n}(x)$. Observe that $X(y) \sim \mathrm{Bin}(|S|,\varepsilon)$, so $\mathbb{E}[X(y)] = \varepsilon|S|$ and, by Lemma 4.2,

$$\mathbb{P}[X(y) \leq 2\varepsilon|S|/3] \leq e^{-\varepsilon|S|/18} \leq e^{-\varepsilon\delta n/18}.$$

Observe, furthermore, that $N_{\mathcal{Q}^n}(x)$ is an independent set in $\mathcal{Q}^n$, hence the variables $\{X(y) : y \in N_{\mathcal{Q}^n}(x)\}$ are mutually independent. It follows that the probability that at least $100/(\varepsilon\delta)$ vertices $y \in N_{\mathcal{Q}^n}(x)$ do not satisfy the bound is at most $\binom{n}{100/(\varepsilon\delta)}e^{-5n}$. Finally, by a union bound over all choices of S and x, we conclude that the statement fails with probability at most $2^{3n}e^{-5n} = o(1)$. $\square$

We are also interested in the number of subcubes in which each vertex lies. Given a graph G, a vertex v and any $\ell \in \mathbb{N}$, we denote the number of copies of $\mathcal{Q}^\ell$ in G which contain v by $d^\ell_G(v)$. It is easy to give trivial upper bounds for this number by considering its value in $\mathcal{Q}^n$. Indeed, for all $v \in \{0,1\}^n$ we have that

$$d^\ell_{\mathcal{Q}^n_\varepsilon}(v) \leq \binom{n}{\ell}. \tag{5.2}$$

Lemma 5.12. *Let $\varepsilon \in (0,1)$, $a \in (1/2,1)$ and $\ell \in \mathbb{N}$. Then, a.a.s. all but at most $2^n e^{-n^{2a-1}/(6\varepsilon)}$ vertices $v \in V(\mathcal{Q}^n_\varepsilon)$ satisfy*

$$d^\ell_{\mathcal{Q}^n_\varepsilon}(v) = (1 \pm \mathcal{O}(n^{a-1}))\frac{\varepsilon^{2^{\ell-1}\ell}}{\ell!}n^\ell. \tag{5.3}$$

Proof. Throughout this proof, we will write $d(v)$ for $d_{\mathcal{Q}^n_\varepsilon}(v)$, $N(v)$ for $N_{\mathcal{Q}^n_\varepsilon}(v)$ and $d^\ell(v)$ for $d^\ell_{\mathcal{Q}^n_\varepsilon}(v)$.

Fix a vertex $v \in \{0,1\}^n$, reveal all the edges incident to v, and condition on the event that $d(v) = \varepsilon n \pm n^a$. We then have that

$$\mathbb{E}[d^\ell(v)] = \binom{\varepsilon n \pm n^a}{\ell}\varepsilon^{(2^{\ell-1}-1)\ell} = (1 \pm \mathcal{O}(n^{a-1}))\frac{n^\ell}{\ell!}\varepsilon^{2^{\ell-1}\ell}. \tag{5.4}$$

Let $\mathcal{D}(v) \subseteq \mathcal{D}(\mathcal{Q}^n)$ be the set of directions such that $N(v) = v + \mathcal{D}(v)$. Consider the graph $\Gamma_\ell(v) := \mathcal{Q}^n[v + \ell(\mathcal{D}(v) \cup \{\mathbf{0}\})]$. For each $i \in [\ell]$, let $L_i \subseteq E(\Gamma_\ell(v))$ be the set of edges which are at distance $i-1$ from v in $\mathcal{Q}^n$. Note that these sets partition $E(\Gamma_\ell(v))$ and that $|L_i| = i\binom{d(v)}{i}$. Let $m := |E(\Gamma_\ell(v))|$ and $m_j := \sum_{i=1}^j |L_i|$ for all $j \in [\ell]$. Label the edges of $\Gamma_\ell(v)$ as $e_1, \dots, e_m$ in such a way that all the edges in L_1 come first, then the edges in L_2, and so on, until covering all the edges in L_ℓ. For each $j \in [m]$, let X_j be the indicator random variable that $e_j \in E(\mathcal{Q}^n_\varepsilon)$ (recall that we condition on the neighbourhood of v being revealed and v being good). We now consider an edge-exposure martingale given by the variables $Y_j := \mathbb{E}[d^\ell(v) \mid X_1, \dots, X_j]$, for $j \in [m]_0$. This is a Doob martingale with $Y_{d(v)} = \mathbb{E}[d^\ell(v)]$ and $Y_m = d^\ell(v)$.

We must now bound the differences $|Y_j - Y_{j-1}|$, for all $j \in [m]$. Observe that the maximum change in the expected number of ℓ-dimensional cubes in $\mathcal{Q}^n_\varepsilon$ containing v when a new edge e_i is revealed is bounded from above by the number of such cubes in $\Gamma_\ell(v)$ containing e_i. Given any $k \in [\ell] \setminus \{1\}$ and any $i \in [m_k] \setminus [m_{k-1}]$, we claim that the number of copies of $\mathcal{Q}^\ell$ in $\Gamma_\ell(v)$ containing e_i is bounded by $\binom{d(v)-k}{\ell-k}$ (recall that for all $i \in [m_1]$ we have that $Y_i = Y_{i-1}$). Indeed, let $e_i = \{x,y\}$ with $\operatorname{dist}(x,v) = k-1$, and let $\mathcal{D}_k(x) \subseteq \mathcal{D}(v)$ be the set of $k-1$ directions such that $\mathcal{Q}^n(v, \mathcal{D}_k(x))$ contains x. Then, any copy $\mathcal{Q}$ of $\mathcal{Q}^\ell$ in $\Gamma_\ell(v)$ containing e_i must satisfy that $\mathcal{D}(\mathcal{Q}) \subseteq \mathcal{D}(v)$ contains $\mathcal{D}_k(x)$, the direction given by $y - x$, and any other $\ell - k$ of the directions in $\mathcal{D}(v)$, for which there is the claimed number of choices. Therefore, we conclude that

$$\sum_{i=1}^m |Y_i - Y_{i-1}|^2 \leq \sum_{k=2}^\ell k\binom{d(v)}{k}\binom{d(v)-k}{\ell-k}^2 = \frac{1}{((\ell-2)!)^2}d(v)^{2\ell-2}\left(1 \pm \mathcal{O}\left(\frac{1}{d(v)}\right)\right). \tag{5.5}$$

Hence, by Lemma 4.5, for n sufficiently large we have that

$$\mathbb{P}\left[|d^\ell(v) - \mathbb{E}[d^\ell(v)]| \geq \sqrt{\frac{2}{\varepsilon}}\frac{1}{(\ell-2)!}d(v)^{\ell-1/2}\right] \leq 2.1^{-n}.$$

Finally, by Remark 5.8 combined with a union bound on all vertices v such that $d(v) = \varepsilon n \pm n^a$, we conclude that a.a.s. all such vertices satisfy that $d^\ell(v) = (1 \pm \mathcal{O}(n^{a-1}))n^\ell\varepsilon^{2^{\ell-1}\ell}/\ell!$. $\square$

Remark 5.13. *In particular, the proof of Lemma 5.12 shows that a.a.s. all vertices $v \in \{0,1\}^n$ which satisfy $d_{\mathcal{Q}^n_\varepsilon}(v) = \varepsilon n \pm n^a$ also satisfy (5.3). Therefore, by Lemma 5.9, for any $r \in \mathbb{N}$ and $a \in (2/3, 1)$, a.a.s. in any ball of radius r, all but at most $n^{2-2a+\eta}$ vertices satisfy (5.3), where $\eta > 0$ is an arbitrarily small constant.*

In more generality, we will need to bound the number of subcubes which contain a given pair of vertices. Given a graph G, two vertices u and v, and any $\ell \in \mathbb{N}$, we denote the number of copies of $\mathcal{Q}^\ell$ in G which contain both u and v by $d^\ell_G(u,v)$. Again, we can easily give upper bounds for this number in $\mathcal{Q}^n_\varepsilon$ by considering its value in $\mathcal{Q}^n$. Indeed, for all $u, v \in \{0,1\}^n$ we have that

$$d^\ell_{\mathcal{Q}^n}(u,v) \leq \binom{n}{\ell - \mathrm{dist}(u,v)} \leq n^{\ell - \mathrm{dist}(u,v)} \tag{5.6}$$

(here, we understand that $\binom{n}{a} = 0$ for all $a < 0$).

We will also need the property that the cubes containing a given vertex use different directions quite evenly. More precisely, given any graph $G \subseteq \mathcal{Q}^n$, any set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$, two vertices $x, y \in \{0,1\}^n$, an integer $\ell \in \mathbb{N}$ and a real $t \in \mathbb{R}$, we denote by $d^\ell_{G,S,t,x}(y)$ the number of copies C of $\mathcal{Q}^\ell$ which contain y, do not contain x, and satisfy $|\mathcal{D}(C) \cap S| \geq t$.

Lemma 5.14. *Let $0 < 1/\ell \ll \delta < 1$, with $\ell \in \mathbb{N}$. Let $\varepsilon, \eta \in (0,1)$ and $a \in (2/3, 1)$. Then, a.a.s. the following holds for every $x \in \{0,1\}^n$: for any set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$, all but at most $n^{2(1+\eta-a)}$ vertices $y \in N_{\mathcal{Q}^n}(x)$ satisfy*

$$d^\ell_{\mathcal{Q}^n_\varepsilon, S, \ell^{1/2}, x}(y) \geq \frac{\varepsilon^{2^{\ell-1}\ell}}{2\ell!} n^\ell.$$

Proof. Throughout this proof, for any $x \in \{0,1\}^n$ and any $y \in N_{\mathcal{Q}^n}(x)$, we will write $N(y)$ for $N_{\mathcal{Q}^n_\varepsilon}(y)$, $d(y)$ for $d_{\mathcal{Q}^n_\varepsilon}(y)$, $d_{S,x}(y)$ for $d_{\mathcal{Q}^n_\varepsilon,S,x}(y)$, $d^\ell(y)$ for $d^\ell_{\mathcal{Q}^n_\varepsilon}(y)$ and $d^\ell_{S,\ell^{1/2},x}(y)$ for $d^\ell_{\mathcal{Q}^n_\varepsilon,S,\ell^{1/2},x}(y)$.

Fix a set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$. Let $D := \varepsilon^{2^{\ell-1}\ell} n^\ell/\ell!$ and fix a vertex $x \in \{0,1\}^n$. Consider any $y \in N_{\mathcal{Q}^n}(x)$; reveal all edges incident to y and condition on the event that $d(y) = \varepsilon n \pm n^a$ and $d_{S,x}(y) \geq \varepsilon |S|/2$. By Lemmas 5.9 and 5.11, we have that a.a.s. all but at most $n^{2(1+\eta-a)}$ vertices $y \in N_{\mathcal{Q}^n}(x)$ satisfy this event.

Let $\mathcal{D}(y)$ be the set of directions such that $N(y) \setminus \{x\} = y + \mathcal{D}(y)$. Thus, $|\mathcal{D}(y)| = d(y) \pm 1$. Let $\alpha := |S \cap \mathcal{D}(y)|/n$, and note that $\varepsilon\delta/2 \leq \alpha \leq \varepsilon + n^{a-1}$. Similar to the proof of (5.4), we have that

$$\mathbb{E}[d^\ell_{S,\ell^{1/2},x}(y)] = \varepsilon^{(2^{\ell-1}-1)\ell} \sum_{i=\lceil \ell^{1/2} \rceil}^{\ell} \binom{\alpha n}{i} \binom{\varepsilon n - \alpha n \pm (n^a + 1)}{\ell - i} \geq 3D/4.$$

Consider the graph $\Gamma_\ell(y) := \mathcal{Q}^n[y + \ell(\mathcal{D}(y) \cup \{\mathbf{0}\})]$. For each $i \in [\ell]$, let $L_i \subseteq E(\Gamma_\ell(y))$ be the set of edges which are at distance $i-1$ from y. Note that these sets partition $E(\Gamma_\ell(y))$ and that $|L_i| = i\binom{d(y)\pm 1}{i}$. Let $m := |E(\Gamma_\ell(y))|$ and $m_j := \sum_{i=1}^j |L_i|$ for all $j \in [\ell]$. Label the edges of $\Gamma_\ell(y)$ as $e_1, \ldots, e_m$ in such a way that all the edges in L_1 come first, then the edges in L_2, and so on, until covering all the edges in L_ℓ. For each $j \in [m]$, let X_j be the indicator random variable that $e_j \in E(\mathcal{Q}^n_\varepsilon)$. We now consider an edge-exposure martingale given by the variables $Y_j := \mathbb{E}[d^\ell_{S,\ell^{1/2},x}(y) \mid X_1, \ldots, X_j]$, for $j \in [m]_0$. This is a Doob martingale with $Y_{d(y)} = \mathbb{E}[d^\ell_{S,\ell^{1/2},x}(y)]$ and $Y_m = d^\ell_{S,\ell^{1/2},x}(y)$.

In order to bound the differences $|Y_j - Y_{j-1}|$, for all $j \in [m]$, observe that the maximum change in the expected number of ℓ-dimensional cubes in $\mathcal{Q}^n_\varepsilon$ containing y when a new edge e_i is revealed is bounded from above by the number of such cubes in $\Gamma_\ell(y)$ containing e_i. In particular, this is an upper bound for the maximum change in the expected number of ℓ-dimensional cubes in $\mathcal{Q}^n_\varepsilon$ containing y, not containing x, and whose directions intersect S in a set of size at least $\ell^{1/2}$, when a new edge e_i is revealed. Thus, similarly as in (5.5), it follows that

$$\sum_{i=1}^m |Y_i - Y_{i-1}|^2 \leq \frac{1}{((\ell-2)!)^2} d(y)^{2\ell-2} \left(1 \pm \mathcal{O}\left(\frac{1}{d(y)}\right)\right).$$

Hence, by Lemma 4.5, we have that

$$\mathbb{P}\left[d^{\ell}_{S,\ell^{1/2},x}(y) \leq D/2\right] \leq e^{-\Theta(n^2)}.$$

Finally, the statement follows by a union bound on all sets $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$, on all vertices $x \in \{0,1\}^n$, and on all vertices $y \in N_{\mathcal{Q}^n}(x)$ such that $d(y) = \varepsilon n \pm n^a$ and $d_{S,x}(y) \geq \varepsilon |S|/2$. $\square$

6. Tiling random subgraphs of the hypercube with small cubes

Throughout this section, we will consider auxiliary hypergraphs to obtain information about subgraphs of the n-dimensional hypercube. The general idea will be to apply the so-called 'Rödl nibble' to achieve this. Roughly speaking, the Rödl nibble is a randomised iterative process which, given an almost regular uniform hypergraph with small codegrees, finds a matching covering all but a small proportion of the vertices. The basic idea is the following. Let H be an almost regular uniform hypergraph with small codegrees. Consider a random subset of the edges $E \subseteq E(H)$, where each edge is taken independently with the same probability. If this probability is chosen carefully, then one can show that, with high probability, E is 'almost' a matching and that the hypergraph resulting after the deletion of all vertices covered by E is still almost regular and has small codegrees. This allows one to iterate the process until all but a small fraction of the vertices have been covered. This approach is the basis for the proof of our main result in this section, Theorem 6.6. The main auxiliary result is Lemma 6.5, which shows that in each iteration of the process we have the properties we require. In particular, we require our matching to satisfy several additional 'local' properties. This means our application of the nibble will require strong concentration results, as well as the use of the Lovász local lemma. It is also worth noting that our result relies strongly on the geometry of the hypercube, and cannot be stated for general hypergraphs.

6.1. **The Rödl nibble.** Given $\ell \in \mathbb{N}$ and any graph $G \subseteq \mathcal{Q}^n$, we will denote by $H_\ell(G)$ the 2^ℓ-uniform hypergraph with vertex set $V(G)$ where a set of vertices $W \subseteq \{0,1\}^n$ with $|W| = 2^\ell$ forms a hyperedge if and only if $G[W] \cong \mathcal{Q}^\ell$. Observe that the vertex set of $H_\ell(G)$ is (a subset of) $\{0,1\}^n$. Hence, we can use the underlying notation of directions we have considered for hypercubes so far. In particular, given any pair of vertices $x, y \in V(H_\ell(G))$, any set $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ and a real $t \in \mathbb{R}$, we denote by $d_{H_\ell(G),S,t,x}(y)$ the number of hyperedges $e \in E(H_\ell(G))$ which contain y, do not contain x, and satisfy $|\mathcal{D}(G[e]) \cap S| \geq t$. Note that, with the notation from Lemma 5.14, $d_{H_\ell(G),S,t,x}(y) = d^{\ell}_{G,S,t,x}(y)$. In order to simplify notation, for any vertex $x \in \{0,1\}^n$ and any sets $Y \subseteq N_{\mathcal{Q}^n}(x)$ and $E \subseteq E(H_\ell(G))$, we let $E_x(Y) := \{e \in E : \mathrm{dist}(x,e) = 1, e \cap Y \neq \varnothing\}$. If E is the set of all edges of a given hypergraph $H \subseteq H_\ell(G)$, we may sometimes denote this by $E_x(H, Y)$. Furthermore, it is worth noting that, for hypergraphs $H_\ell(G)$ defined as above, the inequality $d_H(x) \leq \sum_{y \in V(H)\setminus\{x\}} d_H(x,y)$, which holds for all hypergraphs H and all $x \in V(H)$, can be improved to the following: for every $\ell \geq 2$ and every $x \in V(H_\ell(G))$,

$$d_{H_\ell(G)}(x) \leq \sum_{y \in V(H_\ell(G)) \cap N_{\mathcal{Q}^n}(x)} d_{H_\ell(G)}(x,y). \tag{6.1}$$

The following observations will also come in useful.

Remark 6.1. *Let $\ell, t \in \mathbb{N}$ and $G \subseteq \mathcal{Q}^n$, and let $H := H_\ell(G)$. Let $x \in V(H)$ and $e \in E(H)$ be such that $\mathrm{dist}(x,e) = t$. Then, there is a unique vertex $y \in e$ such that $\mathrm{dist}(x,y) = t$. Furthermore, for every $e' \in E(H)$ such that $x \in e'$ we have that $e \cap e' \neq \varnothing$ if and only if $y \in e'$. In particular, $|\{e' \in E(H) : x \in e', e \cap e' \neq \varnothing\}| = d_H(x,y)$.*

Remark 6.2. *Let $\ell, t \in \mathbb{N}$ and $G \subseteq \mathcal{Q}^n$, and let $H := H_\ell(G)$. Let $x \in V(H)$ and $Y \subseteq N_{\mathcal{Q}^n}(x)$. Let $e \in E(H)$ be such that $\mathrm{dist}(x,e) = \mathrm{dist}(Y,e) = t$. Let $Y' := \{y \in Y : \mathrm{dist}(y,e) = t\}$. Then, $|Y'| \leq \ell$ and none of the edges in $E_x(H, Y \setminus Y')$ intersects e.*

Remark 6.3. *Let $\ell \in \mathbb{N}$ and $G \subseteq \mathcal{Q}^n$, and let $H := H_\ell(G)$. Let $x \in V(H)$ and $Y \subseteq N_{\mathcal{Q}^n}(x)$. Then, for any $e \in E_x(H, Y)$, we have $|\{e' \in E_x(H, Y \setminus e) : e \cap e' \neq \varnothing\}| = \mathcal{O}(n^{\ell-1})$.*

Remark 6.4. *Let $k, n \in \mathbb{N}$ and $A \subseteq \{0,1\}^n$. Then,*

$$|\{v \in \{0,1\}^n : \mathrm{dist}(v, A) = k\}| \leq |A|\binom{n}{k}.$$

Consider $\ell \in \mathbb{N}$, $G \subseteq \mathcal{Q}^n$ and $H := H_\ell(G)$. Recall that each edge of H corresponds to an ℓ-dimensional subcube of G. Let $e \in E(H)$, $E \subseteq E(H)$ and $S \subseteq \mathcal{D}(\mathcal{Q}^n)$. We define the *significance of* e *in* S as $\sigma(e, S) := |\mathcal{D}(e) \cap S|$. Given any $t \in \mathbb{R}$, we say that e is t*-significant in* S if $\sigma(e, S) \geq t$. We define the *significance of* E *in* S as $\sigma(E, S) := \sum_{e \in E} \sigma(e, S)$. We denote $\Sigma(E, S, t) := \{e \in E : \sigma(e, S) \geq t\}$. In particular, $\sigma(E, S) \geq t|\Sigma(E, S, t)|$.

With this, we are now ready to state the main auxiliary result in this section. This shows that, given $H = H_\ell(G)$, under suitable conditions about the degrees, the codegrees and the local distribution of the edges of H along the directions of the cube (namely, that the edges are significant in every large set of directions), one iteration of our nibble process will yield a subset of edges which is locally close to a matching, satisfies several local properties that we require of our matching (namely, the edges given by the nibble are sufficiently significant in large sets of directions, and not too significant in any given direction), and its deletion yields a hypergraph which still satisfies almost the same suitable conditions for further iterations.

Lemma 6.5. *Let $\ell, k, K \in \mathbb{N}$ with $k > \ell \geq 2$ and let $\beta \in (0, 1]$. Let $G \subseteq \mathcal{Q}^n$ and let $H := H_\ell(G)$. Fix $x \in \{0,1\}^n$. Let $A_0 := N_{\mathcal{Q}^n}(x)$ and, for each $i \in [K]$, let $A_i \subseteq A_0$ be a set of size $|A_i| \geq \beta n$. Assume that there exist two constants $a \in (3/4, 1)$ and $\gamma \in (0, 1]$, and $D = \Theta(n^\ell)$, such that*

(P1) *for every $y \in V(H) \cap B^k_{\mathcal{Q}^n}(x)$ we have $d_H(y) = (1 \pm \mathcal{O}(n^{a-1}))D$;*
(P2) *for every $i \in [K]_0$ we have $|V(H) \cap A_i| = (1 \pm \mathcal{O}(n^{a-1}))\gamma|A_i|$.*

Then, for all $\varepsilon \ll 1/\ell$, the following holds.

Let $E' \subseteq E(H)$ be a random subset of $E(H)$ obtained by adding each edge with probability ε/D, independently of every other edge. Let $E'' \subseteq E'$ be the set of all edges not intersecting any other edge of E'. Then, E', E'', $V' := V(H) \setminus V(E')$ and $H' := H[V']$ satisfy the following:

(N1) *with probability at least $1 - e^{-\Theta(n^{1/2})}$, for every $i \in [K]_0$ we have*
$$|V' \cap A_i| = (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|;$$
(N2) *with probability at least $1 - e^{-\Theta(n^{1/2})}$, for every $i \in [K]_0$ we have*
$$|V(E'') \cap A_i| \geq \varepsilon(1 - 2^{\ell+1}\varepsilon)\gamma|A_i|;$$
(N3) *with probability at least $1 - e^{-\Theta(n^{1/2})}$, for every $\hat{e} \in \mathcal{D}(\mathcal{Q}^n)$ we have*
$$|\Sigma(E'_x(A_0), \{\hat{e}\}, 1)| = o(n^{1/2});$$
(N4) *with probability at least $1 - e^{-\Theta(n^{2a-1})}$, for every $y \in V(H') \cap B^{k-\ell}_{\mathcal{Q}^n}(x)$ we have*
$$d_{H'}(y) = (1 \pm \mathcal{O}(n^{a-1}))e^{-(2^\ell - 1)\varepsilon}D.$$

If, in addition to (P1) and (P2), there exist $c, \delta \in (0, 1]$ such that

(P3) *for every $i \in [K]_0$ and every $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$ we have*
$$|\Sigma(E_x(H, A_i), S, \ell^{1/2})| \geq (1 - \mathcal{O}(n^{a-1}))c\gamma|A_i|D,$$

then E' and H' also satisfy the following:

(N5) *with probability at least $1 - e^{-\Theta(n^{1/2})}$, for every $i \in [K]_0$ and every $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$ we have*
$$|\Sigma(E_x(H', A_i), S, \ell^{1/2})| \geq (c - \varepsilon)e^{-(2^\ell - 1)\varepsilon}\gamma|A_i|D;$$
(N6) *for any fixed $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$, with probability at least $1 - e^{-\varepsilon c\gamma\beta n/100}$, for every $i \in [K]_0$ and n sufficiently large we have*
$$|V(\Sigma(E'_x(A_i), S, \ell^{1/2})) \cap A_i| \geq \varepsilon c^2\gamma|A_i|/8.$$

Proof. We begin by noting that, since $H = H_\ell(G)$, the following two properties hold:

(P4) for all $y, z \in V(H)$ such that $\mathrm{dist}(y, z) > \ell$, we have that $d_H(y, z) = 0$;

(P5) for each $i \in [\ell]$, for all $y, z \in V(H)$ with $\mathrm{dist}(y, z) = i$, we have that $d_H(y, z) = \mathcal{O}(D/n^i)$.

Indeed, by the definition of H, both follow from (5.6). These will be used repeatedly throughout the proof.

We next observe another simple property which will be useful later in the proof. Fix any $i \in [K]_0$. Note that, by (P1) and (P2), $|E_x(H, A_i)| = \sum_{y\in A_i\cap V(H)} d_H(y) \pm \ell n^\ell = (1\pm\mathcal{O}(n^{a-1}))\gamma|A_i|D$. Therefore, $\mathbb{E}[|E'_x(A_i)|] = (1 \pm \mathcal{O}(n^{a-1}))\varepsilon\gamma|A_i|$ and, by Lemma 4.2,

$$\mathbb{P}[|E'_x(A_i)| \neq (1 \pm \mathcal{O}(n^{a-1}))\varepsilon\gamma|A_i|] = e^{-\Theta(n^{2a-1})}. \tag{6.2}$$

By a union bound over all $i \in [K]_0$, we conclude that, with probability $1 - e^{-\Theta(n^{2a-1})}$, we have $|E'_x(A_i)| = (1 \pm \mathcal{O}(n^{a-1}))\varepsilon\gamma|A_i|$ for all $i \in [K]_0$.

(N1)**: On the number of vertices in A_i remaining in H'.**

In order to prove that (N1) holds, fix $i \in [K]_0$. Let $Y_i := V(H) \cap A_i$ and fix any vertex $y \in Y_i$. By (P1) we have that $d_H(y) = (1 \pm \mathcal{O}(n^{a-1}))D$. Therefore,

$$\mathbb{P}[y \in V'] = (1 - \varepsilon/D)^{(1\pm\mathcal{O}(n^{a-1}))D} = (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}.$$

Thus, by (P2), $\mathbb{E}[|V' \cap A_i|] = (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|$. We must now prove that $|V' \cap A_i|$ concentrates with high probability. However, the events $\{y \notin V(E')\}_{y\in Y_i}$ are not necessarily independent.

In order to consider independent events, let $E^* := \{e \in E(H) : x \notin e, e \cap A_i \neq \varnothing\}$ and, for each $y \in Y_i$, let $d^*_H(y) := |\{e \in E^* : y \in e\}|$. By (P5), we have $d^*_H(y) = d_H(y) \pm \mathcal{O}(D/n) = (1 \pm \mathcal{O}(n^{a-1}))D$. Let $V^* := Y_i \setminus V(E' \cap E^*)$. For every $y \in Y_i$ we have that

$$\mathbb{P}[y \notin V(E' \cap E^*)] = (1 - \varepsilon/D)^{(1\pm\mathcal{O}(n^{a-1}))D} = (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon},$$

hence $\mathbb{E}[|V^*|] = (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|$. Furthermore, the events $\{y \notin V(E' \cap E^*)\}_{y\in Y_i}$ are mutually independent, so by Lemma 4.2 we have that

$$\mathbb{P}[|V^*| \neq (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|] \leq e^{-\Theta(n^{2a-1})}. \tag{6.3}$$

As $V' \cap A_i \subseteq V^*$, we conclude that

$$\mathbb{P}[|V' \cap A_i| \geq (1 + \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|] \leq e^{-\Theta(n^{2a-1})}. \tag{6.4}$$

In order to obtain the lower tail concentration, observe that

$$|V' \cap A_i| = |V^*| - |V^* \cap V(E' \setminus E^*)|,$$

so it will suffice to show that the last term in the previous expression is small with high probability. Let $\hat{E} := \{e \in E(H) : |e \cap A_i| > 1\}$. Note that $|V^* \cap V(E' \setminus E^*)| \leq 2^\ell|\hat{E} \cap E'|$, so it suffices to bound this quantity. By (P5), $|\hat{E}| = \mathcal{O}(D)$. Since edges are picked independently, we have that $\mathbb{E}[|\hat{E} \cap E'|] = \mathcal{O}(1)$ and, by Lemma 4.4, $\mathbb{P}[|\hat{E} \cap E'| > n^{1/2}] \leq e^{-\Theta(n^{1/2})}$. Combining this with (6.3), we conclude that $\mathbb{P}[|V' \cap A_i| \leq (1 - \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|] \leq e^{-\Theta(n^{1/2})}$. Together with (6.4), the previous yields

$$\mathbb{P}[|V' \cap A_i| \neq (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i|] \leq e^{-\Theta(n^{1/2})}. \tag{6.5}$$

The statement of (N1) follows by a union bound over all $i \in [K]_0$.

(N2)**: On the number of vertices in A_i covered by the matching.**

We now prove (N2). Fix $i \in [K]_0$ and let $Y_i := V(H) \cap A_i$. Observe that

$$|V(E'') \cap A_i| = |(V(H) \setminus V') \cap A_i| - |V(E' \setminus E'') \cap A_i|. \tag{6.6}$$

By (6.5) and (P2) we have that $|(V(H)\setminus V')\cap A_i| = (1\pm\mathcal{O}(n^{a-1}))(1-e^{-\varepsilon})\gamma|A_i|$ with probability at least $1 - e^{-\Theta(n^{1/2})}$, so let us consider the last term in (6.6).

Given any vertex $y \in Y_i$, by abusing notation, let $d_{E'}(y) := |\{e \in E' : y \in e\}|$. Observe that $y \in V(E' \setminus E'')$ if and only if $d_{E'}(y) \geq 2$ or $d_{E'}(y) = 1$ and, for the edge $e \in E'$ such that $y \in e$, there exists $z \in e \setminus \{y\}$ such that $d_{E'\setminus\{e\}}(z) \geq 1$. Let $\mathcal{B}(y)$ be the event that, conditioned on $d_{E'}(y) = 1$, there exists such a vertex z. Then, for any $y \in Y_i$ we have

$$\mathbb{P}[y \in V(E' \setminus E'')] \leq \mathbb{P}[d_{E'}(y) \geq 2] + \mathbb{P}[d_{E'}(y) = 1]\mathbb{P}[\mathcal{B}(y)]. \tag{6.7}$$

Observe that, by (P1), $d_{E'}(y) \sim \mathrm{Bin}((1 \pm \mathcal{O}(n^{a-1}))D, \varepsilon/D)$. Then, it is easy to check that

$$\mathbb{P}[d_{E'}(y) = 1] = (1 \pm \mathcal{O}(n^{a-1}))\varepsilon e^{-\varepsilon} \quad \text{and} \quad \mathbb{P}[d_{E'}(y) \geq 2] = 1 - (1 \pm \mathcal{O}(n^{a-1}))(1+\varepsilon)e^{-\varepsilon}. \tag{6.8}$$

By a union bound and the fact that $\mathbb{P}[d_{E'\setminus e}(z) \geq 1] \leq \mathbb{P}[d_{E'}(z) \geq 1]$ for every $e \in E(H)$ and $z \in e \setminus \{y\}$, we also have that

$$\mathbb{P}[\mathcal{B}(y)] \leq (1 \pm \mathcal{O}(n^{a-1}))(2^\ell - 1)(1 - e^{-\varepsilon}). \tag{6.9}$$

Combining (6.7)–(6.9), for n sufficiently large we have that $\mathbb{P}[y \in V(E' \setminus E'')] \leq 2^\ell \varepsilon^2$. Hence, by considering all $y \in Y_i$ and (P2), we conclude that

$$\mathbb{E}[V(E' \setminus E'') \cap A_i] \leq (1 + \mathcal{O}(n^{a-1}))2^\ell \varepsilon^2 \gamma |A_i|. \tag{6.10}$$

In order to prove concentration, we will resort to Talagrand's inequality. Consider $X \coloneqq |V(E' \setminus E'') \cap A_i|$. This is a random variable on the probability space given by the product of the probability spaces associated with each edge of H being present in E'. In this setting, it is easy to see that X is a $\ell 2^\ell$-Lipschitz function. Furthermore, X is h-certifiable for $h\colon \mathbb{N} \to \mathbb{N}$ given by $h(s) = 2s$. Thus, by Lemma 4.7, for any real values b and t we have that

$$\mathbb{P}\left[X \leq b - t\ell 2^\ell \sqrt{2b}\right] \mathbb{P}[X \geq b] \leq e^{-t^2/4}.$$

By considering the change of variables $c = b - t\ell 2^\ell \sqrt{2b}$, we conclude that, for any reals c and t,

$$\mathbb{P}\left[X \leq c\right] \mathbb{P}\left[X \geq \left(t\ell 2^{\ell+1/2} + \left(t^2\ell^2 2^{2\ell+1} + 4c\right)^{1/2}\right)^2 /4\right] \leq e^{-t^2/4}. \tag{6.11}$$

Let $c \coloneqq 3 \cdot 2^{\ell-1}\varepsilon^2\gamma|A_i|$ and $t \coloneqq \Theta(n^{a-1/2})$. By Markov's inequality, we have that $\mathbb{P}[X \leq c] \geq 1/4$ for n sufficiently large. By substituting these into (6.11), we conclude that

$$\mathbb{P}\left[X \geq (1 + \mathcal{O}(n^{a-1}))c\right] \leq e^{-\Theta(n^{2a-1})}. \tag{6.12}$$

From this and (6.5), it follows that

$$\mathbb{P}[|V(E'') \cap A_i| \leq \varepsilon(1 - 2^{\ell+1}\varepsilon)\gamma|A_i|] \leq e^{-\Theta(n^{1/2})}.$$

The statement of (N2) follows by a union bound over all $i \in [K]_0$.

(N3)**: On the significance of E' in any direction.**

In order to prove (N3), we first observe that there are 'few' edges in $E_x(H, A_0)$ which use any given direction. Indeed, given any vertex $y \in V(H) \cap A_0$ and any direction $\hat{e} \in \mathcal{D}(\mathcal{Q}^n)$, the number of edges $e \in E(H)$ containing y and such that $\hat{e} \in \mathcal{D}(e)$ equals the codegree of y and $y + \hat{e}$. Therefore, by (P5), there are $\mathcal{O}(D/n)$ such edges and, adding over all vertices $y \in V(H) \cap A_0$, we conclude that $|\Sigma(E_x(H, A_0), \{\hat{e}\}, 1)| = \mathcal{O}(D)$. Since $|\Sigma(E'_x(A_0), \{\hat{e}\}, 1)| \sim \mathrm{Bin}(|\Sigma(E_x(H, A_0), \{\hat{e}\}, 1)|, \varepsilon/D)$, it immediately follows that $\mathbb{E}[|\Sigma(E'_x(A_0), \{\hat{e}\}, 1)|] = \mathcal{O}(1)$ and, by Lemma 4.4,

$$\mathbb{P}[|\Sigma(E'_x(A_0), \{\hat{e}\}, 1)| = \Omega(n^{1/2})] \leq e^{-\Theta(n^{1/2})}.$$

The statement of (N3) follows by a union bound over all directions $\hat{e} \in \mathcal{D}(\mathcal{Q}^n)$.

(N4)**: On the degrees in H'.**

We now want to bound the degrees of vertices in H' in order to prove (N4). Consider any vertex $y \in V(H)$ such that $\mathrm{dist}(x, y) \leq k - \ell$. Condition on the event that $y \in V'$. First, observe that, by (P1) and (P5),

$$\mathbb{E}[d_{H'}(y)] = (1 \pm \mathcal{O}(n^{a-1}))D(1 - \varepsilon/D)^{(2^\ell-1)(1\pm\mathcal{O}(n^{a-1}))D} = (1 \pm \mathcal{O}(n^{a-1}))e^{-(2^\ell-1)\varepsilon}D. \tag{6.13}$$

In order to bound the probability that $d_{H'}(y)$ deviates from its expectation, we will apply Lemma 4.6. Observe that the value of $d_{H'}(y)$ is determined by the presence or absence of the edges of $E^\bullet \coloneqq \{e \in E(H) : \text{there exists } e' \in E(H) \text{ such that } y \in e' \setminus e, e \cap e' \neq \varnothing\}$ in E'. Note that, for each $e \in E^\bullet$, the maximum possible change in the value of $d_{H'}(y)$ due to the presence or absence of e is $c_e \coloneqq |\{e' \in E(H) : y \in e', e \cap e' \neq \varnothing\}|$. Let $C \coloneqq \max_{e \in E^\bullet} c_e$ and $\sigma^2 \coloneqq \sum_{e \in E^\bullet}(\varepsilon/D)(1 - \varepsilon/D)c_e^2$. We must now estimate the value of σ.

Partition $E^\bullet$ into sets E_i, $i \in [\ell]$, given by $E_i := \{e \in E^\bullet : \mathrm{dist}(y, e) = i\}$. Observe that, by (P1) and Remark 6.4, for all $i \in [\ell]$ we have

$$|E_i| = \mathcal{O}(n^i D). \tag{6.14}$$

Furthermore, for each $i \in [\ell]$ and each $e \in E_i$, it follows from Remark 6.1 and (P5) that

$$c_e = \mathcal{O}(D/n^i). \tag{6.15}$$

In order to apply Lemma 4.6, we will need to show that σ is not too small. For this, we claim that

$$\text{there exist } \Theta(nD) \text{ edges } e \in E_1 \text{ such that } c_e = \Theta(D/n). \tag{6.16}$$

Indeed, an averaging argument using (6.1) together with (P1) shows that there are $\Theta(n)$ vertices $z \in V(H) \cap N_{\mathcal{Q}^n}(y)$ such that $d_H(y, z) = \Theta(D/n)$. Let Z be the set of all those vertices z. For each $z \in V(H) \cap N_{\mathcal{Q}^n}(y)$, let $E_1(z) := \{e \in E_1 : z \in e\}$. By Remark 6.1, for every $e \in E_1(z)$, z is the unique vertex in e such that $\mathrm{dist}(y, z) = 1$, so this gives a partition of E_1. By (P1) and (P5), we have that $|E_1(z)| = \Theta(D)$ for every $z \in Z$. Again by Remark 6.1, for every $z \in X$ and every $e \in E_1(z)$ we have that $c_e = d_H(y, z)$. (6.16) now follows.

In particular, (6.16) combined with (6.15) shows that $C = \Theta(D/n)$. Combining (6.14)–(6.16), it follows that

$$\sigma^2 = \Theta(D^2/n).$$

Now, by setting $\alpha := \Theta(n^{a-1/2})$, we observe that $\alpha = o(\sigma/C)$ and, thus, by Lemma 4.6 and (6.13),

$$\mathbb{P}[d_{H'}(y) \neq (1 \pm \mathcal{O}(n^{a-1}))e^{-(2^\ell-1)\varepsilon} D] \leq e^{-\Theta(n^{2a-1})}.$$

The statement of (N4) follows by a union bound over all vertices $y \in V(H) \cap B^{k-\ell}_{\mathcal{Q}^n}(x)$.

(N5)**: On the significance of H' in any large set of directions.**

We now turn our attention to (N5). Fix $i \in [K]_0$ and $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$. By (6.2) we have that

$$|E'_x(A_i)| = (1 \pm \mathcal{O}(n^{a-1}))\varepsilon\gamma|A_i| \tag{6.17}$$

with probability at least $1 - e^{-\Theta(n^{2a-1})}$. Furthermore, by (6.5), we have that

$$|V' \cap A_i| = (1 \pm \mathcal{O}(n^{a-1}))e^{-\varepsilon}\gamma|A_i| \tag{6.18}$$

with probability at least $1 - e^{-\Theta(n^{1/2})}$. Reveal $E'_x(A_i)$, as well as all edges in E' which contain x and intersect A_i, and condition on the event that (6.17) and (6.18) hold (note that this event is determined by the edges we have revealed). For the remainder of the proof of (N5), all probabilistic statements refer to probabilities when revealing all other edges in E'.

Let $X' := |\Sigma(E_x(H', A_i), S, \ell^{1/2})|$. Note that X' is a sum of indicator random variables, one for each edge in $\Sigma(E_x(H, A_i), S, \ell^{1/2})$; we will refer to those edges $e \in \Sigma(E_x(H, A_i), S, \ell^{1/2})$ for which we have $\mathbb{P}[e \in \Sigma(E_x(H', A_i), S, \ell^{1/2})] \neq 0$ as *potential edges*. The set of potential edges is denoted by E_P. We now want to prove a lower bound on $|E_P|$. We know that $\Sigma(E_x(H', A_i), S, \ell^{1/2}) \subseteq \Sigma(E_x(H, A_i), S, \ell^{1/2})$. By (P3) we have that $|\Sigma(E_x(H, A_i), S, \ell^{1/2})| \geq (1 - \mathcal{O}(n^{a-1}))c\gamma|A_i|D$. Any edge of $\Sigma(E_x(H, A_i), S, \ell^{1/2})$ whose endpoint in A_i does not lie in V' is not a potential edge. By (6.18), (P1) and (P2), the number of such edges is at most $(1 + \mathcal{O}(n^{a-1}))(1 - e^{-\varepsilon})\gamma|A_i|D$. Furthermore, some of the edges in $E'_x(A_i)$ may intersect other edges in $\Sigma(E_x(H, A_i), S, \ell^{1/2})$ (and, if this happens, the latter are not potential edges). By Remark 6.3 and (6.17), the number of such non-potential edges is $\mathcal{O}(D)$. Combining these bounds, we conclude that $|E_P| \geq (1 - \mathcal{O}(n^{a-1}))(c - (1 - e^{-\varepsilon}))\gamma|A_i|D$. Now, each of these potential edges contributes to X' if and only if none of its vertices lie in any edge in E'. By (P1) and (P5), it follows that, for each $e \in E_P$, $\mathbb{P}[e \in \Sigma(E_x(H', A_i), S, \ell^{1/2})] = (1 \pm \mathcal{O}(n^{a-1}))e^{-(2^\ell-1)\varepsilon}$ and, therefore,

$$\mathbb{E}[X'] \geq (1 - \mathcal{O}(n^{a-1}))(c - (1 - e^{-\varepsilon}))e^{-(2^\ell-1)\varepsilon}\gamma|A_i|D. \tag{6.19}$$

In order to prove concentration we will resort once more to Lemma 4.6. Let $E_i^\bullet := \{e \in E(H) : e \cap A_i = \varnothing \text{ and there exists } e' \in \Sigma(E_x(H, A_i), S, \ell^{1/2}) \text{ such that } e \cap e' \neq \varnothing\}$. The value of X' is determined uniquely by the presence or absence of the edges of $E_i^\bullet$ in E'. For each

$e \in E_i^\bullet$, the maximum change in the value of X' due to the presence or absence of e can be bounded by $c_e := |\{e' \in \Sigma(E_x(H, A_i), S, \ell^{1/2}) : e' \cap e \neq \varnothing\}|$. Let $C := \max_{e \in E_i^\bullet} c_e$ and $\sigma^2 := \sum_{e \in E_i^\bullet}(\varepsilon/D)(1-\varepsilon/D)c_e^2$. We must now estimate the value of σ.

Partition $E_i^\bullet$ into sets E_i^j, $j \in [\ell]$, given by $E_i^j := \{e \in E_i^\bullet : \mathrm{dist}(e, A_i) = j\}$. Observe that, by (P1) and Remark 6.4, for all $j \in [\ell]$ we have

$$|E_i^j| = \mathcal{O}(n^{j+1}D). \tag{6.20}$$

Furthermore, for each $j \in [\ell]$ and each $e \in E_i^j$, it follows from Remarks 6.1 and 6.2 and (P5) that

$$c_e = \mathcal{O}(D/n^j). \tag{6.21}$$

In particular, we claim that

$$\text{there exist } \Theta(n^2D) \text{ edges } e \in E_i^1 \text{ such that } c_e = \Theta(D/n). \tag{6.22}$$

Indeed, by (P1), (P2) and (P3), there are at least $c\gamma|A_i|/2$ vertices $y \in A_i \cap V(H)$ such that $d_{H,S,\ell^{1/2},x}(y) \geq cD/2$. Let U_i denote the set of these vertices. Then, an averaging argument using (6.1) together with (P1) shows that, for each $y \in U_i$, there are $\Theta(n)$ vertices $z \in V(H) \cap (N_{\mathcal{Q}^n}(y) \setminus \{x\})$ such that $d_{H,S,\ell^{1/2},x}(y,z) = \Theta(D/n)$. For each $y \in U_i$, let $Z_i(y)$ be the set of such vertices. Now, fix any $y \in U_i$ and, for each $z \in Z_i(y)$, let $E_i^1(z) := \{e \in E_i^1 : z \in e\}$. By (P1) and (P5), we have that $|E_i^1(z)| = \Theta(D)$ for every $z \in Z_i(y)$; furthermore, by Remark 6.1, for every $e \in E_i^1(z)$, z is the unique vertex in e such that $\mathrm{dist}(y,z) = \mathrm{dist}(y,e) = 1$. Then, for every $z \in Z_i(y)$ and every $e \in E_1(z)$ we have that

$$c_e \geq d_{H,S,\ell^{1/2},x}(y,z) = \Theta(D/n).$$

(6.22) now follows by considering all vertices $y \in U_i$.

In particular, (6.22) combined with (6.21) shows that $C = \Theta(D/n)$. Combining (6.20)–(6.22), it follows that

$$\sigma^2 = \Theta(D^2).$$

Now, by setting $\alpha := n^{a-1}\mathbb{E}[X']/\sigma = \Theta(n^a)$, we observe that $\alpha = o(\sigma/C)$ and, thus, by Lemma 4.6 and (6.19),

$$\mathbb{P}[X' < (c-\varepsilon)e^{-(2^\ell-1)\varepsilon}\gamma|A_i|D] \leq e^{-\Theta(n^{2a})}.$$

Since this holds for every $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \delta n$, by a union bound we conclude that the same holds simultaneously for every such set S. Recall, however, that this holds after conditioning on the event that (6.17) and (6.18) hold, which happens with probability $1 - e^{-\Theta(n^{1/2})}$. Taking this into account and using a union bound over all choices of $i \in [K]_0$, the statement of (N5) follows.

(N6)**: On the significance of E' in a large fixed set of directions.**

We finally turn our attention to (N6). Fix $i \in [K]_0$. Let U_i denote the set of vertices $y \in A_i \cap V(H)$ such that $d_{H,S,\ell^{1/2},x}(y) \geq cD/2$. By (P1), (P2) and (P3) we have $|U_i| \geq c\gamma|A_i|/2$. Let $V_i := V(\Sigma(E'_x(A_i), S, \ell^{1/2})) \cap U_i$. For each $y \in U_i$ we have that

$$\mathbb{P}[y \notin V_i] \leq (1-\varepsilon/D)^{cD/2} = (1 \pm \mathcal{O}(1/D))e^{-\varepsilon c/2}.$$

Thus, we conclude that

$$\mathbb{E}[|V_i|] \geq (1-\mathcal{O}(1/D))(1-e^{-\varepsilon c/2})c\gamma|A_i|/2.$$

Note that the events $\{y \notin V_i\}_{y \in U_i}$ are mutually independent. Hence, by Lemma 4.2,

$$\mathbb{P}[|V_i| \leq (1-e^{-\varepsilon c/2})c\gamma|A_i|/3] \leq e^{-\varepsilon c\gamma\beta n/99}.$$

Finally, note that $(1-e^{-\varepsilon c/2})c\gamma|A_i|/3 \geq (\varepsilon c/2 - \varepsilon^2c^2/8)c\gamma|A_i|/3 \geq \varepsilon c^2\gamma|A_i|/8$. The statement of (N6) follows by a union bound over all $i \in [K]_0$. □

6.2. **Iterating the nibble.** By making use of Lemma 6.5, we can now prove the main result of this section. Roughly speaking, Theorem 6.6 states that, for any constant $\varepsilon > 0$ and $\ell \in \mathbb{N}$, with high probability the random graph $\mathcal{Q}^n_\varepsilon$ contains a set of ℓ-dimensional cubes which are vertex-disjoint, cover all but a small proportion of the vertices of $\mathcal{Q}^n_\varepsilon$, and are 'sufficiently significant' with respect to every large set of directions, while not being 'too significant' with respect to any given direction.

By analogy with the notation introduced before Lemma 6.5, given any $\ell \in \mathbb{N}$, any $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ and any copy C of $\mathcal{Q}^\ell$ with $C \subseteq \mathcal{Q}^n$, we define the *significance of C in S* as $\sigma(C,S) := |\mathcal{D}(C) \cap S|$. Similarly, given any set $\mathcal{C}$ of ℓ-dimensional cubes in $\mathcal{Q}^n$, we define the *significance of $\mathcal{C}$ in S* as $\sigma(\mathcal{C},S) := \sum_{C\in\mathcal{C}} \sigma(C,S)$. We also denote $\Sigma(\mathcal{C},S,t) := \{C \in \mathcal{C} : \sigma(C,S) \geq t\}$. Given any $x \in \{0,1\}^n$ and any $Y \subseteq N_{\mathcal{Q}^n}(x)$, we denote $\mathcal{C}_x(Y) := \{C \in \mathcal{C} : \mathrm{dist}(x,C) = 1, V(C) \cap Y \neq \varnothing\}$. In particular, we will write $\mathcal{C}_x := \mathcal{C}_x(N_{\mathcal{Q}^n}(x))$.

Theorem 6.6. *Let $\varepsilon, \delta, \alpha, \beta \in (0,1)$ and $K, \ell \in \mathbb{N}$ be such that $1/\ell \ll \alpha \ll \beta$. For each $x \in \{0,1\}^n$, let $A_0(x) := N_{\mathcal{Q}^n}(x)$ and, for each $i \in [K]$, let $A_i(x) \subseteq A_0(x)$ be a set of size $|A_i(x)| \geq \beta n$. Then, the graph $\mathcal{Q}^n_\varepsilon$ a.a.s. contains a collection $\mathcal{C}$ of vertex-disjoint copies of $\mathcal{Q}^\ell$ such that the following properties are satisfied for every $x \in \{0,1\}^n$:*

(M1) $|A_0(x) \cap V(\mathcal{C})| \geq (1-\delta)n$;
(M2) *for every $\hat{e} \in \mathcal{D}(\mathcal{Q}^n)$ we have* $|\Sigma(\mathcal{C}_x, \{\hat{e}\}, 1)| = o(n^{1/2})$;
(M3) *for every $i \in [K]_0$ and every $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $\alpha n/2 \leq |S| \leq \alpha n$ we have*

$$|\Sigma(\mathcal{C}_x(A_i(x)), S, \ell^{1/2})| \geq |A_i(x)|/3000.$$

Proof. Let $n_0, k \in \mathbb{N}$ and $\varepsilon' > 0$ be such that $1/n_0 \ll 1/k \ll \varepsilon' \ll 1/\ell, \delta$, and let $n \geq n_0$. Let $H := H_\ell(\mathcal{Q}^n_\varepsilon)$. Observe that, with the notation from Lemmas 5.12 and 5.14, for any $x \in \{0,1\}^n$, $y \in A_0(x)$, $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ and $t \in \mathbb{R}$ we have that $d_H(x) = d^\ell_{\mathcal{Q}^n_\varepsilon}(x)$ and $d_{H,S,t,x}(y) = d^\ell_{\mathcal{Q}^n_\varepsilon,S,t,x}(y)$. Let $D_1 := \varepsilon^{2^{\ell-1}\ell} n^\ell/\ell!$.

Claim 6.1. *We a.a.s. have that, for every $x \in \{0,1\}^n$ and every $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $|S| \geq \alpha n/2$,*

(C1) $|\{y \in B^{2\ell}_{\mathcal{Q}^n}(x) : d_H(y) \neq (1 \pm \mathcal{O}(n^{-1/8}))D_1\}| \leq n^{7/8}$, *and*
(C2) $|\{y \in A_0(x) : d_{H,S,\ell^{1/2},x}(y) < D_1/2\}| \leq n^{3/4}$.

Proof. (C1) holds a.a.s. by Remark 5.13 applied with $a = 7/8$ and 2ℓ playing the role of r. (C2) holds a.a.s. by applying Lemma 5.14 with $a = 3/4$ and $\alpha/2$ playing the role of δ. ◀

Now, we condition on (C1) and (C2) and will show that there exists a collection $\mathcal{C}$ of vertex-disjoint copies of $\mathcal{Q}^\ell$ in $\mathcal{Q}^n_\varepsilon$ satisfying (M1)–(M3), as desired. In order to do this, we would like to apply Lemma 6.5 to H with $a = 7/8$ and $D = D_1$. However, H does not satisfy all the required properties. It is worth noting that it does satisfy (P4) and (P5), which follow immediately from (5.6). The argument now will be to modify H slightly so that Lemma 6.5 can be applied, independently of the choice of $x \in \{0,1\}^n$, and then iterate.

Claim 6.2. *There exists $H_1 \subseteq H$ which satisfies (P1)–(P3) with $7/8$, D_1, 1, $1/2$ and $\alpha/2$ playing the roles of a, D, γ, c and δ, respectively, for every $x \in \{0,1\}^n$ and every value of $k > \ell$.*

Proof of Claim 6.2. We construct H_1 by removing from H all vertices $y \in \{0,1\}^n$ such that $d_H(y) \neq (1 \pm \mathcal{O}(n^{-1/8}))D_1$. We first need to show that this deletion does not substantially decrease the degrees of other vertices. In fact, we claim that, for any $y \in \{0,1\}^n$,

$$\text{if } d_H(y) = (1 \pm \mathcal{O}(n^{-1/8}))D_1, \text{ then } d_{H_1}(y) = (1 \pm \mathcal{O}(n^{-1/8}))D_1. \tag{6.23}$$

Indeed, consider any vertex $y \in \{0,1\}^n$ which satisfies $d_H(y) = (1 \pm \mathcal{O}(n^{-1/8}))D_1$. By (P4), the removal of any vertex z such that $\mathrm{dist}(y,z) > \ell$ does not affect the degree of y. Furthermore, by (C1), the number of vertices $z \in B^\ell_{\mathcal{Q}^n}(y)$ such that $d_H(z) \neq (1 \pm \mathcal{O}(n^{-1/8}))D_1$ is at most $n^{7/8}$. Let Z be the set of all such vertices. By (P5), the number of edges incident to y which are removed because of some fixed $z \in Z$ is $\mathcal{O}(D_1/n)$. By adding over all $z \in Z$, we have that the number of edges incident to y which have been removed is $\mathcal{O}(D_1 n^{-1/8})$, and the claim follows. One can similarly prove that, for any $y \in V(H_1)$ and any $S \subseteq \mathcal{D}(\mathcal{Q}^n)$,

$$\text{if } d_{H,S,\ell^{1/2},x}(y) \geq D_1/2, \text{ then } d_{H_1,S,\ell^{1/2},x}(y) \geq (1 - \mathcal{O}(n^{-1/8}))D_1/2. \tag{6.24}$$

By (6.23), we now have that H_1 satisfies (P1) with the parameters stated in Claim 6.2 for every $x \in \{0,1\}^n$. It also follows that (P2) holds for every $x \in \{0,1\}^n$ since, by (C1), at most $n^{7/8}$ vertices are removed from the neighbourhood of any vertex in H. Finally, by (6.24) and (C2), we also have that H_1 satisfies (P3) for every $x \in \{0,1\}^n$. ◀

Therefore, we are in a position to apply Lemma 6.5. The argument now will be as follows. In order to prove that a collection of cubes as described in the statement exists, we will take a random collection of cubes in an iterative manner. We will prove that such a collection satisfies the desired properties locally with high probability. Then, we will apply the local lemma to extend the properties to the whole hypergraph. For this, it is important to define the probability space we are working with.

Fix a vertex $x \in \{0,1\}^n$. We now proceed iteratively. Let G_1 be the graph obtained by deleting from $\mathcal{Q}^n_\varepsilon$ all vertices $y \in \{0,1\}^n$ which do not satisfy $d^\ell_{\mathcal{Q}^n_\varepsilon}(y) = (1 \pm \mathcal{O}(n^{-1/8}))D_1$. Note that $H_1 = H_\ell(G_1)$. Let $i \in [k]$ and suppose that we have already defined G_i and H_i, where $H_i = H_\ell(G_i)$. Choose a random set of edges $E_i \subseteq E(H_i)$ by adding each edge in $E(H_i)$ to E_i independently with probability ε'/D_i. Then, define $H_{i+1} \coloneqq H_i - V(E_i)$. (Observe that $H_{i+1} = H_\ell(G_{i+1})$, where $G_{i+1} \coloneqq G_i - V(E_i)$.) Finally, let $D_{i+1} \coloneqq e^{-(2^\ell-1)\varepsilon'} D_i$, and iterate for k steps.

The randomized process above defines a probability space on the sequences of outcomes of each iteration of the process. Formally, the process, when iterated, results in a random sequence $E^k \coloneqq (E_1, \ldots, E_k)$ of sets of edges of H_1. Note that, for each $i \in [k]$, the hypergraph H_{i+1} is uniquely determined by $(E_1, \ldots, E_i)$, and H_1 does not depend on any of these sets; thus, the sequence E^k encodes all the information about the outcome of the iterative process. For any $i \in [k]$, we will write $E^i \coloneqq (E_1, \ldots, E_i)$. We will write $\mathbb{P}_{E^0}[E_1] \coloneqq \mathbb{P}[\text{process outputs } E_1 \text{ on input } H_1]$ and, for each $i \in [k] \setminus \{1\}$, we will write $\mathbb{P}_{E^{i-1}}[E_i] \coloneqq \mathbb{P}[\text{process outputs } E_i \text{ on input } H_i]$ (where H_i is determined by E^{i-1} for all $i \geq 2$). Whenever needed, we will treat E^0 as an empty sequence. Note that the choice of the process in any iteration affects the probability distribution on all subsequent iterations. For each $i \in [k]_0$, let Ω^i be the set of all sequences $E^i = (E_1, \ldots, E_i)$ such that, for all $j \in [i]$, $\mathbb{P}_{E^{j-1}}[E_j] > 0$, and let $\Omega \coloneqq \Omega^k$. Given any $\boldsymbol{\omega} = E^k = (E_1, \ldots, E_k) \in \Omega$, we write $\boldsymbol{\omega}^i \coloneqq E^i$. Consider any $\boldsymbol{\omega} = E^k = (E_1, \ldots, E_k) \in \Omega$. The probability distribution on the outputs of the iterative process is given by $\mathbb{P}_\Omega[\boldsymbol{\omega}] \coloneqq \prod_{j=1}^k \mathbb{P}_{E^{j-1}}[E_j]$. Similarly, the distribution on the outputs after i iterations of the process is given by $\mathbb{P}_{\Omega^i}[\boldsymbol{\omega}^i] \coloneqq \prod_{j=1}^i \mathbb{P}_{E^{j-1}}[E_j]$. Observe that, given $i \in [k]$ and $\boldsymbol{\omega}' \in \Omega^i$, we have that

$$\mathbb{P}_\Omega[\boldsymbol{\omega}^i = \boldsymbol{\omega}'] = \mathbb{P}_{\Omega^i}[\boldsymbol{\omega}']. \tag{6.25}$$

In particular, we wish to apply Lemma 6.5 in each iteration of the process. In order to do so, we will restrict ourselves to a suitable subspace of Ω by conditioning (again, in an iterative way). Let $k_1 \coloneqq \lfloor 1/(3\varepsilon') \rfloor$, $\gamma_1 \coloneqq 1$ and $c_1 \coloneqq 1/2$. For each $i \in [k_1]$, we proceed as follows. Given $E^{i-1} \in \Omega^{i-1}$ (and thus the hypergraph H_i), let $\mathcal{A}_i(E^{i-1})$ be the event that E_i, $E_i' \coloneqq \{e \in E_i : e \cap V(E_i \setminus \{e\}) = \varnothing\}$, $V_i \coloneqq V(H_i) \setminus V(E_i)$ and $H_{i+1} = H_i[V_i]$ satisfy (N1)–(N5) with D_i, $(k-i+1)\ell+1$, γ_i, c_i, ε' and $\alpha/2$ playing the roles of D, k, γ, c, ε and δ, respectively (in all iterations we will use $a = 7/8$). Then, let $\gamma_{i+1} \coloneqq e^{-\varepsilon'}\gamma_i$ and $c_{i+1} \coloneqq c_i - \varepsilon'$, and iterate.

Claim 6.3. *For any $i \in [k_1]$, let $E^{i-1} \in \Omega^{i-1}$ be such that H_i satisfies (P1)–(P3) with D_i, $(k-i+1)\ell+1$, γ_i, c_i and $\alpha/2$ playing the roles of D, k, γ, c and δ, respectively. Then,*

$$\mathbb{P}_{E^{i-1}}[\mathcal{A}_i(E^{i-1})] \geq 1 - e^{-\Theta(n^{1/2})}.$$

Proof. This follows immediately from Lemma 6.5. ◀

This will naturally lead us into applying Lemma 6.5 iteratively. Indeed, in any given iteration, assume that H_i satisfies (P1)–(P3) with D_i, $(k-i+1)\ell+1$, γ_i, c_i and $\alpha/2$ playing the roles of

D, k, γ, c and δ, respectively. Note that, for $i=1$, by Claim 6.2, these properties hold for every choice of $x\in\{0,1\}^n$ (but recall that we have now fixed x). If $\mathcal{A}_i(E^{i-1})$ holds, then, because of (N1), (N4) and (N5), the next hypergraph H_{i+1} satisfies (P1)–(P3) with D_{i+1}, $(k-i)\ell+1$, γ_{i+1}, c_{i+1} and $\alpha/2$ playing the roles of D, k, γ, c and δ, respectively, so Lemma 6.5 can be applied again.

As discussed above, in order to apply Lemma 6.5 fully in each iteration, we must condition on the event that certain properties are satisfied after the previous iteration (namely, the corresponding event $\mathcal{A}_i$ holds). For each $j\in[k_1]_0$, let $\Omega_*^j:=\{(E_1,\dots,E_j)\in\Omega^j : E_i\in\mathcal{A}_i(E^{i-1})\text{ for all } i\in[j]\}$. We denote $\Omega_*:=\Omega_*^{k_1}$. Using Claim 6.3, it now easily follows by induction that, for any $i\in[k_1]$,

$$\mathbb{P}_\Omega[\boldsymbol{\omega}^i\in\Omega_*^i]\geq 1-e^{-\Theta(n^{1/2})}. \tag{6.26}$$

Now fix a set of directions $S\subseteq\mathcal{D}(\mathcal{Q}^n)$ with $\alpha n/2\leq|S|\leq\alpha n$. For each $i\in[k_1]$, let $\mathcal{B}_i(S)\subseteq\Omega^i$ be the event that $|V(\Sigma(E_{ix}(A_j(x)),S,\ell^{1/2}))\cap A_j(x)|<\varepsilon' c_i^2\gamma_i|A_j(x)|/8$ for some $j\in[K]_0$. In other words, $\mathcal{B}_i(S)$ is the event that, in the i-th iteration, (N6) fails for S. Let

$$\begin{aligned}
a_i &:= \max_{E^{i-1}\in\Omega_*^{i-1}}\mathbb{P}_{\Omega^i}[\boldsymbol{\omega}\in\mathcal{B}_i(S)\mid\boldsymbol{\omega}^{i-1}=E^{i-1}]=\max_{E^{i-1}\in\Omega_*^{i-1}}\mathbb{P}_{E^{i-1}}[\mathcal{B}_i(S)],\\
b_i &:= \max_{E^{i-1}\in\Omega_*^{i-1}}\mathbb{P}_{\Omega^i}[\boldsymbol{\omega}\in\mathcal{B}_i(S)\cap\mathcal{A}_i(E^{i-1})\mid\boldsymbol{\omega}^{i-1}=E^{i-1}]=\max_{E^{i-1}\in\Omega_*^{i-1}}\mathbb{P}_{E^{i-1}}[\mathcal{B}_i(S)\cap\mathcal{A}_i(E^{i-1})],\\
c_i &:= \max_{E^{i-1}\in\Omega_*^{i-1}}\mathbb{P}_{\Omega^i}[\mathcal{A}_i(E^{i-1})\mid\boldsymbol{\omega}^{i-1}=E^{i-1}]=\max_{E^{i-1}\in\Omega_*^{i-1}}\mathbb{P}_{E^{i-1}}[\mathcal{A}_i(E^{i-1})].
\end{aligned}$$

By Lemma 6.5(N6), for each $i\in[k_1]$ we have that

$$b_i\leq a_i\leq e^{-\varepsilon'\beta n/900}=:d.$$

Let $\mathcal{I}(S)$ be the set of indices $i\in[k_1]$ in which $\mathcal{B}_i(S)$ holds. Note that, for any set $\mathcal{I}\subseteq[k_1]$, by (6.25) we have that $\mathbb{P}_\Omega[\mathcal{I}(S)=\mathcal{I}]=\mathbb{P}_{\Omega^{k_1}}[\mathcal{I}(S)=\mathcal{I}]$. Using the definitions above and induction on k_1, it follows that

$$f:=\mathbb{P}_{\Omega^{k_1}}[(\mathcal{I}(S)=\mathcal{I})\wedge\Omega_*]\leq\prod_{i\in\mathcal{I}}b_i\prod_{i\in[k_1]\setminus\mathcal{I}}c_i\leq d^{|\mathcal{I}|}.$$

Let $X=X(S):=|\mathcal{I}(S)|$. By adding over all sets $\mathcal{I}\subseteq[k_1]$ with $|\mathcal{I}|\geq k_1/2$, we conclude that

$$\mathbb{P}_{\Omega^{k_1}}[(X\geq k_1/2)\wedge\Omega_*]\leq 2^{k_1}d^{k_1/2}\leq e^{-\beta n/7000}. \tag{6.27}$$

Let $\mathcal{B}\subseteq\Omega^{k_1}$ be the event that there exists a set $S\subseteq\mathcal{D}(\mathcal{Q}^n)$ with $\alpha n/2\leq|S|\leq\alpha n$ such that the event $\mathcal{B}_i(S)$ holds in at least $k_1/2$ iterations. A union bound on (6.27) over all choices of $S\subseteq\mathcal{D}(\mathcal{Q}^n)$ with $\alpha n/2\leq|S|\leq\alpha n$ allows us to conclude that $\mathbb{P}_{\Omega^{k_1}}[\mathcal{B}\wedge\Omega_*]\leq e^{-\Theta(n)}$. Finally, combining this with (6.25) and (6.26), we have that

$$\mathbb{P}_\Omega[\boldsymbol{\omega}^{k_1}\in\mathcal{B}]\leq\mathbb{P}_\Omega[\boldsymbol{\omega}^{k_1}\in\mathcal{B}\mid\boldsymbol{\omega}^{k_1}\in\Omega_*]+\mathbb{P}_\Omega[\boldsymbol{\omega}^{k_1}\in\overline{\Omega_*}]\leq e^{-\Theta(n^{1/2})}. \tag{6.28}$$

Now, for all $i\in[k]\setminus[k_1]$, we iterate as above with the difference that we no longer require (P3). Thus, we can no longer guarantee that (N5) or (N6) hold with high probability, but we still have (N1)–(N4) as above. To be more precise, given any $i\in[k]\setminus[k_1]$ and $E^{i-1}\in\Omega^{i-1}$ (and thus the hypergraph H_i), let $\mathcal{A}_i(E^{i-1})$ be the event that E_i, $E_i':=\{e\in E_i : e\cap V(E_i\setminus\{e\})=\varnothing\}$, $V_i:=V(H_i)\setminus V(E_i)$ and $H_{i+1}=H_i[V_i]$ satisfy (N1)–(N4) with D_i, $(k-i+1)\ell+1$, ε' and γ_i playing the roles of D, k, ε and γ, respectively. Then, let $\gamma_{i+1}:=e^{-\varepsilon'}\gamma_i$ and iterate. Similarly to Claim 6.3, we can now show the following.

Claim 6.4. *For any $i\in[k]\setminus[k_1]$, let E^{i-1} be such that H_i satisfies (P1) and (P2) with D_i, $(k-i+1)\ell+1$ and γ_i playing the roles of D, k and γ, respectively. Then,*

$$\mathbb{P}_{E^{i-1}}[\mathcal{A}_i(E^{i-1})]\geq 1-e^{-\Theta(n^{1/2})}.$$

Proof. This follows immediately from Lemma 6.5. ◀

Now, assume that H_i satisfies (P1) and (P2) with D_i, $(k-i+1)\ell+1$ and γ_i playing the roles of D, k and γ, respectively. Note that, for $i = k_1+1$, we have these properties by conditioning on $\mathcal{A}_{k_1}(E^{k_1-1})$. Then, if $\mathcal{A}_i(E^{i-1})$ holds, because of (N1) and (N4), the hypergraph H_{i+1} satisfies (P1) and (P2) with D_{i+1}, $(k-i)\ell+1$ and γ_{i+1} playing the roles of D, k and γ, respectively, so we may apply Lemma 6.5 again.

Let $\mathcal{A} \coloneqq \{(E_1,\ldots,E_k) \in \Omega : E^{k_1} \in \Omega_* \cap \overline{\mathcal{B}}, E_i \in \mathcal{A}_i(E^{i-1})$ for all $i \in [k]\setminus[k_1]\}$. By combining (6.26), (6.28) and Claim 6.4, observe that $\mathbb{P}_\Omega[\mathcal{A}] \geq 1 - e^{-\Theta(n^{1/2})}$. For any $(E_1,\ldots,E_k) \in \mathcal{A}$, let $E \coloneqq \bigcup_{i=1}^k E_i$ and $E' \coloneqq \bigcup_{i=1}^k E'_i$. Note that E' is a matching by construction, that is, it corresponds to a collection $\mathcal{C}'$ of vertex-disjoint copies of $\mathcal{Q}^\ell$ in $\mathcal{Q}^n_\varepsilon$. We will now show that $\mathcal{C}'$ satisfies (M1)–(M3) for our fixed vertex x. Indeed, (M1) and (M2) hold for x since $(E_1,\ldots,E_k) \in \mathcal{A}$ implies that (N2) and (N3) hold in each iteration (note that (M1) follows from the case $i = 0$ of (N2)). In order to prove (M3) for x, consider the following. For each $S \subseteq \mathcal{D}(\mathcal{Q}^n)$ with $\alpha n/2 \leq |S| \leq \alpha n$, there are at least $k_1/2$ iterations $i \in [k_1]$ in which (N6) holds (for all $j \in [K]_0$). For each such set S, let $\mathcal{I}^*(S) \subseteq [k_1]$ be the set of indices of all such iterations. In particular, for each $i \in \mathcal{I}^*(S)$ we have that $|V(\Sigma(E_{ix}(A_j(x)), S, \ell^{1/2})) \cap A_j(x)| \geq \varepsilon' c_i^2\gamma_i|A_j(x)|/8 \geq \varepsilon'|A_j(x)|/432$ for all $j \in [K]_0$. Furthermore, by (N1) and (N2), we know that the number of vertices of $A_j(x)$ covered by $E_i \setminus E'_i$ satisfies $|V(E_i \setminus E'_i) \cap A_j(x)| \leq 2^{\ell+2}(\varepsilon')^2|A_j(x)|$, so $|V(\Sigma(E'_{ix}(A_j(x)), S, \ell^{1/2})) \cap A_j(x)| \geq (1/432 - 2^{\ell+2}\varepsilon')\varepsilon'|A_j(x)|$ for all $j \in [K]_0$. By adding this over all $i \in \mathcal{I}^*(S)$, we conclude that $|V(\Sigma(E'_x(A_j(x)), S, \ell^{1/2})) \cap A_j(x)| \geq |A_j(x)|/3000$ for all $j \in [K]_0$, as we wanted to see.

In order to show that there exists a matching which satisfies (M1)–(M3) simultaneously for all $x \in \{0,1\}^n$, let $\mathcal{E}(x)$ be the event that they hold for x. By the above discussion, we have that $\mathbb{P}_\Omega[\overline{\mathcal{E}(x)}] \leq \mathbb{P}_\Omega[\overline{\mathcal{A}}] \leq e^{-\Theta(n^{1/2})}$ for each $x \in \{0,1\}^n$. Furthermore, throughout the iterative process, the presence or absence in E of any edges e such that $\mathrm{dist}(x, e) > k\ell + 1$ does not have any effect on $\mathcal{E}(x)$, so $\mathcal{E}(x)$ is mutually independent of all events $\mathcal{E}(y)$ with $\mathrm{dist}(x, y) \geq 3k\ell$. Thus, by Lemma 4.8, we conclude that there is a choice of E which satisfies (M1)–(M3) for every $x \in \{0,1\}^n$. □

7. Near-spanning trees in random subgraphs of the hypercube

In this section we present our results on bounded degree near-spanning trees in $\mathcal{Q}^n_\varepsilon$. In Section 7.1 we prove the main result of this section (Theorem 7.1). This implies that with high probability there exists a near-spanning bounded degree tree in $\mathcal{Q}^n_\varepsilon$, which covers most of the neighbourhood of every vertex whilst avoiding a small random set of vertices, to which we refer as a reservoir. In Section 7.2 we prove Theorem 7.19, which allows us to extend the tree using vertices of the reservoir such that (amongst others) the proportion of uncovered vertices is even smaller. Finally, in Lemma 7.20 we show that, if some number of small local obstructions is prescribed, the tree given by Theorem 7.19 can be slightly modified to avoid these obstructions. For convenience, throughout this section, we move away from the algebraic notation for the hypercube to a more combinatorial notation.

We (re-)define the hypercube by setting $V(\mathcal{Q}^n) \coloneqq \mathcal{P}([n])$ and joining two vertices $u, v \in \mathcal{P}([n])$ by an edge if and only if $||u| - |v|| = 1$ and $u \subseteq v$ or $v \subseteq u$. In this setting, directions correspond to the elements in $[n]$, and following a direction $i \in [n]$ from a vertex $v \in \mathcal{P}([n])$ means adding i to v if $i \notin v$, or deleting it from v if $i \in v$. Note that there is a natural partition of $V(\mathcal{Q}^n)$ into sets such that every vertex of a set has the same size. Given any set $S \subseteq [n]$, we denote $S^{(t)} \coloneqq \{X \subseteq S : |X| = t\}$. We will denote by L_i, for $i \in [n]_0$, the set of all vertices $v \in V(\mathcal{Q}^n) = \mathcal{P}([n])$ with $|v| = i$ (that is, $L_i = [n]^{(i)}$), and we will refer to these sets as *levels*. This notation is especially useful because of the natural notion of containment of vertices, which provides a partial order on the vertices of $\mathcal{Q}^n$. Given any graph $G \subseteq \mathcal{Q}^n$, for a vertex $x \in L_i$, we refer to the neighbours of x in G lying in L_{i+1} as *up-neighbours*, and to the neighbours of x in L_{i-1} as *down-neighbours*, and denote these sets by $N^\uparrow_G(x)$ and $N^\downarrow_G(x)$, respectively. We write $d^\uparrow_G(x) \coloneqq |N^\uparrow_G(x)|$ and $d^\downarrow_G(x) \coloneqq |N^\downarrow_G(x)|$. Whenever the subscript is omitted, we mean that

$G = \mathcal{Q}^n$. We will say that a path $P = v_1 \dots v_k$ in $\mathcal{Q}^n$ is a *chain* if its vertices satisfy the relation $v_1 \subseteq \dots \subseteq v_k$, and refer to it as a v_1-v_k chain.

In more generality, because of the symmetries of the hypercube, this notation can be extended with respect to any vertex $v \in V(\mathcal{Q}^n)$ by defining, for each $i \in [n]_0$, $L_i(v) \coloneqq \{u \in V(\mathcal{Q}^n) : \mathrm{dist}(u,v) = i\}$. One can then define up-neighbours and down-neighbours with respect to v, and use the notations $N^{\uparrow}_G(x,v)$, $N^{\downarrow}_G(x,v)$, $d^{\uparrow}_G(x,v)$ and $d^{\downarrow}_G(x,v)$, for $G \subseteq \mathcal{Q}^n$. We say that a path $P = v_1 \dots v_k$ in $\mathcal{Q}^n$ is a *chain with respect to* v if its vertices satisfy that, if $v_1 \in L_j(v)$ for some $j \in [n]_0$, then for all $\ell \in [k] \setminus \{1\}$ we have $v_\ell \in L_{j+\ell-1}(v)$, and refer to it as a v_1-v_k chain. Given any graph $G \subseteq \mathcal{Q}^n$, for any $i \in [n]$ and $v \in V(\mathcal{Q}^n)$, we will write $E_G(L_{i-1}(v), L_i(v))$ for the set of edges of G whose endpoints lie in the levels $L_{i-1}(v)$ and $L_i(v)$, respectively. We will drop the subscript whenever $G = \mathcal{Q}^n$.

7.1. **Constructing a bounded degree near-spanning tree.** Our goal in this subsection is to prove Theorem 7.1 below. Given a graph G and $\delta \in [0,1]$, let $\mathit{Res}(G,\delta)$ be a probability distribution on subsets of $V(G)$, where $R \sim \mathit{Res}(G,\delta)$ is obtained by adding each vertex $v \in V(G)$ to R with probability δ, independently of every other vertex. We will refer to this set R as a *reservoir*.

Theorem 7.1. *Let $0 < 1/D, \delta \ll \varepsilon' \leq 1/2$, and let $\varepsilon, \gamma \in (0,1]$ and $k \in \mathbb{N}$. Then, the following holds a.a.s. Let $\mathcal{A} \subseteq V(\mathcal{Q}^n)$ with the following two properties:*

(P1) *for any distinct $x, y \in \mathcal{A}$ we have $\mathrm{dist}(x,y) \geq \gamma n$, and*

(P2) $B^{k+2}_{\mathcal{Q}^n}(\mathcal{A}) \cap \{\varnothing, [n], [\lceil n/2 \rceil], [n] \setminus [\lceil n/2 \rceil]\} = \varnothing$.

Let $R \sim \mathit{Res}(\mathcal{Q}^n, \delta)$. Then, there exists a tree $T \subseteq \mathcal{Q}^n_\varepsilon - (R \cup B^k_{\mathcal{Q}^n}(\mathcal{A}))$ such that

(T1) $\Delta(T) < D$,

(T2) *for all $x \in V(\mathcal{Q}^n) \setminus B^k_{\mathcal{Q}^n}(\mathcal{A})$, we have that $|N_{\mathcal{Q}^n}(x) \cap V(T)| \geq (1-\varepsilon')n$.*

The set $\mathcal{A}$ will be important in the proof of Theorems 1.5 and 1.8, where it will play the role of the set $\mathcal{U}$ of vertices of small degree. In the proof of Theorems 1.1 and 1.7 we can take $\mathcal{A} = \varnothing$.

To prove Theorem 7.1, we will consider suitable 'branching-like' processes which start at the 'bottom' of the hypercube, and grow 'upwards'. The tree will be formed by considering unions of such processes. The precise definition of the model we use is given in Definition 7.3. Crucially, there is a joint distribution of this branching-like process model and the binomial model $\mathcal{Q}^n_\varepsilon$. These processes are analyzed and constructed in the results leading up to Lemma 7.12. Subgraphs obtained from the processes are then connected into a tree in Lemma 7.17.

We begin with a formal description of our model. We denote by $\mathbf{p} = (p_0, \dots, p_{n-1}) \in [0,1]^n$ an n-component vector of probabilities. We now describe a distribution on subgraphs of $\mathcal{Q}^n$ which is biased with respect to the number of edges between different levels of the hypercube.

Definition 7.2 (Level-biased subgraphs of $\mathcal{Q}^n$)**.** *Given $n \in \mathbb{N}$ and $\mathbf{p} = (p_0, \dots, p_{n-1}) \in [0,1]^n$, let $\mathcal{W}^n_{\mathbf{p}}$ be a distribution on subgraphs of $\mathcal{Q}^n$ where $W \sim \mathcal{W}^n_{\mathbf{p}}$ is generated as follows: we set $V(W) \coloneqq V(\mathcal{Q}^n)$ and, for each $i \in [n-1]_0$, each $e \in E(L_i, L_{i+1})$ is included in W with probability p_i, independently of all other edges.*

Roughly speaking, the above model has the advantage that, by choosing our probabilities p_i appropriately, it will allow us to generate subgraphs of $\mathcal{Q}^n$ where each vertex has the same number of up-neighbours in expectation. Moreover, note that there is a joint distribution of $\mathcal{W}^n_{\mathbf{p}}$ and $\mathcal{Q}^n_p$ such that we have $\mathcal{W}^n_{\mathbf{p}} \subseteq \mathcal{Q}^n_p$, where p is the maximum component of $\mathbf{p}$.

We are now in a position to define one further distribution on subgraphs of $\mathcal{Q}^n$. We will search for a near-spanning tree for $\mathcal{Q}^n_\varepsilon$ in the graphs generated according to this distribution.

Definition 7.3 (Percolation graph $\mathcal{P}(n, \mathbf{p}, M)$)**.** *Given $n, M \in \mathbb{N}$ and $\mathbf{p} = (p_0, \dots, p_{n-1}) \in [0,1]^n$, we define $\mathcal{P}(n, \mathbf{p}, M)$ to be a distribution on subgraphs of $\mathcal{Q}^n$ where $P \sim \mathcal{P}(n, \mathbf{p}, M)$ is generated as follows. Let $R \sim \mathit{Res}(\mathcal{Q}^n, 1/100)$ and $W \sim \mathcal{W}^n_{\mathbf{p}}$. For each $x \in V(\mathcal{Q}^n)$, if $d^{\uparrow}_W(x) \geq M$, let $B(x) \subseteq N^{\uparrow}_W(x)$ be a uniformly random set of size M (otherwise, let $B(x) \coloneqq \varnothing$), and let $E(x)$ be the set of edges joining x to each $y \in B(x)$. Let W' be the spanning subgraph of W with edge set $\bigcup_{x \in V(\mathcal{Q}^n)} E(x)$. The graph $P \subseteq \mathcal{Q}^n$ is then given by setting $P \coloneqq W' - R$.*

Remark 7.4. *Observe that, given any two distinct edges $e, e' \in E(\mathcal{Q}^n)$, the events $e \in E(W')$ and $e' \in E(W')$ are mutually dependent if and only if for some $i \in [n]$ we have $e, e' \in E(L_{i-1}, L_i)$ with $e \cap e' = \{v\}$ for some $v \in L_{i-1}$. Otherwise, these events are independent. In particular, if $e \in E(L_{i-1}, L_i)$ and $e' \in E(L_{j-1}, L_j)$ with $i \neq j$, then these events are always independent.*

Note that $\mathcal{P}(n, \mathbf{p}, M) \subseteq \mathcal{W}^n_{\mathbf{p}}$ by definition, and therefore we have a joint distribution of $\mathcal{P}(n, \mathbf{p}, M)$ and $\mathcal{Q}^n_p$ such that $\mathcal{P}(n, \mathbf{p}, M) \subseteq \mathcal{Q}^n_p$, where p is the maximum component of $\mathbf{p}$.

Definition 7.5 (Feasible $(n, \mathbf{p}, M)$)**.** *We say that the tuple $(n, \mathbf{p}, M)$ is* feasible *if*

(i) *$p_i = 0$ for all $9n/10 < i < n$,*
(ii) *$max_{i\in[n-1]_0} p_i < 1/10$ and $M > 1600$,*
(iii) *there exists $t \in \mathbb{R}$ with $600 < t < 100M$ such that $P \sim \mathcal{P}(n, \mathbf{p}, M)$ satisfies $\mathbb{P}[e \in E(P)] = t/n$ for all $e \in \bigcup_{i=0}^{\lfloor 9n/10\rfloor} E(L_i, L_{i+1})$.*

Remark 7.6. *Let $(n, \mathbf{p}, M)$ be feasible, where $\mathbf{p} = (p_0, \ldots, p_{n-1})$. Note that p_0 determines the value of p_i for all $i \in [\lfloor 9n/10\rfloor]$. Furthermore, let $P \sim \mathcal{P}(n, \mathbf{p}, M)$. We can generate P by first sampling $W \sim \mathcal{W}^n_{\mathbf{p}}$ and $R \sim Res(\mathcal{Q}^n, 1/100)$, and then defining the graph W' as described in Definition 7.3. Let $t' \coloneqq t/(\frac{99}{100})^2$, where t is as in Definition 7.5(iii). Since $(n, \mathbf{p}, M)$ is feasible, for all $e \in \bigcup_{i=0}^{\lfloor 9n/10\rfloor} E(L_i, L_{i+1})$ we have*

$$\mathbb{P}[e \in E(W')] = t'/n.$$

Furthermore, for all $i \in [\lfloor 9n/10\rfloor]$, given $e, e' \in E(L_i, L_{i+1})$ with $e \neq e'$, we have that

$$\mathbb{P}[e, e' \in E(W')] \leq \mathbb{P}[e \in E(W')]^2 = (t'/n)^2.$$

From here on, where it is clear from the context, we will use $p_0, \ldots, p_{n-1}$ to denote the components of each probability vector $\mathbf{p}$, and will use t to denote the value t in Definition 7.5 and t' to denote the value t' in Remark 7.6.

Proposition 7.7. *For all $\varepsilon \in (0, 1/10)$, $M > 1600$, and $n \in \mathbb{N}$ such that $0 < 1/n \ll 1/M, \varepsilon$ there exists a tuple $(n, \mathbf{p}, M)$ which is feasible and such that $p_i \leq \varepsilon$ for all $i \in [n-1]_0$.*

Proof. Let $P \sim \mathcal{P}(n, \mathbf{p}, M)$, for some $\mathbf{p}$ which will be determined later. We generate P by first sampling $W \sim \mathcal{W}^n_{\mathbf{p}}$ and $R \sim Res(\mathcal{Q}^n, 1/100)$. Let $j \in [\lfloor 9n/10\rfloor]_0$ be fixed and let $e \in E(L_j, L_{j+1})$. Let $x \in L_j$ be incident with e. Let $\mathcal{A}$ be the event that $e \in E(W)$. For each $k \in [n-j]_0$, let $\mathcal{B}_k$ be the event that $d^{\uparrow}_W(x) = k$. Let $\mathcal{C}$ be the event that $e \in E(P)$. For each $i \in [n-M]_0$, let

$$f_i(y) \coloneqq \left(\frac{99}{100}\right)^2 \frac{M}{n-i} \sum_{k=M}^{n-i} \binom{n-i}{k} y^k (1-y)^{n-i-k}.$$

Then, we have that

$$\begin{aligned}\mathbb{P}[\mathcal{C}] &= \sum_{k=M}^{n-j} \mathbb{P}[\mathcal{C} \mid \mathcal{A} \wedge \mathcal{B}_k]\, \mathbb{P}[\mathcal{A} \mid \mathcal{B}_k]\, \mathbb{P}[\mathcal{B}_k] \\ &= \sum_{k=M}^{n-j} \left(\frac{99}{100}\right)^2 \frac{M}{k} \frac{k}{n-j} \binom{n-j}{k} p_j^k (1-p_j)^{n-j-k} = f_j(p_j).\end{aligned}$$

Let $m \coloneqq \min_{i\in[\lfloor 9n/10\rfloor]_0} f_i(\varepsilon)$.

Claim 7.1. *We have $\frac{600}{n} < m < \frac{100M}{n}$.*

Proof of Claim 7.1. Let $i \in [\lfloor 9n/10\rfloor]_0$ be such that $f_i(\varepsilon) = m$. Clearly,

$$f_i(\varepsilon) \leq \frac{100M}{n} \sum_{k=M}^{n-i} \binom{n-i}{k} \varepsilon^k (1-\varepsilon)^{n-i-k} < \frac{100M}{n}.$$

Moreover, we have that

$$f_i(\varepsilon) = \left(\frac{99}{100}\right)^2 \frac{M}{n-i} \sum_{k=M}^{n-i} \binom{n-i}{k} \varepsilon^k (1-\varepsilon)^{n-i-k} > \frac{1}{2}\left(\frac{99}{100}\right)^2 \frac{M}{n},$$

as $M \leq \varepsilon(n-i)/2$. ◀

For each $i \in [\lfloor 9n/10 \rfloor]_0$ such that $f_i(\varepsilon) > m$, since $f_i(x)$ is continuous, by the intermediate value theorem we can choose some $p_i \in (0, \varepsilon)$ such that $f_i(p_i) = m$. This determines the probability vector $\mathbf{p} = (p_0, \dots, p_{n-1})$. By Claim 7.1, the tuple $(n, \mathbf{p}, M)$ is feasible, hence the statement is satisfied. □

In order to construct the near-spanning tree, we will generate a graph $P \sim \mathcal{P}(n, \mathbf{p}, M)$, for some feasible $(n, \mathbf{p}, M)$, and will be interested in whether or not there exists a chain in P from some vertex $x \in L_m$ to some vertex $y \in L_{m'}$, for $m' > m$ and $x \subseteq y$. Note that the presence and absence of such chains in P are highly dependent. Thus, in order to show that such chains exist with high probability, we will consider the number of x-y chains and bound its variance. We do so in the following lemma. In order to state it, we first need to set up some notation.

Given $x \in L_m$ and $y \in L_{m'}$ with $m' \geq m$, we denote by $\mathcal{X}_{x,y}$ the collection of x-y chains in $\mathcal{Q}^n$. For each $X \in \mathcal{X}_{x,y}$ and any graph $G \subseteq \mathcal{Q}^n$, let $Y_X(G)$ be the corresponding indicator variable which takes value 1 if $X \subseteq G$ and 0 otherwise. Let $Y_{x,y}(G) \coloneqq \sum_{X \in \mathcal{X}_{x,y}} Y_X(G)$. Whenever G is clear from the context, we will simply write $Y_{x,y}$. We define

$$\Delta(Y_{x,y}) \coloneqq \sum_{\substack{(X,X') \in \mathcal{X}_{x,y}^2 \\ X \neq X'}} \mathrm{Cov}[Y_X Y_{X'}],$$

so $\mathrm{Var}[Y_{x,y}] = \Delta(Y_{x,y}) + \sum_{X \in \mathcal{X}_{x,y}} \mathrm{Var}[Y_X]$.

Lemma 7.8. *Let $P \sim \mathcal{P}(n, \mathbf{p}, M)$, where $(n, \mathbf{p}, M)$ is feasible with $0 < 1/n \ll 1/M$. Let $1 \leq m < m' \leq 9n/10$ with $m' - m + 1 \geq n/4 - 1$. Let $x \in L_m$ and $y \in L_{m'}$ with $x \subseteq y$. Then,*

$$\Delta(Y_{x,y}) \leq 2\mathbb{E}[Y_{x,y}]^2.$$

The proof of Lemma 7.8 makes use of the analysis in the proof of a similar lemma of Kohayakawa, Kreuter and Osthus [40, Lemma 7]. In order to shorten our analysis here, we first state a partial result which follows from the analysis of [40]. For this, we first need to give some more definitions.

Fix $x \in L_m$ and $y \in L_{m'}$ with $m' \geq m$ and $x \subseteq y$. Observe that $|\mathcal{X}_{x,y}| = \mathrm{dist}(x,y)! = (m'-m)!$ depends only on the distance between x and y. For each $k \in [n]$, let $R_k \coloneqq (k-1)!$. Given any $X, X' \in \mathcal{X}_{x,y}$ with $X \neq X'$, let $i(X,X') \coloneqq |V(X) \cap V(X')| - 2$, let $s(X,X')$ be the number of connected components of $X - V(X')$, and let $\ell(X,X')$ be the largest order over these components.

Next, we define the set of possible intersection patterns for two chains. Let $k \coloneqq m' - m + 1$. Given any chains $X, X' \in \mathcal{X}_{x,y}$, let $A(X,X')$ be the collection of indices $a \in [k-2]$ for which X and X' agree on their $(a+1)$-th elements (where we consider x to be the first element of X and X'). An *admissible* (i,ℓ,s)*-pattern* is a set $A \subseteq [k-2]$ with $|A| = i$ such that the longest interval of consecutive elements in $[k-2] \setminus A$ contains exactly ℓ elements and such that the number of maximal intervals of consecutive elements in $[k-2] \setminus A$ is exactly s. We denote by $\mathcal{A}_{i,\ell,s}$ the set of all admissible (i,ℓ,s)-patterns. Furthermore, we define $C_{i,\ell,s} \coloneqq |\mathcal{A}_{i,\ell,s}|$. Note that any pair of chains $X, X' \in \mathcal{X}_{x,y}$ with $i(X,X') = i$, $\ell(X,X') = \ell$ and $s(X,X') = s$ define an admissible (i,ℓ,s)-pattern $A(X,X') \in \mathcal{A}_{i,\ell,s}$.

Given a chain $X \in \mathcal{X}_{x,y}$ and a pattern $A \in \mathcal{A}_{i,\ell,s}$, let $F(A)$ be the number of chains $X' \in \mathcal{X}_{x,y}$ such that $A(X,X') = A$. (Note that the definition of $F(A)$ is independent of X.) Let $F_{i,\ell,s} \coloneqq \max_{A \in \mathcal{A}_{i,\ell,s}} F(A)$. Observe that $F_{i,\ell,s}$ is an upper bound on the number of chains X' with $A(X,X') = A$.

Finally, for each triple $(i,\ell,s)\in[k-3]_0\times[k-2]^2$, let

$$\Delta_{i,\ell,s}:=\sum_{\substack{(X,X')\in\mathcal{X}^2_{x,y},\,X\neq X'\\ i(X,X')=i,\,\ell(X,X')=\ell,\,s(X,X')=s}}\mathbb{E}[Y_XY_{X'}].$$

Furthermore, let

$$\Delta_0(Y_{x,y}):=\sum_{\substack{(X,X')\in\mathcal{X}^2_{x,y}\\ i(X,X')=0}}\mathrm{Cov}[Y_XY_{X'}]\qquad\text{and}\qquad\Delta_1(Y_{x,y}):=\sum_{\substack{(X,X')\in\mathcal{X}^2_{x,y}\\ i(X,X')\in[k-3]}}\mathrm{Cov}[Y_XY_{X'}].$$

Thus, $\Delta(Y_{x,y})=\Delta_0(Y_{x,y})+\Delta_1(Y_{x,y})$. Note that, by summing $\Delta_{i,\ell,s}$ over all triples $(i,\ell,s)\in[k-3]\times[k-2]^2$, we obtain an upper bound for $\Delta_1(Y_{x,y})$.

Lemma 7.9 ([40]). *For all $M>100$ there exists n_0 such that, for all $n\geq n_0$, the following holds. Let $x\in L_1$ and $y\in L_{n-1}$ with $x\subseteq y$. Let $p\geq M/(2n)$. Let $Q\subseteq\mathcal{Q}^n$ be a random subgraph chosen according to any distribution such that*

$$\frac{\Delta_{i,\ell,s}}{\mathbb{E}[Y_{x,y}]^2}\leq\frac{C_{i,\ell,s}F_{i,\ell,s}}{R_{n-1}p^i},$$

for each possible choice of $(i,\ell,s)\in[k-3]\times[k-2]^2$. Then,

$$\Delta_1(Y_{x,y})\leq\frac{100}{M}\mathbb{E}[Y_{x,y}]^2.$$

With this, we are finally ready to prove Lemma 7.8.

Proof of Lemma 7.8. Let $P\sim\mathcal{P}(n,\mathbf{p},M)$, where $(n,\mathbf{p},M)$ is feasible. Recall, from Definition 7.3, that P is generated by first sampling a set $R\sim \mathit{Res}(\mathcal{Q}^n,1/100)$ and a graph $W\sim\mathcal{W}^n_{\mathbf{p}}$. We then generate the graph W' by choosing, for each $v\in\bigcup_{i=0}^{\lfloor 9n/10\rfloor}L_i$, a set of M up-neighbours uniformly at random from the set of up-neighbours v has in W, provided $d^{\uparrow}_{W'}(v)\geq M$ (and by setting $d^{\uparrow}_{W'}(v):=0$ otherwise). Let $t':=t/(\frac{99}{100})^2$. Thus, for all $e\in\bigcup_{i=0}^{\lfloor 9n/10\rfloor}E(L_i,L_{i+1})$ we have by Remark 7.6 that

$$\mathbb{P}[e\in W']=t'/n.$$

Let $k:=m'-m+1$, and let X be a fixed x-y chain in $\mathcal{Q}^n$. By Remark 7.4 it follows that

$$\mathbb{E}[Y_{x,y}]=R_k\mathbb{P}[X\subseteq P]=R_k(t'/n)^{k-1}\Big(\frac{99}{100}\Big)^k.\tag{7.1}$$

Furthermore, for all $(i,\ell,s)\in[k-3]_0\times[k-2]^2$, we have that

$$\Delta_{i,\ell,s}\leq R_kC_{i,\ell,s}F_{i,\ell,s}\left(\frac{99t'}{100n}\right)^{2k-i-2}.\tag{7.2}$$

To see this, note that we may first choose an x-y chain X, for which there are R_k choices. Next, we choose an admissible (i,ℓ,s)-pattern $A\in\mathcal{A}_{i,\ell,s}$, of which there are $C_{i,\ell,s}$. We then have at most $F_{i,\ell,s}$ choices for x-y chains X' with $A(X,X')=A$. Next, we bound the number of vertices and edges of $X\cup X'$. It is clear that X has k vertices and $k-1$ edges, and $|V(X')\setminus V(X)|=k-i-2$. Moreover, observe that $|E(X')\setminus E(X)|\geq k-i-1$. The bound finally follows by considering the probability that all these vertices and edges are present in P and by Remark 7.6.

We are going to compute bounds for $\Delta_0(Y_{x,y})$ and $\Delta_1(Y_{x,y})$ separately, and then combine them to obtain the result. We begin with a bound for $\Delta_1(Y_{x,y})$. Combining (7.1) and (7.2), it follows that, for all $(i,\ell,s)\in[k-3]\times[k-2]^2$,

$$\frac{\Delta_{i,\ell,s}}{\mathbb{E}[Y_{x,y}]^2}\leq\frac{C_{i,\ell,s}F_{i,\ell,s}}{R_k}\left(\frac{n}{t'}\right)^i\left(\frac{1}{\frac{99}{100}}\right)^{i+2}\leq\frac{C_{i,\ell,s}F_{i,\ell,s}}{R_k}\left(\frac{n}{t'(\frac{99}{100})^3}\right)^i.$$

Note that $(\frac{m'-m+2}{n})t'(\frac{99}{100})^3 > 100$. It follows that we can apply Lemma 7.9 with $(\frac{m'-m+2}{n})t'(\frac{99}{100})^3$ and $m'-m+2$ playing the roles of M and n and $p = t'(\frac{99}{100})^3/n$ to obtain that

$$\Delta_1(Y_{x,y}) \leq \frac{100}{(\frac{m'-m+2}{n})t'(\frac{99}{100})^3}\mathbb{E}[Y_{x,y}]^2 \leq \mathbb{E}[Y_{x,y}]^2. \tag{7.3}$$

We now turn our attention to $\Delta_0(Y_{x,y})$. For any two chains $X, X' \in \mathcal{X}_{x,y}$ such that $i(X,X') = 0$, we have that $X \cup X'$ has $2k-2$ vertices and the same number of edges. Therefore, by Remarks 7.4 and 7.6 we have $\mathbb{E}[Y_X Y_{X'}] \leq (\frac{99t'}{100n})^{2k-2}$, and by (7.1) we have that

$$\Delta_0(Y_{x,y}) \leq R_k^2\left(\frac{99t'}{100n}\right)^{2k-2}\left(1-\left(\frac{99}{100}\right)^2\right) \leq \mathbb{E}[Y_{x,y}]^2. \tag{7.4}$$

The conclusion follows immediately by combining (7.3) and (7.4). $\square$

In order to proceed further, we will consider unions of independent graphs $P \sim \mathcal{P}(n, \mathbf{p}, M)$.

Definition 7.10. *Let $n, M, C \in \mathbb{N}$ and $\mathbf{p} \in [0,1]^n$. We define $\mathcal{P}^C(n, \mathbf{p}, M)$ to be a distribution on subgraphs of $\mathcal{Q}^n$ such that $P \sim \mathcal{P}^C(n, \mathbf{p}, M)$ is generated by taking C independently generated graphs $P_i \sim \mathcal{P}(n, \mathbf{p}, M)$ and setting $P := \bigcup_{i=1}^C P_i$. For each $i \in [C]$, there is a set $R_i \sim$ Res$(\mathcal{Q}^n, 1/100)$ associated with P_i. Let $R := \bigcap_{i=1}^C R_i$. We say that R is the* reservoir associated with *P.*

It follows from Definitions 7.3 and 7.10 that there is a joint distribution of $\mathcal{P}^C(n, \mathbf{p}, M)$ and $\mathcal{Q}^n_{\min\{1,Cp\}}$ such that $\mathcal{P}^C(n, \mathbf{p}, M) \subseteq \mathcal{Q}^n_{\min\{1,Cp\}}$, where $p = \max_{i\in[n-1]_0} p_i$. Note that for all $x \in V(\mathcal{Q}^n)$ we have that $\mathbb{P}[x \in R] = (1/100)^C$.

Our next goal is to prove that, by choosing constants appropriately, there is a high probability that there exists an x-y chain in $P \sim \mathcal{P}^C(n, \mathbf{p}, M)$, even if we restrict the set of 'valid' chains to a significant subset of the total. For this, we will make use of Lemma 7.8. Given any vertices $x \in L_m$ and $y \in L_{m'}$ with $x \subseteq y$, any set $\mathcal{Z} \subseteq \mathcal{X}_{x,y}$, and any graph $G \subseteq \mathcal{Q}^n$, we denote the number of x-y chains $X \in \mathcal{Z}$ such that $X \subseteq G$ by $Y(\mathcal{Z}, G)$.

Corollary 7.11. *For $n, C \in \mathbb{N}$ and $\eta, \alpha > 0$ such that $0 < 1/n \ll 1/C \ll \eta, \alpha$ and any feasible $(n, \mathbf{p}, M)$ with $0 < 1/n \ll 1/M$, the following holds. Let $1 \leq m < m' \leq 9n/10$ with $m'-m+1 \geq n/4-1$. Let $x \in L_m$ and $y \in L_{m'}$ with $x \subseteq y$. Let $\mathcal{Z}_{x,y} \subseteq \mathcal{X}_{x,y}$ be such that $|\mathcal{Z}_{x,y}| \geq \alpha|\mathcal{X}_{x,y}|$. Let $P \sim \mathcal{P}^C(n, \mathbf{p}, M)$. Then,*

$$\mathbb{P}[Y(\mathcal{Z}_{x,y}, P) > 0] \geq 1 - \eta.$$

Proof. For each $i \in [C]$, let $P_i \sim \mathcal{P}(n, \mathbf{p}, M)$, and let $P := \bigcup_{i=1}^C P_i$. Let $Y_i := Y_{x,y}(P_i)$ and $Z_i := Y(\mathcal{Z}_{x,y}, P_i)$, and let

$$\Delta(Z_i) := \sum_{\substack{(X,X')\in\mathcal{Z}_{x,y}^2\\ X\neq X'}} \mathrm{Cov}[Y_X(P_i)Y_{X'}(P_i)].$$

Note that

$$\mathbb{E}[Y_i^2] \leq \Delta(Y_i) + \mathbb{E}[Y_i] + \mathbb{E}[Y_i]^2. \tag{7.5}$$

We also have

$$\mathbb{E}[Z_i^2] \leq \mathbb{E}[Y_i^2] \tag{7.6}$$

and, since all x-y chains are equiprobable,

$$\mathbb{E}[Z_i]^2 \geq \alpha^2\mathbb{E}[Y_i]^2. \tag{7.7}$$

Let $k := m'-m+1$. By (7.1), we have that $\mathbb{E}[Y_i] = R_k(t'/n)^{k-1}(99/100)^k$, where t' is the value given in Remark 7.6. Recall that $R_k = |\mathcal{X}_{x,y}| = (k-1)!$. We have by Stirling's formula that $\mathbb{E}[Y_i] > 1$. Therefore, $\mathbb{E}[Y_i] \leq \mathbb{E}[Y_i]^2$. Moreover, it follows by Lemma 7.8 that $\Delta(Y_i) \leq 2\mathbb{E}[Y_i]^2$. So $\mathbb{E}[Y_i^2] \leq 4\mathbb{E}[Y_i]^2$ by (7.5). Combining this with (7.6), (7.7) and Proposition 4.1 we obtain

$$\mathbb{P}[Z_i = 0] \leq 1 - \frac{\mathbb{E}[Z_i]^2}{\mathbb{E}[Z_i^2]} \leq 1 - \frac{\alpha^2\mathbb{E}[Y_i]^2}{\mathbb{E}[Z_i^2]} \leq 1 - \frac{\alpha^2\mathbb{E}[Y_i]^2}{\mathbb{E}[Y_i^2]} \leq 1 - \alpha^2/4.$$

It follows that

$$\mathbb{P}[Y(\mathcal{Z}_{x,y}, P) = 0] = \prod_{i\in[C]} \mathbb{P}[Z_i = 0] \leq (1 - \alpha^2/4)^C \leq \eta. \qquad \square$$

When performing our analysis on the structure of P, the dependence of chains on each other becomes difficult to take into account. In order to deal with this issue, we will show that, with high probability, it suffices to consider only chains which lie in some large subsets of the total sets of chains, with the property that the presence or absence of a chain in one of these large subsets is independent from chains of all other subsets. (Note that Corollary 7.11 works for these sets of chains as long as they are not too small.) The next two lemmas guarantee the existence of such sets. In Lemma 7.12 we prove that, assuming $x, x' \in L_m$, and $y, y' \in L_{m'}$, where y, y' are far apart, one can construct very large sets of chains between the pairs x, y and x', y', which are independent in the sense described above. Then, in Lemma 7.16 we will prove that we can pick many endpoints $y \in L_{m'}$ in such a way that they are suitably far apart.

Given $0 \leq m < m' \leq n$, let $x, x' \in L_m$ and $y, y' \in L_{m'}$ with $x \subseteq y$ and $x' \subseteq y'$. We denote by $\mathcal{X}_{x,y}^{\neg x',y'}$ the collection of chains $X \in \mathcal{X}_{x,y}$ for which there is no $X' \in \mathcal{X}_{x',y'}$ with $V(X) \cap V(X') \neq \varnothing$.

Lemma 7.12. *For all $n \geq 100$, the following holds. Let $1 \leq m < m' \leq n - 1$ be such that $n/4 - 1 \leq k := m' - m + 1 \leq n/2$. Let $x, x' \in L_m$ and $y, y' \in L_{m'}$ with $x \subseteq y$ and $x' \subseteq y'$ be such that* $\mathrm{dist}(x, x') = 2$ *and* $\mathrm{dist}(y, y') \geq 9k^2/(10n)$*. Then,*

$$|\mathcal{X}_{x,y}^{\neg x',y'}| \geq \left(1 - \frac{60000}{n}\right) |\mathcal{X}_{x,y}|.$$

Proof. We may assume that $x \cup x' \subseteq y \cap y'$, since otherwise $\mathcal{X}_{x,y}^{\neg x',y'} = \mathcal{X}_{x,y}$. Let $b := |y \cap y'|$. We have that $b \leq m' - 9k^2/(20n)$. Let H denote the smallest subcube of $\mathcal{Q}^n$ which contains both $x \cup x'$ and $y \cap y'$. For each $i \in [b] \setminus [m]$, let $\mathcal{X}_{x,y}^i \subseteq \mathcal{X}_{x,y}$ be the set of chains $X \in \mathcal{X}_{x,y}$ such that $V(X) \cap L_i \cap V(H) \neq \varnothing$. Note that $\mathcal{X}_{x,y}^{\neg x',y'} \supseteq \mathcal{X}_{x,y} \setminus \bigcup_{i\in[b]\setminus[m]} \mathcal{X}_{x,y}^i$ and

$$|\mathcal{X}_{x,y}^i| = \binom{b - m - 1}{i - m - 1} (m' - i)!(i - m)!. \tag{7.8}$$

Indeed, there are $\binom{b-m-1}{i-m-1}$ choices to fix an element $z \in V(H) \cap L_i$. (To see this, consider that H is itself a cube of dimension $b - m - 1$, and we are choosing a vertex z from the $(i - m - 1)$-th level of this cube.) Then, there are $(i - m)!$ x-z chains, and $(m' - i)!$ z-y chains.

Recall that $|\mathcal{X}_{x,y}| = (k - 1)!$. By comparing this with (7.8) and simplifying, for all $i \in [b] \setminus [m]$ we obtain

$$\frac{|\mathcal{X}_{x,y}^i|}{|\mathcal{X}_{x,y}|} = \frac{i - m}{m' - m} \prod_{j=1}^{i-m-1} \frac{b - m - j}{m' - m - j} \leq \frac{i - m}{k - 1}. \tag{7.9}$$

We now split the analysis into two cases. First, when i is small, we bound (7.9) directly. For all $i \in [b] \setminus [m]$ with $i \leq m + 64$, it follows from (7.9) that

$$\frac{|\mathcal{X}_{x,y}^i|}{|\mathcal{X}_{x,y}|} \leq \frac{64}{n/4 - 1} \leq \frac{300}{n}. \tag{7.10}$$

On the other hand, for each $i \in [b - 1] \setminus [m]$, by (7.9) we have that

$$\frac{|\mathcal{X}_{x,y}^{i+1}|}{|\mathcal{X}_{x,y}|} = \frac{|\mathcal{X}_{x,y}^i|}{|\mathcal{X}_{x,y}|} \frac{i + 1 - m}{i - m} \cdot \frac{b - i}{m' - i} \leq \frac{39}{40} \frac{i + 1 - m}{i - m} \frac{|\mathcal{X}_{x,y}^i|}{|\mathcal{X}_{x,y}|}.$$

For all $i \in [b - 1] \setminus [m + 64]$, this yields

$$\frac{|\mathcal{X}_{x,y}^{i+1}|}{|\mathcal{X}_{x,y}|} \leq \frac{99}{100} \frac{|\mathcal{X}_{x,y}^i|}{|\mathcal{X}_{x,y}|}. \tag{7.11}$$

Finally, by combining (7.10) and (7.11), and considering a geometric series we conclude that

$$\frac{|\mathcal{X}_{x,y}^{\neg x',y'}|}{|\mathcal{X}_{x,y}|} \geq \frac{|\mathcal{X}_{x,y}| - \sum_{i=m+1}^{b} |\mathcal{X}_{x,y}^{i}|}{|\mathcal{X}_{x,y}|} \geq 1 - \frac{60000}{n}. \qquad \square$$

Remark 7.13. *Lemma 7.12 holds similarly if* $\mathrm{dist}(x,x') \geq 9k^2/(10n)$ *and* $\mathrm{dist}(y,y') = 2$.

Proposition 7.14. *Let* $0 < 1/n \ll \gamma, 1/k \leq 1$, *where* $n, k \in \mathbb{N}$, *and let* $S \subseteq V(\mathcal{Q}^n)$ *be such that, for all distinct* $x, x' \in S$, *we have* $\mathrm{dist}(x,x') \geq \gamma n$. *Then, for any* $y \in L_m$ *such that* $m \geq n/8$, *and for every* $\gamma m/2 \leq t \leq (1-\gamma/2)m$, *we have* $|y^{(t)} \cap B^k_{\mathcal{Q}^n}(S)| \leq |y^{(t)}|2^{-\gamma n/200}$.

Proof. Let $m \geq n/8$ and $y \in L_m$. Let $\gamma m/2 \leq t \leq (1-\gamma/2)m$ and let $S' \subseteq S$ be the set of all those $x \in S$ for which $B^k_{\mathcal{Q}^n}(x) \cap y^{(t)} \neq \varnothing$. We have that

$$|B^k_{\mathcal{Q}^n}(S) \cap y^{(t)}| \leq \sum_{x \in S'} |B^k_{\mathcal{Q}^n}(x) \cap y^{(t)}| \leq 2n^k|S'|. \tag{7.12}$$

Moreover, for every $x, x' \in S'$ we have that $B^{\gamma n/3}_{\mathcal{Q}^n}(x) \cap B^{\gamma n/3}_{\mathcal{Q}^n}(x') = \varnothing$, and, therefore,

$$|S'|(\min_{x \in S'} |B^{\gamma n/3}_{\mathcal{Q}^n}(x) \cap y^{(t)}|) \leq |y^{(t)}|. \tag{7.13}$$

Claim 7.2. *For every* $x \in S'$ *we have* $|B^{\gamma n/3}_{\mathcal{Q}^n}(x) \cap y^{(t)}| \geq 2^{\gamma m/20}$.

Proof. Let $x' \in B^k_{\mathcal{Q}^n}(x) \cap y^{(t)}$. Let $z \subseteq x'$ be such that $z \in L_{\gamma m/7}$ (recall that $t \geq \gamma m/2$). Since $y \in L_m$ we have that $|y \setminus x'| = m - t$. Let $z' \subseteq y \setminus x'$ be such that $z' \in L_{\gamma m/7}$ (recall that $t \leq (1-\gamma/2)m$). It follows that $(x' \setminus z) \cup z' \in B^{\gamma n/3}_{\mathcal{Q}^n}(x) \cap y^{(t)}$. Note that there are $\binom{t}{\gamma m/7}$ choices for z and $\binom{m-t}{\gamma m/7}$ choices for z'. It follows that

$$\left|B^{\gamma n/3}_{\mathcal{Q}^n}(x) \cap y^{(t)}\right| \geq \binom{m-t}{\gamma m/7}\binom{t}{\gamma m/7} \geq 2^{\gamma m/20}. \qquad \blacktriangleleft$$

Combining (7.12), (7.13) and the above claim we have

$$|B^k_{\mathcal{Q}^n}(S) \cap y^{(t)}| \leq \frac{2n^k|y^{(t)}|}{\min_{x \in S'}|B^{\gamma n/3}_{\mathcal{Q}^n}(x) \cap y^{(t)}|} \leq 2n^k|y^{(t)}|2^{-\gamma m/20} \leq |y^{(t)}|2^{-\gamma n/200}. \qquad \square$$

Given $x, y \in V(\mathcal{Q}^n)$ with $x \subseteq y$ and $S \subseteq V(\mathcal{Q}^n)$, we denote by $X^{\neg S}_{x,y}$ the collection of chains $X \in \mathcal{X}_{x,y}$ for which $V(X) \cap S = \varnothing$.

Lemma 7.15. *Let* $0 < 1/n \ll \gamma, 1/k \leq 1$ *where* $n, k \in \mathbb{N}$, *and let* $S \subseteq V(\mathcal{Q}^n)$ *be such that for all* $x, x' \in S$ *we have* $\mathrm{dist}(x,x') \geq \gamma n$. *Let* $x, y \in V(\mathcal{Q}^n) \setminus B^k_{\mathcal{Q}^n}(S)$ *with* $x \subseteq y$ *and* $m := \mathrm{dist}(x,y) \geq n/8$. *Then,* $|\mathcal{X}^{\neg B^k_{\mathcal{Q}^n}(S)}_{x,y}| \geq 3m!/4$.

Proof. We may assume that $x = \varnothing$ and $y = [m]$, where $m \geq n/8$. Let $\mathcal{X}^i_{x,y}$ denote the collection of chains $X \in \mathcal{X}_{x.y}$ for which $V(X) \cap L_i \cap B^k_{\mathcal{Q}^n}(S) \neq \varnothing$. We have

$$|\mathcal{X}_{x,y} \setminus \mathcal{X}^{\neg B^k_{\mathcal{Q}^n}(S)}_{x,y}| \leq \sum_{i=1}^{m-1} |\mathcal{X}^i_{x,y}|. \tag{7.14}$$

Furthermore, by Proposition 7.14 (with $\gamma/2$ playing the role of γ), for all $\gamma m/4 \leq i \leq (1-\gamma/4)m$ we have that

$$|\mathcal{X}^i_{x,y}| \leq \binom{m}{i} 2^{-\gamma m/400} i!(m-i)! = 2^{-\gamma m/400} m!. \tag{7.15}$$

Next, we consider the case $i \in [\gamma m/4]$, where first we prove the following claim.

Claim 7.3. *For all* $i \in [4k]$ *we have* $|\mathcal{X}^i_{x,y}| \leq (2n)^{i-1}(k+1)i!(m-i)!$.

Proof. Observe that $|S \cap \bigcup_{i=1}^{\gamma n/2-1} L_i| \leq 1$. If $S \cap \bigcup_{i=1}^{\gamma n/2-1} L_i = \varnothing$, then $\mathcal{X}_{x,y}^i = \varnothing$ for all $i \in [4k]$, so assume $|S \cap \bigcup_{i=1}^{\gamma n/2-1} L_i| = 1$. Let v be the unique vertex in $S \cap \bigcup_{i=1}^{\gamma n/2-1} L_i$. Then, $B_{\mathcal{Q}^n}^k(v) \cap L_i = B_{\mathcal{Q}^n}^k(S) \cap L_i$ for each $i \in [4k]$. Thus, in order to prove Claim 7.3, it suffices to show that $|B_{\mathcal{Q}^n}^k(v) \cap L_i| \leq (2n)^{i-1}(k+1)$ for each $i \in [4k]$.

We will proceed by induction on i. Since $\varnothing = x \notin B_{\mathcal{Q}^n}^k(v)$, it follows that $|B_{\mathcal{Q}^n}^k(v) \cap L_1| \leq k+1$, so the base case holds.

Now, suppose that $|B_{\mathcal{Q}^n}^k(v) \cap L_{i-1}| \leq (2n)^{i-2}(k+1)$ for some $2 \leq i \leq 4k$. Consider first the case where $v \in L_j$ for some $i \leq j \leq i+k$. In this case, any $u \in L_i \cap B_{\mathcal{Q}^n}^k(v)$ satisfies either

(i) $u \subseteq v$ or
(ii) there is a v-u path of length at most k whose penultimate vertex lies in L_{i-1}.

There are $\binom{j}{i} \leq \binom{k+i}{i}$ choices for u satisfying (i), whereas by applying induction to the penultimate vertex in such paths it follows that there are at most $n(2n)^{i-2}(k+1)$ choices for u satisfying (ii). Altogether, we have

$$|B_{\mathcal{Q}^n}^k(v) \cap L_i| \leq \binom{k+i}{i} + n(2n)^{i-2}(k+1) \leq (2n)^{i-1}(k+1).$$

The case where $v \in L_j$ for some $i-k \leq j < i$ is handled similarly. This completes the induction step and the proof of the claim. ◀

Recall that $|S \cap B_{\mathcal{Q}^n}^{\gamma m/2-1}(x)| \leq 1$. It follows that for all $i \in [\gamma m/3]$ we have that $|B_{\mathcal{Q}^n}^k(S) \cap L_i| \leq n^k$ and, therefore, $|\mathcal{X}_{x,y}^i| \leq n^k i!(m-i)!$. Suppose $|S \cap B_{\mathcal{Q}^n}^{\gamma m/2-1}(x)| = 1$, and let v be the unique vertex in $S \cap B_{\mathcal{Q}^n}^{\gamma m/2-1}(x)$. Let $j \in [\gamma m/2-1]$ be such that $v \in L_j$. It follows by Claim 7.3 that

$$\begin{aligned}\sum_{i=1}^{\gamma m/3} |\mathcal{X}_{x,y}^i| \leq \sum_{i=j-k}^{j+k} |\mathcal{X}_{x,y}^i| &\leq \begin{cases} \sum_{i=j-k}^{j+k} (2n)^{i-1}(k+1)i!(m-i)! & \text{if } j \leq 3k, \\ \sum_{i=j-k}^{j+k} n^k i!(m-i)! & \text{if } 3k < j < \gamma m/2 \end{cases} \\ &\leq \sum_{i=2k}^{4k} (2n)^{i-1}(k+1)i!(m-i)!. \end{aligned} \tag{7.16}$$

If $S \cap B_{\mathcal{Q}^n}^{\gamma m/2}(x) = \varnothing$, then this trivially holds too. By the symmetry of the hypercube, we also have that

$$\sum_{i=m-\gamma m/3}^{m} |\mathcal{X}_{x,y}^i| \leq \sum_{i=2k}^{4k} (2n)^{i-1}(k+1)i!(m-i)!. \tag{7.17}$$

Therefore, by (7.14)–(7.17) we have

$$|\mathcal{X}_{x,y} \setminus \mathcal{X}_{x,y}^{\neg B_{\mathcal{Q}^n}^k(S)}| \leq \sum_{i=\gamma m/3}^{m-\gamma m/3} 2^{-\gamma m/400} m! + 2\sum_{i=2k}^{4k} (2n)^{i-1}(k+1)i!(m-i)! \leq m!/4. \qquad \square$$

Lemma 7.16. *Let $0 < 1/n \ll \eta, 1/k', \gamma \leq 1$ and $n/2 \leq k < n$ with $n, k', k \in \mathbb{N}$. Let $\mathcal{A} \subseteq V(\mathcal{Q}^n)$ be such that for all $x \in V(\mathcal{Q}^n)$, we have that $|B_{\mathcal{Q}^n}^{\gamma n}(x) \cap \mathcal{A}| \leq 1$. Let $y \in L_k$ and let $s := \lfloor (k+1)/2 \rfloor$. Then, there exists three sets of vertices $A = \{a_1, \ldots, a_{(1-\eta)n}\} \subseteq L_1$, $B = \{b_1, \ldots, b_{(1-\eta)n}\} \subseteq N_{\mathcal{Q}^n}(y)$ and $C = \{c_1, \ldots, c_{(1-\eta)n}\} \subseteq L_s$ such that*

(i) *for each pair $i, j \in [(1-\eta)n]$ with $i \neq j$ we have $\mathrm{dist}(c_i, c_j) \geq 9s^2/(10n)$,*
(ii) *$B_{\mathcal{Q}^n}^{k'}(\mathcal{A}) \cap C = \varnothing$, and*
(iii) *for each $i \in [(1-\eta)n]$ we have $a_i \subseteq c_i \subseteq b_i$.*

Proof. Choose k vertices $c_1, \ldots, c_k \in y^{(s)}$ independently and uniformly at random. Then, choose $n-k$ vertices $c'_{k+1}, \ldots, c'_n \in y^{(s-1)}$ independently and uniformly at random. For each $i \in [n] \setminus [k]$, choose an element $a_i \in [n] \setminus y$ such that all the a_i are distinct, and let $c_i := c'_i \cup \{a_i\} \in L_s$. For each $i \in [n] \setminus [k]$, let $b_i \in N^{\uparrow}(y)$ be the unique vertex such that $a_i \in b_i$, so that when viewing each a_i now as a 1-element set, we have $a_i \subseteq c_i \subseteq b_i$ for all $i \in [n] \setminus [k]$.

Note that, for each pair $i, j \in [n]$ with $i \neq j$, we have that

$$\mathbb{E}[|c_i \cap c_j|] \leq s^2/k. \tag{7.18}$$

Assume that we reveal each c_i in turn. We then have that, for each $i \in [n] \setminus \{1\}$, the variables $|c_i \cap c_j|$ with $j \in [i-1]$ are hypergeometric. Thus, by Lemma 4.3 and (7.18), for each pair $i, j \in [n]$ with $i \neq j$ we have that

$$\mathbb{P}[|c_i \cap c_j| \geq 21s^2/(20k)] \leq e^{-n/25000}.$$

By a union bound, it follows that a.a.s. for all pairs $i, j \in [n]$ with $i \neq j$ we have $|c_i \cap c_j| < 21s^2/(20k)$ and, thus, $\mathrm{dist}(c_i, c_j) \geq 9s/10$. In particular, (i) holds a.a.s.

Next, let $S_1 := y^{(s)} \cap B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ and $S_2 := y^{(s-1)} \cap B^{k'+1}_{\mathcal{Q}^n}(\mathcal{A})$. By applying Proposition 7.14 first with k', $\mathcal{A}$ and s playing the roles of k, S and t, respectively, and then with $k'+1$, $\mathcal{A}$ and $s-1$ playing the roles of k, S and t, respectively, we obtain that $|S_1| \leq \binom{k}{s}2^{-\gamma n/200}$ and $|S_2| \leq \binom{k}{s-1}2^{-\gamma n/200}$. Therefore, for all $i \in [k]$ we have

$$\mathbb{P}[c_i \in S_1] \leq \frac{\binom{k}{s}2^{-\gamma n/200}}{\binom{k}{s}} = 2^{-\gamma n/200}.$$

For all $i \in [n] \setminus [k]$ we have $\mathbb{P}[c_i \in S_1] \leq \mathbb{P}[c_i' \in S_2]$ and similarly we have $\mathbb{P}[c_i' \in S_2] \leq 2^{-\gamma n/200}$. It now follows by a union bound that (ii) holds a.a.s.

Next, consider an auxiliary bipartite graph H with parts $y^{(1)}$ and $\{c_1, \dots, c_k\}$ and the following edge set. For each $i \in [k]$ and $a \in y^{(1)}$, let $\{a, c_i\}$ be an edge whenever $a \in c_i$. Thus, for each $i \in [k]$ we have that $d_H(c_i) = s$. Furthermore, it follows by Lemma 4.2 that a.a.s. for all $a \in y^{(1)}$ we have $d_H(a) = (1 \pm \eta/2)s$. Condition on this. Then, for each $X \subseteq y^{(1)}$, since we have $e_H(N_H(X), y^{(1)}) \geq e_H(N_H(X), X)$, it follows that $|N(X)| \geq |X| - \eta k/2$. Therefore, by Lemma 5.1 we have a matching of size $(1-\eta/2)k$ in H.

Similarly, a.a.s. we have a matching of size $(1-\eta/2)k$ in the analogous bipartite graph H' with parts $N^{\downarrow}(y)$ and $\{c_1, \dots, c_k\}$, where for each $i \in [k]$ and $b \in N^{\downarrow}(y)$ we have that $\{b, c_i\}$ is an edge whenever $c_i \subseteq b$. By concatenating these matchings (and relabelling the indices if necessary), it follows that a.a.s. there is an ordering $\{a_1, \dots, a_k\}$ of the elements of y and an ordering $\{b_1, \dots, b_k\}$ of the vertices of $N^{\downarrow}(y)$ such that, for all $i \in [(1-\eta)k]$, we have $a_i \subseteq c_i \subseteq b_i$. Furthermore, as explained before, by construction, for all $i \in [n] \setminus [k]$, we have $a_i \subseteq c_i \subseteq b_i$. Thus, (iii) holds a.a.s.

Finally, given that each of (i), (ii), (iii) holds a.a.s., there must exist a choice of $c_1, \dots, c_n$ such that (i)-(iii) hold simultaneously. □

We are now in a position to combine the results we have shown so far to prove the following key lemma, which is used to provide a base structure for the near-spanning tree which we seek.

Lemma 7.17. *Let $0 < 1/n \ll 1/C \ll \varepsilon' \leq 1/2$, and $0 < 1/n \ll 1/k', \gamma \leq 1/2$, where $n, k', C \in \mathbb{N}$, and let $(n, \mathbf{p}, M)$ be feasible with $0 < 1/n \ll 1/M$. Moreover, let $\mathcal{A} \subseteq V(\mathcal{Q}^n)$ be such that, for all $x \in V(\mathcal{Q}^n)$, we have $|B^{\gamma n}_{\mathcal{Q}^n}(x) \cap \mathcal{A}| \leq 1$ and $\varnothing \notin B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$. Then, with probability at least $1 - e^{-50n}$ we have that $P \sim \mathcal{P}^C(n, \mathbf{p}, M)$ satisfies the following: for all $y \in \bigcup_{i=\lceil n/2 \rceil}^{\lfloor 9n/10 \rfloor} L_i \setminus B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$, there exists a collection of chains $\mathcal{X}_y$ such that, for all $X \in \mathcal{X}_y$, we have $X \subseteq P - B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$, one of the endpoints of X belongs to L_1, and*

$$\left| N_{\mathcal{Q}^n}(y) \cap \bigcup_{X \in \mathcal{X}_y} V(X) \right| \geq (1 - \varepsilon')n.$$

Proof. Fix $\eta > 0$ such that $0 < 1/n \ll \eta \ll \varepsilon'$, and let $m := 480000$. Fix a vertex $y \in L_k \setminus B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ for some $n/2 \leq k \leq 9n/10$. Let $s := \lfloor (k+1)/2 \rfloor$. By Lemma 7.16 with $\eta/2$ playing the role of η, there exists a collection of vertices $\{c_1, \dots, c_{(1-\eta/2)n}\} \subseteq L_s$ such that $B^{k'}_{\mathcal{Q}^n}(c_i) \cap \mathcal{A} = \varnothing$ and $\mathrm{dist}(c_i, c_j) \geq 9s^2/(10n)$ for all pairs $i, j \in [(1-\eta/2)n]$ with $i \neq j$; an ordering $b_1, \dots, b_n$ of $N_{\mathcal{Q}^n}(y)$, and an ordering $a_1, \dots, a_n$ of L_1, such that for all $i \in [(1-\eta/2)n]$ we have $a_i \subseteq c_i \subseteq b_i$. For each $i \in [(1-\eta/2)n]$, we call (a_i, b_i, c_i) a *triple*. Note that $|B^{k'}_{\mathcal{Q}^n}(\mathcal{A}) \cap (L_1 \cup N_{\mathcal{Q}^n}(y))| \leq 2(k'+1)$, and hence we may assume for each $i \in [(1-\eta)n]$ that (a_i, b_i, c_i) forms a triple where $a_i, b_i \notin B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$. We denote by $\mathcal{T}$ the collection of all such triples. Partition $[(1-\eta)n]$ into two sets $\mathcal{I}_1 \coloneqq \{i \in [(1-\eta)n] : b_i \in N^{\downarrow}(y)\}$ and $\mathcal{I}_2 \coloneqq [(1-\eta)n] \setminus \mathcal{I}_1$. Let $\mathcal{A}_1 \coloneqq \{a_i : i \in \mathcal{I}_1\}$, $\mathcal{A}_2 \coloneqq \{a_i : i \in \mathcal{I}_2\}$, $\mathcal{B}_1 \coloneqq \{b_i : i \in \mathcal{I}_1\}$, $\mathcal{B}_2 \coloneqq \{b_i : i \in \mathcal{I}_2\}$, $\mathcal{C}_1 \coloneqq \{c_i : i \in \mathcal{I}_1\}$ and $\mathcal{C}_2 \coloneqq \{c_i : i \in \mathcal{I}_2\}$. Note that $k - \eta n \leq |\mathcal{C}_1| \leq k$.

We first turn our attention to $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$. Partition $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$ into sets $\mathcal{A}^1, \ldots, \mathcal{A}^m$, $\mathcal{B}^1, \ldots, \mathcal{B}^m$ and $\mathcal{C}^1, \ldots, \mathcal{C}^m$, respectively, each of size at least $\lfloor (k-\eta n)/m \rfloor$ and at most $2\lfloor (k-\eta n)/m \rfloor$, and such that, for every triple $(a,b,c) \in \mathcal{T}$ there exists $j \in [m]$ such that $a \in \mathcal{A}^j$, $b \in \mathcal{B}^j$ and $c \in \mathcal{C}^j$. For each $i \in [m]$, write $\mathcal{A}^i = \{a^i_1, \ldots, a^i_{|\mathcal{A}^i|}\}$, $\mathcal{B}^i = \{b^i_1, \ldots, b^i_{|\mathcal{A}^i|}\}$ and $\mathcal{C}^i = \{c^i_1, \ldots, c^i_{|\mathcal{A}^i|}\}$, where the labeling is such that $(a^i_j, b^i_j, c^i_j) \in \mathcal{T}$ for each $j \in [|\mathcal{A}^i|]$. For each $i \in [m]$ and $j \in [|\mathcal{A}^i|]$, we define the set $\mathcal{Z}_{a^i_j, c^i_j} \subseteq \mathcal{X}_{a^i_j, c^i_j}$ as the set of all chains $X \in \mathcal{X}_{a^i_j, c^i_j}$ which, for all $j' \in [|\mathcal{A}^i|] \setminus \{j\}$, neither intersect any chain $X' \in \mathcal{X}_{a^i_{j'}, c^i_{j'}}$, nor $B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$. By Lemmas 7.12 and 7.15 and the definition of m, we have that

$$|\mathcal{Z}_{a^i_j, c^i_j}| \geq \frac{1}{2} |\mathcal{X}_{a^i_j, c^i_j}|. \tag{7.19}$$

For each triple $(a,b,c) \in \mathcal{T}$ and any graph $G \subseteq \mathcal{Q}^n$, let $I_{a,c}(G)$ take value 1 if $Y(\mathcal{Z}_{a,c}, G) > 0$, and 0 otherwise. (Recall that $Y(\mathcal{Z}_{a,c}, G)$ denotes the number of chains $X \in \mathcal{Z}_{a,c}$ with $X \subseteq G$.) For each $i \in [m]$, let $I_i(G) \coloneqq \sum_{j \in [|\mathcal{A}^i|]} I_{a^i_j, c^i_j}(G) = \sum_{j \in [|\mathcal{A}^i|]} I_{a^i_j, c^i_j}(G - B^{k'}_{\mathcal{Q}^n}(\mathcal{A}))$.

We are now in a position to consider $P \sim \mathcal{P}^C(n, \mathbf{p}, M)$. Recall that P is generated by sampling C independent graphs P_i, where $P_i \sim \mathcal{P}(n, \mathbf{p}, M)$. In each P_i we can give bounds on the probability that certain chains appear. Note that, for each $i \in [C]$ and each fixed $i' \in [m]$ we have that, for every pair $j, j' \in [|\mathcal{A}^{i'}|]$ with $j \neq j'$, the variables $Y(\mathcal{Z}_{a^{i'}_j, c^{i'}_j}, P_i)$ and $Y(\mathcal{Z}_{a^{i'}_{j'}, c^{i'}_{j'}}, P_i)$ are independent (and, therefore, $I_{a^{i'}_j, c^{i'}_j}(P_i)$ and $I_{a^{i'}_{j'}, c^{i'}_{j'}}(P_i)$ are independent too). Since C is a large constant, this independence will allow us to boost the probability that these chains appear in $P - B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$. The analysis is broken into two steps.

Claim 7.4. *With probability at least $1 - 2e^{-75n}$, the graph $P \sim P^C(n, \mathbf{p}, M)$ satisfies the following.*

(1) *$P - B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains an a-c chain for at least $(1-\varepsilon'/2)k$ of the triples $(a,b,c) \in \mathcal{T}$ with $c \in \mathcal{C}_1$.*
(2) *$P - B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains a c-b chain for at least $(1-\varepsilon'/2)k$ of the triples $(a,b,c) \in \mathcal{T}$ with $c \in \mathcal{C}_1$.*

Proof. We show that (1) and (2) each hold with probability $1 - e^{-75n}$. The result then follows by a union bound.

For (1), let $C' \coloneqq \sqrt{C}$. By (7.19), we can apply Corollary 7.11 with $(\varepsilon')^2$, $1/2$ and C' playing the roles of η, α and C, respectively. Thus, for $P' \sim \mathcal{P}^{C'}(n, \mathbf{p}, M)$, for all $i \in [m]$ and $j \in [|\mathcal{A}^i|]$ we have that

$$\mathbb{P}[I_{a^i_j, c^i_j}(P') = 1] = \mathbb{P}[Y(\mathcal{Z}_{a^i_j, c^i_j}, P') > 0] \geq 1 - (\varepsilon')^2.$$

It follows that for all $i \in [m]$ we have $\mathbb{E}[I_i(P')] \geq (1-(\varepsilon')^2)|\mathcal{A}^i|$ and, therefore, by Lemma 4.2,

$$\mathbb{P}[I_i(P') > |\mathcal{A}^i|(1-(\varepsilon')^{3/2})] > 1 - e^{-(\varepsilon')^3 n/(25 \cdot 10^6)}. \tag{7.20}$$

Let $P \sim \mathcal{P}^C(n, \mathbf{p}, M)$, and note that P can be generated by sampling C' independent graphs $P'_j \sim \mathcal{P}^{C'}(n, \mathbf{p}, M)$ and considering their union. For each $i \in [m]$, let $\mathcal{E}^i$ be the event that $I_i(P) > |\mathcal{A}^i|(1-(\varepsilon')^{3/2})$. It follows from (7.20) that, for each $i \in [m]$, we have $\mathbb{P}[\mathcal{E}^i] > 1 - e^{-100n}$. Now let $\mathcal{E}$ be the event that, for all $i \in [m]$, $\mathcal{E}_i$ holds. It follows by a union bound that

$$\mathbb{P}[\mathcal{E}] \geq 1 - e^{-75n}.$$

Thus, with probability at least $1 - e^{-75n}$ the graph $P - B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains an a-c chain for at least $(1-(\varepsilon')^{3/2})|\mathcal{C}_1|$ of the triples $(a,b,c) \in \mathcal{T}$ with $c \in \mathcal{C}_1$. Since $|\mathcal{C}_1| \geq (1-2\eta)k$, $P - B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains an a-c chain for at least $(1-\varepsilon'/2)k$ of the triples $(a,b,c) \in \mathcal{T}$ with $c \in \mathcal{C}_1$..

To show (2), for each triple $(a,b,c)\in\mathcal{T}$ with $c\in\mathcal{C}_1$, one can consider the set $\mathcal{X}_{c,b}$ and define sets $\mathcal{Z}_{c,b}$ and variables $I_{c,b}(G)$ analogously to the proof of (1). Then, by Corollary 7.11, Lemma 7.12 together with Remark 7.13, and Lemma 7.15, the same argument as above shows that, with probability at least $1-e^{-75n}$, the graph $P-B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains a c-b chain for at least $(1-\varepsilon'/2)k$ of the triples $(a,b,c)\in\mathcal{T}$ with $c\in\mathcal{C}_1$. ◀

It follows by Claim 7.4 that with probability at least $1-2e^{-75n}$ we have that $P-B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains an a-b chain for at least $(1-\varepsilon')k$ of the triples $(a,b,c)\in\mathcal{T}$ with $c\in\mathcal{C}_1$. We can prove an analogous result for the sets $\mathcal{A}_2$, $\mathcal{B}_2$ and $\mathcal{C}_2$. More specifically, we can show that with probability at least $1-2e^{-75n}$, for $P\sim P^C(n,\mathbf{p},M)$, the graph $P-B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains an a-b chain for at least $(1-\varepsilon')(n-k)$ of the triples $(a,b,c)\in\mathcal{T}$ with $c\in\mathcal{C}_2$. Combining this with the previous, it follows that, with probability at least $1-4e^{-75n}$, $P-B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$ contains an a-b chain for at least $(1-\varepsilon')n$ of the triples $(a,b,c)\in\mathcal{T}$. Finally, the result follows by a union bound over all $y\in\bigcup_{i=\lceil n/2\rceil}^{\lfloor 9n/10\rfloor}L_i\setminus B^{k'}_{\mathcal{Q}^n}(\mathcal{A})$. □

Let F be the union of all chains given by Lemma 7.17 (applied with $k':=k$). Then, F satisfies (T2) in Theorem 7.1 for all vertices $x\in\bigcup_{i=\lceil n/2\rceil}^{\lfloor 9n/10\rfloor}L_i\setminus B^{k}_{\mathcal{Q}^n}(\mathcal{A})$. However, we need this property to hold for every $x\in V(\mathcal{Q}^n)\setminus B^{k}_{\mathcal{Q}^n}(\mathcal{A})$. Recall the discussion in the beginning of this section where, due to the symmetries in the hypercube, we can 'redefine' any vertex $v\in V(\mathcal{Q}^n)$ to be the empty set $\varnothing$. As discussed, this leads to a redefined notion of levels in the hypercube where, for each $i\in[n]_0$, we let $L_i(v):=\{u\in V(\mathcal{Q}^n):\operatorname{dist}(u,v)=i\}$. The notion of a chain in this setting was also discussed.

When we consider this generalised setting, by replacing L_i with $L_i(v)$ in Definitions 7.2, 7.3 and 7.10, we obtain a distribution on subgraphs of $\mathcal{Q}^n$ which we denote by $\mathcal{P}^C_v(n,\mathbf{p},M)$. (Note, again, that there is a joint distribution of $\mathcal{P}^C_v(n,\mathbf{p},M)$ and $\mathcal{Q}^n_{\min\{1,Cp\}}$ such that $\mathcal{P}^C_v(n,\mathbf{p},M)\subseteq\mathcal{Q}^n_{\min\{1,Cp\}}$, where $p=\max_{i\in[n-1]_0}p_i$.) Then, for any fixed $v\in V(\mathcal{Q}^n)$, Lemma 7.17 holds in this setting by replacing chains by chains with respect to v. Intuitively, we may think of this simply as growing several branching processes rooted at different vertices of the hypercube. This will be crucial in proving (T2).

Note that F may have unbounded degrees and also may be disconnected. To turn F into a bounded degree forest we will later delete suitable edges. To make it connected without significantly raising any vertex degrees we will apply the following lemma.

Lemma 7.18. *For $n\in\mathbb{N}$ such that $0<1/n\ll\delta\leq 1/50$ and $0<\varepsilon<1/2$, the following holds a.a.s. Let $R\sim Res(\mathcal{Q}^n,\delta)$. Then, there exists a cycle in $\mathcal{Q}^n_\varepsilon[(L_1\cup L_2)\setminus R]$ which covers $L_1\setminus R$.*

Proof. Let $R\sim Res(\mathcal{Q}^n,\delta)$. Let $\mathcal{A}$ be the event that $|R\cap L_1|\geq n/4$. By Lemma 4.4 we have that $\mathbb{P}[\mathcal{A}]\leq e^{-\Theta(n)}$. Expose $R\cap L_1$ and condition on the event that $\mathcal{A}$ does not occur.

Note that for each pair of vertices $x,y\in L_1$ there exists a unique vertex $z\in L_2\cap N_{\mathcal{Q}^n}(x)\cap N_{\mathcal{Q}^n}(y)$ (in particular, $z=x\cup y$). Let H be an auxiliary graph with vertex set $L_1\setminus R$, where we include an edge between x and y if $x\cup y\notin R$ and $\{x,x\cup y\},\{y,x\cup y\}\in E(\mathcal{Q}^n_\varepsilon)$. By definition, a Hamilton cycle in H would correspond uniquely to a cycle in $\mathcal{Q}^n_\varepsilon[(L_1\cup L_2)\setminus R]$ covering $L_1\setminus R$. Note that H has the same distribution as a binomial random graph $G\sim G_{n-|R\cap L_1|,p}$, where $p=(1-\delta)\varepsilon^2$. Let $\mathcal{B}$ be the event that there exists a Hamilton cycle in H. As, after conditioning on $\mathcal{A}$ not holding, $G_{n-|R\cap L_1|,p}$ is a.a.s. Hamiltonian (see e.g. [42, 50]), it follows that

$$\mathbb{P}[\mathcal{B}]\geq\mathbb{P}[\mathcal{B}\mid\overline{\mathcal{A}}]\mathbb{P}[\overline{\mathcal{A}}]\geq(1-o(1))(1-e^{-\Theta(n)})=1-o(1).$$ □

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. Choose constants $M,C\in\mathbb{N}$ such that $1/D,\delta\ll 1/C,1/M\ll\varepsilon'$. By Proposition 7.7, there exists a tuple $(n,\mathbf{p},M)$ which is feasible and such that $\max_{i\in[n-1]_0}p_i\leq\varepsilon/(5C)$. Let $x_1:=\varnothing$, $x_2:=[\lceil n/2\rceil]$, $x_3:=[n]\setminus x_2$ and $x_4:=[n]$. For each $j\in[4]$, let $P_j\sim\mathcal{P}^C_{x_j}(n,\mathbf{p},M)$ be sampled independently, and let R_j be the reservoir associated with P_j. Let $R:=\bigcap_{j\in[4]}R_j$, and note that $R\sim Res(\mathcal{Q}^n,1/10^{8C})$. Finally, let $Q\sim\mathcal{Q}^n_{\varepsilon/5}$ be independent of all other previous choices. Recall that, for each $j \in [4]$, there is a joint distribution of $\mathcal{P}^C_{x_j}(n,\mathbf{p},M)$ and $\mathcal{Q}^n_{\varepsilon/5}$ such that $\mathcal{P}^C_{x_j}(n,\mathbf{p},M) \subseteq \mathcal{Q}^n_{\varepsilon/5}$ (see the discussion after Definition 7.10). It follows that there is a joint distribution of $\bigtimes_{j=1}^4 \mathcal{P}^C_{x_j}(n,\mathbf{p},M) \times \mathcal{Q}^n_{\varepsilon/5}$ and $\mathcal{Q}^n_\varepsilon$ such that $P_1 \cup P_2 \cup P_3 \cup P_4 \cup Q \subseteq \mathcal{Q}^n_\varepsilon$. Therefore, it suffices to show that we can find the desired tree T in $(P_1 \cup P_2 \cup P_3 \cup P_4 \cup (Q - R)) - B^k_{\mathcal{Q}^n}(\mathcal{A})$.

For each $j \in [4]$, let $A_j := \bigcup_{i=\lceil n/2\rceil}^{\lfloor 9n/10\rfloor} L_i(x_j) \setminus B^k_{\mathcal{Q}^n}(\mathcal{A})$, and let $\mathcal{E}_j$ be the event that, for all $y \in A_j$, the graph $P_j - B^k_{\mathcal{Q}^n}(\mathcal{A})$ contains a collection $\mathcal{X}^j_y$ of chains with respect to x_j, where each chain $X \in \mathcal{X}^j_y$ has an endpoint in $L_1(x_j)$ (and thus in $L_1(x_j) \setminus (R_j \cup B^k_{\mathcal{Q}^n}(\mathcal{A}))$), and such that at least $(1-\varepsilon')n$ of the neighbours of y in $\mathcal{Q}^n$ are covered by the union of the chains in $\mathcal{X}^j_y$. Note that $\mathcal{E}_j$ is equivalent to saying that the union of the chains in $\mathcal{X}^j_y$ satisfies (T2) for all $y \in A_j$. For each $j \in [4]$ we have by Lemma 7.17 that $\mathbb{P}[\mathcal{E}_j] \geq 1 - e^{-50n}$. Condition on the event that $\mathcal{E}_j$ holds for all $j \in [4]$.

For each $j \in [4]$, let $F_j \subseteq \mathcal{Q}^n$ be given by $F_j := \bigcup_{y\in A_j}\bigcup_{X\in\mathcal{X}^j_y} X$. For each $j \in [4]$, let $G_j \subseteq F_j$ be defined by removing, for each $y \in V(F_j) \setminus \{x_j\}$, all edges of F_j joining y to its down-neighbours with respect to x_j except for one (if y has one such down-neighbour in F_j). In particular, it follows that each connected component of G_j is a tree and contains one vertex in $L_1(x_j)$, and that $\Delta(G_j) \leq CM + 1$. Since G_j has the same vertex set as F_j, we have that G_j satisfies (T2) for all $y \in A_j$. Furthermore, note that $V(\mathcal{Q}^n) \setminus B^k_{\mathcal{Q}^n}(\mathcal{A}) = \bigcup_{j=1}^4 A_j$. Therefore, the graph $G := \bigcup_{j\in[4]} G_j$ satisfies (T2) and $\Delta(G) \leq 4CM + 4$.

Since $B^{k+2}_{\mathcal{Q}^n}(\mathcal{A}) \cap \{\varnothing, [n], [\lceil n/2\rceil], [n]\setminus[\lceil n/2\rceil]\} = \varnothing$ it follows that $B^k_{\mathcal{Q}^n}(\mathcal{A}) \cap (L_1(x_j) \cup L_2(x_j)) = \varnothing$ for each $j \in [4]$. Let $\mathcal{E}_5$ be the event that, for each $j \in [4]$, $Q[L_1(x_j) \cup L_2(x_j)] - R$ contains a cycle C_j which covers $L_1(x_j) \setminus R$. By four applications of Lemma 7.18 (applied with x_j playing the role of $\varnothing$) we have that $\mathbb{P}[\mathcal{E}_5] = 1 - o(1)$. Condition on the event that this occurs.

Let $H := G \cup \bigcup_{j\in[4]} C_j$. It follows that H is connected and $\Delta(H) \leq 4CM + 6$. In order to complete the proof, let $T \subseteq H$ be a spanning tree of H. □

7.2. **Extending the tree.** Roughly speaking, in Theorem 7.1 we showed that, for any $\varepsilon > 0$, given a reservoir chosen at random, the random graph $\mathcal{Q}^n_\varepsilon$ a.a.s. contains a bounded-degree tree T' which avoids the reservoir and satisfies the local property that, for every vertex $x \in V(\mathcal{Q}^n)$, all but a fixed small proportion of its neighbours are covered by T'. Our goal in this section is to show that T' can be extended into a tree T where the proportion of uncovered vertices (in each neighbourhood) is even smaller, while still retaining the bounded degree property. The precise statement is the following.

Theorem 7.19. *For all $0 < 1/n \ll 1/\ell, \varepsilon \leq 1$, where $n, \ell \in \mathbb{N}$, the following holds. Let $R, W \subseteq V(\mathcal{Q}^n)$ and let $T' \subseteq \mathcal{Q}^n - (R \cup W)$ be a tree. For each $x \in V(\mathcal{Q}^n) \setminus W$, let $Z(x) \subseteq N_{\mathcal{Q}^n}(x) \cap V(T')$ be such that $|Z(x)| \geq 3n/4$. Then, a.a.s. there exists a tree T with $T' \subseteq T \subseteq (\mathcal{Q}^n_\varepsilon \cup T') - W$ such that*

- (TC1) $\Delta(T) \leq \Delta(T') + 1$;
- (TC2) *for all $x \in V(\mathcal{Q}^n)$, we have that $|B^\ell_{\mathcal{Q}^n}(x) \setminus (V(T) \cup W)| \leq n^{3/4}$, and*
- (TC3) *for each $x \in V(T) \cap R$, we have that $d_T(x) = 1$ and the unique neighbour x' of x in T is such that $x' \in Z(x)$.*

Proof. Let $Q \sim \mathcal{Q}^n_\varepsilon$. For each $x \in V(\mathcal{Q}^n) \setminus W$ we have $3\varepsilon n/4 \leq \mathbb{E}[e_Q(x, Z(x))] \leq \varepsilon n$. Let $S_1 := \{x \in V(\mathcal{Q}^n) : d_Q(x) > 11\varepsilon n/10\}$, $S_2 := \{x \in V(\mathcal{Q}^n) \setminus W : e_Q(x, Z(x)) < 2\varepsilon n/3\}$ and $S := S_1 \cup S_2$. Let $\mathcal{E}_1$ be the event that there exists no vertex $x \in V(\mathcal{Q}^n)$ such that $|B^\ell_{\mathcal{Q}^n}(x) \cap S_1| \geq n^{1/2}$. By Lemma 5.10 we have that $\mathbb{P}[\mathcal{E}_1] \geq 1 - e^{-4n}$. Similarly, let $\mathcal{E}_2$ be the event that there exists no vertex $x \in V(\mathcal{Q}^n)$ such that $|B^\ell_{\mathcal{Q}^n}(x) \cap S_2| \geq n^{1/2}$. By Lemma 5.10 we have that $\mathbb{P}[\mathcal{E}_2] \geq 1 - e^{-4n}$. Condition on $\mathcal{E}_1 \wedge \mathcal{E}_2$ holding, that is, that there is no vertex $x \in V(\mathcal{Q}^n)$ such that $|B^\ell_Q(x) \cap S| \geq n^{3/4}$.

Given $\mathcal{E}_1 \wedge \mathcal{E}_2$, let H be an auxiliary bipartite graph with parts $A := V(T') \setminus S$ and $B := V(\mathcal{Q}^n) \setminus (V(T') \cup W \cup S)$, where we include an edge between $a \in A$ and $b \in B$ whenever

$\{a, b\} \in E(Q)$ and $a \in Z(b)$. By definition of S we have for all $a \in A$ that

$$d_H(a) \leq 11\varepsilon n/10 - 2\varepsilon n/3 < \varepsilon n/2.$$

Furthermore, we have for all $b \in B$ that

$$d_H(b) \geq 2\varepsilon n/3 - n^{3/4} > \varepsilon n/2.$$

Since for all $X \subseteq B$ we have $e_H(N_H(X), B) \geq e_H(X, N_H(X))$, it follows that $|N_H(X)| \geq |X|$. Thus, by Lemma 5.1, H contains a matching covering all of B. This corresponds to a matching in $\mathcal{Q} \sim \mathcal{Q}^n_\varepsilon$. The statement follows by setting T to be the union of T' and this matching. $\square$

7.3. **The repatching lemma.** Later we will apply Theorem 7.1 to obtain a tree T and a reservoir R in $\mathcal{Q}^n_\varepsilon$ which is disjoint from $V(T)$. To carry out the absorption step later on, it will be important that for each vertex some proportion of its neighbourhood consists of vertices in R. However, the tree produced by Theorem 7.1 (and the subsequent application of Theorem 7.19) will result in a small number of vertices with few or no neighbours in R. The following repatching lemma will be called on to deal with such vertices, by slightly altering T.

Given a graph P and $S \subseteq V(P)$ we say that S *is connected in* P if the vertices of S lie in the same component of P.

Lemma 7.20. *Let $0 < 1/n \ll c, \varepsilon, 1/f, 1/D$ where $f, D \in \mathbb{N}$. Given a fixed $x \in V(\mathcal{Q}^n)$, let $C(x) \subseteq N_{\mathcal{Q}^n}(x) \times N_{\mathcal{Q}^n}(x)$ be such that $|C(x)| \geq cn$ and such that, for all distinct $(y_1, z_1), (y_2, z_2) \in C(x)$, we have $\{y_1, z_1\} \cap \{y_2, z_2\} = \varnothing$. Furthermore, for each $(y, z) \in C(x)$, let $B(y, z) \subseteq (N_{\mathcal{Q}^n}(y) \cup N_{\mathcal{Q}^n}(z)) \setminus \{x\}$ with $|B(y, z)| < D$. Then, with probability at least $1 - e^{-5n}$, for every $F \subseteq V(\mathcal{Q}^n)$ with $|F| \leq f$, there exist a pair $(y, z) \in C(x)$ with $y, z \notin F$ and a graph $P \subseteq \mathcal{Q}^n_\varepsilon - \{y, z\}$ with $|V(P)| < 5D$ such that*

- (R1) *$B(y, z) \cap N_{\mathcal{Q}^n}(y)$ is connected in P, and so is $B(y, z) \cap N_{\mathcal{Q}^n}(z)$.*
- (R2) *$V(P) \cap F = \varnothing$.*

Proof. We provide a counting argument to show there exist edge-disjoint graphs $P_1, \ldots, P_{\varepsilon' n} \subseteq \mathcal{Q}^n$ such that, if any is present in $\mathcal{Q}^n_\varepsilon$, then it would satisfy (R1) and (R2) for some $(y, z) \in C(x)$. We will then prove that, with high probability, one of the P_i must be present in $\mathcal{Q}^n_\varepsilon$. Note that we may assume $x = \varnothing$. By passing to a subset of $C(x)$ and replacing c with $c/(30D)$ if necessary, we may also assume that $|C(x)| = cn$ and $2Dc < 1/10$. Similarly, by passing to a suitable subset of $C(x)$, we may assume that, for any distinct $(y, z), (y', z') \in C(x)$, we have that $B(y, z) \cap B(y', z') = \varnothing$.

Fix any $F \subseteq V(\mathcal{Q}^n)$ with $|F| \leq f$. We update $C(x)$ by removing any pair $(y, z) \in C(x)$ for which $(\{y, z\} \cup B(y, z)) \cap F \neq \varnothing$. It follows that $|C(x)| \geq cn - 2f$. Now, for each $(y, z) \in C(x)$ and for each $w \in \{y, z\}$, let $A_w := N_{\mathcal{Q}^n}(w) \cap B(y, z)$, and let $x_1^w, \ldots, x_{|A_w|}^w$ be the vertices of A_w.

Claim 7.5. *For each $e = (y, z) \in C(x)$, $w \in \{y, z\}$ and $i \in [|A_w| - 1]$, there exists a collection $\mathcal{P}_i^w$ of subgraphs of $\mathcal{Q}^n$ such that the following hold:*

- (RC1) *$|\mathcal{P}_i^w| \geq n/2$ and for each $P \in \mathcal{P}_i^w$ we have $V(P) \cap (F \cup \{y, z\}) = \varnothing$.*
- (RC2) *Every $P \in \mathcal{P}_i^w$ is an (x_i^w, x_{i+1}^w)-path of length 4.*
- (RC3) *The graphs in $\mathcal{P}_i^w$ are pairwise edge-disjoint.*
- (RC4) *For every $e' = (y', z') \in C(x)$ with $e' \neq e$, every $w' \in \{y', z'\}$ and every $j \in [|A_{w'}| - 1]$, the graphs in $\mathcal{P}_i^w$ are edge-disjoint from those in $\mathcal{P}_j^{w'}$.*

(Note that we do not require the paths in $\mathcal{P}_i^w$ to be edge-disjoint from those in $\mathcal{P}_{i'}^{w'}$ when $w, w' \in \{y, z\}$ are distinct and $i \in [|A_w| - 1]$, $i' \in [|A_{w'}| - 1]$.)

Proof of Claim 7.5. Let $e_1, \ldots, e_{cn}$ be an ordering of the elements of $C(x)$, where for each $k \in [cn]$ we have that $e_k = (y_k, z_k)$. Note that, for each $k \in [cn]$, each $w \in \{y_k, z_k\}$ and all $i \in [|A_w|]$, we have that $|x_i^w| = 2$, and for each $i, j \in [|A_w|]$ with $i \neq j$ we have that $\mathrm{dist}(x_i^w, x_j^w) = 2$, with $x_i^w \cap x_j^w = w$.

Suppose that, for some $1 < k \leq cn$, every $j \in [k-1]$, every $w \in \{y_j, z_j\}$ and every $i \in [|A_w| - 1]$, we have found a collection $\mathcal{P}_i^w$ which satisfies (RC1)–(RC4). We now show that, for each $w \in \{y_k, z_k\}$ and each $i \in [|A_w| - 1]$, a suitable choice for $\mathcal{P}_i^w$ exists. We construct the set $\mathcal{P}_i^w$ as follows. Let $v_1 := x_i^w \setminus w$ and $v_2 := x_{i+1}^w \setminus w$. For each $d \in [n] \setminus (x_i^w \cup x_{i+1}^w)$, let $P_d \subseteq \mathcal{Q}^n$ be the path which passes through the following vertices in successive order:

$$x_i^w, x_i^w \cup \{d\}, x_i^w \cup \{d\} \cup v_2, (x_i^w \cup \{d\} \cup v_2) \setminus v_1 = x_{i+1}^w \cup \{d\}, x_{i+1}^w.$$

Note that each path P_d has length 4 and that $V(P_d) \cap \{y_k, z_k\} = \varnothing$. Furthermore, for any distinct $d, d' \in [n] \setminus (x_i^w \cup x_{i+1}^w)$, it is clear that P_d and $P_{d'}$ are internally disjoint, and hence, are edge-disjoint. To avoid F as well as the edges of any previously chosen paths we set

$$\mathcal{P}_i^w := \left\{ P_d : d \in [n] \setminus (x_i^w \cup x_{i+1}^w); x_i^w \cup \{d\}, x_{i+1}^w \cup \{d\} \notin N\left(\bigcup_{j=1}^{k-1} B(y_j, z_j)\right); V(P_d) \cap F = \varnothing \right\}.$$

It follows that $\mathcal{P}_i^w$ satisfies (RC2) and (RC3). Recall that $V(P_d) \cap \{y_k, z_k\} = \varnothing$. Therefore, to see that (RC1) holds, note that, for all distinct $(y', z'), (y'', z'') \in C(x)$ and all $x' \in B(y', z'), x'' \in B(y'', z'')$, since $B(y', z') \cap B(y'', z'') = \varnothing$, we have that x' and x'' contain at most one common neighbour in the third level L_3 of $\mathcal{Q}^n$. Since $|\bigcup_{j=1}^{k-1} B(y_j, z_j)| < Dcn$, there are at most $2Dcn < n/10$ choices for d such that $x_i^w \cup \{d\} \in N(\bigcup_{j=1}^{k-1} B(y_j, z_j))$, or $x_{i+1}^w \cup \{d\} \in N(\bigcup_{j=1}^{k-1} B(y_j, z_j))$. Furthermore, since $|F| \leq f$, it follows that there are still at least $n/2$ suitable choices for d, that is, (RC1) holds as desired. Additionally, (RC4) holds by construction; indeed, since neither the second nor the fourth vertex of each path in $\mathcal{P}_i^w$ lies in some path in $\bigcup_{j \in [k-1]} \bigcup_{w' \in \{y_j, z_j\}} \bigcup_{i' \in [|A_{w'}|-1]} \mathcal{P}_{i'}^{w'}$, the paths in $\mathcal{P}_i^w$ must be edge-disjoint from all the paths in $\bigcup_{j \in [k-1]} \bigcup_{w' \in \{y_j, z_j\}} \bigcup_{i' \in [|A_{w'}|-1]} \mathcal{P}_{i'}^{w'}$. Thus, we can proceed by induction and create a suitable collection $\mathcal{P}_i^w$ for each $k \in [cn]$, $w \in \{y_k, z_k\}$ and $i \in [|A_w| - 1]$. ◀

For each $e = (y, z) \in C(x)$, $w \in \{y, z\}$ and $i \in [|A_w| - 1]$, let $\mathcal{P}_i^w$ be the collection of subgraphs given by Claim 7.5. Note that, for any choice of $P_1 \in \mathcal{P}_1^w, \dots, P_{|A_w|-1} \in \mathcal{P}_{|A_w|-1}^w$, we have that A_w is connected in $P_w := \bigcup_{j=1}^{|A_w|-1} P_j$. To complete the proof, we now show that, on passing to $\mathcal{Q}_\varepsilon^n$, with high probability there will exist some $e = (y, z) \in C(x)$ and some P_y and P_z of the above form such that $P_y \cup P_z \subseteq \mathcal{Q}_\varepsilon^n$. Moreover, note that each such choice of $P_y \cup P_z$ satisfies (R1) and (R2) for our fixed F and $|P_y \cup P_z| \leq 5D$. Since $P_y \cup P_z \subseteq B_{\mathcal{Q}^n}^4(x)$, Lemma 7.20 will then follow by a union bound over all choices of $F \subseteq B_{\mathcal{Q}^n}^4(x)$ with $|F| \leq f$.

Let $Q \sim \mathcal{Q}_\varepsilon^n$. Consider $e = (y, z) \in C(x), w \in \{y, z\}$ and $i \in [|A_w| - 1]$. Let $P \in \mathcal{P}_i^w$ and recall that P has length 4. It follows that $\mathbb{P}[P \nsubseteq Q] = 1 - \varepsilon^4$. Let $\mathcal{E}_i^w$ be the event that there exists some $P \in \mathcal{P}_i^w$ such that $P \subseteq Q$. Since $|\mathcal{P}_i^w| \geq n/2$ and paths in $\mathcal{P}_i^w$ are edge-disjoint by (RC3), we have that $\mathbb{P}[\mathcal{E}_i^w] \geq 1 - (1 - \varepsilon^4)^{n/2}$. Let $\mathcal{E}_e := \bigwedge_{w \in \{y, z\}} \bigwedge_{i \in [|A_w|-1]} \mathcal{E}_i^w$. Since $|A_y| + |A_z| \leq 2D$, we have that

$$\mathbb{P}[\mathcal{E}_e] \geq 1 - 2D(1 - \varepsilon^4)^{n/2} \geq 1 - e^{-\varepsilon^4 n/4}.$$

Finally, let $\mathcal{E}$ be the event that there exists some $e \in C(x)$ such that the event $\mathcal{E}_e$ occurs. It follows by (RC4) that, for $e, e' \in C(x)$ with $e \neq e'$, the event $\mathcal{E}_e$ is independent of $\mathcal{E}_{e'}$. Therefore, since $|C(x)| \geq cn$, we have that

$$\mathbb{P}[\mathcal{E}] \geq 1 - e^{-\varepsilon^4 cn^2/4}.$$

Recall that by (RC2) it now suffices to consider a union bound over all choices of $F \subseteq B_{\mathcal{Q}^n}^4(x)$ with $|F| \leq f$. The result follows since

$$1 - f\binom{n^4}{f} e^{-\varepsilon^4 cn^2/4} > 1 - e^{-5n}. \qquad \square$$

8. Hamilton cycles in randomly perturbed dense subgraphs of the hypercube

In this section, we introduce a few more auxiliary lemmas and combine them with the tools we have developed so far to prove the following result.

Theorem 8.1. *For every $\varepsilon, \alpha \in (0, 1]$ and $c > 0$, there exists $\Phi \in \mathbb{N}$ such that the following holds. Let $H \subseteq \mathcal{Q}^n$ be a spanning subgraph with $\delta(H) \geq \alpha n$ and let $G \sim \mathcal{Q}_\varepsilon^n$. Then, a.a.s. there* *is a subgraph $G' \subseteq G$ with $\Delta(G') \leq \Phi$ such that, for every $F \subseteq \mathcal{Q}^n$ with $\Delta(F) \leq c\Phi$, the graph $((H \cup G) \setminus F) \cup G'$ is Hamiltonian.*

Note that Theorem 8.1 trivially implies the case $k = 1$ of Theorem 1.7. In fact, in Section 8.5 we will use Theorem 8.1 to prove Theorem 1.7 in full generality. For this derivation, we will need the stronger conditions imposed in the statement of Theorem 8.1. More precisely, the formulation of Theorem 8.1 involving a 'forbidden' graph F and a 'protected' graph G' is designed to make repeated applications of Theorem 8.1 possible in order to take out k edge-disjoint Hamilton cycles. When finding the i-th Hamilton cycle, the protected graph will contain all the essential ingredients for this, while the forbidden graph will contain all previously chosen Hamilton cycles as well as the protected graphs for the entire set of Hamilton cycles (see Section 8.5 for details).

The first step of the proof of Theorem 8.1 will be to consider a particular partition of the hypercube into subcubes. The structure of this partition will be used extensively throughout the rest of the paper, so we first introduce the necessary notation in the next subsection. Then, in Section 8.2 we prove several results regarding this structure, concerning its properties in $\mathcal{Q}^n_\varepsilon$ and with respect to a reservoir $R \sim \mathit{Res}(\mathcal{Q}^n, \delta)$. In Section 8.3, we will prove our *connecting lemmas*, which provide sets of paths in (sub)cubes which (roughly speaking) link up pairs of vertices and, together, span all vertices of these (sub)cubes. We prove Theorem 8.1 in Section 8.4. Finally, we deduce Theorems 1.1, 1.2 and 1.7 from Theorem 8.1 in Section 8.5.

8.1. **Layers, molecules, atoms and absorbing structures.** Throughout this section, given any two vectors u and v, we will write uv for their concatenation. Consider $\mathcal{Q}^n$ and some $s \in \mathbb{N}$, with $s < n$. We divide $\mathcal{Q}^n$ into 2^s vertex-disjoint copies of $\mathcal{Q}^{n-s}$ as follows: for each $a \in \{0,1\}^s$, we consider the set of vertices $V_a \coloneqq \{av : v \in \{0,1\}^{n-s}\}$, and consider the graph $\mathcal{Q}(a) \coloneqq \mathcal{Q}^n[V_a]$. We will refer to each $\mathcal{Q}(a)$ as an *s-layer* of $\mathcal{Q}^n$ (s will be dropped whenever clear from the context). Given $\ell \leq n - s$, we will refer to any copy of a cube $\mathcal{Q}^\ell$ in one of the s-layers as an *ℓ-atom* (again, ℓ will be dropped whenever clear from the context).

Fix a Hamilton cycle $\mathcal{C}$ of $\mathcal{Q}^s$. By abusing notation, whenever necessary, we assume that the coordinate vector of each vertex of $\mathcal{C}$ is concatenated with $n - s$ 0's. $\mathcal{C}$ induces a cyclical ordering on $\{0,1\}^s$, which we will label as $a_1, \ldots, a_{2^s}$. In turn, this gives a cyclical ordering on the set of layers. In this section, for each $i \in [2^s]$, we denote $L_i \coloneqq \mathcal{Q}(a_i)$ (as opposed to Section 7, where L_i denoted the i-th level of the hypercube). Given an ℓ-atom $\mathcal{A}$ in an s-layer $\mathcal{Q}(a)$, we refer to $\mathcal{M}(\mathcal{A}) \coloneqq \mathcal{A} + V(\mathcal{C})$ as an *(s, ℓ)-molecule* (again, the parameters will be dropped when clear from the context). Thus $\mathcal{M}(\mathcal{A})$ is the vertex-disjoint union of 2^s copies of $\mathcal{Q}^\ell$. We refer to an $(s, 1)$-molecule as a *vertex molecule* and an (s, ℓ)-molecule for $\ell > 1$ as a *cube molecule*. Observe that, if we label the atoms in a molecule cyclically following the labelling of the layers, then $\mathcal{Q}^n$ contains a perfect matching between any two consecutive atoms where all edges are in the same direction as the corresponding edge in $\mathcal{C}$. Whenever we work with molecules, we consider this cyclical order implicitly. In particular, whenever we refer to a molecule $\mathcal{M} = \mathcal{M}(\mathcal{A}) = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{2^s}$, the cyclical order $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{2^s}$ of the $\mathcal{A}_i$ is that induced by $\mathcal{C}$. Given a molecule $\mathcal{M}(\mathcal{A}) = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{2^s}$, a *slice* $\mathcal{M}^* \subseteq \mathcal{M}(\mathcal{A})$ will consist of the subgraph of $\mathcal{M}(\mathcal{A})$ induced by its intersection with some number of consecutive layers, i.e. $\mathcal{M}^* = \mathcal{A}_{a+1} \cup \cdots \cup \mathcal{A}_{a+t}$ for some $a, t \in [2^s]$. Alternatively, given any $a \in V(\mathcal{C})$, any path $P \subseteq \mathcal{C}$ and any atom $\mathcal{A} \subseteq \mathcal{Q}(a)$, P determines a slice of $\mathcal{M}(\mathcal{A})$ by setting $\mathcal{M}^* \coloneqq \mathcal{A} + V(P)$.

Consider $i \in [2^s]$ and the cyclical ordering of the layers given by $\mathcal{C}$. Given any subgraph $G \subseteq \mathcal{Q}^n$, we will often denote the restriction of G to the i-th layer by $L_i(G)$, that is, $L_i(G) \coloneqq G[V(L_i)]$. Given any $v \in \{0,1\}^{n-s}$, we will refer to the vertex $a_i v$ as the i-th *clone* of v. In general, when it is clear from the context, we will also refer to the i-th clone of a cube $C \subseteq \mathcal{Q}^{n-s}$ (as well as other subgraphs), which, analogously, will be the corresponding copy in L_i of C. In particular, the i-th layer L_i is the i-th clone of $\mathcal{Q}^{n-s}$.

As we already discussed in Section 2, in order to prove our results we will first construct a near-spanning cycle and then absorb the remaining vertices into this cycle. We will achieve this by using the following absorbing structure.

Definition 8.2 (Absorbing ℓ-cube pair)**.** *Let $\ell, n \in \mathbb{N}$, and let $G \subseteq \mathcal{Q}^n$. Given a vertex $x \in V(\mathcal{Q}^n)$, an* absorbing ℓ-cube pair *for x in G, which we denote by (C^l, C^r), is a subgraph of G which consists of two vertex-disjoint ℓ-dimensional cubes $C^l, C^r \subseteq G$ and three edges $e, e^l, e^r \in E(G)$ satisfying the following properties:*

(AP1) $|V(C^l) \cap N_{\mathcal{Q}^n}(x)| = |V(C^r) \cap N_{\mathcal{Q}^n}(x)| = 1$;
(AP2) *e^l and e^r are the unique edges from x to C^l and C^r, respectively;*
(AP3) *the unique vertex $y \in V(C^l) \cap N_{\mathcal{Q}^n}(x)$ satisfies* $\mathrm{dist}(y, C^r) = 1$, *and*
(AP4) *e is the unique edge from y to C^r.*

We will refer to C^l as the left absorption cube *and to C^r as the* right absorption cube*. Given an absorbing ℓ-cube pair (C^l, C^r) we refer to y as the* left absorber tip*, and to the unique vertex $z \in V(C^r) \cap N_{\mathcal{Q}^n}(x)$ as the* right absorber tip*. We refer to the unique vertex $z' \in e \setminus \{y\}$ as the* third absorber vertex*.*

8.2. **Bondless and bondlessly surrounded molecules.** Given any graph $G \subseteq \mathcal{Q}^n$, we will say that an (s, ℓ)-molecule $\mathcal{M} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{2^s} \subseteq \mathcal{Q}^n$, where $\mathcal{A}_i$ is the i-th clone of some ℓ-cube $\mathcal{A} \subseteq \mathcal{Q}^{n-s}$, is *bonded* in G if, for all $i \in [2^s]$, G contains at least 100 edges between $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ whose endpoint in $\mathcal{A}_i$ has even parity and at least 100 such edges whose endpoint in $\mathcal{A}_i$ has odd parity. Otherwise, we call it *bondless* in G. Furthermore, given a collection $\mathcal{U}$ of (s, ℓ)-molecules in G, we say that $\mathcal{M} \in \mathcal{U}$ is *bondlessly surrounded* in G (with respect to $\mathcal{U}$) if there exists some vertex $v \in V(\mathcal{M})$ which has at least $n/2^{\ell+5s}$ neighbours in $\mathcal{Q}^n$ which are part of (s, ℓ)-molecules of $\mathcal{U}$ which are bondless in G. Both bondless and bondlessly surrounded molecules create difficulties in applying the rainbow matching lemma (Lemma 5.5), which in turn is used to assign absorption structures to vertices. Therefore, it will become important that we bound the number of each, and show that they are well spread out.

Lemma 8.3. *Let $\varepsilon > 0$ and $\ell, s, n \in \mathbb{N}$ be such that $s < n$, $\ell \leq n - s$ and $1/\ell \ll \varepsilon$. Then, for any (s, ℓ)-molecule $\mathcal{M} \subseteq \mathcal{Q}^n$, the probability that it is bondless in $\mathcal{Q}^n_\varepsilon$ is at most $2^{s+1-\varepsilon 2^\ell/4}$.*

Proof. Fix an (s, ℓ)-molecule $\mathcal{M} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{2^s} \subseteq \mathcal{Q}^n$. Consider a pair of consecutive atoms $\mathcal{A}_i, \mathcal{A}_{i+1} \subseteq \mathcal{M}$, for some $i \in [2^s]$. Let X_i be the number of edges between $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ in $\mathcal{Q}^n_\varepsilon$ whose endpoint in $\mathcal{A}_i$ is odd, and let Y_i be the number of such edges whose endpoint in $\mathcal{A}_i$ is even. We have that $X_i, Y_i \sim \mathrm{Bin}(2^{\ell-1}, \varepsilon)$. By Lemma 4.2, it follows that

$$\mathbb{P}[X_i < 100] \leq 2^{-\varepsilon 2^\ell/4},$$

and the same bound holds for $\mathbb{P}[Y_i < 100]$. By a union bound over all $i \in [2^s]$, we conclude that

$$\mathbb{P}[\mathcal{M} \text{ is bondless in } \mathcal{Q}^n_\varepsilon] \leq 2^{s+1-\varepsilon 2^\ell/4}. \qquad \square$$

Lemma 8.4. *Let $\varepsilon \in (0,1)$ and $\ell, n \in \mathbb{N}$ with $0 < 1/n \ll 1/\ell \ll \varepsilon$, and let $s \coloneqq 10\ell$. Let $\mathfrak{M}$ be a collection of vertex-disjoint (s, ℓ)-molecules $\mathcal{M} \subseteq \mathcal{Q}^n$. For each $x \in V(\mathcal{Q}^n)$, let $N^{\mathfrak{M}}(x) \coloneqq \{\mathcal{M} \in \mathfrak{M} : \mathrm{dist}(x, \mathcal{M}) = 1\}$. Assume that the following holds for every $x \in V(\mathcal{Q}^n)$:*

(BS) *for any direction $\hat{e} \in \mathcal{D}(\mathcal{Q}^n)$, there are at most $\sqrt{n}$ molecules $\mathcal{M} \in N^{\mathfrak{M}}(x)$ such that $\hat{e} \in \mathcal{D}(\mathcal{A})$ for all atoms $\mathcal{A} \in \mathcal{M}$.*

Then, with probability at least $1 - 2^{-n^{9/8}}$, for every $x \in V(\mathcal{Q}^n)$ we have that $B^{\ell^2}_{\mathcal{Q}^n}(x)$ intersects at most $n^{1/3}$ molecules from $\mathfrak{M}$ which are bondlessly surrounded in $\mathcal{Q}^n_\varepsilon$.

Proof. We begin by fixing an arbitrary vertex $x \in V(\mathcal{Q}^n)$ and an arbitrary set $B \subseteq \mathfrak{M}$ of $n^{1/3}$ molecules which intersect $B^{\ell^2}_{\mathcal{Q}^n}(x)$. We will estimate the probability that all of the molecules in B are bondlessly surrounded in $\mathcal{Q}^n_\varepsilon$, by considering the neighbourhoods of the different vertices which make up these molecules. If the probability of being bondlessly surrounded was independent over different molecules and vertices, then this would be a straightforward calculation. However, there are dependencies which we must consider: namely, when two different molecules have edges to the same third molecule. We will first bound the number of such configurations in $\mathcal{Q}^n$. Since the molecules in $\mathfrak{M} \supseteq B$ are vertex-disjoint, it follows that, if two of these molecules are adjacent in $\mathcal{Q}^n$, then all of their atoms are pairwise adjacent in each of the layers, via clones of the same edges. Thus, we can restrict the analysis to a single layer.

Fix a layer L and let $\mathfrak{A}$ be the collection of atoms obtained by intersecting each molecule $\mathcal{M} \in \mathfrak{M}$ with L. Let $\mathfrak{A}_B \subseteq \mathfrak{A}$ be the set of such atoms whose molecules lie in B. Fix an atom $\mathcal{A} \in \mathfrak{A}_B$, and let $y \in V(\mathcal{A})$ be a fixed vertex. We say an atom $\mathcal{A}' \in \mathfrak{A}$ is *y-dependent* if there exists $\mathcal{A}'' \in \mathfrak{A}_B$, $\mathcal{A}'' \neq \mathcal{A}$, such that $\mathrm{dist}(y, \mathcal{A}') = \mathrm{dist}(\mathcal{A}', \mathcal{A}'') = 1$. The following claim will allow us to bound the number of y-dependent atoms.

Claim 8.1. *Fix $\mathcal{A}'' \in \mathfrak{A}_B$ with $\mathcal{A}'' \neq \mathcal{A}$. Then, the number of $\mathcal{A}' \in \mathfrak{A}$ for which $\mathrm{dist}(y, \mathcal{A}') = \mathrm{dist}(\mathcal{A}', \mathcal{A}'') = 1$ is at most $2^\ell(2 + \sqrt{n})$.*

Proof. Let $z \in V(\mathcal{A}'')$ and let $\hat{e} \in \mathcal{D}(y, z)$. Let $\mathcal{A}' \in \mathfrak{A}$ be such that $\mathrm{dist}(y, \mathcal{A}') = \mathrm{dist}(z, \mathcal{A}') = 1$. Suppose first that $\hat{e} \notin \mathcal{D}(\mathcal{A}')$. Then we must have either $y + \hat{e} \in V(\mathcal{A}')$ or $z + \hat{e} \in V(\mathcal{A}')$. Since all the atoms in $\mathfrak{A}$ are vertex-disjoint, this leaves only two possibilities for $\mathcal{A}'$. Alternatively, suppose $\hat{e} \in \mathcal{D}(\mathcal{A}')$. Then, by (BS) applied with y playing the role of x, we have at most $\sqrt{n}$ possibilities for $\mathcal{A}'$. Finally, by considering all $z \in V(\mathcal{A}'')$ we prove the claim. ◀

By considering all possibilities for $\mathcal{A}'' \in \mathfrak{A}_B$, since $|\mathfrak{A}_B| = n^{1/3}$, it follows by Claim 8.1 that the number of y-dependent atoms is at most $n^{6/7}$. For each $y \in V(\mathcal{A})$, let $N'(y) \subseteq N^{\mathfrak{M}}(y)$ be given by removing from $N^{\mathfrak{M}}(y)$ all molecules which contain a y-dependent atom. It follows that $|N'(y)| = |N^{\mathfrak{M}}(y)| - o(n)$ for every $y \in V(\mathcal{A})$.

Let $\mathcal{M}_{\mathcal{A}} \in \mathfrak{M}$ be the molecule containing $\mathcal{A}$. For each vertex $y \in V(\mathcal{A})$, let $\mathcal{E}_y$ be the event that $N'(y)$ contains at least $n/2^{\ell+5s+1}$ molecules $\mathcal{M} \in \mathfrak{M}$ which are bondless in $\mathcal{Q}^n_\varepsilon$. Then, $|N'(y)| \geq n/2^{\ell+5s+2}$. Moreover, we only consider here those vertices $y \in V(\mathcal{A})$ for which $|N^{\mathfrak{M}}(y)| \geq n/2^{\ell+5s+1}$, since otherwise y cannot contribute towards $\mathcal{M}_{\mathcal{A}}$ being bondlessly surrounded. Fix such a vertex y. Let Y be the number of atoms $\mathcal{A} \in N'(y)$ which correspond to molecules which are bondless in $\mathcal{Q}^n_\varepsilon$. Note that Y is a sum of independent indicator variables. By Lemma 8.3, we have that $\mathbb{E}[Y] \leq 2^{s+1-\varepsilon 2^\ell/4}n$. In order to derive a lower bound for $\mathbb{E}[Y]$, note that the probability that an (s, ℓ)-molecule $\mathcal{M}$ is bondless can be bounded from below by the probability that there are no edges between two fixed consecutive atoms $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{M}$, whose endpoints in $\mathcal{A}_1$ are even. This occurs with probability $(1-\varepsilon)^{2^{\ell-1}}$. Thus,

$$\mathbb{E}[Y] \geq (1-\varepsilon)^{2^{\ell-1}}|N'(y)| \geq (1-\varepsilon)^{2^{\ell-1}}(n/2^{\ell+5s+2}).$$

By Lemma 4.4, we have that $\mathbb{P}[\mathcal{E}_y] \leq 2^{-cn}$, for some constant $c > 0$ which depends on ℓ and ε. For each atom $\mathcal{A} \in \mathfrak{A}_B$, let $\mathcal{B}_{\mathcal{A}}$ be the event that there exists a vertex $y \in V(\mathcal{A})$ such that $\mathcal{E}_y$ holds. Let $\mathcal{B} := \bigwedge_{\mathcal{A} \in \mathfrak{A}_B} \mathcal{B}_{\mathcal{A}}$. Note that the definition of $N'(y)$ ensures that the events $\mathcal{B}_{\mathcal{A}}$ with $\mathcal{A} \in \mathfrak{A}_B$ are pairwise independent. Thus,

$$\mathbb{P}[\mathcal{E}] \leq (2^{\ell-cn})^{n^{1/3}} < 2^{-n^{5/4}}.$$

In turn, this means that the probability that all molecules $\mathcal{M} \in B$ are bondlessly surrounded is bounded from above by $2^{-n^{5/4}}$. Lemma 8.4 now follows by a union bound over the 2^n choices for x and the at most $\binom{n^{\ell^2}}{n^{1/3}}$ choices for B. □

Finally, we will show that 'scant' molecules are not too clustered. (We will later define a vertex molecule as 'scant' –with respect to a graph H and a reservoir R– if one of its vertices v_i has the property that few of its neighbours lie in the i-th clone of R.)

Lemma 8.5. *Let $C, s, n \in \mathbb{N}$ such that $0 < 1/n \ll 1/C \ll \alpha, \delta \leq 1$ and $1/n \ll 1/s$. Let $H \subseteq \mathcal{Q}^n$ be such that $\delta(H) \geq \alpha n$. For each $v \in V(\mathcal{Q}^{n-s})$ and each $i \in [2^s]$, let v_i be the i-th clone of v, and let $\mathcal{M}_v := \{v_i : i \in [2^s]\}$. Let $R \sim Res(\mathcal{Q}^{n-s}, \delta)$ and, for each $i \in [2^s]$, let R_i be the i-th clone of R. Let*

$$B := \{\mathcal{M}_v \mid v \in V(\mathcal{Q}^{n-s}),\ \text{there exists } i \in [2^s] : e_H(v_i, R_i) < \alpha\delta n/4\}.$$

Let $\mathcal{E}$ be the event that there exists some $u \in V(\mathcal{Q}^{n-s})$ such that $B^{10\ell}_{\mathcal{Q}^{n-s}}(u)$ contains more than C vertices $v \in V(\mathcal{Q}^{n-s})$ with $\mathcal{M}_v \in B$. Then, $\mathbb{P}[\mathcal{E}] < e^{-n}$.

Proof. Let $u \in V(\mathcal{Q}^{n-s})$ and let $D \subseteq B^{10\ell}_{\mathcal{Q}^{n-s}}(u)$ be a set of C vertices. Let $D' \coloneqq \bigcup_{x,y\in D: x\neq y} N_{\mathcal{Q}^{n-s}}(x) \cap N_{\mathcal{Q}^{n-s}}(y)$. Since any pair of distinct vertices share at most two neighbours, we have that $|D'| \leq 2\binom{C}{2}$. For each $i \in [2^s]$, we denote the i-th clone of D' by D'_i, and let $R'_i \coloneqq R_i \setminus D'_i$.

For each $x \in V(\mathcal{Q}^n)$, let $i(x)$ be the unique index $i \in [2^s]$ such that $x \in V(L_i)$. Observe that $e_H(x, V(L_{i(x)})) > 2\alpha n/3$ for every $x \in V(\mathcal{Q}^n)$. For each $x \in V(\mathcal{Q}^n)$, let $\mathcal{E}_x$ be the event that $e_H(x, R_{i(x)}) \leq \alpha\delta n/4$, and let $\mathcal{E}'_x$ be the event that $e_H(x, R'_{i(x)}) \leq \alpha\delta n/4$. It follows by Lemma 4.2 that $\mathbb{P}[\mathcal{E}'_x] \leq e^{-\alpha\delta n/16}$ for all $x \in V(\mathcal{Q}^n)$. For each $v \in V(\mathcal{Q}^{n-s})$, let $\mathcal{E}_v$ and $\mathcal{E}'_v$ be the events that there exists $i \in [2^s]$ such that $\mathcal{E}_{v_i}$ and $\mathcal{E}'_{v_i}$ hold, respectively. By a union bound, it follows that $\mathbb{P}[\mathcal{E}'_v] \leq 2^s e^{-\alpha\delta n/16}$ for all $v \in V(\mathcal{Q}^{n-s})$. Finally, let $\mathcal{E}_D$ and $\mathcal{E}'_D$ be the events that $\mathcal{E}_v$ and $\mathcal{E}'_v$, respectively, hold for every $v \in D$. Note that the events in the collection $\{\mathcal{E}'_v : v \in V(\mathcal{Q}^{n-s})\}$ are mutually independent. Furthermore, since the event $\mathcal{E}_x$ implies $\mathcal{E}'_x$ for all $x \in V(\mathcal{Q}^n)$, we have that

$$\mathbb{P}[\mathcal{E}_D] \leq \mathbb{P}[\mathcal{E}'_D] \leq (2^s e^{-\alpha\delta n/16})^C < e^{-5n}.$$

Taking a union bound over all vertices u and over all choices of D we obtain the result. □

8.3. **Connecting cubes.** The hypercube satisfies some robust connectivity properties. The problem of (almost) covering $\mathcal{Q}^n$ with disjoint paths has been extensively studied.

In order to create a long cycle, which can be used to absorb all remaining vertices, while preserving the absorbing structure, we will make use of the robust connectivity properties of the hypercube. In particular, we will need several results which guarantee that, given any prescribed pairs of vertices in a slice, there is a spanning collection of vertex-disjoint paths, each of which uses the vertices of one of the given pairs as endpoints. We will also need similar results for almost spanning collections of paths, where these paths avoid a given prescribed vertex. Throughout this subsection we denote by uv the edge between two given adjacent vertices u and v (instead of $\{u, v\}$).

The following lemma will be essential for us. It follows from some results of Dvořák and Gregor [24, Corollary 5.2].

Lemma 8.6. *For all $n \geq 100$, the graph $\mathcal{Q}^n$ satisfies the following.*

(i) *Let $m \in [25]$ and let $\{u_i, v_i\}_{i\in[m]}$ be disjoint pairs of vertices with $u_i \neq_{\mathrm{p}} v_i$ for all $i \in [m]$. Then, there exist m vertex-disjoint paths $\mathcal{P}_1, \dots, \mathcal{P}_m \subseteq \mathcal{Q}^n$ such that, for each $i \in [m]$, $\mathcal{P}_i$ is a (u_i, v_i)-path, and $\bigcup_{i\in[m]} V(\mathcal{P}_i) = V(\mathcal{Q}^n)$.*

(ii) *Let $x \in V(\mathcal{Q}^n)$. Let $m \in [25]$ and let $\{u_i, v_i\}_{i\in[m]}$ be disjoint pairs of vertices of $\mathcal{Q}^n - \{x\}$ such that $u_1, v_1 \neq_{\mathrm{p}} x$ and $u_i \neq_{\mathrm{p}} v_i$ for all $i \in [m] \setminus \{1\}$. Then, there exist m vertex-disjoint paths $\mathcal{P}_1, \dots, \mathcal{P}_m \subseteq \mathcal{Q}^n$ such that, for each $i \in [m]$, $\mathcal{P}_i$ is a (u_i, v_i)-path, and $\bigcup_{i\in[m]} V(\mathcal{P}_i) = V(\mathcal{Q}^n) \setminus \{x\}$.*

(iii) *Let $\{u_i, v_i\}_{i\in[2]}$ be disjoint pairs of vertices with $u_i =_{\mathrm{p}} v_i$ for all $i \in [2]$ and $u_1 \neq_{\mathrm{p}} u_2$. Then, there exist two vertex-disjoint paths $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{Q}^n$ such that, for each $i \in [2]$, $\mathcal{P}_i$ is a (u_i, v_i)-path, and $V(\mathcal{P}_1) \cup V(\mathcal{P}_2) = V(\mathcal{Q}^n)$.*

We now motivate the statement (as well as the proof) of Lemma 8.8, which is the main result of this subsection. We are given a slice $\mathcal{M}^*$ of a molecule $\mathcal{M} \subseteq \mathcal{Q}^n$ which is bonded in a graph $G \subseteq \mathcal{Q}^n$. Furthermore, we are given collections of vertices L, R (which are part of absorbing cube structures), and S (which, when constructing a long cycle, will be used to enter and leave $\mathcal{M}^*$). More specifically, we have that

- L will have size 0 or 2, and will consist of left absorber tips. If it has size 2, the vertices will have opposite parities. These must be avoided by our connecting paths, so that we can make use of the absorbing structures we have put in place (see the discussion in Section 2).
- R will consist of the pairs of right absorber tip and third absorber vertex. These must be connected via an edge with the paths we find.
- S will consist of a set of pairs of vertices $\{u, v\}$ with $u \neq_{\mathrm{p}} v$. Later, when creating a long cycle, u will be a vertex through which we enter $\mathcal{M}^*$ from a different molecule, and v

will be the next vertex from which we leave $\mathcal{M}^*$ (with respect to some ordering). Each of our paths will be a (u,v)-path, for some such pair $\{u,v\}$.

In order to find our paths, we will call on Lemma 8.6. To illustrate this, suppose $\mathcal{M}^*$ consists of the atoms $\mathcal{A}_1,\dots,\mathcal{A}_t$, for some $t\in\mathbb{N}$. Suppose that $S=\{u,v\}$ with $u\in V(\mathcal{A}_1)$ and $v\in V(\mathcal{A}_t)$. Furthermore, suppose that $L,R=\varnothing$. To construct a path from u to v, we will first specify the edges used to pass between different atoms. For all $k\in[t-1]$, we choose an edge $v_k^{\uparrow}u_{k+1}^{\uparrow}$ from $\mathcal{A}_k$ to $\mathcal{A}_{k+1}$, thus $v_k^{\uparrow}\neq_{\mathrm{p}} u_{k+1}^{\uparrow}$. For technical reasons, we aim to have all the vertices $u_{k+1}^{\uparrow}$ of the same parity as u. We can then apply Lemma 8.6 to find a path from $u_{k+1}^{\uparrow}$ to $v_{k+1}^{\uparrow}$ which covers all of $V(\mathcal{A}_{k+1})$. Together with the edges $v_k^{\uparrow}u_{k+1}^{\uparrow}$, all these paths will form a single path from u to v which spans $V(\mathcal{M}^*)$. In the more general setting where $u\in V(\mathcal{A}_i)$ and $v\in V(\mathcal{A}_j)$ with $1<i<j<t$, the (u,v)-path we construct would first pass down to $\mathcal{A}_1$, then up to $\mathcal{A}_t$ and, finally, back down to $\mathcal{A}_j$.

When $L\neq\varnothing$, due to vertex parities, the following issue can arise. Suppose $L=\{x,y\}$ with $x\in V(\mathcal{A}_1)$, $u\in V(\mathcal{A}_2)$, $y\in V(\mathcal{A}_3)$ and $v\in V(\mathcal{A}_j)$ for some $j>3$ (and $R=\varnothing$). Furthermore, suppose that both u and x have odd parity. In line with the above description, the vertex $u_1^{\downarrow}$, through which we enter $\mathcal{A}_1$, would have odd parity. It follows that, since x also has odd parity, we cannot hope to construct a path which starts at $u_1^{\downarrow}$ and covers all of $V(\mathcal{A}_1)\setminus\{x\}$. The solution will be instead to pass up to $\mathcal{A}_3$ first (and, in general, to whichever atom contains y). Recall that, since x has odd parity, y must have even parity. We specify a vertex $u_3^{\uparrow}$ of odd parity, through which we enter $\mathcal{A}_3$, but then also specify a vertex $v_3^{\downarrow}$ of odd parity from which we will leave $\mathcal{A}_3$ to reenter $\mathcal{A}_2$. We now arrive back in $\mathcal{A}_2$ with a vertex $u_2^{\downarrow}$ of even parity. We will specify another vertex $v_2^{\downarrow}$ of odd parity from which we leave $\mathcal{A}_2$ and a vertex $u_1^{\downarrow}$ of even parity through which we enter $\mathcal{A}_1$. In this way, we can now apply Lemma 8.6 to find a path which starts at $u_1^{\downarrow}$ and covers all of $V(\mathcal{A}_1)\setminus\{x\}$, and which can be extended into a path from u to v covering all of $V(\mathcal{M}^*)\setminus L$.

There are several other instances which must be dealt with in a similar way. This is formalised by Lemma 8.8. Before proving this lemma, however, we need the following definition.

Definition 8.7 ((u,j,F,R)-alternating parity sequence)**.** *Let $\ell,s,t,n\in\mathbb{N}$ with $t\leq 2^s$ and $2\leq\ell\leq n-s$. Let $G\subseteq\mathcal{Q}^n$. Let $\mathcal{M}=\mathcal{A}_1\cup\dots\cup\mathcal{A}_{2^s}\subseteq\mathcal{Q}^n$ be an (s,ℓ)-molecule and let $\mathcal{M}^*=\mathcal{A}_{a+1}\cup\dots\cup\mathcal{A}_{a+t}$, for some $a\in[2^s]$, be a slice of $\mathcal{M}$. Let $u\in V(\mathcal{A}_i)$, for some $i\in[a+t]\setminus[a]$. Let $j\in[a+t]\setminus[a]$, and let $F,R\subseteq V(\mathcal{M}^*)$. Suppose $i\leq j$. Let $I_R\coloneqq\{k\in[j-i]_0:|R\cap V(\mathcal{A}_{i+k})|\geq 1\}$. Assume that the following properties hold:*

- *For all $k\in[j-i]_0$ we have that $|R\cap V(\mathcal{A}_{i+k})|\in\{0,2\}$.*
- *For each $k\in I_R$, the vertices in $R\cap V(\mathcal{A}_{i+k})$ are adjacent in $\mathcal{Q}^n$, and we write $R\cap V(\mathcal{A}_{i+k})=\{w_k,z_k\}$ so that $w_k\neq_{\mathrm{p}} u$.*

Let $\mathcal{S}'=(u_0,v_1,u_1,\dots,v_{j-i},u_{j-i})$ be a sequence of vertices satisfying the following properties:

(P0) *If $u\in R$, then $u_0\coloneqq w_0$; otherwise, $u_0\coloneqq u$.*
(P1) *For each $k\in[j-i]$ we have that $u_k=_{\mathrm{p}} u$.*
(P2) *For each $k\in[j-i]$ we have that $v_k\in V(\mathcal{A}_{i+k-1})$, $u_k\in V(\mathcal{A}_{i+k})$ and $v_ku_k\in E(G)$.*
(P3) *The vertices of $\mathcal{S}'$ other than u_0 avoid $F\cup R$.*

A (u,j,F,R)-alternating parity sequence $\mathcal{S}$ in G *is a sequence obtained from any sequence $\mathcal{S}'$ which satisfies (P0)–(P3) as follows. For each $k\in I_R\cap[j-i]$, replace each segment (v_k,u_k) of $\mathcal{S}'$ by (v_k,u_k,w_k,z_k).*

The case $i>j$ is defined similarly by replacing each occurrence of $[j-i]$ and $[j-i]_0$ in the above by $[i-j]$ and $[i-j]_0$, and each occurrence of $\mathcal{A}_{i+k}$ and $\mathcal{A}_{i+k-1}$ by $\mathcal{A}_{i-k}$ and $\mathcal{A}_{i-k+1}$.

Given an alternating parity sequence $\mathcal{S}$, we will denote by $\mathcal{S}^-$ the sequence obtained from $\mathcal{S}$ by deleting its initial element.

Lemma 8.8. *Let $n,s,\ell\in\mathbb{N}$ be such that $s\geq 4$ and $100\leq\ell\leq n-s$. Let $G\subseteq\mathcal{Q}^n$ and consider any (s,ℓ)-molecule $\mathcal{M}=\mathcal{A}_1\cup\dots\cup\mathcal{A}_{2^s}\subseteq\mathcal{Q}^n$ which is bonded in G. Let $\mathcal{M}^*=\mathcal{A}_{a+1}\cup\dots\cup\mathcal{A}_{a+t}$, for some $a\in[2^s]$ and $t\geq 10$, be a slice of $\mathcal{M}$. Moreover, consider the following sets.*

(C1) *Let* $L \subseteq V(\mathcal{M}^*)$ *be a set of size* $|L| \in \{0,2\}$ *such that, if* $L = \{x,y\}$*, then* $x \in V(\mathcal{A}_i)$ *and* $y \in V(\mathcal{A}_j)$ *with* $i \neq j$ *and* $x \neq_{\mathrm{p}} y$*.*

(C2) *Let* $R \subseteq V(\mathcal{M}^*) \setminus L$ *be a (possibly empty) set of vertices with* $|R| \leq 10$ *such that, for all* $k \in [a+t] \setminus [a]$*, we have* $|R \cap V(\mathcal{A}_k)| \in \{0,2\}$ *and, if* $|R \cap V(\mathcal{A}_k)| = 2$*, then* $R \cap V(\mathcal{A}_k) = \{w_k, z_k\}$ *satisfies that* $w_k z_k \in E(\mathcal{M}^*)$ *and, if* $|L| = 2$*, then* $k \notin \{i,j\}$*.*

(C3) *Let* $m \in [14]$ *and consider* m *vertex-disjoint pairs* $\{u_r, v_r\}_{r\in[m]}$*, where* $u_r, v_r \in V(\mathcal{M}^*) \setminus L$ *and* $u_r \neq_{\mathrm{p}} v_r$ *for all* $r \in [m]$*, such that, for each* $r \in [m]$*, we have* $u_r \in V(\mathcal{A}_{i_r})$ *and* $v_r \in V(\mathcal{A}_{j_r})$*. Assume, furthermore, that for each* $t' \in [t]$ *we have that* $|\bigcup_{r\in[m]}\{u_r, v_r\} \cap V(\mathcal{A}_{a+t'}) \cap R| \leq 1$*.*

Then, there exist vertex-disjoint paths $\mathcal{P}_1, \dots, \mathcal{P}_m \subseteq \mathcal{M}^* \cup G$ *such that, for each* $r \in [m]$*,* $\mathcal{P}_r$ *is a* (u_r, v_r)*-path,* $\bigcup_{r\in[m]} V(\mathcal{P}_r) = V(\mathcal{M}^*) \setminus L$*, and every pair* $\{w_k, z_k\}$ *with* $k \in [a+t] \setminus [a]$ *is an edge of some* $\mathcal{P}_r$*.*

Proof. By relabelling the atoms, we may assume that $\mathcal{M}^* = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_t$. Let $S \coloneqq \{u_r, v_r : r \in [m]\}$. By relabelling the vertices, we may assume that $i_r \leq j_r$ for all $r \in [m]$ and (if $L \neq \varnothing$) $i < j$. Let $I_L \coloneqq \{k \in [t] : L \cap V(\mathcal{A}_k) \neq \varnothing\}$, $I_R \coloneqq \{k \in [t] : R \cap V(\mathcal{A}_k) \cap S \neq \varnothing\}$ and $R^* \coloneqq R \setminus \bigcup_{k\in I_R} V(\mathcal{A}_k)$. Note that $I_L = \varnothing$ or $I_L = \{i,j\}$ and $I_L \cap I_R = \varnothing$. For each $r \in [m]$, let $I_R^r \coloneqq \{k \in \{i_r, j_r\} : R \cap V(\mathcal{A}_k) \cap \{u_r, v_r\} \neq \varnothing\}$, so that $I_R = \bigcup_{r=1}^m I_R^r$. Without loss of generality, we may also assume that, for each $r \in [m]$, if $u_r \in R$, then $u_r = z_{i_r}$, and if $v_r \in R$, then $v_r = w_{j_r}$. Similarly, for each $k \in [t] \setminus I_R$, if $R \cap V(\mathcal{A}_k) = \{w_k, z_k\}$, we may assume that $w_k \neq_{\mathrm{p}} u_1$.

For each $r \in [m]$, we will create a list $\mathcal{L}_r$ of vertices. We will refer to $\mathcal{L}_r$ as the *skeleton* for $\mathcal{P}_r$. We will later use these skeletons to construct the vertex-disjoint paths via Lemma 8.6. For each $r \in [m]$, we will write L_r^* for the (unordered) set of vertices in $\mathcal{L}_r$. In order to construct each $\mathcal{L}_r$, we will start with an empty list and update it in (possibly) several steps, by concatenating alternating parity sequences. Whenever $\mathcal{L}_r$ is updated, we implicitly update L_r^*. In the end, for each $r \in [m]$ we will have a list of vertices $\mathcal{L}_r = (x_1^r, \dots, x_{\ell_r}^r)$. For each $r \in [m]$ and $k \in [t]$, let $I_r(k) \coloneqq \{h \in [\ell_r - 1] : 2 \nmid h \text{ and } x_h^r, x_{h+1}^r \in V(\mathcal{A}_k)\}$. We will require the $\mathcal{L}_r$ to be pairwise vertex-disjoint. Furthermore, we will require that they satisfy the following properties:

($\mathcal{L}$1) For all $r \in [m]$ we have that ℓ_r is even.

($\mathcal{L}$2) For all $r \in [m]$ and $h \in [\ell_r - 1]$, if h is odd, then $x_h^r, x_{h+1}^r \in V(\mathcal{A}_k)$, for some $k \in [t]$; if h is even, then $x_h^r x_{h+1}^r \in E(G \cup \mathcal{M}^*)$.

($\mathcal{L}$3) For all $k \in [t]$ we have that $1 \leq |I_1(k)| \leq 6$ and $|I_r(k)| \leq 1$ for all $r \in [m] \setminus \{1\}$.

($\mathcal{L}4)_1$ For each $k \in [t] \setminus (I_L \cup I_R^1)$ and each $h \in I_1(k)$, we have $x_h^1 \neq_{\mathrm{p}} x_{h+1}^1$. For each $k \in I_L \cup I_R^1$, for all but one $h \in I_1(k)$ we have $x_h^1 \neq_{\mathrm{p}} x_{h+1}^1$, while for the remaining index $h \in I_1(k)$ we have that $x_h^1 =_{\mathrm{p}} x_{h+1}^1$ and their parity is opposite to that of the unique vertex in $L \cap V(\mathcal{A}_k)$ if $k \in I_L$ and to that of the unique vertex in $\{w_k, z_k\} \cap \{u_1, v_1\}$ if $k \in I_R^1$.

($\mathcal{L}4)_r$ For each $r \in [m] \setminus \{1\}$, the following holds. For each $k \in [t] \setminus I_R^r$ and each $h \in I_r(k)$, we have $x_h^r \neq_{\mathrm{p}} x_{h+1}^r$. For each $k \in I_R^r$, for all but one $h \in I_r(k)$ we have $x_h^r \neq_{\mathrm{p}} x_{h+1}^r$, while for the remaining index $h \in I_r(k)$ we have that $x_h^r =_{\mathrm{p}} x_{h+1}^r$ and their parity is opposite to that of the unique vertex in $\{w_k, z_k\} \cap \{u_r, v_r\}$.

($\mathcal{L}$5) For each $r \in [m]$, we have the following. If $u_r \notin R$, then $u_r = x_1^r$. If $v_r \notin R$, then $v_r = x_{\ell_r}^r$. If $u_r \in R$ (and thus $u_r = z_{i_r}$), then $w_{i_r} = x_1^r$ and $u_r \notin L_1^* \cup \dots \cup L_m^*$. If $v_r \in R$ (and thus $v_r = w_{j_r}$), then $z_{j_r} = x_{\ell_r}^r$ and $v_r \notin L_1^* \cup \dots \cup L_m^*$.

($\mathcal{L}$6) Every pair (w_k, z_k) with $\{w_k, z_k\} \subseteq R^*$ is contained in $\mathcal{L}_1$ and z_k directly succeeds w_k.

We begin by constructing $\mathcal{L}_1$. Let $\mathcal{L}_1 \coloneqq \varnothing$ and let $F \coloneqq L \cup R \cup S$. If $i_1 = 1$ and $R^* \cap V(\mathcal{A}_1) = \{w_1, z_1\}$, then let $\mathcal{S}_1 \coloneqq (u_1, w_1, z_1)$. If $i_1 = 1$ and $u_1 \in R$, then let $\mathcal{S}_1 \coloneqq (u_1)$. Otherwise, let $\mathcal{S}_1$ be a $(u_1, 1, F, (R \cap V(\mathcal{A}_{i_1})) \cup (R^* \cap V(\mathcal{A}_1)))$-alternating parity sequence. Let $\mathcal{L}_1 \coloneqq \mathcal{S}_1$. Note that the existence of such a sequence $\mathcal{S}_1$ is guaranteed by our assumption that $\mathcal{M}$ is bonded in G. To see this, note that all edges of G required by $\mathcal{S}_1$ (that is, the pairs $\{v_k, u_k\}$ in Definition 8.7) need to be chosen so that they do not have an endpoint in F; given any particular pair of consecutive atoms, this forbids at most 30 edges between these two atoms (26 because of S and 4 because of $L \cup R$).

We will now update $\mathcal{L}_1$. While doing so, we will update F and consider several alternating parity sequences. The existence of each of these follows a similar argument to the above. For any given pair of consecutive atoms, every time we update F, the set of forbidden edges will increase its size by at most 3. We will update F at most four times, so F will forbid at most 42 edges between any pair of consecutive atoms. Thus, by the definition of bondedness, each of the alternating parity sequences required below actually exists.

Let $u_1^{\downarrow}$ be the last vertex in $\mathcal{L}_1$. Note that $u_1^{\downarrow} =_{\mathrm{p}} u_1$ by Definition 8.7(P1). We update F as $F := F \cup L_1^*$. For the next step in the construction of $\mathcal{L}_1$, there are three cases to consider, depending on the size of L and, if $|L| = 2$, the relative parities of x and u_1. If $i_1 = 1$ and $u_1 \in R$, let $R^\diamond := R^* \cup \{w_1, z_1\}$; otherwise, let $R^\diamond := R^*$.

Case 1: $L = \varnothing$.
Let $\mathcal{S}_2$ be a $(u_1^{\downarrow}, t, F, R^\diamond)$-alternating parity sequence. If $i_1 = 1$ and $u_1 \in R$, update $\mathcal{L}_1$ as $\mathcal{L}_1 := \mathcal{S}_2$. Otherwise, update $\mathcal{L}_1$ as $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_2^-$. Update $F := F \cup L_1^*$.

Case 2: $|L| = 2$ and $x \neq_{\mathrm{p}} u_1$.
Let $\mathcal{S}_2$ be a $(u_1^{\downarrow}, i, F, R^\diamond)$-alternating parity sequence. If $i_1 = 1$ and $u_1 \in R$, update $\mathcal{L}_1$ as $\mathcal{L}_1 := \mathcal{S}_2$. Otherwise, update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_2^-$. Update $F := F \cup L_1^*$. Choose any vertex $u_i^* \in V(\mathcal{A}_i)$ with $u_i^* \neq_{\mathrm{p}} u_1$, and let $\mathcal{S}_3$ be a $(u_i^*, j, F, R^\diamond)$-alternating parity sequence. Update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_3^-$ and $F := F \cup L_1^*$. Let v^- be the final vertex of $\mathcal{S}_2$, and let v^+ be the second vertex of $\mathcal{S}_3$. Note that v^- and v^+ appear consecutively in $\mathcal{L}_1$ and that $v^- =_{\mathrm{p}} v^+ =_{\mathrm{p}} u_1 \neq_{\mathrm{p}} x$. Finally, choose any vertex $u_j^* \in V(\mathcal{A}_j)$ with $u_j^* =_{\mathrm{p}} u_1$, let $\mathcal{S}_4$ be a $(u_j^*, t, F, R^\diamond)$-alternating parity sequence, and update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_4^-$ and $F := F \cup L_1^*$. Let w^- be the final vertex of $\mathcal{S}_3$, and let w^+ be the second vertex of $\mathcal{S}_4$. We then have that w^- and w^+ appear consecutively in $\mathcal{L}_1$, and $w^- =_{\mathrm{p}} w^+ \neq_{\mathrm{p}} y, u_1$. Moreover, the final vertex $u_t^{\uparrow}$ of $\mathcal{L}_1$ satisfies $u_t^{\uparrow} =_{\mathrm{p}} u_1$.

Case 3: $|L| = 2$ and $x =_{\mathrm{p}} u_1$.
Let $\mathcal{S}_2$ be a $(u_1^{\downarrow}, j, F, R^\diamond)$-alternating parity sequence. If $i_1 = 1$ and $u_1 \in R$, update $\mathcal{L}_1$ as $\mathcal{L}_1 := \mathcal{S}_2$; otherwise, update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_2^-$. Update $F := F \cup L_1^*$. Next, let $u_j^* \in V(\mathcal{A}_j)$ be a vertex with $u_j^* \neq_{\mathrm{p}} u_1$ and let $\mathcal{S}_3$ be a $(u_j^*, i, F, \varnothing)$-alternating parity sequence. Update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_3^-$ and $F := F \cup L_1^*$. Finally, let $u_i^* \in V(\mathcal{A}_i)$ be a vertex with $u_i^* =_{\mathrm{p}} u_1$ and let $\mathcal{S}_4$ be a $(u_i^*, t, F, R^* \cap \bigcup_{k=j+1}^{t} V(\mathcal{A}_k))$-alternating parity sequence. Update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_4^-$ and $F := F \cup L_1^*$.

In each of the three cases, let $u_t^{\uparrow}$ denote the last vertex in $\mathcal{L}_1$. Note that, by Definition 8.7(P1), we have $u_t^{\uparrow} =_{\mathrm{p}} u_1$, and recall that $v_1 \neq_{\mathrm{p}} u_1$. Let $\mathcal{S}_5$ be a $(u_t^{\uparrow}, j_1, F, \varnothing)$-alternating parity sequence. Update $\mathcal{L}_1 := \mathcal{L}_1\mathcal{S}_5^-$. Again by Definition 8.7(P1), we have that the final vertex u^* of $\mathcal{L}_1$ is such that $u^* =_{\mathrm{p}} u_t^{\uparrow} =_{\mathrm{p}} u_1 \neq_{\mathrm{p}} v_1$. Finally, if $v_1 \in R$, update $\mathcal{L}_1 := \mathcal{L}_1(z_{j_1})$; otherwise, update it as $\mathcal{L}_1 := \mathcal{L}_1(v_1)$. Observe that $\mathcal{L}_1$ satisfies $(\mathcal{L}1)$–$(\mathcal{L}3)$, $(\mathcal{L}4)_1$, $(\mathcal{L}5)$ and $(\mathcal{L}6)$ for the case $r = 1$ by construction.

We now construct $\mathcal{L}_r$ for all $r \in [m] \setminus \{1\}$. For each $r \in [m] \setminus \{1\}$, we proceed iteratively as follows. Let $\mathcal{L}_r := \varnothing$ and $F_r := L \cup R \cup S \cup \bigcup_{r' \in [r-1]} L_{r'}^*$. Let $\mathcal{S}^r$ be a $(u_r, j_r, F_r, R \cap V(\mathcal{A}_{i_r}))$-alternating parity sequence and update $\mathcal{L}_r$ as $\mathcal{L}_r := \mathcal{S}^r$. If $v_r \in R$, update $\mathcal{L}_r := \mathcal{L}_r(z_{j_r})$; otherwise, update $\mathcal{L}_r := \mathcal{L}_r(v_r)$. Note that each sequence $\mathcal{S}^r$ requires the existence of at most one edge of G, which has to avoid F_r, between any pair of consecutive atoms of $\mathcal{M}^*$. In a similar way to what was discussed above, at most three choices of such edges can be forbidden every time we add a new alternating parity sequence to F. Since for each $r \in [m] \setminus \{1\}$ we consider one new sequence, by the time we consider F_m we have increased the number of forbidden edges by at most $3(m-1) \leq 39$. This gives a total of at most 81 forbidden edges and, thus, the existence of the sequences $\mathcal{S}^r$ is guaranteed by the assumption that $\mathcal{M}$ is bonded in G. Moreover, the lists $\mathcal{L}_1, \dots, \mathcal{L}_r$ now satisfy $(\mathcal{L}1)$–$(\mathcal{L}6)$.

We are now in a position to apply Lemma 8.6. For each $k \in [t]$, let $t_k := \sum_{r \in [m]} |I_r(k)|$. Furthermore, for any $r \in [m]$ and $k \in [t]$, for each $h \in I_r(k)$, we refer to the pair x_h^r, x_{h+1}^r as a *matchable pair*. By $(\mathcal{L}3)$, $(\mathcal{L}4)_1$, $(\mathcal{L}4)_r$ and Lemma 8.6(i), each atom $\mathcal{A}_k$ with $k \in [t] \setminus (I_L \cup I_R)$ can be covered by t_k vertex-disjoint paths, each of whose endpoints are a matchable pair contained in $\mathcal{A}_k$. Similarly, by ($\mathcal{L}3$), ($\mathcal{L}4)_1$, ($\mathcal{L}4)_r$ and Lemma 8.6(ii), each atom $\mathcal{A}_k$ with $k \in I_L \cup I_R$ contains t_k vertex-disjoint paths, each of whose endpoints are a matchable pair in $\mathcal{A}_k$ such that the union of these t_k paths covers precisely $V(\mathcal{A}_k) \setminus (L \cup (S \cap R))$. (Recall that by (C2) and (C3) the set $V(\mathcal{A}_k) \cap (L \cup (S \cap R))$ consists of a single vertex if $k \in I_L \cup I_R$.) For each matchable pair x^r_h, x^r_{h+1} in $\mathcal{A}_k$, let us denote the corresponding path by $\mathcal{P}_{x^r_h, x^r_{h+1}}$.

The paths $\mathcal{P}_1, \dots, \mathcal{P}_m$ required for Lemma 8.8 can now be constructed as follows. For each $r \in [m]$, let $\mathcal{P}_r$ be the path obtained from the concatenation of the paths $\mathcal{P}_{x^r_h, x^r_{h+1}}$, for each odd $h \in [\ell_r]$, via the edges $x^r_h x^r_{h+1}$ for $h \in [\ell_r - 1]$ even. By ($\mathcal{L}5$), if $\mathcal{P}_r$ does not contain u_r, then $\mathcal{P}_r$ starts in w_{i_r}, and u_r does not lie in any other path; therefore, we can update $\mathcal{P}_r$ as $\mathcal{P}_r := u_r\mathcal{P}_r$. Similarly, if $\mathcal{P}_r$ does not contain v_r, then $\mathcal{P}_r$ ends in z_{j_r} and v_r does not lie in any other path, and thus we can update $\mathcal{P}_r$ as $\mathcal{P}_r := \mathcal{P}_r v_r$. It follows that $\bigcup_{r\in[m]} V(\mathcal{P}_r) = V(\mathcal{M}^*) \setminus L$, and thus the paths $\mathcal{P}_r$ are as required in Lemma 8.8. □

We also need the following simpler result. Its proof follows similar ideas as those present in the proof of Lemma 8.8. For the sake of completeness, we include the proof of Lemma 8.9 in Appendix A. We point out here that Lemma 8.6(iii) is only needed for this proof.

Lemma 8.9. *Let $n, s, \ell \in \mathbb{N}$ be such that $4 \le s$ and $100 \le \ell \le n - s$. Let $G \subseteq \mathcal{Q}^n$ and consider any (s, ℓ)-molecule $\mathcal{M} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_{2^s} \subseteq \mathcal{Q}^n$ which is bonded in G. Let $\mathcal{M}^* = \mathcal{A}_{a+1} \cup \dots \cup \mathcal{A}_{a+t}$, for some $a \in [2^s]$ and $t \ge 10$, be a slice of $\mathcal{M}$. Moreover, consider the following sets.*

- (C′1) *Let $L \subseteq V(\mathcal{M}^*)$ be a set of size $|L| \in \{0, 2\}$ such that, if $L = \{x, y\}$, then $x \in V(\mathcal{A}_i)$ and $y \in V(\mathcal{A}_j)$, with $i \ne j$ and $x \ne_{\mathrm{p}} y$.*
- (C′2) *Let $R \subseteq V(\mathcal{M}^*) \setminus L$ be a (possibly empty) set of vertices with $|R| \le 10$ such that, for all $k \in [a + t] \setminus [a]$, we have $|R \cap V(\mathcal{A}_k)| \in \{0, 2\}$ and, if $|R \cap V(\mathcal{A}_k)| = 2$, then $R \cap V(\mathcal{A}_k) = \{w_k, z_k\}$ satisfies that $w_k z_k \in E(\mathcal{M}^*)$ and, if $|L| = 2$, then $k \notin \{i, j\}$.*
- (C′3) *Consider two vertex-disjoint pairs $\{u_r, v_r\}_{r\in[2]}$ with $u_1, u_2 \in V(\mathcal{A}_{a+1}) \setminus L$ and $v_1, v_2 \in V(\mathcal{A}_{a+t}) \setminus L$ such that $u_1 \ne_{\mathrm{p}} u_2$, $v_1 \ne_{\mathrm{p}} v_2$, $u_1 =_{\mathrm{p}} v_1$, and $|\{u_1, u_2\} \cap R|, |\{v_1, v_2\} \cap R| \le 1$.*

Then, there exist two vertex-disjoint paths $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{M}^* \cup G$ such that, for each $r \in [2]$, $\mathcal{P}_r$ is a (u_r, v_r)-path, $V(\mathcal{P}_1) \cup V(\mathcal{P}_2) = V(\mathcal{M}^*) \setminus L$, and every pair of the form $\{w_k, z_k\} \subseteq R$ with $k \in [a + t] \setminus [a]$ is an edge of either $\mathcal{P}_1$ or $\mathcal{P}_2$.

8.4. **Proof of Theorem 8.1.**

Proof of Theorem 8.1. Let $1/D, \delta' \ll 1$, and let

$$0 < 1/n_0 \ll \delta \ll 1/\ell \ll 1/k^*, \alpha' \ll \beta, 1/S' \ll 1/c, 1/D, \delta', \varepsilon, \alpha,$$

where $n_0, \ell, k^*, S', D \in \mathbb{N}$. Our proof assumes that n tends to infinity; in particular, $n \ge n_0$. Let $s := 10\ell$, $\Phi := 12\ell$ and and $\Psi := c\Phi$.

Observe that $\mathcal{Q}^n[\{0, 1\}^s \times \{0\}^{n-s}] \cong \mathcal{Q}^s$ contains a Hamilton cycle. We fix an ordering of the layers $L_1, \dots, L_{2^s}$ of $\mathcal{Q}^n$ induced by this Hamilton cycle (as defined in Section 8.1). If we view these layers as different subgraphs on the vertex set of $\mathcal{Q}^{n-s}$, we can define the *intersection graph* of the layers $I := \bigcap_{i=1}^{2^s} L_i$ (note that $I \cong \mathcal{Q}^{n-s}$) and, for any $G \subseteq \mathcal{Q}^n$, we denote $I(G) := \bigcap_{i=1}^{2^s} L_i(G)$. Note that, if $\mathcal{G} \subseteq I(G)$, then there is a clone of $\mathcal{G}$ in $L_i(G)$, for each $i \in [2^s]$. For each layer L, we denote by $\mathcal{G}_L$ the clone of $\mathcal{G}$ in $L(G)$. Observe that, for any $\eta \in [0, 1]$, we have $I(\mathcal{Q}^n_\eta) \sim \mathcal{Q}^{n-s}_{\eta^{2^s}}$. We will sometimes write G_I for the subgraph of I where, for each $e \in E(I)$, we have $e \in E(G_I)$ whenever G contains some clone of e (thus, G_I is the 'union' of the subgraphs that G induces on each layer).

For each $i \in [7]$, let $\varepsilon_i := \varepsilon/7$ and let $G_i \sim \mathcal{Q}^n_{\varepsilon_i}$, where these graphs are taken independently. It is easy to see that $\bigcup_{i=1}^7 G_i \sim \mathcal{Q}^n_{\varepsilon'}$ for some $\varepsilon' < \varepsilon$. Thus, it suffices to show that a.a.s. there is a graph $G' \subseteq \bigcup_{i=1}^7 G_i$ with $\Delta(G') \le \Phi$ such that, for every $F \subseteq \mathcal{Q}^n$ with $\Delta(F) \le \Psi$, the graph $((H \cup \bigcup_{i=1}^7 G_i) \setminus F) \cup G'$ is Hamiltonian. We now split our proof into several steps.

Step 1: Finding a tree and a reservoir. Consider the probability space $\Omega := \mathcal{Q}^{n-s}_{\varepsilon_1^{2^s}} \times \mathit{Res}(\mathcal{Q}^{n-s}, \delta')$ (with the latter defined as in Section 7.1), so that, given $R \sim \mathit{Res}(I, \delta')$, we have that $(I(G_1), R) \sim \Omega$.

Let $\mathcal{E}_1$ be the event that there exists a tree $T \subseteq I(G_1) - R$ such that the following hold:

(TR1) $\Delta(T) < D$, and

(TR2) for all $x \in V(I)$, we have that $|N_I(x) \cap V(T)| \geq 4(n-s)/5$.

It follows from Theorem 7.1, with $n-s$, $\varepsilon_1^{2^s}$, $\delta', \varnothing$ and $1/5$ playing the roles of n, ε, $\delta, \mathcal{A}$ and ε', respectively, that $\mathbb{P}_\Omega[\mathcal{E}_1] = 1 - o(1)$.

Step 2: Identifying scant molecules. For each $v \in V(I)$, let $\mathcal{M}_v$ denote the vertex molecule $\mathcal{M}_v := \{av : a \in \{0,1\}^s\}$. We say a vertex molecule $\mathcal{M}_v$ is *scant* if there exist some layer L and some vertex $x \in V(\mathcal{M}_v \cap L)$ such that $d_H(x, R_L) < \alpha\delta' n/10$, where R_L is the clone of R in L. Let $\mathcal{E}_2$ be the event that there exists some $x \in V(I)$ such that there are more than S' vertices $v \in B_I^{10\ell}(x)$ satisfying that $\mathcal{M}_v$ is scant. It follows from Lemma 8.5 with S' and δ' playing the roles of C and δ that $\mathbb{P}_\Omega[\mathcal{E}_2] < e^{-n}$. Let $\mathcal{E}_1^* := \mathcal{E}_1 \wedge \overline{\mathcal{E}_2}$. Therefore, $\mathbb{P}_\Omega[\mathcal{E}_1^*] = 1 - o(1)$.

Condition on $\mathcal{E}_1^*$ holding. Then, G_1 satisfies the following: there exist a set $R \subseteq V(I)$ and a tree $T \subseteq I(G_1) - R$ such that the following hold:

(T1) $\Delta(T) < D$;

(T2) for all $x \in V(I)$, we have that $|N_I(x) \cap V(T)| \geq 4(n-s)/5$, and

(T3) for every $x \in V(I)$, $B_I^{10\ell}(x)$ contains at most S' vertices v such that $\mathcal{M}_v$ is scant.

Recall this implies clones of T and R satisfying (T1)–(T3) exist simultaneously in each layer of G_1.

Step 3: Finding robust matchings for each slice. Recall from Section 2.5 that we will absorb vertices in pairs, where each pair consists of two clones x', x'' of the same vertex $x \in V(I)$. In this step, for each $x \in V(I)$ and for each set of clones of x that may need to be absorbed, we find a pairing of these clones so that we can later build suitable absorbing ℓ-cube pairs for each such pair of clones. We will find this pairing separately for each slice of the vertex molecule $\mathcal{M}_x$. Considering each slice separately has the advantage that the chosen pairs are 'localised'. This will be convenient later when linking up the paths used to absorb these vertices. Accordingly, we now partition the set of layers into sets of consecutive layers as follows. Let

$$q := 2^{10Dk^*} \qquad \text{and let} \qquad t := 2^s/q. \tag{8.1}$$

For each $j \in [t]$, let $S_j := \bigcup_{i=(j-1)q+1}^{jq} L_i$. Given any molecule $\mathcal{M}$, we consider the slices $\mathcal{S}_j(\mathcal{M}) := S_j \cap \mathcal{M}$. We denote by $\mathcal{S}(\mathcal{M})$ the collection of all these slices of $\mathcal{M}$.

Let $V_{\text{sc}} \subseteq V(I)$ be the set of all vertices $x \in V(I)$ such that $\mathcal{M}_x$ is scant. Recall $G_2 \sim \mathcal{Q}^n_{\varepsilon_2}$. For each $v \in V(I) \setminus V_{\text{sc}}$ and each $\mathcal{S} \in \mathcal{S}(\mathcal{M}_v)$, we define the following auxiliary bipartite graphs. Let $H(\mathcal{S}) := (V(\mathcal{S}), N_I(v), E_H)$, where E_H is defined as follows. Consider $v' \in V(\mathcal{S})$ and let $L^{v'}$ be the layer which contains v'. Let $w \in N_I(v)$, and let $w_{L^{v'}}$ be the clone of w in $L^{v'}$. Then, $\{v', w\} \in E_H$ if and only if $w \in R$ and $\{v', w_{L^{v'}}\} \in E(H)$. Note that $d_{H(\mathcal{S})}(v') \geq \alpha\delta' n/10$ for all $v' \in V(\mathcal{S})$ since $\mathcal{S}$ is a slice of a vertex molecule which is not scant. Similarly, we define $G_2(\mathcal{S}) := (V(\mathcal{S}), N_I(v), E_{G_2})$, where $\{v', w\} \in E_{G_2}$ if and only if $\{v', w_{L^{v'}}\} \in E(G_2)$.

Note that the partition of $V(\mathcal{S})$ into vertices of even and odd parity is a balanced bipartition. Define the graph $\Gamma^\beta_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S}))$ as in Section 5.1. Note that, by definition, we have that $V(\Gamma^\beta_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S}))) = V(\mathcal{S})$. Furthermore, by definition,

(RM) given any $w_1, w_2 \in V(\mathcal{S})$, we have that $\{w_1, w_2\} \in E(\Gamma^\beta_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S})))$ if and only if $|N_{H(\mathcal{S})}(w_1) \cap N_{G_2(\mathcal{S})}(w_2)| \geq \beta(n-s)$ or $|N_{G_2(\mathcal{S})}(w_1) \cap N_{H(\mathcal{S})}(w_2)| \geq \beta(n-s)$.

By applying Lemma 5.2 with $d = 24D$, $r = 0$, $\alpha = \alpha\delta'/10$, $\varepsilon = \varepsilon_2$, $n = n-s$, $k = q = 2^{10Dk^*}$, $\beta = \beta$ and $G = H(\mathcal{S})$, we obtain that, with probability at least $1 - 2^{-10(n-s)} \geq 1 - 2^{-8n}$, the graph $\Gamma^\beta_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S}))$ is $24D$-robust-parity-matchable with respect to the partition of $V(\mathcal{S})$ into vertices of even and odd parity.

We would like to proceed as above for slices in scant molecules; however, recall that scant molecules contain vertices with few or no neighbours in the reservoir, and therefore we must adapt our approach. For each $v \in V_{\text{sc}}$ and each $\mathcal{S} \in \mathcal{S}(\mathcal{M}_v)$, we define an auxiliary bipartite graph $H(\mathcal{S})$ and $G_2(\mathcal{S})$ as above, except that we omit the condition that $w \in R$ for the existence of an edge in $H(\mathcal{S})$. By applying Lemma 5.2 again, we obtain that, with probability at least $1-2^{-8n}$, the graph $\Gamma^{\beta}_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S}))$ is $24D$-robust-parity-matchable with respect to the partition of $V(\mathcal{S})$ into vertices of even and odd parity.

By a union bound over all $v \in V(I)$ and all slices $\mathcal{S} \in \mathcal{S}(\mathcal{M}_v)$, we have that a.a.s. the graph $\Gamma^{\beta}_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S}))$ is $24D$-robust-parity-matchable (with respect to the partition of $V(\mathcal{S})$ into vertices of even and odd parity) for every slice $\mathcal{S}$, where $H(\mathcal{S})$ is as defined above in each case. We condition on this event holding and call it $\mathcal{E}_2^*$. Thus, for each slice $\mathcal{S}$ and each set $S \subseteq V(\mathcal{S})$ with $|S| \leq 24D$ which contains as many odd vertices as even vertices, there exists a perfect matching $\mathfrak{M}(\mathcal{S},S)$ in the bipartite graph with parts consisting of the even and odd vertices of $V(\mathcal{S}) \setminus S$, respectively, and edges given by $\Gamma^{\beta}_{H(\mathcal{S}),G_2(\mathcal{S})}(V(\mathcal{S}))$. For each slice $\mathcal{S}$, we denote by $\mathfrak{M}(\mathcal{S})$ the set of edges contained in the union (over all S) of the matchings $\mathfrak{M}(\mathcal{S},S)$ (without multiplicity). Furthermore, for each $e=\{w_{\mathrm{e}},w_{\mathrm{o}}\} \in \mathfrak{M}(\mathcal{S})$, we let $N(e) \coloneqq (N_{H(\mathcal{S})}(w_{\mathrm{e}}) \cap N_{G_2(\mathcal{S})}(w_{\mathrm{o}})) \cup (N_{G_2(\mathcal{S})}(w_{\mathrm{e}}) \cap N_{H(\mathcal{S})}(w_{\mathrm{o}}))$. By (RM), we have $|N(e)| \geq \beta(n-s) \geq \beta n/2$. For each $v \in V(I)$, let $\mathfrak{M}(v) \coloneqq \bigcup_{\mathcal{S}\in\mathcal{S}(\mathcal{M}_v)} \mathfrak{M}(\mathcal{S})$. Let $K \coloneqq \max_{v\in V(I)} |\mathfrak{M}(v)|$. In particular, we have that $K \leq \binom{2^s}{2}$.

Step 4: Obtaining an appropriate cube factor via the nibble. For each $x \in V(I)$, consider the multiset $\mathfrak{A}(x) \coloneqq \{N(e) : e \in \mathfrak{M}(x)\}$. If $|\mathfrak{A}(x)| < K$, we artificially increase its size to K by repeating any of its elements. Label the sets in $\mathfrak{A}(x)$ arbitrarily as $\mathfrak{A}(x) = \{A_1(x),\ldots,A_K(x)\}$. Thus, if $x \in V(I) \setminus V_{\mathrm{sc}}$, then $A_i(x) \subseteq R$ for all $i \in [K]$.

Let $\mathcal{C}$ be any collection of subgraphs C of I such that $C \cong \mathcal{Q}^\ell$ for all $C \in \mathcal{C}$. For any vertex $x \in V(I)$ and any set $Y \subseteq N_I(x)$, let $\mathcal{C}_x(Y) \subseteq \mathcal{C}$ be the set of all $C \in \mathcal{C}$ such that $x \notin V(C)$ and $Y \cap V(C) \neq \varnothing$, and let $\mathcal{C}_x \coloneqq \mathcal{C}_x(N_I(x))$.

Recall $G_3 \sim \mathcal{Q}^n_{\varepsilon_3}$ and $I(G_3) \sim \mathcal{Q}^{n-s}_{\varepsilon_3^{2^s}}$. We now apply Theorem 6.6 to the graph $I(G_3)$, with $\varepsilon_3^{2^s}, \alpha', \delta/2$, $\beta/2$, K and ℓ playing the roles of ε, α, δ, β, K and ℓ, respectively, and using the sets $A_i(x)$ given above, for each $x \in V(I)$ and $i \in [K]$. Thus, a.a.s. we obtain a collection $\mathcal{C}$ of vertex-disjoint copies of $\mathcal{Q}^\ell$ in $I(G_3)$, such that the following properties hold for every $x \in V(I)$:

(N1) $|\mathcal{C}_x| \geq (1-\delta)n$.
(N2) For every direction $\hat{e} \in \mathcal{D}(I)$ we have that $|\Sigma(\mathcal{C}_x,\{\hat{e}\},1)| = o(n^{1/2})$.
(N3) For every $i \in [K]$ and every $S \subseteq \mathcal{D}(I)$ with $\alpha'(n-s)/2 \leq |S| \leq \alpha'(n-s)$ we have

$$|\Sigma(\mathcal{C}_x(A_i(x)),S,\ell^{1/2})| \geq |A_i(x)|/3000 \geq \beta n/6000.$$

Condition on the above event holding and call it $\mathcal{E}_3^*$.

Step 5: Absorption cubes. For each $x \in V(I)$ and $i \in [K]$, we define an auxiliary digraph $\mathfrak{D} = \mathfrak{D}(A_i(x))$ on vertex set $A_i(x) - \{x\}$ (seen as a set of directions of $\mathcal{D}(I)$) by adding a directed edge from $\hat{e}$ to $\hat{e}'$ if there is a cube $C^r \in \mathcal{C}_x(A_i(x))$ such that $x+\hat{e} \in V(C^r)$ and $\hat{e}' \in \mathcal{D}(C^r)$. In this way, an edge from $\hat{e}$ to $\hat{e}'$ in $\mathcal{D}$ indicates that the cube C^r could be used as a right absorber cube for x, if combined with a vertex-disjoint left absorber cube with tip $x+\hat{e}'$. Observe that, for all $\hat{e} \in A_i(x)-\{x\}$,

$$d^+_{\mathfrak{D}}(\hat{e}) \in [\ell]_0. \tag{8.2}$$

Furthermore, it follows by (N3) that any set $S \subseteq V(\mathfrak{D})$ with $|S| = \alpha' n/2$ satisfies

$$e_{\mathfrak{D}}(V(\mathfrak{D}),S) \geq \ell^{1/2}\beta n/6000 > \ell^{1/2}\beta^2 n. \tag{8.3}$$

Recall that $A_i(x) = N(\{x_1,x_2\})$ for some $\{x_1,x_2\} \in \mathfrak{M}(\mathcal{S})$, where $\mathcal{S} \in \mathcal{S}(\mathcal{M}_x)$ is some slice of $\mathcal{M}_x$. Note that $x_1,x_2 \in \mathcal{M}_x$, and let L^j be the layer containing x_j for each $j \in [2]$. We say that x_1 and x_2 are the vertices (or clones of x) which *correspond* to the pair (x,i). Let $(\hat{e},\hat{e}') \in E(\mathfrak{D})$ and, for each $j \in [2]$, let e_j be the clone of $\{x+\hat{e}', x+\hat{e}'+\hat{e}\}$ in L^j. It follows that there is a cube $C^r \in \mathcal{C}_x(A_i(x))$ such that e_j connects the clone C_j of C^r to the clone of $x+\hat{e}'$ in L^j.

Recall $G_4 \sim \mathcal{Q}^n_{\varepsilon_4}$. Let $\mathfrak{D}' \subseteq \mathfrak{D}$ be the subdigraph which retains each edge $(\hat{e},\hat{e}') \in E(\mathfrak{D})$ if and only if the edges e_1,e_2 described above are both present in G_4. Note that each edge of $\mathfrak{D}$ is therefore retained independently of every other edge with probability ε_4^2. By Lemma 4.2, (8.2) and (8.3), it follows that $\mathfrak{D}'$ satisfies the following with probability at least $1-e^{-10n}$:

(DG1) for every $A \subseteq V(\mathfrak{D})$ with $|A| = \alpha' n/2$ we have $\sum_{v\in A} d^-_{\mathfrak{D}'}(v) \geq \varepsilon_4^3\beta^2\ell^{1/2}n$, and
(DG2) for every $B \subseteq V(\mathfrak{D})$ we have that $\sum_{v\in B} d^+_{\mathfrak{D}'}(v) \leq \ell|B|$.

Recall that $\mathfrak{D} = \mathfrak{D}(A_i(x))$. By a union bound, (DG1) and (DG2) hold a.a.s. for all $x \in V(I)$ and $i \in [K]$. We condition on this event and call it $\mathcal{E}_4^*$.

For each $x \in V(I)$ and $i \in [K]$, recall that (RM) and the definition of $A_i(x)$ imply that $|A_i(x)| \geq \beta(n-s)$. Thus, it follows by Lemma 5.4 with $|A_i(x)|$, $2\alpha'/\beta$, $\varepsilon_4^3\beta^3\ell^{1/2}/(2\alpha')$ and ℓ playing the roles of n, α, c and C, respectively, that there exists a matching $M''(A_i(x))$ of size at least $\frac{\varepsilon_4^3\beta^2}{2\ell^{1/2}}|A_i(x)| \geq \varepsilon_4^3\beta^3 n/(3\ell^{1/2})$ in $\mathfrak{D}'(A_i(x))$.

Next, for each $x \in V(I)$ and $i \in [K]$, we remove from $M''(A_i(x))$ all edges $(\hat{e},\hat{e}') \in M''(A_i(x))$ such that $x+\hat{e}'$ does not lie in any cube of $\mathcal{C}_x(A_i(x))$. We denote the resulting matching by $M'(A_i(x))$. Note that, by (N1), we have

$$|M'(A_i(x))| \geq \varepsilon_4^3\beta^3 n/(3\ell^{1/2}) - \delta n \geq n/\ell. \tag{8.4}$$

Consider $A_i(x)$, for some $x \in V(I)$ and $i \in [K]$, and let x_1, x_2 be the clones of x which correspond to (x,i). As before, for each $j \in [2]$, let L^j be the layer containing x_j. Recall Definition 8.2 and note that, by construction, we have the following.

(AB1) For each edge $(\hat{e},\hat{e}') \in M'(A_i(x))$, there is an absorbing ℓ-cube pair (C^l, C^r) for x in I such that, for each $j \in [2]$, the clone (C^l_j, C^r_j) of (C^l, C^r) in L^j is an absorbing ℓ-cube pair for x_j in $H \cup G_2 \cup G_3 \cup G_4$. In particular, the edge joining the left absorber tip to the third absorber vertex lies in G_4. Moreover, $C^l, C^r \in \mathcal{C}_x(A_i(x)) \subseteq \mathcal{C}$ and (C^l, C^r) has left and right absorber tips $x+\hat{e}'$ and $x+\hat{e}$, respectively. Furthermore, for each $x \in V(I)\setminus V_{\mathrm{sc}}$, these tips lie in R. We refer to (C^l_1, C^r_1) and (C^l_2, C^r_2) as the absorbing ℓ-cube pairs for x_1 and x_2 *associated* with $(\hat{e},\hat{e}')$.

Thus, the graph $H\cup G_2\cup G_3\cup G_4$, contains at least n/ℓ absorbing ℓ-cube pairs for each of the clones x_1 and x_2 of x associated with edges in $M'(A_i(x)) \subseteq \mathfrak{D}(A_i(x))$. Moreover, since $M'(A_i(x))$ is a matching, for each $j \in [2]$ these absorbing ℓ-cube pairs for x_j are pairwise vertex-disjoint apart from x_j.

For ease of notation, we will often consider the absorbing ℓ-cube pair (C^l, C^r) for x in I which (C^l_1, C^r_1) and (C^l_2, C^r_2) are clones of, and use it as a placeholder for both of its clones. By slightly abusing notation, we will refer to (C^l, C^r) as the *absorbing ℓ-cube pair associated* with $(\hat{e},\hat{e}')$. Note, however, that (C^l, C^r) is not necessarily an absorbing ℓ-cube pair for x in $I(H\cup G_2\cup G_3\cup G_4)$.

Step 6: Removing bondless molecules. Recall $G_5 \sim \mathcal{Q}^n_{\varepsilon_5}$. In this step, we consider the edges between the different layers.

For each $C \in \mathcal{C}$, let $\mathcal{M}_C$ denote the cube molecule consisting of the clones of C. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the set of cubes $C \in \mathcal{C}$ for which $\mathcal{M}_C$ is bonded in G_5. By an application of Lemma 8.3, for each $C \in \mathcal{C}$ we have that

$$\mathbb{P}[C \notin \mathcal{C}'] = \mathbb{P}[\mathcal{M}_C \text{ is bondless in } G_5] \leq 2^{s+1-\varepsilon_5 2^\ell/4} \leq 2^{-\varepsilon 2^\ell/30}.$$

For each $x \in V(I)$, let $A_0(x) \coloneqq N_I(x)$. For each $i \in [K]_0$, let $\mathcal{E}(x,i)$ be the event that $|\mathcal{C}_x(A_i(x))\setminus\mathcal{C}'| > n/\ell^4$. Since the cubes $C \in \mathcal{C}$ are vertex-disjoint, the events that the molecules $\mathcal{M}_C$ are bondless in G_5 are independent. Therefore, we have that

$$\mathbb{P}[\mathcal{E}(x,i)] \leq \binom{n}{n/\ell^4}(2^{-\varepsilon 2^\ell/30})^{n/\ell^4} \leq 2^{-10n}.$$

Let $\mathcal{E}_4 \coloneqq \bigvee_{x\in V(I)}\bigvee_{i\in[K]_0}\mathcal{E}(x,i)$. By a union bound over all $x \in V(I)$ and $i \in [K]_0$, it follows that

$$\mathbb{P}[\mathcal{E}_4] \leq 2^{-8n}. \tag{8.5}$$

Let $\mathcal{C}_{\mathrm{bs}} \subseteq \mathcal{C}$ be the set of all $C \in \mathcal{C}$ such that $\mathcal{M}_C$ is bondlessly surrounded in G_5 (with respect to $\{\mathcal{M}_{C'} : C' \in \mathcal{C}\}$). For each $x \in V(I)$, let $\mathcal{E}(x)$ be the event that there are more than

$n^{1/3}$ cubes $C \in \mathcal{C}_{\mathrm{bs}}$ which intersect $B_I^{\ell^2}(x)$. Let $\mathcal{E}_5 := \bigvee_{x \in V(I)} \mathcal{E}(x)$. By (N2), we may apply Lemma 8.4 with ε_5 playing the role of ε to conclude that

$$\mathbb{P}[\mathcal{E}_5] \leq 2^{-n^{9/8}}. \tag{8.6}$$

Now let $\mathcal{E}_5^* := \overline{\mathcal{E}_4} \wedge \overline{\mathcal{E}_5}$. It follows from (8.5) and (8.6) that $\mathcal{E}_5^*$ occurs a.a.s. Condition on this event.

Let $\mathcal{C}'' := \mathcal{C}' \setminus \mathcal{C}_{\mathrm{bs}}$. For each $x \in V(I)$ and each $i \in [K]$, let

(AB2) $M(A_i(x)) \subseteq M'(A_i(x))$ consist of all edges $(\hat{e}, \hat{e}') \in M'(A_i(x))$ whose associated absorbing ℓ-cube pair (C^l, C^r) satisfies that $C^r, C^l \in \mathcal{C}''$.

By combining (8.4) with the further conditioning, it follows that, for each $x \in V(I)$ and each $i \in [K]$,

$$|M(A_i(x))| \geq n/\ell - n/\ell^4 - n^{1/3} \geq n/\ell^2. \tag{8.7}$$

Consider any $x \in V(I)$ and $i \in [K]$, and let x_1, x_2 be the two clones of x corresponding to (x, i). Then, at this point, for each $j \in [2]$, $H \cup G_2 \cup G_3 \cup G_4$ contains at least n/ℓ^2 vertex-disjoint (apart from x_j) absorbing ℓ-cube pairs for x_j such that each of these absorbing ℓ-cube pairs (C^l, C^r) is associated with an edge of $M(A_i(x))$, and for each $C \in \{C^l, C^r\}$ the corresponding cube molecule $\mathcal{M}_C$ is bonded in G_5 and (within the collection $\{\mathcal{M}_{C'} : C' \in \mathcal{C}\}$ of all cube molecules) $\mathcal{M}_C$ is not bondlessly surrounded in G_5.

Step 7: Extending the tree $\boldsymbol{T}$. For each $x \in V(I)$, let $Z(x) := N_I(x) \cap V(T) \cap \big(\bigcup_{C \in \mathcal{C}''} V(C)\big)$. It follows by (T2), (N1) and our conditioning on the event $\mathcal{E}_5^*$ that, for each $x \in V(I)$, we have that

$$|Z(x)| \geq 4(n-s)/5 - \delta n - n/\ell^4 - n^{1/3} \geq 3n/4.$$

Recall $G_6 \sim \mathcal{Q}^n_{\varepsilon_6}$. We apply Theorem 7.19 with $\varepsilon_6^{2^s}$, 2, T, $R, \varnothing$ and the sets $Z(x)$ playing the roles of ε, ℓ, T', R, W and $Z(x)$, respectively. Combining this with (T1), we conclude that a.a.s. there exists a tree T' such that $T \subseteq T' \subseteq I(G_6) \cup T$ and the following hold:

(ET1) $\Delta(T') < D + 1$;
(ET2) for all $x \in V(I)$, we have that $|B_I^2(x) \setminus V(T')| \leq n^{3/4}$;
(ET3) for each $x \in V(T') \cap R$, we have that $d_{T'}(x) = 1$ and the unique neighbour x' of x in T' is such that $x' \in Z(x)$.

We condition on the above event holding and call it $\mathcal{E}_6^*$.

At this point, for each $x \in V(I)$ and each $i \in [K]$, we redefine the set $M(A_i(x))$.

(AB3) Let $M(A_i(x))$ retain only those edges whose associated absorbing ℓ-cube pair (C^l, C^r) satisfies that both C^l and C^r intersect T' in at least 2 vertices.

It follows from (8.7) and (ET2) that

$$|M(A_i(x))| \geq n/\ell^2 - n^{3/4} > 4n/\ell^3. \tag{8.8}$$

Step 8: Fixing a collection of absorbing $\boldsymbol{\ell}$-cube pairs for the vertices in scant molecules. Recall $G_7 \sim \mathcal{Q}^n_{\varepsilon_7}$. Consider any $x \in V_{\mathrm{sc}}$ and $j \in [K]$. Recall from Step 3 that the tips of the cubes of the absorbing ℓ-cube pair associated with a given edge in $M(A_j(x))$ may not lie in the reservoir R. Roughly speaking, we will alter T' so that the tips are relocated from the tree T' to the reservoir R.

We start by redefining the matchings $M(A_j(x))$ as follows: for each $x \in V_{\mathrm{sc}}$ and each $j \in [K]$, remove from $M(A_j(x))$ all edges $(\hat{e}, \hat{e}')$ such that $N_{T'}(x) \cap \{x + \hat{e}, x + \hat{e}'\} \neq \varnothing$. It follows from (8.8) and (ET1) that, for all $x \in V(I)$ and $j \in [K]$,

$$|M(A_j(x))| \geq 4n/\ell^3 - D > 2n/\ell^3. \tag{8.9}$$

For each $x \in V_{\mathrm{sc}}$, each $j \in [K]$ and each matching $M' \subseteq M(A_j(x))$ with $|M'| \geq n/\ell^3$, let $\mathcal{E}'(x, j, M')$ be the following event:

for every set $B \subseteq V(I)$ with $|B| < 2^{\ell+s+3}\Psi KS'$, there exists an edge $\vec{e} \in M'$, whose associated absorbing ℓ-cube pair (C^l, C^r) has tips x^l and x^r, for which there exists a subgraph $P(\vec{e}, B) \subseteq I(G_7) - \{x^l, x^r\}$ such that $|V(P(\vec{e}, B))| < 21D/2$, $V(P(\vec{e}, B)) \cap B = \varnothing$, and both $N_{T'}(x^l)$ and $N_{T'}(x^r)$ are connected in $P(\vec{e}, B)$.

For a graph $P(\vec{e}, B)$ as above, we will refer to x^l and x^r as the tips *associated* with $P(\vec{e}, B)$, and refer to (C^l, C^r) as the absorbing ℓ-cube pair *associated* with $P(\vec{e}, B)$. (Recall that, if $\vec{e} = (\hat{e}, \hat{e}')$, then $x^l = x + \hat{e}$ and $x^r = x + \hat{e}'$.)

By invoking Lemma 7.20 with $n-s$, $\varepsilon_7^{2^s}$, $1/\ell^3$, $2^{\ell+s+3}\Psi KS'$, $2D+2$ and the sets $\{(x+\hat{e}, x+\hat{e}') : (\hat{e}, \hat{e}') \in M'\}$ and $(N_{T'}(x+\hat{e}) \cup N_{T'}(x+\hat{e}'))_{(\hat{e},\hat{e}') \in M'}$ playing the roles of n, ε, c, f, D, $C(x)$ and $(B(y,z))_{(y,z) \in C(x)}$, respectively, we have that $\mathcal{E}'(x, j, M')$ holds with probability at least $1 - 2^{-5(n-s)}$. Let $\mathcal{E}_7^* := \bigwedge_{x \in V_{\mathrm{sc}}} \bigwedge_{j \in [K]} \bigwedge_{M' \subseteq M(A_j(x)) : |M'| \geq n/\ell^3} \mathcal{E}'(x, j, M')$. By a union bound over all $x \in V_{\mathrm{sc}}$, $j \in [K]$ and $M' \subseteq M(A_j(x))$ such that $|M'| \geq n/\ell^3$, it follows that $\mathbb{P}[\mathcal{E}_7^*] \geq 1 - 2^{-2n}$.

Condition on the event that $\mathcal{E}_7^*$ holds. It follows that, for each $x \in V_{\mathrm{sc}}$, $j \in [K]$, $M' \subseteq M(A_j(x))$ with $|M'| \geq n/\ell^3$ and any $B \subseteq V(I)$ with $|B| < 2^{\ell+s+3}\Psi KS'$, there exists a subgraph $P(x, j, M', B) \subseteq I(G_7)$ with $|V(P(x, j, M', B))| < 21D/2$ which avoids $B \cup \{x^l, x^r\}$, where x^l and x^r are the tips associated with $P(x, j, M', B)$, and such that both $N_{T'}(x^l)$ and $N_{T'}(x^r)$ are connected in $P(x, j, M', B)$. Moreover, by choosing $P(x, j, M', B)$ minimal, we may assume that it consists of at most two components, and each such component contains either $N_{T'}(x^l)$ or $N_{T'}(x^r)$.

Let $\iota := |V_{\mathrm{sc}}|$ and let $x_1, \ldots, x_\iota$ be an ordering of V_{sc}. For each $i \in [\iota]$, $j \in [K]$ and $k \in [2^{s+1}\Psi]$, by ranging over i first, then j, and then k, we will iteratively fix a graph $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$ as above. In particular, this graph will have an absorbing ℓ-cube pair with tips $x^l_{i,j,k}$ and $x^r_{i,j,k}$ associated with it. After the graph $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$ is fixed, so are these tips. Let $\mathcal{J}_{i,j,k} := ([i-1] \times [K] \times [2^{s+1}\Psi]) \cup \{(i, j', k') : (j', k') \in [j-1] \times [2^{s+1}\Psi]\} \cup \{(i, j, k'') : k'' \in [k-1]\}$ and suppose that we have already fixed $P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'})$ for all $(i', j', k') \in \mathcal{J}_{i,j,k}$ such that these $P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'})$ are vertex-disjoint from each other and from the set $\{x^l_{i',j',k'}, x^r_{i',j',k'} : (i', j', k') \in \mathcal{J}_{i,j,k}\}$ of tips associated with all these $P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'})$. In order to fix $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$, we first define the sets $B_{i,j,k}$ and $M'_{i,j,k}$. Let $M'_{i,j,k}$ be obtained from $M(A_j(x_i))$ as follows. Remove all edges whose associated absorbing ℓ-cube pair (C^l, C^r) satisfies $(V(C^l) \cup V(C^r)) \cap \{x^l_{i',j',k'}, x^r_{i',j',k'} : (i', j', k') \in \mathcal{J}_{i,j,k}\} \neq \varnothing$. Remove all edges $(\hat{e}, \hat{e}') \in M(A_j(x_i))$ such that $\{x_i + \hat{e}, x_i + \hat{e}'\} \cap \bigcup_{(i',j',k') \in \mathcal{J}_{i,j,k}} V(P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'})) \neq \varnothing$ too. Note that, by (8.9) and (T3), it follows that $|M'_{i,j,k}| \geq n/\ell^3$. Let $B_{i,j,k}$ be the set of vertices $y \in B_I^{\ell/2}(x_i)$ such that at least one of the following holds:

(P1) there exists $(i', j', k') \in \mathcal{J}_{i,j,k}$ such that $y \in V(P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'}))$;

(P2) there exists $(i', j', k') \in \mathcal{J}_{i,j,k}$ such that y lies in the absorbing ℓ-cube pair associated with $P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'})$.

Note that $|B_{i,j,k}| < 2^{s+\ell+3}\Psi KS'$ by (T3). We then fix $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$ to be the graph guaranteed by our conditioning on $\mathcal{E}_7^*$. Observe that, by the choice of $B_{i,j,k}$, we have that $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$ is vertex-disjoint from $\bigcup_{(i',j',k') \in \mathcal{J}_{i,j,k}} P(x_{i'}, j', k', M'_{i',j',k'}, B_{i',j',k'})$. We denote by $(C^l(x_i, j, k), C^r(x_i, j, k))$ the absorbing ℓ-cube pair for x_i associated with $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$. By the choice of $M'_{i,j,k}$, we have that

(CD) for all $(i', j', k') \in \mathcal{J}_{i,j,k}$, $C^l(x_i, j, k)$ and $C^r(x_i, j, k)$ are both vertex-disjoint from $C^l(x_{i'}, j', k')$ and $C^r(x_{i'}, j', k')$.

Let $\mathcal{C}_1^{\mathrm{sc}} := \{(C^l(x_i, j, k), C^r(x_i, j, k)) : (i, j, k) \in [\iota] \times [K] \times [2^{s+1}\Psi]\}$. Let $P' := \{x^l_{i,j,k}, x^r_{i,j,k} : (i, j, k) \in [\iota] \times [K] \times [2^{s+1}\Psi]\}$ and $P := \bigcup_{i \in [\iota], j \in [K], k \in [2^{s+1}\Psi]} P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$. Recall that $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$ avoids the tips $x^l_{i,j,k}$ and $x^r_{i,j,k}$ associated with it. It follows from this, (P2), and the definition of $M'_{i,j,k}$ that $P' \cap V(P) = \varnothing$. Let $T''' := T'[V(T') \setminus P'] \cup P$. Note that T''' is connected by the definition of $\mathcal{E}'(x, j, M')$. Let T'' be a spanning tree of T'''. By (ET1) and the fact that the graphs $P(x_i, j, k, M'_{i,j,k}, B_{i,j,k})$ are vertex-disjoint and satisfy

$|V(P(x_i,j,k,M'_{i,j,k},B_{i,j,k}))|<21D/2$, it follows that

$$\Delta(T'')\leq 12D. \tag{8.10}$$

Define the (new) reservoir $R':=(R\cup P')\setminus V(P)$.

At this point, for each $x\in V(I)\setminus V_{\mathrm{sc}}$ and each $i\in[K]$, we redefine the set $M(A_i(x))$ as follows.

(AB4) Let $M(A_i(x))$ retain only those edges whose associated absorbing ℓ-cube pair (C^l,C^r) satisfies that both C^l and C^r are vertex-disjoint from both cubes of all absorbing ℓ-cube pairs of $\mathcal{C}_1^{\mathrm{sc}}$ and both tips x^l and x^r satisfy that $x^l,x^r\in R\setminus V(P)\subseteq R'$.

Note that, by (T3), we have $|B_I^{\ell+1}(x)\cap V(P)|\leq 21\cdot 2^s\Psi DKS'$ and $|B_I^{\ell+1}(x)\cap V(\bigcup_{(C^l,C^r)\in\mathcal{C}_1^{\mathrm{sc}}}(C^l\cup C^r))|\leq 4\cdot 2^{\ell+s}\Psi KS'$. Combining this with (8.8) and (AB1), it follows that

$$|M(A_i(x))|\geq 4n/\ell^3-(21D+4\cdot 2^\ell)2^s\Psi KS'>n/\ell^3. \tag{8.11}$$

Step 9: Fixing a collection of absorbing ℓ-cube pairs for the vertices in non-scant molecules. At this point, we still do not know which vertices will need to be absorbed eventually into an almost spanning cycle, but we can already determine the vertices in I whose clones the vertices to be absorbed will be (the reason for this will be apparent later, see Step 13). Recall that $\mathcal{C}'$ and $\mathcal{C}''$ were defined in Step 6. Let $\mathcal{C}''':=\{C\in\mathcal{C}':V(C)\cap V(T'')\neq\varnothing\}$ and let $V_{\mathrm{abs}}:=V(I)\setminus\bigcup_{C\in\mathcal{C}'''}V(C)$. We will now fix a collection of absorbing ℓ-cube pairs for all vertices in each vertex molecule $\mathcal{M}_x$ with $x\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}$.

First, recall from (T3) that, for all $x\in V(I)$, we have that $|B_I^{10\ell}(x)\cap V_{\mathrm{sc}}|\leq S'$. Thus, in constructing T'', we removed at most $2^{s+2}\Psi KS'$ vertices in $B_I^\ell(x)$ from T'. Therefore, it follows from (ET2) that, for all $x\in V(I)$, we have

$$|B_I^2(x)\setminus V(T'')|\leq 2n^{3/4}. \tag{8.12}$$

For all $x\in\bigcup_{C\in\mathcal{C}''}V(C)$, we claim that

$$|N_I(x)\cap V(T'')\cap\bigcup_{C\in\mathcal{C}'}V(C)|\geq(1-2^{1-\ell-5s})n. \tag{8.13}$$

To see that this holds, combine (N1), (8.12) and the definition of bondlessly surrounded molecules.

Recall also the definition of $\mathfrak{M}(x)$ from Step 3.

Claim 8.2. *For each $x\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}$ and each $e\in\mathfrak{M}(x)$, there exists a set $\mathcal{C}_1^{\mathrm{abs}}(e)$ of $2^{s+1}\Psi$ absorbing ℓ-cube pairs $(C_k^l(e),C_k^r(e))\subseteq I$, for $k\in[2^{s+1}\Psi]$, which satisfies the following:*

(i) *for all $x\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}$, $e\in\mathfrak{M}(x)$ and $k\in[2^{s+1}\Psi]$, the absorbing ℓ-cube pair $(C_k^l(e),C_k^r(e))$ is associated with some edge in $M(A_j(x))$, for some $j\in[K]$, and*

(ii) *for all $x,x'\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}$, all $e\in\mathfrak{M}(x)$ and $e'\in\mathfrak{M}(x')$, and all $k,k'\in[2^{s+1}\Psi]$ with $(x,e,k)\neq(x',e',k')$, the absorbing ℓ-cube pairs $(C_k^l(e),C_k^r(e))$ are vertex-disjoint (except for x in the case when $x=x'$).*

Proof. Let $\mathcal{V}:=\bigcup_{x\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}}\mathfrak{M}(x)$. Let $K':=|\mathcal{V}|$, and let $f_1,\ldots,f_{K'}$ be an ordering of the edges in $\mathcal{V}$. Given any $i\in[K']$, the edge f_i corresponds to a pair $(x,j(i))$ (in the sense that $A_{j(i)}(x)=N(f_i)$, see Step 4), where $x\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}$ and $j(i)\in[K]$. Let $\mathfrak{C}_i$ be the collection of at least n/ℓ^3 absorbing ℓ-cube pairs for x in I guaranteed by (8.11). In particular, each of these absorbing ℓ-cube pairs (C^l,C^r) is associated with an edge of $M(A_{j(i)}(x))$ and, by (AB2), satisfies $C^l,C^r\in\mathcal{C}''$.

Let $\mathcal{H}$ be the $2^{s+1}\Psi K'$-edge-coloured auxiliary multigraph with $V(\mathcal{H}):=\mathcal{C}''$, which contains an edge between C and C' of colour $(i,k)\in[K']\times[2^{s+1}\Psi]$ whenever $(C,C')\in\mathfrak{C}_i$ or $(C',C)\in\mathfrak{C}_i$. In particular, $\mathcal{H}$ contains at least n/ℓ^3 edges of each colour. We now bound $\Delta(\mathcal{H})$. Consider any $C\in V(\mathcal{H})$. Note that, for each edge e of $\mathcal{H}$ incident to C, there exists some $x=x(e)\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}$ such that C together with some other cube $C'\in V(\mathcal{H})$ forms an absorbing ℓ-cube pair for x. In particular, x must be adjacent to C in I. Moreover, if e has colour (i,k), then $f_i\in\mathfrak{M}(x)$ (and it has corresponding pair $(x,j(i))$ for some $j(i)\in[K]$). Since $f_i\in\mathfrak{M}(x)$ and $|\mathfrak{M}(x)|\leq\binom{2^s}{2}$, it follows that each vertex y which is adjacent to C in I can play the role of x for at most $2^{s+1}\Psi \cdot 2^{2s}$ edges of $\mathcal{H}$ incident to C. Thus, $d_{\mathcal{H}}(C)$ is at most $2^{s+1}\Psi \cdot 2^{2s}$ times the number of vertices $y \in V_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ which are adjacent to C in I. Recall that $V_{\mathrm{abs}} = V(I) \setminus \bigcup_{C\in\mathcal{C}'''} V(C)$. Together with (8.13), this implies that the number of vertices in V_{abs} which are adjacent to C is at most $|C|n/2^{\ell+5s-1}$. Thus, $d_{\mathcal{H}}(C) \leq 2^{s+1}\Psi 2^{2s}|C|n/2^{\ell+5s-1} \leq n/\ell^4$.

Since each colour class has size at least n/ℓ^3 and $\Delta(\mathcal{H}) \leq n/\ell^4$, by Lemma 5.5, $\mathcal{H}$ contains a rainbow matching of size $2^{s+1}\Psi K'$. For each $(i,k) \in [K'] \times [2^{s+1}\Psi]$, let $(C_k^l(f_i), C_k^r(f_i)) \in \mathfrak{C}_i$ be the absorbing ℓ-cube pair of colour (i,k) in this rainbow matching. ◀

Recall that, for any $x \in V(I)$, each index $i \in [K]$ is given by a unique edge $e \in \mathfrak{M}(x)$ via the relation $N(e) = A_i(x)$. For each $x \in V_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and each $i \in [K]$, let $\mathcal{C}_1^{\mathrm{abs}}(x,i) \coloneqq \mathcal{C}_1^{\mathrm{abs}}(e)$, where e is the unique edge given by the relation above, be the set of absorbing ℓ-cube pairs guaranteed by Claim 8.2. Similarly, for each $k \in [2^{s+1}\Psi]$, let $(C_k^l(x,i), C_k^r(x,i)) \coloneqq (C_k^l(e), C_k^r(e))$.

Let $G \coloneqq \bigcup_{i=1}^7 G_i$. For each $x \in V_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and each $i \in [K]$, let $G^*(x,i) \subseteq I$ be the graph consisting of all edges between the left absorber tip and third absorber vertex of every absorbing ℓ-cube pair in $\mathcal{C}_1^{\mathrm{abs}}(x,i)$. Let $G^\bullet \subseteq I$ be the graph consisting of all edges between the left absorber tip and third absorber vertex of every absorbing ℓ-cube pair in $\mathcal{C}_1^{\mathrm{sc}}$. Let $G^* \coloneqq G^\bullet \cup \bigcup_{x\in V_{\mathrm{abs}}\setminus V_{\mathrm{sc}}} \bigcup_{i\in[K]} G^*(x,i) \subseteq I$. Recall that, given any graph $\mathcal{G} \subseteq I$, for each layer L, we denote by $\mathcal{G}_L$ the clone of $\mathcal{G}$ in L. Let $G_4^* \coloneqq G_4 \cap \bigcup_{i=1}^{2^s} G^*_{L_i}$. Furthermore, let $G_5^* \subseteq G_5$ consist of all edges of G_5 which have endpoints in different layers. We let $G' \subseteq G$ be the spanning subgraph with edge set

$$E(G') \coloneqq E(G_4^*) \cup E(G_5^*) \cup \bigcup_{C\in\mathcal{C}'} E(\mathcal{M}_C) \cup \bigcup_{i=1}^{2^s} E(T''_{L_i}).$$

Note that, using (8.10), we have that $\Delta(G') \leq \Phi$.

Now, let $F \subseteq \mathcal{Q}^n$ be any graph with $\Delta(F) \leq \Psi$. Recall that we denote by $F_I \subseteq I$ the graph which contains every edge $\{x,y\} \in E(I)$ such that there exists an edge $e = \{x',y'\} \in E(F)$ with $x' \in \mathcal{M}_x$ and $y' \in \mathcal{M}_y$.

Note that $T'' \subseteq I(G')$, $R' \subseteq V(I)$, and $C \subseteq I(G')$ for every $C \in \mathcal{C}'$. Recall the definitions of $\mathcal{C}''$ from Step 6 and $\mathcal{C}'''$ from Step 9. Combining all the previous steps, we claim that the following hold (conditioned on the events $\mathcal{E}_1^*, \dots, \mathcal{E}_7^*$, which occur a.a.s.).

- (C1) $\Delta(T'') \leq 12D$.
- (C2) Any vertex $x \in R' \cap V(T'')$ is a leaf of T''. Furthermore, if $x \in R' \cap V(T'')$, then its unique neighbour x' in T'' satisfies that $x' \in Z(x)$ (where $Z(x)$ is as defined in Step 7).
- (C3) For all $x \in V(I)$, we have that $|N_I(x) \cap V(T'') \cap \bigcup_{C\in\mathcal{C}''} V(C)| \geq (1 - 2/\ell^4)n$.
- (C4) For each $x \in V_{\mathrm{sc}}$ and $i \in [K]$, there is an absorbing ℓ-cube pair $(C^l(x,i), C^r(x,i))$ for x in I, which is associated with some edge $e \in M(A_i(x))$. In particular, $(C^l(x,i), C^r(x,i))$ is as described in (AB1) (recall also (AB2)), that is, there are two absorbing ℓ-cube pairs $(C_1^l(x,i), C_1^r(x,i))$ and $(C_2^l(x,i), C_2^r(x,i))$ in $H \cup G$, associated with $e \in M(A_i(x))$, for the clones x_1 and x_2 of x which correspond to (x,i). Additionally, each of these absorbing ℓ-cube pairs $(C^l(x,i), C^r(x,i))$ satisfies the following:
  - (C4.1) $(C_1^l(x,i), C_1^r(x,i)) \cup (C_2^l(x,i), C_2^r(x,i)) - V(\mathcal{M}_x) \subseteq G'$;
  - (C4.2) the tips x^l of $C^l(x,i)$ and x^r of $C^r(x,i)$ lie in $R' \setminus V(T'')$, and $\{x,x^l\}, \{x,x^r\} \notin E(F_I)$; in particular, the tips x_1^l, x_1^r of $(C_1^l(x,i), C_1^r(x,i))$ and x_2^l, x_2^r of $(C_2^l(x,i), C_2^r(x,i))$ satisfy that $\{x_1,x_1^l\}, \{x_1,x_1^r\}, \{x_2,x_2^l\}, \{x_2,x_2^r\} \in E((H\cup G)\setminus F)$;
  - (C4.3) $C^l(x,i), C^r(x,i) \in \mathcal{C}'' \cap \mathcal{C}'''$, and
  - (C4.4) for any $x' \in V_{\mathrm{sc}}$ and $i' \in [K]$ with $(x',i') \neq (x,i)$ we have that $C^l(x,i)$, $C^r(x,i)$, $C^l(x',i')$ and $C^r(x',i')$ are vertex-disjoint.

  Let $\mathcal{C}^{\mathrm{sc}}$ denote the collection of these absorbing ℓ-cube pairs.
- (C5) For each $x \in V_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i \in [K]$, there is an absorbing ℓ-cube pair $(C^l(x,i), C^r(x,i))$ for x in I, which is associated with some edge in $M(A_i(x))$. In particular, $(C^l(x,i), C^r(x,i))$ is as described in (AB1), that is, there are two absorbing ℓ-cube pairs $(C_1^l(x,i), C_1^r(x,i))$

and $(C_2^l(x,i), C_2^r(x,i))$ in $H \cup G$, associated with $e \in M(A_i(x))$, for the clones x_1 and x_2 of x which correspond to (x,i). Moreover, each of these absorbing ℓ-cube pairs $(C^l(x,i), C^r(x,i))$ satisfies the following:

(C5.1) $(C_1^l(x,i), C_1^r(x,i)) \cup (C_2^l(x,i), C_2^r(x,i)) - V(\mathcal{M}_x) \subseteq G'$;

(C5.2) the tips x_i^l of $C^l(x,i)$ and x_i^r of $C^r(x,i)$ lie in R', and $\{x, x_i^l\}, \{x, x_i^r\} \notin E(F_I)$; in particular, the tips x_1^l, x_1^r of $(C_1^l(x,i), C_1^r(x,i))$ and x_2^l, x_2^r of $(C_2^l(x,i), C_2^r(x,i))$ satisfy that $\{x_1, x_1^l\}, \{x_1, x_1^r\}, \{x_2, x_2^l\}, \{x_2, x_2^r\} \in E((H \cup G) \setminus F)$;

(C5.3) $C^l(x,i), C^r(x,i) \in \mathcal{C}'' \cap \mathcal{C}'''$;

(C5.4) for any $x' \in V_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i' \in [K]$ with $(x', i') \neq (x, i)$ we have that $C^l(x,i)$, $C^r(x,i)$, $C^l(x',i')$ and $C^r(x',i')$ are vertex-disjoint, and

(C5.5) both $C^l(x,i)$ and $C^r(x,i)$ are vertex-disjoint from all cubes of absorbing ℓ-cube pairs in $\mathcal{C}^{\mathrm{sc}}$.

Let $\mathcal{C}^{\neg\mathrm{sc}}$ denote the collection of these absorbing ℓ-cube pairs.

Indeed, (C1) is given in (8.10). (C2) holds by (ET3) and the fact that $P' \cap V(T'') = \varnothing$. (C3) follows by combining (N1), the conditioning on $\mathcal{E}_5^*$, and (8.12). (C4) follows from the construction of P and T'' in Step 8. Indeed, for each $x \in V_{\mathrm{sc}}$ and $i \in [K]$, consider the collection of absorbing ℓ-cube pairs $\{(C^l(x,i,k), C^r(x,i,k))\}_{k \in [2^{s+1}\Psi]}$ defined in Step 8. Since $\Delta(F) \leq \Psi$, it follows that $d_{F_I}(x) \leq 2^s\Psi$, and thus there must exist some absorbing ℓ-cube pair in this collection such that the edges joining its tips to x do not belong to F_I. Fix one such absorbing ℓ-cube pair and call it $(C^l(x,i), C^r(x,i))$. Then, (C4.1) holds by the definition of G' combined with (AB1), and (C4.2) holds by the definition of R' and T'' combined with (AB1), while (C4.4) holds by (CD). On the other hand, (C4.3) follows because of the definition of the set $M(A_i(x))$ in (AB2) and (AB3). Finally, consider (C5). For each $x \in V_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i \in [K]$, consider the collection $\mathcal{C}_1^{\mathrm{abs}}(x,i)$ of $2^{s+1}\Psi$ absorbing ℓ-cube pairs for x in I guaranteed by Claim 8.2. For each of these absorbing ℓ-cube pairs we have that (C5.3) holds by (AB2), (AB3) and the fact that, by (AB4), their intersection with T'' contains their intersection with T'. Similarly, (C5.4) holds by Claim 8.2, and (C5.5) holds because of (AB4). Finally, note that $\Delta(F_I) \leq 2^s\Psi$. It follows that there exists a choice of $(C^l(x,i), C^r(x,i)) \in \mathcal{C}_1^{\mathrm{abs}}(x,i)$ such that $\{x, x_i^l\}, \{x, x_i^r\} \notin E(F_I)$. Then, (C5.1) and (C5.2) hold by the definition of G', (AB1) and (AB4).

Step 10: Constructing auxiliary trees T^* and τ_0. From this point on, every step will be deterministic. Let T^* be obtained from T'' by removing all leaves of T'' which lie in R'.

We will now construct an auxiliary tree τ_0, which will be used in the construction of an almost spanning cycle. We start by defining an auxiliary multigraph Γ' as follows. First, let $\Gamma_1 := T^* \cup \bigcup_{C \in \mathcal{C}'} C$. (Recall that $\mathcal{C}'$ is the collection of all $C \in \mathcal{C}$ for which $\mathcal{M}_C$ is bonded in G_5, see Step 6.) Let Γ_2 be the graph obtained by iteratively removing all leaves from Γ_1 until all vertices have degree at least 2. Observe that, after this is achieved, the resulting graph still contains all cubes $C \in \mathcal{C}'$. Let Γ_3 be obtained from Γ_2 by removing all connected components which consist of a single cube $C \in \mathcal{C}'$. Now, let Γ' be the multigraph obtained by contracting each cube $C \in \mathcal{C}'$ such that $C \subseteq \Gamma_3$ into a single vertex. We refer to the vertices resulting from contracting such cubes as *atomic vertices*, and to the remaining vertices in Γ' as *inner tree vertices*. Given $C \in \mathcal{C}$ and $j \in [2^s]$, we call $\mathcal{A} = \mathcal{M}_C \cap L_j$ an *atom*. We continue to identify each inner tree vertex v with the vertex $v \in V(I)$ from which it originated in Γ_1. Observe that Γ' is connected, and (C1) implies that

$$d_{\Gamma'}(v) \leq 12D \text{ for all inner tree vertices, and } \Delta(\Gamma') \leq 12 \cdot 2^\ell D. \tag{8.14}$$

Given an atomic vertex $v \in V(\Gamma')$, let $C(v) \in \mathcal{C}$ be the cube which was contracted to v in the construction of Γ', and let $\mathcal{M}(v) := \mathcal{M}_{C(v)}$. Furthermore, for each $j \in [2^s]$, let $\mathcal{A}_j(v) := \mathcal{M}(v) \cap L_j$. Similarly, for any $v \in V(\Gamma')$ which is an inner tree vertex, we define $\mathcal{M}(v) := \mathcal{M}_v$. Observe that every edge $e \in E(\Gamma')$ corresponds to a unique edge $e' \in I(G')$. We say that e *originates* from e'. We denote by $D(e) \in \mathcal{D}(I)$ the direction of e' in I. By abusing notation, we will sometimes also view $D(e)$ as a direction in $\mathcal{Q}^n$.

Next, we fix any atomic vertex $v_0 \in V(\Gamma')$. We define an auxiliary labelled rooted tree $\tau_0 = \tau_0(v_0)$ by performing a depth-first search on Γ' rooted at v_0 and then iteratively removing all leaves which are inner tree vertices. This results in a tree τ_0 rooted at an atomic vertex v_0 and all whose leaves are atomic vertices. Let $m := |V(\tau_0)| - 1$, and let the vertices of τ_0 be labelled as $v_0, v_1, \ldots, v_m$, with the labelling given by the order in which each vertex is explored by the depth-first search performed on Γ'. For each $i \in [m]$, we define τ_i as the maximal subtree of τ_0 which contains v_i and all whose vertices have labels which are at least as large as i. Given any vertex $x \in V(I)$, we say that x is *represented* in τ_0 if $x \in V(\tau_0)$ or there exists some atomic vertex $v \in V(\tau_0)$ such that $x \in V(C(v))$. Similarly, we say that a cube $C \in \mathcal{C}$ is *represented* in τ_0 if there exists an atomic vertex $v \in V(\tau_0)$ such that $C = C(v)$. We will sometimes also say that $\mathcal{M}_x$ or $\mathcal{M}_C$ are represented in τ_0, respectively.

The tree τ_0 will be the backbone upon which we construct our long cycle. First, we need to set up some more notation. For each $i \in [m]_0$, let $p_i := d_{\tau_i}(v_i)$ and let $N_{\tau_i}(v_i) = \{u_1^i, \ldots, u_{p_i}^i\}$. It follows from (8.14) that

$$p_i \leq 12D - 1 \text{ if } v_i \text{ is an inner tree vertex, and } \Delta(\tau_0) \leq 12 \cdot 2^\ell D. \tag{8.15}$$

For each $i \in [m]_0$ and $k \in [p_i]$, let $e_k^i := \{v_i, u_k^i\}$, let $f_k^i := D(e_k^i)$, and let j_k^i be the label of u_k^i in τ_0, that is, $u_k^i = v_{j_k^i}$. For any $k \in [p_i]$, we will sometimes refer to i as the *parent index* of j_k^i. Furthermore, for each $i \in [m]_0$ such that v_i is an atomic vertex, and for each $k \in [p_i]$, consider the edge in $I(G')$ from which e_k^i originates and let ν_k^i be its endpoint in $C(v_i)$. Finally, for each $i \in [m]_0$, we define a parameter $\Delta(v_i)$ recursively by setting

$$\Delta(v_i) := \begin{cases} 0 & \text{if } v_i \text{ is an atomic vertex which is a leaf of } \tau_0, \\ \sum_{k=1}^{p_i} \Delta(u_k^i) & \text{if } v_i \text{ is an atomic vertex which is not a leaf of } \tau_0, \\ p_i + 1 + \sum_{k=1}^{p_i} \Delta(u_k^i) & \text{if } v_i \text{ is an inner tree vertex.} \end{cases} \tag{8.16}$$

This parameter $\Delta(v_i)$ will be used to keep track of parities throughout the following steps. Note that $\Delta(v_i)$ counts the number of times a depth first search of τ_i (starting and ending at v_i) traverses an inner tree vertex.

Consider the partition of all molecules into slices of size q introduced at the beginning of Step 3, where q is as defined in (8.1). Given any $v \in V(\tau_0)$, we denote the slices of its molecule by $\mathcal{M}_1(v), \ldots, \mathcal{M}_t(v)$, where t is as defined in (8.1). Thus, for each $i \in [t]$ we have that $\mathcal{M}_i(v) = \bigcup_{j=(i-1)q+1}^{iq} \mathcal{A}_j(v)$. For each $i \in [m]_0$, we are going to assign an *input slice* $\mathcal{M}_{b(i)}(v_i)$ to each vertex v_i. We do so by recursively assigning an *input index* $b(i) \in [t]$ to each $i \in [m]_0$. We begin by letting $b(0) := 1$. Then, for each $i \in [m]_0$ and each $k \in [p_i]$, we set

$$b(j_k^i) := \begin{cases} b(i) & \text{if } v_i \text{ is an inner tree vertex,} \\ b(i) + k - 1 \pmod t & \text{if } v_i \text{ is an atomic vertex.} \end{cases}$$

Note that the bound on $\Delta(\tau_0)$ in (8.15) and the definition of t in (8.1) imply that $b(j_k^i) \neq b(j_{k'}^i)$ whenever v_i is an atomic vertex and $k \neq k'$.

Step 11: Finding an external skeleton for T^*. Our next goal is to find an almost spanning cycle in G' by using τ_0 to explore different molecules in a given order. For this, we are going to generate a *skeleton*; this will be an ordered list of vertices which we will denote by $\mathcal{L}$. In order to construct $\mathcal{L}$, we will construct disjoint *partial skeletons* $\mathcal{L}_i$ and $\hat{\mathcal{L}}_i$ for all $i \in [m]$ in an inductive way. Each of these skeletons will start and end in the input slice for the vertex v_i which is being considered. These partial skeletons will depend on the starting and ending vertices of $\mathcal{M}_{b(i)}(v_i)$ which are provided for each of them. Therefore, given two distinct starting vertices $x, \hat{x} \in V(\mathcal{M}_{b(i)}(v_i))$ and two distinct ending vertices $y, \hat{y} \in V(\mathcal{M}_{b(i)}(v_i))$, we will denote the partial skeletons by $\mathcal{L}_i(x, y)$ and $\hat{\mathcal{L}}_i(\hat{x}, \hat{y})$, respectively.

The first step in the construction of $\mathcal{L}$ is to construct a set of vertices $L^\bullet$, to which we will refer as an *external skeleton*, and for which we will in turn construct *partial external skeletons* in an inductive way. The external skeleton will be essential in determining which vertices will not be covered by the almost spanning cycle, and hence need to be absorbed. Roughly speaking, the external skeleton will contain

(i) all vertices where the almost spanning cycle enters and leaves each cube molecule represented in τ_0, and
(ii) all vertices which are not in cube molecules and are needed to connect cube molecules to each other (that is, some clones of inner tree vertices).

On the other hand, all vertices in a vertex molecule represented in τ_0 by an inner tree vertex which do not belong to the external skeleton will have to be absorbed.

For each $i \in [m]$, given the starting and ending vertices $x, y, \hat{x}, \hat{y} \in V(\mathcal{M}_{b(i)}(v_i))$ for $\mathcal{L}_i(x, y)$ and $\hat{\mathcal{L}}_i(\hat{x}, \hat{y})$, we will denote the corresponding partial external skeleton by $L_i^\bullet(x, y, \hat{x}, \hat{y})$.

The external skeleton is constructed recursively. The partial external skeletons are the result of each recursive step, assuming that the starting and ending points have been defined. Roughly speaking, for each $i \in [m]$, we will define partial external skeletons for any possible starting and ending vertices. The starting and ending vertices which we actually use are then fixed by the partial external skeleton whose index is the parent of i. Ultimately, all of them will be fixed when defining the external skeleton $L^\bullet$.

Let $\mathcal{M}_{\mathrm{Res}} \subseteq V(\mathcal{Q}^n)$ be the union of all the clones of R'. We will construct an external skeleton $L^\bullet$ which satisfies the following properties:

(ES1) For each $i \in [m]$ such that v_i is an inner tree vertex, $L^\bullet \cap V(\mathcal{M}_{b(i)}(v_i))$ contains exactly $2p_i + 2$ vertices, half of them of each parity, and $L^\bullet \cap (V(\mathcal{M}(v_i)) \setminus V(\mathcal{M}_{b(i)}(v_i))) = \varnothing$.
(ES2) For each $i \in [m]$ such that v_i is an atomic vertex, $L^\bullet \cap V(\mathcal{M}(v_i))$ contains exactly $4p_i + 4$ vertices. If v_i is not a leaf of τ_0, eight of these vertices (four of each parity) lie in $V(\mathcal{M}_{b(i)}(v_i))$, and four (two of each parity) lie in each $V(\mathcal{M}_{b(i)+k}(v_i))$ with $k \in [p_i - 1]$. If v_i is a leaf, then all four of these vertices lie in $V(\mathcal{M}_{b(i)}(v_i))$.
(ES3) $L^\bullet \cap V(\mathcal{M}(v_0))$ contains exactly $4p_0$ vertices, four of them (two of each parity) lying in each $V(\mathcal{M}_k(v_0))$ with $k \in [p_0]$.
(ES4) The sets described in (ES1)–(ES3) partition $L^\bullet$.
(ES5) $L^\bullet \cap \mathcal{M}_{\mathrm{Res}} = \varnothing$.

We now proceed to define the partial external skeletons formally. The construction proceeds by induction on $i \in [m]$ in decreasing order, starting with $i = m$. We define a *valid connection sequence* $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ *for* v_i as any set of distinct vertices $x^i, y^i, \hat{x}^i, \hat{y}^i \in V(\mathcal{M}_{b(i)}(v_i))$ which satisfy the following:

(V1) $x^i \neq_{\mathrm{p}} y^i$ if $\Delta(v_i)$ is even, and $x^i =_{\mathrm{p}} y^i$ otherwise;
(V2) $\hat{x}^i \neq_{\mathrm{p}} x^i$, and
(V3) $\hat{y}^i \neq_{\mathrm{p}} y^i$.

Given any valid connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$, we will refer to x^i and $\hat{x}^i$ as *starting* vertices, and to y^i and $\hat{y}^i$ as *ending* vertices. Throughout the construction ahead, observe that, every time we use a partial external skeleton to build a larger one, its starting and ending vertices form a valid connection sequence by construction. The vertices x^i, y^i, etc. will be part of $\mathcal{L}_i(x^i, y^i)$, and the vertices $\hat{x}^i$, $\hat{y}^i$, etc. will be part of $\hat{\mathcal{L}}_i(\hat{x}^i, \hat{y}^i)$. The vertices x^i, y^i, $\hat{x}^i$, $\hat{y}^i$ will be used by the skeleton to move from the molecule represented by v_i in τ_0 to the molecule represented by its parent. Given these vertices, the following construction provides the vertices w_k^i and $\hat{w}_k^i$ (as well as z_k^i and $\hat{z}_k^i$, if applicable) which are used to move to molecules represented by the children of v_i. Given any vertices $(x, y, \hat{x}, \hat{y})$ in $\mathcal{Q}^n$ and any direction $f \in \mathcal{D}(\mathcal{Q}^n)$, we write $f + (x, y, \hat{x}, \hat{y}) = (f + x, f + y, f + \hat{x}, f + \hat{y})$.

Now suppose that $i \in [m]$ and that, for each $i' \in [m] \setminus [i]$, we have already constructed a partial external skeleton $L_{i'}^\bullet(x^{i'}, y^{i'}, \hat{x}^{i'}, \hat{y}^{i'})$ for $v_{i'}$ and every valid connection sequence $(x^{i'}, y^{i'}, \hat{x}^{i'}, \hat{y}^{i'})$ for $v_{i'}$. We will now construct a partial external skeleton for v_i and every valid connection sequence for v_i. We consider several cases.

Case 1: $v_i \in V(\tau_0)$ is a leaf of τ_0. Assume that $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ is a valid connection sequence for v_i. Then, the partial external skeleton for this connection sequence is given by $L_i^\bullet(x^i, y^i, \hat{x}^i, \hat{y}^i) \coloneqq \{x^i, y^i, \hat{x}^i, \hat{y}^i\}$.

Case 2: $v_i \in V(\tau_0)$ is an inner tree vertex. We construct a set of partial external skeletons for v_i as follows.

1. Suppose $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ is a valid connection sequence for v_i. Let $w_0^i \coloneqq x^i$, $w_{p_i}^i \coloneqq y^i$, $\hat{w}_0^i \coloneqq \hat{x}^i$ and $\hat{w}_{p_i}^i \coloneqq \hat{y}^i$. Let $W_0^i \coloneqq \{w_0^i, w_{p_i}^i, \hat{w}_0^i, \hat{w}_{p_i}^i\}$.
2. For each $k \in [p_i - 1]$, iteratively choose two vertices $w_k^i, \hat{w}_k^i \in V(\mathcal{M}_{b(i)}(v_i)) \setminus W_{k-1}^i$ such that $f_k^i + (w_{k-1}^i, w_k^i, \hat{w}_{k-1}^i, \hat{w}_k^i)$ is a valid connection sequence for u_k^i, and let $W_k^i \coloneqq W_{k-1}^i \cup \{w_k^i, \hat{w}_k^i\}$.

Note that the definition of q in (8.1) and the bound on p_i in (8.15) ensure that we have sufficiently many vertices to choose from (similar comments apply in the other cases). Moreover, (8.16) implies that $f_{p_i}^i + (w_{p_i-1}^i, w_{p_i}^i, \hat{w}_{p_i-1}^i, \hat{w}_{p_i}^i)$ is a valid connection sequence for $u_{p_i}^i$. The partial external skeleton for v_i and connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ is defined as

$$L_i^\bullet(x^i, y^i, \hat{x}^i, \hat{y}^i) \coloneqq \{x^i, \hat{x}^i\} \cup \bigcup_{k=1}^{p_i} \left(\{w_k^i, \hat{w}_k^i\} \cup L_{j_k^i}^\bullet(f_k^i + (w_{k-1}^i, w_k^i, \hat{w}_{k-1}^i, \hat{w}_k^i))\right).$$

Case 3: $v_i \in V(\tau_0)$ is an atomic vertex which is not a leaf. We construct a set of partial external skeletons for v_i as follows.

1. Assume $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ is a valid connection sequence for v_i. Let $w_0^i \coloneqq x^i$.
2. For each $k \in [p_i]$, iteratively choose distinct vertices $z_k^i, w_k^i, \hat{z}_k^i, \hat{w}_k^i \in (V(\mathcal{M}_{b(i)+k-1}(v_i)) \cap V(\mathcal{M}_{\nu_k^i})) \setminus \{x^i, y^i, \hat{x}^i, \hat{y}^i\}$ satisfying that $z_k^i \neq_{\mathrm{p}} w_{k-1}^i$ and $f_k^i + (z_k^i, w_k^i, \hat{z}_k^i, \hat{w}_k^i)$ is a valid connection sequence for u_k^i.

Then, the partial external skeleton for v_i and connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ is defined as

$$L_i^\bullet(x^i, y^i, \hat{x}^i, \hat{y}^i) \coloneqq \{x^i, y^i, \hat{x}^i, \hat{y}^i\} \cup \bigcup_{k=1}^{p_i} \left(\{z_k^i, w_k^i, \hat{z}_k^i, \hat{w}_k^i\} \cup L_{j_k^i}^\bullet(f_k^i + (z_k^i, w_k^i, \hat{z}_k^i, \hat{w}_k^i))\right).$$

After having constructed all these partial external skeletons for all v_i with $i \in [m]$, we are now ready to construct $L^\bullet$.

1. Choose any vertex $w_0^0 \in V(\mathcal{A}_1(v_0))$.
2. For each $k \in [p_0]$, iteratively choose four distinct vertices $z_k^0, \hat{z}_k^0, w_k^0, \hat{w}_k^0 \in (V(\mathcal{M}_k(v_0)) \cap V(\mathcal{M}_{\nu_k^0}))$ satisfying that $z_k^0 \neq_{\mathrm{p}} w_{k-1}^0$ and $f_k^0 + (z_k^0, w_k^0, \hat{z}_k^0, \hat{w}_k^0)$ is a valid connection sequence for u_k^0.

Then, we define

$$L^\bullet \coloneqq \bigcup_{k=1}^{p_0} \left(\{z_k^0, w_k^0, \hat{z}_k^0, \hat{w}_k^0\} \cup L_{j_k^0}^\bullet(f_k^0 + (z_k^0, w_k^0, \hat{z}_k^0, \hat{w}_k^0))\right).$$

Observe that (ES1)–(ES4) hold by construction. In turn, (ES5) holds because of the definition of τ_0. Indeed, observe that $V(T^*) \cap R' = \varnothing$ by (C2). Moreover, by the construction above, all vertices in $L^\bullet$ are incident to some edge in a clone of the tree T^*, and thus, they cannot lie in $\mathcal{M}_{\mathrm{Res}}$.

Step 12: Constructing an auxiliary tree τ_0'. In order to extend the external skeleton into the skeleton and construct an almost spanning cycle, we first need to extend τ_0 to a new auxiliary tree τ_0' which encodes information about some additional molecules.

We construct τ_0' by appending some new leaves to τ_0. Note that τ_0 was built by encoding all the information about T^*, and τ_0' will encode the information about T''. In particular, by (C2), each cube $C \in \mathcal{C}'$ which intersects T'' and does not intersect T^* contains at least one vertex u which is joined to T^* by an edge $e' = \{u, v\} \in E(T'')$ such that $v \in V(C')$, where $C \neq C' \in \mathcal{C}''$. Note that the construction of τ_0 implies that C' is represented in τ_0. For each such cube C, choose one such vertex u and append a new vertex to the atomic vertex representing C' in τ_0 via an edge e which originates as $e' \in E(T'')$. We say that this newly added vertex is atomic and *represents* C. The resulting tree after all these leaves are appended is τ_0'. In particular, $\tau_0 \subseteq \tau_0'$, and it now follows that precisely the $C \in \mathcal{C}'''$ are represented in τ_0', where $\mathcal{C}'''$ is as defined in Step 9. Furthermore, it follows from (C1) that

$$\begin{aligned} &d_{\tau_0'}(v) \leq 12D \text{ for all } v \in V(\tau_0') \text{ which are inner tree vertices, and} \\ &\Delta(\tau_0') \leq 12 \cdot 2^\ell D. \end{aligned} \tag{8.17}$$

For all vertices of τ_0', we will use the same notation for the vertices, cubes and molecules that they represent as we did for the vertices of τ_0. Note that, by (C4.3) and (C5.3),

(CP) every cube C belonging to some absorbing ℓ-cube pair in $\mathcal{C}^{\mathrm{sc}} \cup \mathcal{C}^{\neg\mathrm{sc}}$ is represented in τ_0'.

It will be important for us that τ_0' represents 'most' vertices of the hypercube. In particular, for each $x \in V(I)$, let $\lambda(x)$ denote the number of vertices $y \in N_I(x)$ which are represented in τ_0' by atomic vertices. By (C3), we have that

$$\lambda(x) \geq (1 - 2/\ell^4)n. \tag{8.18}$$

By an averaging argument, it follows that at least $(1-2/\ell^4)2^{n-s}$ vertices $x \in V(I)$ are represented in τ_0' by atomic vertices. We will construct an almost spanning cycle in G' which contains all the clones of these vertices.

Let $m' := |V(\tau_0')| - 1$. Label $V(\tau_0') \setminus V(\tau_0) = \{v_{m+1}, \dots, v_{m'}\}$ arbitrarily. For each $i \in [m]$, we define τ_i' as the maximal subtree of τ_0' which contains v_i and all of whose vertices have labels at least as large as i. For each $i \in [m]_0$, let $p_i' := d_{\tau_i'}(v_i)$ and let $N_{\tau_i'}(v_i) = \{u_1^i, \dots, u_{p_i'}^i\}$ (where the labelling is consistent with that of $N_{\tau_i}(v_i)$). For each $i \in [m]_0$ and $k \in [p_i'] \setminus [p_i]$, let $e_k^i := \{v_i, u_k^i\}$, let $f_k^i := D(e_k^i)$, and let j_k^i be the label of u_k^i in τ_0'. Furthermore, for each $i \in [m]_0$ such that v_i is an atomic vertex, and for each $k \in [p_i'] \setminus [p_i]$, consider the unique edge which e_k^i originates from in $I(G')$ and let ν_k^i be its endpoint in $C(v_i)$. Finally, for each $i \in [m'] \setminus [m]$ we set $\Delta(v_i) := 0$.

As in Step 10, we consider the partition into slices for the new molecules arising from the newly added cubes represented by τ_0'. For each $i \in [m'] \setminus [m]$, we assign an input index $b(i) \in [t]$. To do so, for each $i \in [m]_0$ such that v_i is an atomic vertex and each $k \in [p_i'] \setminus [p_i]$, we set $b(j_k^i) := b(i) + k - 1 \pmod t$. Similarly to Step 10, (8.1) and (8.17) imply that in this case $b(j_k^i) \neq b(j_{k'}^i)$ for all $k \neq k'$. For each $i \in [m'] \setminus [m]$, let ℓ_i be the label in τ_0' of the unique vertex adjacent to v_i (i.e., the parent label of i), and let m_i be the label of v_i in $N_{\tau_{\ell_i}'}(v_{\ell_i})$. Note that $b(i) = b(\ell_i) + m_i - 1$.

Step 13: Fixing absorbing ℓ-cube pairs for vertices that need to be absorbed. At this point, we can determine every vertex in $V(\mathcal{Q}^n)$ that will have to be absorbed into the almost spanning cycle we are going to construct. For every vertex $x \in V(I)$ not represented in τ_0', we will have to absorb all vertices in $\mathcal{M}_x$. Furthermore, for each $v \in V(\tau_0)$ which is an inner tree vertex, we will also need to absorb all vertices in $\mathcal{M}_v \setminus L^\bullet$. By (ES1), this means that, in each such molecule $\mathcal{M}_v$, the same number of vertices of each parity need to be absorbed. Recall the definition of V_{abs} from Step 9. This is precisely the set of vertices which are not represented in τ_0' by an atomic vertex and, therefore, it is the set of all vertices $x \in V(I)$ such that some clone of x needs to be absorbed. It follows from (8.18) that

$$|V_{\mathrm{abs}}| \leq 2^{n-s+1}/\ell^4. \tag{8.19}$$

Now, for each $x \in V_{\mathrm{abs}}$, we will pair the vertices in each slice which need to be absorbed (each pair consisting of one vertex of each parity) and fix an absorbing ℓ-cube pair for each such pair of vertices. The absorbing ℓ-cube pair that we fix will be the one given by (C4) or (C5) for this pair of vertices, depending on whether $x \in V_{\mathrm{sc}}$ or not.

For each $x \in V_{\mathrm{abs}}$ and $\mathcal{S} \in \mathcal{S}(\mathcal{M}_x)$, let $S(x, \mathcal{S}) := V(\mathcal{S}) \cap L^\bullet$. It follows by (ES1)–(ES4) that $|S(x, \mathcal{S})| \leq 24D$ and $S(x, \mathcal{S})$ contains the same number of vertices of each parity. (Here we also use that $p_i \leq 12D - 1$ for every inner tree vertex v_i by (8.15) and (8.17).) Therefore, the matching $\mathfrak{M}(\mathcal{S}, S(x, \mathcal{S}))$ defined in Step 3 is well defined. Recall that each edge $e \in \mathfrak{M}(\mathcal{S}, S(x, \mathcal{S}))$ gives rise to a unique index $i \in [K]$ via the relation $N(e) = A_i(x)$. (Here we ignore all those indices $i' \in [K]$ arising by artificially increasing the size of $\mathfrak{A}(x)$, see the beginning of Step 4.) For each $x \in V_{\mathrm{abs}}$, let $\mathfrak{I}_x \subseteq [K]$ be the set of indices $i \in [K]$ which correspond to edges in $\bigcup_{\mathcal{S} \in \mathcal{S}(\mathcal{M}_x)} \mathfrak{M}(\mathcal{S}, S(x, \mathcal{S}))$.

For each $x \in V_{\mathrm{abs}}$ and $i \in \mathfrak{I}_x$, as stated in (C4) and (C5), we have already fixed an absorbing ℓ-cube pair for the clones of x corresponding to (x, i). Let

$$V^{\mathrm{abs}} := \bigcup_{x \in V_{\mathrm{abs}}} V(\mathcal{M}_x) \setminus L^{\bullet}.$$

As discussed above, this is the set of all vertices that need to be absorbed. Recall that G' was defined before (C1)–(C5). It follows from (C4) and (C5) that $((H \cup G) \setminus F) \cup G'$ contains a set $\mathcal{C}^{\mathrm{abs}} = \{(C^l(u), C^r(u)) : u \in V^{\mathrm{abs}}\}$ of absorbing ℓ-cube pairs such that

(C_1) for all distinct $u, v \in V^{\mathrm{abs}}$, the absorbing ℓ-cube pairs $(C^l(u), C^r(u))$ and $(C^l(v), C^r(v))$ for u and v are vertex-disjoint and $(C^l(u), C^r(u)) \cup (C^l(v), C^r(v)) - \{u, v\} \subseteq G'$;

(C_2) there exists a pairing $\mathcal{U} = \{f_1, \ldots, f_{K'}\}$ of V^{abs} such that

- ($\mathrm{C}_{2.1}$) for all $i \in [K']$, if $f_i = \{u_i, u_i'\}$, then $u_i \neq_{\mathrm{p}} u_i'$;
- ($\mathrm{C}_{2.2}$) if $f_i = \{u_i, u_i'\}$, then there is a vertex $v \in V_{\mathrm{abs}}$ such that u_i and u_i' are clones of v which lie in the same slice of $\mathcal{M}_v$, and $(C^l(u_i), C^r(u_i))$ and $(C^l(u_i'), C^r(u_i'))$ are clones of the same absorbing ℓ-cube pair for v in I such that $(C^l(u_i), C^r(u_i))$ lies in the same layer as u_i and $(C^l(u_i'), C^r(u_i'))$ lies in the same layer as u_i';
- ($\mathrm{C}_{2.3}$) if $u, u' \in V^{\mathrm{abs}}$ do not form a pair $f \in \mathcal{U}$, then $(C^l(u), C^r(u))$ and $(C^l(u'), C^r(u'))$ are clones of vertex-disjoint absorbing ℓ-cube pairs in I (except in the case when u, u' are clones of the same vertex $v \in V_{\mathrm{abs}}$, in which case $(C^l(u), C^r(u))$ and $(C^l(u'), C^r(u'))$ are clones of absorbing ℓ-cube pairs in I which intersect only in v);

(C_3) if we let $\mathcal{C}^* := \bigcup_{(C^l(u), C^r(u)) \in \mathcal{C}^{\mathrm{abs}}} \{C^l(u), C^r(u)\}$, then $\mathcal{C}^*$ contains either two or no clones of each cube $C \in \mathcal{C}'' \cap \mathcal{C}'''$, and every cube in $\mathcal{C}^*$ is a clone of some cube $C \in \mathcal{C}'' \cap \mathcal{C}'''$.

The pairing described in (C_2) is given by the matchings $\mathfrak{M}(\mathcal{S}, S(x, \mathcal{S}))$. Furthermore, it follows from (C4.2), (C5.2) and (ES5) that

(C_4) the set of all tips of the absorbing ℓ-cube pairs in $\mathcal{C}^{\mathrm{abs}}$ is disjoint from $L^{\bullet}$.

We denote by $\mathfrak{L}$, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the collections of all left absorber tips, right absorber tips, and third absorber vertices, respectively, of the absorbing ℓ-cube pairs in $\mathcal{C}^{\mathrm{abs}}$. Observe that the following properties are satisfied:

(C^*1) For all $i \in [m']_0$ such that v_i is an atomic vertex and all $j \in [t]$, we have that $|\mathfrak{L} \cap V(\mathcal{M}_j(v_i))| \in \{0, 2\}$ and, if $|\mathfrak{L} \cap V(\mathcal{M}_j(v_i))| = 2$, then these two vertices u, u' lie in different atoms of the slice and satisfy that $u \neq_{\mathrm{p}} u'$.

(C^*2) For all $i \in [m']_0$ such that v_i is an atomic vertex and all $j \in [t]$, we have that $|(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}_j(v_i))| \in \{0, 4\}$. If $|(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}_j(v_i))| = 4$, then these four vertices form two pairs such that one vertex of each pair belongs to $\mathfrak{R}_1$ and the other to $\mathfrak{R}_2$. Each of these pairs lies in a different atom of the slice and satisfies that its two vertices are adjacent in G'.

(C^*3) For all $i \in [m']_0$ such that v_i is an atomic vertex and all $j \in [t]$, if $\mathfrak{L} \cap V(\mathcal{M}_j(v_i)) \neq \varnothing$, then $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}_j(v_i)) = \varnothing$.

(C^*4) The sets described in (C^*1) and (C^*2) partition $\mathfrak{L}$ and $\mathfrak{R}_1 \cup \mathfrak{R}_2$, respectively.

Indeed, (C^*1)–(C^*3) follow from (C_2) and (C_3), and (C^*4) follows by (CP).

For each $u \in V^{\mathrm{abs}}$, we denote the edge consisting of the right absorber tip and the third absorber vertex of $(C^l(u), C^r(u))$ by $e_{\mathrm{abs}}(u)$, and we denote by $\mathcal{P}^{\mathrm{abs}}(u)$ the path of length three formed by the third absorber vertex, the left absorber tip, u, and the right absorber tip, visited in this order. Note that $e_{\mathrm{abs}}(u) \in E(G')$ by (C_1). Moreover, recall that $\mathcal{C}^{\mathrm{abs}}$ consists of absorbing ℓ-cube pairs in $((H \cup G) \setminus F) \cup G')$. Thus, $\mathcal{P}^{\mathrm{abs}}(u) \subseteq ((H \cup G) \setminus F) \cup G'$.

Step 14: Constructing the skeleton. We can now define the skeleton for the almost spanning cycle. Intuitively, this skeleton builds on the external skeleton by adding more structure that the cycle will have to follow. In particular, the skeleton adds the edges used to traverse from each slice in a cube molecule to its neighbouring slices, and it also incorporates the cube molecules represented in τ_0' which were not represented in τ_0. (The reason why these were not incorporated earlier is the following: if we already choose the valid connection sequences for these cube molecules in Step 12, then the tips of the absorbing cubes chosen in Step 13 might have non-empty intersection with the external skeleton, which we want to avoid, see (S7) below.) Furthermore, the skeleton gives an ordering to its vertices, and the cycle will visit the vertices of the skeleton in this order.

We will build a skeleton $\mathcal{L} = (x_1, \dots, x_r)$, for some $r \in \mathbb{N}$, and write $\mathcal{L}^\bullet := \{x_1, \dots, x_r\}$. We will construct $\mathcal{L}$ in such a way that the following properties hold:

(S1) For all distinct $k, k' \in [r]$, we have that $x_k \neq x_{k'}$.
(S2) $\{x_1, x_r\} \in E(G')$.
(S3) For every $k \in [r-1]$, if x_k and x_{k+1} do not both lie in the same slice of a cube molecule represented in τ_0', then $\{x_k, x_{k+1}\} \in E(G')$. Moreover, in this case, if x_{k+1} lies in a cube molecule represented in τ_0', then x_{k+2} lies in the same slice of this cube molecule as x_{k+1}.
(S4) For every $i \in [m']_0$ and every $j \in [t]$, no three consecutive vertices of $\mathcal{L}$ lie in $\mathcal{M}_j(v_i)$ (here $\mathcal{L}$ is viewed as a cyclic sequence of vertices).
(S5) For every $i \in [m']$ such that v_i is an atomic vertex and every $j \in [t]$, we have that $|V(\mathcal{M}_j(v_i)) \cap \mathcal{L}^\bullet|$ is even and $4 \leq |V(\mathcal{M}_j(v_i)) \cap \mathcal{L}^\bullet| \leq 12$. In particular, $|V(\mathcal{M}_t(v_0)) \cap \mathcal{L}^\bullet| = 4$.
(S6) For all $k \in [r]$ except two values, we have that $x_k \neq_{\mathrm{p}} x_{k+1}$. The remaining two values $k_1, k_2 \in [r]$ correspond to two pairs of vertices $x_{k_1}, x_{k_1+1}, x_{k_2}, x_{k_2+1} \in V(\mathcal{M}_t(v_0))$. For these two values, we have that $x_{k_1} \neq_{\mathrm{p}} x_{k_2}$ and either
 (i) $x_{k_1} =_{\mathrm{p}} x_{k_1+1}$ and $x_{k_2} =_{\mathrm{p}} x_{k_2+1}$, or
 (ii) $x_{k_1} \neq_{\mathrm{p}} x_{k_1+1}$ and $x_{k_2} \neq_{\mathrm{p}} x_{k_2+1}$,
 where $x_{k_1}, x_{k_2} \in V(\mathcal{A}_{(t-1)q+1}(v_0))$ and $x_{k_1+1}, x_{k_2+1} \in V(\mathcal{A}_{tq}(v_0))$.
(S7) $\mathcal{L}^\bullet \cap (\mathfrak{L} \cup \mathfrak{R}_1 \cup V^{\mathrm{abs}}) = \varnothing$ and $L^\bullet \subseteq \mathcal{L}^\bullet$.

As happened with the external skeleton, the skeleton is built recursively from partial skeletons, which are defined first for the leaves. This recursive construction means that the overall order in which the molecules are visited will be determined by a depth first search of the tree τ_0'. Moreover, as discussed in Section 2.5, for parity reasons the skeleton will actually traverse τ_0' twice. These two traversals will be 'tied together' in the final step of the construction of the skeleton.

Note that, for each $i \in [m]$, the starting and ending vertices x^i, $\hat{x}^i$, y^i, $\hat{y}^i$ for the partial skeletons for v_i are determined by the external skeleton. For each $i \in [m'] \setminus [m]$, the starting and ending vertices for the partial skeletons of v_i will be determined when constructing the partial skeleton for the parent vertex v_{ℓ_i} of v_i. In particular, when constructing the partial skeleton for v_{ℓ_i}, we will define vertices $z^{\ell_i}_{m_i}, \hat{z}^{\ell_i}_{m_i}, w^{\ell_i}_{m_i}, \hat{w}^{\ell_i}_{m_i} \in \mathcal{M}_{b(i)}(v_{\ell_i})$. Then, the starting and ending vertices for the partial skeleton of v_i will be

$$(x^i, y^i, \hat{x}^i, \hat{y}^i) := f^{\ell_i}_{m_i} + (z^{\ell_i}_{m_i}, w^{\ell_i}_{m_i}, \hat{z}^{\ell_i}_{m_i}, \hat{w}^{\ell_i}_{m_i}). \tag{8.20}$$

(Recall that ℓ_i, m_i, $b(i)$ and $f^{\ell_i}_{m_i}$ were defined at the end of Step 12.)

We are now in a position to define the partial skeletons formally. The construction proceeds by induction on $i \in [m']$ in decreasing order, starting with $i = m'$. Recall from the beginning of Step 11 that, for all $i \in [m]$, $x^i, y^i \in V(\mathcal{M}_{b(i)}(v_i))$ are the starting and ending vertices for the first partial skeleton $\mathcal{L}(x^i, y^i)$ for v_i, respectively, and $\hat{x}^i, \hat{y}^i \in V(\mathcal{M}_{b(i)}(v_i))$ are the starting and ending vertices for the second partial skeleton $\hat{\mathcal{L}}(\hat{x}^i, \hat{y}^i)$ for v_i, respectively. The vertices $x^i, y^i, \hat{x}^i, \hat{y}^i$ were fixed in the construction of the external skeleton, and they form a valid connection sequence. For each $i \in [m'] \setminus [m]$, the vertices $x^i, y^i, \hat{x}^i, \hat{y}^i \in V(\mathcal{M}_{b(i)}(v_i))$ defined in (8.20) will also form a valid connection sequence.

Let $\mathcal{F} := \mathfrak{L} \cup \mathfrak{R}_1 \cup L^\bullet$. For each $k \in [2^s]$, let $\hat{e}_k$ be the direction of the edges in $\mathcal{Q}^n$ between L_k and L_{k+1}. Throughout the following construction, we will often choose vertices which are used to transition between neighbouring slices, all while avoiding the set $\mathcal{F}$. Similarly to the proof of Lemma 8.8, all of these choices can be made by (ES2), (ES3), (C*1), (C*2), and because all cube molecules considered here are bonded in G_5 and, therefore, also in G'. (The latter holds since for each atomic vertex $v \in V(\tau_0')$ the corresponding cube $C(v)$ satisfies $C(v) \in \mathcal{C}'$.) Whenever we mention a vertex that we do not define here, we refer to the vertex with the same notation defined when constructing the external skeleton in Step 11.

Suppose that $i \in [m']$ and that for every $i' \in [m'] \setminus ([i] \cup [m])$ and every valid connection sequence $(x^{i'}, y^{i'}, \hat{x}^{i'}, \hat{y}^{i'})$ for $v_{i'}$ we have already defined two partial skeletons $\mathcal{L}(x^{i'}, y^{i'})$, $\hat{\mathcal{L}}(\hat{x}^{i'}, \hat{y}^{i'})$ for $v_{i'}$ with this connection sequence. (As discussed above, eventually we will only use the two partial skeletons for $v_{i'}$ with connection sequence as defined in (8.20).) Moreover, suppose that for every $i' \in [m] \setminus [i]$ we have already defined two partial skeletons $\mathcal{L}(x^{i'}, y^{i'})$, $\hat{\mathcal{L}}(\hat{x}^{i'}, \hat{y}^{i'})$ for $v_{i'}$ with connection sequence $(x^{i'}, y^{i'}, \hat{x}^{i'}, \hat{y}^{i'})$ (fixed by the external skeleton). If $i \in [m]$, let $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ be the connection sequence for v_i fixed by the external skeleton. If $i \in [m'] \setminus [m]$, let $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ be any connection sequence for v_i. We will now define the two partial skeletons for v_i with connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$. We consider several cases.

Case 1: v_i is a leaf of τ_0'. We construct the partial skeletons as follows. Let $x_0^i := x^i$ and $\hat{x}_0^i := \hat{x}^i$. For each $k \in [t-1]_0$, iteratively choose any two vertices $y_k^i, \hat{y}_k^i \in V(\mathcal{A}_{(b(i)+k)q}(v_i)) \setminus (\mathcal{F} \cup \{x^i, y^i, \hat{x}^i, \hat{y}^i\})$ satisfying that

1. $y_k^i \neq_{\mathrm{P}} x_k^i$ and $\hat{y}_k^i \neq_{\mathrm{P}} \hat{x}_k^i$;
2. $x_{k+1}^i := y_k^i + \hat{e}_{(b(i)+k)q} \notin \mathcal{F} \cup \{x^i, y^i, \hat{x}^i, \hat{y}^i\}$ and $\hat{x}_{k+1}^i := \hat{y}_k^i + \hat{e}_{(b(i)+k)q} \notin \mathcal{F} \cup \{x^i, y^i, \hat{x}^i, \hat{y}^i\}$, and
3. $\{y_k^i, x_{k+1}^i\}, \{\hat{y}_k^i, \hat{x}_{k+1}^i\} \in E(G')$.

Recall that we use $\bigtimes$ to denote the concatenation of sequences. The first and second partial skeletons for v_i with connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ are given by

$$\mathcal{L}_i(x^i, y^i) := (x^i) \left(\bigtimes_{k=0}^{t-1} (y_k^i, x_{k+1}^i) \right) (y^i) \quad \text{and} \quad \hat{\mathcal{L}}_i(\hat{x}^i, \hat{y}^i) := (\hat{x}^i) \left(\bigtimes_{k=0}^{t-1} (\hat{y}_k^i, \hat{x}_{k+1}^i) \right) (\hat{y}^i).$$

Case 2: $v_i \in V(\tau_0)$ is an inner tree vertex. Then, the first and second partial skeletons for v_i with connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ are defined as

$$\mathcal{L}_i(x^i, y^i) := (x^i) \bigtimes_{k=1}^{p_i} (\mathcal{L}_{j_k^i}(x^{j_k^i}, y^{j_k^i}), w_k^i) \quad \text{and} \quad \hat{\mathcal{L}}_i(\hat{x}^i, \hat{y}^i) := (\hat{x}^i) \bigtimes_{k=1}^{p_i} (\hat{\mathcal{L}}_{j_k^i}(\hat{x}^{j_k^i}, \hat{y}^{j_k^i}), \hat{w}_k^i),$$

where j_k^i was defined in Step 10.

Case 3: $v_i \in V(\tau_0)$ is an atomic vertex which is not a leaf of τ_0'. We construct the partial skeletons for v_i as follows. (Recall that, for each $k \in [p_i'] \setminus [p_i]$, the vertex ν_k^i was defined in Step 12.)

1. For each $k \in [p_i]$, iteratively choose distinct vertices $y_k^i, \hat{y}_k^i \in V(\mathcal{A}_{(b(i)+k-1)q}(v_i)) \setminus \mathcal{F}$ such that
   1.1. $y_k^i \neq_{\mathrm{P}} w_k^i$ and $\hat{y}_k^i \neq_{\mathrm{P}} \hat{w}_k^i$;
   1.2. $x_{k+1}^i := y_k^i + \hat{e}_{(b(i)+k-1)q} \notin \mathcal{F}$ and $\hat{x}_{k+1}^i := \hat{y}_k^i + \hat{e}_{(b(i)+k-1)q} \notin \mathcal{F}$, and
   1.3. $\{y_k^i, x_{k+1}^i\}, \{\hat{y}_k^i, \hat{x}_{k+1}^i\} \in E(G')$.
2. If $p_i = 0$, let $x_1^i := x^i$ and $\hat{x}_1^i := \hat{x}^i$. For each $k \in [p_i'] \setminus [p_i]$, iteratively choose distinct vertices $z_k^i, w_k^i, \hat{z}_k^i, \hat{w}_k^i \in (V(\mathcal{M}_{b(i)+k-1}(v_i)) \cap V(\mathcal{M}_{\nu_k^i})) \setminus (\mathcal{F} \cup \{x_k^i, \hat{x}_k^i\})$ and distinct vertices $y_k^i, \hat{y}_k^i \in V(\mathcal{A}_{(b(i)+k-1)q}(v_i)) \setminus (\mathcal{F} \cup \{z_k^i, w_k^i, \hat{z}_k^i, \hat{w}_k^i\})$ satisfying that
   2.1. $z_k^i, \hat{w}_k^i \neq_{\mathrm{P}} x_k^i$ and $\hat{z}_k^i, w_k^i =_{\mathrm{P}} x_k^i$;
   2.2. $x^{j_k^i}, y^{j_k^i}, \hat{x}^{j_k^i}, \hat{y}^{j_k^i} \notin \mathcal{F}$, where $x^{j_k^i}$, $y^{j_k^i}$, $\hat{x}^{j_k^i}$ and $\hat{y}^{j_k^i}$ are defined as in (8.20);
   2.3. $y_k^i \neq_{\mathrm{P}} w_k^i$ and $\hat{y}_k^i \neq_{\mathrm{P}} \hat{w}_k^i$;
   2.4. $x_{k+1}^i := y_k^i + \hat{e}_{(b(i)+k-1)q} \notin \mathcal{F}$ and $\hat{x}_{k+1}^i := \hat{y}_k^i + \hat{e}_{(b(i)+k-1)q} \notin \mathcal{F}$, and
   2.5. $\{y_k^i, x_{k+1}^i\}, \{\hat{y}_k^i, \hat{x}_{k+1}^i\} \in E(G')$.

   As discussed earlier, observe that a choice satisfying 2.2. exists by (C*1), (C*2) and (ES2).
3. For each $k \in [t] \setminus [p_i']$, iteratively choose distinct vertices $y_k^i, \hat{y}_k^i \in V(\mathcal{A}_{(b(i)+k-1)q}(v_i)) \setminus \mathcal{F}$ satisfying that
   3.1. $y_k^i \neq_{\mathrm{P}} x_k^i$ and $\hat{y}_k^i \neq_{\mathrm{P}} \hat{x}_k^i$;
   3.2. $x_{k+1}^i := y_k^i + \hat{e}_{(b(i)+k-1)q} \notin \mathcal{F}$ and $\hat{x}_{k+1}^i := \hat{y}_k^i + \hat{e}_{(b(i)+k-1)q} \notin \mathcal{F}$, and

3.3. $\{y_k^i, x_{k+1}^i\}, \{\hat{y}_k^i, \hat{x}_{k+1}^i\} \in E(G')$.

Then, we may define the first and second partial skeletons for v_i with connection sequence $(x^i, y^i, \hat{x}^i, \hat{y}^i)$ as

$$\mathcal{L}_i(x^i, y^i) \coloneqq (x^i) \left(\bigtimes_{k=1}^{p_i'} (z_k^i, \mathcal{L}_{j_k^i}(x^{j_k^i}, y^{j_k^i}), w_k^i, y_k^i, x_{k+1}^i) \right) \left(\bigtimes_{k=p_i'+1}^{t} (y_k^i, x_{k+1}^i) \right) (y^i),$$

$$\hat{\mathcal{L}}_i(\hat{x}^i, \hat{y}^i) \coloneqq (\hat{x}^i) \left(\bigtimes_{k=1}^{p_i'} (\hat{z}_k^i, \hat{\mathcal{L}}_{j_k^i}(\hat{x}^{j_k^i}, \hat{y}^{j_k^i}), \hat{w}_k^i, \hat{y}_k^i, \hat{x}_{k+1}^i) \right) \left(\bigtimes_{k=p_i'+1}^{t} (\hat{y}_k^i, \hat{x}_{k+1}^i) \right) (\hat{y}^i).$$

We are now ready to construct $\mathcal{L}$. The idea is similar to that of Case 3, except that we now tie together the first and second partial skeletons in Step 1.2 below.

1. Choose any two vertices $x_1^0, \hat{x}_1^0 \in V(\mathcal{A}_1(v_0)) \setminus \mathcal{F}$ such that
   1.1. $x_1^0 =_{\mathrm{p}} w_0^0$ and $\hat{x}_1^0 \neq_{\mathrm{p}} w_0^0$;
   1.2. $y_t^0 \coloneqq \hat{x}_1^0 + \hat{e}_{2^s} \notin \mathcal{F}$ and $\hat{y}_t^0 \coloneqq x_1^0 + \hat{e}_{2^s} \notin \mathcal{F}$, and
   1.3. $\{x_1^0, \hat{y}_t^0\}, \{\hat{x}_1^0, y_t^0\} \in E(G')$.
2. For each $k \in [p_0]$, iteratively choose two distinct vertices $y_k^0, \hat{y}_k^0 \in V(\mathcal{A}_{qk}(v_0)) \setminus \mathcal{F}$ such that
   2.1. $y_k^0 \neq_{\mathrm{p}} w_k^0$ and $\hat{y}_k^0 \neq_{\mathrm{p}} \hat{w}_k^0$;
   2.2. $x_{k+1}^0 \coloneqq y_k^0 + \hat{e}_{kq} \notin \mathcal{F}$ and $\hat{x}_{k+1}^0 \coloneqq \hat{y}_k^0 + \hat{e}_{kq} \notin \mathcal{F}$, and
   2.3. $\{y_k^0, x_{k+1}^0\}, \{\hat{y}_k^0, \hat{x}_{k+1}^0\} \in E(G')$.
3. For each $k \in [p_0'] \setminus [p_0]$, iteratively choose distinct vertices $z_k^0, w_k^0, \hat{z}_k^0, \hat{w}_k^0 \in (V(\mathcal{M}_k(v_0)) \cap V(\mathcal{M}_{\nu_{k+1}^0})) \setminus (\mathcal{F} \cup \{x_k^0, \hat{x}_k^0\})$ and distinct vertices $y_k^0, \hat{y}_k^0 \in V(\mathcal{A}_{kq}(v_0)) \setminus (\mathcal{F} \cup \{z_k^0, w_k^0, \hat{z}_k^0, \hat{w}_k^0\})$ satisfying that
   3.1. $z_k^0, \hat{w}_k^0 \neq_{\mathrm{p}} x_k^0$ and $\hat{z}_k^0, w_k^0 =_{\mathrm{p}} x_k^0$;
   3.2. $x^{j_k^0}, y^{j_k^0}, \hat{x}^{j_k^0}, \hat{y}^{j_k^0} \notin \mathcal{F}$, where $x^{j_k^0}$, $y^{j_k^0}$, $\hat{x}^{j_k^0}$ and $\hat{y}^{j_k^0}$ are defined as in (8.20);
   3.3. $y_k^0 \neq_{\mathrm{p}} w_k^0$ and $\hat{y}_k^0 \neq_{\mathrm{p}} \hat{w}_k^0$;
   3.4. $x_{k+1}^0 \coloneqq y_k^0 + \hat{e}_{kq} \notin \mathcal{F}$ and $\hat{x}_{k+1}^0 \coloneqq \hat{y}_k^0 + \hat{e}_{kq} \notin \mathcal{F}$, and
   3.5. $\{y_k^0, x_{k+1}^0\}, \{\hat{y}_k^0, \hat{x}_{k+1}^0\} \in E(G')$.
4. For each $k \in [t-1] \setminus [p_0']$, iteratively choose any two vertices $y_k^0, \hat{y}_k^0 \in V(\mathcal{A}_{kq}(v_0)) \setminus \mathcal{F}$ satisfying that
   4.1. $y_k^0 \neq_{\mathrm{p}} x_k^0$ and $\hat{y}_k^0 \neq_{\mathrm{p}} \hat{x}_k^0$;
   4.2. $x_{k+1}^0 \coloneqq y_k^0 + \hat{e}_{kq} \notin \mathcal{F}$ and $\hat{x}_{k+1}^0 \coloneqq \hat{y}_k^0 + \hat{e}_{kq} \notin \mathcal{F}$, and
   4.3. $\{y_k^0, x_{k+1}^0\}, \{\hat{y}_k^0, \hat{x}_{k+1}^0\} \in E(G')$.

The final definition of $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L} \coloneqq (x_1^0) &\left(\bigtimes_{k=1}^{p_0'} (z_k^0, \mathcal{L}_{j_k^0}(x^{j_k^0}, y^{j_k^0}), w_k^0, y_k^0, x_{k+1}^0) \right) \left(\bigtimes_{k=p_0'+1}^{t-1} (y_k^0, x_{k+1}^0) \right) (y_t^0, \hat{x}_1^0) \\ &\left(\bigtimes_{k=1}^{p_0'} (\hat{z}_k^0, \hat{\mathcal{L}}_{j_k^0}(\hat{x}^{j_k^0}, \hat{y}^{j_k^0}), \hat{w}_k^0, \hat{y}_k^0, \hat{x}_{k+1}^0) \right) \left(\bigtimes_{k=p_0'+1}^{t-1} (\hat{y}_k^0, \hat{x}_{k+1}^0) \right) (\hat{y}_t^0).\end{aligned}$$

Observe that (S1)–(S6) hold by construction. In particular, (8.16) together with (V1) ensure that in Case 3 the final two vertices of the two partial skeletons satisfy $x_{t+1}^i \neq_{\mathrm{p}} y^i$ and $\hat{x}_{t+1}^i \neq_{\mathrm{p}} \hat{y}^i$. Moreover, the pairs x_t^0, y_t^0 and $\hat{x}_t^0, \hat{y}_t^0$ will play the roles of the pairs x_{k_1}, x_{k_1+1} and x_{k_2}, x_{k_2+1} in the second part of (S6). Similarly, (S7) holds by combining the construction of $\mathcal{L}$, (C_4), (ES5) and the definition of V^{abs}.

Recall that we write $\mathcal{L} = (x_1, \ldots, x_r)$. For each $i \in [m']_0$ such that v_i is an atomic vertex and each $j \in [t]$, let $\mathfrak{J}_{i,j} \coloneqq \{k \in [r] : x_k, x_{k+1} \in V(\mathcal{M}_j(v_i))\}$ and $S_{i,j} \coloneqq \{\{x_k, x_{k+1}\} : k \in \mathfrak{J}_{i,j}\}$.

Step 15: Constructing an almost spanning cycle. We will now apply the connecting lemmas to obtain an almost spanning cycle in G' from $\mathcal{L} = (x_1, \ldots, x_r)$. For each $i \in [m']_0$ such that v_i is an atomic vertex and each $j \in [t]$, except the pair $(0, t)$, we apply Lemma 8.8 to the slice $\mathcal{M}_j(v_i)$ and the graph G', with $\mathfrak{L} \cap V(\mathcal{M}_j(v_i))$, $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}_j(v_i))$ and $S_{i,j}$ playing the roles of L, R and the pairs of vertices described in Lemma 8.8(C3), respectively. Note that the conditions of Lemma 8.8 can be verified as follows. (C1) and (C2) hold by (C*1) and (C*2) combined with (C*3). (C3) holds by (S1) and (S3)–(S7). For $\mathcal{M}_t(v_0)$, we apply Lemma 8.8 or Lemma 8.9 depending on whether (ii) or (i) holds in (S6) (the conditions for Lemma 8.9 can be checked analogously). For each $i \in [m']_0$ such that v_i is an atomic vertex and each $j \in [t]$, this yields $|\mathfrak{I}_{i,j}|$ vertex-disjoint paths $(\mathcal{P}_k^{i,j})_{k\in\mathfrak{I}_{i,j}}$ in $\mathcal{M}_j(v_i) \cup G' = G'$ such that, for each $k \in \mathfrak{I}_{i,j}$,

(i) $\mathcal{P}_k^{i,j}$ is an (x_k, x_{k+1})-path,
(ii) $\bigcup_{k\in\mathfrak{I}_{i,j}} V(\mathcal{P}_k^{i,j}) = V(\mathcal{M}_j(v_i)) \setminus \mathfrak{L}$, and
(iii) any pair of second and third absorber vertices in $\mathfrak{R}_1 \cup \mathfrak{R}_2$ contained in the same atom of $\mathcal{M}_j(v_i)$ form an edge in one of the paths.

Now consider the path obtained as follows by going through $\mathcal{L}$. Start with x_1. For each $k \in [r]$, if there exist $i \in [m']_0$ and $j \in [t]$ such that $\{x_k, x_{k+1}\} \in S_{i,j}$, add $\mathcal{P}_k^{i,j}$ to the path; otherwise, add the edge $\{x_k, x_{k+1}\}$ (this must be an edge of G' by (S3)). Finally, add the edge $\{x_r, x_1\}$ of G' (this is given by (S2)) to the path to close it into a cycle $\mathfrak{H}$ in G'. This cycle satisfies the following properties (recall that $e_{\mathrm{abs}}(u)$ was defined at the end of Step 13):

(HC1) $|V(\mathfrak{H})| \geq (1 - 4/\ell^4)2^n$.
(HC2) $V(\mathfrak{H}) \,\dot\cup\, \mathfrak{L} \,\dot\cup\, V^{\mathrm{abs}}$ partitions $V(\mathcal{Q}^n)$.
(HC3) For all $u \in V^{\mathrm{abs}}$, we have that $e_{\mathrm{abs}}(u) \in E(\mathfrak{H})$.

Indeed, note that $\mathfrak{H}$ covers all vertices in $L^\bullet$ (since $L^\bullet \subseteq \mathcal{L}^\bullet$ by (S7)) as well as all vertices lying in cube molecules represented in τ_0' except for those in $\mathfrak{L}$ (by (ii)). Together with the definition of V^{abs}, this implies (HC2). Moreover, since $|\mathfrak{L}| = |V^{\mathrm{abs}}|$, (HC1) follows from (8.19). Finally, (HC3) follows by (iii).

Step 16: Absorbing vertices to form a Hamilton cycle. For each $u \in V^{\mathrm{abs}}$, replace the edge $e_{\mathrm{abs}}(u)$ by the path $\mathcal{P}_{\mathrm{abs}}(u)$ (recall from the end of Step 13 that $\mathcal{P}_{\mathrm{abs}}(u)$ lies in $((H \cup G) \setminus F) \cup G')$. Clearly, this incorporates all vertices of $\mathfrak{L} \cup V^{\mathrm{abs}}$ into the cycle and, by (HC2) and (HC3), the resulting cycle is Hamiltonian. □

8.5. **Proofs of Theorems 1.1, 1.2 and 1.7.** First, we show that, as a byproduct of the proof of Theorem 8.1, we also have a proof of Theorem 1.2.

Proof of Theorem 1.2. Apply Steps 1, 4, 6, 7, 10, 11, 12, 14 and 15 in succession. In general, any reference to absorbing cubes in these steps (see e.g. the end of Step 6) should be skipped as well. □

Next, we will show how Theorem 8.1 can be used to prove Theorem 1.7.

Proof of Theorem 1.7. Consider a decomposition of H into k edge-disjoint subgraphs $H_1\cup\cdots\cup H_k$ such that, for every $i \in [k]$, we have $\delta(H_i) \geq \alpha n/(2k)$. To see that this is possible, let us randomly partition the edges of H so that each $e \in E(H)$ is assigned to one of the H_i's uniformly at random and independently from all other edges. Thus, for every $i \in [k]$ we have $\mathbb{P}[e \in E(H_i)] = 1/k$. It follows by Lemma 4.2 that, for every vertex $x \in V(\mathcal{Q}^n)$ and every $i \in [k]$,

$$\mathbb{P}[d_{H_i}(x) \leq \alpha n/(2k)] \leq e^{-\alpha n/(8k)}.$$

For each $x \in V(\mathcal{Q}^n)$, let $\mathcal{B}(x)$ be the event that $d_{H_i}(x) \leq \alpha n/(2k)$ for some $i \in [k]$. Hence, $\mathbb{P}[\mathcal{B}(x)] \leq ke^{-\alpha n/(8k)}$ for all $x \in V(\mathcal{Q}^n)$. Observe that $\mathcal{B}(x)$ is independent of the collection of events $\{\mathcal{B}(y) : \mathrm{dist}(x, y) \geq 2\}$. A simple application of Lemma 4.8 shows that

$$\mathbb{P}\left[\bigwedge_{x\in V(\mathcal{Q}^n)} \overline{\mathcal{B}(x)}\right] > 0$$

and, therefore, such a decomposition of H exists.

We now consider a similar decomposition of $\mathcal{Q}^n_\varepsilon$. In particular, given $\mathcal{Q}^n_\varepsilon$, we partition its edges into k edge-disjoint subgraphs, $Q_1 \cup \cdots \cup Q_k$, in such a way that, if $e \in E(\mathcal{Q}^n_\varepsilon)$, then e

is assigned to one of the Q_i chosen uniformly at random and independently of all other edges. Thus, for each $e \in E(\mathcal{Q}^n_\varepsilon)$ we have $\mathbb{P}[e \in E(Q_i)] = 1/k$ for all $i \in [k]$. It follows that, for each $i \in [k]$, we have $Q_i \sim \mathcal{Q}^n_{\varepsilon/k}$.

Let Φ be a constant such that Theorem 8.1 holds with ε/k, $\alpha/(2k)$ and $k+2$ playing the roles of ε, α and c, respectively. For each $i \in [k]$, apply Theorem 8.1 with H_i and Q_i playing the roles of H and G, respectively. We obtain that a.a.s. there exists a subgraph $G_i \subseteq Q_i$ with $\Delta(G_i) \leq \Phi$ such that, for every $F_i \subseteq \mathcal{Q}^n$ with $\Delta(F_i) \leq (k+2)\Phi$, the graph $((H_i \cup Q_i) \setminus F_i)) \cup G_i$ is Hamiltonian. Condition on the event that this holds for all $i \in [k]$ simultaneously (which holds a.a.s. by a union bound).

We are now going to find k edge-disjoint Hamilton cycles $C_1, \ldots, C_k$ iteratively. For each $i \in [k]$, we proceed as follows. Let $F_i := \bigcup_{j=1}^{k} G_j \cup \bigcup_{j=1}^{i-1} C_j$. It is clear by construction that $\Delta(F_i) \leq k(\Phi+2) \leq (k+2)\Phi$. By the conditioning above, there must be a Hamilton cycle $C_i \subseteq ((H_i \cup Q_i) \setminus F_i)) \cup G_i$. Take any such C_i and proceed.

It remains to prove that $C_1, \ldots, C_k$ are pairwise edge-disjoint. In order to see this, suppose that there exist $i, j \in [k]$ with $i < j$ such that $E(C_i) \cap E(C_j) \neq \varnothing$, and let $e \in E(C_i) \cap E(C_j)$. In order to have $e \in E(C_i)$, since $G_j \subseteq F_i \setminus G_i$, we must have $e \notin E(G_j)$. However, since $e \in F_j$ by definition, we must have $e \in E(G_j)$, a contradiction. □

Now, Theorem 1.1 follows as an immediate corollary.

Proof of Theorem 1.1. It is well known (see e.g. [12]) and easy to show that $\mathcal{Q}^n_{1/2-\varepsilon}$ a.a.s. contains isolated vertices. So it suffices to consider $\mathcal{Q}^n_{1/2+\varepsilon}$ for any fixed $\varepsilon > 0$ and show that a.a.s. it contains k edge-disjoint Hamilton cycles. Let $0 < \delta \ll \varepsilon \leq 1/2$. Let $H \sim \mathcal{Q}^n_{1/2+\varepsilon/2}$ and $G \sim \mathcal{Q}^n_{\varepsilon/2}$. Note that $H \cup G \sim \mathcal{Q}^n_\eta$, for some $\eta \leq 1/2+\varepsilon$. Furthermore, by Lemma 5.6, a.a.s. $\delta(H) \geq \delta n$. Applying Theorem 1.7 to $H \cup G$, we obtain the desired result. □

9. Hitting time result

In this section we prove Theorem 1.8 and Theorem 1.5. The proof of Theorem 1.8 closely follows that of Theorem 1.7, with some necessary changes to deal with the possibility that some vertices have degree $o(n)$. We will describe here all necessary changes. It is worth noting that that the set $\mathcal{A}$ in Theorem 7.1 is only needed for this section.

Theorem 1.8 will actually follow from a slightly more general result (see Theorem 9.6), which we prove in Section 9.3. Similarly as for Theorem 1.7, we will first prove Theorem 1.8 for the property of being Hamiltonian (that is, the case $k=2$), and then use this to prove the general result.

Let $s \in \mathbb{N}$. Throughout this section, we consider a similar setup to that in Section 8. In particular, we have a partition of the vertex set of the hypercube into layers $L_1, \ldots, L_{2^s}$, where the labelling of the layers is given by some Hamilton cycle of $\mathcal{Q}^s$. We fix any such partition, and will write $\hat{e}_i$ for the direction of the edges between the layers L_i and L_{i+1}. We will use the notation for atoms, molecules and slices in the same way. We will also use part of the notation introduced at the beginning of the proof of Theorem 8.1; in particular, we will consider the intersection graph $I \cong \mathcal{Q}^{n-s}$ of the layers, and for any $G \subseteq \mathcal{Q}^n$ we define $I(G)$ analogously.

9.1. **Absorbing structures for vertices with small degree.** The main reason why the proof of Theorem 8.1 does not work for the case $k=2$ of Theorem 1.8 is the existence of vertices of very low degree (as low as degree 2). We cannot hope to absorb these via absorbing ℓ-cube pairs as in the proof of Theorem 8.1, as they may not have any neighbours which lie in the cube factor we construct. Thus, we first construct alternative absorbing structures for these vertices.

To be more precise, recall that in the proof of Theorem 8.1 we absorbed vertices in pairs, in order to compensate the parities (within a vertex molecule). Here, we will need something similar. To achieve this, we will define several paths. One path will contain the vertex of low degree, while the others are used to compensate the parities of vertices in this first path. Moreover, the paths will be constructed in such a way that they end in vertices which can be paired up so that each pair consists of clones of a given vertex of the intersection graph I and these two clones lie in a (bonded) cube molecule. These cube molecules can then be used to connect these paths. It is worth mentioning, however, that we cannot guarantee that the pairing of the vertices can be done within a single slice, so we need to alter our approach to deal with this.

These special absorbing structures will be used in a somewhat different way to the absorbing ℓ-cube pairs (which are still used to absorb all other vertices). For the latter, recall that we enforce a condition on the near-spanning cycle $\mathfrak{H}$ (namely, that it contains certain edges $e_{\mathrm{abs}}(u)$) so that we can absorb the required vertices $u \in V^{\mathrm{abs}}$. For the special absorbing structures, however, we will actually enforce that the vertices of very low degree are already incorporated into $\mathfrak{H}$. In particular, consider the cube molecules which contain the endpoints of paths of one of these special absorbing structures. When constructing the skeleton, we will add extra segments in these molecules which will be used to connect these paths in such a way that, when we apply the connecting lemmas, we can completely incorporate each special absorbing structure into $\mathfrak{H}$.

Suppose $x \in V(\mathcal{Q}^n)$ is incident to one edge with direction a and one edge with direction b. We will construct three different types of special absorbing structures, to handle the cases where both, only one, or none of these two edges lie in the same layer as x. Representations of these three types of special absorbing structures can be found in Figure 1. Let L be the layer such that $x \in V(L)$. Given a path $P = x_1x_2\dots x_k$ in $\mathcal{Q}^n$, we define $\mathrm{end}(P) \coloneqq \{x_1, x_k\}$.

Type I. Assume that $x + a, x + b \in V(L)$. Let $f\colon V(\mathcal{Q}^n) \to V(\mathcal{Q}^n)$ be defined as follows: for each $i \in [2^s]$ and each $y \in V(L_i)$, if i is even, we set $f(y) \coloneqq y+\hat{e}_{i-1}$; otherwise, we set $f(y) \coloneqq y+\hat{e}_i$. By abusing notation, for any $F \subseteq L_i$, we also consider the graph $f(F)$, where for each edge $e = \{y, z\} \in E(F)$ we define $f(e) \coloneqq \{f(y), f(z)\}$. Let $(c, d, d_1, d_2, d_3, d_4) \in (\mathcal{D}(L) \setminus \{a, b\})^6$ be a tuple of distinct directions and define the following paths in $\mathcal{Q}^n$:

- $P_1 \coloneqq (x + a + d_1, x + a, x, x + b, x + b + d_2)$;
- $P_2 \coloneqq (f(x + b + d_2), f(x + b), f(x + b + c))$;
- $P_3 \coloneqq (x + c + b, x + c, x + c + d_3)$;
- $P_4 \coloneqq (f(x + c + d_3), f(x + c), f(x), f(x + d), f(x + d + d_4))$;
- $P_5 \coloneqq (x + d + d_4, x + d, x + d + a)$;
- $P_6 \coloneqq (f(x + a + d), f(x + a), f(x + a + d_1))$.

Observe that, for every $y \in V(P_1 \cup \dots \cup P_6)$, we have that $f(y) = f^{-1}(y) \in V(P_1 \cup \dots \cup P_6)$. We say that $CS(x, a, b) \coloneqq (P_1, \dots, P_6)$ is an (x, a, b)-*consistent system of paths*, and let $\mathrm{end}(CS(x, a, b)) \coloneqq \bigcup_{i=1}^6 \mathrm{end}(P_i)$.

Let $(P_1, \dots, P_6)$ be an (x, a, b)-consistent system of paths. Let $\mathcal{D} \coloneqq \{c, d, d_1, d_2, d_3, d_4\} \subseteq \mathcal{D}(L)$ be the set of directions such that $P_1, \dots, P_6$ are as defined above. Let $\mathbf{C} \coloneqq \{C_1, \dots, C_{12}\}$ be a collection of vertex-disjoint ℓ-cubes which satisfy the following:

(PI.1) for all $i \in [12]$, we have that either $C_i \subseteq L$ or $C_i \subseteq f(L)$;
(PI.2) for all $i \in [6]$, we have $f(C_{2i}) = C_{2i+1}$, where indices are taken modulo 12;
(PI.3) for all $i \in [6]$, C_{2i-1} contains the first vertex of P_i, and C_{2i} contains the last vertex of P_i;
(PI.4) for all $i \in [12]$, we have $\mathcal{D}(C_i) \cap (\mathcal{D} \cup \{a, b\}) = \varnothing$.

We say that $SA(x, a, b) \coloneqq (P_1, \dots, P_6, C_1, \dots, C_{12})$ is an (x, a, b)-*special absorbing structure*.

Finally, given any graph $G \subseteq \mathcal{Q}^n$ and an (x, a, b)-consistent system of paths $CS(x, a, b) = (P_1, \dots, P_6)$, we say that $CS(x, a, b)$ *extends* to an (x, a, b)-special absorbing structure in G if there is a collection $\mathbf{C} = \{C_1, \dots, C_{12}\}$ with $C_i \subseteq G$ such that $SA(x, a, b) = (P_1, \dots, P_6, C_1, \dots, C_{12})$ is an (x, a, b)-special absorbing structure.

Type II. Assume now that $x + a, x + b \notin V(L)$. Let $(d_1, d_2) \in (\mathcal{D}(L))^2$ be a pair of distinct directions and define the following paths in $\mathcal{Q}^n$:

- $P_1 \coloneqq (x + a + d_1, x + a, x, x + b, x + b + d_2)$;
- $P_2 \coloneqq (x + a + b + d_2, x + a + b, x + a + b + d_1)$.

We say that $CS(x, a, b) \coloneqq (P_1, P_2)$ is an (x, a, b)-*consistent system of paths*, and let $\mathrm{end}(CS(x, a, b)) \coloneqq \mathrm{end}(P_1) \cup \mathrm{end}(P_2)$.

Let (P_1, P_2) be an (x, a, b)-consistent system of paths. Let $\mathcal{D} \coloneqq \{d_1, d_2\} \subseteq \mathcal{D}(L)$ be the set of directions such that P_1 and P_2 are as defined above. Let L_{ab}, L_a and L_b be the layers such that

$x+a+b \in V(L_{ab})$, $x+a \in V(L_a)$ and $x+b \in V(L_b)$, respectively. Let $\mathbf{C} := \{C_1, C_2, C_3, C_4\}$ be a collection of vertex-disjoint ℓ-cubes which satisfy the following:

(PII.1) $C_1 \subseteq L_a$, $C_2 \subseteq L_b$ and $C_3, C_4 \subseteq L_{ab}$;
(PII.2) $C_1 + b = C_4$ and $C_2 + a = C_3$;
(PII.3) for all $i \in [2]$, C_{2i-1} contains the first vertex of P_i, and C_{2i} contains the last vertex of P_i;
(PII.4) for all $i \in [4]$, we have $\mathcal{D}(C_i) \cap \mathcal{D} = \varnothing$.

We say that $SA(x,a,b) := (P_1, P_2, C_1, \dots, C_4)$ is an (x,a,b)-*special absorbing structure.*

Finally, given any graph $G \subseteq \mathcal{Q}^n$ and an (x,a,b)-consistent system of paths $CS(x,a,b) = (P_1, P_2)$, we say that $CS(x,a,b)$ *extends* to an (x,a,b)-special absorbing structure in G if there is a collection $\mathbf{C} = \{C_1, \dots, C_4\}$ with $C_i \subseteq G$ such that $SA(x,a,b) = (P_1, P_2, C_1, \dots, C_4)$ is an (x,a,b)-special absorbing structure.

Type III. Finally, assume that $x+a \notin V(L)$ and $x+b \in V(L)$. For each vertex $y \in V(L)$, let $f(y) := y+a$. By abusing notation, for any $F \subseteq L$, we also consider the graph $f(F)$, where for each edge $e = \{y,z\} \in E(F)$ we define $f(e) := \{f(y), f(z)\}$. Let $(d_1, d_2, d_3) \in (\mathcal{D}(L) \setminus \{b\})^3$ be a tuple of distinct directions and define the following paths in $\mathcal{Q}^n$:

- $P_1 := (f(x+d_1+d_2), f(x+d_1), f(x), x, x+b, x+b+d_3)$;
- $P_2 := (f(x+b+d_3), f(x+b), f(x+b+d_1))$;
- $P_3 := (x+d_1+b, x+d_1, x+d_1+d_2)$.

We say that $CS(x,a,b) := (P_1, P_2, P_3)$ is an (x,a,b)-*consistent system of paths*, and let $\mathrm{end}(CS(x,a,b)) := \mathrm{end}(P_1) \cup \mathrm{end}(P_2) \cup \mathrm{end}(P_3)$.

Let (P_1, P_2, P_3) be an (x,a,b)-consistent system of paths. Let $\mathcal{D} := \{d_1, d_2, d_3\} \subseteq \mathcal{D}(L)$ be the set of directions such that P_1, P_2 and P_3 are as defined above. Let $\mathbf{C} := \{C_1, \dots, C_6\}$ be a set of vertex-disjoint ℓ-cubes which satisfy the following:

(PIII.1) for all $i \in [6]$, we have that $C_i \subseteq L$ or $C_i \subseteq f(L)$;
(PIII.2) for all $i \in [3]$, we have $C_{2i} = f(C_{2i+1})$, where indices are taken modulo 6;
(PIII.3) for all $i \in [3]$, C_{2i-1} contains the first vertex of P_i, and C_{2i} contains the last vertex of P_i;
(PIII.4) for all $i \in [6]$, we have $\mathcal{D}(C_i) \cap (\mathcal{D} \cup \{b\}) = \varnothing$.

We say that $SA(x,a,b) := (P_1, P_2, P_3, C_1, \dots, C_6)$ is an (x,a,b)-*special absorbing structure.*

Finally, given any graph $G \subseteq \mathcal{Q}^n$ and an (x,a,b)-consistent system of paths $CS(x,a,b) = (P_1, P_2, P_3)$, we say that $CS(x,a,b)$ *extends* to an (x,a,b)-special absorbing structure in G if there is a collection $\mathbf{C} = \{C_1, \dots, C_6\}$ with $C_i \subseteq G$ such that $SA(x,a,b) = (P_1, P_2, P_3, C_1, \dots, C_6)$ is an (x,a,b)-special absorbing structure.

Whenever x, a and b are clear from the context, we will simply write CS and SA instead of $CS(x,a,b)$ and $SA(x,a,b)$. Given any consistent system of paths CS, we let $\mathrm{endmol}(CS)$ be the set of vertices $v \in V(I)$ such that some clone of v lies in $\mathrm{end}(CS)$. We write $\mathcal{D}(CS)$ to denote the set of directions $\mathcal{D} \cup \{a,b\}$ used to define the paths which comprise CS as above. If CS extends to a special absorbing structure SA, we denote $\mathrm{end}(SA) := \mathrm{end}(CS)$. Moreover, we denote by $\mathbf{C}(SA)$ the collection of cubes associated with SA. Observe that (PI.4), (PII.4) and (PIII.4) imply that

(AS) each cube $C \in \mathbf{C}(SA)$ is vertex-disjoint from the paths in SA except for the unique vertex in $\mathrm{end}(SA)$ contained in C.

We will sometimes abuse notation and treat CS and SA as graphs; in particular, we will write $V(CS)$ to denote the vertices of the union of the paths which comprise CS, and $V(SA)$ to denote the vertices of the union of the paths and cubes which comprise SA, and similarly for $E(CS)$ and $E(SA)$.

9.2. **Auxiliary lemmas.** We now state and prove some auxiliary lemmas. When taking random subgraphs of the hypercube, we will need to guarantee that, given a vertex in I and a large collection of cubes in I incident to this vertex, some of the cube molecules given by these cubes are bonded.

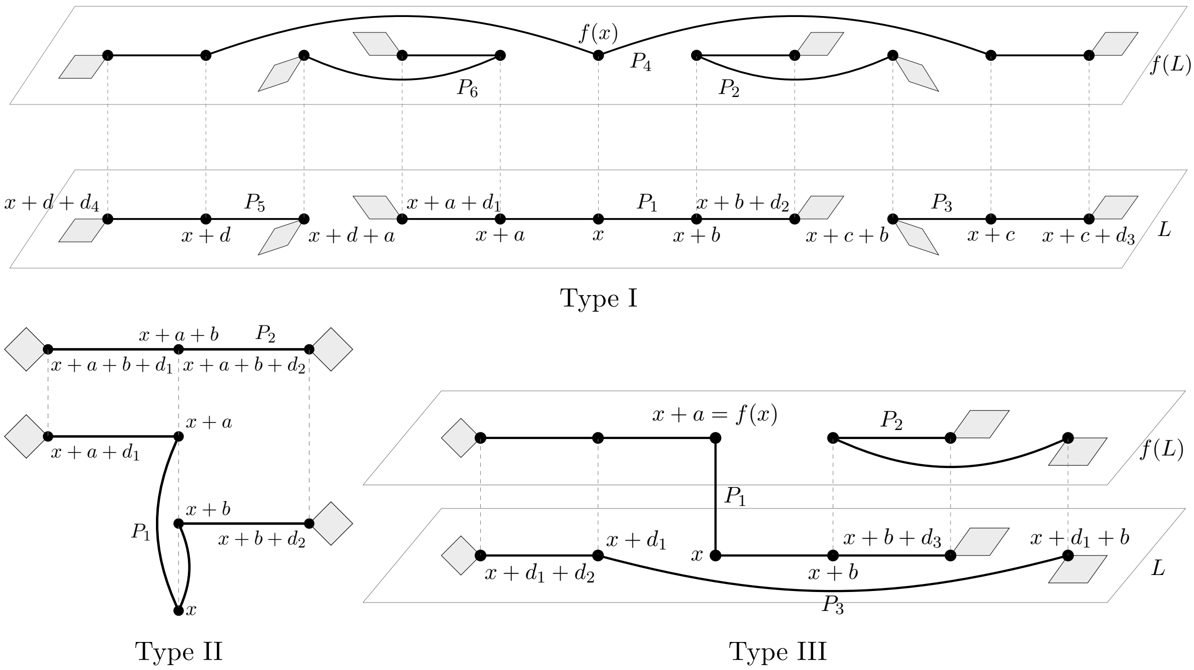

FIGURE 1. A representation of the special absorbing structures. In each case, vertices (or cubes) represented in vertical lines are clones of the same vertex (or cube) of the intersection graph I, and any vertices in the same horizontal line lie in the same layer of $\mathcal{Q}^n$.

Lemma 9.1. *Let $\varepsilon, \gamma \in (0,1)$ and $\ell, n \in \mathbb{N}$ with $0 < 1/n \ll 1/\ell \ll \varepsilon, \gamma$, and let $s := 10\ell$. Let $x \in V(I)$ and let $\mathcal{C}$ be a collection of ℓ-cubes $C \subseteq I$ such that $|\mathcal{C}| \geq \gamma n^\ell$ and, for all $C \in \mathcal{C}$, we have $x \in V(C)$. For each $C \in \mathcal{C}$, let $\mathcal{M}_C$ denote the cube molecule of C in $\mathcal{Q}^n$. For any graph $G \subseteq \mathcal{Q}^n$, let*

$$B(G) := \{C \in \mathcal{C} : \mathcal{M}_C \text{ is bonded in } G\}.$$

Then, with probability at least $1 - 2^{-10n}$, we have $|B(\mathcal{Q}^n_\varepsilon)| \geq \gamma n^\ell/4$.

Proof. Let $G \sim \mathcal{Q}^n_\varepsilon$ and let $\mathcal{C}' := \{C - (N_I(x) \cup \{x\}) : C \in \mathcal{C}\}$. Given $C' \in \mathcal{C}'$, for each $i \in [2^s]$, let C'_i be the i-th clone of C'. We denote $\mathcal{M}_{C'} := C'_1 \cup \cdots \cup C'_{2^s}$, and refer to it as the molecule of C' in $\mathcal{Q}^n$. We say that $\mathcal{M}_{C'}$ is *bonded in* G if, for each $i \in [2^s]$, the graph G contains at least 100 edges between C'_i and C'_{i+1} whose endpoints in C'_i have odd parity and 100 edges whose endpoints in C'_i have even parity, where indices are taken cyclically. Note that, if $C' = C - (N_I(x) \cup \{x\})$ for some $C \in \mathcal{C}$ and C' is bonded in G, then C must be bonded in G. Moreover, $|V(C')| = |V(C)| - \ell - 1 > 9|V(C)|/10$. Therefore, similarly to the proof of Lemma 8.3, by Lemma 4.2 and a union bound we have that

$$\mathbb{P}[\mathcal{M}_{C'} \text{ is bondless in } G] \leq 2^{s - \varepsilon 2^\ell/100} \leq 1/2.$$

Let $X := |\{C' \in \mathcal{C}' : \mathcal{M}_{C'} \text{ is bonded in } G\}|$. It follows that

$$\mathbb{E}[|B(G)|] \geq \mathbb{E}[X] \geq \gamma n^\ell/2. \tag{9.1}$$

Let $V := \bigcup_{C' \in \mathcal{C}'} V(\mathcal{M}_{C'})$. Let $e_1, \ldots, e_m$ be an arbitrary ordering of the edges of $E := \bigcup_{i \in [2^s]} E_{\mathcal{Q}^n}(L_i \cap V, L_{i+1} \cap V)$. For each $j \in [m]$, let X_j be the indicator variable which takes value 1 if $e_j \in E(G)$ and 0 otherwise. Consider the edge-exposure martingale $Y_j := \mathbb{E}[X \mid X_1, \ldots, X_j]$ for $j \in [m]_0$. This is a Doob martingale with $Y_0 = \mathbb{E}[X]$ and $Y_m = X$.

We will now bound the differences $|Y_j - Y_{j-1}|$, for all $j \in [m]$. For each $i \in [\ell] \setminus \{1\}$, let $N^i(x) := \{y \in \bigcup_{C' \in \mathcal{C}'} V(C') : \mathrm{dist}(x, y) = i\}$. Let $E^i \subseteq E$ be the collection of edges $e = (u, v)$ where both u and v are clones of a vertex $z \in N^i(x)$. Note that the sets $E^2, \ldots, E^\ell$ partition E. Moreover, for each $j \in [m]$, if $e_j \in E^i$, then $|Y_j - Y_{j-1}| \leq n^{\ell - i}$. Furthermore,

$|E^i| = 2^s|N^i(x)| \leq 2^s n^i$ and, thus,

$$\sum_{j\in[m]} |Y_j - Y_{j-1}|^2 \leq \sum_{i=2}^{\ell} |E^i(x)| n^{2\ell-2i} \leq \sum_{i=2}^{\ell} 2^s n^i n^{2\ell-2i} = \mathcal{O}(n^{2\ell-2}).$$

Therefore, we can apply Lemma 4.5 and combine it with (9.1) to obtain

$$\mathbb{P}[|B(G)| < \gamma n^\ell/4] \leq \mathbb{P}[X < \gamma n^\ell/4] \leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]/2] \leq e^{-10n}. \qquad \square$$

The following observation will also be used repeatedly.

Remark 9.2. *Let $n, \ell \in \mathbb{N}$ and $\eta, \eta' \in (0,1)$ with $1/n \ll \eta' < \eta$. Let $x \in V(\mathcal{Q}^n)$. Let $\mathcal{C}$ be a collection of ℓ-cubes $C \subseteq \mathcal{Q}^n$ such that $x \in V(C)$ for all $C \in \mathcal{C}$ and $|\mathcal{C}| \geq \eta n^\ell$. Let $\mathcal{D}' \subseteq \mathcal{D}(\mathcal{Q}^n)$ be a set of directions with $|\mathcal{D}'| \leq \eta' n$. Then, there exists a cube $C \in \mathcal{C}$ with $\mathcal{D}(C) \cap \mathcal{D}' = \varnothing$.*

Proof. Observe that the number of ℓ-cubes $C \subseteq \mathcal{Q}^n$ with $x \in C$ and $\mathcal{D}(C) \cap \mathcal{D}' \neq \varnothing$ is at most $\eta' n \cdot n^{\ell-1}$. Since $\eta > \eta'$, we are done. $\square$

As discussed in Section 2.6, a crucial requirement for the proof of Theorem 1.8 will be that vertices of very low degree in $\mathcal{Q}^n_{1/2-\varepsilon}$ are few and far apart. Moreover, we will also require some more properties about the distribution of these vertices, and that, for all of them, we can find many candidates for special absorbing structures. We express all this information in the following definition.

Definition 9.3. *Let $n, s, \ell \in \mathbb{N}$ with $1/n \ll 1/s \leq 1/\ell$, and let $\varepsilon_1, \varepsilon_2, \gamma \in [0,1]$. Fix an ordering of the layers $L_1, \ldots, L_{2^s}$ of $\mathcal{Q}^n$ induced by any Hamilton cycle in $\mathcal{Q}^s$ (as defined in Section 8.1). Let $G \subseteq \mathcal{Q}^n$ be a spanning subgraph. For any $\varepsilon > 0$, let $\mathcal{U}(G,\varepsilon) \coloneqq \{x \in V(\mathcal{Q}^n) : d_G(x) < \varepsilon n\}$. Let $\mathcal{U} \subseteq V(\mathcal{Q}^n)$ be a set of size $|\mathcal{U}| \leq 2^{\varepsilon_2 n}$. We say that G is* $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U})$-robust *if the following properties are satisfied:*

- (R1) *$\mathcal{U}(G, \varepsilon_1) \subseteq \mathcal{U}$.*
- (R2) *For all $x \in \mathcal{U}$ and every $y \in B^{s+5\ell}_{\mathcal{Q}^n}(x) \setminus \{x\}$, we have $d_G(y) \geq \gamma n$.*
- (R3) *For all $x \in V(\mathcal{Q}^n)$, we have $|\mathcal{U} \cap B^{\gamma n}_{\mathcal{Q}^n}(x)| \leq 1$.*
- (R4) *For all $x \in \mathcal{U}$ and any distinct directions $a, b \in \mathcal{D}(\mathcal{Q}^n)$, there exists a collection $\mathfrak{C}(x,a,b)$ of (x,a,b)-consistent systems of paths in $G \cup \{\{x,x+a\},\{x,x+b\}\}$ which satisfies the following. Let L be the layer containing x.*
    - (R4.I) *Suppose $x+a, x+b \in V(L)$. Then, there exists a collection $\mathcal{D}^{(2)}(x,a,b)$ of disjoint pairs of distinct directions $c, d \in \mathcal{D}(L) \setminus \{a,b\}$ such that $|\mathcal{D}^{(2)}(x,a,b)| \geq \gamma n$ and, for every $(c,d) \in \mathcal{D}^{(2)}(x,a,b)$, there is a collection $\mathcal{D}^{(4)}(x,a,b,c,d)$ of disjoint 4-tuples of distinct directions in $\mathcal{D}(L) \setminus \{a,b,c,d\}$ with $|\mathcal{D}^{(4)}(x,a,b,c,d)| \geq \gamma n$ satisfying the following property: for each $(c,d) \in \mathcal{D}^{(2)}(x,a,b)$ and each $(d_1,d_2,d_3,d_4) \in \mathcal{D}^{(4)}(x,a,b,c,d)$, the (x,a,b)-consistent system of paths $CS(c,d,d_1,d_2,d_3,d_4) = (P_1,\ldots,P_6)$ defined as in Section 9.1 belongs to $\mathfrak{C}(x,a,b)$.*
    - (R4.II) *Suppose $x+a, x+b \notin V(L)$. Then, there exists a collection $\mathcal{D}^{(2)}(x,a,b)$ of disjoint pairs of distinct directions $d_1, d_2 \in \mathcal{D}(L)$ such that $|\mathcal{D}^{(2)}(x,a,b)| \geq \gamma n$ and, for every $(d_1,d_2) \in \mathcal{D}^{(2)}(x,a,b)$, the (x,a,b)-consistent system of paths $CS(d_1,d_2) = (P_1,P_2)$ defined as in Section 9.1 belongs to $\mathfrak{C}(x,a,b)$.*
    - (R4.III) *Suppose $x+a \notin V(L)$ and $x+b \in V(L)$. Then, there exists a set $\mathcal{D}(x,a,b)$ of directions $d_1 \in \mathcal{D}(L)$ such that $|\mathcal{D}(x,a,b)| \geq \gamma n$ and, for every $d_1 \in \mathcal{D}(x,a,b)$, there exists a collection $\mathcal{D}^{(2)}(x,a,b,d_1)$ of disjoint pairs of distinct directions in $\mathcal{D}(L) \setminus \{b, d_1\}$ with $|\mathcal{D}^{(2)}(x,a,b,d_1)| \geq \gamma n$ satisfying the following property: for each $d_1 \in \mathcal{D}^{(2)}(x,a,b)$ and each $(d_2,d_3) \in \mathcal{D}^{(2)}(x,a,b,d_1)$, the (x,a,b)-consistent system of paths $CS(d_1,d_2,d_3) = (P_1,P_2,P_3)$ defined as in Section 9.1 belongs to $\mathfrak{C}(x,a,b)$.*
- (R5) *Let $x_1 \coloneqq \{0\}^n$, $x_2 \coloneqq \{1\}^n$, $x_3 \coloneqq \{1\}^{\lceil n/2 \rceil}\{0\}^{n-\lceil n/2 \rceil}$ and $x_4 \coloneqq \{0\}^{\lceil n/2 \rceil}\{1\}^{n-\lceil n/2 \rceil}$. Then, for each $i \in [4]$ we have $\mathcal{U} \cap B^{s+\ell}_{\mathcal{Q}^n}(x_i) = \varnothing$.*

Lemma 9.4. *Let $1/n \ll 1/s \leq 1/\ell \ll \varepsilon_1 \ll \varepsilon \ll \varepsilon_2 \ll \gamma \ll 1/r$ with $n, s, \ell, r \in \mathbb{N}$. Then,* *(i)* *a.a.s.* $G \sim \mathcal{Q}^n_{1/2-\varepsilon}$ *is* $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U}(G, \varepsilon_1))$*-robust, and*

(ii) *given any* $\mathcal{U} \subseteq V(\mathcal{Q}^n)$ *with* $|\mathcal{U}| \leq 2^{\varepsilon_2 n}$ *and any* $H \subseteq \mathcal{Q}^n$ *which is* $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U})$*-robust, there exists an edge-decomposition* $H = H_1 \dot\cup \cdots \dot\cup H_r$ *such that for each* $i \in [r]$ *we have that* $H_i \subseteq H$ *is spanning and* $(s, \ell, \varepsilon_1/(2r), \varepsilon_2, \gamma/r^{10}, \mathcal{U})$*-robust.*

Proof. We begin with a proof of (i). Let $\mathcal{U} := \mathcal{U}(G, \varepsilon_1)$. For any $x \in V(\mathcal{Q}^n)$, we have that

$$\mathbb{P}[x \in \mathcal{U}] \leq \binom{n}{\varepsilon_1 n}(1/2+\varepsilon)^{n-\varepsilon_1 n} < 2^{-n+20\varepsilon n}. \tag{9.2}$$

It follows that $\mathbb{E}[|\mathcal{U}|] < 2^n 2^{-n+20\varepsilon n} = 2^{20\varepsilon n}$. Therefore, by Markov's inequality we have that

$$\mathbb{P}[|\mathcal{U}| \geq 2^{\varepsilon_2 n}] < 2^{20\varepsilon n}/2^{\varepsilon_2 n} < 2^{-\varepsilon_2 n/2},$$

so a.a.s. $|\mathcal{U}| \leq 2^{\varepsilon_2 n}$. (R1) holds trivially by the choice of $\mathcal{U}$. Furthermore, we have that

$$\left| \bigcup_{i \in [4]} B^{s+\ell}_{\mathcal{Q}^n}(x_i) \right| \leq 5n^{s+\ell},$$

so, by (9.2) and a union bound, (R5) also holds a.a.s.

To see that (R2) holds, fix $x \in V(\mathcal{Q}^n)$ and $y \in B^{s+5\ell}_{\mathcal{Q}^n}(x)$. Then, $\mathbb{E}[d_{G-\{x\}}(y)] = (1/2-\varepsilon)n \pm 1$. Thus, by Lemma 4.2,

$$\mathbb{P}[d_{G-\{x\}}(y) \leq \gamma n - 1] \leq 2^{-n/20}.$$

Therefore, by (9.2), we have that $\mathbb{P}[x \in \mathcal{U} \wedge d_{G-\{x\}}(y) \leq \gamma n - 1] \leq 2^{-n+20\varepsilon n - n/20} \leq 2^{-31n/30}$. A union bound over all $x \in V(\mathcal{Q}^n)$ and over all $y \in B^{s+5\ell}_{\mathcal{Q}^n}(x)$ shows that (R2) holds a.a.s.

The fact that (R3) holds a.a.s. can be shown similarly.

Finally, consider (R4). Let $x \in V(\mathcal{Q}^n)$, and suppose $x \in V(L)$, for some layer L. First, let $a, b \in \mathcal{D}(L)$ be distinct. We are going to show that a.a.s. we can find the desired collection of (x, a, b)-consistent systems of paths.

Recall that an (x, a, b)-consistent system of paths $(P_1, \ldots, P_6)$, as defined in Section 9.1, is determined uniquely by a 6-tuple of directions $(c, d, d_1, d_2, d_3, d_4)$. In order to show that (R4.I) is satisfied, we will first consider the directions c and d, and then the rest of the tuple. Recall that all (x, a, b)-consistent systems of paths contain the two edges $\{x, x+a\}$ and $\{x, x+b\}$. Then, once c and d are fixed, this determines a total of 6 more edges. The remaining 8 edges will be determined by the choice of (d_1, d_2, d_3, d_4).

Consider a collection $\mathcal{W}$ of disjoint pairs of distinct directions (c, d) with $c, d \in \mathcal{D}(L) \setminus \{a, b\}$ such that $|\mathcal{W}| \geq n/4$. For each $(c, d) \in \mathcal{W}$, let $E^*(c, d) \subseteq E(\mathcal{Q}^n)$ be the set of six edges of an (x, a, b)-consistent system of paths determined by these two directions. Observe that, since the pairs in $\mathcal{W}$ are disjoint, it follows that, for any distinct $(c, d), (c', d') \in \mathcal{W}$, we have $E^*(c, d) \cap E^*(c', d') = \varnothing$. Now let $\mathcal{W}_G := \{(c, d) \in \mathcal{W} : E^*(c, d) \subseteq E(G)\}$ and $X := |\mathcal{W}_G|$. We have that $\mathbb{E}[X] \geq (1/2-\varepsilon)^6 n/4$ and, by Lemma 4.2, it follows that $\mathbb{P}[X \leq \gamma n] \leq 2^{-\gamma n}$.

For each $(c, d) \in \mathcal{W}$, let $\mathcal{V}(c, d)$ be a collection of disjoint 4-tuples of distinct directions (d_1, d_2, d_3, d_4) with $d_1, d_2, d_3, d_4 \in \mathcal{D}(L) \setminus \{a, b, c, d\}$ such that $|\mathcal{V}(c, d)| \geq n/5$. For each $(d_1, d_2, d_3, d_4) \in \mathcal{V}(c, d)$, let $E^*(c, d, d_1, d_2, d_3, d_4) \subseteq E(\mathcal{Q}^n)$ be the set of eight edges of an (x, a, b)-consistent system of paths determined by $(c, d, d_1, d_2, d_3, d_4)$ which are not in $E^*(c, d) \cup \{\{x, x+a\}, \{x, x+b\}\}$. In particular, since the tuples in $\mathcal{V}(c, d)$ are disjoint, it follows that, for any distinct $(d_1, d_2, d_3, d_4), (d'_1, d'_2, d'_3, d'_4) \in \mathcal{V}(c, d)$, we have $E^*(c, d, d_1, d_2, d_3, d_4) \cap E^*(c, d, d'_1, d'_2, d'_3, d'_4) = \varnothing$. Now let $\mathcal{V}_G(c, d) := \{(d_1, d_2, d_3, d_4) \in \mathcal{V}(c, d) : E^*(c, d, d_1, d_2, d_3, d_4) \subseteq E(G)\}$, and let $Y(c, d) := |\mathcal{V}_G(c, d)|$. We then have that $\mathbb{E}[Y(c, d)] \geq (1/2-\varepsilon)^8 n/5$ and, again by Lemma 4.2, it follows that $\mathbb{P}[Y(c, d) \leq \gamma n] \leq 2^{-\gamma n}$. Thus, by a union bound, with probability at least $1 - 2^{-\gamma n/2}$, for every $(c, d) \in \mathcal{W}$ we have $Y(c, d) \geq \gamma n$.

Let $\mathcal{E}(x, a, b)$ be the event that $G \cup \{\{x, x+a\}, \{x, x+b\}\}$ contains a collection $\mathfrak{C}(x, a, b)$ of (x, a, b)-consistent systems of paths satisfying (R4.I). By combining all the above, it follows that

$$\mathbb{P}[\mathcal{E}(x, a, b)] \geq 1 - 2^{-\gamma n/4}. \tag{9.3}$$

The same bound can be proved for the cases where $a \notin \mathcal{D}(L), b \in \mathcal{D}(L)$ and $a, b \notin \mathcal{D}(L)$. Observe that, for any $x \in V(\mathcal{Q}^n)$ and $a, b \in \mathcal{D}(\mathcal{Q}^n)$, the event $\mathcal{E}(x, a, b)$ is independent of the event that $x \in \mathcal{U}$. Now, by combining (9.3) with (9.2) and a union bound over all choices of $a, b \in \mathcal{D}(\mathcal{Q}^n)$, and then a union bound over all $x \in V(\mathcal{Q}^n)$, we conclude that (R4) holds a.a.s.

The proof of (ii) is similar. Let $H \subseteq \mathcal{Q}^n$ be given and consider a random partition of $E(H)$ into r parts $H_1, \dots, H_r$, in such a way that each edge is assigned to one of the parts uniformly and independently of all other edges. For each $x \in V(\mathcal{Q}^n) \setminus \mathcal{U}$, let $\mathcal{B}(x)$ be the event that there exists some $i \in [r]$ such that $d_{H_i}(x) < d_H(x)/(2r)$. Observe that, if $\overline{\mathcal{B}(x)}$ holds for all $x \in V(\mathcal{Q}^n) \setminus \mathcal{U}$, then no vertex outside $\mathcal{U}$ will be contained in $\mathcal{U}(H_i, \varepsilon_1/(2r))$ for any $i \in [r]$. It would then follow that (R1), (R2), (R3) and (R5) all hold with the desired constants for each H_i. Fix $x \in V(\mathcal{Q}^n) \setminus \mathcal{U}$ and $i \in [r]$. Let $X := d_{H_i}(x)$. Then, $\mathbb{E}[X] = d_H(x)/r \geq \varepsilon_1 n/r$. Thus, by Lemma 4.2, $\mathbb{P}[X \leq \mathbb{E}[X]/2] \leq e^{-\varepsilon_1^2 n}$. A union bound over all $i \in [r]$ shows that $\mathbb{P}[\mathcal{B}(x)] \leq re^{-\varepsilon_1^2 n} \leq e^{-\varepsilon_1^3 n}$.

We now consider the property (R4). For each $x \in \mathcal{U}$, let $\mathcal{B}(x)$ be the event that there exist $i \in [r]$ and distinct directions $a, b \in \mathcal{D}(\mathcal{Q}^n)$ such that (R4) does not hold for H_i with γ/r^{10} playing the role of γ.

Fix $x \in \mathcal{U}$, $i \in [r]$ and distinct directions $a, b \in \mathcal{D}(\mathcal{Q}^n)$. Similarly as in (i), using Lemma 4.2 one can show that the probability that H_i does not satisfy (R4) for x with γ/r^{10} playing the role of γ is at most $2^{-\gamma^2 n} + 2^{-\gamma^3 n}$. Therefore, by a union bound over all choices of $a, b \in \mathcal{D}(\mathcal{Q}^n)$ and over each $i \in [r]$, we have that $\mathbb{P}[\mathcal{B}(x)] \leq rn^2(2^{-\gamma^2 n} + 2^{-\gamma^3 n}) \leq e^{-\varepsilon_1^3 n}$.

Finally, we are interested in the event where $\mathcal{B}(x)$ does not occur for any $x \in V(\mathcal{Q}^n)$. We will invoke Lemma 4.8. Note that each event $\mathcal{B}(x)$ is mutually independent of all but at most n^{10} other events. We have that $\mathbb{P}[\mathcal{B}(x)] \leq e^{-\varepsilon_1^3 n}$ for every $x \in V(\mathcal{Q}^n)$ and $e \cdot e^{-\varepsilon_1^3 n}(n^{10} + 1) < 1$, so by Lemma 4.8 there exists an edge-decomposition of H with the desired properties. $\square$

Finally, we need to show a result analogous to Lemma 8.5 for robust graphs, that is, that scant molecules are not too clustered. Recall that $\mathit{Res}(\mathcal{Q}^n, \delta)$ was defined in Section 7.1.

Lemma 9.5. *Let $0 < 1/n \ll 1/C \ll \varepsilon_1, \varepsilon_2 \ll \gamma, \delta \leq 1$ and $1/n \ll 1/s \leq 1/\ell$, where $n, C, s, \ell \in \mathbb{N}$. Let $H \subseteq \mathcal{Q}^n$ and $\mathcal{U} \subseteq V(\mathcal{Q}^n)$ be such that H is $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U})$-robust. Let $\mathcal{U}_I \subseteq V(I)$ be the set of vertices $u \in V(I)$ such that $\mathcal{U}$ contains some clone of u. For each $v \in V(I)$ and each $i \in [2^s]$, let v_i be the i-th clone of v, and let $\mathcal{M}_v := \{v_i : i \in [2^s]\}$. Let $R \sim \mathit{Res}(I, \delta)$ and, for each $i \in [2^s]$, let R_i be the i-th clone of R. Let*

$$B := \{v \in V(I) : \text{there exists } i \in [2^s] \text{ with } v_i \notin \mathcal{U} \text{ and } e_H(v_i, R_i) < \varepsilon_1 \delta n/4\}.$$

Let $\mathcal{E}_1$ be the event that there exists some $u \in V(I)$ such that $|B_I^{10\ell}(u) \cap B| \geq C$. Let $\mathcal{E}_2$ be the event that there exists some $u \in \mathcal{U}_I$ such that $|B_I^{5\ell}(u) \cap B| \geq 1$. Then, $\mathbb{P}[\mathcal{E}_1 \vee \mathcal{E}_2] \leq 1/n$.

Proof. Let $u \in V(I)$ and let $D \subseteq B_I^{10\ell}(u) \setminus \mathcal{U}_I$ be a set of C vertices. Let $D' := \bigcup_{x,y \in D : x \neq y} N_I(x) \cap N_I(y)$. Since any pair of distinct vertices in I share at most two neighbours, we have that $|D'| \leq 2\binom{C}{2}$. For each $i \in [2^s]$, we denote the i-th clone of D' by D'_i, and let $R'_i := R_i \setminus D'_i$.

For each $x \in V(\mathcal{Q}^n)$, let $i(x)$ be the unique index $i \in [2^s]$ such that $x \in V(L_i)$. Observe that, by (R1), we have $e_H(x, V(L_{i(x)})) > 2\varepsilon_1 n/3$ for every $x \in V(\mathcal{Q}^n) \setminus \mathcal{U}$. For each $x \in V(\mathcal{Q}^n)$, let $\mathcal{E}_x$ be the event that $e_H(x, R_{i(x)}) \leq \varepsilon_1 \delta n/4$, and let $\mathcal{E}'_x$ be the event that $e_H(x, R'_{i(x)}) \leq \varepsilon_1 \delta n/4$. It follows by Lemma 4.2 that $\mathbb{P}[\mathcal{E}'_x] \leq e^{-\varepsilon_1 \delta n/16}$ for all $x \in V(\mathcal{Q}^n) \setminus \mathcal{U}$. For each $v \in V(I)$, let $\mathcal{E}_v$ and $\mathcal{E}'_v$ be the events that there exists $i \in [2^s]$ with $v_i \notin \mathcal{U}$ such that $\mathcal{E}_{v_i}$ and $\mathcal{E}'_{v_i}$ hold, respectively. By a union bound, it follows that $\mathbb{P}[\mathcal{E}'_v] \leq 2^s e^{-\varepsilon_1 \delta n/16}$ for all $v \in V(I)$. Finally, let $\mathcal{E}_D$ and $\mathcal{E}'_D$ be the events that $\mathcal{E}_v$ and $\mathcal{E}'_v$, respectively, hold for every $v \in D$. Note that the events in the collection $\{\mathcal{E}'_v : v \in V(I)\}$ are mutually independent. Furthermore, since the event $\mathcal{E}_x$ implies $\mathcal{E}'_x$ for all $x \in V(\mathcal{Q}^n)$, we have that

$$\mathbb{P}[\mathcal{E}_D] \leq \mathbb{P}[\mathcal{E}'_D] \leq (2^s e^{-\varepsilon_1 \delta n/16})^C < e^{-5n}.$$

By a union bound over all $u \in V(I)$ and over all choices of D, we have $\mathbb{P}[\mathcal{E}_1] \leq e^{-n}$.

Consider now any $u \in \mathcal{U}_I$. Observe that, if $v \in B_I^{5\ell}(u)$, then for every $i, j \in [2^s]$ we have that $\operatorname{dist}(u_i, v_j) \leq 5\ell + s$. Therefore, by (R2), for all $v \in B_I^{5\ell}(u)$ and $i \in [2^s]$ such that $v_i \notin \mathcal{U}$, we have $d_H(v_i) \geq \gamma n$. For each $v \in B_I^{5\ell}(u)$ and each $i \in [2^s]$ with $v_i \notin \mathcal{U}$, let $\mathcal{F}_{v_i}$ be the event that

$e_H(v_i, R_i) \leq \varepsilon_1\delta n/4$, and let $\mathcal{F}_v$ be the event that there exists some $i \in [2^s]$ with $v_i \notin \mathcal{U}$ such that $\mathcal{F}_{v_i}$ holds. By Lemma 4.2 and a union bound, it follows that $\mathbb{P}[\mathcal{F}_v] \leq 2^{-\gamma\delta n/16}$. Then, by a union bound over all $u \in \mathcal{U}_I$ and $v \in B_I^{5\ell}(u)$,

$$\mathbb{P}[\mathcal{E}_2] \leq |\mathcal{U}_I| \cdot |B_I^{5\ell}(\mathcal{U}_I)| \cdot \mathbb{P}[\mathcal{F}_v] \leq 2n^{5\ell}2^{\varepsilon_2 n}2^{-\gamma\delta n/16}. \qquad \square$$

9.3. **Hamilton cycles in robust subgraphs of the cube.** It will be useful to prove the following result, which (together with Lemma 9.4) directly implies the case $k = 2$ of Theorem 1.8 (by choosing H in Theorem 1.8 to play the role of H' in Theorem 9.6 and F to be empty). As with Theorem 8.1, the formulation of Theorem 9.6 is designed so that the case $k > 2$ can be derived easily (see Section 9.4). To state the result, we need the following notation.

Given any integers $s \leq n$, we say that $d \in \mathcal{D}(\mathcal{Q}^n)$ is an *s-direction* if, for any $x \in V(\mathcal{Q}^n)$, x and $x + d$ differ only by one of the first s coordinates. Given a graph $F \subseteq \mathcal{Q}^n$, a set $\mathcal{U} \subseteq V(\mathcal{Q}^n)$ and $\ell, s \in \mathbb{N}$, we say that F is *$(\mathcal{U}, \ell, s)$-good* if, for each $x \in \mathcal{U}$, the set $E_F(x) := \{e \in E(F) : e \cap N_{\mathcal{Q}^n}(x) \neq \varnothing\}$ satisfies that, for each $d \in \mathcal{D}(\mathcal{Q}^n)$ which is not an s-direction, we have $|\{e \in E_F(x) : \mathcal{D}(e) = d\}| \leq n/\ell$. Thus, a graph is good if locally the directions of its edges are not too correlated (ignoring s-directions). The goodness of the 'forbidden' graph F below will be needed when finding the special absorbing structures (see Step 11).

Theorem 9.6. *Let $0 < 1/\ell \ll \varepsilon_1 \ll \varepsilon_2 \ll \gamma \leq 1$ and $1/\ell \ll \eta, 1/c \leq 1$, with $\ell \in \mathbb{N}$. Let $s := 10\ell$ and $n \in \mathbb{N}$. Then, there exists $\Phi \in \mathbb{N}$ such that the following holds.*

Let $H \subseteq \mathcal{Q}^n$ and $\mathcal{U} \subseteq V(\mathcal{Q}^n)$ be such that H is an $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U})$-robust subgraph, and let $Q \sim \mathcal{Q}^n_\eta$. Then, a.a.s. there is a $(\mathcal{U}, \ell^2, s)$-good subgraph $Q' \subseteq Q$ with $\Delta(Q') \leq \Phi$ such that

- *for every $H' \subseteq \mathcal{Q}^n$, where $d_{H'}(x) \geq 2$ for every $x \in \mathcal{U}$, and*
- *for every $F \subseteq \mathcal{Q}^n$ with $\Delta(F) \leq c\Phi$ which is $(\mathcal{U}, \ell, s)$-good,*

we have that $((H \cup Q) \setminus F) \cup H' \cup Q'$ contains a $(\mathcal{U}, \ell^2, s)$-good Hamilton cycle C such that, for all $x \in \mathcal{U}$, both edges of C incident to x belong to H'.

As we have already discussed, the proof of Theorem 9.6 builds on that of Theorem 8.1. Thus, we will avoid repeating all the details which are analogous, and we will often refer back the proof of Theorem 8.1. A full sequential proof can be found in [28].

Proof of Theorem 9.6. Let $1/D, \delta' \ll 1$, let $\varepsilon_1 \ll \varepsilon_2 \ll \gamma, \delta' \leq 1$, and let

$$0 < 1/n_0 \ll \delta, \lambda \ll 1/\ell \ll 1/k^*, \alpha' \ll \beta, 1/S' \ll 1/c, 1/D, \eta, \varepsilon_1, \delta',$$

where $n_0, \ell, k^*, S', D \in \mathbb{N}$. Our proof assumes that n tends to infinity; in particular, $n \geq n_0$. Let $\Phi := 60\ell^4$ and $\Psi := c\Phi$.

We define the layers $L_1, \ldots, L_{2^s}$ of $\mathcal{Q}^n$, the intersection graph I and, for each $G \subseteq \mathcal{Q}^n$, the graphs $I(G)$ and G_I as in the proof of Theorem 8.1. Similarly, for any layer L and $\mathcal{G} \subseteq I$, we keep the notation $\mathcal{G}_L$ for the clone of $\mathcal{G}$ in L.

Let $\mathcal{U}$ be as in the statement of Theorem 9.6. In particular, $\mathcal{U}(H, \varepsilon_1) \subseteq \mathcal{U}$ by (R1). Let $\mathcal{U}_I \subseteq V(I)$ be the set of vertices $x \in V(I)$ such that there is some clone x' of x with $x' \in \mathcal{U}$. Note that, by property (R2), for each $x \in \mathcal{U}_I$, there is exactly one clone x' of x with $x' \in \mathcal{U}$.

For each $i \in [8]$, let $\eta_i := \eta/8$ and $G_i \sim \mathcal{Q}^n_{\eta_i}$, where these graphs are chosen independently. We have that $\bigcup_{i=1}^8 G_i \sim \mathcal{Q}^n_{\eta'}$ for some $\eta' < \eta$, so it suffices to show that a.a.s. there is a $(\mathcal{U}, \ell^2, s)$-good subgraph $Q' \subseteq \bigcup_{i=1}^8 G_i$ with $\Delta(Q') \leq \Phi$ and such that, for every $H' \subseteq \mathcal{Q}^n$, where $d_{H'}(x) \geq 2$ for every $x \in \mathcal{U}$, and every $F \subseteq \mathcal{Q}^n$ with $\Delta(F) \leq \Psi$ which is $(\mathcal{U}, \ell, s)$-good, the graph $((H \cup \bigcup_{i=1}^8 G_i) \setminus F) \cup H' \cup Q'$ contains a Hamilton cycle of the form in the statement of the theorem. We now split our proof into several steps.

Step 1. Finding a tree and a reservoir. As in the proof of Theorem 8.1, consider the probability space $\Omega := \mathcal{Q}^{n-s}_{\eta_1^{2^s}} \times Res(\mathcal{Q}^{n-s}, \delta')$ and let $R \sim Res(I, \delta')$. Let $\mathcal{E}_1$ be the event that there exists a tree $T \subseteq I(G_1) - (R \cup B_I^5(\mathcal{U}_I))$ such that the following hold:

(TR1) $\Delta(T) < D$, and

(TR2) for all $x \in V(I) \setminus B_I^5(\mathcal{U}_I)$, we have that $|N_I(x) \cap V(T)| \geq 4(n - s)/5$.

Note that, by (R3), for all $x, y \in \mathcal{U}_I$ we have that $\mathrm{dist}(x,y) \geq \gamma n/2$. Furthermore, by (R5), we have that, if we see $x_1 := \{0\}^{n-s}$, $x_2 := \{1\}^{n-s}$, $x_3 := \{1\}^{\lceil (n-s)/2 \rceil}\{0\}^{n-s-\lceil (n-s)/2 \rceil}$ and $x_4 := \{0\}^{\lceil (n-s)/2 \rceil}\{1\}^{n-s-\lceil (n-s)/2 \rceil}$ as vertices of I, then $\mathcal{U}_I \cap B_I^{\ell}(x_i) = \varnothing$ for all $i \in [4]$. Thus, it follows from Theorem 7.1, with $n-s$, D, δ', $1/5$, $\eta_1^{2^s}$, $\gamma/2$, 5 and $\mathcal{U}_I$ playing the roles of n, D, δ, ε', ε, γ, k and $\mathcal{A}$, respectively, that $\mathbb{P}_\Omega[\mathcal{E}_1] = 1 - o(1)$.

Step 2. Identifying scant molecules. For each $v \in V(I)$, let $\mathcal{M}_v$ denote the vertex molecule of all clones of v in $\mathcal{Q}^n$. We say $\mathcal{M}_v$ is *scant* if there exist some layer L and some vertex $x \in V(\mathcal{M}_v \cap L) \setminus \mathcal{U}$ such that $e_H(x, R_L) < \varepsilon_1 \delta' n/10$, where R_L is the clone of R in L. Let $\mathcal{E}_2$ be the event that there exists some $x \in V(I)$ such that there are at least S' vertices $v \in B_I^{10\ell}(x)$ with the property that $\mathcal{M}_v$ is scant. Let $\mathcal{E}_3$ be the event that there exist $x \in \mathcal{U}_I$ and $v \in B_I^{5\ell}(x)$ such that $\mathcal{M}_v$ is scant. It follows from Lemma 9.5 with S' and δ' playing the roles of C and δ that $\mathbb{P}_\Omega[\mathcal{E}_2 \vee \mathcal{E}_3] = o(1)$. Let $\mathcal{E}_1^* := \mathcal{E}_1 \wedge \overline{\mathcal{E}_2} \wedge \overline{\mathcal{E}_3}$. Then, $\mathbb{P}_\Omega[\mathcal{E}_1^*] = 1 - o(1)$.

Condition on $\mathcal{E}_1^*$ holding. Then, there exist a set $R \subseteq V(I)$ and a tree $T \subseteq I(G_1) - (R \cup B_I^5(\mathcal{U}_I))$ such that the following hold:

(T1) $\Delta(T) < D$;
(T2) for all $x \in V(I) \setminus B_I^5(\mathcal{U}_I)$, we have that $|N_I(x) \cap V(T)| \geq 4(n-s)/5$;
(T3) for every $x \in V(I)$, we have $|\{v \in B_I^{10\ell}(x) : \mathcal{M}_v \text{ is scant}\}| \leq S'$, and
(T4) for every $x \in \mathcal{U}_I$ and every $v \in B_I^{5\ell}(x)$, we have that $\mathcal{M}_v$ is not scant.

Recall this implies clones of T and R satisfying (T1)–(T4) exist simultaneously in each layer of G_1.

Step 3: Finding clustered robust matchings for each molecule. As in the proof of Theorem 8.1, the aim is to find auxiliary matchings which can later be used to pair up vertices which need to be absorbed. In the proof of Theorem 8.1, we were able to carry out this pairing within each slice. However, we cannot guarantee that the vertices of each special absorbing structure of Type II and III will lie within a single slice. There will be an even number of these vertices (zero, two or four) within each vertex molecule, but there might be exactly one within a slice of this molecule, making it impossible to pair up vertices within a slice. Thus we now consider the entire vertex molecule when finding the auxiliary matching (rather than each slice separately). This would normally make it much more difficult to link up vertices of the skeleton in Step 18. We are able to overcome this problem by considering matchings which are 'clustered', i.e. the endpoints of each matching edge either lie in a common slice or in two consecutive slices.

Let $q := 2^{10Dk^*}$ and $t := 2^s/q$. For each $j \in [t]$, let $S_j := \bigcup_{i=(j-1)q+1}^{jq} L_i$. Given any molecule $\mathcal{M}$, we define the slices $\mathcal{S}_j(\mathcal{M}) := S_j \cap \mathcal{M}$. We denote by $\mathcal{S}(\mathcal{M})$ the collection of all these slices of $\mathcal{M}$.

Let $V_{\mathrm{sc}} \subseteq V(I)$ be the set of all vertices $x \in V(I)$ such that $\mathcal{M}_x$ is scant. In particular, by (T4) we have that $V_{\mathrm{sc}} \cap \mathcal{U}_I = \varnothing$. Recall $G_2 \sim \mathcal{Q}^n_{\eta_2}$. For each $v \in V(I) \setminus (V_{\mathrm{sc}} \cup \mathcal{U}_I)$, we define auxiliary bipartite graphs $H(v) := (V(\mathcal{M}_v), N_I(v), E_H)$ and $G_2(v) := (V(\mathcal{M}_v), N_I(v), E_{G_2})$, where E_H and E_{G_2} are defined as in Step 3 of the proof of Theorem 8.1 for vertices $v \in V(I) \setminus V_{\mathrm{sc}}$ (but now the first vertex class of $H(v)$ and $G_2(v)$ is $V(\mathcal{M}_v)$ rather than $V(\mathcal{S})$ for some slice $\mathcal{S}$). For each $v \in V_{\mathrm{sc}}$, we similarly define $H(v)$ and $G_2(v)$ as we did in Step 3 of the proof of Theorem 8.1, with the same modifications as above.

For each $v \in \mathcal{U}_I$, we also define two such auxiliary graphs. Let $H(v) := (V(\mathcal{M}_v), N_I(v), E_H^*)$, where E_H^* is defined as follows. Consider $v' \in V(\mathcal{M}_v)$ and let $L^{v'}$ be the layer which contains v'. Let $w \in N_I(v)$, and let $w_{L^{v'}}$ be the clone of w in $L^{v'}$. Then, if $v' \in \mathcal{U}$, we add $\{v', w\}$ to E_H^* (these can be seen as purely auxiliary edges, and we will ignore their effect later). Otherwise, $\{v', w\} \in E_H^*$ if and only if $w \in R$ and $\{v', w_{L^{v'}}\} \in E(H)$. In particular, $d_{H(v)}(v') \geq \varepsilon_1 \delta' n/10$ for all $v' \in V(\mathcal{M}_v)$ since $\mathcal{M}_v$ is a not a scant molecule. We define $G_2(v) := (V(\mathcal{M}_v), N_I(v), E_{G_2})$, where $\{v', w\} \in E_{G_2}$ if and only if $\{v', w_{L^{v'}}\} \in E(G_2)$.

For every $v \in V(I)$ and every slice $\mathcal{S} \in \mathcal{S}(\mathcal{M}_v)$, note that the partition of $V(\mathcal{S})$ into vertices of even and odd parity is a balanced bipartition. Define the graph $\Gamma^{\beta}_{H(v),G_2(v)}(V(\mathcal{M}_v))$ as in Section 5.1. Thus, $V(\Gamma^{\beta}_{H(v),G_2(v)}(V(\mathcal{M}_v))) = V(\mathcal{M}_v)$. Furthermore, by definition, (RM) given any $w_1, w_2 \in V(\mathcal{M}_v)$, we have that $\{w_1, w_2\} \in E(\Gamma^{\beta}_{H(v),G_2(v)}(V(\mathcal{M}_v)))$ if and only if $|N_{H(v)}(w_1) \cap N_{G_2(v)}(w_2)| \geq \beta(n-s)$ or $|N_{G_2(v)}(w_1) \cap N_{H(v)}(w_2)| \geq \beta(n-s)$.

For each $i \in [t]$, let $\mathfrak{A}_i(v)$ consist of all vertices of $V(\mathcal{S}_i(\mathcal{M}_v))$ of even parity, and let $\mathfrak{B}_i(v)$ consist of those of odd parity. By applying Corollary 5.3 with $d = 100D$, $\alpha = \varepsilon_1\delta'/10$, $\varepsilon = \eta_2$, $n = n-s$, $k = q = 2^{10Dk^*}$, $\beta = \beta$, $t = t$, $G = H(\mathcal{M}_v)$ and $V(\mathcal{S}_1(\mathcal{M}_v)) \cup \ldots \cup V(\mathcal{S}_t(\mathcal{M}_v))$ as a partition of $V(\mathcal{M}_v)$, we obtain that, with probability at least $1 - 2^{-9(n-s)} \geq 1 - 2^{-8n}$, the graph $\Gamma^{\beta}_{H(v),G_2(v)}(V(\mathcal{M}_v))$ is $100D$-robust-parity-matchable with respect to $(\bigcup_{i=1}^t \mathfrak{A}_i(v), \bigcup_{i=1}^t \mathfrak{B}_i(v))$ clustered in $(V(\mathcal{S}_1(\mathcal{M}_v)), \ldots, V(\mathcal{S}_t(\mathcal{M}_v)))$.

By a union bound over all $v \in V(I)$, a.a.s. $\Gamma^{\beta}_{H(v),G_2(v)}(V(\mathcal{M}_v))$ is $100D$-robust-parity-matchable with respect to $(\bigcup_{i=1}^t \mathfrak{A}_i(v), \bigcup_{i=1}^t \mathfrak{B}_i(v))$ clustered in $(V(\mathcal{S}_1(\mathcal{M}_v)), \ldots, V(\mathcal{S}_t(\mathcal{M}_v)))$ for every $v \in V(I)$. We condition on this event holding and call it $\mathcal{E}_2^*$. Thus, for each $v \in V(I)$ and each set $S \subseteq V(\mathcal{M}_v)$ with $|S| \leq 100D$ which contains as many odd vertices as even vertices, there exists a perfect matching $\mathfrak{M}(\mathcal{M}_v, S)$ in the bipartite graph with parts consisting of the even and odd vertices of $V(\mathcal{M}_v) \setminus S$, respectively, and edges given by $\Gamma^{\beta}_{H(v),G_2(v)}(V(\mathcal{M}_v))$, with the property that, for each $e = \{w_{\mathrm{e}}, w_{\mathrm{o}}\} \in \mathfrak{M}(\mathcal{M}_v, S)$, if $w_{\mathrm{e}} \in V(\mathcal{S}_i(\mathcal{M}_v))$ for some $i \in [t]$, then $w_{\mathrm{o}} \in V(\mathcal{S}_{i-1}(\mathcal{M}_v)) \cup V(\mathcal{S}_i(\mathcal{M}_v)) \cup V(\mathcal{S}_{i+1}(\mathcal{M}_v))$ (where indices are taken cyclically). When we apply this in Step 15, we will have $\mathcal{U} \cap V(\mathcal{M}_v) \subseteq S$.

For each $v \in V(I) \setminus \mathcal{U}_I$, we denote by $\mathfrak{M}(v)$ the set of edges contained in the union (over all S) of the matchings $\mathfrak{M}(\mathcal{M}_v, S)$ (without multiplicity). For each $v \in \mathcal{U}_I$, we let $u(v)$ be the unique vertex in $\mathcal{M}_v$ such that $u(v) \in \mathcal{U}$, and we denote by $\mathfrak{M}(v)$ the set of edges contained in the union of the matchings $\mathfrak{M}(\mathcal{M}_v, S)$ over all S such that $u(v) \in S$ (again, without multiplicity). Furthermore, for each $v \in V(I)$ and each $e = \{w_{\mathrm{e}}, w_{\mathrm{o}}\} \in \mathfrak{M}(v)$, we let $N(e) \coloneqq (N_{H(v)}(w_{\mathrm{e}}) \cap N_{G_2(v)}(w_{\mathrm{o}})) \cup (N_{G_2(v)}(w_{\mathrm{e}}) \cap N_{H(v)}(w_{\mathrm{o}}))$. By (RM), we have $|N(e)| \geq \beta(n-s) \geq \beta n/2$. Let $K \coloneqq \max_{v \in V(I)} |\mathfrak{M}(v)|$. Thus, $K \leq \binom{2^s}{2}$.

Step 4: Obtaining an appropriate cube factor via the nibble. For each $x \in V(I)$, define the sets $A_1(x), \ldots, A_K(x)$ as in Step 4 of the proof of Theorem 8.1. Recall that $G_3 \sim \mathcal{Q}^n_{\eta_3}$ and $I(G_3) \sim \mathcal{Q}^{n-s}_{\eta_3^{2^s}}$. Apply Theorem 6.6 to the graph $I(G_3)$, so that a.a.s. we obtain a collection $\mathcal{C}$ of vertex-disjoint copies of $\mathcal{Q}^\ell$ in $I(G_3)$ satisfying (N1)–(N3) in the proof of Theorem 8.1. Condition on the event that such a collection $\mathcal{C}$ exists and call it $\mathcal{E}_3^*$.

Step 5: Absorption cubes. Recall $G_4 \sim \mathcal{Q}^n_{\eta_4}$. We define the event $\mathcal{E}_4^*$ in the same way as in Step 5 of the proof of Theorem 8.1. Upon conditioning on this event, for each $x \in V(I)$ and $i \in [K]$, we construct the matching $M'(A_i(x))$ in exactly the same way as well.

Consider $A_i(x)$, for some $x \in V(I)$ and $i \in [K]$, and let x_1, x_2 be the clones of x which correspond to (x, i). For each $j \in [2]$, let L^j be the layer containing x_j. Similarly as in the proof of Theorem 8.1, the following holds.

(AB1) For each edge $(\hat{e}, \hat{e}') \in M'(A_i(x))$, there is an absorbing ℓ-cube pair (C^l, C^r) for x in I such that, for each $j \in [2]$, the clone (C^l_j, C^r_j) of (C^l, C^r) in L^j is an absorbing ℓ-cube pair for x_j in $H \cup G_2 \cup G_3 \cup G_4$. In particular, the edge joining the left absorber tip to the third absorber vertex lies in G_4. Moreover, $C^l, C^r \in \mathcal{C}_x(A_i(x)) \subseteq \mathcal{C}$ and (C^l, C^r) has left and right absorber tips $x + \hat{e}'$ and $x + \hat{e}$, respectively. Furthermore, for each $x \in V(I) \setminus V_{\mathrm{sc}}$, these tips lie in R. We refer to (C^l_1, C^r_1) and (C^l_2, C^r_2) as the absorbing ℓ-cube pairs for x_1 and x_2 *associated* with $(\hat{e}, \hat{e}')$.

Recall by the construction in Step 3 that, for each $x \in V(I)$ and $i \in [K]$, we have $x_1, x_2 \notin \mathcal{U}$. In particular, this means that we do not choose absorbing ℓ-cube pairs for the vertices in $\mathcal{U}$ and, thus, the auxiliary edges at the vertices in $\mathcal{U}$ introduced in the definition of the sets E^*_H in Step 3 will never be used. As discussed before, the vertices in $\mathcal{U}$ will instead be incorporated into the Hamilton cycle using the special absorbing structures introduced in Section 9.1.

Step 6: Removing bondless molecules. Recall $G_5 \sim \mathcal{Q}^n_{\eta_5}$. We define the collections $\mathcal{C}'$, $\mathcal{C}''$ and $\mathcal{C}_{\mathrm{bs}}$ in the same way as in Step 6 of the proof of Theorem 8.1, and we also define the event $\mathcal{E}_5^*$ in the same way. We condition on this event and, as before, for each $x \in V(I)$ and each $i \in [K]$, we modify the matching $M'(A_i(x))$ into a matching $M(A_i(x))$ described in (AB2).

Step 7: Extending the tree T. For each $x \in V(I) \setminus B^5_I(\mathcal{U}_I)$, let $Z(x) := N_I(x) \cap V(T) \cap (\bigcup_{C\in\mathcal{C}''} V(C))$. As in the proof of Theorem 8.1, we have that

$$|Z(x)| \geq 3n/4. \tag{9.4}$$

Recall $G_6 \sim \mathcal{Q}^n_{\eta_6}$. We apply Theorem 7.19 in the same way as in the proof of Theorem 8.1, but this time with 2ℓ and $B^5_I(\mathcal{U}_I)$ playing the roles of ℓ and W, respectively. Combining this with (T1), we conclude that a.a.s. there exists a tree T' such that $T \subseteq T' \subseteq (I(G_6) \cup T) - B^5_I(\mathcal{U}_I)$ and the following hold:

(ET1) $\Delta(T') < D + 1$;
(ET2) for all $x \in V(I)$, we have that $|B^{2\ell}_I(x) \setminus (V(T') \cup B^5_I(\mathcal{U}_I))| \leq n^{3/4}$;
(ET3) for each $x \in V(T') \cap R$, we have that $d_{T'}(x) = 1$ and the unique neighbour x' of x in T' is such that $x' \in Z(x)$.

We condition on the above event holding and call it $\mathcal{E}_6^*$.

As in the proof of Theorem 8.1, for each $x \in V(I)$ and each $i \in [K]$, we now redefine the set $M(A_i(x))$ so that (AB3) holds. It again follows that

$$|M(A_i(x))| \geq n/\ell^2 - n^{3/4} \geq 4n/\ell^3. \tag{9.5}$$

Step 8. Consistent systems of paths and cubes. Recall $G_7 \sim \mathcal{Q}^n_{\eta_7}$. For each $v \in V(I)$, let $\mathcal{C}(v) := \{C \subseteq I(G_7) : C \cong \mathcal{Q}^\ell, v \in V(C)\}$. Let $\mathcal{P} := \{v \in V(I) : |\mathcal{C}(v)| \geq \lambda n^\ell\}$. By Remark 5.13 (applied with $r = 10$ and $\eta_7^{2^s}$ playing the role of ε), the following property holds a.a.s.

(D1) For every $v \in V(I)$ we have $|B^{10}_I(v) \setminus \mathcal{P}| \leq n^{7/8}$.

For each $v \in \mathcal{P}$, a straightforward application of Lemma 9.1 with λ and $\mathcal{C}(v)$ playing the roles of γ and $\mathcal{C}$, respectively, shows that the following holds with probability at least $1 - 2^{-10n}$: there exists a subcollection $\mathcal{C}'(v) \subseteq \mathcal{C}(v)$, with $|\mathcal{C}'(v)| \geq \lambda n^\ell/4$, with the property that, for every $C \in \mathcal{C}'(v)$, the molecule $\mathcal{M}_C$ is bonded in G_7. By a simple union bound over all vertices in $\mathcal{P}$, we obtain that the following holds a.a.s.

(D2) For every $v \in \mathcal{P}$, there exists a collection $\mathcal{C}'(v) \subseteq \mathcal{C}(v)$ with $|\mathcal{C}'(v)| \geq \lambda n^\ell/4$ such that, for every $C \in \mathcal{C}'(v)$, we have that $\mathcal{M}_C$ is bonded in G_7.

Condition on the event that (D1) and (D2) hold and call it $\mathcal{E}_7^*$.

We will show that we may extend many of the consistent systems of paths given by (R4) into special absorbing structures. Recall that, since H is $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U})$-robust, for every $x \in \mathcal{U}$ and each pair of directions $a, b \in \mathcal{D}(\mathcal{Q}^n)$, there exists a collection $\mathfrak{C}(x, a, b)$ of (x, a, b)-consistent systems of paths in $H \cup \{\{x, x + a\}\{x, x + b\}\}$ satisfying (R4). By (D1), for every $x \in \mathcal{U}$ and $a, b \in \mathcal{D}(\mathcal{Q}^n)$

(CS) there exists a subcollection $\mathfrak{C}'(x, a, b) \subseteq \mathfrak{C}(x, a, b)$ which satisfies (R4) with $\gamma/2$ playing the role of γ and such that, for every $CS \in \mathfrak{C}'(x, a, b)$, we have $\mathrm{endmol}(CS) \subseteq \mathcal{P}$.

Let L be any layer of $\mathcal{Q}^n$. For each $x \in \mathcal{U}$, let $\mathrm{End}(x) := \bigcup_{a,b\in\mathcal{D}(\mathcal{Q}^n)} \bigcup_{CS\in\mathfrak{C}'(x,a,b)} \mathrm{endmol}(CS)$, let $\mathrm{End\,II}(x) := \bigcup_{a,b\in\mathcal{D}(\mathcal{Q}^n)\setminus\mathcal{D}(L)} \bigcup_{CS\in\mathfrak{C}'(x,a,b)} \mathrm{endmol}(CS)$, and let $x_I \in V(I)$ be such that x is a clone of x_I. Given any tree $T^\bullet \subseteq I$ and any cube $C \subseteq V(I)$, we say that C *meets* $T^\bullet$ if $V(C) \cap V(T^\bullet) \neq \varnothing$. Recall that, for any $y, z \in V(I)$, we denote the set of directions in a shortest path in I between y and z by $\mathcal{D}(y, z)$.

Claim 9.1. *For each $x \in \mathcal{U}$ and each $z \in \mathrm{End}(x)$, there exists a collection of cubes $\mathcal{C}''(z) \subseteq \mathcal{C}'(z)$ with $|\mathcal{C}''(z)| = 20\ell$ which satisfies the following properties.*

(i) *Every $C \in \mathcal{C}''(z)$ meets T'.*
(ii) *For every $C \in \mathcal{C}''(z)$, we have $\mathcal{D}(C) \cap \mathcal{D}(x_I, z) = \varnothing$.*

(iii) *For every* $C_1, C_2 \in \mathcal{C}''(z)$*, we have* $V(C_1) \cap V(C_2) = \{z\}$*.*
(iv) *For each* $x \in \mathcal{U}$*, let* $\boldsymbol{\mathcal{C}''}(x) \coloneqq \bigcup_{z \in \mathrm{End\,II}(x)} \mathcal{C}''(z)$*. For each* $d \in \mathcal{D}(L)$ *and each* $x \in \mathcal{U}$*, let* $\alpha(d, x) \coloneqq |\{C \in \boldsymbol{\mathcal{C}''}(x) : d \in \mathcal{D}(C)\}|$*. For all* $d \in \mathcal{D}(L)$ *and* $x \in \mathcal{U}$*, we have that* $\alpha(d, x) \leq n/\ell^3$*.*

Proof. Fix any vertex $x \in \mathcal{U}$. Let $\mathfrak{m} \coloneqq |\mathrm{End}(x)|$ and $m \coloneqq |\mathrm{End\,II}(x)|$, and consider an arbitrary labelling $z_1, \ldots, z_m, z_{m+1}, \ldots, z_{\mathfrak{m}}$ of the vertices in $\mathrm{End}(x)$ such that all vertices in $\mathrm{End\,II}(x)$ come first. Observe that, by the definition of Type II consistent systems of paths (see Section 9.1), we have that $m \leq n$. We will now iteratively define the sets $\mathcal{C}''(z_i)$ for each $i \in [\mathfrak{m}]$.

Fix first any $i \in [m]$, and suppose that a set $\mathcal{C}''(z_j)$ satisfying the claim is already defined for all $j \in [i-1]$. Let $\boldsymbol{\mathcal{C}''}(x, i) \coloneqq \bigcup_{j=1}^{i-1} \mathcal{C}''(z_j)$. For each $d \in \mathcal{D}(L)$, let $\alpha(d, x, i) \coloneqq |\{C \in \boldsymbol{\mathcal{C}''}(x, i) : d \in \mathcal{D}(C)\}|$. Let $\mathcal{D}^* \coloneqq \{d \in \mathcal{D}(L) : \alpha(d, x, i) \geq n/\ell^3\}$. Observe that $|\mathcal{D}^*| \leq 20\ell^5$. Let $\mathcal{C}'''(z_i)$ be the set of all cubes $C \in \mathcal{C}'(z_i)$ such that C meets T' and $\mathcal{D}(C) \cap (\mathcal{D}(x_I, z_i) \cup \mathcal{D}^*) = \varnothing$ (i.e., they satisfy (i) and (ii) and, if added to the collection, would not violate (iv)). We claim that $|\mathcal{C}'''(z_i)| \geq \lambda n^\ell/5$. Indeed, observe that $\mathrm{dist}(x_I, z_i) \leq 2$ and, thus, the number of cubes $C \in \mathcal{C}'(z_i)$ such that $\mathcal{D}(C) \cap (\mathcal{D}(x_I, z_i) \cup \mathcal{D}^*) \neq \varnothing$ is at most $(20\ell^5 + 2)n^{\ell-1}$, and, by (ET2) and (R3), the number of such cubes which do not meet T' is at most $n^{\ell-2}$. The bound then follows by (D2).

We can now construct $\mathcal{C}''(z_i)$ by obtaining cubes $C_1(z_i), \ldots, C_{20\ell}(z_i)$ iteratively. Note that, for any pair of cubes $C_1, C_2 \in \mathcal{C}'(z_i)$, we have that $z \in V(C_1) \cap V(C_2)$. Then, (iii) is equivalent to having that $\mathcal{D}(C_1) \cap \mathcal{D}(C_2) = \varnothing$. For each $k \in [20\ell]$, we proceed as follows. Let $\mathcal{D}'_k \coloneqq \bigcup_{j=1}^{k-1} \mathcal{D}(C_j(z_i))$. Note that $|\mathcal{D}'_k| \leq 20\ell^2 \leq \lambda n/8$. Now, applying Remark 9.2 with $n - s$, z, $\lambda/5$, $\lambda/8$ and $\mathcal{D}'_k$ playing the roles of n, x, η, η' and $\mathcal{D}'$, respectively, we deduce that there is a cube $C_k(z_i) \in \mathcal{C}'''(z_i)$ with $\mathcal{D}(C_k(z_i)) \cap \mathcal{D}'_k = \varnothing$. By enforcing that (iii) holds, it follows that each direction is used at most once in the cubes that were added in this step. It then follows that (iv) holds as well.

Consider now any $i \in [\mathfrak{m}] \setminus [m]$, and suppose that a set $\mathcal{C}''(z_j)$ satisfying the claim is already defined for all $j \in [i-1]$. Let $\mathcal{C}'''(z_i)$ be the set of all cubes $C \in \mathcal{C}'(z_i)$ such that C meets T' and $\mathcal{D}(C) \cap \mathcal{D}(x_I, z_i) = \varnothing$ (i.e., they satisfy (i) and (ii)). As above, we claim that $|\mathcal{C}'''(z_i)| \geq \lambda n^\ell/5$. Indeed, the number of cubes $C \in \mathcal{C}'(z_i)$ such that $\mathcal{D}(C) \cap \mathcal{D}(x_I, z_i) \neq \varnothing$ is at most $2n^{\ell-1}$, and, again, the number of such cubes which do not meet T' is at most $n^{\ell-2}$. The bound then follows by (D2).

We can now construct $\mathcal{C}''(z_i)$ as above. For each $k \in [20\ell]$, we proceed as follows. Let $\mathcal{D}'_k \coloneqq \bigcup_{j=1}^{k-1} \mathcal{D}(C_j(z_i))$. Note that $|\mathcal{D}'_k| \leq 20\ell^2 \leq \lambda n/8$. Now, applying Remark 9.2 with $n - s$, z, $\lambda/5$, $\lambda/8$ and $\mathcal{D}'_k$ playing the roles of n, x, η, η' and $\mathcal{D}'$, respectively, we deduce that there is a cube $C_k(z_i) \in \mathcal{C}'''(z_i)$ with $\mathcal{D}(C_k(z_i)) \cap \mathcal{D}'_k = \varnothing$. ◀

Let $J^1 \coloneqq \bigcup_{x \in \mathcal{U}} \bigcup_{z \in \mathrm{End}(x)} \bigcup_{C \in \mathcal{C}''(z)} \mathcal{M}_C$, where $\mathcal{C}''(z)$ are the sets given by Claim 9.1, and let $G_7^* \subseteq G_7$ consist of all edges of G_7 which have endpoints in different layers.

Claim 9.2. *$J^1 \cup G_7^*$ is $(\mathcal{U}, \ell^3, s)$-good and $\Delta(J^1 \cup G_7^*) \leq 50\ell^4$.*

Proof. In order to see that $J^1 \cup G_7^*$ is $(\mathcal{U}, \ell^3, s)$-good, observe first that the edges of G_7^* do not affect this definition, so it suffices to see that J^1 is $(\mathcal{U}, \ell^3, s)$-good. By Claim 9.1(ii), for all $x \in \mathcal{U}$, $z \in \mathrm{End}(x)$ and $C \in \mathcal{C}''(z)$ we have that $\mathrm{dist}(x_I, C) = \mathrm{dist}(x_I, z)$. In particular, by the definition of the different consistent systems of paths (see Section 9.1), it follows that the only cubes which affect whether J^1 is $(\mathcal{U}, \ell^3, s)$-good or not are those of the collection $\boldsymbol{\mathcal{C}''}(x)$ described in Claim 9.1(iv). But then, by Claim 9.1(iv), we have that no direction $d \in \mathcal{D}(L)$ is used more than n/ℓ^3 times, as required.

Note that $\Delta(G_7^*) \leq s = 10\ell$, by construction. We will now show that $\Delta(J^1) \leq 45\ell^4$. Observe that J^1 does not contain any edges with endpoints in different layers. In particular, J^1 consists of clones of the same subgraph of I, that is $I(J^1) = \bigcup_{x \in \mathcal{U}} \bigcup_{z \in \mathrm{End}(x)} \bigcup_{C \in \mathcal{C}''(z)} C$. By this observation, it is enough to show that $\Delta(I(J^1)) \leq 45\ell^4$.

Recall that, by (R3), given any distinct $x, y \in \mathcal{U}_I$, we have that $\mathrm{dist}(x, y) \geq \gamma n/2$. In particular, by this observation and Claim 9.1(ii), it follows that, for all $x \in \mathcal{U}_I$, we have $d_{I(J^1)}(x) = 0$. We also note that, for every $z \in V(I)$ for which $\mathrm{dist}(z, \mathcal{U}_I) \geq \ell + 3$, we have $d_{I(J^1)}(z) = 0$. Now, fix any $x \in \mathcal{U}_I$ and $z \in V(I)$ such that $\mathrm{dist}(z,x) = t$, for some $t \in [\ell+2]$. We claim that $d_{I(J^1)}(z) \leq 2t^2 \cdot 20\ell^2 \leq 45\ell^4$.

Suppose first that $t = 1$. Then, by Claim 9.1(ii), for every edge $e \in E(I(J^1))$ incident with z, we have $e \in E(C)$ for some $C \in \mathcal{C}''(z)$, and hence $d_{I(J^1)}(z) \leq \ell|\mathcal{C}''(z)| = 20\ell^2$, as we wanted to show. Suppose now that $t \geq 2$ and let $\mathcal{D}(z,x) = \{d_1, \ldots, d_t\}$. Every edge e incident with z must come from the edges of a cube $C \in \mathcal{C}''(w)$, for some $w \in \mathrm{End}(x)$. Moreover, by Claim 9.1(ii), we must have $\mathcal{D}(x,w) \subseteq \mathcal{D}(x,z)$. As there are at most $t + t^2 \leq 2t^2$ vertices $w \in V(I)$ such that $\mathcal{D}(x,w) \subseteq \mathcal{D}(x,z)$ and $\mathrm{dist}(x,w) \in [2]$, we have have that $d_{I(J^1)}(z) \leq 2t^2 20\ell^2$, which concludes the proof of the claim. ◀

Step 9: Fixing a collection of absorbing ℓ-cube pairs for the vertices in scant molecules. In this step, we use $G_8 \sim \mathcal{Q}^n_{\eta_8}$ to alter T' so that any tips of absorbing ℓ-cube pairs for vertices $x \in V_{\mathrm{sc}}$ which do not lie in R are relocated from the tree T' to the reservoir. We follow the same approach as in Step 8 of the proof of Theorem 8.1. In particular, we define an event $\mathcal{E}_8^*$ (which is analogous to $\mathcal{E}_7^*$ in the proof of Theorem 8.1) and condition that it holds. This then gives a set of absorbing ℓ-cube pairs $\mathcal{C}_1^{\mathrm{sc}} = \{(C^l(x,j,k), C^r(x,j,k)) \subseteq I : x \in V_{\mathrm{sc}}, j \in [K], k \in [2^{s+1}\Psi]\}$, where each $(C^l(x,j,k), C^r(x,j,k))$ is an absorbing ℓ-cube pair for x, which satisfies that

(CD) for all distinct $(x,j,k),(x',j',k') \in V_{\mathrm{sc}} \times [K] \times [2^{s+1}\Psi]$, $C^l(x,j,k)$ and $C^r(x,j,k)$ are both vertex-disjoint from $C^l(x',j',k')$ and $C^r(x',j',k')$.

We define P' and P as in Step 8 of the proof of Theorem 8.1. Observe that (T4) implies that $(P' \cup V(P)) \cap B^{2\ell}(\mathcal{U}_I) = \varnothing$. Let $T^{\mathrm{IV}} := T'[V(T') \setminus P'] \cup P$, which is connected, and let T'' be a spanning tree of T^{IV}. In particular, it follows from the above and the definitions of T and T' in Steps 1 and 7 that

$$T'' \subseteq I(G_1 \cup G_6 \cup G_8) - B_I^5(\mathcal{U}_I) \subseteq I - B_I^5(\mathcal{U}_I). \tag{9.6}$$

Furthermore, as in Step 8 of the proof of Theorem 8.1, we have that

$$\Delta(T'') \leq 12D. \tag{9.7}$$

Define the (new) reservoir $R' := (R \cup P') \setminus V(P)$.

For each $x \in V(I) \setminus B_I^5(\mathcal{U}_I)$, let $Z'(x) := Z(x) \cap V(T'')$ (where $Z(x)$ is as defined in Step 7). It follows by (9.4) and (T3) that

$$|Z'(x)| \geq 3n/4 - 4 \cdot 2^s\Psi K S' \geq n/2.$$

Choose any vertex $x \in V(I)$ with $\mathrm{dist}(x,\mathcal{U}_I) \geq 3\ell$. Again by (T3), there are at most $4 \cdot 2^{s+\ell}\Psi K S'$ vertices in $Z'(x)$ which lie in cubes of absorbing ℓ-cube pairs of $\mathcal{C}_1^{\mathrm{sc}}$. Choose any vertex $y \in Z'(x)$ which does not lie in any of those cubes. Denote the cube $C \in \mathcal{C}''$ which contains y by $C^\bullet$.

For each $x \in V(I) \setminus V_{\mathrm{sc}}$ and each $i \in [K]$, we now redefine the set $M(A_i(x))$ as follows.

(AB4) Let $M(A_i(x))$ retain only those edges whose associated absorbing ℓ-cube pair (C^l, C^r) satisfies that both C^l and C^r are different from $C^\bullet$ and vertex-disjoint from both cubes of all absorbing ℓ-cube pairs of $\mathcal{C}_1^{\mathrm{sc}}$, and both tips x^l and x^r satisfy that $x^l, x^r \in R \setminus V(P) \subseteq R'$.

Note that, by (T3), we have $|B_I^{\ell+1}(x) \cap V(P)| \leq 21 \cdot 2^s \Psi D K S'$ and $|B_I^{\ell+1}(x) \cap V(\bigcup_{(C^l,C^r) \in \mathcal{C}_1^{\mathrm{sc}}}(C^l \cup C^r))| \leq 4 \cdot 2^{\ell+s}\Psi K S'$. Combining this with (9.5) and (AB1), it follows that

$$|M(A_i(x))| \geq 4n/\ell^3 - (21D + 4 \cdot 2^\ell)2^s\Psi K S' - 1 \geq 2n/\ell^3. \tag{9.8}$$

Step 10: Fixing a collection of absorbing ℓ-cube pairs for vertices in non-scant molecules and vertices near $\mathcal{U}$. At this point, it is not yet clear which vertices will need to eventually be absorbed into the long cycle we construct. For vertices in I which are 'far' from $\mathcal{U}_I$, we can already determine those which will have clones that will need to be absorbed (though we cannot yet determine the precise clones). However, for vertices which are 'near' $\mathcal{U}_I$, we still cannot say which of them will have clones that need to be absorbed (this depends on the special absorbing structure which is fixed once the edges of H' are revealed). As a result, we proceed as if all of the clones of vertices in I near $\mathcal{U}_I$ will need to be absorbed. Recall that $\mathcal{C}'$ and $\mathcal{C}''$ were introduced in Step 6. Let

$$\mathcal{C}''' \coloneqq \{C \in \mathcal{C}' : V(C) \cap V(T'') \neq \varnothing\} \quad \text{and} \quad V'_{\mathrm{abs}} \coloneqq (V(I) \setminus \bigcup_{C\in\mathcal{C}'''} V(C)) \cup B^{3\ell}_I(\mathcal{U}_I).$$

We will now fix a collection of absorbing ℓ-cube pairs for all vertices in each vertex molecule $\mathcal{M}_x$ with $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$, except for the vertices of $\mathcal{U}$.

Similarly as in (8.12), we have

$$|B^{2\ell}_I(x) \setminus (V(T'') \cup B^5_I(\mathcal{U}_I))| \leq 2n^{3/4}. \tag{9.9}$$

For all $x \in \bigcup_{C\in\mathcal{C}''} V(C)$, by combining (N1), (9.9) and (R3) with the definition of bondlessly surrounded molecules, we have that

$$|\mathcal{C}_x \cap \mathcal{C}'''| \geq (1 - 2^{-\ell-5s+1})n, \tag{9.10}$$

where $\mathcal{C}_x$ is the collection of all those $C \in \mathcal{C}$ such that $x \notin V(C)$ and $N_I(x) \cap V(C) \neq \varnothing$.

Recall that, for any $x \in V(I)$, each index $i \in [K]$ is given by a unique edge $e \in \mathfrak{M}(x)$ via the relation $N(e) = A_i(x)$. Recall also the definition of $\mathfrak{M}(x)$ from Step 3. We now prove the following claim, which is similar to Claim 8.2 (apart from the new property (ii)).

Claim 9.3. *For each $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and each $e \in \mathfrak{M}(x)$, there exists a set $\mathcal{C}^{\mathrm{abs}}_1(e)$ of $2^{s+1}\Psi$ absorbing ℓ-cube pairs $(C^l_k(e), C^r_k(e)) \subseteq I$, one for each $k \in [2^{s+1}\Psi]$, which satisfies the following:*

(i) *for all $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$, $e \in \mathfrak{M}(x)$ and $k \in [2^{s+1}\Psi]$, the absorbing ℓ-cube pair $(C^l_k(e), C^r_k(e))$ is associated with some edge in $M(A_j(x))$, for some $j \in [K]$;*

(ii) *for all $x \in B^{3\ell}_I(\mathcal{U}_I)$, $e \in \mathfrak{M}(x)$ and $k \in [2^{s+1}\Psi]$, the absorbing ℓ-cube pair $(C^l_k(e), C^r_k(e))$ has tips $x^l_k(e)$ and $x^r_k(e)$ which satisfy that $\mathrm{dist}(x,\mathcal{U}_I) < \mathrm{dist}(x^l_k(e),\mathcal{U}_I)$ and $\mathrm{dist}(x,\mathcal{U}_I) < \mathrm{dist}(x^r_k(e),\mathcal{U}_I)$, and*

(iii) *for all $x, x' \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$, all $e \in \mathfrak{M}(x)$ and $e' \in \mathfrak{M}(x')$, and all $k, k' \in [2^{s+1}\Psi]$ with $(x, e, k) \neq (x', e', k')$, the absorbing ℓ-cube pairs $(C^l_k(e), C^r_k(e))$ and $(C^l_{k'}(e'), C^r_{k'}(e'))$ satisfy that $(V(C^l_k(e)) \cup V(C^r_k(e))) \cap (V(C^l_{k'}(e')) \cup V(C^r_{k'}(e'))) = \varnothing$.*

Proof. Let $\mathcal{V} \coloneqq \bigcup_{x\in V'_{\mathrm{abs}}\setminus V_{\mathrm{sc}}} \mathfrak{M}(x)$. Let $K' \coloneqq |\mathcal{V}|$, and let $f_1, \dots, f_{K'}$ be an ordering of the edges in $\mathcal{V}$. Given any $i \in [K']$, the edge f_i corresponds to a pair $(x, j(i))$, where $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $j(i) \in [K]$. If $x \notin B^{3\ell}_I(\mathcal{U}_I)$, let $\mathfrak{C}_i$ be the collection of absorbing ℓ-cube pairs for x in I associated with some edge of $M(A_{j(i)}(x))$. Otherwise, let $\mathfrak{C}_i$ be the same collection, after removing all those absorbing ℓ-cube pairs for which (ii) does not hold. Since each $x \in B^{3\ell}_I(\mathcal{U}_I)$ has at most 3ℓ neighbours $y \in N_I(x)$ such that $\mathrm{dist}(x,\mathcal{U}_I) \geq \mathrm{dist}(y,\mathcal{U}_I)$, it follows by (9.8) that $|\mathfrak{C}_i| \geq n/\ell^3$ for all $i \in [K']$. In particular, by (AB2), each of the absorbing ℓ-cube pairs (C^l, C^r) in any of the collections $\mathfrak{C}_i$ satisfies that $C^l, C^r \in \mathcal{C}''$.

Let $\mathcal{H}$ be the $2^{s+1}\Psi K'$-edge-coloured auxiliary multigraph with $V(\mathcal{H}) \coloneqq \mathcal{C}''$, which contains one edge of colour $(i, k) \in [K'] \times [2^{s+1}\Psi]$ between C and C' whenever $(C, C') \in \mathfrak{C}_i$ or $(C', C) \in \mathfrak{C}_i$. In particular, $\mathcal{H}$ contains at least n/ℓ^3 edges of each colour. We now bound $\Delta(\mathcal{H})$. Consider any $C \in V(\mathcal{H})$. Note that, for each edge e of $\mathcal{H}$ incident to C, there exists some $x = x(e) \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ such that C together with some other cube $C' \in V(\mathcal{H})$ forms an absorbing ℓ-cube pair for x. In particular, x must be adjacent to C in I. Let $\overline{\partial}(C) \coloneqq \{x(e) : e \in E(\mathcal{H}) \text{ is incident to } C\}$. Moreover, if e has colour (i, z), then $f_i \in \mathfrak{M}(x)$ (and f_i has corresponding pair $(x, j(i))$ for some $j(i) \in [K]$). Since $f_i \in \mathfrak{M}(x)$ and $|\mathfrak{M}(x)| \leq \binom{2^s}{2}$, it follows that each vertex y which is adjacent to C in I can play the role of x for at most $2^{s+1}\Psi \cdot 2^{2s}$ edges of $\mathcal{H}$ incident to C. Thus, $d_{\mathcal{H}}(C) \leq 2^{3s+1}\Psi|\overline{\partial}(C)|$.

Fix a cube $C \in V(\mathcal{H})$. In order to bound $|\overline{\partial}(C)|$, consider first $|\overline{\partial}(C) \cap B^{3\ell}_I(\mathcal{U}_I)|$. Recall that, by (R3), there is at most one vertex $z \in \mathcal{U}_I \cap B^{3\ell}_I(V(C))$. Furthermore, since the property described in (ii) holds for all absorbing ℓ-cube pairs for vertices in $B^{3\ell}_I(\mathcal{U}_I)$ represented in $\mathcal{H}$, it follows that each vertex $x \in V(C)$ has at most $3\ell + 1$ neighbours in $B^{3\ell}_I(\mathcal{U}_I) \cap \overline{\partial}(C)$. Thus, in total, $|\overline{\partial}(C) \cap B^{3\ell}_I(\mathcal{U}_I)| \leq (3\ell + 1)2^\ell$. Consider now $|\overline{\partial}(C) \setminus B^{3\ell}_I(\mathcal{U}_I)|$. Note that $\overline{\partial}(C) \setminus B^{3\ell}_I(\mathcal{U}_I) \subseteq (V'_{\mathrm{abs}} \cap N_I(V(C))) \setminus B^{3\ell}_I(\mathcal{U}_I) \subseteq N_I(V(C)) \setminus \bigcup_{C'\in\mathcal{C}'''} V(C')$. By (9.10), the number of vertices in

$V'_{\mathrm{abs}} \setminus B_I^{3\ell}(\mathcal{U}_I)$ which are adjacent to C is at most $2|C|n/2^{\ell+5s}$, that is, $|\overline{\partial}(C) \setminus B_I^{3\ell}(\mathcal{U}_I)| \leq 2n/2^{5s}$. We conclude that $|\overline{\partial}(C)| \leq 3n/2^{5s}$ and, thus, $d_{\mathcal{H}}(C) \leq 2^{3s+1}\Psi 3n/2^{5s} \leq n/\ell^4$.

Since each colour class has size at least n/ℓ^3 and $\Delta(\mathcal{H}) \leq n/\ell^4$, by Lemma 5.5, $\mathcal{H}$ contains a rainbow matching of size $2^{s+1}\Psi K'$. For each $(i,z) \in [K'] \times [2^{s+1}\Psi]$, let $(C_z^l(f_i), C_z^r(f_i)) \in \mathfrak{C}_i$ be the absorbing ℓ-cube pair of colour (i,z) in this rainbow matching. This ensures that (iii) holds, while (i) and (ii) follow by construction. ◀

For each $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and each $i \in [K]$, let $\mathcal{C}_1^{\mathrm{abs}}(x,i) \coloneqq \mathcal{C}_1^{\mathrm{abs}}(e)$ be the set of absorbing ℓ-cube pairs guaranteed by Claim 9.3, where $e \in \mathfrak{M}(x)$ is the unique edge such that $A_i(x) = N(e)$. Similarly, for each $k \in [2^{s+1}\Psi]$, let $(C^l(x,i,k), C^r(x,i,k)) \coloneqq (C_k^l(e), C_k^r(e))$. Let $\mathcal{C}_1^{\mathrm{abs}} \coloneqq \bigcup_{x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}} \bigcup_{i\in[K]} \mathcal{C}_1^{\mathrm{abs}}(x,i)$.

Let $G \coloneqq \bigcup_{i=1}^{8} G_i$. Recall that G_7^* and J^1 were defined in Step 8. We let $Q' \subseteq G$ be the spanning subgraph with edge set

$$E(Q') \coloneqq E(J^1) \cup E(G_4^*) \cup E(G_5^*) \cup E(G_7^*) \cup \bigcup_{C \in \mathcal{C}'} E(\mathcal{M}_C) \cup \bigcup_{i=1}^{2^s} E(T''_{L_i}),$$

where G_4^* and G_5^* are as defined in Step 9 of the proof of Theorem 8.1 (but with V'_{abs} playing the role of V_{abs}). Note that, using Claim 9.2 and (9.7), we have that $\Delta(Q') \leq \Phi$.

Claim 9.4. *Q' is $(\mathcal{U}, 2\ell^2, s)$-good.*

Proof. Indeed, observe that this fact only depends on those edges contained within a layer which are incident to a neighbour of x in $\mathcal{Q}^n$, for some $x \in \mathcal{U}$. Therefore, the graph G_5^* has no effect here. By property (C3) below, the graph $\bigcup_{i=1}^{2^s} T''_{L_i}$ also has no effect. Now, by Claim 9.2 we have that $J^1 \cup G_7^*$ is $(\mathcal{U}, \ell^3, s)$-good, and $\bigcup_{C\in\mathcal{C}'} \mathcal{M}_C$ is $(\mathcal{U}, \ell^3, s)$-good by (N2) combined with (R3). Finally, consider G_4^*. For each $x \in V'_{\mathrm{abs}} \cup V_{\mathrm{sc}}$, $i \in [K]$ and $k \in [2^{s+1}\Psi]$, let $e(x,i,k)$ be the edge between the left absorber tip and the third absorber vertex of $(C^l(x,i,k), C^r(x,i,k)) \in \mathcal{C}_1^{\mathrm{sc}} \cup \mathcal{C}_1^{\mathrm{abs}}$. Observe that, for all $x \in V'_{\mathrm{abs}} \cup V_{\mathrm{sc}}$ such that $\mathrm{dist}(x, \mathcal{U}_I) \geq 5$, all $i \in [K]$ and all $k \in [2^{s+1}\Psi]$, we have that $e(x,i,k)$ does not affect whether Q' is $(\mathcal{U}, 2\ell^2, s)$-good or not. In particular, by (T4), this is true for all $x \in V_{\mathrm{sc}}$. Now consider each $x \in V'_{\mathrm{abs}}$ such that $\mathrm{dist}(x, \mathcal{U}_I) < 5$. By Claim 9.3(ii), it follows that $e(x,i,k)$ only affects our claim when $x \in \mathcal{U}_I$. Observe that, for each $i \in [K]$ and $k \in [2^{s+1}\Psi]$, the direction of $e(x,i,k)$ is the same as that of the edge $e'(x,i,k)$ joining x to the right absorber tip of $(C^l(x,i,k), C^r(x,i,k))$. By Claim 9.3(iii), all cubes of absorbing ℓ-cube pairs in $\bigcup_{i\in[K]} \mathcal{C}_1^{\mathrm{abs}}(x,i)$ are vertex disjoint, which implies that each edge $e'(x,i,k)$ with $i \in [K]$ and $k \in [2^{s+1}\Psi]$ uses a different direction. Hence, G_4^* is $(\mathcal{U}, n, s)$-good, and the claim follows. ◀

Note that $T'' \subseteq I(Q')$, $R' \subseteq V(I)$, and $C \subseteq I(Q')$ for all $C \in \mathcal{C}'$. Recall the definitions of C'' from Step 6 and C''' from Step 10. For any $u \in \mathcal{U}$, recall the definitions of $\mathrm{End}(x)$ given in Step 8. Recall also the definitions of P, P' and $C^\bullet$ from Step 9. Combining all the previous steps, we claim that the following hold (conditioned on the events $\mathcal{E}_1^*, \dots, \mathcal{E}_8^*$, which occur a.a.s.).

- (C1) $\Delta(T'') \leq 12D$.
- (C2) Any vertex $x \in R' \cap V(T'')$ is a leaf of T''. Furthermore, if $x \in R' \cap V(T'')$, then $x \notin V(T)$ and its unique neighbour x' in T'' satisfies that $x' \in Z(x)$ (where $Z(x)$ is as defined in Step 7).
- (C3) $B_I^5(\mathcal{U}_I) \cap V(T'') = \varnothing$.
- (C4) For all $x \in V(I)$ we have that $|\mathcal{C}_x \cap \mathcal{C}'''| \geq (1 - 3/2\ell^4)n$.
- (C5) For each $x \in V_{\mathrm{sc}}$ and $i \in [K]$, there is a collection $\mathcal{C}_1^{\mathrm{sc}}(x,i)$ of $2^{s+1}\Psi$ absorbing ℓ-cube pairs $(C^l(x,i,k), C^r(x,i,k))$ for x in I (defined in Step 9), each of which is associated with some edge $e \in M(A_i(x))$. In particular, $(C^l(x,i,k), C^r(x,i,k))$ is as described in (AB1) (recall also (AB2)), that is, there are two absorbing ℓ-cube pairs $(C_1^l(x,i,k), C_1^r(x,i,k))$ and $(C_2^l(x,i,k), C_2^r(x,i,k))$ in $H \cup G$, associated with $e \in M(A_i(x))$, for the clones x_1 and x_2 of x which correspond to (x,i). Moreover, each of these absorbing ℓ-cube pairs $(C^l(x,i,k), C^r(x,i,k))$ satisfies the following:

(C5.1) $(C_1^l(x,i,k), C_1^r(x,i,k)) \cup (C_2^l(x,i,k), C_2^r(x,i,k)) - V(\mathcal{M}_x) \subseteq Q'$;
(C5.2) the tips of $C^l(x,i,k)$ and $C^r(x,i,k)$ lie in $R' \setminus V(T'')$;
(C5.3) $C^l(x,i,k), C^r(x,i,k) \in \mathcal{C}'' \cap \mathcal{C}'''$, and
(C5.4) for any $x' \in V_{\mathrm{sc}}$, $i' \in [K]$ and $k' \in [2^{s+1}\Psi]$ with $(x',i',k') \neq (x,i,k)$, we have that $C^l(x,i,k)$, $C^r(x,i,k)$, $C^l(x',i',k')$ and $C^r(x',i',k')$ are vertex-disjoint.

(C6) For each $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i \in [K]$, there is a collection $\mathcal{C}_1^{\mathrm{abs}}(x,i)$ of $2^{s+1}\Psi$ absorbing ℓ-cube pairs $(C^l(x,i,k), C^r(x,i,k))$ for x in I (defined in Step 10), each of which is associated with an edge $e \in M(A_i(x))$. In particular, $(C^l(x,i,k), C^r(x,i,k))$ is as described in (AB1) (recall also (AB2)), that is, there are two absorbing ℓ-cube pairs $(C_1^l(x,i,k), C_1^r(x,i,k))$ and $(C_2^l(x,i,k), C_2^r(x,i,k))$ in $H \cup G$, associated with $e \in M(A_i(x))$, for the clones x_1 and x_2 of x which correspond to (x,i). Moreover, each of these absorbing ℓ-cube pairs $(C^l(x,i,k), C^r(x,i,k))$ satisfies the following:
(C6.1) $(C_1^l(x,i,k), C_1^r(x,i,k)) \cup (C_2^l(x,i,k), C_2^r(x,i,k)) - V(\mathcal{M}_x) \subseteq Q'$;
(C6.2) the tips of $C^l(x,i,k)$ and $C^r(x,i,k)$ lie in R';
(C6.3) $C^l(x,i,k), C^r(x,i,k) \in \mathcal{C}'' \cap \mathcal{C}'''$;
(C6.4) for any $x' \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$, $i' \in [K]$ and $k' \in [2^{s+1}\Psi]$ with $(x',i',k') \neq (x,i,k)$, we have that $C^l(x,i,k)$, $C^r(x,i,k)$, $C^l(x',i',k')$ and $C^r(x',i',k')$ are vertex-disjoint, and
(C6.5) both $C^l(x,i,k)$ and $C^r(x,i,k)$ are vertex-disjoint from all cubes of absorbing ℓ-cube pairs in $\mathcal{C}_1^{\mathrm{sc}}$.

(C7) For every $x \in \mathcal{U}$ and every $z \in \mathrm{End}(x)$, there exists a collection of cubes $\mathcal{C}''(z)$ in $I(Q')$ with $|\mathcal{C}''(z)| = 20\ell$ which satisfies the following properties:
(C7.1) for every $C \in \mathcal{C}''(z)$, we have that $z \in V(C)$;
(C7.2) for every $C \in \mathcal{C}''(z)$, the molecule $\mathcal{M}_C$ is bonded in Q';
(C7.3) every $C \in \mathcal{C}''(z)$ meets T'';
(C7.4) for every $C_1, C_2 \in \mathcal{C}''(z)$, we have $V(C_1) \cap V(C_2) = \{z\}$, and
(C7.5) for every $C^* \in \mathcal{C}_1^{\mathrm{sc}}$ and $C \in \mathcal{C}''(z)$, we have $V(C) \cap V(C^*) = \varnothing$.

(C8) $C^\bullet$ intersects $V(T) \cap V(T'')$ and is different from all cubes described in (C5), (C6) and (C7).

Indeed, (C1) is given by (9.7). (C2) holds by (ET3) and the fact that $P' \cap V(T'') = \varnothing$. (C3) follows directly by (9.6). (C4) follows by combining (N1), the conditioning on $\mathcal{E}_5^*$, (9.9) and (R3). (C5) follows from the construction of P and T'' in Step 9. Indeed, (C5.1) follows from the definition of Q' combined with (AB1), and (C5.2) holds by the definition of R' and T'' combined with (AB1), while (C5.3) follows because of the definition of the set $M(A_i(x))$ in (AB2) and (AB3), and (C5.4) holds by (CD). Consider now (C6). For each $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i \in [K]$, consider $\mathcal{C}_1^{\mathrm{abs}}(x,i)$. All absorbing ℓ-cube pairs of $\mathcal{C}_1^{\mathrm{abs}}(x,i)$ satisfy (C6.1) and (C6.2) by the definition of Q', (AB1) and (AB4). Similarly, they satisfy (C6.3) by (AB2), (AB3) and the fact that, by (AB4), their intersection with T'' contains their intersection with T'. Moreover, (C6.4) holds by Claim 9.3, and (C6.5) holds because of (AB4). Now, (C7) holds by Claim 9.1 and (T4). Indeed, let $\mathcal{C}''(z)$ be the collection of cubes given by Claim 9.1, so (C7.1), (C7.2) and (C7.4) follow directly. (C7.3) follows by using again the observation that, by (T4) and the construction of P, for any $x \in \mathcal{U}_I$, we have that T' and T'' coincide in $B_I^{2\ell}(x)$. Now recall that, by (T4), all vertices $x \in \mathcal{U}_I$ are at distance at least 5ℓ from V_{sc}, so (C7.5) follows by construction. Finally, consider (C8). The fact that $C^\bullet$ intersects $V(T) \cap V(T'')$ follows by its definition in Step 9, as does the fact that it is different from all cubes described in (C5). The fact that it is different from all cubes in (C6) follows by (AB4). Finally, the fact that it is different from the cubes in (C7) follows since $\mathrm{dist}(V(C^\bullet), \mathcal{U}_I) \geq \ell$ by the definition of $C^\bullet$ in Step 9.

Step 11: Fixing special absorbing structures. From this point onward, every step will be deterministic. Let $F \subseteq \mathcal{Q}^n$ be any graph with $\Delta(F) \leq \Psi$ which is $(\mathcal{U}, \ell, s)$-good, that is, for each $x \in \mathcal{U}$, the set $E_F(x) := \{e \in E(F) : e \cap N_{\mathcal{Q}^n}(x) \neq \varnothing\}$ satisfies the following:

(F^*) for each layer L of $\mathcal{Q}^n$ and all $d \in \mathcal{D}(L)$, we have $|\{e \in E_F(x) : \mathcal{D}(e) = d\}| \leq n/\ell$.

Let $H' \subseteq \mathcal{Q}^n$ be any graph such that, for every $x \in \mathcal{U}$, we have $d_{H'}(x) \geq 2$. For each $x \in \mathcal{U}$, let $\{x, x+a(x)\}, \{x, x+b(x)\} \in E(H')$, where $a(x), b(x) \in \mathcal{D}(\mathcal{Q}^n)$. Our goal is to find a

$(\mathcal{U}, \ell^2, s)$-good Hamilton cycle in $((H \cup G) \setminus F) \cup H' \cup Q'$ which, for each $x \in \mathcal{U}$, contains the edges $\{x, x + a(x)\}$ and $\{x, x + b(x)\}$. Recall that $\mathfrak{C}'(x, a(x), b(x))$ was defined in Step 8.

Claim 9.5. *For every $x \in \mathcal{U}$, there exists an $(x, a(x), b(x))$-consistent system of paths $CS(x) \in \mathfrak{C}'(x, a(x), b(x))$ such that $(E(CS(x)) \setminus \{\{x, x + a(x)\}, \{x, x + b(x)\}\}) \cap E(F) = \varnothing$.*

Proof. Suppose $x \in V(L)$, for some layer L. Suppose first that $x + a(x), x + b(x) \in V(L)$. Thus, we must show the existence of an $(x, a(x), b(x))$-consistent system of paths of Type I in $\mathfrak{C}'(x, a(x), b(x))$ with the desired property. Recall all the notation for consistent systems of paths introduced in Section 9.1, as well as Definition 9.3. By (CS), there is a collection $\mathcal{D}^{(2)}(x, a(x), b(x))$ of at least $\gamma n/2$ disjoint pairs of distinct directions $c, d \in \mathcal{D}(L) \setminus \{a(x), b(x)\}$ such that, for each $(c, d) \in \mathcal{D}^{(2)}(x, a(x), b(x))$, there is a collection $\mathcal{D}^{(4)}(x, a(x), b(x), c, d)$ of at least $\gamma n/2$ disjoint 4-tuples of distinct directions in $\mathcal{D}(L) \setminus \{a(x), b(x), c, d\}$ satisfying the following: for each $(c, d) \in \mathcal{D}^{(2)}(x, a(x), b(x))$ and each $(d_1, d_2, d_3, d_4) \in \mathcal{D}^{(4)}(x, a(x), b(x), c, d)$, $\mathfrak{C}'(x, a(x), b(x))$ contains the $(x, a(x), b(x))$-consistent system of paths $CS(c, d, d_1, d_2, d_3, d_4)$ defined as in Section 9.1. We will now show that there are many such consistent systems of paths which avoid F.

The choice of $(c, d) \in \mathcal{D}^{(2)}(x, a(x), b(x))$ determines six edges of the consistent system of paths: $e_1 := \{f(x + a(x)), f(x + a(x) + d)\}$, $e_2 := \{f(x + b(x)), f(x + b(x) + c)\}$, $e_3 := \{f(x + c), f(x)\}$, $e_4 := \{f(x), f(x + d)\}$, $e_5 := \{x + c, x + c + b(x)\}$ and $e_6 := \{x + d, x + d + a(x)\}$. Since $f(x)$, $f(x + a(x))$ and $f(x + b(x))$ are fixed and $\Delta(F) \leq \Psi$, for each $i \in [4]$ there are at most Ψ choices of $(c, d) \in \mathcal{D}^{(2)}(x, a(x), b(x))$ such that $e_i \in E(F)$. Furthermore, by (F^*), for each $i \in \{5, 6\}$ there are at most n/ℓ choices $(c, d) \in \mathcal{D}^{(2)}(x, a(x), b(x))$ such that $e_i \in E(F)$. Thus, there exist at least $\gamma n/2 - 4\Psi - 2n/\ell \geq \gamma n/4$ choices $(c, d) \in \mathcal{D}^{(2)}(x, a(x), b(x))$ such that $e_i \in E(H) \setminus E(F)$ for all $i \in [6]$. For any such choice of (c, d), the choice of $(d_1, d_2, d_3, d_4) \in \mathcal{D}^{(4)}(x, a(x), b(x), c, d)$ now determines the remaining eight edges of an $(x, a(x), b(x))$-consistent system of paths, each with a unique endpoint in $\{x + a(x), x + b(x), f(x + b(x)), x + c, f(x + c), f(x + d), x + d, f(x + a(x))\}$. It now follows by the fact that $\Delta(F) \leq \Psi$ that there are at most 8Ψ choices of $(d_1, d_2, d_3, d_4) \in \mathcal{D}^{(4)}(x, a(x), b(x), c, d)$ such that some of these eight edges lies in $E(F)$. In particular, we may fix a consistent system of paths $CS(x) \in \mathfrak{C}'(x, a(x), b(x))$ which satisfies the statement of the claim.

The cases where $x + a(x) \notin L$, $x + b(x) \in L$ and where $x + a(x), x + b(x) \notin L$ can be shown similarly. ◀

Note that $CS(x) \subseteq (H \setminus F) \cup H'$ for each $x \in \mathcal{U}$.

Claim 9.6. *For every $x \in \mathcal{U}$, we can extend $CS(x)$ into an $(x, a(x), b(x))$-special absorbing structure $SA(x)$ such that the following hold:*

(SA$_{\mathrm{i}}$) *for every $C \in \mathbf{C}(SA(x))$, we have that $\mathcal{M}_C \subseteq Q'$ and $\mathcal{M}_C$ is bonded in Q', and*
(SA$_{\mathrm{ii}}$) *every $C \in \mathbf{C}(SA(x))$ meets T''.*

Proof. For each $x \in \mathcal{U}$, we iterate through each $z \in \mathrm{endmol}(CS(x))$ fixing a cube $C(z) \in \mathcal{C}''(z)$. This will then determine $\mathbf{C}(SA(x))$, by taking the appropriate clones of $C(z)$. To see that this can be done, note that $|\mathrm{endmol}(CS(x))| \leq 6$. For each $z \in \mathrm{endmol}(CS(x))$, by (C7), there exist at least 20ℓ choices of $C(z) \in \mathcal{C}''(z)$ for which (SA$_{\mathrm{i}}$) and (SA$_{\mathrm{ii}}$) hold. Finally, by (C7.4) it follows that we can fix $C(z) \in \mathcal{C}''(z)$ such that $\mathcal{D}(C(z)) \cap \mathcal{D}(CS(x)) = \varnothing$. In particular, this implies that $C(z)$ is vertex-disjoint from all $C(z')$ already fixed with $z \neq z' \in \mathrm{endmol}(CS(x))$ and, therefore, this process forms a valid extension of $CS(x)$ into an $(x, a(x), b(x))$-special absorbing structure. ◀

For each $x \in \mathcal{U}$, let $SA(x)$ be an $(x, a(x), b(x))$-special absorbing structure which extends $CS(x)$, as determined by Claim 9.6. Note that, by (R3) and the fact that $V(SA(x)) \subseteq B^{2\ell}_{\mathcal{Q}^n}(x)$, the special absorbing structures in the collection $\{SA(x) : x \in \mathcal{U}\}$ are pairwise vertex-disjoint. Denote by $SA^v := \bigcup_{x \in \mathcal{U}} V(SA(x))$. Recall that, for any $C \in \mathbf{C}(SA(x))$, $C_I \subseteq I$ denotes the cube which C is a clone of. Given any tree $T^\bullet \subseteq I$ and any $x \in \mathcal{U}$, we say that $SA(x)$ *meets* $T^\bullet$ if, for all $C \in \mathbf{C}(SA(x))$, we have $V(C_I) \cap V(T^\bullet) \neq \varnothing$.

Recall that $\mathcal{C}'$ and $\mathcal{C}''$ were defined in Step 6. Let $\mathcal{C}^*_1 := \{C \in \mathcal{C}' : V(\mathcal{M}_C) \cap SA^v \neq \varnothing\}$. Note that, by (R3), (CB) for each $x \in \mathcal{U}$, there are at most $2^{2\ell}$ ℓ-cubes $C \in \mathcal{C}_1^*$ such that $V(C) \cap B^{10\ell}(x) \neq \varnothing$.

Let $\mathcal{C}_1' \coloneqq \mathcal{C}' \setminus \mathcal{C}_1^*$ and $\mathcal{C}_1'' = \mathcal{C}'' \setminus \mathcal{C}_1^*$. We now define a tree $T''' \subseteq T''$ in the following way. Consider each $x \in R' \cap V(T'')$ such that $x \in V(C)$ for some $C \in \mathcal{C}_1'$. By (C2), we have that x has a unique neighbour x' in T'', and $x' \in Z(x)$. By the definition of $Z(x)$ (see Step 7), it follows that $x' \in V(C')$ for some $C' \in \mathcal{C}''$. If $C' \notin \mathcal{C}_1''$, then we remove x from T''. We denote the resulting tree by T'''. Let $\mathcal{C}_1''' \coloneqq \{C \in \mathcal{C}_1' : V(C) \cap V(T''') \neq \varnothing\}$. By using (C1)–(C6), the definition of $\mathcal{C}_1'$, $\mathcal{C}_1''$ and $\mathcal{C}_1'''$, the construction of T''', and the maximum degree of F, we claim that the following now hold.

(C′1) $\Delta(T''') \leq 12D$.

(C′2) Any vertex $x \in R' \cap V(T''')$ is a leaf of T'''. Furthermore, if $x \in R' \cap V(T''')$, then $x \notin V(T)$ and its unique neighbour x' in T''' satisfies that $x' \in N_I(x) \cap V(T) \cap (\bigcup_{C \in \mathcal{C}_1''} V(C)) \subseteq Z(x)$.

(C′3) $B_I^5(\mathcal{U}_I) \cap V(T''') = \varnothing$.

(C′4) For all $x \in V(I)$ we have that $|\mathcal{C}_x \cap \mathcal{C}_1'''| \geq (1 - 2/\ell^4)n$.

(C′5) For each $x \in V_{\mathrm{sc}}$ and $i \in [K]$, there is an absorbing ℓ-cube pair $(C^l(x,i), C^r(x,i))$ for x in I, which is associated with some edge $e \in M(A_i(x))$. In particular, $(C^l(x,i), C^r(x,i))$ is such that there are two absorbing ℓ-cube pairs $(C_1^l(x,i), C_1^r(x,i))$ and $(C_2^l(x,i), C_2^r(x,i))$ in $H \cup G$, associated with $e \in M(A_i(x))$, for the clones x_1 and x_2 of x which correspond to (x,i). Additionally, each of these absorbing ℓ-cube pairs $(C^l(x,i), C^r(x,i))$ satisfies the following:

- (C′5.1) $(C_1^l(x,i), C_1^r(x,i)) \cup (C_2^l(x,i), C_2^r(x,i)) - V(\mathcal{M}_x) \subseteq Q'$;
- (C′5.2) the tips x^l of $C^l(x,i)$ and x^r of $C^r(x,i)$ lie in $R' \setminus V(T''')$, and $\{x, x^l\}, \{x, x^r\} \notin E(F_I)$; in particular, the tips x_1^l, x_1^r of $(C_1^l(x,i), C_1^r(x,i))$ and x_2^l, x_2^r of $(C_2^l(x,i), C_2^r(x,i))$ satisfy that $\{x_1, x_1^l\}, \{x_1, x_1^r\}, \{x_2, x_2^l\}, \{x_2, x_2^r\} \in E((H \cup G) \setminus F)$;
- (C′5.3) $C^l(x,i), C^r(x,i) \in \mathcal{C}_1'' \cap \mathcal{C}_1'''$, and
- (C′5.4) for any $x' \in V_{\mathrm{sc}}$ and $i' \in [K]$ with $(x',i') \neq (x,i)$, we have that $C^l(x,i)$, $C^r(x,i)$, $C^l(x',i')$ and $C^r(x',i')$ are vertex-disjoint.

Let $\mathcal{C}^{\mathrm{sc}}$ denote the collection of these absorbing ℓ-cube pairs.

(C′6) For each $x \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i \in [K]$, there is an absorbing ℓ-cube pair $(C^l(x,i), C^r(x,i))$ for x in I, which is associated with an edge $e \in M(A_i(x))$. In particular, $(C^l(x,i), C^r(x,i))$ is such that there are two absorbing ℓ-cube pairs $(C_1^l(x,i), C_1^r(x,i))$ and $(C_2^l(x,i), C_2^r(x,i))$ in $H \cup G$, associated with $e \in M(A_i(x))$, for the clones x_1 and x_2 of x which correspond to (x,i). Moreover, each of these absorbing ℓ-cube pairs $(C^l(x,i), C^r(x,i))$ satisfies the following:

- (C′6.1) $(C_1^l(x,i), C_1^r(x,i)) \cup (C_2^l(x,i), C_2^r(x,i)) - V(\mathcal{M}_x) \subseteq Q'$;
- (C′6.2) the tips x^l of $C^l(x,i)$ and x^r of $C^r(x,i)$ lie in R', and $\{x, x^l\}, \{x, x^r\} \notin E(F_I)$; in particular, the tips x_1^l, x_1^r of $(C_1^l(x,i), C_1^r(x,i))$ and x_2^l, x_2^r of $(C_2^l(x,i), C_2^r(x,i))$ satisfy that $\{x_1, x_1^l\}, \{x_1, x_1^r\}, \{x_2, x_2^l\}, \{x_2, x_2^r\} \in E((H \cup G) \setminus F)$;
- (C′6.3) $C^l(x,i), C^r(x,i) \in \mathcal{C}_1'' \cap \mathcal{C}_1'''$;
- (C′6.4) for any $x' \in V'_{\mathrm{abs}} \setminus V_{\mathrm{sc}}$ and $i' \in [K]$ with $(x',i') \neq (x,i)$, we have that $C^l(x,i)$, $C^r(x,i)$, $C^l(x',i')$ and $C^r(x',i')$ are vertex-disjoint, and
- (C′6.5) both $C^l(x,i)$ and $C^r(x,i)$ are vertex-disjoint from all cubes of absorbing ℓ-cube pairs in $\mathcal{C}^{\mathrm{sc}}$.

Let $\mathcal{C}^{\neg\mathrm{sc}}$ denote the set of these absorbing ℓ-cube pairs.

(C′7) For every $x \in \mathcal{U}$, there is an $(x, a(x), b(x))$-consistent system of paths $CS(x)$ in $(H \setminus F) \cup H'$ which extends into an $(x, a(x), b(x))$-special absorbing structure $SA(x)$ which meets T''' and with the property that, for every $C \in \mathbf{C}(SA(x,a,b))$, we have that $\mathcal{M}_{C_I} \subseteq Q'$ and $\mathcal{M}_{C_I}$ is bonded in Q'. Moreover, $\{x, x + a(x)\}, \{x, x + b(x)\} \in E(H')$.

(C′8) $C^\bullet$ intersects $V(T) \cap V(T''')$ (so, in particular, $C^\bullet \in \mathcal{C}_1'''$) and is different from all cubes described in (C′5), (C′6) and (C′7).

(C′9) $\bigcup_{C \in \mathcal{C}''' \setminus \mathcal{C}_1'''} V(C) \subseteq B_I^{3\ell}(\mathcal{U}_I)$.

Indeed, since $T''' \subseteq T''$, (C′1)–(C′3) follow immediately by (C1)–(C3), respectively. (C′4) follows from (C4), (R3) and (CB). Now, for each $x \in V_{\mathrm{sc}}$ and $i \in [K]$, consider the set $\mathcal{C}_1^{\mathrm{sc}}(x,i)$ described in (C5). We first remove from this set all absorbing ℓ-cube pairs any of whose cubes do not belong to $\mathcal{C}_1'$. Then, we remove all absorbing ℓ-cube pairs any of whose cubes do not intersect T'''. Finally, we remove all absorbing ℓ-cube pairs such that any of the edges joining its tips to x belong to F_I. Observe that by (CB) and (C′1) it follows that, for any $y \in V(I)$, we have $|B_I^{10\ell}(y) \cap (V(T'') \setminus V(T'''))| \leq 12D \cdot 2^{3\ell}$. Using this fact, (CB), and the fact that $\Delta(F) \leq \Psi$ (and, thus, $\Delta(F_I) \leq 2^s\Psi$), it follows that there is at least one absorbing ℓ-cube pair remaining in the collection. Let $(C^l(x,i), C^r(x,i))$ be such an absorbing ℓ-cube pair. Then, (C′5.2) and (C′5.3) hold by the choice above, and (C′5.1) and (C′5.4) hold by (C5.1) and (C5.4), respectively. For each $x \in V_{\text{abs}}' \setminus V_{\text{sc}}$ and $i \in [K]$, we proceed similarly from the set $\mathcal{C}_1^{\text{abs}}(x,i)$ to fix an absorbing ℓ-cube pair $(C^l(x,i), C^r(x,i)) \in \mathcal{C}_1^{\text{abs}}(x,i)$ which satisfies (C′6.2) and (C′6.3). Then, (C′6.1), (C′6.4) and (C′6.5) hold by (C6.1), (C6.4) and (C6.5), respectively. Furthermore, (C′7) holds by Claim 9.5, Claim 9.6 and the construction of T''' above. (Indeed, to see that each $SA(x)$ still meets T''', note that $(V(T'') \setminus V(T''')) \cap SA^v = \varnothing$.) For (C′8), by construction $C^\bullet \in \mathcal{C}'$ and $V(C^\bullet) \cap B_I^{3\ell/2}(\mathcal{U}_I) = \varnothing$. Therefore, $C^\bullet \in \mathcal{C}_1'$. The fact that $C^\bullet$ intersects $V(T) \cap V(T''')$ follows by (C8) and the fact that, in constructing T''', none of the leaves which are removed are vertices of T. Thus, in particular, $C^\bullet \in \mathcal{C}_1'''$. The rest of (C′8) follows immediately from (C8). Finally, (C′9) follows by the definition of $\mathcal{C}_1^*$. Indeed, consider the set $SA_I^v \subseteq V(I)$ of vertices such that each vertex in SA^v is a clone of some vertex in SA_I^v. It follows by construction (see Section 9.1) that for any $x \in SA_I^v$ we have $\operatorname{dist}(x, \mathcal{U}_I) \leq \ell + 2$. The claim follows since any ℓ-cube $C \in \mathcal{C}_1^*$ must intersect SA_I^v and any two vertices in C are at distance at most ℓ.

Let $\mathcal{C}_2' \coloneqq \bigcup_{x\in\mathcal{U}} \{C_I : C \in \mathbf{C}(SA(x))\}$ and $R'' \coloneqq R' \setminus \bigcup_{C\in\mathcal{C}_2'} V(C)$. Finally, let $\mathcal{C}_3''' \coloneqq \mathcal{C}_1''' \cup \mathcal{C}_2'$. Note that, by construction, any two cubes in $\mathcal{C}_3'''$ are vertex-disjoint.

Step 12: Constructing auxiliary trees T^* and τ_0. Let T^* be obtained from T''' by removing all leaves of T''' which lie in R''. In particular, by (C′2) and (C′8), we have that $C^\bullet$ intersects T^*.

We now construct an auxiliary tree τ_0, which will be used in the construction of an almost spanning cycle. The construction of τ_0 is identical to that in Step 10 of the proof of Theorem 8.1, except that $\mathcal{C}_3'''$ plays the role of $\mathcal{C}'$ in the definition of $\Gamma_1 \coloneqq T^* \cup \bigcup_{C\in\mathcal{C}_3'''} C$ and the subsequent steps, and that, for the depth-first search on Γ', the root vertex $v_0 \in V(\Gamma')$ is chosen to be the vertex which resulted from contracting $C^\bullet$.

Let $m \coloneqq |V(\tau_0)| - 1$. We define $v_1, \dots, v_m$, $\mathcal{M}(v_i)$, $\mathcal{A}_j(v_i)$, τ_i, p_i, $u_1^i, \dots, u_{p_i}^i$, e_k^i, f_k^i, j_k^i, ν_k^i, $\Delta(v_i)$ and $b(i)$ analogously to Step 10 of the proof of Theorem 8.1. In particular, we again have that

$$p_i \leq 12D - 1 \text{ if } v_i \text{ is an inner tree vertex, and } \Delta(\tau_0) \leq 12 \cdot 2^\ell D. \tag{9.11}$$

Step 13: Finding an external skeleton for T^*. We now generate an external skeleton, following Step 11 of the proof of Theorem 8.1. Using this external skeleton, we will construct a first skeleton in Step 16 and then extend it in Step 17 by incorporating the special absorbing structures for the vertices in $\mathcal{U}$.

Let $\mathcal{M}_{\text{Res}} \subseteq V(\mathcal{Q}^n)$ be the union of all the clones of R''. For each $x \in \mathcal{U}$, consider the graph $CS(x)_I \subseteq I$, and let $\mathcal{M}_{CS} \subseteq \mathcal{Q}^n$ be the union of all the clones of $\bigcup_{x\in\mathcal{U}} CS(x)_I$. We construct an external skeleton $L^\bullet$ which satisfies properties (ES1)–(ES4) as in the proof of Theorem 8.1 and the following variant of (ES5):

(ES5) $L^\bullet \cap (\mathcal{M}_{\text{Res}} \cup V(\mathcal{M}_{CS})) = \varnothing$.

The construction of $L^\bullet$ is identical to Step 11 of the proof of Theorem 8.1. The new (ES5) holds because of the definition of τ_0. Indeed, by (C′2) and (C′3) together with the definition of R'' and T^*, observe that $V(T^*) \cap (R'' \cup \bigcup_{x\in\mathcal{U}} V(CS(x)_I)) = \varnothing$. Moreover, by construction, all vertices in $L^\bullet$ are incident to some edge in a clone of the tree T^*, and thus they cannot lie in $\mathcal{M}_{\text{Res}} \cup V(\mathcal{M}_{CS})$.

Step 14: Constructing an auxiliary tree τ_0'. We now extend τ_0 to a new auxiliary tree τ_0' which encodes information about all cube molecules which intersect T'''. This is done as in

Step 12 of the proof of Theorem 8.1, except that, again, $\mathcal{C}_3'''$ plays the role of $\mathcal{C}'''$ and T''' plays the role of T''. Then, the cubes represented in τ_0' are precisely all those in $\mathcal{C}_3'''$.

Analogously to the proof of Theorem 8.1, it follows from (C′1) that

$$\begin{aligned}&d_{\tau_0'}(v) \leq 12D \text{ for all } v \in V(\tau_0') \text{ which are inner tree vertices, and}\\ &\Delta(\tau_0') \leq 12 \cdot 2^{\ell} D.\end{aligned} \tag{9.12}$$

By (C′5.3) and (C′6.3), we have that

(CP) every cube C belonging to some absorbing ℓ-cube pair in $\mathcal{C}^{\mathrm{sc}} \cup \mathcal{C}^{\neg\mathrm{sc}}$ is represented in τ_0'.

Finally, for each $x \in V(I)$, let $\zeta(x)$ denote the number of vertices $y \in N_I(x)$ which are represented in τ_0' by atomic vertices. By (C′4), we have that

$$\zeta(x) \geq (1 - 2/\ell^4)n. \tag{9.13}$$

Let $m' := |V(\tau_0')| - 1$ and label $V(\tau_0') \setminus V(\tau_0) = \{v_{m+1}, \dots, v_{m'}\}$ arbitrarily. We define τ_i', p_i', $u_1^i, \dots, u_{p_i'}^i$, e_k^i, f_k^i, j_k^i, ν_k^i, $\Delta(v_i)$, $b(i)$, ℓ_i and m_i as in Step 12 of the proof of Theorem 8.1.

Step 15: Fixing absorbing ℓ-cube pairs for the vertices that need to be absorbed. We can now determine every vertex in $V(\mathcal{Q}^n)$ that will have to be absorbed via absorbing ℓ-cube pairs into the almost spanning cycle we are going to construct. Recall from Step 11 that $SA^v = \bigcup_{y \in \mathcal{U}} V(SA(y))$. For every vertex $x \in V(I)$ not represented in τ_0', we will have to absorb all vertices in $\mathcal{M}_x \setminus SA^v$. Furthermore, for each $v \in V(\tau_0)$ which is an inner tree vertex, we will also need to absorb all vertices in $\mathcal{M}_v \setminus L^\bullet = \mathcal{M}_v \setminus (L^\bullet \cup SA^v)$. (The fact that $\mathcal{M}_v \cap SA^v = \varnothing$ follows by (C′3).) Recall the definition of V_{abs}' from Step 10. Let $V_{\mathrm{abs}} \subseteq V(I)$ be the set of all vertices which are not represented in τ_0' by an atomic vertex. Therefore, V_{abs} is the set of all vertices $x \in V(I)$ such that some clone of x needs to be absorbed. Moreover, $V_{\mathrm{abs}} = V(I) \setminus \bigcup_{C \in \mathcal{C}_3'''} V(C)$ and, thus, (C′9) and the definition of $\mathcal{C}_3'''$ at the end of Step 11 imply that $V_{\mathrm{abs}} \subseteq V_{\mathrm{abs}}'$. It follows from (9.13) that

$$|V_{\mathrm{abs}}| \leq 2^{n-s+1}/\ell^4. \tag{9.14}$$

Now, for each $x \in V_{\mathrm{abs}}$, we will pair all those vertices in $\mathcal{M}_x$ which need to be absorbed (each pair consisting of one vertex of each parity) and fix an absorbing ℓ-cube pair for each such pair of vertices. Recall that a difference between this pairing and the pairing in Step 13 of the proof of Theorem 8.1 is that, in Theorem 8.1, we could guarantee that each pair was contained in one of the slices defined in Step 3. Since a special absorbing structure might not lie in a single slice, we now cannot guarantee this anymore. Instead, we can impose that each pair either lies in a slice or in two consecutive slices (with respect to their labelling). The absorbing ℓ-cube pair that we fix for each pair will be the one given by (C′5) or (C′6), depending on whether $x \in V_{\mathrm{sc}}$ or not.

For each $x \in V_{\mathrm{abs}}$, let $S(x) := V(\mathcal{M}_x) \cap (L^\bullet \cup SA^v) = V(\mathcal{M}_x) \cap (L^\bullet \cup V(\mathcal{M}_{CS}))$. It follows by (ES1)–(ES5), (R3) and the definition of our special absorbing structures that $|S(x)| \leq 25D$ and $S(x)$ contains the same number of vertices of each parity. (Here we also use that $p_i \leq 12D - 1$ for every inner tree vertex v_i by (9.11) and (9.12).) Therefore, the matching $\mathfrak{M}(\mathcal{M}_x, S(x))$ defined in Step 3 is well defined, and we can use this matching to define our pairing of the vertices in $V(\mathcal{M}_x) \setminus S(x)$. Recall that each edge $e \in \mathfrak{M}(\mathcal{M}_x, S(x))$ gives rise to a unique index $i \in [K]$ via the relation $N(e) = A_i(x)$. (Here we ignore all those indices $i' \in [K]$ arising by artificially increasing the size of $\mathfrak{A}(x)$, see of Step 4 as well as Step 4 in the proof of Theorem 8.1.) For each $x \in V_{\mathrm{abs}}$, let $\mathfrak{I}_x \subseteq [K]$ be the set of indices $i \in [K]$ which correspond to edges in $\mathfrak{M}(\mathcal{M}_x, S(x))$.

For each $x \in V_{\mathrm{abs}}$ and $i \in \mathfrak{I}_x$, as stated in (C′5) and (C′6), we have already fixed an absorbing ℓ-cube pair for the clones of x corresponding to (x, i). Let

$$V^{\mathrm{abs}} := \bigcup_{x \in V_{\mathrm{abs}}} V(\mathcal{M}_x) \setminus (L^\bullet \cup SA^v).$$

(Thus, in particular, $V^{\mathrm{abs}} \cap \mathcal{U} = \varnothing$.) As discussed above, this is the set of all vertices that need to be absorbed via absorbing ℓ-cube pairs. Recall that Q' was defined before (C1)–(C8). It follows from (C′5) and (C′6) that $((H \cup G) \setminus F) \cup Q'$ contains a set $\mathcal{C}^{\mathrm{abs}} = \{(C^l(u), C^r(u)) : u \in V^{\mathrm{abs}}\}$

of absorbing ℓ-cube pairs such that (C$_1$)–(C$_4$) in the proof of Theorem 8.1 hold with Q', $\mathcal{C}_1''$ and $\mathcal{C}_1'''$ playing the roles of G', $\mathcal{C}''$ and $\mathcal{C}'''$, except that (C$_{2.2}$) is now replaced by the following:

(C$_{2.2}$) if $f_i = \{u_i, u_i'\}$, then there is a vertex $v \in V_{\rm abs}$ such that u_i and u_i' are clones of v which lie in either the same or consecutive slices of $\mathcal{M}_v$, and $(C^l(u_i), C^r(u_i))$ and $(C^l(u_i'), C^r(u_i'))$ are clones of the same absorbing ℓ-cube pair for v in I such that $(C^l(u_i), C^r(u_i))$ lies in the same layer as u_i and $(C^l(u_i'), C^r(u_i'))$ lies in the same layer as u_i'.

We denote by $\mathfrak{L}$, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the collections of all left absorber tips, right absorber tips, and third absorber vertices, respectively, of the absorbing ℓ-cube pairs in $\mathcal{C}^{\rm abs}$. Observe that by (C$_1$)–(C$_3$) the following properties are satisfied:

(C*1) For all $i \in [m']_0$ such that v_i is an atomic vertex, we have that $|\mathfrak{L} \cap V(\mathcal{M}(v_i))| \in \{0, 2\}$. If $|\mathfrak{L} \cap V(\mathcal{M}(v_i))| = 2$, then these two vertices u, u' lie in different atoms of either the same or consecutive slices of $\mathcal{M}(v_i)$, and satisfy that $u \neq_{\rm p} u'$.

(C*2) For all $i \in [m']_0$ such that v_i is an atomic vertex, we have that $|(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}(v_i))| \in \{0, 4\}$. If $|(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}(v_i))| = 4$, then these four vertices form two pairs such that one vertex of each pair belongs to $\mathfrak{R}_1$ and the other to $\mathfrak{R}_2$. Each of these pairs lies in a different atom of the same or consecutive slices of $\mathcal{M}(v_i)$ and satisfies that its two vertices are adjacent in Q'.

(C*3) For all $i \in [m']_0$ such that v_i is an atomic vertex, if $\mathfrak{L} \cap V(\mathcal{M}(v_i)) \neq \varnothing$, then $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}(v_i)) = \varnothing$.

(C*4) The sets described in (C*1) and (C*2) partition $\mathfrak{L}$ and $\mathfrak{R}_1 \cup \mathfrak{R}_2$, respectively.

Indeed, (C*1)–(C*3) follow from (C$_1$)–(C$_3$), and (C*4) follows by (CP).

For each $u \in V^{\rm abs}$, we denote the edge consisting of the right absorber tip and the third absorber vertex of $(C^l(u), C^r(u))$ by $e_{\rm abs}(u)$, and we denote by $\mathcal{P}^{\rm abs}(u)$ the path of length three formed by the third absorber vertex, the left absorber tip, u, and the right absorber tip, visited in this order. Note that $e_{\rm abs}(u) \in E(Q')$ by (C$_1$). Moreover, recall that $\mathcal{C}^{\rm abs}$ consists of absorbing ℓ-cube pairs in $((H \cup G) \setminus F) \cup Q'$. Thus, $\mathcal{P}^{\rm abs}(u) \subseteq ((H \cup G) \setminus F) \cup Q'$.

Step 16: Constructing the skeleton. As in Step 14 of the proof of Theorem 8.1, we now define the skeleton $\mathcal{L} = (x_1, \dots, x_r)$ for the almost spanning cycle. We again let $\mathcal{L}^\bullet \coloneqq \{x_1, \dots, x_r\}$. The skeleton $\mathcal{L}$ again satisfies (S1)–(S7) in the proof of Theorem 8.1, except that Q' now plays the role of G' and, (S3) and (S7) are now replaced by the following.

(S3) For every $k \in [r-1]$, if x_k and x_{k+1} do not both lie in the same slice of a cube molecule represented in $\mathcal{T}_0'$, then $\{x_k, x_{k+1}\} \in E(Q')$. Moreover, in this case, if x_{k+1} lies in a cube molecule represented in $\mathcal{T}_0'$, then x_{k+2} lies in the same slice of this cube molecule as x_{k+1}.

(S7) $\mathcal{L}^\bullet \cap (\mathfrak{L} \cup \mathfrak{R}_1 \cup V^{\rm abs} \cup V(\mathcal{M}_{CS})) = \varnothing$ and $L^\bullet \subseteq \mathcal{L}^\bullet$.

The construction of $\mathcal{L}$ is identical to that in Theorem 8.1. The only difference is that the 'forbidden set $\mathcal{F}$' which the skeleton has to avoid is replaced by $\mathcal{F} \coloneqq \mathfrak{L} \cup \mathfrak{R}_1 \cup L^\bullet \cup V(\mathcal{M}_{CS})$ (this is required to ensure that (S7) holds). For each $i \in [m']_0$ such that v_i is an atomic vertex and each $j \in [t]$, let $\mathfrak{J}_{i,j} \coloneqq \{k \in [r] : x_k, x_{k+1} \in V(\mathcal{M}_j(v_i))\}$ and $S_{i,j} \coloneqq \{\{x_k, x_{k+1}\} : k \in \mathfrak{J}_{i,j}\}$.

Step 17. Incorporating special absorbing structures into the skeleton. In this step, we are going to incorporate all special absorbing structures fixed in Step 11 into the skeleton we just constructed. Note that all cube molecules referred to are represented by atomic vertices in $\mathcal{T}_0'$, and all slices referred to are one of the t slices of each of these molecules defined in Step 3. For each $x \in \mathcal{U}$, consider the consistent system of paths $CS(x)$ and the special absorbing structure $SA(x)$ given by (C'7). By (S7), we have that $\mathcal{L}$ avoids $CS(x)$. Moreover, by the definition of $\mathcal{C}_3'''$ at the end of Step 11, each $C' \in \mathbf{C}(SA(x))$ is a clone of some $C \in \mathcal{C}_3'''$. Thus, by (S5) we have that $\mathcal{L}$ has positive intersection with each slice which contains a vertex of $\mathrm{end}(CS(x))$.

Recall that a special absorbing structure is a tuple of paths and cubes (see Section 9.1). For each $z \in \mathcal{U}$, let P_1^z denote the first path of $SA(z)$. Let $x(z) \in \mathcal{L}^\bullet$ be the first vertex in $\mathcal{L}$ that is contained in the slice which contains the first vertex of P_1^z. Let $x'(z)$ be the successor of $x(z)$ in $\mathcal{L}$ (in particular, by (S3), both $x(z)$ and $x'(z)$ lie in the same slice). Now, for each $z \in \mathcal{U}$, depending on the type of the special absorbing structure $SA(z)$, we will update $\mathcal{L}$ in different ways.

(I) If $SA(z)$ is a special absorbing structure of Type I, proceed as follows. Let $P_1^z,\dots,P_6^z$ be the six paths of $SA(z)$. Let $S := \bigtimes_{i=1}^{6} P_i^z$ and let S^{-1} denote the same sequence of vertices in reverse order. If $x(z)$ has opposite parity to the initial vertex of P_1^z, then we replace the segment $(x(z),x'(z))$ of $\mathcal{L}$ by $(x(z),S,x'(z))$; otherwise, we we replace the segment $(x(z),x'(z))$ by $(x(z),S^{-1},x'(z))$.

(II) If $SA(z)$ is a special absorbing structure of Type II, we proceed as follows. Let P_1^z and P_2^z be the two paths of $SA(z)$. Let y^1 and x_0^1 be the first and last vertices of P_1^z, and let y^2 and x_0^2 be the first and last vertices of P_2^z, respectively. Let $v,v'\in V(\tau_0')$ and $t_1,t_2,t_3\in[t]$ be such that $y^1\in V(\mathcal{M}_{t_1}(v))$, $x_0^1\in V(\mathcal{M}_{t_2}(v'))$, and $y^2\in V(\mathcal{M}_{t_3}(v'))$ (this implies $x_0^2\in V(\mathcal{M}_{t_3}(v))$).

We now define two sequences of vertices S_1^z and S_2^z following similar ideas to Step 16. Recall that, for each $i\in[2^s]$, we use $\hat{e}_i$ to denote the direction of the edges between L_i and L_{i+1}. If $t_3\geq t_2$, let $m_1:=t_3-t_2$; otherwise, let $m_1:=t-(t_2-t_3)$. For each $k\in[m_1-1]_0$, iteratively choose a vertex $y_k^1\in V(\mathcal{A}_{(t_2+k)q}(v'))\setminus\mathcal{L}^\bullet$ satisfying that

1. $y_k^1\neq_{\mathrm{p}} x_k^1$;
2. $x_{k+1}^1 := y_k^1+\hat{e}_{(t_2+k)q}\notin\mathcal{L}^\bullet$, and
3. $\{y_k^1,x_{k+1}^1\}\in E(Q')$.

We set $S_1^z:=\bigtimes_{k=0}^{m_1-1}(y_k^1,x_{k+1}^1)$.

In order to construct S_2^z, we proceed similarly. If $t_3\geq t_1$, let $m_2:=t_3-t_1$; otherwise, let $m_2:=t-(t_1-t_3)$. For each $k\in[m_2-1]_0$, iteratively choose a vertex $y_k^2\in V(\mathcal{A}_{(t_3-k-1)q+1}(v))\setminus\mathcal{L}^\bullet$ satisfying that

1. $y_k^2\neq_{\mathrm{p}} x_k^2$;
2. $x_{k+1}^2 := y_k^2+\hat{e}_{(t_3-k-1)q}\notin\mathcal{L}^\bullet$, and
3. $\{y_k^2,x_{k+1}^2\}\in E(Q')$.

Now, let $S_2^z:=\bigtimes_{k=0}^{m_2-1}(y_k^2,x_{k+1}^2)$.

Let $S:=P_1^z\times S_1^z\times P_2^z\times S_2^z$, and let S^{-1} denote the same sequence of vertices in reverse order. Finally, we replace the segment $(x(z),x'(z))$ of $\mathcal{L}$ by $(x(z),S,x'(z))$ if $x(z)$ has parity opposite to the initial vertex of P_1^z; otherwise, we replace $(x(z),x'(z))$ by $(x(z),S^{-1},x'(z))$.

(III) If $SA(z)$ is a special absorbing structure of Type III, we proceed as follows. Let P_1^z, P_2^z and P_3^z be the three paths of $SA(z)$. For each $i\in[3]$, let y^i and x_0^i be the first and last vertices of P_i^z, respectively. Let $v_1,v_2,v_3\in V(\tau_0')$ and $t_1,t_2\in[t]$ be such that $y^1\in V(\mathcal{M}_{t_1}(v_1))$, $y^2\in V(\mathcal{M}_{t_1}(v_2))$ and $y^3\in V(\mathcal{M}_{t_2}(v_3))$ (note this implies that $x_0^1\in V(\mathcal{M}_{t_2}(v_2))$, $x_0^2\in V(\mathcal{M}_{t_1}(v_3))$ and $x_0^3\in V(\mathcal{M}_{t_2}(v_1))$).

For each $i\in[3]$, we define a sequence S_i^z as follows. If $t_2\geq t_1$, let $m^*:=t_2-t_1$; otherwise, let $m^*:=t-(t_1-t_2)$. For each $k\in[m^*-1]_0$, iteratively choose three vertices $y_k^1\in V(\mathcal{A}_{(t_2-k-1)q+1}(v_2))\setminus\mathcal{L}^\bullet$, $y_k^2\in V(\mathcal{A}_{(t_1+k)q}(v_3))\setminus\mathcal{L}^\bullet$ and $y_k^3\in V(\mathcal{A}_{(t_2-k-1)q+1}(v_1))\setminus\mathcal{L}^\bullet$ satisfying that

1. $y_k^i\neq_{\mathrm{p}} x_k^i$ for all $i\in[3]$;
2. $x_{k+1}^1:=y_k^1+\hat{e}_{(t_2-k-1)q}\notin\mathcal{L}^\bullet$, $x_{k+1}^2:=y_k^2+\hat{e}_{(t_1+k)q}\notin\mathcal{L}^\bullet$ and $x_{k+1}^3:=y_k^3+\hat{e}_{(t_2-k-1)q}\notin\mathcal{L}^\bullet$, and
3. $\{y_k^i,x_{k+1}^i\}\in E(Q')$ for all $i\in[3]$.

Then, for each $i\in[3]$, we define $S_i^z:=\bigtimes_{k=0}^{m^*-1}(y_k^i,x_{k+1}^i)$.

Let $S:=\bigtimes_{i=1}^{3}(P_i^z\times S_i^z)$, and let S^{-1} denote the same sequence in reverse order. Finally, we replace the segment $(x(z),x'(z))$ of $\mathcal{L}$ by $(x(z),S,x'(z))$ if $x(z)$ has parity opposite to the initial vertex of P_1^z; otherwise, we replace $(x(z),x'(z))$ by $(x(z),S^{-1},x'(z))$.

Write $\mathcal{L}=(x_1,\dots,x_r)$, for some $r\in\mathbb{N}$, for the extended skeleton into which all the special absorbing structures $SA(z)$ for $z\in\mathcal{U}$ have been incorporated, and let $\mathcal{L}^\bullet:=\{x_1,\dots,x_r\}$. It follows from (S1)–(S7) and the construction above (together with the choice of v_0 in Step 12) that the following properties hold:

(S′1) For all distinct $k, k' \in [r]$, we have that $x_k \neq x_{k'}$.
(S′2) $\{x_1, x_r\} \in E(Q')$.
(S′3) For every $k \in [r-1]$, if x_k and x_{k+1} do not both lie in the same slice of a cube molecule represented in $\mathcal{T}'_0$, then $\{x_k, x_{k+1}\} \in E((H \setminus F) \cup H' \cup Q')$. Moreover, in this case $\{x_k, x_{k+1}\} \in E(Q')$ unless both x_k and x_{k+1} lie in SA^v.
(S′4) For every $i \in [m']_0$ and every $j \in [t]$, no three consecutive vertices of $\mathcal{L}$ lie in $\mathcal{M}_j(v_i)$ (here $\mathcal{L}$ is viewed as a cyclic sequence of vertices).
(S′5) For every $i \in [m']$ such that v_i is an atomic vertex and every $j \in [t]$, we have that $|V(\mathcal{M}_j(v_i)) \cap \mathcal{L}^\bullet|$ is even and $4 \leq |V(\mathcal{M}_j(v_i)) \cap \mathcal{L}^\bullet| \leq 14$. Moreover, $|V(\mathcal{M}_t(v_0)) \cap \mathcal{L}^\bullet| = 4$.
(S′6) For all $k \in [r]$ except two values, we have that $x_k \neq_{\mathrm{p}} x_{k+1}$. The remaining two values $k_1, k_2 \in [r]$ correspond to two pairs of vertices $x_{k_1}, x_{k_1+1}, x_{k_2}, x_{k_2+1} \in V(\mathcal{M}_t(v_0))$. For these two values, we have that $x_{k_1} \neq_{\mathrm{p}} x_{k_2}$ and either
 (i) $x_{k_1} =_{\mathrm{p}} x_{k_1+1}$ and $x_{k_2} =_{\mathrm{p}} x_{k_2+1}$, or
 (ii) $x_{k_1} \neq_{\mathrm{p}} x_{k_1+1}$ and $x_{k_2} \neq_{\mathrm{p}} x_{k_2+1}$,
 where $x_{k_1}, x_{k_2} \in V(\mathcal{A}_{(t-1)q+1}(v_0))$ and $x_{k_1+1}, x_{k_2+1} \in V(\mathcal{A}_{tq}(v_0))$.
(S′7) $\mathcal{L}^\bullet \cap (\mathfrak{L} \cup \mathfrak{R}_1 \cup V^{\mathrm{abs}}) = \varnothing$ and $L^\bullet \subseteq \mathcal{L}^\bullet$.

Indeed, for (S′3)–(S′6) we make use of the properties of the paths P^z_j defined in Section 9.1 as well as (AS) (see Section 9.1) and (C′7). We also use that the set of all cube molecules represented in $\mathcal{T}'_0$ is precisely $\mathcal{C}'''_3 = \mathcal{C}'''_1 \,\dot\cup\, \mathcal{C}'_2$. Recall that, by the definition of $\mathcal{C}'''_1$, for each $C \in \mathcal{C}'''_1$ we have $V(\mathcal{M}_C) \cap SA^v = \varnothing$. Moreover, by (R3) and (AS), for each $C \in \mathcal{C}'_2$ we have $|V(\mathcal{M}_C) \cap SA^v| = 2$, and these two vertices $x^1_C, x^2_C \in V(\mathcal{M}_C) \cap SA^v$ satisfy the following properties:

(i) there is some $z \in \mathcal{U}$ and two consecutive paths P^z_i, P^z_{i+1} in $SA(z)$ (with indices taken cyclically) such that x^1_C is the final vertex of P^z_i and x^2_C is the first vertex of P^z_{i+1}, and
(ii) $\mathbf{C}(SA(z))$ contains two clones C_1, C_2 of C, where $x^1_C \in V(C_1)$ and $x^2_C \in V(C_2)$.

Moreover, to check (S′5) for case (I), note that the definition of f in a consistent system of paths of Type I in Section 9.1 implies that, for each $i \in [6]$, the final vertex of P^z_i and the first vertex of P^z_{i+1} lie in the same slice.

Step 18: Constructing an almost spanning cycle. Similarly to Step 15 in the proof of Theorem 8.1, we will now apply the connecting lemmas to obtain an almost spanning cycle in $(H \setminus F) \cup H' \cup Q'$ from $\mathcal{L} = (x_1, \ldots, x_r)$. For each $i \in [m']$ such that v_i is an atomic vertex, by (C*1) there is at most one value $k(i) \in [t]$ such that $|\mathfrak{L} \cap V(\mathcal{M}_{k(i)}(v_i))| = 1$ and $|\mathfrak{L} \cap V(\mathcal{M}_{k(i)+1}(v_i))| = 1$. If such $k(i)$ exists, then we denote by $k^*(i)$ an additional index not in $[t]$ and let $\mathfrak{J}_{i,k^*(i)} := \mathfrak{J}_{i,k(i)} \cup \mathfrak{J}_{i,k(i)+1}$, $S_{i,k^*(i)} := S_{i,k(i)} \cup S_{i,k(i)+1}$ and $\mathcal{M}_{k^*(i)}(v_i) := \mathcal{M}_{k(i)}(v_i) \cup \mathcal{M}_{k(i)+1}(v_i)$. Let

$$\mathfrak{T}(i) := \begin{cases} [t] & \text{if there is no } k(i) \text{ as above,} \\ ([t] \cup \{k^*(i)\}) \setminus \{k(i), k(i)+1\} & \text{otherwise.} \end{cases}$$

Observe that the definition of v_0 in Step 12 together with (C′8) ensures that $\mathfrak{T}(0) = [t]$.

For each $i \in [m']_0$ such that v_i is an atomic vertex and for each $j \in \mathfrak{T}(i)$, except the pair $(0, t)$, we apply Lemma 8.8 to $\mathcal{M}_j(v_i)$ and the graph Q', with $\mathfrak{L} \cap V(\mathcal{M}_j(v_i))$, $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap V(\mathcal{M}_j(v_i))$ and $S_{i,j}$ playing the roles of L, R and the pairs of vertices described in Lemma 8.8(C3), respectively. For $\mathcal{M}_t(v_0)$, we apply Lemma 8.8 or Lemma 8.9 depending on whether (ii) or (i) holds in (S′6). For each $i \in [m']_0$ such that v_i is an atomic vertex and each $j \in \mathfrak{T}(i)$, this yields $|\mathfrak{J}_{i,j}|$ vertex-disjoint paths $(\mathcal{P}^{i,j}_k)_{k \in \mathfrak{J}_{i,j}}$ in $\mathcal{M}_j(v_i) \cup Q' = Q'$ such that, for each $k \in \mathfrak{J}_{i,j}$, properties (i)–(iii) in Step 15 of the proof of Theorem 8.1 hold.

Now consider the path obtained as follows by going through $\mathcal{L}$. Start with x_1. For each $k \in [r]$, if there exist $i \in [m']_0$ and $j \in \mathfrak{T}(i)$ such that $\{x_k, x_{k+1}\} \in S_{i,j}$, add $\mathcal{P}^{i,j}_k$ to the path; otherwise, add the edge $\{x_k, x_{k+1}\}$ (this must be an edge of $(H \setminus F) \cup H' \cup Q'$ by (S′3)). Finally, add the edge $\{x_r, x_1\}$ of Q' (this is given by (S′2)) to the path to close it into a cycle $\mathfrak{H}$ in $(H \setminus F) \cup H' \cup Q'$. This cycle $\mathfrak{H}$ satisfies (HC1)–(HC3) as in the proof of Theorem 8.1 as well as the following:

(HC4) For all $x \in \mathcal{U}$, we have that $\{x, x + a(x)\}, \{x, x + b(x)\} \in E(\mathfrak{H})$.

(HC5) For all $x \in V(\mathfrak{H}) \setminus SA^v$, each of the two edges of $\mathfrak{H}$ incident to x lies in Q'.

Indeed, (HC4) follows immediately by the definition of P_1 in each of the three types of special absorbing structures defined in Section 9.1, and (HC5) follows from (S′3).

Step 19: Absorbing vertices to form a Hamilton cycle. Similarly as in Step 16 of the proof of Theorem 8.1, for each $u \in V^{\mathrm{abs}}$ we now replace the edge $e_{\mathrm{abs}}(u)$ by the path $\mathcal{P}_{\mathrm{abs}}(u)$ (recall from the end of Step 15 that $\mathcal{P}_{\mathrm{abs}}(u)$ lies in $((H \cup G) \setminus F) \cup Q')$. Clearly, this incorporates all vertices of $\mathfrak{L} \cup V^{\mathrm{abs}}$ into the cycle and, by (HC2) and (HC3), the resulting cycle $\mathfrak{H}'$ is Hamiltonian. Moreover, since by (C_3) the endvertices of each edge $e_{\mathrm{abs}}(u)$ lie in cubes belonging to $\mathcal{C}_1'''$, all these endvertices avoid $\mathcal{U}$. Thus, by (HC4), for each $x \in \mathcal{U}$ the edges at x in $\mathfrak{H}'$ are still $\{x, x + a(x)\}$ and $\{x, x + b(x)\}$, and so, in particular, by (C′7) these edges belong to H'.

It now remains to show that $\mathfrak{H}'$ is $(\mathcal{U}, \ell^2, s)$-good. Fix any vertex $x \in \mathcal{U}$. Let $Y_x \coloneqq N_{\mathcal{Q}^n}(x) \setminus (V(SA(x)) \cup V^{\mathrm{abs}})$ (that is, by (C′3), Y_x is the set of all vertices in $N_{\mathcal{Q}^n}(x) \setminus SA^v = N_{\mathcal{Q}^n}(x) \setminus V(SA(x))$ which lie in clones of cubes which are represented in τ_0' by atomic vertices). By (9.13), we have that $|Y_x| \geq (1 - 2/\ell^4)n - |V(SA(x))| \geq (1 - 1/\ell^3)n$. Claim 9.3(ii) implies that $Y_x \cap (\mathfrak{L} \cup \mathfrak{R}_1 \cup \mathfrak{R}_2) = \varnothing$, so by definition we have that $Y_x \cap \bigcup_{u \in V^{\mathrm{abs}}} V(\mathcal{P}_{\mathrm{abs}}(u)) = \varnothing$. It then follows by (HC5) that, for each $y \in Y_x$, each of the two edges of $\mathfrak{H}'$ incident to y lies in Q'. But Q' is $(\mathcal{U}, 2\ell^2, s)$-good by Claim 9.4. Now, even if all the edges incident to the remaining vertices $y \in N_{\mathcal{Q}^n}(x) \setminus Y_x$ used the same pair of directions, it follows that the edges of $\mathfrak{H}'$ incident to the vertices in $N_{\mathcal{Q}^n}(x)$ use each direction of $\mathcal{Q}^n$ which is not an s-direction at most $n/\ell^3 + n/(2\ell^2) \leq n/\ell^2$ times. □

9.4. **Proofs of Theorem 1.8 and Theorem 1.5.** We now deduce Theorems 1.5 and 1.8 from Theorem 9.6.

Proof of Theorem 1.8. Let $0 < 1/n \ll 1/\ell \ll \varepsilon_1 \ll \varepsilon \ll \varepsilon_2 \ll \gamma \ll 1/k \leq 1$. Let $s \coloneqq 10\ell$. Let $H^* \sim \mathcal{Q}^n_{1/2-2\varepsilon}$ and $Q \sim \mathcal{Q}^n_{\varepsilon}$. Observe that $H^* \cup Q \sim \mathcal{Q}^n_{1/2-\varepsilon'}$ for some $\varepsilon' \geq \varepsilon$, so it suffices to prove that $H \cup H^* \cup Q$ contains the desired Hamilton cycles and perfect matchings.

By Lemma 9.4 with 2ε playing the role of ε, we have that a.a.s. H^* is $(s, \ell, \varepsilon_1, \varepsilon_2, \gamma, \mathcal{U}(H^*, \varepsilon_1))$-robust. Condition on this event and let $\mathcal{U} \coloneqq \mathcal{U}(H^*, \varepsilon_1)$. By an application of Lemma 9.4(ii), it follows that there exists a decomposition of H^* into $r \coloneqq \lceil k/2 \rceil$ edge-disjoint spanning subgraphs $H_1^*, \ldots, H_r^*$ such that, for every $i \in [r]$, we have that H_i^* is $(s, \ell, \varepsilon_1/(2r), \varepsilon_2, \gamma/r^{10}), \mathcal{U})$-robust.

Consider a random decomposition of Q into r edge-disjoint spanning subgraphs $Q_1, \ldots, Q_r$ in such a way that, if $e \in Q$, then e is assigned to one of the Q_i chosen uniformly at random and independently of all other edges. It follows that, for all $i \in [r]$, we have $Q_i \sim \mathcal{Q}^n_{\varepsilon/r}$.

Let Φ be a constant such that Theorem 9.6 holds with $\varepsilon_1/(2r)$, γ/r^{10}, ε/r and $r+2$ playing the roles of ε_1, γ, η and c, respectively. (In particular, $\Phi \geq r$.) For each $i \in [r]$, apply Theorem 9.6 with H_i^*, Q_i, $\varepsilon_1/(2r)$, γ/r^{10}, ε/r and $r+2$ playing the roles of H, Q, ε_1, γ, η and c, respectively, to conclude that a.a.s. there is a $(\mathcal{U}, \ell^2, s)$-good subgraph $Q_i' \subseteq Q_i$ with $\Delta(Q_i') \leq \Phi$ such that, for every $H' \subseteq \mathcal{Q}^n$ such that $d_{H'}(x) \geq 2$ for every $x \in \mathcal{U}$, and every $F \subseteq \mathcal{Q}^n$ with $\Delta(F) \leq (r+2)\Phi$ which is $(\mathcal{U}, \ell, s)$-good, we have that $((H_i^* \cup Q_i) \setminus F) \cup H' \cup Q_i'$ contains a $(\mathcal{U}, \ell^2, s)$-good Hamilton cycle C such that, for all $x \in \mathcal{U}$, both edges of C incident to x belong to H'. Condition on the event that this holds for all $i \in [r]$ (which holds a.a.s. by a union bound).

Now consider the graph H from the statement of Theorem 1.8. By (R3) in Definition 9.3, we can greedily find r edge-disjoint subgraphs $H_1, \ldots, H_r \subseteq H$ such that

(i) for each $i \in [\lfloor k/2 \rfloor]$, we have that $|E(H_i)| = 2|\mathcal{U}|$ and $d_{H_i}(x) = 2$ for every $x \in \mathcal{U}$, and
(ii) if $2r = k + 1$, then H_r is a matching of size $|\mathcal{U}|$ such that $d_{H_r}(x) = 1$ for all $x \in \mathcal{U}$.

Suppose first that $2r = k$. We are going to find r edge-disjoint $(\mathcal{U}, \ell^2, s)$-good Hamilton cycles $C_1, \ldots, C_r$ with $H_i \subseteq C_i$ iteratively. Suppose that for some $i \in [r]$ we have already found $C_1, \ldots, C_{i-1}$. Let $F_i \coloneqq \bigcup_{j=1}^r Q_j' \cup \bigcup_{j=1}^{i-1} C_j$. It follows by construction that F_i is $(\mathcal{U}, \ell, s)$-good and $\Delta(F_i) \leq r(\Phi+2) \leq (r+2)\Phi$. Then, by the conditioning above, the graph $((H_i^* \cup Q_i) \setminus F_i) \cup H_i \cup Q_i'$ must contain a $(\mathcal{U}, \ell^2, s)$-good Hamilton cycle C_i such that, for each $u \in \mathcal{U}$, both edges of C_i incident to x belong to H_i. In particular, $H_i \subseteq C_i$. Take one such cycle and proceed.

In order to see that these r cycles are pairwise edge-disjoint, suppose that there exist $i, j \in [r]$ with $i < j$ such that $E(C_i) \cap E(C_j) \neq \varnothing$, and let $e \in E(C_i) \cap E(C_j)$. Observe that $e \notin E(H_i) \cup E(H_j)$ because, otherwise, we would have e incident to some vertex $x \in \mathcal{U}$, and we know that both edges incident to x in C_i and C_j belong to H_i and H_j, respectively, which are edge-disjoint. Therefore, since $e \in E(C_i)$ and $Q'_j \subseteq F_i \setminus Q'_i$, we must have that $e \notin E(Q'_j)$. However, since $e \in E(C_j)$ and $e \in E(F_j)$ by definition, we must have $e \in E(Q'_j)$, a contradiction.

Suppose now that $2r = k + 1$. Let $F_1 := \bigcup_{j=1}^{r} Q'_j$, so $\Delta(F_1) \leq r\Phi$ and it is $(\mathcal{U}, \ell, s)$-good. By the conditioning above, $((H_1^* \cup Q_1) \setminus F_1) \cup H_1 \cup Q'_1$ contains a $(\mathcal{U}, \ell^2, s)$-good Hamilton cycle C with $H_1 \subseteq C$. We split C into two perfect matchings $M_1 \cup M_2$ (observe that both of them are $(\mathcal{U}, \ell^2, s)$-good) and redefine $H_r := H_r \cup \{e \in M_2 : \mathcal{U} \cap e \neq \varnothing\}$, so that H_r now satisfies (i). Now, for each $i \in \{2, \dots, r\}$, we proceed as follows. Let $F_i := M_1 \cup \bigcup_{j=1}^{r} Q'_j \cup \bigcup_{j=2}^{i-1} C_j$. It follows by construction that F_i is $(\mathcal{U}, \ell, s)$-good and $\Delta(F_i) \leq r(\Phi + 2) \leq (r + 2)\Phi$. Then, by the conditioning above, the graph $((H_i^* \cup Q_i) \setminus F_i) \cup H_i \cup Q'_i$ must contain a $(\mathcal{U}, \ell^2, s)$-good Hamilton cycle C_i with $H_i \subseteq C_i$. Take one such cycle and proceed. The fact that the graphs $M_1, C_2, \dots, C_r$ are pairwise edge-disjoint can be proved as in the previous case. □

We now prove Theorem 1.5. Recall from Section 1.4 that, for any $k \in \mathbb{N}$ and any graph $G \subseteq \mathcal{Q}^n$, we say that $G \in \boldsymbol{\delta}k$ if $\delta(G) \geq k$, and $G \in \mathcal{HM}k$ if it contains $\lfloor k/2 \rfloor$ edge-disjoint Hamilton cycles and $k - 2\lfloor k/2 \rfloor$ perfect matchings which are edge-disjoint from these cycles. We say that $G \in \mathcal{P}k$ if, for every spanning subgraph $H \subseteq \mathcal{Q}^n$ with $H \in \boldsymbol{\delta}k$, we have $G \cup H \in \mathcal{HM}k$.

Proof of Theorem 1.5. The case $k = 1$ of the statement was proved by Bollobás [12], so we may assume $k \geq 2$. Let $0 < \varepsilon \ll 1/k$ and $G \sim \mathcal{Q}^n_{1/2-\varepsilon}$. By Theorem 1.8, we have $\mathbb{P}[G \in \mathcal{P}k] = 1 - o(1)$. Also note that, by Lemma 4.2, we have that $\mathbb{P}[e(G) \geq (1/2 - \varepsilon/2)n2^{n-1}] = o(1)$. Hence,

$$\mathbb{P}[\{G \in \mathcal{P}k\} \wedge \{e(G) < (1/2 - \varepsilon/2)n2^{n-1}\}] = 1 - o(1).$$

Thus, by a simple conditioning argument, there exists a positive integer $m < (1/2 - \varepsilon/2)n2^{n-1}$ such that

$$\mathbb{P}[G \in \mathcal{P}k \mid e(G) = m] = 1 - o(1). \tag{9.15}$$

Let $G_m \subseteq \mathcal{Q}^n$ be a uniformly random subgraph of $\mathcal{Q}^n$ with exactly m edges. Since $\mathbb{P}[G \in \mathcal{P}k \mid e(G) = m] = \mathbb{P}[G_m \in \mathcal{P}k]$, by (9.15) we have $\mathbb{P}[G_m \in \mathcal{P}k] = 1 - o(1)$. Now, because a.a.s. $\tau_{\boldsymbol{\delta}k}(\tilde{\mathcal{Q}}^n(\sigma)) \geq (1/2 - \varepsilon/4)n2^{n-1}$, the result follows. □

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Appendix A. Proof of Lemma 8.9

Proof of Lemma 8.9. The proof is similar (but easier) to that of Lemma 8.8. By relabelling the atoms, we may assume that $\mathcal{M}^* = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_t$. Without loss of generality, we may assume that, for each $r \in [2]$, if $u_r \in R$, then $u_r = z_1$, and if $v_r \in R$, then $v_r = w_t$. Moreover, we may assume that $x \neq_{\mathrm{p}} u_1$. Let $S \coloneqq \{u_1, v_1, u_2, v_2\}$. Let $I_R \coloneqq \{k \in [t] : R \cap V(\mathcal{A}_k) \cap S \neq \varnothing\}$, $R^* \coloneqq R \setminus \bigcup_{k \in I_R} V(\mathcal{A}_k)$ and $I_{R^*} \coloneqq \{k \in [t] : R^* \cap V(\mathcal{A}_k) \neq \varnothing\}$. For each $r \in [2]$, let $I_R^r \subseteq \{1, t\}$ be such that $1 \in I_R^r$ if and only if $u_r \in R$ and $t \in I_R^r$ if and only if $v_r \in R$. Note that $I_R = I_R^1 \cup I_R^2$. Fix an index $t^* \in [t-1] \setminus (I_{R^*} \cup \{1\})$. If $|L| = 2$, let $I_L^1 \coloneqq \{i\}$, $I_L^2 \coloneqq \{j\}$ and $I_L \coloneqq \{i, j\}$; otherwise, let $I_L^1 \coloneqq I_L^2 \coloneqq I_L \coloneqq \{t^*\}$.

For each $r \in [2]$, we create an ordered list $\mathcal{L}_r$ of vertices, which will be used to construct the vertex-disjoint paths $\mathcal{P}_r$. Given any list of vertices $\mathcal{L}_r$, we write L_r^* to denote the (unordered) set of vertices in $\mathcal{L}_r$, and whenever $\mathcal{L}_r$ is updated, we implicitly update L_r^*. In the end, for each $r \in [2]$ we will have a list of vertices $\mathcal{L}_r = (x_1^r, \ldots, x_{\ell_r}^r)$. For each $r \in [2]$ and $k \in [t]$, let $I_r(k) \coloneqq \{h \in [\ell_r - 1] : 2 \nmid h \text{ and } x_h^r, x_{h+1}^r \in V(\mathcal{A}_k)\}$. We will require $\mathcal{L}_1$ and $\mathcal{L}_2$ to be vertex-disjoint and to satisfy the following properties:

($\mathcal{L}'1$) ℓ_1 and ℓ_2 are even.
($\mathcal{L}'2$) For each $r \in [2]$, for all $h \in [\ell_r - 1]$, if h is odd, then $x_h^r, x_{h+1}^r \in V(\mathcal{A}_k)$, for some $k \in [t]$; if h is even, then $x_h^r x_{h+1}^r \in E(G \cup \mathcal{M}^*)$.
($\mathcal{L}'3$) For all $k \in [t]$ we have that $|I_1(k)|, |I_2(k)| \geq 1$ and $2 \leq |I_1(k)| + |I_2(k)| \leq 3$.
($\mathcal{L}'4$) For each $r \in [2]$, the following holds. For each $k \in [t] \setminus (I_L^r \cup I_R^r)$ and each $h \in I_r(k)$, we have $x_h^r \neq_{\mathrm{p}} x_{h+1}^r$. For each $k \in I_L^r \cup I_R^r$, we have that $|I_r(k)| = 1$ and for the unique index $h \in I_r(k)$ we have $x_h^r =_{\mathrm{p}} x_{h+1}^r$, with the same parity as u_r in the case when $k \in I_L^r$, and with parity opposite to that of the unique vertex in $\{w_k, z_k\} \cap \{u_r, v_r\}$ in the case when $k \in I_R^r$.
($\mathcal{L}'5$) For each $r \in [2]$, we have the following. If $u_r \notin R$, then $u_r = x_1^r$. If $v_r \notin R$, then $v_r = x_{\ell_r}^r$. If $u_r \in R$ (and thus $u_r = z_1$), then $w_1 = x_1^r$ and $u_r \notin L_1^* \cup L_2^*$. If $v_r \in R$ (and thus $v_r = w_t$), then $z_t = x_{\ell_r}^r$ and $v_r \notin L_1^* \cup L_2^*$.
($\mathcal{L}'6$) Every pair (w_k, z_k) with $\{w_k, z_k\} \subseteq R^*$ is contained in $\mathcal{L}_1$ and z_k directly succeeds w_k or vice versa.

If $R^* \cap V(\mathcal{A}_1) = \{w_1, z_1\}$, then let $\mathcal{L}_1 \coloneqq (u_1, w_1, z_1)$, where we assume that $w_1 \neq_{\mathrm{p}} u_1$; otherwise, let $\mathcal{L}_1 \coloneqq (u_1)$. Observe once more that, in what follows, the existence of each alternating parity sequence follows from the bondedness of $\mathcal{M}$.

Let $F_1 \coloneqq L \cup R \cup S$ and let $t_1^\bullet \in I_L^1$. Let $\mathcal{S}_1$ be a $(u_1, t_1^\bullet, F_1, R)$-alternating parity sequence. If $u_1 \in R$, update $\mathcal{L}_1 \coloneqq \mathcal{S}_1$; otherwise, update $\mathcal{L}_1 \coloneqq \mathcal{L}_1 \mathcal{S}_1^-$. Choose any vertex $u_{t_1^\bullet} \in V(\mathcal{A}_{t_1^\bullet})$ with $u_{t_1^\bullet} \neq_{\mathrm{p}} u_1$, and let $\mathcal{S}_2$ be a $(u_{t_1^\bullet}, t, F_1, R^*)$-alternating parity sequence. Update $\mathcal{L}_1 \coloneqq \mathcal{L}_1 \mathcal{S}_2^-$. If $v_1 \in R$, update $\mathcal{L}_1 \coloneqq \mathcal{L}_1(z_t)$. Otherwise, update $\mathcal{L}_1 \coloneqq \mathcal{L}_1(v_1)$.

Next, let $F_2 \coloneqq F_1 \cup L_1^*$ and let $t_2^\bullet \in I_L^2$. Let $\mathcal{S}_3$ be a $(u_2, t_2^\bullet, F_2, R \cap V(\mathcal{A}_1))$-alternating parity sequence, and let $\mathcal{L}_2 \coloneqq \mathcal{S}_3$. Choose any vertex $u'_{t_2^\bullet} \in V(\mathcal{A}_{t_2^\bullet})$ with $u'_{t_2^\bullet} \neq_{\mathrm{p}} u_2$, and let $\mathcal{S}_4$ be a $(u'_{t_2^\bullet}, t, F_2, \varnothing)$-alternating parity sequence. Update $\mathcal{L}_2 \coloneqq \mathcal{L}_2 \mathcal{S}_4^-$. Finally, if $v_2 \in R$, update $\mathcal{L}_2 \coloneqq \mathcal{L}_2(z_t)$. Otherwise, update $\mathcal{L}_2 \coloneqq \mathcal{L}_2(v_2)$.

Observe that $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy ($\mathcal{L}'1$)–($\mathcal{L}'6$). We are now in a position to apply Lemma 8.6. For each $k \in [t]$, let $t_k \coloneqq |I_1(k)| + |I_2(k)|$. Again, for any $r \in [2]$ and $k \in [t]$, for each $h \in I_r(k)$, we refer to the pair x_h^r, x_{h+1}^r as a *matchable pair*. By ($\mathcal{L}'3$), ($\mathcal{L}'4$) and Lemma 8.6(i), each $\mathcal{A}_k$ with $k \in [t] \setminus (I_L \cup I_R)$ can be covered by t_k vertex-disjoint paths, each of whose endpoints are a matchable pair contained in $\mathcal{A}_k$. Similarly, by $(\mathcal{L}'3)$, $(\mathcal{L}'4)$ and Lemma 8.6(ii), each $\mathcal{A}_k$ with $k \in I_R$ contains t_k vertex-disjoint paths, each of whose endpoints are a matchable pair in $\mathcal{A}_k$, such that the union of these t_k paths covers precisely $V(\mathcal{A}_k) \setminus (S \cap R)$. Similarly, if $L \neq \varnothing$ and $k \in I_L$, then $\mathcal{A}_k$ contains t_k paths, each of whose endpoints are a matchable pair in $\mathcal{A}_k$, such that the union of these t_k paths covers precisely $V(\mathcal{A}_k) \setminus L$. Finally, by $(\mathcal{L}'3)$, $(\mathcal{L}'4)$ and Lemma 8.6(iii), if $L = \varnothing$ and $k \in I_L$ (that is, $k = t^*$), then $\mathcal{A}_k$ can be covered by t_k paths, each of whose endpoints are a matchable pair in $\mathcal{A}_k$. For each matchable pair x_h^r, x_{h+1}^r in $\mathcal{A}_k$, let us denote the corresponding path by $\mathcal{P}_{x_h^r, x_{h+1}^r}$.

The paths $\mathcal{P}_r$ required for Lemma 8.9 can now be constructed as follows. For each $r \in [2]$, let $\mathcal{P}_r$ be the path obtained from the concatenation of the paths $\mathcal{P}_{x_h^r, x_{h+1}^r}$, for each odd $h \in [\ell_r]$, via the edges $x_h^r x_{h+1}^r$ for $h \in [\ell_r - 1]$ even. By $(\mathcal{L}'5)$, if $\mathcal{P}_r$ does not contain u_r, then $\mathcal{P}_r$ starts in w_1, and u_r does not lie in any other path; therefore, we can update $\mathcal{P}_r$ as $\mathcal{P}_r \coloneqq u_r\mathcal{P}_r$. Similarly, if $\mathcal{P}_r$ does not contain v_r, then $\mathcal{P}_r$ ends in z_t and v_r does not lie in any other path, hence we can update $\mathcal{P}_r$ as $\mathcal{P}_r \coloneqq \mathcal{P}_r v_r$. It follows that $V(\mathcal{P}_1 \cup \mathcal{P}_2) = V(\mathcal{M}^*) \setminus L$, and thus the paths $\mathcal{P}_r$ are as required for Lemma 8.9. □