HAMILTONIAN DEGREE SEQUENCES IN DIGRAPHS

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ABSTRACT. We show that for each $\eta > 0$ every digraph G of sufficiently large order n is Hamiltonian if its out- and indegree sequences $d_1^+ \leq \cdots \leq d_n^+$ and $d_1^- \leq \cdots \leq d_n^-$ satisfy (i) $d_i^+ \geq i + \eta n$ or $d_{n-i-\eta n}^- \geq n - i$ and (ii) $d_i^- \geq i + \eta n$ or $d_{n-i-\eta n}^+ \geq n - i$ for all $i < n/2$. This gives an approximate solution to a problem of Nash-Williams [22] concerning a digraph analogue of Chvátal's theorem. In fact, we prove the stronger result that such digraphs G are pancyclic.

1. INTRODUCTION

Since it is unlikely that there is a characterization of all those graphs which contain a Hamilton cycle it is natural to ask for sufficient conditions which ensure Hamiltonicity. One of the most general of these is Chvátal's theorem [9] that characterizes all those degree sequences which ensure the existence of a Hamilton cycle in a graph: Suppose that the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. This condition on the degree sequence is best possible in the sense that for any degree sequence violating this condition there is a corresponding graph with no Hamilton cycle. More precisely, if $d_1 \leq \cdots \leq d_n$ is a graphical degree sequence (i.e. there exists a graph with this degree sequence) then there exists a non-Hamiltonian graph G whose degree sequence $d_1' \leq \cdots \leq d_n'$ is such that $d_i' \geq d_i$ for all $1 \leq i \leq n$.

A special case of Chvátal's theorem is Dirac's theorem, which states that every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle. An analogue of Dirac's theorem for digraphs was proved by Ghouila-Houri [10]. (The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices.) Nash-Williams [22] raised the question of a digraph analogue of Chvátal's theorem quite soon after the latter was proved.

For a digraph G it is natural to consider both its outdegree sequence $d_1^+, \ldots, d_n^+$ and its indegree sequence $d_1^-, \ldots, d_n^-$. Throughout this paper we take the convention that $d_1^+ \leq \cdots \leq d_n^+$ and $d_1^- \leq \cdots \leq d_n^-$ without mentioning this explicitly. Note that the terms d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of G.

Conjecture 1 (Nash-Williams [22]). *Suppose that G is a strongly connected digraph on $n \geq 3$ vertices such that for all $i < n/2$*

  (i) $d_i^+ \geq i + 1$ *or* $d_{n-i}^- \geq n - i$,

(ii) $d_i^- \geq i + 1$ *or* $d_{n-i}^+ \geq n - i$.

Then G contains a Hamilton cycle.

No progress has been made on this conjecture so far (see also [4]). It is even an open problem whether the conditions imply the existence of a cycle through any pair of given vertices (see [5]).

As discussed in Section 2, one cannot omit the condition that G is strongly connected. At first sight one might also try to replace the degree condition in Chvátal's theorem by

- $d_i^+ \geq i + 1$ or $d_{n-i}^+ \geq n - i$,
- $d_i^- \geq i + 1$ or $d_{n-i}^- \geq n - i$.

However, Bermond and Thomassen [5] observed that the latter conditions do not guarantee Hamiltonicity. Indeed, consider the digraph obtained from the complete digraph K on $n - 2 \geq 4$ vertices by adding two new vertices v and w which both send an edge to every vertex in K and receive an edge from one fixed vertex $u \in K$.

The following example shows that the degree condition in Conjecture 1 would be best possible in the sense that for all $n \geq 3$ and all $k < n/2$ there is a non-Hamiltonian strongly connected digraph G on n vertices which satisfies the degree condition except that $d_k^+, d_k^- \geq k + 1$ are replaced by $d_k^+, d_k^- \geq k$ in the kth pair of conditions. To see this, take an independent set I of size $k < n/2$ and a complete digraph K of order $n - k$. Pick a set X of k vertices of K and add all possible edges (in both directions) between I and X. The digraph G thus obtained is strongly connected, not Hamiltonian and

$$\underbrace{k, \ldots, k}_{k \text{ times}}, \underbrace{n-1-k, \ldots, n-1-k}_{n-2k \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k \text{ times}}$$

is both the out- and indegree sequence of G. A more detailed discussion of extremal examples is given in Section 2.

In this paper we prove the following approximate version of Conjecture 1 for large digraphs.

Theorem 2. *For every $\eta > 0$ there exists an integer $n_0 = n_0(\eta)$ such that the following holds. Suppose G is a digraph on $n \geq n_0$ vertices such that for all $i < n/2$*

- $d_i^+ \geq i + \eta n$ or $d_{n-i-\eta n}^- \geq n - i$,
- $d_i^- \geq i + \eta n$ or $d_{n-i-\eta n}^+ \geq n - i$.

Then G contains a Hamilton cycle.

Instead of proving Theorem 2 directly, we will prove the existence of a Hamilton cycle in a digraph satisfying a certain expansion property (Theorem 16). We defer the precise statement to Section 6.

The following weakening of Conjecture 1 was posed earlier by Nash-Williams [20, 21]. It would yield a digraph analogue of Pósa's theorem which states that a graph G on $n \geq 3$ vertices has a Hamilton cycle if its degree sequence $d_1, \ldots, d_n$ satisfies $d_i \geq i + 1$ for all $i < (n-1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd [23]. Note that this is much stronger than Dirac's theorem but is a special case of Chvátal's theorem.

Conjecture 3 (Nash-Williams [20, 21]). *Let G be a digraph on $n \geq 3$ vertices such that $d_i^+, d_i^- \geq i + 1$ for all $i < (n-1)/2$ and such that additionally $d_{\lceil n/2 \rceil}^+, d_{\lceil n/2 \rceil}^- \geq \lceil n/2 \rceil$ when n is odd. Then G contains a Hamilton cycle.*

The previous example shows that the degree condition would be best possible in the same sense as described there. The assumption of strong connectivity is not necessary in Conjecture 3, as it follows from the degree conditions. The following approximate version of Conjecture 3 is an immediate consequence of Theorem 2.

Corollary 4. *For every $\eta > 0$ there exists an integer $n_0 = n_0(\eta)$ such that every digraph G on $n \geq n_0$ vertices with $d_i^+, d_i^- \geq i + \eta n$ for all $i < n/2$ contains a Hamilton cycle.*

In Section 4 we give a construction which shows that for oriented graphs there is no analogue of Pósa's theorem. (An oriented graph is a digraph with no 2-cycles.)

It will turn out that the conditions of Theorem 2 even guarantee the digraph G to be *pancyclic*, i.e. G contains a cycle of length t for all $t = 2, \ldots, n$.

Corollary 5. *For every $\eta > 0$ there exists an integer $n_0 = n_0(\eta)$ such that the following holds. Suppose G is a digraph on $n \geq n_0$ vertices such that for all $i < n/2$*

- *$d_i^+ \geq i + \eta n$ or $d_{n-i-\eta n}^- \geq n - i$,*
- *$d_i^- \geq i + \eta n$ or $d_{n-i-\eta n}^+ \geq n - i$.*

Then G is pancyclic.

Thomassen [25] proved an Ore-type condition which implies that every digraph with minimum in- and outdegree $> n/2$ is pancyclic. (The complete bipartite digraph whose vertex class sizes are as equal as possible shows that the latter bound is best possible.) Alon and Gutin [1] observed that one can use Ghouila-Houri's theorem to show that every digraph G with minimum in- and outdegree $> n/2$ is even vertex-pancyclic. Here a digraph G is called *vertex-pancyclic* if every vertex of G lies on a cycle of length t for all $t = 2, \ldots, n$. In Proposition 9 we show that one cannot replace pancyclicity by vertex-pancyclicity in Corollary 5. Minimum degree conditions for (vertex-) pancyclicity of oriented graphs are discussed in [15].

Our result on Hamilton cycles in expanding digraphs (Theorem 16) is used as a tool in [16] to prove an approximate version of Sumner's universal tournament conjecture. Theorem 16 also has an application to a conjecture of Thomassen on tournaments. A *tournament* is an orientation of a complete graph. We say that a tournament is *regular* if every vertex has equal in- and outdegree. Thus regular tournaments contain an odd number n of vertices and each vertex has in- and outdegree $(n-1)/2$. It is easy to see that every regular tournament contains a Hamilton cycle. Thomassen [27] conjectured that even if we remove a number of edges from a regular tournament G, the remaining oriented graph still contains a Hamilton cycle.

Conjecture 6 (Thomassen [27]). *If G is a regular tournament on n vertices and A is any set of less than $(n-1)/2$ edges of G, then $G - A$ contains a Hamilton cycle.*

In Section 7 we prove Conjecture 6 for sufficiently large regular tournaments. Note that Conjecture 6 is a weakening of the following conjecture of Kelly (see e.g. [4, 6, 19]).

Conjecture 7 (Kelly). *Every regular tournament on n vertices can be decomposed into $(n-1)/2$ edge-disjoint Hamilton cycles.*

In [18] we showed that every sufficiently large regular tournament can be 'almost' decomposed into edge-disjoint Hamilton cycles, thus giving an approximate solution to Kelly's conjecture.

This paper is organized as follows. We first give a more detailed discussion of extremal examples for Conjecture 1. After introducing some basic notation, in Section 3 we then deduce Corollary 5 from Theorem 2 and show that one cannot replace pancyclicity by vertex-pancyclicity. Our proof of Theorem 2 uses the Regularity lemma for digraphs which, along with other tools, is introduced in Section 5. The proof of Theorem 2 is included in Section 6. It relies on a result (Lemma 12) from joint work [12] of the first two authors with Keevash on an analogue of Dirac's theorem for oriented graphs. A related result was proved earlier in [14].

It is a natural question to ask whether the 'error terms' in Theorem 2 and Corollary 4 can be eliminated using an 'extremal case' or 'stability' analysis. However, this seems quite difficult as there are many different types of digraphs which come close to violating the conditions in Conjectures 1 and 3 (this is different e.g. to the situation in [12]). As a step in this direction, very recently it was shown in [7] that the degrees in the first parts of the conditions in Theorem 2 can be capped at $n/2$, i.e. the conditions can be replaced by

- $d_i^+ \geq \min\{i + \eta n, n/2\}$ or $d_{n-i-\eta n}^- \geq n - i$,
- $d_i^- \geq \min\{i + \eta n, n/2\}$ or $d_{n-i-\eta n}^+ \geq n - i$.

The proof of this result is considerably more difficult than that of Theorem 2. A (parallel) algorithmic version of Chvátal's theorem for undirected graphs was recently considered in [24] and for directed graphs in [8].

2. Extremal examples for Conjecture 1 and a weaker conjecture

The example given in the introduction does not quite imply that Conjecture 1 would be best possible, as for some k it violates both (i) and (ii) for $i = k$. Here is a slightly more complicated example which only violates one of the conditions for $i = k$ (unless n is odd and $k = \lfloor n/2 \rfloor$).

Suppose $n \geq 5$ and $1 \leq k < n/2$. Let K and K' be complete digraphs on $k - 1$ and $n - k - 2$ vertices respectively. Let G be the digraph on n vertices obtained from the disjoint union of K and K' as follows. Add all possible edges from K' to K (but no edges from K to K') and add new vertices u and v to the digraph such that there are all possible edges from K' to u and v and all possible edges from u and v to K. Finally, add a vertex w that sends and receives edges from all other vertices of G (see Figure 1). Thus G is strongly connected, not Hamiltonian and has outdegree sequence

$$\underbrace{k-1, \ldots, k-1}_{k-1 \text{ times}}, k, k, \underbrace{n-1, \ldots, n-1}_{n-k-1 \text{ times}}$$
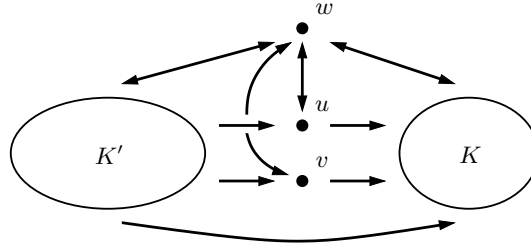
and indegree sequence

FIGURE 1. An extremal example for Conjecture 1

$$\underbrace{n-k-2,\ldots,n-k-2}_{n-k-2\text{ times}},n-k-1,n-k-1,\underbrace{n-1,\ldots,n-1}_{k\text{ times}}.$$

Suppose that either n is even or, if n is odd, we have that $k < \lfloor n/2 \rfloor$. One can check that G then satisfies the conditions in Conjecture 1 except that $d_k^+ = k$ and $d_{n-k}^- = n - k - 1$. (When checking the conditions, it is convenient to note that our assumptions on k and n imply $n - k - 1 \geq \lceil n/2 \rceil$. Hence there are at least $\lceil n/2 \rceil$ vertices of outdegree $n-1$ and so (ii) holds for all $i < n/2$.) If n is odd and $k = \lfloor n/2 \rfloor$ then conditions (i) and (ii) both fail for $i = k$. We do not know whether a similar construction as above also exists for this case. It would also be interesting to find an analogous construction as above for Conjecture 3.

Here is also an example which shows that the assumption of strong connectivity in Conjecture 1 cannot be omitted. Let $n \geq 4$ be even. Let K and K' be two disjoint copies of a complete digraph on $n/2$ vertices. Obtain a digraph G from K and K' by adding all possible edges from K to K' (but none from K' to K). It is easy to see that G is neither Hamiltonian, nor strongly connected, but satisfies the condition on the degree sequences given in Conjecture 1.

As it stands, the additional connectivity assumption means that Conjecture 1 does not seem to be a precise digraph analogue of Chvátal's theorem: in such an analogue, we would ask for a complete characterization of *all* digraph degree sequences which force Hamiltonicity. However, it turns out that it makes sense to replace the strong connectivity assumption with an additional degree condition (condition (iii) below). If true, the following conjecture would provide the desired characterization.

Conjecture 8. *Suppose that G is a digraph on $n \geq 3$ vertices such that for all $i < n/2$*

    (i) $d_i^+ \geq i + 1$ *or* $d_{n-i}^- \geq n - i$,
    (ii) $d_i^- \geq i + 1$ *or* $d_{n-i}^+ \geq n - i$,

and such that (iii) $d_{n/2}^+ \geq n/2$ *or* $d_{n/2}^- \geq n/2$ *if n is even. Then G contains a Hamilton cycle.*

Conjecture 8 would actually follow from Conjecture 1. To see this, it of course suffices to check that the conditions in Conjecture 8 imply strong connectivity. This in turn is easy to verify, as the degree conditions imply that for any vertex set S

with $|S| \leq n/2$ we have $|N^-(S) \cup S| > |S|$ and $|N^+(S) \cup S| > |S|$. (We need (iii) to obtain this assertion precisely for those S with $|S| = n/2$.)

It remains to check that Conjecture 8 would indeed characterize all digraph degree sequences which force a Hamilton cycle. Unless n is odd and $k = \lfloor n/2 \rfloor$, the construction at the beginning of the section already gives non-Hamiltonian graphs which satisfy all the degree conditions (including (iii)) except (i) for $i = k$. To cover the case when n is odd and $k = \lfloor n/2 \rfloor$, let G be the digraph obtained from two disjoint cliques K and K' of orders $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ by adding all edges from K to K'. If $i = k = \lfloor n/2 \rfloor$ then G satisfies (ii) (because $d^+_{n-k} = n - 1$) but not (i). For all other i, both conditions are satisfied. Finally, the example immediately preceding Conjecture 8 gives a graph on an even number n of vertices which satisfies (i) and (ii) for all $i < n/2$ but does not satisfy (iii).

Nash-Williams observed that Conjecture 1 would imply Chvátal's theorem. (Indeed, given an undirected graph G satisfying the degree condition in Chvátal's theorem, obtain a digraph by replacing each undirected edge with a pair of directed edges, one in each direction. This satisfies the degree condition in Conjecture 1. It is also strongly connected, as it is easy to see that G must be connected.) A disadvantage of Conjecture 8 is that it would not imply Chvátal's theorem in the same way: consider a graph G which is obtained from $K_{n/2,n/2}$ by removing a perfect matching and adding a spanning cycle in one of the two vertex classes. The degree sequence of this G satisfies the conditions of Chvátal's theorem. However, the digraph obtained by doubling the edges of G does not satisfy (iii) in Conjecture 8.

3. NOTATION AND THE PROOF OF COROLLARY 5

We begin this section with some notation. Given two vertices x and y of a digraph G, we write xy for the edge directed from x to y. The order $|G|$ of G is the number of its vertices. We denote by $N^+_G(x)$ and $N^-_G(x)$ the out- and the in-neighbourhood of x and by $d^+_G(x)$ and $d^-_G(x)$ its out- and indegree. We will write $N^+(x)$ for example, if this is unambiguous. Given $S \subseteq V(G)$, we write $N^+_G(S)$ for the union of $N^+_G(x)$ for all $x \in S$ and define $N^-_G(S)$ analogously. The *minimum semi-degree* $\delta^0(G)$ of G is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$.

Proof of Corollary 5. Our first aim is to prove the existence of a vertex $x \in V(G)$ such that $d^+(x) + d^-(x) \geq n$. Such a vertex exists if there is an index j with $d^+_j + d^-_{n-j} \geq n$. Indeed, at least $n - j + 1$ vertices of G have outdegree at least d^+_j and at least $j + 1$ vertices have indegree at least d^-_{n-j}. Thus there will be a vertex x with $d^+(x) \geq d^+_j$ and $d^-(x) \geq d^-_{n-j}$.

To prove the existence of such an index j, suppose first that there is an i with $2 \leq i < n/2$ and such that $d^+_{i-1} \geq i$ but $d^+_i = i$. Then $d^-_{n-i} \geq n - i$ and so $d^+_i + d^-_{n-i} \geq n$ as required. The same argument works if there is an i with $2 \leq i < n/2$ and such that $d^-_{i-1} \geq i$ but $d^-_i = i$. Suppose next that $d^+_1 \leq 1$. Then $d^-_{n-1} \geq n - 1$ and so $d^+_1 = 1$. Thus we can take $j := 1$. Again, the same argument works if $d^-_1 \leq 1$.

Thus we may assume that $d^+_{\lceil n/2 \rceil - 1}, d^-_{\lceil n/2 \rceil - 1} \geq \lceil n/2 \rceil$. But in this case we can take $j := \lfloor n/2 \rfloor$.

Now let x be a vertex with $d^+(x) + d^-(x) \geq n$, set $G' := G - x$ and $n' := |G'|$. Let $d^+_{1,G'}, \ldots, d^+_{n',G'}$ and $d^-_{1,G'}, \ldots, d^-_{n',G'}$ denote the out- and the indegree sequences of G'. Given some $i \leq n'$ and $s > 0$, if $d^+_i \geq s$ then at least $n + 1 - i$ vertices in G have outdegree at least s. Thus at least $n - i = n' + 1 - i$ vertices in G' have outdegree at least $s - 1$ and so $d^+_{i,G'} \geq s - 1$. Thus for all $i < n/2$ the degree sequences of G' satisfy

- $d^+_{i,G'} \geq i + \eta n - 1$ or $d^-_{n-i-\eta n, G'} \geq n - i - 1$,
- $d^-_{i,G'} \geq i + \eta n - 1$ or $d^+_{n-i-\eta n, G'} \geq n - i - 1$

and so

- $d^+_{i,G'} \geq i + \eta n'/2$ or $d^-_{n'-i-\eta n'/2, G'} \geq n' - i$,
- $d^-_{i,G'} \geq i + \eta n'/2$ or $d^+_{n'-i-\eta n'/2, G'} \geq n' - i$.

Hence we can apply Theorem 2 with η replaced by $\eta/2$ to obtain a Hamilton cycle $C = x_1 \ldots x_{n'}$ in G'. We now apply the same trick as in [1] to obtain a cycle (through x) in G of the desired length, t say (where $2 \leq t \leq n$): Since $d^+_G(x) + d^-_G(x) \geq n > n'$ there exists an i such that $x_i \in N^+_G(x)$ and $x_{i+t-2} \in N^-_G(x)$ (where we take the indices modulo n'). But then $x x_i \ldots x_{i+t-2} x$ is the required cycle of length t. $\qquad \square$

Note that the proof of Corollary 5 shows that if Conjecture 1 holds and G is a strongly 2-connected digraph with

- $d^+_i \geq i + 2$ or $d^-_{n-i-1} \geq n - i$,
- $d^-_i \geq i + 2$ or $d^+_{n-i-1} \geq n - i$

for all $i < n/2$ then G is pancyclic.

The next result implies that we cannot replace pancyclicity with vertex-pancyclicity in Corollary 5.

Proposition 9. *Given any $k \geq 3$ there are $\eta = \eta(k) > 0$ and $n_0 = n_0(k)$ such that for every $n \geq n_0$ there exists a digraph G on n vertices with $d^+_i, d^-_i \geq i + \eta n$ for all $i < n/2$, but such that some vertex of G does not lie on a cycle of length less than k.*

Proof. Let $\eta := 1/(k3^k)$ and suppose that n is sufficiently large. Let G be the digraph obtained from the disjoint union of $k - 2$ independent sets $V_1, \ldots, V_{k-2}$ with $|V_i| = 3^i \lceil \eta n \rceil$ and a complete digraph K on $n - 1 - |V_1 \cup \cdots \cup V_{k-2}|$ vertices as follows. Add a new vertex x which sends an edge to all vertices in V_1 and receives an edge from all vertices in K. Add all possible edges from V_i to V_{i+1} (but no edges from V_{i+1} to V_i) for each $i \leq k - 3$. Finally, add all possible edges going from vertices in K to other vertices and add all edges from V_{k-2} to K. Then $d^-_i \geq |K| \geq 2n/3$ and $d^+_i \geq i + \eta n$ for all $i < n/2$ with room to spare. However, if C is a cycle containing x then the inneighbour of x on C must lie in K. But the shortest path from x to K has length $k - 1$ and so $|C| \geq k$, as required. $\qquad \square$

4. DEGREE SEQUENCES FOR HAMILTON CYCLES IN ORIENTED GRAPHS

In Section 1 we mentioned Ghouila-Houri's theorem which gives a bound on the minimum semi-degree of a digraph G guaranteeing a Hamilton cycle. A natural question raised by Thomassen [26] is that of determining the minimum semi-degree which ensures a Hamilton cycle in an oriented graph. Häggkvist [11] conjectured that every oriented graph G of order $n \geq 3$ with $\delta^0(G) \geq (3n-4)/8$ contains a Hamilton cycle. The bound on the minimum semi-degree would be best possible. The first two authors together with Keevash [12] confirmed this conjecture for sufficiently large oriented graphs.

Pósa's theorem implies the existence of a Hamilton cycle in a graph G even if G contains a significant number of vertices of degree much less than $n/2$, i.e. of degree much less than the minimum degree required to force a Hamilton cycle. In particular, Pósa's theorem is much stronger than Dirac's theorem. In the same sense, Conjecture 3 would be much stronger than Ghouila-Houri's theorem. The following proposition implies that we cannot strengthen Häggkvist's conjecture in this way: there are non-Hamiltonian oriented graphs which contain just a bounded number of vertices whose semi-degree is (only slightly) smaller than $3n/8$. To state this proposition we need to introduce the notion of dominating sequences: Given sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of numbers we say that $y_1, \ldots, y_n$ *dominates* $x_1, \ldots, x_n$ if $x_i \leq y_i$ for all $1 \leq i \leq n$.

Proposition 10. *For every $0 < \alpha < 3/8$, there is an integer $c = c(\alpha)$ and infinitely many oriented graphs G whose in- and outdegree sequences both dominate*

$$\underbrace{\alpha|G|, \ldots, \alpha|G|}_{c \text{ times}}, 3|G|/8, \ldots, 3|G|/8$$

but such that G does not contain a Hamilton cycle.

Proof. Define $c := 4t$ where $t \in \mathbb{N}$ is chosen such that $3 - 1/t > 8\alpha$. Let n be sufficiently large and such that $8t$ divides n and define vertex sets A, B, C, D and E of sizes $n/4, n/8, n/8 - 1, n/4 + 1$ and $n/4$ respectively.

Let G be the oriented graph obtained from the disjoint union of A, B, C, D and E by defining the following edges: G contains all possible edges from A to B, B to C, C to D, A to C, B to D and D to A. E sends out all possible edges to A and B and receives all possible edges from C and D. B and C both induce tournaments that are as regular as possible (see Figure 2). So certainly $d_G^+(x), d_G^-(x) \geq 3n/8$ for all $x \in B \cup C \cup E$. Furthermore, currently, $d_G^+(a) = n/4 - 1$, $d_G^-(a) = n/2 + 1$, $d_G^+(d) = n/2$ and $d_G^-(d) = n/4 - 1$ for all $a \in A$ and all $d \in D$.

Partition A into A' and A'' where $|A''| = c$ and thus $|A'| = n/4 - c$. Write $A' =: \{x_1, x_2, \ldots, x_{n/8-c/2}, y_1, y_2, \ldots, y_{n/8-c/2}\}$ and $A'' =: \{z_1, \ldots, z_{2t}, w_1, \ldots, w_{2t}\}$. Let A' induce a tournament that is as regular as possible. In particular, every vertex in A' sends out at least $n/8 - c/2 - 1$ edges to other vertices in A'. We define the edges between A' and A'' as follows: Add the edges $x_i z_j, y_i w_j$ to G for all $1 \leq i \leq n/8 - c/2$ and $1 \leq j \leq 2t$. Note that we can partition both $\{x_1, \ldots, x_{n/8-c/2}\}$ and $\{y_1, \ldots, y_{n/8-c/2}\}$ into t sets of size $s := n/(2c) - 2$. For each
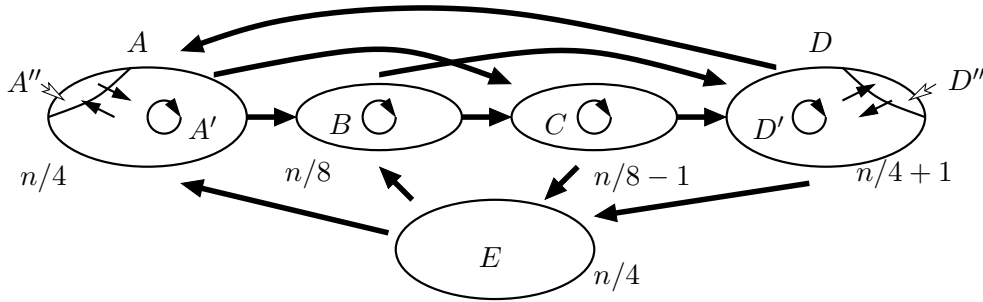
FIGURE 2. The oriented graph G in Proposition 10

$0 \le i \le t-1$ add all possible edges from $\{x_{si+1}, \ldots, x_{s(i+1)}\}$ to $\{w_{2i+1}, w_{2i+2}\}$ and from $\{y_{si+1}, \ldots, y_{s(i+1)}\}$ to $\{z_{2i+1}, z_{2i+2}\}$. If $a' \in A'$ and $a'' \in A''$ are such that the edge $a'a''$ has not been included into G so far then add the edge $a''a'$ to G. Thus, $d_G^+(a') \ge (n/4 - 1) + (n/8 - c/2 - 1) + c/2 + 2 = 3n/8$ for all $a' \in A'$ and

$$d_G^+(a'') \ge (n/4 - 1) + (n/8 - c/2 - s) = 3n/8 - c/2 - n/(2c) + 1 \ge \alpha n$$

for all $a'' \in A''$.

Partitioning D into D' and D'' (where $|D''| = c$) and defining edges inside D in a similar fashion to those inside A, we can ensure that $d_G^-(d') \ge 3n/8$ for all $d' \in D'$ and $d_G^-(d'') \ge \alpha n$ for all $d'' \in D''$. So indeed G has the desired degree sequences.

E is an independent set, so if G contains a Hamilton cycle H then the inneighbour of each vertex in E on H must lie in $C \cup D$ while its outneighbour lies in $A \cup B$. So H contains at least $|E| = n/4$ disjoint edges going from $A \cup B$ to $C \cup D$. However, all such edges in G have at least one endvertex in $B \cup C$. So there are at most $|B| + |C| = n/4 - 1 < |E|$ such disjoint edges in G. Thus G does not contain a Hamilton cycle (in fact, G does not contain a 1-factor). □

5. The Diregularity lemma and other tools

In the proof of Theorem 2 we will use the directed version of Szemerédi's Regularity lemma. Before we can state it we need some more definitions. The *density* of an undirected bipartite graph $G = (A, B)$ with vertex classes A and B is defined to be

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

We will write $d(A, B)$ if this is unambiguous. Given any $\varepsilon > 0$ we say that G is *ε-regular* if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have that $|d(X, Y) - d(A, B)| < \varepsilon$.

Given disjoint vertex sets A and B in a digraph G, we write $(A, B)_G$ for the oriented bipartite subgraph of G whose vertex classes are A and B and whose edges are all the edges from A to B in G. We say $(A, B)_G$ is *ε-regular and has density d* if the underlying bipartite graph of $(A, B)_G$ is ε-regular and has density d. (Note that the ordering of the pair (A, B) is important here.)

The Diregularity lemma is a variant of the Regularity lemma for digraphs due to Alon and Shapira [2]. Its proof is similar to the undirected version. We will use the degree form of the Diregularity lemma which is derived from the standard version in the same manner as the undirected degree form (see e.g. the survey [17] for a sketch of the undirected version).

Lemma 11 (Degree form of the Diregularity lemma). *For every $\varepsilon \in (0,1)$ and every integer M' there are integers M and n_0 such that if G is a digraph on $n \geq n_0$ vertices and $d \in [0,1]$ is any real number, then there is a partition of the vertex set of G into $V_0, V_1, \ldots, V_k$ and a spanning subdigraph G' of G such that the following holds:*

- *$M' \leq k \leq M$,*
- *$|V_0| \leq \varepsilon n$,*
- *$|V_1| = \cdots = |V_k| =: m$,*
- *$d_{G'}^+(x) > d_G^+(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$,*
- *$d_{G'}^-(x) > d_G^-(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$,*
- *for all $i = 1, \ldots, k$ the digraph $G'[V_i]$ is empty,*
- *for all $1 \leq i, j \leq k$ with $i \neq j$ the pair $(V_i, V_j)_{G'}$ is ε-regular and has density either 0 or density at least d.*

We call $V_1, \ldots, V_k$ *clusters*, V_0 the *exceptional set* and the vertices in V_0 *exceptional vertices*. We refer to G' as the *pure digraph*. The last condition of the lemma says that all pairs of clusters are ε-regular in both directions (but possibly with different densities). The *reduced digraph R of G with parameters ε, d and M'* is the digraph whose vertices are $V_1, \ldots, V_k$ and in which $V_i V_j$ is an edge precisely when $(V_i, V_j)_{G'}$ is ε-regular and has density at least d.

Given $0 < \nu \leq \tau < 1$, we call a digraph G a *(ν, τ)-outexpander* if $|N^+(S)| \geq |S| + \nu|G|$ for all $S \subseteq V(G)$ with $\tau|G| < |S| < (1-\tau)|G|$. The main tool in the proof of Theorem 2 is the following result from [12].

Lemma 12. *Let M', n_0 be positive integers and let $\varepsilon, d, \eta, \nu, \tau$ be positive constants such that $1/n_0 \ll 1/M' \ll \varepsilon \ll d \ll \nu \leq \tau \ll \eta < 1$. Let G be an oriented graph on $n \geq n_0$ vertices such that $\delta^0(G) \geq 2\eta n$. Let R be the reduced digraph of G with parameters ε, d and M'. Suppose that there exists a spanning oriented subgraph R^* of R with $\delta^0(R^*) \geq \eta|R^*|$ which is a (ν, τ)-outexpander. Then G contains a Hamilton cycle.*

Here we write $0 < a_1 \ll a_2 \ll a_3 \leq 1$ to mean that we can choose the constants a_1, a_2, a_3 from right to left. More precisely, there are increasing functions f and g such that, given a_3, whenever we choose some $a_2 \leq f(a_3)$ and $a_1 \leq g(a_2)$, all calculations needed in the proof of Lemma 12 are valid.

Our next aim is to show that any digraph G as in Theorem 2 is an outexpander. In fact, we will show that even the 'robust outneighbourhood' of any set $S \subseteq V(G)$ of reasonable size is significantly larger than S. More precisely, let $0 < \nu \leq \tau < 1$. Given any digraph G and $S \subseteq V(G)$, the *ν-robust outneighbourhood $RN_{\nu,G}^+(S)$ of S* is the set of all those vertices x of G which have at least $\nu|G|$ inneighbours in S. G is called a *robust (ν, τ)-outexpander* if $|RN_{\nu,G}^+(S)| \geq |S| + \nu|G|$ for all $S \subseteq V(G)$ with $\tau|G| < |S| < (1-\tau)|G|$.

Lemma 13. *Let n_0 be a positive integer and τ, η be positive constants such that $1/n_0 \ll \tau \ll \eta < 1$. Let G be a digraph on $n \geq n_0$ vertices with*

    (i) *$d_i^+ \geq i + \eta n$ or $d_{n-i-\eta n}^- \geq n - i$,*

    (ii) *$d_i^- \geq i + \eta n$ or $d_{n-i-\eta n}^+ \geq n - i$*

for all $i < n/2$. Then $\delta^0(G) \geq \eta n$ and G is a robust (τ^2, τ)-outexpander.

Proof. Clearly, if $d_1^+ \geq 1 + \eta n$ then $\delta^+(G) \geq \eta n$. If $d_1^+ < 1 + \eta n$ then (i) implies that $d_{n-1-\eta n}^- \geq n - 1$. Thus G has at least $\eta n + 1$ vertices of indegree $n - 1$ and so $\delta^+(G) \geq \eta n$. It follows similarly that $\delta^-(G) \geq \eta n$.

Consider any non-empty set $S \subseteq V(G)$ with $\tau n < |S| < (1 - \tau)n$ and $|S| \neq n/2 + \lfloor \tau n \rfloor$. Let us first deal with the case when $d_{|S|-\lfloor \tau n \rfloor}^+ \geq |S| - \lfloor \tau n \rfloor + \eta n \geq |S| + \eta n/2$. Then S contains a set X of $\lfloor \tau n \rfloor$ vertices, each having outdegree at least $|S| + \eta n/2$. Let Y be the set of all those vertices of G that have at least $\tau^2 n$ inneighbours in X. Then

$$|X|(|S| + \eta n/2) \leq |Y||X| + (n - |Y|)\tau^2 n \leq |Y||X| + \tau^2 n^2$$

and so $|RN_{\tau^2, G}^+(S)| \geq |Y| \geq |S| + 2\tau^2 n$.

So suppose next that $d_{|S|-\lfloor \tau n \rfloor}^+ < |S| - \lfloor \tau n \rfloor + \eta n$. Since $\delta^-(G) \geq \eta n$ we may assume that $|S| \leq (1 - \eta + \tau^2)n < n - 1 - \eta n + \lfloor \tau n \rfloor$ (otherwise $RN_{\tau^2, G}^+(S) = V(G)$ and we are done). Thus

$$d_{n-|S|+\lfloor \tau n \rfloor-\eta n}^- \geq n - |S| + \lfloor \tau n \rfloor \geq n - |S| + \tau^2 n$$

by (i) and (ii). (Here we use that $|S| \neq n/2 + \lfloor \tau n \rfloor$.)

So G contains at least $|S| - \lfloor \tau n \rfloor + \eta n \geq |S| + \eta n/2$ vertices x of indegree at least $n - |S| + \tau^2 n$. If $|RN_{\tau^2, G}^+(S)| < |S| + 2\tau^2 n$ then $V(G) \setminus RN_{\tau^2, G}^+(S)$ contains such a vertex x. But then x has at least $\tau^2 n$ neighbours in S, i.e. $x \in RN_{\tau^2, G}^+(S)$, a contradiction.

If $|S| = n/2 + \lfloor \tau n \rfloor$ then considering the outneighbourhood of a subset of S of size $|S| - 1$ shows that $|RN_{\tau^2, G}^+(S)| \geq |S| - 1 + 2\tau^2 n \geq |S| + \tau^2 n$. $\qquad\square$

The next result implies that the property of a digraph G being a robust outexpander is 'inherited' by the reduced digraph of G. For this (and for Lemma 15) we need that G is a robust outexpander, rather than just an outexpander.

Lemma 14. *Let M', n_0 be positive integers and let $\varepsilon, d, \eta, \nu, \tau$ be positive constants such that $1/n_0 \ll \varepsilon \ll d \ll \nu, \tau, \eta < 1$ and such that $M' \ll n_0$. Let G be a digraph on $n \geq n_0$ vertices with $\delta^0(G) \geq \eta n$ and such that G is a robust (ν, τ)-outexpander. Let R be the reduced digraph of G with parameters ε, d and M'. Then $\delta^0(R) \geq \eta|R|/2$ and R is a robust $(\nu/2, 2\tau)$-outexpander.*

Proof. Let G' denote the pure digraph, $k := |R|$, let $V_1, \ldots, V_k$ be the clusters of G (i.e. the vertices of R) and V_0 the exceptional set. Let $m := |V_1| = \cdots = |V_k|$. Then

$$\delta^0(R) \geq (\delta^0(G') - |V_0|)/m \geq (\delta^0(G) - (d + 2\varepsilon)n)/m \geq \eta k/2.$$

Consider any $S \subseteq V(R)$ with $2\tau k \leq |S| \leq (1 - 2\tau)k$. Let S' be the union of all the clusters belonging to S. Then $\tau n \leq |S'| \leq (1 - 2\tau)n$. Since $|N_{G'}^-(x) \cap S'| \geq |N_G^-(x) \cap S'| - (d + \varepsilon)n \geq \nu n/2$ for every $x \in RN_{\nu,G}^+(S')$ this implies that

$$|RN_{\nu/2,G'}^+(S')| \geq |RN_{\nu,G}^+(S')| \geq |S'| + \nu n \geq |S|m + \nu mk.$$

However, in G' every vertex $x \in RN_{\nu/2,G'}^+(S') \setminus V_0$ receives edges from vertices in at least $|N_{G'}^-(x) \cap S'|/m \geq (\nu n/2)/m \geq \nu k/2$ clusters $V_i \in S$. Thus by the final property of the partition in Lemma 11 the cluster V_j containing x is an outneighbour of each such V_i (in R). Hence $V_j \in RN_{\nu/2,R}^+(S)$. This in turn implies that

$$|RN_{\nu/2,R}^+(S)| \geq (|RN_{\nu/2,G'}^+(S')| - |V_0|)/m \geq |S| + \nu k/2,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The strategy of the proof of Theorem 2 is as follows. By Lemma 13 our given digraph G is a robust outexpander and by Lemma 14 this also holds for the reduced digraph R of G. The next result gives us a spanning oriented subgraph R^* of R which is still an outexpander. The somewhat technical property concerning the subdigraph $H \subseteq R$ in Lemma 15 will be used to guarantee an oriented subgraph G^* of G which has linear minimum semidegree and such that R^* is a reduced digraph of G^*. (G^* will be obtained from the spanning subgraph of the pure digraph G' which corresponds to R^* by modifying the neighbourhoods of a small number of vertices.) Finally, we will apply Lemma 12 with R^* playing the role of both R and R^* and G^* playing the role of G to find a Hamilton cycle in G^* and thus in G.

Lemma 15. *Given positive constants $\nu \leq \tau \leq \eta$, there exists a positive integer n_0 such that the following holds. Let R be a digraph on $n \geq n_0$ vertices which is a robust (ν, τ)-outexpander. Let H be a spanning subdigraph of R with $\delta^0(H) \geq \eta n$. Then R has a spanning oriented subgraph R^* which is a robust $(\nu/12, \tau)$-outexpander and such that $\delta^0(R^* \cap H) \geq \eta n/4$.*

Proof. Consider a random spanning oriented subgraph R^* of R obtained by deleting one of the edges xy, yx (each with probability $1/2$) for every pair $x, y \in V(R)$ for which $xy, yx \in E(R)$, independently from all other such pairs. Given a vertex x of R, we write $N_R^\pm(x)$ for the set of all those vertices of R which are both out- and inneighbours of x and define $N_H^\pm(x)$ similarly. Let $H^* := H \cap R^*$. Clearly, $d_{H^*}^+(x), d_{H^*}^-(x) \geq \eta n/4$ if $|N_H^\pm(x)| \leq 3\eta n/4$. So suppose that $|N_H^\pm(x)| \geq 3\eta n/4$. Let $X := |N_H^\pm(x) \cap N_{H^*}^+(x)|$. Then $\mathbb{E}X \geq 3\eta n/8$ and so a standard Chernoff estimate (see e.g. [3, Cor. A.14]) implies that

$$\mathbb{P}(d_{H^*}^+(x) < \eta n/4) \leq \mathbb{P}(X < \eta n/4) \leq \mathbb{P}(X < 2\mathbb{E}X/3) < 2\mathrm{e}^{-c\mathbb{E}X} \leq 2\mathrm{e}^{-3c\eta n/8},$$

where c is an absolute constant (i.e. it does not depend on ν, τ or η). Similarly it follows that $\mathbb{P}(d_{H^*}^-(x) < \eta n/4) \leq 2\mathrm{e}^{-3c\eta n/8}$.

Consider any set $S \subseteq V(R^*) = V(R)$. Let $ERN_{\nu/3,R}^+(S) := RN_{\nu/3,R}^+(S) \setminus S$ and define $ERN_{\nu/12,R^*}^+(S)$ similarly. We say that S is *good* if all but at most $\nu n/6$ vertices

in $ERN^+_{\nu/3,R}(S)$ are contained in $ERN^+_{\nu/12,R^*}(S)$. Our next aim is to show that

$$(1) \qquad\qquad \mathbb{P}(S \text{ is not good}) \leq \mathrm{e}^{-n}.$$

To prove (1), write $ERN^{\pm}_R(S)$ for the set of all those vertices $x \in ERN^+_{\nu/3,R}(S)$ for which $|N^{\pm}_R(x) \cap S| \geq \nu n/4$. Note that every vertex in $ERN^+_{\nu/3,R}(S) \setminus ERN^{\pm}_R(S)$ will automatically lie in $ERN^+_{\nu/12,R^*}(S)$. We say that a vertex $x \in ERN^{\pm}_R(S)$ *fails* if $x \notin ERN^+_{\nu/12,R^*}(S)$. The expected size of $N^-_{R^*}(x) \cap N^{\pm}_R(x) \cap S$ is at least $\nu n/8$. So as before, a Chernoff estimate gives

$$\mathbb{P}(x \text{ fails}) \leq \mathbb{P}(|N^-_{R^*}(x) \cap N^{\pm}_R(x) \cap S| < \nu n/12) \leq 2\mathrm{e}^{-c\nu n/8} =: p.$$

Let Y be the number of all those vertices $x \in ERN^{\pm}_R(S)$ which fail. Then $\mathbb{E}Y \leq p|ERN^{\pm}_R(S)| \leq pn$. Note that the failure of distinct vertices is independent (which is the reason we are only considering vertices in the external neighbourhood of S). So we can apply the following Chernoff estimate (see e.g. [3, Theorem A.12]): If $C \geq \mathrm{e}^2$ we have

$$\mathbb{P}(Y \geq C\mathbb{E}Y) \leq \mathrm{e}^{(C-C\ln C)\mathbb{E}Y} \leq \mathrm{e}^{-C(\ln C)\mathbb{E}Y/2}.$$

Setting $C := \nu n/(6\mathbb{E}Y) \geq \nu/(6p)$ this gives

$$\mathbb{P}(S \text{ is not good}) = \mathbb{P}(Y > \nu n/6) = \mathbb{P}(Y > C\mathbb{E}Y) \leq \mathrm{e}^{-C(\ln C)\mathbb{E}Y/2} = \mathrm{e}^{-\nu n(\ln C)/12}$$
$$\leq \mathrm{e}^{-n}.$$

(The last inequality follows since $p \ll \nu$ if n is sufficiently large.) This completes the proof of (1).

Since $4n\mathrm{e}^{-3c\eta n/8} + 2^n\mathrm{e}^{-n} < 1$ (if n is sufficiently large) this implies that there is an outcome for R^* such that $\delta^0(R^* \cap H) \geq \eta n/4$ and such that every set $S \subseteq V(R)$ is good. We will now show that the latter property implies that such an R^* is a robust $(\nu/12, \tau)$-outexpander. So consider any set $S \subseteq V(R)$ with $\tau n < |S| < (1-\tau)n$. Let $EN := ERN^+_{\nu,R}(S)$ and $N := RN^+_{\nu,R}(S) \cap S$. So $EN \cup N = RN^+_{\nu,R}(S)$. Since S is good and $EN \subseteq ERN^+_{\nu/3,R}(S)$ all but at most $\nu n/6$ vertices in EN are contained in $ERN^+_{\nu/12,R^*}(S) \subseteq RN^+_{\nu/12,R^*}(S)$.

Now consider any partition of S into S_1 and S_2 such that every vertex $x \in N$ satisfies $|N^-_R(x) \cap S_i| \geq \nu n/3$ for $i = 1,2$. (The existence of such a partition follows by considering a random partition.) Then $S_1 \cap N \subseteq ERN^+_{\nu/3,R}(S_2)$. But since S_2 is good this implies that all but at most $\nu n/6$ vertices in $S_1 \cap N$ are contained in $ERN^+_{\nu/12,R^*}(S_2) \subseteq RN^+_{\nu/12,R^*}(S)$. Similarly, since S_1 is good, all but at most $\nu n/6$ vertices in $S_2 \cap N$ are contained in $ERN^+_{\nu/12,R^*}(S_1) \subseteq RN^+_{\nu/12,R^*}(S)$. Altogether this shows that

$$|RN^+_{\nu/12,R^*}(S)| \geq |EN \cup (S_1 \cap N) \cup (S_2 \cap N)| - \frac{3\nu n}{6} = |RN^+_{\nu,R}(S)| - \frac{\nu n}{2} \geq |S| + \frac{\nu n}{2},$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

6. Proof of Theorem 2

As indicated in Section 1, instead of proving Theorem 2 directly, we will prove the following stronger result. It immediately implies Theorem 2 since by Lemma 13 any digraph G as in Theorem 2 is a robust outexpander and satisfies $\delta^0(G) \geq \eta n$.

Theorem 16. *Let n_0 be a positive integer and ν, τ, η be positive constants such that $1/n_0 \ll \nu \leq \tau \ll \eta < 1$. Let G be a digraph on $n \geq n_0$ vertices with $\delta^0(G) \geq \eta n$ which is a robust (ν, τ)-outexpander. Then G contains a Hamilton cycle.*

Proof. Pick a positive integer M' and additional constants ε, d such that $1/n_0 \ll 1/M' \ll \varepsilon \ll d \ll \nu$. Apply the Regularity lemma (Lemma 11) with parameters ε, d and M' to G to obtain clusters $V_1, \ldots, V_k$, an exceptional set V_0 and a pure digraph G'. Then $\delta^0(G') \geq (\eta - (d + \varepsilon))n$ by Lemma 11. Let R be the reduced digraph of G with parameters ε, d and M'. Lemma 14 implies that $\delta^0(R) \geq \eta k/2$ and that R is a robust $(\nu/2, 2\tau)$-outexpander.

Let H be the spanning subdigraph of R in which $V_i V_j$ is an edge if $V_i V_j \in E(R)$ and the density $d_{G'}(V_i, V_j)$ of the oriented subgraph $(V_i, V_j)_{G'}$ of G' is at least $\eta/4$. We will now give a lower bound on $\delta^+(H)$. So consider any cluster V_i and let $m := |V_i|$. Writing $e_{G'}(V_i, V(G) \setminus V_0)$ for the number of all edges from V_i to $V(G) \setminus V_0$ in G', we have

$$\sum_{V_j \in N_R^+(V_i)} d_{G'}(V_i, V_j)m^2 = e_{G'}(V_i, V(G) \setminus V_0) \geq \delta^0(G')m - |V_0|m \geq (\eta - 2d)nm.$$

It is easy to see that this implies that there are at least $\eta k/4$ outneighbours V_j of V_i in R such that $d_{G'}(V_i, V_j) \geq \eta/4$. But each such V_j is an outneighbour of V_i in H and so $\delta^+(H) \geq \eta k/4$. It follows similarly that $\delta^-(H) \geq \eta k/4$. We now apply Lemma 15 to find a spanning oriented subgraph R^* of R which is a (robust) $(\nu/24, 2\tau)$-outexpander and such that $\delta^0(R^* \cap H) \geq \eta k/16$. Let $H^* := H \cap R^*$.

Our next aim is to modify the pure digraph G' into a spanning oriented subgraph of G having minimum semi-degree at least $\eta^2 n/100$. Let G^* be the spanning subgraph of G' which corresponds to R^*. So G^* is obtained from G' by deleting all those edges xy that join some cluster V_i to some cluster V_j with $V_i V_j \in E(R) \setminus E(R^*)$. Note that $G^* - V_0$ is an oriented graph. However, some vertices of $G^* - V_0$ may have small degrees. We will show that there are only a few such vertices and we will add them to V_0 in order to achieve that the out- and indegrees of all the vertices outside V_0 are large. So consider any cluster V_i. For any cluster $V_j \in N_{H^*}^+(V_i)$ at most εm vertices in V_i have less than $(d_{G'}(V_i, V_j) - \varepsilon)m \geq \eta m/5$ outneighbours in V_j (in the digraph G'). Call all these vertices of V_i *useless for V_j*. Thus on average any vertex of V_i is useless for at most $\varepsilon|N_{H^*}^+(V_i)|$ clusters $V_j \in N_{H^*}^+(V_i)$. This implies that at most $\sqrt{\varepsilon}m$ vertices in V_i are useless for more than $\sqrt{\varepsilon}|N_{H^*}^+(V_i)|$ clusters $V_j \in N_{H^*}^+(V_i)$. Let $U_i^+ \subseteq V_i$ be a set of size $\sqrt{\varepsilon}m$ which consists of all these vertices and some extra vertices from V_i if necessary. Similarly, we can choose a set $U_i^- \subseteq V_i \setminus U_i^+$ of size $\sqrt{\varepsilon}m$ such that for every vertex $x \in V_i \setminus U_i^-$ there are at most $\sqrt{\varepsilon}|N_{H^*}^-(V_i)|$ clusters $V_j \in N_{H^*}^-(V_i)$ such that x has less than $\eta m/5$ inneighbours in V_j. For each $i = 1, \ldots, k$ remove all the vertices in $U_i^+ \cup U_i^-$ and

add them to V_0. We still denote the subclusters obtained in this way by $V_1, \ldots, V_k$ and the exceptional set by V_0. Thus we now have that $|V_0| \leq 3\sqrt{\varepsilon}n$. Moreover,

$$\delta^0(G^* - V_0) \geq \frac{\eta m}{5}(1 - \sqrt{\varepsilon})\delta^0(H^*) - |V_0| \geq \frac{\eta m}{5}\frac{\eta k}{17} - 3\sqrt{\varepsilon}n \geq \frac{\eta^2 n}{100}.$$

We now modify G^* by altering the neighbours of the exceptional vertices: For every $x \in V_0$ we select a set of $\eta n/2$ outneighbours of x in G and a set of $\eta n/2$ inneighbours such that these two sets are disjoint and add the edges between x and the selected neighbours to G^*. We still denote the oriented graph thus obtained from G^* by G^*. Then $\delta^0(G^*) \geq \eta^2 n/100$. Since the partition $V_0, V_1, \ldots, V_k$ of $V(G^*)$ is as described in the Regularity lemma (Lemma 11) with parameters $3\sqrt{\varepsilon}$, $d - \varepsilon$ and M' (where G^* plays the role of G' and G) we can say that R^* is a reduced digraph of G^* with these parameters. Thus we may apply Lemma 12 with R^* playing the role of both R and R^* and G^* playing the role of G to find a Hamilton cycle in G^* and thus in G. $\qquad\square$

7. Hamilton cycles in regular tournaments

In this section we prove Conjecture 6 for sufficiently large regular tournaments. The following observation of Keevash and Sudakov [13] will be useful for this.

Proposition 17. *Let $0 < c < 10^{-4}$ and let G be an oriented graph on n vertices such that $\delta^0(G) \geq (1/2 - c)n$. Then for any (not necessarily disjoint) $S, T \subseteq V(G)$ of size at least $(1/2 - c)n$ there are at least $n^2/60$ directed edges from S to T.*

We now show that Theorem 16 implies Conjecture 6 for sufficiently large regular tournaments.

Corollary 18. *There exists an integer n_0 such that the following holds. Given any regular tournament G on $n \geq n_0$ vertices and a set A of less than $(n-1)/2$ edges of G, then $G - A$ contains a Hamilton cycle.*

Proof. Let $0 < \nu \ll \tau \ll \eta \ll 1$. It is not difficult to show that G is a robust (ν, τ)-outexpander. Indeed, if $S \subseteq V(G)$ and $(1/2 + \tau)n < |S| < (1 - \tau)n$ then $RN_{\nu,G}^+(S) = V(G)$. If $\tau n < |S| < (1/2 - \tau)n$ then it is easy to see that $|RN_{\nu,G}^+(S)| \geq (1 - \tau)n/2 \geq |S| + \nu n$. So consider the case when $(1/2 - \tau)n \leq |S| \leq (1/2 + \tau)n$. Suppose $|RN_{\nu,G}^+(S)| < |S| + \nu n \leq (1/2 + 2\tau)n$. Then by Proposition 17 there are at least $n^2/60$ directed edges from S to $V(G) \backslash RN_{\nu,G}^+(S)$. By definition each vertex $x \in V(G) \backslash RN_{\nu,G}^+(S)$ has less than νn inneighbours in S, a contradiction. So $|RN_{\nu,G}^+(S)| \geq |S| + \nu n$ as desired.

Since $|A| < (n-1)/2$ and n is sufficiently large, $G - A$ must be a robust $(\nu/2, \tau)$-outexpander. Thus if $\delta^0(G-A) \geq \eta n$ then by Theorem 16, $G-A$ contains a Hamilton cycle.

If $\delta^0(G - A) < \eta n$ then there exists precisely one vertex $x \in V(G - A)$ such that either $d_{G-A}^+(x) < \eta n$ or $d_{G-A}^-(x) < \eta n$. Without loss of generality we may assume that $d_{G-A}^+(x) < \eta n$. Note that $d_{G-A}^+(x) \geq 1$ and let $y \in N_{G-A}^+(x)$. Let G' be the digraph obtained from $G - A$ by removing x and y from $G - A$ and adding a new

vertex z so that $N^+_{G'}(z) := N^+_{G-A}(y)$ and $N^-_{G'}(z) := N^-_{G-A}(x)$. So $\delta^0(G') \geq \eta n - 2 \geq \eta n/2$ and G' is a robust $(\nu/3, 2\tau)$-outexpander. Thus by Theorem 16 G' contains a Hamilton cycle which corresponds to one in G. $\qquad\square$

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