MSM2B Complex Variable Theory 2005

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A Review of Complex Numbers

Definition *For* $z = x + iy$*, the real number* x *is the* **real part** *of* z *and the real number* y *is the* **imaginary part** *of* z*. We denote the real part (of* z*) by* $Re(z)$ *and the imaginary part (of* z*) by* $Im(z)$*.*

We have $(x+iy)(v+iw) = (xy-yw)+i(xw+yv)$ and $(x+iy)+(v+iw) = (x+v)+i(y+w)$. In particular, $i^2 = -1$.

Argand Plane: A Geometrical Interpretation

Suppose that $z = x + iy$ is a complex number. Then we may "plot" z on the **Argand Plane**. The Argand Plane is nothing more than the familiar **Cartesian Plane** or $x - y$ axes on which we are accustomed to plotting coordinates. Thus we represent $z = x + iy$ as the coordinate (x, y). We plot real values on the **real axis** and imaginary values on the **imaginary axis**.

We often refer to complex numbers as points on the Argand plane.

Definition *The* **modulus** *or* **absolute value** *of a complex number* $z = x + iy$ *is the non-negative real number* $\sqrt{x^2 + y^2}$ *and is denoted by* $|z|$*.*

Thus we have
$$|z| = \sqrt{x^2 + y^2}.$$

Notice that on the Argand Plane $|z|$ is the distance from the origin (the complex number 0) to z.

If we consider the set of complex numbers $\{z \mid |z| = 1\}$ and plot the points on the Argand Plane then we plot all the solutions to the equation $x^2 + y^2 = 1$, this equation determines a circle of radius 1 about the origin. Further, if z_1 and z_2 are two complex numbers then $|z_1 - z_2|$ is geometrically the distance between the points z_1 and z_2.

Definition *The* **complex conjugate** *of a complex number* $z = x + iy$ *is defined to be the complex number* $\overline{z} = x - iy$.

Graphically, (meaning represented on the Argand plane) $\overline{z}$ is the reflection of z in the real axis.

Proposition 0.1 *The following statements about complex conjugates and moduli hold:*

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
2. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
3. $\overline{z}z = |z^2|$.
4. $|z_1 z_2| = |z_1||z_2|$ *and* $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$.
5. $|z_1 + z_2| \leq |z_1| + |z_2|$. *(The* **Triangle Inequality***)*
6. *The multiplicative inverse* z^{-1} *of* $z = x + iy$ *is* $\frac{\overline{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$.

The Polar Form

Let (x, y) be a point on the Argand Plane and let r and θ be the polar coordinates of that point.

Then by basic trigonometry $x = r\cos\theta$ and $y = r\sin\theta$. Hence we may choose to express a complex number in polar form:

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

This is Euler's formula from 1748. Note that $r = |z|$. We write *arg* $z = \theta$. Note that *arg* z is only defined up to a mutliple of 2π (i.e. it is not unique). But often we want a unique argument θ. This we do by simply requiring that θ lies in the interval $(-\pi, \pi]$. That is $-\pi < \theta \leq \pi$.

Definition *Suppose that* z *is a non-zero complex number. Then the* **Principal Value** *of arg* z *is denoted by Arg* z *and is the unique argument of* z *that lies in* $(-\pi, \pi]$.

Suppose that $z = r(\cos\theta + i\sin\theta)$. Then it is often convenient to express z in a more compressed form:

$$z = re^{i\theta}$$

So $e^{i\theta}$ represents $\cos\theta + i\sin\theta$. It will be important later on to observe that

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

The notation $e^{i\theta}$ is partly prompted because of the fortunate behaviour of the exponential form under (complex) multiplication:

Proposition 0.2 *Suppose that* $z_1 = r_1 e^{i\theta_1}$ *and* $z_2 = r_2 e^{i\theta_2}$ *are complex numbers. Then*

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Proof: Use the definition plus the formula for $\cos(a + b)$ and $\sin(a + b)$. □

1 Functions, Limits and Continuity

1.1 Functions

We have already met functions in real variable theory and in set theory. Recall that every function, f, comes with a **domain** and a **range**. Then f maps the members of the domain to the members of the range. More precisely, f maps a single member of the domain to a single member of the range. Functions **do not** map one member of a domain to more than one member of the range.

Example $f(x) = \sqrt{x}$ is only a function if we specify that $\sqrt{x}$ means the positive square root of x.

We will be interested in functions f which map a set of complex numbers to another set of complex numbers. We call such functions **functions of complex variable**.

For a function to be **well-defined** you must give both

1. A domain of definition (where it maps from); and
2. A rule that which when given an element of the domain determines the unique element of the range. (i.e square it, or something of the sort, or a constant function)

Note that we will be interested in functions which map a set of complex numbers $S \subseteq \mathbb{C}$ to another set of complex numbers $T \subseteq \mathbb{C}$.

Example

(a) Let $S = \mathbb{C}$ and $T = \mathbb{C}$. For $z \in S$ we define $f(z) = z^2 \in T$. Then f is a function. We can also express this function as follows:
$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

(b) Let $S = \mathbb{C}$ and $T = \mathbb{C}$. Define $f(z) = Re(z)$. Alternatively $f(x + iy) = x$.

(c) $S = \mathbb{C} \setminus \{0\}$ and $T = \{z \in \mathbb{C} \mid |z| = 1\}$. Define $f(z) = \frac{z}{|z|}$. Alternatively $f(x+iy) = \frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}$.

(d) Let $S = \mathbb{C}$ and $T = \mathbb{C}$. Define, for $z = x + iy \in S$,
$$f(z) = f(x + iy) = \cos y + i(\sin y + \cos x).$$

The last example indicates a way in which we often express a function from the complex numbers to the complex numbers:
$$f(x + iy) = u(x, y) + iv(x, y).$$
Here $u(x, y)$ and $v(x, y)$ are both functions of two real variables mapping a pair of real numbers (x, y) (the i is missing!) to a further real number. This way of expressing a function of a complex variable is extremely important.

Definition *Suppose that* $n \geq 0$ *is an integer and* $a_0,\ a_1, \ldots, a_n$ *are complex constants (just complex numbers that are fixed.) Then*
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots a_1 z + a_0$$
(a function with domain $\mathbb{C}$*) is called a* **polynomial of degree** n. *If* $P(z)$ *and* $R(z)$ *are polynomials, then the function* $Q(z) = P(z)/R(z)$ *is called a* **rational** *function.*

One of the difficulties with functions of a complex variable is that they are very difficult to visualise; you can't just draw a graph. There are of course instances when you can visualise something.

Example

(a) The function $f(z) = z + 1$ is visualised as translation of the point z on S to the point one unit to the right in T. That is translation 1 unit to the right.

(b) The function $f(z) = iz = -y + ix$ a non-zero point in S is transformed to a point of T by rotating anti-clockwise through $\pi/2$ radians.

(c) More generally if $a = e^{i\theta}$, then the function

$$f(z) = az$$

rotates the position of z on the Argand diagram through θ radians (anticlockwise).

(d) $f(z) = \overline{z}$ is represented by reflection in the real axis.

(e) $f(z) = 1/z$ with domain $\mathbb{C} \setminus \{0\}$ maps points z with $|z| > 1$ to points w with $|w| < 1$ and vice versa.

1.2 Limits

Limits of complex functions are defined siimilarly as for (functions of) real variables:

Definition *Let* $S, T \subseteq \mathbb{C}$, $f : S \to T$ *be a function and suppose that* $z_0 \in \mathbb{C}$ *Then*

$$\lim_{z \to z_0} f(z) = w_0,$$

means that for each real $\epsilon > 0$ *there exists a real* $\delta > 0$ *such that*

$$|f(z) - w_0| < \epsilon \textit{ whenever } z \in S \textit{ and } 0 < |z - z_0| < \delta.$$

IMPORTANT Domains of definition for functions can be missing certain points. And in fact the definition of a limit specifically does not require that the function f is defined at the point z_0. Furthermore, if f is defined at z_0, then $\lim_{z \to z_0} f(z) = w_0$ does not imply that $f(z_0) = w_0$.

If we want to think of this graphically we may do this as in the last section. For this, we first introduce a mathematical way of expressing that two different complex numbers are close together:

Definition *Let* z_0 *be a complex number and* $\epsilon > 0$ *be a real number. Then the set*

$$B(z_0, \epsilon) = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$$

is called an ϵ-**neighbourhood** *of* z_0.

On the Argand Plane $B(z_0, \epsilon)$ is simply the ball of radius ϵ about z_0. So B is for ball. We will often simply use the expression:

Suppose that z *is in some neighbourhood of* z_0.

This means that there exists an $\epsilon > 0$ such that $z \in B(z_0, \epsilon)$.

Now $\lim_{z \to z_0} f(z) = w_0$ implies that given a ball, B, of arbitrary non-zero radius about the point w_0, we may find a ball of non-zero radius about our point z_0 for which every interior point not equal to z_0 maps into the interior of B.

Example

(a) Suppose that $S = \{z \in \mathbb{C} \mid |z| < 1\}$ is the domain of definition of $f(z) = iz/2$. Then prove that $\lim_{z\to 1} f(z) = \frac{i}{2}$ even though $1 \notin S$.

We clearly have for $z \in S$

$$\left|f(z) - \frac{i}{2}\right| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \left|\frac{i}{2}(z-1)\right| = \left|\frac{i}{2}\right| |z-1| = \frac{|z-1|}{2}.$$

Thus to *prove* that the limit is as stated we suppose that $\epsilon > 0$ is given and we *choose* $\delta = 2\epsilon > 0$. Then if $|z-1| = |z - z_0| < \delta$ we get

$$\left|f(z) - \frac{i}{2}\right| = \frac{|z-1|}{2} < \delta/2 = \epsilon.$$

Therefore, the limit of $f(z)$ as z tends to 1 is $i/2$ as claimed.

(b) Suppose that $f(z) = \mathrm{Arg}(z)$. Show that $f(z)$ does not possess a limit on the negative real axis.

Let z_0 be a complex number which lies on the negative real axis in the Argand Plane. Then for any neighbourhood $B(z_0, \epsilon)$ of z_0 there are complex numbers (points), z with $\mathrm{Arg}(z)$ arbitrarily close to π and points with argument arbitrarily close to $-\pi$ Thus, since $f(z)$ tends to two different values as z approaches z_0, we see that the limit does not exist.

In fact the last example raises a question! Is it true that limits are unique? Can there possibly be a situation in which there are two limits of a given function? This question is answered by

Lemma 1.1 *Suppose that f is a function. If $\lim_{z\to z_0} f(z)$ exists, then it is unique.*

Proof: (Omitted in lectures) Suppose that $\lim_{z\to z_0} f(z) = w_0$ and w_1. Then the distance between w_1 and w_0 is γ say. That is $|w_1 - w_0| = \gamma$. Pick $0 < \epsilon < \frac{\gamma}{2}$.

Then by the definition of a limit there exists $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever } |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \epsilon \quad \text{whenever } |z - z_0| < \delta_1.$$

Choose $\delta = \min(\delta_0, \delta_1)$. Then for $z \in B(z_0, \delta)$ and $z \neq z_0$

$$f(z) \in B(w_0, \epsilon) \cap B(w_1, \epsilon) = \emptyset$$

which is of course against the definition of a function. □

Theorem 1.2 *Suppose that*

$$f(z) = u(x,y) + iv(x,y)$$

$z_0 = x_0 + iy_0$ *and* $w_0 = u_0 + iv_0$. *Then*

$$\lim_{z\to z_0} f(z) = w_0 \textit{ if and only if } \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \textit{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

Proof: (Omitted in lectures.) Assume first that the left hand side of the statement holds. We show that the right hand side holds. From the definition of a limit we have, for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

This is the same as

$$|(u(x, y) - u_0) + i(v(x, y) - v_0)| < \epsilon \text{ whenever } 0 < |(x - x_0) + i(y - y_0)| < \delta.$$

Now we always have $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$ (since $\sqrt{x^2} \leq \sqrt{x^2 + y^2}$). So

$$|u(x, y) - u_0| \leq |(u(x, y) - u_0) + i(v(x, y) - v_0)| < \epsilon$$

and

$$|v(x, y) - v_0| \leq |(u(x, y) - u_0) + i(v(x, y) - v_0)| < \epsilon.$$

Therefore, as $|(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = d((x, y), (x_0, y_0))$, we conclude that $|u(x, y) - u_0| < \epsilon$ whenever $d((x, y), (x_0, y_0)) < \delta$ and $|v(x, y) - v_0| < \epsilon$ whenever $d((x, y), (x_0, y_0)) < \delta$. Thus the right hand side is true.

Now suppose that the right hand side is true. Select $\epsilon > 0$ we want to show that the properties of a limit hold for f at $z_0 = x_0 + iy_0$.

The right hand side tells us that for each $\epsilon_1 = \epsilon/2$ there exits $\delta_u > 0$ and $\delta_v > 0$ such that

$$|u(x, y) - u_0| < \epsilon_1 \text{ whenever } d((x, y), (x_0, y_0)) < \delta_u$$

and

$$|v(x, y) - v_0| < \epsilon_1 \text{ whenever } d((x, y), (x_0, y_0)) < \delta_v.$$

Choose $\delta = \min(\delta_u, \delta_v)$. Then, using the Triangle Inequality,

$$|(u(x, y) - u_0) + i(v(x, y) - v_0)| \leq |u(x, y) - u_0| + |i(v(x, y) - v_0)|$$

$$= |u(x, y) - u_0| + |i||(v(x, y) - v_0)| = |u(x, y) - u_0| + |(v(x, y) - v_0)|$$

$$< \epsilon_1 + \epsilon_1 = 2\epsilon_1 = \epsilon$$

whenever $|(x + iy) - (x_0 + iy_0)| < \delta$. So the left hand side holds. □

The following theorem will be left as an exercise.

Theorem 1.3 *Suppose that* f *and* g *are functions with* $\lim_{z \to z_0} f(z) = \phi$, $\lim_{z \to z_0} g(z) = \rho$ *and* c *is a complex constant. Then the following hold:*

1. $\lim_{z \to z_0} cf(z) = c\phi$
2. $\lim_{z \to z_0} f(z) + g(z) = \phi + \rho$.
3. $\lim_{z \to z_0} f(z)g(z) = \phi\rho$.
4. *if* $\rho \neq 0$, *then* $\lim_{z \to z_0} f(z)/g(z) = \phi/\rho$.

Theorem 1.3 tells us, for example, that polynomials behave particularly nicely. Indeed it is easy to show that the function $f(z) = z$ satisfies

$$\lim_{z \to z_0} f(z) = z_0$$

for all z_0. Therefore, Theorem 1.3 (3) shows that $g(z) = f(z)^n = z^n$ satisfies

$$\lim_{z \to z_0} g(z) = z_0^n$$

for all integers $n \geq 0$. So using Theorem 1.3 (1) and (2) for any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

we get

$$\lim_{z \to z_0} P(z) = P(z_0).$$

This is a fact that you can regularly use.

One final handy tool before we get to some examples is

Lemma 1.4 *If* $\lim_{z \to z_0} f(z) = w_0$, *then* $\lim_{z \to z_0} |f(z)| = |w_0|$.

Proof: We use the following inequality:

$$||z_1| - |z_2|| \leq |z_1 - z_2|.$$

(To prove this inequality, we use the Triangle Inequality to see that

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$

so $|z_1| - |z_2| \leq |z_1 - z_2|$ and we are done if $|z_1| - |z_2|$ is positive. Otherwise we consider $z_2 - z_1 + z_1$.) Now the proof of the lemma is an easy exercise. □

Example 1.4 Prove that $\lim_{z \to 1-i} [x + i(2x + y)]] = 1 + i$.

We will do this in two ways. First directly: we have $f(z) = x + i(2x + y)$, $z_0 = 1 - i$, $w_0 = 1 + i$. Let $\epsilon > 0$ and select $\delta = \frac{\epsilon}{4}$. So if $|z - z_0| < \delta$, then

$$|(x + iy) - (1 - i)| = |(x - 1) + i(y + 1)|.$$

Then as $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$ we have $|x - 1| < \delta$ and $|y + 1| < \delta$. Therefore, whenever $0 < |z - z_0| < \delta$ we have (using the Triangle Inequality twice)

$$\begin{aligned} |f(z) - w_0| &= |x + i(2x + y) - 1 - i| = |(x - 1) + i(2(x - 1) + y + 1)| \\ &\leq |x - 1| + |2(x - 1) + y + 1| \leq |x - 1| + 2|(x - 1)| + |y + 1| \\ &= 3|x - 1| + |y + 1| < 3\delta + \delta = \epsilon. \end{aligned}$$

Therefore, given $\epsilon > 0$ we can find $\delta = \epsilon/4 > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and this completes our first verification of this particular limit.

Now for the second justification. We will use all the tools that we have. They are the above Theorems. As above we have $f(z) = x + i(2x + y)$. So $u(x, y) = x$ and $v(x, y) = 2x + y$. By Theorem 1.2 it suffices to investigate

$$\lim_{(x,y) \to (1,-1)} u(x, y) \text{ and } \lim_{(x,y) \to (1,-1)} v(x, y)$$

and then add the results together. As $u(x, y) = x$ is a polynomial we have $\lim_{(x,y) \to (1,-1)} u(x, y) = 1$ and similarly

$$\lim_{(x,y) \to (1,-1)} v(x, y) = v(1, -1) = 2 - 1 = 1.$$

Hence

$$\lim_{z \to 1-i} [x + i(2x + y)]] = \lim_{(x,y) \to (1,-1)} u(x, y) + i \lim_{(x,y) \to (1,-1)} v(x, y) = 1 + i$$

and we are done (for free).

1.3 The Riemann Sphere

We have a small problem with infinity. On the real line we choose the two opposite ends and denote them by $\pm\infty$. In the complex plane the situation is less comfortable. What we think of as the infinite seems to be a big place (the border of the complex plane.)

This problem is solved by viewing the complex plane as a **sphere**. This is done by mapping every complex number to a unique point of the sphere as follows:

To each point z of the complex plane we have a unique point of the sphere obtained by placing your unit sphere with the equator on the $\{z \mid |z| = 1\}$ circle and then joining the point z to the north pole of the sphere. The unique point Z thus determined is the point of intersection of the sphere and the line. Notice that the effect of this map is that all the points a long way from the 0 of the complex plane are mapped closer and closer to the north pole of the sphere. In this way we can think of infinity as a single point (i.e. the north pole of the sphere). When we do this we have really added a new point to the complex plane. We call the result the **extended complex plane** and denote it by $\hat{\mathbb{C}}$ The sphere is called the **Riemann Sphere** after the German mathematician Riemann and the function between the plane and the sphere is called a **Stereographic Projection**. (note that it distorts the distances).

Note that $|z| > 1$ goes to Z in the northern hemisphere, and $|z| < 1$ goes to Z in the southern hemisphere. The origin is mapped to the south pole.

So that we can use ∞ like any other complex number we must be told how to add, subtract, multiply and divide. For this, we have the following conventions: If a is any complex number

1. $a + \infty = \infty$.
2. $a - \infty = \infty$.
3. $a \cdot \infty = \infty$ $(a \neq 0)$.
4. $\frac{a}{\infty} = 0$ $(a \neq \infty)$.
5. $\frac{a}{0} = \infty$ $(a \neq \infty)$.

A neighbourhood of infinity then is a ball about the north pole of the Riemann sphere. This is the visualised back in the Argand Diagram as a circle about the origin which has interior on the 'outside' of the circle.

We now explain what this means for limits via a few examples.

Definition *Suppose that $f(z)$ is a complex function. We have*

$$\lim_{z\to\infty} f(z) = w_0$$

if for each $\epsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \textit{ whenever } |z| > \frac{1}{\delta}.$$

Example Suppose that $f(z) = \frac{1}{z^2}$. Show that $\lim_{z\to\infty} f(z) = 0$.

Given $\epsilon > 0$, then $\delta = \sqrt{\epsilon}$ works. Suppose that $|z| > \frac{1}{\sqrt{\epsilon}}$. Then

$$|f(z) - w_0| = \left|\frac{1}{z^2} - 0\right| = \frac{1}{|z|^2} < \left|\frac{1}{\left(\frac{1}{\sqrt{\epsilon}}\right)^2}\right| = \epsilon.$$

Definition *Suppose that $f(z)$ is a complex function. We have*

$$\lim_{z\to z_0} f(z) = \infty$$

if for each real $\epsilon > 0$ there is a real $\delta > 0$ such that

$$|f(z)| > \frac{1}{\epsilon} \textit{ whenever } 0 < |z - z_0| < \delta.$$

Example

(a) Suppose that $f(z) = \frac{1}{(z-i)^2}$. Show that $\lim_{z\to i} f(z) = \infty$.
Suppose that $\epsilon > 0$ and take $\delta = \sqrt{\epsilon}$.
If $0 < |z - z_0| = |z - i| < \delta$, we have

$$|f(z)| = \left|\frac{1}{(z-i)^2}\right| > \frac{1}{\delta^2} = \frac{1}{\epsilon}$$

as required.

(b) Suppose that $f(z) = \frac{1}{z^2+1}$. Show that $\lim_{z\to -i} f(z) = \infty$.
Suppose that $\epsilon > 0$ and choose δ so that $\delta^2 + 2\delta < \epsilon$. Then, if $0 < |z - z_0| = |z + i| < \delta$, we have

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{(z^2+1)}\right| = \left|\frac{1}{(z+i)(z-i)}\right| = \left|\frac{1}{(z+i)(z+i-2i)}\right| \\
&= \frac{1}{|z+i||(z+i-2i)|} \geq \frac{1}{|z+i|(|z+i| + |2i|)} \\
&= \frac{1}{|z+i|^2 + |z+i|2} > \frac{1}{\delta^2 + 2\delta} > \frac{1}{\epsilon}.
\end{aligned}$$

1.4 Continuity

Definition *A function $f(z)$ is* **continuous** *at a point z_0 provided that*

1. *$f(z_0)$ is defined; and*
2. *$\lim_{z\to z_0} f(z)$ exists and is equal to $f(z_0)$.*

Example

(a) Polynomials $P(z)$ are continuous on the whole of $\mathbb{C}$ (Recall $\lim_{z\to z_0} P(z) = P(z_0)$).

(b) Suppose that $f(z)$ and $g(z)$ are continuous at a point z_0. Then Theorem 1.3 shows that

  (i) $f(z) + g(z)$ is continuous at z_0.

  (ii) $f(z)g(z)$ is continuous at z_0.

  (iii) If $g(z_0) \neq 0$, then $\frac{f(z)}{g(z)}$ is continuous at z_0.

(c) If $f(z)$ is continuous at z_0 and $g(z)$ is continuous at $f(z_0)$, then $g(f(z))$ is continuous at z_0.

(d) If $f(z) = u(x, y) + iv(x, y)$, then $f(z)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0). (See Theorem 1.2.)

(e) The function $f(z) = x^2y - 1 + i(3x^2 - y)$ is continuous at every point in the complex plane because its real and imaginary parts are polynomials in x and y and therefore are continuous at every point z.

(f) The function $f(z) = e^{x^2+y} + i\cos(x - y)$ is continuous for all z because of the continuity of the polynomials in x and y and the continuity of the exponential and cosine functions of two real variables.

1.5 Various Types of Sets

Definition *Suppose that S is a set of complex numbers.*

1. *Suppose that $z \in \mathbb{C}$, then one of the following holds:*
   (a) *there is a neighbourhood of z which lies in S. Then we say z is in the interior of S.*
   (b) *there is a neighbourhood of z which contains no points of S. We say that z is in the exterior of S.*
   (c) *Neither of the above, we say that z lies on the boundary of S.*
2. *We call the set of boundary points of S, the* **Boundary of S**.
3. *S is* **open** *if it contains none of its boundary points.*
4. *S is* **closed** *if it contains (all of) its boundary.*
5. *An open set is* **connected** *if every two points z_1, z_2 in S can be joined by a finite polygonal path which lies completely inside of S.*
6. *If S is both open and connected then S is a* **domain**.
7. *A domain is* **simply connected** *if it contains no holes. That is if every circuit can be contracted to a point within the domain.*
8. *A domain with some, none, or all of its boundary points is called a* **region.**

Example

(a) Suppose that S is the set of complex numbers z with $|z| < 1$, i.e. $S = B(0,1)$. Then S is open and simply connected. S is the **interior** of the set $\overline{S} = \{z \mid |z| \leq 1\}$ which we call the **closure** of S. The set $\{z \mid |z| = 1\}$ is the boundary of S.

(b) $B(z,r)$ is open. It is often called the open ball or open neighbourhood of radius r about z.

(c) The set $S = \{z \in \mathbb{C} \mid |z| < 1\} \setminus \{z \in \mathbb{C} \mid |z| \leq \frac{1}{2}\}$. Is open, and connected and so is a domain. It is however, not simply connected. (It has a hole in the middle.)

(d) $S = \{z \in \mathbb{C} \mid |z| < 1\} \cup \{z \in \mathbb{C} \mid |z-3| < 1\}$. Is open but not connected. Therefore, S is not a domain.

(e) $S = \{z \in \mathbb{C} \mid |z| \geq 3\} \cup \{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}$ is not open or closed or connected.

We have the following equivalent definition of an open set (check that it is indeed equivalent!).

Definition *A set* $S \subseteq \mathbb{C}$ *is an open set if for each* $x \in S$ *there exists an* $\epsilon > 0$ *so that* $B(x,\epsilon) \subseteq S$.

1.6 Bounded Functions and Uniform Continuity

Definition *Suppose that R is a region and f is a complex function defined on R. Then f is continuous on R provided it is continuous at every point of R.*

We say that a region R is **bounded** if there exists a real number r such that $R \subseteq B(0,r)$. (Every point of R lies inside some circle $|z| = r$.) Otherwise it is unbounded.

We say that a function $f(z)$ is **bounded** in the region R if there exists a real number M such that

$$|f(z)| \leq M$$

for all $z \in R$.

The following result illustrates how bounded regions and functions are related. This result is proved in a similar way as the analogous Theorem 1.7.8 in MSM2Ba (exercise!).

Theorem 1.5 *Suppose that $f(z) = u(x,y) + iv(x,y)$ is continuous in a region R which is both closed and bounded. Then the function*

$$|f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2}$$

is continuous and bounded on R and this bound is attained.

Definition *A function $f(z)$ which is continuous in a region R is* **uniformly continuous** *there if for all ϵ there is a δ so that for all z_0 in R we have $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.*

Uniform continuity is a more restrictive notion than continuity because it allows us to choose the same δ for all z_0 in R.

The function $f(z) = 1/z$ for example is uniformly continuous on the domain $R = \{z \mid |z| \geq 1\}$. On the domain $R = \{z \mid z \neq 0\}$, it is continuous but not uniformly continuous. The next theorem tells us that the two notions coincide if we restrict ourselves to domains which are closed and bounded (exercise: show that both the conditions of being "closed" and "bounded" are necessary!).

Theorem 1.6 *A function $f(z)$ which is continuous in a region R which is both closed and bounded is* **uniformly continuous** *there.*

2 Derivatives and Analytic Functions

2.1 Derivatives

Definition *Let f be a function whose domain of definition contains a neighbourhood of a point z_0. Then the* **derivative** *of f at z_0 is defined by*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{1}$$

If this limit exists then the function is said to be **differentiable** *at z_0. If it does not exist then the derivative of f at z_0 does not exist and f is not* **differentiable** *at z_0.*

If we write

$$\Delta z = z - z_0,$$

then we can rewrite (1) as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2}$$

Example Suppose that $f(z) = z^2$. Determine the derivative of f at the point $z_0 \in \mathbb{C}$.

By definition we have

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0. \end{aligned}$$

This last example shows that the derivative of $f(z) = z^2$ exists at all points $z_0 \in \mathbb{C}$ and it is in fact true that the derivative of all functions $f(z) = z^n$ exist at all points of $\mathbb{C}$ for all positive integers n (and equals nz^{n-1}).

This next example illustrates that nice looking functions may not be as nice as they appear.

Example Let $f(z) = |z|^2$. Investigate its derivatives.

Again by definition we have

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0 + \Delta z}) - z_0\overline{z_0}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0\overline{z}_0 + z_0\overline{\Delta z} + \overline{z}_0\Delta z + \Delta z\overline{\Delta z} - z_0\overline{z}_0}{\Delta z} \\ &= \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} + z_0\frac{\overline{\Delta z}}{\Delta z}. \end{aligned}$$

So much for the manipulation. What does it tell us? Well if $z_0 = 0$ we get a further reduction

$$f'(0) = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

Thus at $z_0 = 0$ f is differentiable with derivative 0.

Next assume that $z_0 \neq 0$. Then what can we say? Consider how Δz might tend to 0. First of all suppose that it approaches 0 along the real axis. That is assume that $\Delta z = \Delta x + i0$. Then $\overline{\Delta z} = \Delta x - i0 = \Delta z$ and so we can again reduce our expression for f' to obtain

$$f'(z_0) = \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} + z_0 = \overline{z_0} + z_0 = x_0 - iy_0 + x_0 + iy_0 = 2x_0.$$

Now suppose that we approach 0 through purely imaginary values. That is let $\Delta z = 0 + i\Delta y$. Then $\overline{\Delta z} = 0 - i\Delta y = -\Delta z$ and so this time we get

$$f'(z_0) = \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} - z_0 = \overline{z_0} - z_0 = -2iy_0.$$

Because $z_0 \neq 0$, the above two limits can never be equal. But they both should be the limit as Δz tends to 0 of the same expression. Thus Lemma 1.1 (if a limit exists it is unique) implies that the limit does not exist at z_0. Thus we conclude that f is differentiable only at the point $z = 0$.

On the other hand, since $|z|^2 = x^2 + y^2$, $|z|^2$ is continuous at all points of $\mathbb{C}$. Thus we get the following useful fact:

Continuity at a point does not imply the existence of a derivative at that point.

However, we do have

Lemma 2.1 *If the derivative of a function f exists at a point z_0, then f is continuous at z_0.*

Proof: We have to show that $\lim_{z \to z_0} f(z) = f(z_0)$. So we use Theorem 1.3, which implies that

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} z - z_0 = f'(z_0) \cdot 0 = 0$$

which is what we required. $\square$

Recall from the course on real variable theory that the above lemma and the fact preceding it also hold for real valued functions.

2.2 The Cauchy-Riemann Equations

The point about limits in the complex plane is that the values of z approach z_0 from an infinite number of directions. In the last example we saw that when $z_0 \neq 0$ we obtained different values for the limit when we approached z_0 through strictly real values (i.e Δz was real) compared to approaching via strictly imaginary values (i.e Δz was strictly imaginary.) We then concluded that f was not differentiable at any non-zero z_0. We will see in the next three theorems that this idea of approaching z_0 from two different directions forces very severe restrictions on the relationship between the real and imaginary parts of a differentiable function f.

Theorem 2.2 *Suppose that $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$. Then*

$$f'(z_0) = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} + i\left.\frac{\partial v}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} = \left.\frac{\partial v}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} - i\left.\frac{\partial u}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}.$$

Proof: So the plan was outlined above. First we let Δz tend to 0 through real values, that is along the real axis. That is we suppose that

$$\Delta z = x - x_0 - i(y_0 - y_0) = \Delta x.$$

From the definition

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

We express the top line of the limit as

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= f(x_0 + \Delta x + iy_0) - f(x_0 + iy_0) \\ &= u(x_0 + \Delta x, y_0) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0)). \end{aligned}$$

Therefore in this case we have

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

Hence as Δz approaches 0 through real values and as f is differentiable at z_0 we have that

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}. \end{aligned}$$

Next we approach z_0 through purely imaginary values. So

$$\Delta z = 0 + i\Delta y.$$

As above we get

$$f'(z_0) = \lim_{\Delta z \to \infty} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y}.$$

Thus

$$f'(z_0) = -i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}.$$

As by hypothesis f is differentiable the two values we calculated for $f'(z_0)$ must be equal. ☐

Definition *Suppose that f is a complex function. If f', the derivative of f, exists at z_0 and at all the points of some open neighbourhood of z_0, then we say that f is* **analytic** *at z_0. If U is a domain in $\mathbb{C}$ and f is defined on U. Then f is* **analytic** *in U if and only if f is analytic at each point of U.*

(Note that in some books analytic goes under the names of **regular** and **holomorphic**.)

If I speak of a function, f, as being analytic on some set, S, that is not open, then it is to be understood that there exists some open set bigger than S on which f is analytic.

Example Suppose that $f(z) = |z|^2$. Then as we saw earlier, f has a derivative only at $z_0 = 0$ and not at any other point in $\mathbb{C}$. Therefore, f is not differentiable in any neighbourhood of z_0 and is hence not analytic at z_0.

Definition *If f is analytic on $\mathbb{C}$, then we say that f is an* **entire** *function.* (Sometimes called an **integral function**.)

If a function fails to be analytic at a point z_0, but is analytic at some point of every neighbourhood of z_0, then z_0 is called a **singular point**.

Example $z_0 = 0$ is a singular point of the function $f(z) = \frac{1}{z}$. $f'(z) = -1/z^2$.

Theorem 2.2 now yields the important **Cauchy-Riemann Equations**.

Theorem 2.3 *Suppose that $f(z) = u(x,y) + iv(x,y)$ and f is an analytic function on a domain U. Then the first order partial derivatives of $u(x,y)$ and $v(x,y)$ exist at (x,y) for all $z = x + iy \in U$. Furthermore, they satisfy the Cauchy-Riemann Equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

at all $z \in U$. Furthermore, the derivative at $z = x + iy \in U$ is given by

$$f'(z) = \left.\frac{\partial u}{\partial x}\right|_{(x,y)} + i\left.\frac{\partial v}{\partial x}\right|_{(x,y)} = \left.\frac{\partial v}{\partial y}\right|_{(x,y)} - i\left.\frac{\partial u}{\partial y}\right|_{(x,y)}.$$

Proof: Equate the real and imaginary parts of the expression for f' given in Theorem 2.2.

Theorem 2.3 is rather astounding. It shows that most randomly chosen pairs of functions $u(x,y)$ and $v(x,y)$, no matter how differentiable they might be, cannot be the real and imaginary parts of an analytic function.

Example

(a) Verify that the Cauchy-Riemann equations are satisfied for all values of z by the function $f(z) = z^2$

Well $f(z) = z^2 = x^2 - y^2 + i2xy$, So

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy.$$

Therefore we get

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

and

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y.$$

So we see that the Cauchy-Riemann equations are satisfied.

(b) Show that $f(z) = Re(z) = x$ is not differentiable anywhere.

It suffices to show that the Cauchy-Riemann equations are NOT satisfied for all (x, y).

We have $u(x, y) = x$ and $v(x, y) = 0$. Therefore,

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0.$$

So f is not differentiable at any point.

(c) Show that $f(z) = x^2 + iy^2$ is not analytic anywhere.

We have $u(x, y) = x^2$ and $v(x, y) = y^2$. Thus the Cauchy-Riemann equations give us $2x = 2y$ and $0 = 0$. Thus the Cauchy Riemann equations only hold on the line $y = x$. But if $z = x + ix$ is on the line $y = x$, then any neighbourhood of z contains points at which f is not differentiable. Therefore, f is not analytic at z and so not at any point in $\mathbb{C}$.

It is important to remember that satisfying the Cauchy-Riemann equations at z_0 is NOT a sufficient condition for f to be differentiable at z_0. (They are a necessary condition.)

For real valued functions, following result should already be familiar to you:

Theorem 2.4 *Suppose that $f(z)$ is analytic on a domain D and that $f'(z) = 0$ for all $z \in D$. Then f is a constant function on D.*

Proof: We use Theorem 2.2. We have, for $f = u + iv$,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

and

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0.$$

So equating real and imaginary parts we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0.$$

It follows that $u(x, y)$ and $v(x, y)$ are constant on lines parallel to the x- and y-axis, as D is connected we conclude that $u(x, y) = c_1$ and $v(x, y) = c_2$ are constant on D. Thus $f(z) = c_1 + ic_2$ is also constant on D. ☐

The formulae for differentiation resemble those that we are all familiar with for differentiation of real functions. We will omit the proof of the statements and we will usually apply them without specific reference. Let's sum things up before hand by simply saying that the usual differentiation rules apply.

Differentiation Rules Suppose that $g(z)$ and $f(z)$ are differentiable functions and assume that $c \in \mathbb{C}$ is a constant. Then

1. $\frac{d}{dz}c = 0$.
2. $\frac{d}{dz}cf(z) = cf'(z)$.
3. $\frac{d}{dz}z^n = nz^{n-1}$ for all positive integers n and for negative integers so long as $z \neq 0$.
4. $\frac{d}{dz}\left(f(z) \pm g(z)\right) = f'(z) \pm g'(z)$.
5. $\frac{d}{dz}f(z)g(z) = f(z)g'(z) + f'(z)g(z)$.
6. $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$ so long as $g(z) \neq 0$.
7. $\frac{d}{dz}g(f(z)) = g'(f(z))f'(z)$.

We have the following fundamental result that you must commit to memory. It provides a "converse" to the Cauchy-Riemann equations.

Theorem 2.5 *Suppose that $f(z) = u(x, y) + iv(x, y)$ is a function defined throughout some neighbourhood B containing $z_0 = x_0 + iy_0$. Assume*

1. *$u(x, y)$ and $v(x, y)$ are continuous functions at (x_0, y_0); and*
2. *$\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial v}{\partial x}$ all exist and are also continuous at (x_0, y_0).*

Then if the Cauchy-Riemann equations are satisfied at z_0 then f is differentiable at z_0.

Proof: Let $\Delta z = \Delta x + i\Delta y$ and $z_0 = x_0 + iy_0$. We want to show that $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exists at every point of B.

We have

$$f(z_0 + \Delta z) - f(z_0) = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + iv(x_0 + \Delta x, y_0 + \Delta y) - iv(x_0, y_0).$$

Now we rewrite

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) =$$
$$[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)] + [u(x_0, y_0 + \Delta y) - u(x_0, y_0)].$$

Now we may apply the mean value theorem for functions of a single real variable to each of the parts of the above equation. This yields for each $z_0 + \Delta z \in B$ numbers $|x_1| < |\Delta x|$ and $|y_1| < |\Delta y|$ such that

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \frac{\partial u}{\partial x}\bigg|_{(x_0+x_1, y_0+\Delta y)}$$

and

$$u(x_0, y_0 + \Delta y) - u(x_0, y_0) = \Delta y \frac{\partial u}{\partial y}\bigg|_{(x_0, y_0+y_1)}.$$

Set

$$\phi(\Delta x, \Delta y) = [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] - \left[\Delta x \frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + \Delta y \frac{\partial u}{\partial y}\bigg|_{(x_0,y_0)}\right].$$

The above lines then give

$$\frac{\phi(\Delta x,\Delta y)}{\Delta z}=\frac{\Delta x}{\Delta z}\left[\left.\frac{\partial u}{\partial x}\right|_{(x_0+x_1,y_0+\Delta y)}-\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}\right]+\frac{\Delta y}{\Delta z}\left[\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0+y_1)}-\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}\right].$$

However, $|\Delta x|\le|\Delta z|$, $|\Delta y|\le|\Delta z|$, $|x_1|<|\Delta x|$, $|y_1|<|\Delta y|$ and the fact that $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are continuous delivers (using $|\Delta x|/|\Delta z|\le 1$ and $|\Delta y|/|\Delta z|\le 1$)

$$\lim_{\Delta z\to 0}\frac{\phi(\Delta x,\Delta y)}{\Delta z}=0.$$

Now rearranging

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)=\Delta x\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+\Delta y\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}+\phi(\Delta x,\Delta y).$$

A similar argument applies to $v(x,y)$. By defining $\psi(\Delta x,\Delta y)$ appropriately we get

$$v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)=\Delta x\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}+\Delta y\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}+\psi(\Delta x,\Delta y)$$

where

$$\lim_{\Delta z\to 0}\frac{\psi(\Delta x,\Delta y)}{\Delta z}=0.$$

Now add the above expressions, write $(\frac{\partial v}{\partial x})$ for $\frac{\partial v}{\partial x}\big|_{(x_0,y_0)}$ and use the Cauchy-Riemann equations (in the fourth line) to get

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\\
&= \lim_{\Delta z\to 0}\frac{[u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)]}{\Delta z}\\
&= \lim_{\Delta z\to 0}\frac{\Delta x(\frac{\partial u}{\partial x})+\Delta y(\frac{\partial u}{\partial y})+\phi(\Delta x,\Delta y)+i[\Delta x(\frac{\partial v}{\partial x})+\Delta y(\frac{\partial v}{\partial y})+\psi(\Delta x,\Delta y)]}{\Delta z}\\
&= \lim_{\Delta z\to 0}\frac{\Delta x(\frac{\partial u}{\partial x})-\Delta y(\frac{\partial v}{\partial x})+\phi(\Delta x,\Delta y)+i[\Delta x(\frac{\partial v}{\partial x})+\Delta y(\frac{\partial u}{\partial x})+\psi(\Delta x,\Delta y)]}{\Delta z}\\
&= \lim_{\Delta z\to 0}\frac{\Delta x(\frac{\partial u}{\partial x})+i\Delta y(\frac{\partial u}{\partial x})-\Delta y(\frac{\partial v}{\partial x})+i\Delta x(\frac{\partial v}{\partial x})+\phi(\Delta x,\Delta y)+i\psi(\Delta x,\Delta y)}{\Delta z}\\
&= \lim_{\Delta z\to 0}\frac{\Delta z\frac{\partial u}{\partial x}|_{(x_0,y_0)}}{\Delta z}+\frac{(i\Delta x-\Delta y)\frac{\partial v}{\partial x}|_{(x_0,y_0)}}{\Delta z}+\frac{\phi(\Delta x,\Delta y)+i\psi(\Delta x,\Delta y)}{\Delta z}\\
&= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}+\lim_{\Delta z\to 0}\frac{\phi(\Delta x,\Delta y)+i\psi(\Delta x,\Delta y)}{\Delta z}
\end{aligned}$$

Because the last term in the above expression tends to zero as Δz does we see that $f'(z_0)=\frac{\partial u}{\partial x}\big|_{(x_0,y_0)}+i\,\frac{\partial v}{\partial x}\big|_{(x_0,y_0)}$. Finally, as $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ both exist we have, f is analytic on B. □

Example Suppose that $f(z)=e^x(\cos y+i\sin y)$. Then $u(x,y)=e^x\cos y$ and $v(x,y)=e^x\sin y$. It follows that

$$\frac{\partial u}{\partial x}=e^x\cos y \text{ and } \frac{\partial v}{\partial y}=e^x\cos y$$

and

$$\frac{\partial u}{\partial y}=-e^x\sin y \text{ and } \frac{\partial v}{\partial x}=e^x\sin y.$$

Thus the Cauchy-Riemann equations are satisfied for all (x,y); moreover, u,v and the partial derivatives are continuous. Therefore, Theorem 2.5 implies that $f(z)$ is an analytic function and that $f'(z)=f(z)$. We'll come back to this later.

2.3 Harmonic Functions

Once again we are interested in analytic functions of the form $f(z) = u(x, y) + iv(x, y)$. By Theorem 2.2 the functions u and v satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Suppose that in addition u and v have **continuous second order** partial derivatives: (something that will be shown to be true in Chapter 4) then the Cauchy-Riemann equations give us

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2},$$

so u satisfies **Laplace's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(Alternative notations are $\nabla^2 u = \Delta u = 0$.)

Definition *A function of two real variables which satisfies Laplace's equation and has continuous partial derivatives of the first and second order is called a* **harmonic function***.*

Applications Harmonic functions arise in many practical applications in areas of applied mathematics and physics, such as steady fluid flow, and in finding steady temperatures, electrostatic potentials and gravitational potentials.

The above argument shows that when the partial derivatives of the first and second order are continuous the real part of the analytic function f is a harmonic function.

Similarly we have

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2}.$$

Thus the imaginary part of f is also a harmonic function. So we have

If f is an analytic function, then both u and v are harmonic functions (provided they have continuous second order partial derivatives).

This suggests the following problem: given a harmonic function u defined in a domain D can we find another function v such that $f = u + iv$ is an analytic function?

The answer is mostly yes.

Definition *An ordered pair of harmonic functions u and v which satisfy the Cauchy-Riemann equations are called* **harmonic conjugates** *(nothing to do with $\overline{z}$).*

So if f is an analytic function, then v is a harmonic conjugate of u.

If v is a harmonic conjugate of u, then it is NOT necessarily the case that u is a harmonic conjugate of v. In fact it is left as an easy exercise for you to verify:

Lemma 2.6 *Suppose that v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.*

Proof: Exercise.

Example Assume that we are given a function $u(x, y) = 2x - x^3 + 3xy^2$. Find an analytic function which has u as the real part.

Before we start seeking such an analytic function we begin by noting that u does satisfy Laplace's equation. We have

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \text{ and } \frac{\partial^2 u}{\partial x^2} = -6x$$

as well as

$$\frac{\partial u}{\partial y} = 6xy \text{ and } \frac{\partial^2 u}{\partial y^2} = 6x.$$

So Laplace's equation is satisfied at every point of the complex plane. Hence u is a harmonic function. Now see if we can find a harmonic conjugate $v(x, y)$ of $u(x, y)$.

Suppose that v is the imaginary part of our supposed analytic function f. Then u and v must satisfy the Cauchy–Riemann equations. Whence

$$\frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2.$$

Now integrate to give

$$v = 2y - 3x^2y + y^3 + \phi(x).$$

Thus

$$\frac{\partial v}{\partial x} = -6xy + \phi'(x).$$

Now we use the second half of the Cauchy–Riemann equations to get

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6xy.$$

Thus we conclude that $\phi'(x) = 0$ and that $\phi(x)$ is a constant. We can choose that constant to be 0. Then we get that

$$v(x, y) = 2y - 3x^2y + y^3.$$

Then, by Theorem 2.5,

$$f(z) = 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3) = 2z - z^3$$

is an analytic function on the whole of $\mathbb{C}$ that is, f is entire.

3 Transcendental Functions

3.1 The Trigonometric Functions

In this shorter section we extend our favourite functions such as sin, cos, the exponential function e^x and finally the logarithm $\ln x$ from the real line to the whole complex plane.

Recall that in example 2.6 after the statement of Theorem 2.5 we met the function $f(z) = e^x(\cos y + i \sin y)$ and I suggested it would be a good candidate for a complex exponential function. So define $e^{x+iy} = e^x e^{iy}$. Thus

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Now does this function behave like an exponential function should behave? The first thing is that if z is a real number, then $y = 0$ and so $e^{x+iy} = e^x$. So if we restrict to the real line we recover the real exponential function. We have already seen in this example that the Cauchy-Riemann equations are satisfied for all points z and the first order partial derivatives are continuous. Therefore Theorem 2.5 shows that e^z is analytic at every point of the complex plane and is thus an entire function. Moreover, Theorem 2.3 also shows that

$$\frac{de^z}{dz} = e^x \cos y + ie^x \sin y = e^z$$

which is the most important property required of the exponential function. So to reiterate, the exponential function is entire and it is equal to its own derivative.

Now since additive combinations of entire functions remain entire we can define the entire functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Note that that the complex fuctions $\sin z$ and $\cos z$ are unbounded. Similarly we can define complex versions of the hyperbolic functions:

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

and

$$\sinh z = \frac{e^z - e^{-z}}{2}.$$

In the exercises we will have a look at how these functions behave under differentiation. We will also see that all the rules that we knew about traditional trigonometric functions continue to hold for the complex versions.

For example,

$$\frac{\mathrm{d}}{\mathrm{d}z} \cos z = -\sin z, \text{ and } \frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z$$

and the usual trigonomic identities hold, for instance

$$\cos^2 z + \sin^2 z = 1.$$

3.2 The Logarithm Function and Branches

An important fact about the exponential function that we introduced in the last subsection is that it has a certain periodicity. Indeed, for all integers k, we have $e^z = e^{z+2k\pi i}$. Hence the exponential function is not one to one. This is a severe problem when we want to determine the inverse. (Recall that a function maps one complex number to a unique other complex number; however, an inverse to the exponential function has to deal with the fact that each point has to map to an infinite number of complex points.) The problem is again like the problem that we had with the argument of a complex number (see Section 1). In that instance we rescued the situation by choosing a particular argument; the principal argument and we proceed similarly for the logarithm.

Given a nonzero complex number z, a complex number w such that

$$e^w = z$$

is called a logarithm of z, written

$$w = \log z.$$

Suppose that $w = u + iv$ and let $z = e^w = e^{u+iv}$. Then

$$e^u = |z| \quad \text{gives that} \quad u = \ln |z|,$$

where $\ln |z|$ is the natural logarithm, to the base e, of a real number, and

$$v = \arg z = \mathrm{Arg} z + 2k\pi,$$

where k is an integer. (Recall that the principal argument $\mathrm{Arg}z$ takes values only in the interval $-\pi < \mathrm{Arg}z \leq \pi$). Thus if $z \neq 0$, we have

$$\log z = u + iv = \ln |z| + i(\mathrm{Arg}z + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots$$

This has infinitely many values at each point z. It is not a function because $\arg z$ is not a function (it has multiple values). This can be solved by restricting the value that $\arg z$ can take.

Define the **principal logarithm** as

$$\mathrm{Log}z = \ln |z| + i\mathrm{Arg}z, \quad -\pi < \mathrm{Arg}z \leq \pi.$$

This map is now a function. Remember that $\mathrm{Arg}z$ is not continuous and so has no derivative at any point on the negative real axis. So we consider the domain of the principal logarithm to be the whole complex plane with the negative real axis extracted (including the zero). Also notice that this is a domain. Now if all has gone well log should be the inverse function to the exponential function. We check that $e^{\log z} = z$:

$$e^{\mathrm{Log}z} = e^{\ln |z| + i\mathrm{Arg}(z)} = e^{\ln |z|} e^{i\mathrm{Arg}(z)} = |z| e^{i\mathrm{Arg}(z)} = z.$$

Now the last term in the above expression is just the polar form of z. Thus our logarithmic function is indeed an inverse function to the exponential function. So what about analyticy?

Theorem 3.1 *Let U be the complex plane with the negative real axis removed. The principal logarithm is an analytic function on U and its derivative is $\frac{1}{z}$.*

Proof: We work from first principles so select $z \neq z_0 \in U$ and let $w = \mathrm{Log}(z)$ and $w_0 = \mathrm{Log}(z_0)$. Then $z = e^w$ and $z_0 = e^{w_0}$. Hence

$$\frac{\mathrm{Log}z - \mathrm{Log}z_0}{z - z_0} = \frac{w - w_0}{z - z_0} = \frac{w - w_0}{e^w - e^{w_0}} = \left(\frac{e^w - e^{w_0}}{w - w_0} \right)^{-1}.$$

Since $\mathrm{Log}z$ is continuous on U, we have that $w \to w_0$ when $z \to z_0$. Thus

$$\lim_{z \to z_0} \frac{\mathrm{Log}z - \mathrm{Log}z_0}{z - z_0} = \left(\lim_{w \to w_0} \frac{e^w - e^{w_0}}{w - w_0} \right)^{-1}.$$

But

$$\lim_{w\to w_0}\frac{e^w-e^{w_0}}{w-w_0}=\left.\frac{de^z}{dz}\right|_{z=w_0}=\left.e^z\right|_{z=w_0}=e^{w_0}=z_0.$$

Thus

$$\lim_{z\to z_0}\frac{\mathrm{Log}z-\mathrm{Log}z_0}{z-z_0}=\frac{1}{z_0}$$

as required. □

Definition *Suppose that U is a domain of the complex plane and that $f : U \to \mathbb{C}$ is a continuous function such that $z = e^{f(z)}$ for all $z \in U$. Then f is called a* **branch** *of the logarithm in U.*

Thus the principal logarithm is a branch of the logarithm. We will call this the **principal branch**.

So let's emphasise that the branch is the function NOT the domain. Notice also that any branch of the logarithm will be defined on a cut plane.

Example 3.1 Suppose that we insist that the argument must lie in $(0, 2\pi]$. Then once again we get a branch of the logarithm but this time we have to choose the domain to be the whole plane minus the positive real axis.

We could also insist that the argument lie in the interval $(-7\pi/4, \pi/4]$. Then we get a further branch of the logarithm this time with domain the complex plane minus the positive part of the line defined by the equation $x = y$. Different branches may have the same domain.

3.3 The General Exponent

We have already met functions such as $f(z) = z^n$ where n is an integer. We can define such functions for general $\alpha \in \mathbb{C}$. Indeed whenever we choose a branch of the logarithm $\log_\phi$ we can set

$$g_\phi(z) = z^\alpha = e^{\alpha \log_\phi(z)}.$$

Define the **principal exponent** by

$$g(z) = z^\alpha = e^{\alpha \mathrm{Log}(z)}$$

where Log$z$ is the principal branch of the logarithm.

Example 3.2 (Euler 1746). Show that

$$i^i = e^{-(\pi/2+2k\pi)}$$

We have

$$i^i = e^{i\log i} = e^{i(i(\pi/2+2k\pi))} = e^{-(\pi/2+2k\pi)} \quad k = 0, \pm1, \pm2, \ldots.$$

The principal value is $e^{-\pi/2}$.

Definition *A portion of a line or curve consisting of singular points which is introduced in order to define a branch of a multiple-valued function is called a* **branch cut***. A singular point common to all branch cuts for a multiple-valued function is called a* **branch point***.*

The ray $\theta = \pi$ is the branch cut for Log z. The origin is the branch point for log z.

4 Contour Integration and Cauchy's Theorem

4.1 Paths, Arcs and Contours

Definition *Suppose that* $[a, b]$ *is an interval of the real line. Then a* **path** *is a continuous function*

$$\gamma : [a, b] \to \mathbb{C}.$$

Visually we will generally identify a path with its image

$$\Gamma = \{z \in \mathbb{C} \mid \gamma(t) = z \text{ for some } t \in [a, b]\}.$$

The image is called a **Curve**.

We will also give the curve an orientation. The **initial point** is always $\gamma(a)$ and the **terminal point** is $\gamma(b)$.

Example

(a) $\gamma(t) = (2t + 1) + it^3$ where $t \in [0, 3]$ is a path.

(b) The paths $\gamma_1(t) = \cos t + i \sin t$ where $t \in [0, \pi]$ and $\gamma_2(t) = \cos t^2 + i \sin t^2$ where $t \in [0, \sqrt{\pi}]$ define identical curves though the paths are distinct.

Suppose that γ_1 and γ_2 are paths defined on the intervals $[a, b]$ and $[c, d]$ respectively, where we assume $d > c$. Then if $\gamma_1(b) = \gamma_2(c)$, we may define a further path $\gamma_1 + \gamma_2$ defined on $[a, b + (d - c)]$ as follows:

$$\begin{aligned} \gamma_1 + \gamma_2(t) &= \gamma_1(t) \text{ where } t \in [a, b] \\ \gamma_1 + \gamma_2(t) &= \gamma_2(t + c - b) \text{ where } t \in [b, b + d - c] \end{aligned}$$

We may also reverse the orientation of a path to get

$$\gamma^* : [a, b] \to \mathbb{C}$$

$$\gamma^*(t) = \gamma(a + b - t).$$

We denote the curve of γ^* by $-\Gamma$. (This is also referred to as the opposite path and sometimes denoted by $-\gamma$.)

Definition *Suppose that* γ *is a path. Then*

1. *γ is* **simple** *if the curve Γ does not cross itself.*
2. *γ is* **closed** *if $\gamma(a) = \gamma(b)$.*
3. *γ is* **simple closed** *if it is both simple and closed.*
4. *γ is* **smooth** *if γ' exists and is continuous on $[a, b]$.*
5. *γ is a* **contour** *if it is piecewise smooth (i.e. it consists of a finite number of smooth paths).*

We illustrate the first three items of the preceding definition by the following pictures:

In definition (4) the derivative of $\gamma = \gamma_1 + i\gamma_2$ is defined to be $\gamma' = \gamma_1' + i\gamma_2'$ where for $j = 1, 2$, $\gamma_j : [a, b] \to \mathbb{R}$. The derivative γ_j' is just the "standard" real-valued derivative of γ_j.

Example The following are all contours

(a) $\gamma_a : [-1, 1] \to \mathbb{C}$ given by $\gamma_a(t) = t$.

(b) $\gamma_b : [0, \pi] \to \mathbb{C}$ given by $\gamma_b(t) = e^{it} = \cos t + i \sin t$. (Note that $\gamma_b'(t) = ie^{it}$ is continuous.)

(c) $\gamma_a + \gamma_b$ is a simple closed contour.

We have the following important "topological" result:

Theorem 4.1 (Jordan Curve Theorem) *If Γ is a simple closed curve, then the set of points in the plane which do not lie on Γ is the union of two disjoint domains, one of these domains is bounded and is called the* **interior** *of Γ ($Int(\Gamma)$) the other domain is unbounded and is called the exterior of Γ ($Ext(\Gamma)$). In particular,*

$$\mathbb{C} = \Gamma + Int(\Gamma) + Ext(\Gamma).$$

This obvious seeming result is not easily proven!

Convention: By traversing a contour with the interior always to our left hand we can always pick an orientation of the contour. This is the positive direction. Roughly this means that the positive orientation of a contour is anti-clockwise.

4.2 Integrals on Paths

Let $F : [a,b] \to \mathbb{C}$ be a function with $F(t) = A(t) + iB(t)$ for all $t \in [a,b]$ where A and B are both real valued functions. Then define

$$\int_a^b F(t)dt = \int_a^b A(t)dt + i\int_a^b B(t)dt.$$

where the integrals on the right-hand side are the ones that we all know and love. Then $\int_a^b F(t)dt$ obeys all the rules of the traditional real integral (including integration by parts).

Now suppose that f is a function of a complex variable defined on some domain U of $\mathbb{C}$. Let $\gamma : [a,b] \to \mathbb{C}$ be a contour in U (i.e. $\gamma(t) \in U$).

Then we define the integral of f along Γ by

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

These integrals are also known as line integrals.

Example

(a) Suppose that $U = \mathbb{C}$ and $f(z) = z^n$, where $n \neq -1$. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma(t) = e^{it} = \cos t + i \sin t$.

Then γ is a simple closed contour and with $\gamma'(t) = ie^{it}$

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = \int_0^{2\pi} f(e^{it})ie^{it}dt \\ &= \int_0^{2\pi} e^{int}ie^{it}dt = \int_0^{2\pi} ie^{i(n+1)t}dt \\ &= \left[\frac{1}{n+1}e^{i(n+1)t}\right]_0^{2\pi} \\ &= 0.\end{aligned}$$

Thus if $n \neq -1$, then $\int_\Gamma z^n dz = 0$.

(b) Suppose that $U = \mathbb{C} \setminus \{0\}$, $f(z) = 1/z$ and γ is as in (a). Then

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_0^{2\pi} e^{-it}ie^{it}dt \\ &= \int_0^{2\pi} idt \\ &= 2\pi i.\end{aligned}$$

(Note that the integrand is not analytic at $z = 0$.)

From the definition of an integral over a path we get

Proposition 4.2 (Properties) *Suppose that* $\Gamma = \Gamma_1 + \Gamma_2$ *is a contour.*

1. $\int_\Gamma f(z)dz = -\int_{-\Gamma} f(z)dz$.
2. $\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$.
3. $\int_\Gamma cf(z)dz = c\int_\Gamma f(z)dz$ *for all complex constants* c.
4. $\int_\Gamma (f(z)+g(z))dz = \int_\Gamma f(z)dz + \int_\Gamma g(z)dz$.

Example

(a) Evaluate $I_1 = \displaystyle\int_{\Gamma_1} z^2 dz$ where Γ_1 is the line segment from $z = 0$ to $z = 2+i$.

We have the path $\gamma_1(t) = 2t+it$ where $t \in [0,1]$. So $\gamma_1'(t) = 2+i$. On Γ_1, $z^2 = (2t+it)^2 = 4t^2 - t^2 + 4it^2 = 3t^2 + i4t^2$. So

$$\begin{aligned} I_1 &= \int_0^1 (3t^2 + i4t^2)(2+i)dt = \int_0^1 (6t^2 + i3t^2 + i8t^2 - 4t^2)dt \\ &= \int_0^1 (2t^2 + i11t^2)dt = \left[\frac{2}{3}t^3 + i\frac{11}{3}t^3\right]_0^1 = \frac{2}{3} + i\frac{11}{3}. \end{aligned}$$

(b) Let Γ_2 be the line segment from $z = 0$ to $z = 2$. Let Γ_3 be the line segment from $z = 2$ to $z = 2+i$. Evaluate $I_2 = \displaystyle\int_\Gamma z^2 dz$ where $\Gamma = \Gamma_2 + \Gamma_3$.

We have

$$I_2 = \int_{\Gamma_2} z^2 dz + \int_{\Gamma_3} z^2 dz.$$

On Γ_2, $\gamma_2(t) = t$ where $t \in [0,2]$. On Γ_3, $\gamma_3(t) = 2+it$ where $t \in [0,1]$. Therefore,

$$\begin{aligned} I_2 &= \int_0^2 t^2 dt + \int_0^1 (2+it)^2 i dt \\ &= \left[\frac{t^3}{3}\right]_0^2 + i\left[\int_0^1 (4 - t^2 + 4it)dt\right] \\ &= \frac{8}{3} + i\left\{\left[4t - \frac{t^3}{3} + 2it^2\right]_0^1\right\} \\ &= \frac{8}{3} + i\frac{11}{3} - 2 = \frac{2}{3} + i\frac{11}{3}. \end{aligned}$$

So $I_1 = I_2$ so the choice of contour does not matter here. Also the integral of z^2 over the simple closed contour $\Gamma_2 + \Gamma_3 - \Gamma_1$ (which is a triangle) is zero. We will see that this is always the case when the integrand is analytic interior to and on the contour.

4.3 Antiderivatives

Definition *Let f be a function which is continuous throughout a domain D and suppose that there is an analytic function, F, such that $F'(z) = f(z)$ for each $z \in D$. Then F is said to be an* **antiderivative** *or* **primitive** *of f.*

Now suppose that z_1 and z_2 are any two points in a domain U and suppose that $\gamma : [a, b] \to \mathbb{C}$ is a contour stretching from z_1 to z_2.

Then, from the chain rule,

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt \\ &= [F(\gamma(t))]_a^b = F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1).\end{aligned}$$

Thus the value of the integral over Γ depends on the end points alone NOT on the contour chosen.

We reiterate this

Theorem 4.3 *Let f be continuous in the domain U and suppose that f has an antiderivative F. Let z_1 and z_2 be any two points in U and let γ be any contour joining z_1 and z_2 in U. Then*

$$\int_\Gamma f(z)dz = \int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$$

i.e. the value of the integral is independent of the contour taken.

Corollary 4.4 *Suppose that f is as above. If γ is any closed contour in U, then*

$$\int_\Gamma f(z)dz = 0.$$

Proof: Take $z_2 = z_1$ in Theorem 4.3. □

Example Let D be the domain $\{z \in \mathbb{C} \mid |z| > 0, -\pi < \text{Arg}z < \pi\}$. Then using Theorem 4.3 we have

$$\int_{-2i}^{2i} \frac{1}{z}dz = \text{Log}(2i) - \text{Log}(-2i) = ln(2) + i\pi/2 - ln(2) + i\pi/2 = i\pi.$$

Observe this is half what we calculated when we went all the way around 0 on the contour e^{it}!

4.4 The ML-Result

Definition *Let γ be a smooth path defined on $[a, b]$. Then the* **length** *of γ is defined to be*

$$L(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Example Suppose that $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by $\gamma(t) = re^{it}$. Then

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)|dt = \int_0^{2\pi} |rie^{it}|dt = \int_0^{2\pi} rdt = 2\pi r.$$

We will need the following lemma

Lemma 4.5 *Suppose that $\phi : [a, b] \to \mathbb{C}$. Then*

$$\left|\int_a^b \phi(t)dt\right| \leq \int_a^b |\phi(t)|dt.$$

Proof: Define r and θ by $\int_a^b \phi(t)dt = re^{i\theta}$. Then

$$\begin{aligned}
\left|\int_a^b \phi(t)dt\right| &= r = e^{-i\theta}\int_a^b \phi(t)dt \\
&= \int_a^b e^{-i\theta}\phi(t)dt \\
&= \int_a^b Re(e^{-i\theta}\phi(t))dt + i\int_a^b Im(e^{-i\theta}\phi(t))dt \\
&= \int_a^b Re(e^{-i\theta}\phi(t))dt + 0 \quad (\text{ because } r \text{ is real }) \\
&\leq \int_a^b |(e^{-i\theta}\phi(t))|dt \quad (\text{ because } |z| = \sqrt{(Rez)^2 + (Imz)^2} \geq Rez) \\
&= \int_a^b |\phi(t))|dt \quad (\text{ because } |e^{-i\theta}| = 1.)
\end{aligned}$$

□

The following result is very important.

Theorem 4.6 *[ML-Result] Suppose that f is continuous on the domain U and that γ is a contour of length L. If there is a real constant $M > 0$ such that $|f(z)| \leq M$ for all z on Γ, then*

$$\left|\int_\Gamma f(z)dz\right| \leq ML.$$

(This is also known as the ML inequality.)

Proof: We have $\gamma : [a, b] \to U$ for some a, b. We apply Lemma 4.5 in the following calculation.

$$\begin{aligned}
\left|\int_\Gamma f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt \\
&\leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq \int_a^b M|\gamma'(t)|dt \\
&= M\int_a^b |\gamma'(t)|dt = ML.
\end{aligned}$$

□

4.5 The Cauchy-Goursat Theorem

Consider now a function f which is defined and continuous on some domain U. Given z_1 and z_2 in U there are many different contours joing z_1 to z_2 and in general we must expect the integral along such contours to take different values according to which contour is chosen. On the other hand, we have seen in Theorem 4.3 that if f has an antiderivative the value of the integral is dependent only on the end points of the contour. Thus we come to the following problem:

Is there a simple restriction on f and/or U which guarantees that the integral is always independent of the contour taken?

Suppose that γ_1 and γ_2 are two contours in U joining z_1 to z_2.

Then $\gamma_1 + (-\gamma_2)$ is a closed contour in U. We have

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

if and only if

$$0 = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1+(-\gamma_2)} f(z)dz.$$

Thus we can equivalently ask

What conditions guarantee $\int_\Gamma f(z)dz = 0$ where Γ is a closed contour?

It turns out that analyticity suffices. This result will be used many times in the remainder of the course.

Theorem 4.7 (Cauchy-Goursat Theorem) *Let U be a simply connected domain and assume that f is an analytic function on U. Then for all simple closed contours γ in U*

$$\int_\Gamma f(z)dz = 0.$$

We omit the proof (the strategy is to first prove the result for the case where Γ is a triangle and then to use this to prove the result for arbitrary Γ).

Example Suppose that $U = \mathbb{C}$ and that γ is any simple closed contour in $\mathbb{C}$.

(a) $\int_\Gamma e^z dz = 0$.

(b) $\int_\Gamma z^n dz = 0$ for $n > 0$.

(c) Evaluate $\displaystyle\int_\Gamma \frac{\cos z \cosh 2z}{(z^2+16)(z^3-28)}$ where Γ is a circular contour of radius 2 centred at 0.

The function is an analytic function on the domain U of radius 3 about the origin. This is because all the composite parts of the function are analytic and the zeros of the constituent functions in the denominator occur at $z = \pm 4i$ and at the cube roots of 28. Thus U is a simply connected domain, Γ is contained therein and f is analytic on U. Therefore, Theorem 4.7 tells us that

$$\int_\Gamma \frac{\cos z \cosh 2z}{(z^2+16)(z^3-28)} = 0.$$

We comment that in the last example we may be pushed to find the antiderivative.

The theorem extends to an arbitrary (i.e. not necessarily simple) closed contour.

Corollary 4.8 *Let U be a simply connected domain and assume that f is an analytic function on U. Then for all closed contours Γ*

$$\int_{\Gamma} f(z)dz = 0.$$

Proof: If Γ intersects itself a finite number of times it consists of a finite number of simple closed contours.

So we can apply the Cauchy-Goursat theorem to each of them. ☐

We can now deduce our result on path independence.

Corollary 4.9 *Let U be a simply connected domain and assume that f is an analytic function on U. Suppose that z_1 and z_2 are arbitrary points in U and that Γ_1 and Γ_2 are arbitrary contours joining z_1 to z_2. Then*

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

Proof: $\Gamma_1 - \Gamma_2$ is a closed contour inside of U. Thus by Corollary 4.8 $\int_{\Gamma_1 - \Gamma_2} f(z)dz = 0$. Therefore, $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$. ☐

Example Suppose that $f(z) = 1/z$ and that Γ is any (positively oriented) simple closed contour that encloses 0. Show that $\int_{\Gamma} f(z)dz = 2\pi i$.

Recall that in an earlier exercise we have shown that when Γ is a circle centered at 0, then the result holds. Unfortunately, at first glance, Theorem 4.7 is not applicable because f is not analytic at all the points interior to Γ (f is not analytic at 0). This can be overcome by doing some surgery. We will cut out the offending point!

Note that to get the $B(0,\delta)$ contained totally in Γ we are using the fact that $Int\Gamma$ is open (by Jordan Curve Theorem 4.1) and thus does not contain boundary points.

So let Γ_1 be the new contour. Then as $f(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ Theorem 4.7 applies and we get

$$\int_{\Gamma_1} f(z) = 0.$$

But

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma} f(z)dz - \int_{B} f(z)dz + \int_{\eta_1} f(z)dz + \int_{\eta_2} f(z)dz.$$

Also

$$\int_{\eta_1} f(z)dz + \int_{\eta_2} f(z)dz = 0.$$

Therefore we conclude that

$$\int_{\Gamma} f(z)dz = \int_{B} f(z)dz = 2\pi i, \text{as required.}$$

The proof of the following lemma is almost the same as that of the preceding example. We will use it later on in the proof of Cauchy's integral formula.

Lemma 4.10 *Suppose that Γ is any simple closed contour with $z_0 \in Int(\Gamma)$. Then*

$$\int_\Gamma \frac{1}{z-z_0}dz = 2\pi i.$$

Proof: Exercise.

The following result is again proved by using "surgery", i.e. adding auxiliary contours to make the function f analytic on the domain in question.

Theorem 4.11 (Cauchy Goursat for Multiply Connected Domains) *Let Γ be a simple closed contour and let Γ_j $(j = 1, 2, \ldots, n)$ be a finite number of simple closed contours each contained in the interior of Γ and each disjoint (their interiors have no points in common). Let*

$$U = (Int(\Gamma) \cup \Gamma) - \bigcup_{j=1}^{n} Int(\Gamma_j).$$

Suppose that f is analytic at every point of U and let T be the contour $\Gamma - \sum_{j=1}^{n} \Gamma_j$ (all oriented positively). Then

$$\int_T f(z)dz = 0.$$

In particular,

$$\int_\Gamma f(z)dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)dz.$$

Proof: Copy the procedure in the last example n times to get a contour T' on a simply connected domain $U' \subset U$ with $\int_{T'} f(z)dz = \int_T f(z)dz$. Now we can apply the Cauchy-Goursat theorem to obtain $\int_{T'} f(z)dz = 0$. □

4.6 The Cauchy Integral Formula

Theorem 4.12 (Cauchy's integral formula) *Let U be a simply connected domain and let f be analytic on U. Suppose that Γ is a simple closed contour contained in U and z_0 be any point in the interior of Γ. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz.$$

This means that the values of f at points in the interior of Γ are totally determined by the value f takes at the points of Γ! This is remarkable!

So analytic functions are so well behaved that we can predict values that the function will take simply by knowing the values on a closed contour.

By rearranging the above formula, we also get a tool for evaluating some integrals. Indeed we get

$$\int_\Gamma \frac{f(z)}{z-z_0}dz = f(z_0)2\pi i.$$

Proof: We begin with an initial manipulation. We have

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz + \frac{1}{2\pi i}\int_\Gamma \frac{f(z_0)}{z-z_0}dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz + \frac{f(z_0)}{2\pi i}\int_\Gamma \frac{1}{z-z_0}dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz + \frac{f(z_0)}{2\pi i}2\pi i \qquad \text{(using Lemma 4.10)} \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz + f(z_0).
\end{aligned}$$

Therefore, to prove the result it suffices to show that

$$\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz = 0.$$

Now z_0 is contained in the interior which is open. Thus we can find $\delta > 0$ such that

$$B(z_0, \delta) \subseteq Int\Gamma$$

So for any $0 < \alpha < \delta$ the contour

$$\Gamma_\alpha = \{z \mid |z_0 - z| = \alpha\}$$

is contained entirely in the interior of Γ. Now the function $\frac{f(z)-f(z_0)}{z-z_0}$ is analytic at all points inside Γ except at z_0. This gives us a multiply connected domain $\Gamma + Int\Gamma - Int\Gamma_\alpha$. Using the case $n = 1$ of Theorem 4.11 (the Cauchy-Goursat theorem for multiply connected domains) we get that

$$\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz = \int_{\Gamma_\alpha} \frac{f(z)-f(z_0)}{z-z_0}dz.$$

We now want to use the ML-Result (Theorem 4.6) to evaluate the latter integral.

For this we need to know the length of Γ_α. This is $2\pi\alpha$ (see Section 4.4). Next we need to bound the integrand. Now since f is analytic at z_0 there exists $\delta_1 > 0$ such that whenever $0 < |z - z_0| < \delta_1$,

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < 1.$$

Therefore, if $0 < |z - z_0| < \delta_1$, using the triangle inequality,

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leq \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| + |f'(z_0)| < 1 + |f'(z_0)|.$$

Now choosing $\alpha < min\{\delta, \delta_1\}$ we have by Theorem 4.6 that

$$\left|\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz\right| < (1 + |f'(z_0)|)\, 2\pi\alpha.$$

Since we can choose α to be as small as we wish we deduce that

$$\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}dz = 0$$

and the proof is complete. □

Example Suppose that Γ is the circle $|z| = 2$. Evaluate the integral $\int_\Gamma \frac{\cos z}{1+z^2} dz$.

Notice that $f(z) = \frac{\cos z}{1+z^2}$ is infinite at $z = \pm i$ and $\cos z$ is entire. We use partial fractions to re-express $f(z)$ and get

$$f(z) = \frac{\cos z}{1+z^2} = \frac{\cos z}{(z-i)(z+i)} = \frac{\cos z}{2i(z-i)} - \frac{\cos z}{2i(z+i)}.$$

So

$$\int_\Gamma \frac{\cos z}{1+z^2} dz = \frac{1}{2i} \int_\Gamma \frac{\cos z}{z-i} dz - \frac{1}{2i} \int_\Gamma \frac{\cos z}{z+i} dz.$$

Now the Cauchy Integral Formula gives

$$\int_\Gamma \frac{\cos z}{z-i} dz = 2\pi i \cos(i)$$

and

$$\int_\Gamma \frac{\cos z}{z+i} dz = 2\pi i \cos(-i).$$

Now $\cos(i) = \cos(-i) = \cosh(1)$ (check this!), so

$$\int_\Gamma \frac{\cos z}{1+z^2} dz = 2\pi i \left[\frac{\cos(i)}{2i} - \frac{\cos(-i)}{2i} \right] = \frac{2\pi i}{2i} \cosh(1) = \pi \cosh(1).$$

Let's have another look at the Cauchy Integral Formula.

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-z_0} dz.$$

Now differentiate both sides (with respect to z_0) to give

$$f'(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^2} dz$$

and if we do it again

$$f''(z_0) = \frac{2}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^3} dz$$

and so on. This is not justified. How do we know that we can differentiate both sides? Nonetheless the following theorem is true.

Theorem 4.13 [The Cauchy Integral Formula for Higher Derivatives] *Suppose that f is analytic in the simply connected region U and assume that Γ is any simple closed contour in U. Then f has derivatives of all orders in IntΓ each of which is analytic in IntΓ. Moreover,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Proof: Omitted for time reasons. (See the discussion on pages 184 and 185 of Complex Variables with Applications by David Wunsch, 2nd Edition Addison Wesley 1994, also Osbourne pp. 89-91.)

Example Evaluate

$$\int_\Gamma \frac{\cos z}{z^2(z-1)} dz$$

where Γ is the circle $|z| = 1/3$.

Now $f(z) = \dfrac{\cos z}{z-1}$ is analytic on and inside Γ, so by Cauchy's integral formula for higher derivatives (applied with $n = 1$ and $z_0 = 0$)we have

$$\int_\Gamma \frac{\frac{\cos z}{z-1}}{z^2} dz = 2\pi i \left. \frac{d}{dz} \left(\frac{\cos z}{z-1} \right) \right|_{z=0} = 2\pi i \left. \left[-\frac{\sin z}{z-1} - \frac{\cos z}{(z-1)^2} \right] \right|_{z=0} = -2\pi i.$$

Theorem 4.14 *If f is analytic in some simply connected region U, then f has derivatives of all orders throughout U and each derivative is analytic.*

Proof: For any point z in U we can find a simple closed contour in U enclosing z. Then apply Theorem 4.13. □

Theorem 4.15 (Morera's Theorem) *If a function f is continuous throughout a domain D and if $\int_\Gamma f(z)dz = 0$ for all closed contours Γ contained in D, then f is an analytic function on D.*

(This is the converse of Cauchy's theorem.)

Proof: Suppose that z_1 and z_2 are distinct points in D. Then as

$$\int_\Gamma f(z)dz = 0$$

for all closed paths, we see that the integrals along any contour joining z_1 to z_2 always give the same value. Now fix a in D and define a function

$$F : \mathbb{C} \to \mathbb{C}$$

by

$$F(z) = \int_a^z f(\zeta)d\zeta.$$

We now verify that F is differentiable at any point z_0 in D. We have

$$F(z) = \int_a^{z_0} f(\zeta)d\zeta + \int_{z_0}^z f(\zeta)d\zeta.$$

So we get

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\int_{z_0}^z f(\zeta)d\zeta.$$

This in turn delivers

$$\begin{aligned}\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0}\int_{z_0}^z f(\zeta)d\zeta - f(z_0)\\ &= \frac{1}{z - z_0}\int_{z_0}^z \left[f(\zeta) - f(z_0)\right]d\zeta\end{aligned}$$

where the last equality has come from the fact that, by the Anti-derivative Theorem (Theorem 4.3),

$$\int_{z_0}^z f(z_0)d\zeta = f(z_0)\int_{z_0}^z d\zeta = f(z_0)(z - z_0).$$

Thus we get (using the ML-Result Theorem 4.6 and choosing the contour to be a straight line)

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| = \left|\frac{1}{z - z_0}\int_{z_0}^z \left[f(\zeta) - f(z_0)\right]d\zeta\right| \le \frac{|z - z_0|}{|z - z_0|}M = M.$$

Here M is the maximal value taken by $f(z) - f(z_0)$. We know that f is continuous. So given any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all $0 < |z - z_0| < \delta$. Thus as δ tends to zero M can be chosen arbitrarily small. Hence

$$\lim_{z \to z_0}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| = 0.$$

Hence F is differentiable at every point in the domain D and so F is analytic. Moreover, the derivative of F is f. Finally, we apply Theorem 4.14 to give the result that f is analytic in D. □

Notice that in the proof of Theorem 4.15 we actually constructed an antiderivative of f. In fact given f is analytic this method always allows us to construct an antiderivative.

5 Series Expansions of a Function of a Complex Variable

In this chapter we will prove the theorems of Taylor and Laurent; theorems which allow us to express a function as a series:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

where a_n are complex constants.

5.1 Uniform Convergence

Given a sequence complex functions $g_0(z), g_1(z), g_2(z), \ldots$ for all integers $m \geq 0$ we can define a function

$$f_m(z)=\sum_{n=0}^{m}g_n(z).$$

We can also write

$$f(z)=\sum_{n=0}^{\infty}g_n(z).$$

Definition *The sequence of functions* $\{f_m(z)\}$ *is said to* **uniformly converge** *to* $f(z)$ *(where* $f_m(z)$ *and* $f(z)$ *are defined as above) in a region* R *if for any* $\epsilon > 0$ *there exists a number* N *which is independent of* z *so that for all* z *in* R *and* $m > N$

$$|f_m(z)-f(z)|<\epsilon.$$

Note that

$$f_m(z)-f(z)=\sum_{n=m+1}^{\infty}g_n(z)$$

and that if $\{f_m(z)\}$ uniformly converges then the limit $f(z)$ must be finite.

Also, it is crucial that the parameter N depends only on ϵ and NOT on z. (If for $\epsilon > 0$ and z there exists a number N so that for all $m > N$ we have $|f_m(z) - f(z)| < \epsilon$ we still say that the sequence **converges** to $f(z)$ but this is not sufficient to guarantee that it is **uniformly** convergent.)

One useful way to establish uniform convergence is via the M-test:

Theorem 5.1 (Weierstrass M-test) *Let* $\sum_{n=0}^{\infty} M_n$ *be a convergent series whose terms* M_0, $M_1, \ldots$ *are all positive real constants. Then the sequence* $f_m(z) = \sum_{n=0}^{m} g_n(z)$ *converges uniformly in a region* R *if, for all* $z \in R$,

$$|g_n(z)| \leq M_n.$$

The following theorem shows that uniformly convergent series behave very nicely.

Theorem 5.2 *Suppose that* $f_m(z) = \sum_{n=0}^{m} g_n(z)$ *converges uniformly to* $f(z) = \sum_{n=0}^{\infty} g_n(z)$ *in a region* R.

1. *If each* $g_n(z)$ *is continuous in* R*, then* $f(z)$ *is continuous in* R.
2. *If each* $g_n(z)$ *is analytic in* R*, then* $f(z)$ *is analytic in* R.
3. *If each* $g_n(z)$ *is analytic in* R*, then*

$$\frac{df}{dz}=\sum_{n=0}^{\infty}\frac{dg_n}{dz}.$$

4. *Suppose that* Γ *is a contour in* R *and each* $g_n(z)$ *is continuous in* R*. Then*

$$\int_\Gamma f(z)dz = \sum_{n=0}^{\infty} \int_\Gamma g_n(z)dz.$$

We omit the proof of the theorem. Recall that you proved an analogue for third property for real valued power series was proved last term.

Definition *Suppose that* $z_0 \in \mathbb{C}$*. Then*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is called a **power series** *(centered at* z_0*).*

Generally there will be a real number R such that the power series converges for $|z - z_0| < R$ and **diverges** for $|z - z_0| > R$ where the constant R is called the **radius of convergence** of the series. For $|z - z_0| = R$ the series may or may not converge.

Proposition 5.3 *Suppose* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *is a given power series, then the radius of convergence* R *is given by*

$$R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|.$$

We omit the proof. It is similar to the analogous result for real variables which you have seen last term.

5.2 Taylor and Laurent Series

Theorem 5.4 (Taylor's Theorem) *Suppose that* f *is analytic in a domain* $B = B(z_0, R)$*. Then at each point* z *in* $IntB$ *we have*

$$f(z) \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

So Taylor's Theorem says that the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ converges to $f(z)$ whenever $|z - z_0| < R$. If f is entire then the series converges for all $z \in \mathbb{C}$. Notice that the theorem does not say that R is the radius of convergence! Later on, we will deduce Taylor's theorem from the more general Laurent's theorem.

The special case when $z_0 = 0$ is called a **Maclaurin series**.

Example The series that we know and love from real variable theory remain valid for the complex domain.

(a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all z in $\mathbb{C}$.

(b) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots$ for all $z \in \mathbb{C}$; and

(c) $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4!} + \dots$ for all z in $\mathbb{C}$.

(d) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$.

(e) $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ for $|z| < 1$.

Example Find the radius of convergence R of the Maclaurin series representation of the function

$$f(z) = \frac{1 + z + z^2}{(1 + z^2)(3 + z^2)}.$$

(There is no need to calculate the coefficients.)

The series converges to $f(z)$ within the circle about $z = 0$ whose radius is the distance from 0 to the nearest point where f fails to be analytic. Now $f(z)$ has singularities (i.e. points where f is not analytic, but where f is analytic at some point in every neighbourhood of that point) when $z^2 = -1$ and $z^2 = -3$, i.e. at $z = \pm i$ and $z = \pm\sqrt{3}i$. Thus, $R = 1$.

Example Find the Maclaurin series representation of $f(z) = \sin^2 z$.

Now

$$\begin{aligned} f'(z) = 2\sin z \cos z &= \sin 2z \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!} \end{aligned}$$

Thus by Theorem 5.2 we have

$$f(z) = \sum_{n=0}^{\infty} \int_0^z (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!} dz.$$

So we have

$$\sin^2 z = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{z^{2n+2}}{(2n+2)!} \quad \text{(for any } z \text{ with } |z| < \infty.)$$

Definition *A singular point* z_0 *is said to be* **isolated** *if there is some neighbourhood of* z_0 *throughout which* f *is analytic except at the point itself. So* f *is analytic on a domain* $0 < |z - z_0| < R$.

Example

(a) $f(z) = 1/z$ is analytic everywhere except at $z = 0$. So the origin is an isolated singular point of $f(z)$.

(b) $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has three isolated singular points, at $z = 0$ and $z = \pm i$.

(c) $f(z) = \text{Log} z$ has a singular point at $z = 0$. But every neighbourhood of $z = 0$ contains a point on the negative real axis where f is not analytic. So $z = 0$ is not an isolated singular point of Log z.

(d) $f(z) = 1/\sin(\pi/z)$ has singular points at $z = 0$ and $z = 1/n$, $n = \pm 1, \pm 2, \ldots$ all lying on the segment of the real axis from $z = -1$ to $z = 1$. Each singular point except $z = 0$ is isolated. $z = 0$ is not isolated since every neighbourhood of $z = 0$ contains other singular points of $f(z)$.

5.3 Generalised Power Series

If a function fails to be analytic at a point z_0 we cannot apply Taylor's theorem at that point. However, it is often possible to find a series representation for f involving both positive and negative powers of $z - z_0$, i.e. which looks as follows.

$$\sum_{n=-\infty}^{\infty} c_n z^n \quad = \quad \sum_{n=1}^{\infty} c_{-n} z^{-n} + \sum_{n=0}^{\infty} c_n z^n.$$

This converges if both series converge. The first sum converges in a region outside a disc $|z| > R_1$. The second sum converges inside a disc $|z| < R_2$. Thus the region of convergence is the intersection of the two regions

$$D = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\},$$

i.e. an annulus.

The following is a significant extension of Taylor's Theorem:

Theorem 5.5 (Laurent's Theorem) *Suppose that f is analytic in an annulus*

$$D = \{z \mid R_1 < |z - z_0| < R_2\}$$

with centre z_0. Let γ_r be the contour

$$\{z \mid |z - z_0| = r\}$$

where $R_1 < r < R_2$. Then for $z \in Int\ D$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where, for $n = 0, \pm 1, \pm 2 \ldots$,

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(s)}{(s - z_0)^{n+1}} ds.$$

Proof: By considering the function $f(z - z_0)$ instead of $f(z)$, it is easily seen that is suffices to prove the theorem for the case when $z_0 = 0$.

Suppose that $z \in Int\ D$. Select r_1 and r_2 with $R_1 < r_1 < |z| < r_2 < R_2$ and let γ_{r_1} and γ_{r_2} be two circular contours centred at z_0 of radius r_1 and r_2 respectively.

Now add the auxiliary contours η_1 and η_2 as above. After adding the auxiliary contours η_1 and η_2 we get a contour Γ as above. (so that Γ contains η_1, η_2 and almost all of γ_{r_1} and γ_{r_2}). Now, by assumption, f is analytic on the simply connected region $\Gamma + int\ \Gamma$.

Therefore, we may apply the Cauchy Integral Formula to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}}\frac{f(s)}{s-z}ds + \int_{\eta_1}\frac{f(s)}{s-z}ds - \int_{\gamma_{r_1}}\frac{f(s)}{s-z}ds + \int_{\eta_2}\frac{f(s)}{s-z}ds\right) \\ &= \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}}\frac{f(s)}{s-z}ds - \int_{\gamma_{r_1}}\frac{f(s)}{s-z}ds\right) \\ &= \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}}\frac{f(s)}{s-z}\,ds + \int_{\gamma_{r_1}}\frac{f(s)}{z-s}\,ds\right). \end{aligned}$$

We have

$$\frac{1}{s-z} = \frac{1}{s\left(1-\frac{z}{s}\right)} = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{z}{s}\right)^n$$

whenever $|\frac{z}{s}| < 1$.

Similarly

$$\frac{1}{z-s} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{s}{z}\right)^n$$

whenever $|\frac{s}{z}| < 1$.

Substituting these two expressions into the above formula we get

$$f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}}\left(\sum_{n=0}^{\infty} f(s)\frac{1}{s}\left(\frac{z}{s}\right)^n\right)\,ds + \int_{\gamma_{r_1}}\left(\sum_{n=0}^{\infty} f(s)\frac{1}{z}\left(\frac{s}{z}\right)^n\right)\,ds\right).$$

Now we worry about uniform convergence. We use the M-test. We have for s on γ_{r_2}

$$\left|f(s)\frac{z^n}{s^{n+1}}\right| \leq M_1\frac{|z|^n}{r_2^{n+1}}$$

where M_1 is the maximum value of $f(s)$ on γ_{r_2}. Since $|z| < r_2$, the series $\sum_{n=0}^{\infty} M_1\frac{|z|^n}{r_2^{n+1}}$ converges. Thus the M-test indicates that $\sum_{n=0}^{\infty} f(s)\frac{1}{s}(\frac{z}{s})^n$ converges uniformly. Therefore, Theorem 5.2 implies

$$\int_{\gamma_{r_2}}\left(\sum_{n=0}^{\infty} f(s)\frac{1}{s}\left(\frac{z}{s}\right)^n\right)\,ds = \sum_{n=0}^{\infty}\left(\int_{\gamma_{r_2}}\frac{f(s)}{s^{n+1}}z^n\right)\,ds.$$

A similar argument shows that

$$\int_{\gamma_{r_1}}\left(\sum_{n=0}^{\infty} f(s)\frac{1}{z}\left(\frac{s}{z}\right)^n\right)\,ds = \sum_{n=-\infty}^{-1}\left(\int_{\gamma_{r_1}}\frac{f(s)}{s^{n+1}}z^n\right)\,ds.$$

Hence

$$f(z) = \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}\left(\int_{\gamma_{r_2}}\frac{f(s)}{s^{n+1}}z^n ds\right) + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}\left(\int_{\gamma_{r_1}}\frac{f(s)}{s^{n+1}}z^n ds\right)\right).$$

Let $R_1 < r < R_2$. Then by the Cauchy-Goursat Theorem for multiply connected domains we have for each n

$$\int_{\gamma_{r_1}}\frac{f(s)}{s^{n+1}}z^n ds = \int_{\gamma_r}\frac{f(s)}{s^{n+1}}z^n\,ds$$

and

$$\int_{\gamma_{r_2}}\frac{f(s)}{s^{n+1}}z^n ds = \int_{\gamma_r}\frac{f(s)}{s^{n+1}}z^n ds.$$

Thus

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{where} \quad c_n = \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(s)}{s^{n+1}}ds.$$

□

Now we show that Taylor's theorem is indeed a special case of Laurent's theorem.

Proof: [of Taylor's Theorem] Suppose that f is analytic at all points inside with $|z - z_0| < R_2$ (so analytic in the whole disk.) We first claim that the coefficients of the Laurent expansion for negative n are all 0. We have for $n < 0$

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(s)}{(s-z_0)^{n+1}} ds \\ &= \frac{1}{2\pi i}\int_{\gamma_r} f(s)(s-z_0)^{-1-n} ds. \end{aligned}$$

But n is negative so $(s - z_0)^{-n-1}$ is an analytic function on $IntB(z_0, R_2)$ as is f, therefore the product is analytic. But then the Cauchy-Goursat Theorem 4.7 shows that $\int_{\gamma_r} f(s)(s - z_0)^{-1-n} ds = 0$ and we have proved the claim.

For $n \geq 0$, by Theorem 4.13, the coefficient c_n of the Laurent expansion satisfies

$$c_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(s)}{(s-z_0)^{n+1}} ds = \frac{f^{(n)}(z_0)}{n!},$$

as required for the Taylor expansion. ☐

It is also a fact that

Theorem 5.6 *Given an annulus D with centre z_0 the Laurent expansion is the unique expression for f as a power series in $(z - z_0)$.*

Proof: Omitted.

Notice that the Laurent expansion of f depends upon both the point z_0 and the annulus D. We will make this more precise in the next example.

Example Find all possible Laurent expansions of $f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$ about $z_0 = 0$.

We notice that $f(z)$ has singular points (simple poles) at $z = 0, 1$.

First of all consider $0 < |z| < 1$. We have

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

which is convergent for $|z| < 1$ and therefore,

$$f(z) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$$

converges for $0 < |z| < 1$. Now since the Laurent expansion is unique (by Theorem 5.6) this is the Laurent expansion for f about $z_0 = 0$ on the annulus $D = \{z \mid 0 < |z| < 1\}$. Notice that f is indeed analytic on D. There is, however, a further annulus on which f is analytic:

$$D_1 = \{z \mid |z| > 1\}.$$

Thus we will obtain a further Laurent expansion. How do we find it?

Well, since $|z| > 1$, $\sum_{n=0}^{\infty} z^n$ is no longer convergent. So we rearrange things and write

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\frac{1}{(1-\frac{1}{z})}$$

and we can expand $\dfrac{1}{(1-\frac{1}{z})}$ in powers of $\frac{1}{z}$:

$$\frac{1}{(1-\frac{1}{z})} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n .$$

Therefore,

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{-\infty}^{-1} z^n.$$

Thus in total we have $f(z) = \sum_{-\infty}^{-2} z^n$.

6 Singularities and Residues

6.1 Zeros

We begin with some observations on **zeros** of analytic functions. We need the following result (of which we omit the proof) later on:

Lemma 6.1 *Suppose that f is an analytic function in a domain U and that for all $n \geq 1$, $f(p) = f^{(n)}(p) = 0$ for some point p in U. Then $f(z) = 0$ for all $z \in U$.*

Definition *Any point p such that $f(p) = 0$ is called a* **zero** *of f.*

Suppose that f is analytic in a domain U and that p is a zero of f. Then there will be an open neighbourhood $B(p, r)$ in which f is analytic. Then on this open ball we apply Taylor's Theorem 5.4 to get for all $z \in B(p, r)$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)(z-p)^n}{n!}.$$

Using Lemma 6.1 we see that unless f is identically zero in U there exists $N \geq 0$ such that $f^{(N)}(p) \neq 0$. (This means that the coefficient c_N in the Taylor expansion is not zero and I'm assuming that it is the first such.) In this case we say that f **has a zero of order** N **at** p.

To reiterate, if f has a zero of order N at p the first non-zero term in the Taylor expansion is $\frac{f^{(N)}(p)(z-p)^N}{N!}$ and thus we may write

$$f(z) = (z-p)^N G(z)$$

where

$$G(z) = \sum_{n=0}^{\infty} c_{N+n}(z-p)^n$$

in which case $G(p) = c_N \neq 0$.

Conversely if $f(z) = (z-p)^N g(z)$ where $g(z)$ is analytic and $g(p) \neq 0$, then f has a zero of order N at p.

6.2 Singularities

Suppose that p is an isolated singularity. Then by Laurent's Theorem 5.5 f has an expansion about p

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-p)^n$$

for z with $0 < |z-p| < R$.

We now split this sum and write

$$f(z) = \sum_{n=1}^{\infty} c_{-n}(z-p)^{-n} + \sum_{n=0}^{\infty} c_n(z-p)^n.$$

The first sum is called the **principal part** of the Laurent expansion of f.

There are three possibilities:

(i) If the principal part of the expansion of f has only finitely many non-zero terms

$$\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-p)^n} = \frac{c_{-1}}{z-p} + \frac{c_{-2}}{(z-p)^2} + \ldots + \frac{c_{-M}}{(z-p)^M}$$

then we say that the singularity of $f(z)$ at p is a **pole of order** M. Poles of order 1 or of **first order** are called **simple poles**.

Example

(a) $f(z) = \frac{1}{z}$ has a simple pole at $z = 0$.

(b) $f(z) = \frac{z-3}{(z-4)^4}$ has a pole of order 4 at $z = 4$, because the Laurent expansion about 4 is

$$f(z) = \frac{1}{(z-4)^4} + \frac{1}{(z-4)^3}.$$

(ii) If the principal part of the expansion has infinitely many terms, then we say that $f(z)$ has at p an **isolated essential singularity**. (The word 'isolated' is usually omitted.)

Example The expansion of $\sin \frac{1}{z}$ about $z = 0$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}}$ and so $z = 0$ is an isolated essential singularity.

(iii) If the principal part of the expansion vanishes, i.e. $c_{-n} = 0$ for all $n \geq 1$ we say that $f(z)$ has a **removable singularity** at p. If we define $f(p) = c_0$ the function is analytic at $z = p$.

Example Consider $f(z) = \frac{\sin z}{z}$. Then $f(z)$ is not defined at $z = 0$. But the Laurent expansion about $z_0 = 0$ is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots$$

and so the principal part of $f(z)$ is zero. So $f(z)$ has a removable singularity at $z = 0$ and the series does in fact converge at $z = 0$. If we write $f(0) = 1$ then the resulting function is entire.

We have the following result which relates removable singularities with poles.

Theorem 6.2 *The following are equivalent:*

1. *$f(z)$ has a pole of order N at p.*
2. *$\lim_{z \to p}(z-p)^{N+1} f(z) = 0$ and not for any $n < N$.*
3. *$(z-p)^N f(z)$ has a removable singularity at p.*

Proof: Suppose that f has a pole of order N at p. Then, by definition the Laurent expansion of $f(z)$ has the form:

$$f(z) = \frac{c_{-N}}{(z-p)^N} + \dots + c_0 + \dots.$$

Therefore, the function $g(z) = (z-p)^N f(z)$ has (by the Uniqueness result) the Laurent expansion

$$\begin{aligned} g(z) &= (z-p)^N \left(\frac{c_{-N}}{(z-p)^N} + \dots + c_0 + \dots\right) \\ &= c_{-N} + c_{-N+1}(z-p) + \dots + c_0(z-p)^N + \dots. \end{aligned}$$

and $(z-p)^{N+1} f(z)$ has expansion

$$(z-p)^{N+1} f(z) = c_{-N}(z-p) + c_{-N+1}(z-p)^2 + \dots + c_0(z-p)^{N+1} + \dots.$$

Thus we deduce that (1) implies both (2) and (3).

By similar arguments, one can show that (3) implies (2) and (2) implies (1). ☐

Notice that if $f(z)$ has a pole of order $N > 0$ at p, then $\lim_{z \to p} |f(z)| = \infty$.

The behaviour of a function close to an essential isolated singularity is very unpredictable:

Theorem 6.3 (Picard) *In any open ball around an isolated essential singularity an analytic function $f(z)$ takes any value with at most one exception.*

Proof: See e.g. Conway , Functions of One Complex Variable, page 300. ☐

6.3 Residues

Definition *Let f be defined on the domain U and let p be an isolated singularity of f. Suppose that*

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-p)^n$$

is the Laurent expansion of f centred at p. Then the coefficient c_{-1} is called the **residue** *of f at p and is denoted by $Res(f,p)$.*

Thus, from Laurent's Theorem 5.5, we have

$$Res(f,p) = c_{-1} = \frac{1}{2\pi i}\int_{\gamma} f(z)dz$$

where γ is any closed contour (positively oriented) enclosing p with p the only singularity in and on γ.

So already residues look interesting. The definition say that an integral is a multiple of a residue. This expression provides a powerful method for evaluating certain integrals around simple closed contours.

The following five rules are useful (and can be deduced straight from the Laurent expansion of f about p.)

RULE I Suppose that f has a simple pole at p. Then $f(z) + \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + \ldots$ and so

$$Res(f,p) = \lim_{z \to p}(z-p)f(z).$$

RULE II Suppose that $f(z)$ has a simple pole at p and $f(z) = \frac{h(z)}{g(z)}$ where $g(z)$ and $h(z)$ are analytic on U and $g(p) = 0$, then $\lim_{z\to p}(z-p)\frac{h(z)}{g(z)} = \lim_{z\to p}\frac{((z-p)h(z))'}{g'(z)} = \frac{h(p)}{g'(p)}$ by L'Hôpital's rule. Thus

$$Res(f,p) = \frac{h(p)}{g'(p)}.$$

(Actually, you have seen a proof of L'Hôpital's rule only for real valued functions. However, it works also for complex valued functions.)

RULE III Suppose that $f(z)$ has a pole of order m at p and $g(z) = (z-p)^m f(z)$. Then

$$Res(f,p) = \frac{1}{(m-1)!}g^{(m-1)}(p).$$

(because $f(z) = \frac{c_{-m}}{(z-p)^m} + \ldots + \frac{c_{-1}}{z-p} + c_0 + \ldots$ and so $g(z) = c_{-m} + \ldots + c_{-1}(z-p)^{m-1} + \ldots$.

RULE IV If $f(z) = \frac{g(z)}{h(z)}$ (where $h(z)$ and $g(z)$ are analytic on U) has an isolated essential singularity at p, then calculate the first few terms of the Taylor series of $h(z)$ and $g(z)$ and then divide.

RULE V Calculate the Laurent series directly.

Example

(a) Suppose that $f(z) = \frac{\sin z}{z^4}$. Then f has a pole of order 3 (check the Laurent series to see this) at $z = 0$. We can calculate the residue by Rule V.

$$f(z) = \frac{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \ldots.$$

Therefore, $Res(f,0) = -\frac{1}{6}$.

(b) Let $f(z) = \frac{(z+1)}{(z-1)^3(z+3)}$. Then f has a pole of order 3 at $z = 1$. We use Rule III to determine the residue. We have $g(z) = \frac{(z+1)}{(z+3)}$ thus we differentiate twice

$$g'(z) = \frac{z+3-z-1}{(z+3)^2} = \frac{2}{(z+3)^2}$$

$$g''(z) = \frac{-4}{(z+3)^3}.$$

Therefore, the residue at 1 is $Res(f,1) = \frac{g''(1)}{2!} = -\frac{1}{2!16} = -\frac{1}{32}$.

(c) Suppose that $f(z) = z^2 \sin(\frac{1}{z})$. Then f has an isolated essential singularity at $z = 0$. Thus we must use Rule IV or V. We use rule IV. We know that

$$f(z) = z^2 \left(\frac{1}{z} - \frac{1}{3!z^3} + \ldots \right) = z - \frac{1}{3!z} + \ldots$$

Therefore,

$$Res(f,0) = -\frac{1}{6}.$$

Note that the residue at a removable singularity is zero.

6.4 Cauchy's Residue Theorem

Theorem 6.4 (Cauchy's Residue Theorem) *Suppose that γ is a simple closed contour in a simply connected domain U. Suppose that f is analytic on $Int\Gamma$ except for a finite number of isolated singularities $p_1, \ldots, p_n$. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, p_k).$$

Proof: Enclose each of the points p_k in a circular contour γ_k so that each of the contours γ_k and γ do not touch.

Then by the Cauchy-Goursat Theorem for multiply-connected domains (Theorem 4.11) and Laurent's theorem (Theorem 5.5), we get

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, p_k).$$

$\square$

6.5 Evaluation of Real Integrals

One of the main use of the Residue Theorem is to evaluate real integrals. There is a three step approach to this method:

Step 1 Determine the appropriate contour for your particular real integral.

Step 2 Calculate the residues of singularities within your chosen contour.

Step 3 If the integrals are improper make a suitable approximation (using the ML-result or Jordan's Lemma 6.5)

(The example sheet describes a method for evaluating the order of a pole, which is useful in step 2.)

TYPE I:

Integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$$

where f is a function of $\cos\theta$ and $\sin\theta$. When this is the case we use the transformation

$$z \longrightarrow e^{i\theta}.$$

the integrand then becomes a rational function of z and the unit circle becomes the contour. In particular

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

and

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Example Evaluate

$$I = \int_0^{2\pi} \frac{1}{5 - 4\cos\theta} d\theta.$$

Now write $z = e^{i\theta}$. Then

$$dz = \frac{dz}{d\theta} d\theta = ie^{i\theta} d\theta.$$

So we have (where the contour Γ is the unit circle whose centre is 0)

$$I = \int_\Gamma \frac{1}{5 - 2(z + 1/z)} \frac{dz}{iz} = \int_\Gamma \frac{-i}{5z - 2z^2 - 2} dz = \int_\Gamma \frac{i}{(2z-1)(z-2)} dz.$$

The integrand has simple poles at $z = 1/2$ and $z = 2$. Only $z = 1/2$ is inside Γ. Thus,

$$I = 2\pi i Res(f, 1/2) = 2\pi i \lim_{z \to 1/2} \frac{(z - 1/2)i}{2(z - 1/2)(z - 2)} = 2\pi i \frac{i}{2(-3/2)} = \frac{2\pi}{3}.$$

TYPE II: Improper real integrals.

Thesse are integrals of the form

$$\int_{-\infty}^{\infty} f(x)dx$$

or

$$\int_{0}^{\infty} [\text{Even Integrand}]dx.$$

The integrand must be analytic on the Upper Half Pane except for a finite number of poles (none lying on the real axis).

We define the improper integral to be the **Cauchy Principal Value** version, i.e. we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

provided the limit exists.

We immediately observe that if $f(x)$ is an odd function, $\int_{-\infty}^{\infty} f(x)dx = 0$ and if $f(x)$ is an even function, then $\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$ which explains the integral above with the limits 0 and ∞. For such examples we integrate $f(z)$ around the closed contour $\Gamma^R = \Gamma_1^R + \Gamma_2^R$, where

$$\Gamma_1^R = \{z \mid |z| = R \text{ and } Im(z) > 0\} \text{ and } \Gamma_2^R = \{z \mid z = Re(z) \text{ and } -R \le z \le R\}$$

Then we need that the integral round the semicircular part Γ_1^R of the contour tends to zero as R approaches infinity. In this case we get

$$\lim_{R\to\infty} \int_{\Gamma^R} f(z)dz = \lim_{R\to\infty} \left(\int_{\Gamma_1^R} f(z)dz + \int_{\Gamma_2^R} f(z)dz \right) = \int_{-\infty}^{\infty} f(x)dx.$$

The simplest case arises when $|f(z)|$ is $O(|z|^{-k})$ with $k > 1$. (Recall that $f(z)$ is $O(|z|^{-k})$ means that there exists a real constant C such that for all z we have $|\frac{f(z)}{z^k}| \le C$.) Then (since the length of Γ_1^R is πR) by the ML result (Theorem 4.4), we will always have

$$\lim_{R\to\infty} \int_{\Gamma_1^R} f(z)dz = \lim_{R\to\infty} (\pi R)(CR^{-k}) = 0.$$

Example Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+4} dx.$$

We consider

$$f(z) = \frac{z^2}{z^4+5z^2+4} = \frac{z}{(z^2+1)(z^2+4)}.$$

Then f has four isolated singularities at $\pm 2i$, $\pm i$. Each of the singularities is a simple pole and only i and $2i$ are in the Upper Half Plane. Neither of them are on the real axis so we are in business. Let $\Gamma^R = \Gamma_1^R + \Gamma_2^R$ as above.

By the Residue Theorem we have

$$\int_{\Gamma^R} f(z)dz = 2\pi i(Res(f,i) + Res(f,2i)).$$

We calculate the residues by Rule I.

$$Res(f,i) = \lim_{z\to i}(z-i)\frac{z^2}{(z^2+1)(z^2+4)} = \lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{(2i)(3)} = -\frac{1}{6i}$$

and similarly

$$Res(f,2i) = \lim_{z\to 2i}(z-2i)\frac{z^2}{(z^2+1)(z^2+4)} = \frac{-4}{(-3)(4i)} = \frac{1}{3i}.$$

Thus

$$\int_{\Gamma_R} f(z)dz = 2\pi i\left(-\frac{1}{6i}+\frac{1}{3i}\right) = \frac{\pi}{3}.$$

We want to determine $\int_{\Gamma_1^R} f(z)dz$. To do this we use the ML result. We know the length of the contour is πR (a semicircle). Furthermore, on the contour $|z|^2 = R^2$ and $|z^2+1| \geq |z^2| - 1 = R^2 - 1$ (by the triangle inequality) and similarly $|z^2+4| \geq R^2 - 4$, so we get

$$\sup_{z\in\Gamma_1^R} |f(z)| = M \leq \frac{R^2}{(R^2-1)(R^2-4)}.$$

Thus the ML result gives

$$\left|\int_{\Gamma_1^R} f(z)dz\right| \leq \pi R\frac{R^2}{(R^2-1)(R^2-4)} = O(R^{-1}).$$

So as $R\to\infty$ we do indeed have $\int_{\Gamma_1^R} f(z)dz \to 0$.

Finally we get

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\left(\int_{\Gamma_1^R} f(z)dz + \int_{-R}^{R} f(z)dz\right) = \lim_{R\to\infty}\int_{\Gamma^R} f(z)dz = \frac{\pi}{3}.$$

Sometimes it is necessary to use a deeper argument to show that the integral round the semicircle approaches 0 as R approaches ∞.

Lemma 6.5 (Jordan) *Let* $\gamma_R = \{z \mid z = Re^{i\theta}\ 0 \le \theta \le \pi\}$ *and suppose that*

$$M(R) = \sup_{z\in\gamma_R} |f(z)|.$$

If $\lim_{R\to\infty} M(R) = 0$*, then for* $\alpha > 0$

$$\lim_{R\to\infty} \int_{\gamma_R} e^{i\alpha z} f(z)dz = 0.$$

Proof: Put

$$I_R = \int_{\gamma_R} e^{i\alpha z} f(z)dz.$$

On γ_R, we have $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ and so $\frac{dz}{d\theta} = iRe^{i\theta}$. Using the definition of a contour integral we thus get

$$\begin{aligned} |I_R| &= \left| \int_0^\pi e^{i\alpha R\cos\theta - \alpha R\sin\theta} f(Re^{i\theta}) iRe^{i\theta} d\theta \right| \\ &\le \int_0^\pi \left| e^{i\alpha R\cos\theta - \alpha R\sin\theta} f(Re^{i\theta}) iRe^{i\theta} \right| d\theta \end{aligned}$$

where the last inequality comes from Lemma 4.5. Now $|e^{i\alpha R\cos\theta}| = 1$, $|f(Re^{i\theta})| \le M(R)$ (by hypothesis) and $|iRe^{i\theta}| = R$; therefore, since $\sin\theta$ is symmetric about $\frac{\pi}{2}$

$$|I_R| \le RM(R) \int_0^\pi |e^{-\alpha R\sin\theta}| d\theta = 2RM(R) \int_0^{\frac{\pi}{2}} e^{-\alpha R\sin\theta} d\theta.$$

Observe that for $0 \le \theta \le \frac{\pi}{2}$ we have $\sin\theta \ge \frac{2\theta}{\pi}$. (check this by differentiating $\sin\theta/\theta$!)

This implies that for $0 \le \theta \le \frac{\pi}{2}$ we have

$$e^{-\alpha R\sin\theta} \le e^{-2\theta\alpha R/\pi}$$

which is positive. Therefore,

$$\begin{aligned} |I_R| &\le 2RM(R) \int_0^{\frac{\pi}{2}} e^{-2\theta\alpha R/\pi} d\theta = 2RM(R) \left[-\frac{\pi}{2\alpha R} e^{-2\theta\alpha R/\pi} \right]_0^{\frac{\pi}{2}} \\ &= \frac{M(R)\pi}{\alpha} \left(1 - e^{-\alpha R}\right) \le \frac{M(R)\pi}{\alpha}. \end{aligned}$$

Hence $\lim_{R\to\infty} |I_R| \le \lim_{R\to\infty} \frac{M(R)\pi}{\alpha} = 0$ and so $\lim_{R\to\infty} I_R = 0$ as claimed. □

Example Evaluate the following integral for $a > 0$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx.$$

We consider the complex integral

$$I_R = \int_{\Gamma^R} \frac{e^{iz}}{z^2 + a^2} dz$$

where $\Gamma^R = \Gamma_1^R + \Gamma_2^R$ and (exactly as in example 6.7)

$$\Gamma_1^R = \{z \mid |z| = R \text{ and } Im(z) > 0\} \text{ and } \Gamma_2^R = \{z \mid z = Re(z) \text{ and } -R \leq z \leq R\}.$$

Put

$$g(z) = \frac{e^{iz}}{z^2 + a^2}.$$

The integrand has two simple poles one at ia and one at $-ia$. Only ia is inside the contour. Rule I gives:

$$Res(g, ia) = \lim_{z \to ia} (z - ia) \left(\frac{e^{iz}}{z^2 + a^2} \right) = \lim_{z \to ia} \frac{e^{iz}}{z + ia} = \frac{e^{-a}}{2ia}.$$

Hence by Cauchy's Residue Theorem we have

$$I_R = 2\pi i \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a}.$$

To show that $\int_{\Gamma_1^R} g(z)dz \to 0$ as $R \to \infty$, we apply Jordan's Lemma with $f(z) = \frac{1}{z^2+a^2}$. We have

$$M(R) = \sup_{z \in \Gamma^R} |f(z)| \leq \frac{1}{R^2 - a^2}$$

which certainly approaches 0 as R tends to ∞ so we are OK. Jordan's Lemma implies that

$$\lim_{R \to \infty} I_R = \int_{-\infty}^{\infty} g(z)dz = \frac{\pi e^{-a}}{a}.$$

Finally, this in turn implies that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = Re \left(\int_{-\infty}^{\infty} g(z)dz \right) = \frac{\pi e^{-a}}{a}.$$

Type III: Indenting

This time the integrand may have **simple poles** on the real axis. The plan is to skip around them and then to proceed as in the Type II integrals using Jordan's Lemma or the ML-result.

Example With this method, one can show $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$. To prove this, one considers the complex function $f(z) = \dfrac{e^{iz}}{z}$ around the indented contour shown below. (Note that f has a simple pole at $z = 0$.)

Type IV: Branch Cuts

Some functions such as $\log(z)$ and z^{α} are not defined on the whole plane. So when constructing a contour for such functions we must stay within the domain of definition. That is we must avoid at all costs the cut where the function is not defined.

Example Evaluate for $\alpha \in (0, 1)$

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

Here we choose to consider the complex function

$$f(z) = \frac{z^{\alpha-1}}{1-z},$$

(notice $1 - z$ is correct), where $z^{\alpha-1} = e^{(\alpha-1)Log(z)}$ and $Logz$ is the principal branch and as such is only analytic on the plane with the negative real axis deleted. We consider the following contour $\gamma = \gamma_R + AB + CD + \gamma_{\epsilon}$, where

$$\begin{array}{rcl} \gamma_R & = & \{Re^{i\theta} \mid -\pi + \rho \leq \theta \leq \pi - \rho\} \\ AB & = & \{xe^{i(\pi-\rho)} \mid \epsilon \leq x \leq R\} \\ CD & = & \{xe^{i(\rho-\pi)} \mid \epsilon \leq x \leq R\} \\ \gamma_{\epsilon} & = & \{\epsilon e^{i\theta} \mid -\pi + \rho \leq \theta \leq \pi - \rho\} \end{array}$$

It turns out that as $R \to \infty$ and $\epsilon \to 0$, the contributions from the circular parts vanish, whereas both horizontal parts contribute to the final result. After some calculation, one can show that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \alpha\pi}.$$

[This is related to the Gamma function $\Gamma(z)$. In fact it equals $\Gamma(\alpha)\Gamma(1-\alpha)$.]