MSM3P17 Computability – Example Sheet 1

All students will be asked to hand in solutions to the first 3 questions for assessment (not all of which will be marked). 4th year students will additionally be asked to hand in a solution to Question 4. You should place your solutions in my (=Deryk Osthus) pigeonhole by **1pm on Monday 31st January**. The first example class will take place on Tuesday, 25th January at 2pm (there will be a **lecture at 1pm**). You should not consider nondeterministic machines on this example sheet.

1. Consider the following Turing machine M: The states are $Q = \{q_0, q_1, q_e\}$, the alphabet is $\Sigma = \{0, 1, a, \#, \square\}$, the initial state is q_0 and the only accepting state is q_e. The transition function δ is given by

$$\delta(q_0, 1) = (q_1, a, R), \quad \delta(q_0, 0) = (q_0, a, R), \quad \delta(q_0, \#) = (q_e, a, R)$$

   (and as usual δ is not defined for cases not specified explicitly).

   (a) Write down all configurations of the computation if the input is 0101 and 0#101.

   (b) Determine (with justification) the function computed by M. (Note for this, you first have to determine which strings are accepted.)

2. (a) Write down explicitly a Turing machine which computes the function $f(n) = n - 1$, where the input and output is given in binary. Recall that the head is initially at the leftmost position of the input. It suffices if your machine works correctly if the input begins with a 1 (i.e. $n > 0$) and the input consists only of symbols $\{0, 1\}$. Also, leading zeros in the output need not be deleted (e.g. if the answer is 111, it is ok if the output is 0111). But the final head position should be correct. Explain in a few sentences why and how it works.

   (b) Write down all configurations for the input 10 (where 10 is viewed as a binary string).

   (c) Suppose that the number of binary digits of the number n is b. What is the running time of your machine in terms of b? Is it $O(b)$? Is it $O(n)$?

3. Write down explicitly the transitions of a 2-tape Turing machine T which, given a string x on tape 1 which consists entirely of 1's, outputs the string xx on tape 1. The final head position does not matter. Calculate the running time of T (both exactly and using the $O()$ notation) and explain in a couple of sentences how it works.

4. Let $x \in \mathbb{N}$ and recall that $\text{bin}(x)$ denotes the binary expansion of x. Show that

$$\log_2 x < |\text{bin}(x)| \leq 1 + \log_2 x.$$

   Deduce that $|\text{bin}(x)| = 1 + \lfloor \log_2 x \rfloor$. (Hint: exponentiate the inequalities for the first inequality)