Forcing unbalanced complete bipartite minors

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Abstract

Myers conjectured that for every integer s there exists a positive constant C such that for all integers t every graph of average degree at least Ct contains a $K_{s,t}$ minor. We prove the following stronger result: for every $0 < \varepsilon < 10^{-16}$ there exists a number $t_0 = t_0(\varepsilon)$ such that for all integers $t \geq t_0$ and $s \leq \varepsilon^6 t/\log t$ every graph of average degree at least $(1+\varepsilon)t$ contains a $K_{s,t}$ minor. The bounds are essentially best possible. We also show that for fixed s every graph as above even contains $K_s + \overline{K}_t$ as a minor.

1 Introduction

Let $d(s)$ be the smallest number such that every graph of average degree greater than $d(s)$ contains the complete graph K_s as minor. The existence of $d(s)$ was first proved by Mader [4]. Kostochka [3] and Thomason [10] independently showed that the order of magnitude of $d(s)$ is $s\sqrt{\log s}$. Later, Thomason [11] was able to prove that $d(s) = (\alpha + o(1))s\sqrt{\log s}$, where $\alpha = 0.638\ldots$ is an explicit constant. Here the lower bound on $d(s)$ is provided by random graphs. In fact, Myers [6] proved that all extremal graphs are essentially disjoint unions of pseudo-random graphs.

Recently, Myers and Thomason [8] extended the results of [11] from complete minors to H minors for arbitrary dense (and large) graphs H. The extremal function has the same form as $d(s)$, except that $\alpha \leq 0.638\ldots$ is now an explicit parameter depending on H and s is replaced by the order of H. They raised the question of what happens for sparse graphs H. One partial result in this direction was obtained by Myers [7]: he showed that every graph of average degree at least $t+1$ contains a $K_{2,t}$ minor. This is best possible as he observed that for all positive ε there are infinitely many graphs of average degree at least $t+1-\varepsilon$ which do not contain a $K_{2,t}$ minor. (These examples also show that random graphs are not extremal in this case.) More generally, Myers [7] conjectured that for fixed s the extremal function for a $K_{s,t}$ minor is linear in t:

Conjecture 1 (Myers) *Given $s \in \mathbb{N}$, there exists a positive constant C such that for all $t \in \mathbb{N}$ every graph of average degree at least Ct contains a $K_{s,t}$ minor.*

Here we prove the following strengthened version of this conjecture. (It implies that asymptotically the influence of the number of edges on the extremal function is negligible.)

Theorem 2 *For every $0 < \varepsilon < 10^{-16}$ there exists a number $t_0 = t_0(\varepsilon)$ such that for all integers $t \geq t_0$ and $s \leq \varepsilon^6 t/\log t$ every graph of average degree at least $(1+\varepsilon)t$ contains a $K_{s,t}$ minor.*

Theorem 2 is essentially best possible in two ways. Firstly, the complete graph K_{s+t-1} shows that up to the error term εt the bound on the average degree cannot be reduced. Secondly, as we will see in Proposition 9 (applied with $\alpha := 1/3$), the result breaks down if we try to set $s \geq 18t/\log t$. Moreover, Proposition 9 also implies that if $t/\log t = o(s)$ then even a linear average degree (as in Conjecture 1) no longer suffices to force a $K_{s,t}$ minor.

The case where $s = ct$ for some constant $0 < c \leq 1$ is covered by the results of Myers and Thomason [8]. The extremal function in this case is $(\alpha\frac{2\sqrt{c}}{1+c} + o(1))r\sqrt{\log r}$ where $\alpha = 0.638\ldots$ again and $r = s + t$.

For fixed s, we obtain the following strengthening of Theorem 2:

Theorem 3 *For every $\varepsilon > 0$ and every integer s there exists a number $t_0 = t_0(\varepsilon, s)$ such that for all integers $t \geq t_0$ every graph of average degree at least $(1+\varepsilon)t$ contains $K_s + \overline{K}_t$ as a minor.*

This note is organized as follows. We first prove Theorem 2 for graphs whose connectivity is linear in their order (Lemma 8). We then use ideas of Thomason [11] to extend the result to arbitrary graphs. The proof of Theorem 3 is almost the same as that of Theorem 2 and so we only sketch it.

2 Notation and tools

We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order and $d(G) := 2e(G)/|G|$ for its average degree. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. If $P = x_1 \ldots x_\ell$ is a path and $1 \leq i \leq j \leq \ell$, we write $x_i P x_j$ for its subpath $x_i \ldots x_j$.

We say that a graph H is a *minor* of G if for every vertex $h \in H$ there is set $C_h \subseteq V(G)$ such that all the C_h are disjoint, each $G[C_h]$ is connected and G contains a C_h-$C_{h'}$ edge whenever hh' is an edge in H. C_h is called the *branch set corresponding to h.*

We will use the following result of Mader [5].

Theorem 4 *Every graph G contains a $\lceil d(G)/4 \rceil$-connected subgraph.*

Given $k \in \mathbb{N}$, we say that a graph G is *k-linked* if $|G| \geq 2k$ and for every $2k$ distinct vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ of G there exist disjoint paths $P_1, \ldots, P_k$ such that P_i joins x_i to y_i. Jung as well as Larman and Mani independently proved that every sufficiently highly connected graph is k-linked. Later, Bollobás and Thomason [2] showed that a connectivity linear in k suffices. Simplifying the argument in [2], Thomas and Wollan [9] recently obtained an even better bound:

Theorem 5 *Every $16k$-connected graph is k-linked.*

Similarly as in [11], given positive numbers d and k, we shall consider the class $\mathcal{G}_{d,k}$ of graphs defined by

$$\mathcal{G}_{d,k} := \{G \,:\, |G| \geq d,\ e(G) > d|G| - kd\}.$$

We say that a graph G is *minor-minimal in* $\mathcal{G}_{d,k}$ if G belongs to $\mathcal{G}_{d,k}$ but no proper minor of G does. The following lemma states some properties of the minor-minimal elements of $\mathcal{G}_{d,k}$. The proof is simple, its counterpart for digraphs can be found in [11, Section 2]. (The first property follows by counting the number of edges of the complete graph on $\lfloor(2-\varepsilon)d\rfloor$ vertices.)

Lemma 6 *Given $0 < \varepsilon < 1/2$, $d \geq 2/\varepsilon$ and $1/d \leq k \leq \varepsilon d/2$, every minor-minimal graph in $\mathcal{G}_{d,k}$ satisfies the following properties:*

(i) $|G| \geq (2-\varepsilon)d$,

(ii) $e(G) \leq d|G| - kd + 1$,

(iii) *every edge of G lies in more than $d-1$ triangles,*

(iv) *G is $\lceil k \rceil$-connected.*

We will also use the following easy fact, see [11, Lemma 4.2] for a proof.

Lemma 7 *Suppose that x and y are distinct vertices of a k-connected graph G. Then G contains at least $k^2/4|G|$ internally disjoint x-y paths of length at most $2|G|/k$.*

3 Proof of theorems

The strategy of the proof of Theorem 2 is as follows. It is easily seen that to prove Theorem 2 for all graphs of average degree at least $(1+\varepsilon)t =: d$, it suffices to consider only those graphs G which are minor-minimal in the class $\mathcal{G}_{d/2,k}$ for some suitable k. In particular, together with Lemma 6 this implies that we only have to deal with k-connected graphs. If d (and so also k) is linear in the order of G, then a simple probabilistic argument gives us the desired $K_{s,t}$ minor (Lemma 8). In the other case we use that by Lemma 6 each vertex of G together with its neighbourhood induces a dense subgraph of G. We apply this to find 10 disjoint $K_{10s,\lceil d/9\rceil}$ minors which we combine to a $K_{s,t}$ minor.

Lemma 8 *For all $0 < \varepsilon, c < 1$ there exists a number $k_0 = k_0(\varepsilon, c)$ such that for each integer $k \geq k_0$ every k-connected graph G whose order n satisfies $k \geq cn$ contains a $K_{s,t}$ minor where $t := \lceil(1-\varepsilon)n\rceil$ and $s := \lceil c^4\varepsilon n/(32\log n)\rceil$. Moreover, the branch sets corresponding to the vertices in the vertex class of the $K_{s,t}$ of size t can be chosen to be singletons whereas all the other branch sets can be chosen to have size at most $8\log n/c^2$.*

Proof. Throughout the proof we assume that k (and thus also n) is sufficiently large compared with both ε and c for our estimates to hold. Put $a := \lfloor 4 \log s/c \rfloor$. Successively choose as vertices of G uniformly at random without repetitions. Let C_1 be the set of the first a of these vertices, let C_2 be the set of the next a vertices and so on up to C_s. Let C be the union of all the C_i. Given $i \le s$, we call a vertex $x \in G - C$ *good for* i if x has at least one neighbour in C_i. Moreover, we say that x is *good* if it is good for every $i \le s$. Thus

$$\mathbb{P}(x \text{ is not good for } i) \le \left(1 - \frac{d_G(x) - as}{n}\right)^a \le \mathrm{e}^{-a(k-as)/n} \le \mathrm{e}^{-ac/2}$$

and so x is not good with probability at most $s\mathrm{e}^{-ac/2} < \varepsilon/2$. Therefore the expected number of good vertices outside C is at least $(1-\varepsilon/2)|G-C|$. Hence there exists an outcome $C_1, \dots, C_s$ for which at least $(1-\varepsilon/2)|G-C|$ vertices in $G - C$ are good.

We now extend all these C_i to disjoint connected subgraphs of G as follows. Let us start with C_1. Fix a vertex $x_1 \in C_1$. For each $x \in C_1 \setminus \{x_1\}$ in turn we apply Lemma 7 to find an x-x_1 path of length at most $2n/k \le 2/c$ which is internally disjoint from all the paths chosen previously and which avoids $C_2 \cup \dots \cup C_s$. Since Lemma 7 guarantees at least $k^2/4n \ge as \cdot 2/c$ short paths between a given pair of vertices, we are able to extend each C_i in turn to a connected subgraph in this fashion. Denote the graphs thus obtained from $C_1, \dots, C_s$ by $G_1, \dots, G_s$. Thus all the G_i are disjoint.

Note that at most $2as/c$ good vertices lie in some G_i. Thus at least $(1-\varepsilon/2)|G-C|-2as/c \ge (1-\varepsilon)n$ good vertices avoid all the G_i. Hence G contains a $K_{s,t}$ minor as required. (The good vertices avoiding all the G_i correspond to the vertices of the $K_{s,t}$ in the vertex class of size t. The branch sets corresponding to the vertices of the $K_{s,t}$ in the vertex class of size s are the vertex sets of $G_1, \dots, G_s$.) □

Proof of Theorem 2. Let $d := (1+\varepsilon)t$ and $s := \lfloor \varepsilon^6 d/\log d \rfloor$. Throughout the proof we assume that t (and thus also d) is sufficiently large compared with ε for our estimates to hold. We have to show that every graph of average degree at least d contains a $K_{s,t}$ minor. Put $k := \lceil \varepsilon d/4 \rceil$. Since $\mathcal{G}_{d/2,k}$ contains all graphs of average degree at least d, it suffices to show that every graph G which is minor-minimal in $\mathcal{G}_{d/2,k}$ contains a $K_{s,t}$ minor. Let $n := |G|$. As is easily seen, (i) and (iv) of Lemma 6 together with Lemma 8 imply that we may assume that $d \le n/600$. (Lemma 8 is applied with $c := \varepsilon/2400$ and with ε replaced by $\varepsilon/3$.) Let X be the set of all those vertices of G whose degree is at most $2d$. Since by Lemma 6 (ii) the average degree of G is at most d, it follows that $|X| \ge n/2$. Let us first prove the following claim.

Either G contains a $K_{s,t}$ minor or G contains 10 disjoint $\lceil 3d/25 \rceil$-connected subgraphs $G_1, \dots, G_{10}$ such that $3d/25 \le |G_i| \le 3d$ for each $i \le 10$.

Choose a vertex $x_1 \in X$ and let G_1' denote the subgraph of G induced by x_1 and its neighbourhood. Then $|G_1'| = d_G(x_1) + 1 \le 2d + 1$. Since by Lemma 6 (iii) each edge between x_1 and $N_G(x_1)$ lies in at least $d/2-1$ triangles, it follows that the minimum degree of G_1' is at least $d/2-1$. Thus Theorem 4 implies that G_1' contains a $\lceil 3d/25\rceil$-connected subgraph. Take G_1 to be this subgraph. Put $X_1 := X \setminus V(G_1)$ and let X_1' be the set of all those vertices in X_1 which have at least $d/500$ neighbours in G_1.

Suppose first that $|X_1'| \geq |X|/10$. In this case we will find a $K_{s,t}$ minor in G. Since the argument is similar to the proof of Lemma 8, we only sketch it. Set $a := \lfloor 10^4 \log s\rfloor$. This time, we choose the a-element sets $C_1, \ldots, C_s$ randomly inside $V(G_1)$. Since every vertex in X_1' has at least $d/500$ neighbours in G_1, the probability that the neighbourhood of a given vertex $x \in X_1'$ avoids some C_i is at most $s\mathrm{e}^{-a/(3\cdot 10^3)} < \varepsilon$. So the expected number of such bad vertices in X_1' is at most $\varepsilon|X_1'|$. Thus for some choice of $C_1, \ldots, C_s$ there are at least $(1-\varepsilon)|X_1'| \geq (1-\varepsilon)n/20 \geq t$ vertices in X_1' which have a neighbour in each C_i. Since the connectivity of G_1 is linear in its order, we may again apply Lemma 7 to make the C_i into disjoint connected subgraphs of G_1 by adding suitable short paths from G_1. This shows that G contains a $K_{s,t}$ minor.

Thus we may assume that at least $|X_1| - |X|/10 \geq 9|X|/10 - 3d > 0$ vertices in X_1 have at most $d/500$ neighbours in G_1. Choose such a vertex x_2. Let G_2' be the subgraph of G induced by x_2 and all its neighbours outside G_1. Since by Lemma 6 (iii) every edge of G lies in at least $d/2-1$ triangles, it follows that the minimum degree of G_2' is at least $d/2 - 1 - d/500 > 12d/25$. Again, we take G_2 to be a $\lceil 3d/25\rceil$-connected subgraph of G_2' obtained by Theorem 4.

We now put $X_2 := X_1 \setminus (X_1' \cup V(G_2))$ and define X_2' to be the set of all those vertices in X_2 which have at least $d/500$ neighbours in G_2. If $|X_2'| \geq |X|/10$, then as before, we can find a $K_{s,t}$ minor in G. If $|X_2'| \leq |X|/10$ we define G_3 in a similar way as G_2. Continuing in this fashion proves the claim. (Note that when choosing x_{10} we still have $|X_9| - |X|/10 \geq |X|/10 - 9\cdot 3d > 0$ vertices at our disposal since $n \geq 600d$.)

Apply Lemma 8 with $c := 1/25$ to each G_i to find a $K_{10s,\lceil d/9\rceil}$ minor. Let $C_1^i, \ldots, C_s^i, D_1^i, \ldots, D_{9s}^i$ denote the branch sets corresponding to the vertices of the $K_{10s,\lceil d/9\rceil}$ in the vertex class of size $10s$. By Lemma 8 we may assume that all the C_j^i and all the D_j^i have size at most $8\cdot 25^2 \log|G_i| \leq 10^5 \log d$ and that all the branch sets corresponding to the remaining vertices of the $K_{10s,\lceil d/9\rceil}$ are singletons. Let $T^i \subseteq V(G_i)$ denote the union of all these singletons. Let C be the union of all the C_j^i, let D be the union of all the D_j^i and let T be the union of all the T^i.

We will now use these 10 $K_{10s,\lceil d/9\rceil}$ minors to form a $K_{s,t}$ minor in G. Recall that by Lemma 6 (iv) the graph G is $\lceil \varepsilon d/4\rceil$-connected and so by Theorem 5 it is $\lfloor \varepsilon d/64\rfloor$-linked. Thus there exists a set $\mathcal{P}$ of $9s$ disjoint paths in G such that for all $i \leq 9$ and all $j \leq s$ the set C_j^i is joined to C_j^{i+1} by one of these paths and such that no path from $\mathcal{P}$ contains an inner vertex in $C \cup D$. (To see this, use that $\varepsilon d/64 \geq 100s\cdot 10^5 \log d \geq |C \cup D|$.)

The paths in $\mathcal{P}$ can meet T in many vertices. But we can reroute them such that every new path contains at most two vertices from each T^i. For every path $P \in \mathcal{P}$ in turn we will do this as follows. If P meets T^1 in more than 2 vertices, let t and t' denote the first and the last vertex from T^1 on P. Choose some set

D_j^1 and replace the subpath tPt' by some path between t and t' whose interior lies entirely in $G[D_j^1]$. (This is possible since $G[D_j^1]$ is connected and since both t and t' have a neighbour in D_j^1.) Proceed similarly if the path thus obtained still meets some other T^i. Then continue with the next path from $\mathcal{P}$. (The sets D_j^i used for the rerouting are chosen to be distinct for different paths.) Note that the paths thus obtained are still disjoint since D was avoided by all the paths in $\mathcal{P}$.

We now have found our $K_{s,t}$ minor. Each vertex lying in the vertex class of size s of the $K_{s,t}$ corresponds to a set consisting of $C_j^1 \cup \cdots \cup C_j^{10}$ together with the (rerouted) paths joining these sets. For the remaining vertices of the $K_{s,t}$ we can take all the vertices in T which are avoided by the (rerouted) paths. There are at least t such vertices since these paths contain at most $20 \cdot 9s$ vertices from T and $|T| - 180s \geq 10d/9 - 180s \geq t$. $\square$

Proof of Theorem 3 (Sketch). Without loss of generality we may assume that $\varepsilon < 10^{-16}$. The proof of Theorem 3 is almost the same as that of Theorem 2. The only difference is that now we also apply Lemma 7 to find $\binom{s}{2}$ short paths connecting all the pairs of the C_i. This can be done at the point where we extend the C_i's to connected subgraphs. $\square$

The following proposition shows that the bound on s in Theorem 2 is essentially best possible. Its proof is an adaption of a well-known argument of Bollobás, Catlin and Erdős [1].

Proposition 9 *There exists an integer n_0 such that for each integer $n \geq n_0$ and each number $\alpha > 0$ there is a graph G of order n and with average degree at least $n/2$ which does not have a $K_{s,t}$ minor with $s := \lceil 2n/\alpha \log n \rceil$ and $t := \lceil \alpha n \rceil$.*

Proof. Let $p := 1 - 1/\mathrm{e}$. Throughout the proof we assume that n is sufficiently large for our estimates to hold. Consider a random graph G_p of order n which is obtained by including each edge with probability p independently from all other edges. We will show that with positive probability G_p is as required in the proposition. Clearly, with probability $> 3/4$ the average degree of G_p is at least $n/2$. Hence it suffices to show that with probability at most $1/2$ the graph G_p will have the property that its vertex set $V(G_p)$ can be partitioned into disjoint sets $S_1, \dots, S_s$ and $T_1, \dots, T_t$ such that G_p contains an edge between every pair S_i, T_j ($1 \leq i \leq s$, $1 \leq j \leq t$). Call such a partition of $V(G_p)$ *admissible.* Thus we have to show that the probability that G_p has an admissible partition is $\leq 1/2$.

Let us first estimate the probability that a given partition $\mathcal{P}$ is admissible:

$$\begin{aligned}\mathbb{P}(\mathcal{P} \text{ is admissible}) &= \prod_{i,j}\left(1-(1-p)^{|S_i||T_j|}\right) \le \exp\left(-\sum_{i,j}(1-p)^{|S_i||T_j|}\right)\\ &\le \exp\left(-st\prod_{i,j}(1-p)^{|S_i||T_j|(st)^{-1}}\right) \le \exp\left(-st(1-p)^{n^2(st)^{-1}}\right)\\ &\le \exp\left(-\frac{2n^2}{\log n}\cdot n^{-\frac{1}{2}}\right) \le \exp(-n^{\frac{4}{3}}).\end{aligned}$$

(The first expression in the second line follows since the arithmetic mean is at least as large as the geometric mean.) Since the number of possible partitions is at most n^n, it follows that the probability that G_p has an admissible partition is at most $n^n \cdot \mathrm{e}^{-n^{4/3}} < 1/2$, as required. $\square$

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