3-UNIFORM HYPERGRAPHS OF BOUNDED DEGREE HAVE LINEAR RAMSEY NUMBERS

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Abstract. Chvátal, Rödl, Szemerédi and Trotter [1] proved that the Ramsey numbers of graphs of bounded maximum degree are linear in their order. We prove that the same holds for 3-uniform hypergraphs. The main new tool which we prove and use is an embedding lemma for 3-uniform hypergraphs of bounded maximum degree into suitable 3-uniform 'pseudo-random' hypergraphs.

keywords: hypergraphs; regularity lemma; Ramsey numbers; embedding problems

1. Introduction

1.1. **Ramsey numbers.** The *Ramsey number* $R(H)$ of a graph H is defined to be the smallest $N \in \mathbb{N}$ such that in every colouring of the edges of the complete graph on N vertices with two colours one can find a monochromatic copy of H. In general, the best upper bound on $R(H)$ is exponential in $|H|$. However, if H is sparse, then one can sometimes improve considerably on this. A central result in this area was proved by Chvátal, Rödl, Szemerédi and Trotter [1]. They showed that for every Δ there exists a constant $C = C(\Delta)$ such that all graphs H with maximum degree at most Δ satisfy $R(H) \le C|H|$.

Here we prove an analogue of this result for 3-uniform hypergraphs $\mathscr{H}$ of bounded maximum degree. Thus we now consider hyperedges (each consisting of 3 vertices) instead of edges. The *degree of a vertex x* in $\mathscr{H}$ is defined to be the number of hyperedges which contain x. The *maximum degree* $\Delta(\mathscr{H})$ and the *Ramsey number* $R(\mathscr{H})$ of a 3-uniform hypergraph $\mathscr{H}$ are then defined in the obvious way.

Theorem 1. *For every $\Delta \in \mathbb{N}$ there exists a constant $C = C(\Delta)$ such that all 3-uniform hypergraphs $\mathscr{H}$ of maximum degree at most Δ satisfy $R(\mathscr{H}) \le C|\mathscr{H}|$.*

Theorem 1 was also proved independently by Nagle, Olsen, Rödl and Schacht [16]. Kostochka and Rödl [14] had previously shown that Ramsey numbers of k-uniform hypergraphs of bounded maximum degree are 'almost linear' in their orders. More precisely, they showed that for all $\varepsilon, \Delta, k > 0$ there is a constant C such that $R(\mathscr{H}) \le C|\mathscr{H}|^{1+\varepsilon}$ if $\mathscr{H}$ is k-uniform and has maximum degree at most Δ. In a sequel [3] to this paper, we generalised Theorem 1 to k-uniform hypergraphs of bounded degree for arbitrary k. This was also done independently by Ishigami [12]. Another related result is that of Haxell et al. [9, 10], who asymptotically determined the Ramsey numbers of 3-uniform tight and loose cycles. For general 3-uniform hypergraphs the best upper bound is still due to Erdős and Rado [5], which implies that every 3-uniform hypergraph $\mathscr{H}$ satisfies $R(\mathscr{H}) \le 2^{2^{4|\mathscr{H}|}}$.

1.2. **Embedding graphs and hypergraphs.** The proof in [1] which shows that graphs of bounded maximum degree have linear Ramsey numbers proceeds as follows: Given a red/blue colouring of the edges of the complete graph on N vertices, we consider the red

N. Fountoulakis and D. Kühn were supported by the EPSRC, grant no. EP/D50564X/1.

subgraph G and apply Szemerédi's regularity lemma to it to obtain a vertex partition of G into a bounded number of clusters such that almost all of the bipartite subgraphs induced by the clusters are 'pseudo-random'. We now define a reduced graph R whose vertices are the clusters and any two of them are connected by an edge if the corresponding bipartite subgraph of G is 'pseudo-random'. Since R is very dense, by Turán's theorem it contains a large clique K. We now define an edge-colouring of K by colouring an edge red if the density of the corresponding bipartite subgraph of G is large, and blue otherwise. An application of Ramsey's theorem now gives a monochromatic clique of order $k := \Delta(H) + 1$ in K. Without loss of generality, assume it is red. This corresponds to a large complete k-partite subgraph G' of G where all the bipartite subgraphs induced by the vertex classes are 'pseudo-random'. Since the chromatic number of the desired graph H is at most k, one can use the 'pseudo-randomness' of G' to find a copy of H in G'. The tool which enables the final step is often called the 'embedding lemma' or 'key lemma' (see e.g. [13]).

In our proof of Theorem 1, we adopt a similar strategy. Instead of Szemerédi's regularity lemma, we will use the regularity lemma for 3-uniform hypergraphs due to Frankl and Rödl [6]. However, this has the problem that the 'pseudo-random' hypergraph into which we aim to embed our given 3-uniform hypergraph $\mathscr{H}$ of bounded maximum degree could be very sparse and not as 'pseudo-random' as one would like it to be. This means that the proof of the corresponding embedding lemma is considerably more difficult and rather different from that of the graph version, while the adaption of the other steps is comparatively easy. Thus we view the embedding lemma (Lemma 2) as the main result of this paper and also believe that it will have other applications besides Theorem 1. Its precise formulation needs some preparation, so we defer its statement to Section 2.

The strategy for the k-uniform case in [3] is similar to the one used here, but several additional problems arise.

1.3. **Organization of the paper.** In Section 2 we state the embedding lemma (Lemma 2). Our proof proceeds by induction on the order of the hypergraph we aim to embed. This argument yields a significantly stronger result (Lemma 3).

In Section 3 we state several results which are important in the proof of the induction step for Lemma 3. In particular, we will need a variant of the counting lemma, which implies that for any 3-uniform hypergraph $\mathscr{H}$ of bounded size every suitable 'pseudo-random' hypergraph $\mathscr{G}$ contains roughly as many copies of $\mathscr{H}$ as one would expect in a random hypergraph. We will also need an extension lemma, which states that for any 3-uniform hypergraph $\mathscr{H}'$ of bounded size, any induced subhypergraph $\mathscr{H} \subseteq \mathscr{H}'$ and any suitable 'pseudo-random' hypergraph $\mathscr{G}$, almost all copies of $\mathscr{H}$ in $\mathscr{G}$ can be extended to approximately the same number of copies of $\mathscr{H}'$ as one would expect if $\mathscr{G}$ were a random hypergraph. In Section 5 we derive our variant of the counting lemma from that of Nagle, Rödl and Schacht [19]. In Section 6 we will deduce the extension lemma from the counting lemma (which corresponds to the case when $\mathscr{H}$ is empty).

Before this, in Section 4 we use the extension lemma to prove the strengthened version of the embedding lemma mentioned before (Lemma 3). Finally, we use the embedding lemma together with the regularity lemma for 3-uniform hypergraphs due to Frankl and Rödl [6] to prove Theorem 1.

2. The embedding lemma

Before we can state the embedding lemma, we first have to introduce some notation. Given a bipartite graph G with vertex classes A and B, we denote the number of edges of G

by $e(A, B)$. The *density* of G is defined to be

$$d_G(A, B) := \frac{e(A, B)}{|A||B|}.$$

We will also use $d(A, B)$ instead of $d_G(A, B)$ if this is unambiguous. Given $0 < \delta, d \leq 1$, we say that G is (d, δ)*-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \delta|A|$ and $|Y| \geq \delta|B|$ we have $(1 - \delta)d < d(X, Y) < (1 + \delta)d$.

Given $\delta > 0$, we also say that G is δ*-regular* if for any $X \subseteq A$ and $Y \subseteq B$ which satisfy $|X| \geq \delta|A|$, $|Y| \geq \delta|B|$, we have

$$|d(X, Y) - d(A, B)| \leq \delta.$$

It can easily be seen that this definition of regularity is roughly equivalent to (d, δ)-regularity. We say that a *k-partite graph P is (d, δ)-regular* if each of the $\binom{k}{2}$ bipartite subgraphs forming P is (d, δ)-regular or empty.

Given a 3-uniform hypergraph $\mathscr{G}$, we denote by $|\mathscr{G}|$ the number of its vertices and by $E(\mathscr{G})$ the set of its hyperedges. We write $e(\mathscr{G}) := |E(\mathscr{G})|$. We say that vertices $x, y \in \mathscr{G}$ are *neighbours* if x and y lie in a common hyperedge of $\mathscr{G}$.

In order to state the embedding lemma we will now say what we mean by a 'pseudo-random' hypergraph, i.e. we will formally define regularity of 3-uniform hypergraphs. Suppose we are given a 3-partite graph P with vertex classes V_i, V_j, V_k where the three bipartite graphs forming P are denoted by P^{ij}, P^{jk} and P^{ik}. We will often refer to such a 3-partite graph as a *triad*. We write $T(P)$ for the set of all triangles contained in P and let $t(P)$ denote the number of these triangles. Given a 3-partite 3-uniform hypergraph $\mathscr{G}$ with the same vertex classes, we define the *density of P with respect to $\mathscr{G}$* by

$$d_{\mathscr{G}}(P) := \begin{cases} |E(\mathscr{G}) \cap T(P)|/t(P) & \text{if } t(P) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $d_{\mathscr{G}}(P)$ denotes the proportion of all those triangles in P which are hyperedges of $\mathscr{G}$. More generally, suppose that we are given an r-tuple $\vec{Q} = (Q(1), \ldots, Q(r))$ of subtriads of P, where $Q(s) = Q^{ij}(s) \cup Q^{jk}(s) \cup Q^{ik}(s)$, and $Q^{ij}(s) \subseteq P^{ij}$, $Q^{jk}(s) \subseteq P^{jk}$, $Q^{ik}(s) \subseteq P^{ik}$ for all $s \in [r]$, where $[r]$ denotes $\{1, \ldots, r\}$. Put

$$t(\vec{Q}) := \left| \bigcup_{s=1}^{r} T(Q(s)) \right|.$$

The *density of $\vec{Q}$ with respect to $\mathscr{G}$* is defined to be

$$d_{\mathscr{G}}(\vec{Q}) := \begin{cases} |E(\mathscr{G}) \cap \bigcup_{s=1}^{r} T(Q(s))|/t(\vec{Q}) & \text{if } t(\vec{Q}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this definition, the sets $T(Q(s))$ of triangles need not necessarily be disjoint. We say that a triad P is (d_3, δ_3, r)*-regular* with respect to $\mathscr{G}$ if for every r-tuple $\vec{Q} = (Q(1), \ldots, Q(r))$ of subtriads of P with

$$t(\vec{Q}) \geq \delta_3 \cdot t(P)$$

we have

$$|d_3 - d_{\mathscr{G}}(\vec{Q})| < \delta_3.$$

We say that P is (δ_3, r)*-regular* with respect to $\mathscr{G}$ if it is (d, δ_3, r)-regular for some d. More generally, if $k \geq 3$, P is a k-partite graph and $\mathscr{G}$ is a k-partite 3-uniform hypergraph with the same vertex classes, we say that P is (d_3, δ_3, r)*-regular* with respect to $\mathscr{G}$ if each of the triads P' induced by P is either (d_3, δ_3, r)-regular with respect to $\mathscr{G}$ or satisfies $d_{\mathscr{G}}(P') = 0$.

If $\mathscr{H}$ and $\mathscr{G}$ are k-partite 3-uniform hypergraphs with vertex classes $X_1, \dots, X_k$ and $V_1, \dots, V_k$ respectively, and if P is a k-partite graph with vertex classes $V_1, \dots, V_k$, we say that $(\mathscr{G}, P)$ *respects the partition of* $\mathscr{H}$ if, for all $i < j < \ell$, whenever $\mathscr{H}$ contains a hyperedge with vertices in X_i, X_j, X_ℓ, the hypergraph $\mathscr{G}$ contains a hyperedge with vertices in V_i, V_j, V_ℓ which also forms a triangle in P.

Note that if $(\mathscr{G}, P)$ respects the partition of $\mathscr{H}$ and P is (d_3, δ_3, r)-regular with respect to $\mathscr{G}$ then the triad $P[V_i, V_j, V_\ell]$ induced by $V_i \cup V_j \cup V_\ell$ is (d_3, δ_3, r)-regular whenever $\mathscr{H}$ contains a hyperedge with vertices in X_i, X_j, X_ℓ. Thus if P is also graph-regular, $\mathscr{H}$ has bounded maximum degree, $\delta_3 \ll d_3$ and $|X_i| \le |V_i|$ for all i, then one might hope that this regularity can be used to find an embedding of $\mathscr{H}$ in $\mathscr{G}$ (where the vertices in X_i are represented by vertices in V_i).

Lemma 2 (Embedding lemma)**.** *Let Δ, k, r, n_0 be positive integers and let $c, d_2, d_3, \delta_2, \delta_3$ be positive constants such that*

$$1/n_0 \ll 1/r \ll \delta_2 \ll \min\{\delta_3, d_2\} \le \delta_3 \ll d_3, 1/\Delta, 1/k \quad \text{and} \quad c \ll d_2, d_3, 1/\Delta, 1/k.$$

Then the following holds for all integers $n \ge n_0$. Suppose that $\mathscr{H}$ is a k-partite 3-uniform hypergraph of maximum degree at most Δ with vertex classes $X_1, \dots, X_k$ such that $|X_i| \le cn$ for all $i = 1, \dots, k$. Suppose that $\mathscr{G}$ is a k-partite hypergraph with vertex classes $V_1, \dots, V_k$, which all have size n. Suppose that P is a (d_2, δ_2)-regular k-partite graph with vertex classes $V_1, \dots, V_k$ which is (d_3, δ_3, r)-regular with respect to $\mathscr{G}$, and $(\mathscr{G}, P)$ respects the partition of $\mathscr{H}$. Then $\mathscr{G}$ contains a copy of $\mathscr{H}$.

Here we write $0 < a_1 \ll a_2 \ll a_3$ to mean that we can choose the constants a_1, a_2, a_3 from right to left. More precisely, there are increasing functions f and g such that, given a_3, whenever we choose some $a_2 \le f(a_3)$ and $a_1 \le g(a_2)$, all calculations needed in the proof of Lemma 2 are valid. In order to simplify the exposition, we will not determine these functions explicitly. Hierarchies with more constants are defined in the obvious way.

The strategy of our proof of Lemma 2 is to proceed by induction on $|\mathscr{H}|$. So for any vertex h of $\mathscr{H}$, let $\mathscr{H}_h$ denote the hypergraph obtained from $\mathscr{H}$ by removing h. Let v, w be any vertices of $\mathscr{H}$ forming a hyperedge with h. In the induction step, we only want to consider copies of $\mathscr{H}_h$ in $\mathscr{G}$ for which vw is an edge of P (otherwise there is clearly no chance of using the regularity of $\mathscr{G}$ to extend this copy of $\mathscr{H}_h$ to one of $\mathscr{H}$). This motivates the following definition. A *complex* $\mathscr{H}$ consists of vertices, edges and hyperedges such that the set of edges is a subset of the set of unordered pairs of vertices and the set of hyperedges is a subset of the set of unordered triples of vertices. Moreover, each pair of vertices of $\mathscr{H}$ lying in a common hyperedge has to form an edge of $\mathscr{H}$. Thus we can make every 3-uniform hypergraph $\mathscr{H}$ into a complex by adding an edge between every pair of vertices that lies in a common hyperedge of $\mathscr{H}$. We will often denote this complex by $\mathscr{H}$ again.

Instead of Lemma 2, we will prove an embedding lemma for complexes. In order to state it, we need to introduce some more notation. Given a complex $\mathscr{H}$, we let $V(\mathscr{H})$ denote the set of its vertices, we write $E_2(\mathscr{H})$ for the set of its edges and $E_3(\mathscr{H})$ for the set of its hyperedges. Note that each hyperedge of a complex $\mathscr{H}$ forms a triangle in the underlying graph (whose vertex set is $V(\mathscr{H})$ and whose set of edges is $E_2(\mathscr{H})$). We set $|\mathscr{H}| := |V(\mathscr{H})|$, and $e_i(\mathscr{H}) := |E_i(\mathscr{H})|$ for $i = 2, 3$. We say that a complex $\mathscr{H}$ is *k-partite* if its underlying graph is k-partite. The *degree of a vertex x in a complex $\mathscr{H}$* is the maximum of the degree of x in the underlying graph and its degree in the underlying hypergraph (whose vertex set is $V(\mathscr{H})$ and whose set of hyperedges is $E_3(\mathscr{H})$). The *maximum degree of $\mathscr{H}$* is then defined in the obvious way. We say that vertices x and y are *neighbours in $\mathscr{H}$* if they are neighbours in the underlying graph. Subcomplexes of $\mathscr{H}$ and subcomplexes induced by some vertex set $X \subseteq V(\mathscr{H})$ are defined in the natural way. Also, the symbol $K_k^{(3)}$ will denote either the complete complex or the complete 3-uniform hypergraph on k vertices. It will be clear from the context which of the two is intended. Note that the complete complex $K_1^{(3)}$ is just a vertex and $K_2^{(3)}$ consists of two vertices joined by an edge.

Given k-partite complexes $\mathscr{H}$ and $\mathscr{G}$ with vertex classes $X_1, \dots, X_k$ and $V_1, \dots, V_k$, we say that *$\mathscr{G}$ respects the partition of $\mathscr{H}$* if it satisfies the following two properties. First, for all $i < j < \ell$, the complex $\mathscr{G}$ contains a hyperedge with vertices in V_i, V_j, V_ℓ whenever $\mathscr{H}$ contains a hyperedge with vertices in X_i, X_j, X_ℓ. Second, for all $i < j$, the complex $\mathscr{G}$ contains an edge between V_i and V_j whenever $\mathscr{H}$ contains an edge between X_i and X_j.

We say that a k-partite complex $\mathscr{G}$ is *$(d_3, \delta_3, d_2, \delta_2, r)$-regular* if its underlying graph P is (d_2, δ_2)-regular and P is (d_3, δ_3, r)-regular with respect to the underlying hypergraph of $\mathscr{G}$.

Suppose that we have k-partite complexes $\mathscr{H}$ and $\mathscr{G}$ with vertex classes $X_1, \dots, X_k$ and $V_1, \dots, V_k$ respectively. A *labelled partition-respecting copy of $\mathscr{H}$ in $\mathscr{G}$* is a labelled subcomplex of $\mathscr{G}$ which is isomorphic to $\mathscr{H}$ such that the corresponding isomorphism maps X_i to a subset of V_i. This definition naturally extends to labelled partition-respecting copies of subcomplexes $\mathscr{H}'$ of $\mathscr{H}$ in $\mathscr{G}$. Given any subcomplex $\mathscr{H}'$ of $\mathscr{H}$, we write $|\mathscr{H}'|_{\mathscr{G}}$ for the number of labelled partition-respecting copies of $\mathscr{H}'$ in $\mathscr{G}$.

Instead of Lemma 2 we will prove the following result, which implies it immediately.

Lemma 3 (Embedding lemma for complexes)**.** *Let Δ, k, r, n_0 be positive integers and let $c, \alpha, d_2, d_3, \delta_2, \delta_3$ be positive constants such that*

$$1/n_0 \ll 1/r \ll \delta_2 \ll \min\{\delta_3, d_2\} \leq \delta_3 \ll \alpha \ll d_3, 1/\Delta, 1/k \quad \text{and} \quad c \ll \alpha, d_2.$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathscr{H}$ is a k-partite complex of maximum degree at most Δ with vertex classes $X_1, \dots, X_k$ such that $|X_i| \leq cn$ for all $i = 1, \dots, k$. Suppose also that $\mathscr{G}$ is a k-partite $(d_3, \delta_3, d_2, \delta_2, r)$-regular complex with vertex classes $V_1, \dots, V_k$, all of size n, which respects the partition of $\mathscr{H}$. Then for every vertex $h \in V(\mathscr{H})$ we have that

$$|\mathscr{H}|_{\mathscr{G}} \geq (1-\alpha) n d_2^{e_2(\mathscr{H}) - e_2(\mathscr{H}_h)} d_3^{e_3(\mathscr{H}) - e_3(\mathscr{H}_h)} |\mathscr{H}_h|_{\mathscr{G}},$$

where $\mathscr{H}_h$ denotes the induced subcomplex of $\mathscr{H}$ obtained by removing h. In particular, $\mathscr{G}$ contains at least $((1-\alpha)n)^{|\mathscr{H}|} d_2^{e_2(\mathscr{H})} d_3^{e_3(\mathscr{H})}$ labelled partition-respecting copies of $\mathscr{H}$.

Note that we would expect almost $n d_2^{e_2(\mathscr{H}) - e_2(\mathscr{H}_h)} d_3^{e_3(\mathscr{H}) - e_3(\mathscr{H}_h)} |\mathscr{H}_h|_{\mathscr{G}}$ labelled partition-respecting copies of $\mathscr{H}$ if $\mathscr{G}$ were a random complex. As indicated above, we will prove Lemma 3 by induction on $|\mathscr{H}|$. In the induction step, it will be extremely useful to assume the existence of the expected number of copies of any proper subcomplex $\mathscr{H}'$ of $\mathscr{H}$ in $\mathscr{G}$ and not just the existence of one such copy.

3. Tools

In our proof of Lemma 3 we will use the so-called counting lemma.

Lemma 4 (Counting lemma)**.** *Let k, r, t, n_0 be positive integers and let $\beta, d_2, d_3, \delta_2, \delta_3$ be positive constants such that*

$$1/n_0 \ll 1/r \ll \delta_2 \ll \min\{\delta_3, d_2\} \leq \delta_3 \ll \beta, d_3, 1/k, 1/t.$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathscr{H}$ is a k-partite complex on t vertices with vertex classes $X_1, \dots, X_k$. Suppose also that $\mathscr{G}$ is a k-partite $(d_3, \delta_3, d_2, \delta_2, r)$-regular complex with vertex classes $V_1, \dots, V_k$, all of size n, which respects the partition of $\mathscr{H}$. Then $\mathscr{G}$ contains

$$(1 \pm \beta) n^t d_2^{e_2(\mathscr{H})} d_3^{e_3(\mathscr{H})}$$

labelled partition-respecting copies of $\mathscr{H}$.

The lower bound in Lemma 4 for $K_k^{(3)}$'s was proved by Nagle and Rödl [17] (a short proof was given later in [18]). Nagle, Rödl and Schacht [19] generalized this lower bound to arbitrary k-uniform complexes (Lemma 7 in Section 5). In a slightly different setup, this was also proved independently by Gowers [7]. The upper bound in Lemma 4 can easily be derived from the lower bound. This was done for $K_k^{(3)}$'s in [17]. In Section 5 we show how one can derive Lemma 4 from Lemma 7.

Note that Lemma 3 is a generalization of the lower bound in Lemma 4. As a special case, Lemma 4 includes the counting lemma for graphs, which is an easy consequence of the definition of graph regularity.

The following result is another strengthening of Lemma 4. We will need it in the proof of Lemma 3. Given complexes $\mathscr{H} \subseteq \mathscr{H}'$ such that $\mathscr{H}$ is induced, it states that $\mathscr{G}$ not only contains about the expected number of copies of $\mathscr{H}'$, but also that almost all copies of $\mathscr{H}$ in $\mathscr{G}$ are extendible to about the expected number of copies of $\mathscr{H}'$. The special case when $\mathscr{H}$ is a hyperedge was proved earlier by Haxell, Nagle and Rödl [11].

Lemma 5 (Extension lemma)**.** *Let k, r, t, t', n_0 be positive integers, where $t < t'$, and let $\beta, d_2, d_3, \delta_2, \delta_3$ be positive constants such that*

$$1/n_0 \ll 1/r \ll \delta_2 \ll \min\{\delta_3, d_2\} \leq \delta_3 \ll \beta, d_3, 1/k, 1/t'.$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathscr{H}'$ is a k-partite complex on t' vertices with vertex classes $X_1, \dots, X_k$ and let $\mathscr{H}$ be an induced subcomplex of $\mathscr{H}'$ on t vertices. Suppose also that $\mathscr{G}$ is a k-partite $(d_3, \delta_3, d_2, \delta_2, r)$-regular complex with vertex classes $V_1, \dots, V_k$, all of size n, which respects the partition of $\mathscr{H}'$. Then all but at most $\beta |\mathscr{H}|_{\mathscr{G}}$ labelled partition-respecting copies of $\mathscr{H}$ in $\mathscr{G}$ are extendible to

$$(1 \pm \beta) n^{t'-t} d_2^{e_2(\mathscr{H}') - e_2(\mathscr{H})} d_3^{e_3(\mathscr{H}') - e_3(\mathscr{H})}$$

labelled partition-respecting copies of $\mathscr{H}'$ in $\mathscr{G}$.

Lemmas 4 and 5 differ from Lemma 3 in that the positions of t and t' in the hierarchy mean we can only look at complexes $\mathscr{H}, \mathscr{H}'$ of bounded size. In particular, in the proof of Lemma 3 we will apply these lemmas to complexes whose order is some function of Δ and so does not depend on n. Lemma 5 will be proved in Section 6.

4. Proof of the embedding lemma

We first outline the proof a graph version of the embedding lemma, before going on to prove the hypergraph version. The graph version is not necessary for the arguments in the rest of the paper; it is included only to give the reader an introduction to the ideas used in the more complicated hypergraph version.

In both cases, whenever we refer to a particular copy of a certain subgraph or subcomplex $\mathscr{H}'$ of $\mathscr{H}$ in $\mathscr{G}$, we mean that this copy is labelled and partition-respecting without mentioning it explicitly. We usually denote such a copy by H' (i.e. by the corresponding italic letter).

4.1. **The graph case.** We use much of the same notation as in the hypergraph case. So now $\mathscr{H}$ will be a graph which we wish to embed in a graph $\mathscr{G}$. For any subgraph $\mathscr{H}'$ of $\mathscr{H}$, the number of labelled partition-respecting copies of $\mathscr{H}'$ in $\mathscr{G}$ is denoted by $|\mathscr{H}'|_{\mathscr{G}}$.

Claim 6. *Let Δ, k, n_0 be positive integers and let c, α, d_2, δ_2 be positive constants such that*

$$0 < 1/n_0 \ll \delta_2 \ll \alpha \ll 1/\Delta, 1/k \ \text{ and } \ c, \delta_2 \ll \alpha, d_2$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathscr{H}$ is a k-partite graph of maximum degree $\Delta(\mathscr{H}) \leq \Delta$ with vertex classes $X_1, \ldots, X_k$ such that $|X_i| \leq cn$ for all $i = 1, \ldots, k$. Suppose that $\mathscr{G}$ is a k-partite (d_2, δ_2)-regular graph with vertex classes $V_1, \ldots, V_k$, all of size n, which respects the partition of $\mathscr{H}$. Then for every vertex $h \in V(\mathscr{H})$, we have that

$$|\mathscr{H}|_{\mathscr{G}} \geq (1-\alpha) n d_2^{e(\mathscr{H})-e(\mathscr{H}_h)} |\mathscr{H}_h|_{\mathscr{G}},$$

where $\mathscr{H}_h$ denotes the induced subgraph of $\mathscr{H}$ obtained by removing h. In particular, $\mathscr{G}$ contains at least $((1-\alpha)n)^{|\mathscr{H}|} d_2^{e(\mathscr{H})}$ labelled partition-respecting copies of $\mathscr{H}$.

Proof. We prove the claim by induction on $|\mathscr{H}|$. We will assume that the component of $\mathscr{H}$ which contains h has size at least Δ^2. (If it does not, a simple application of the counting lemma, as in the hypergraph case, proves the claim.) This covers the base case of the induction. It also ensures that the second neighbourhood of $\mathscr{H}$ is nonempty, which will be important later on.

Now let $\mathscr{N}_h$ be the graph induced by the neighbours of h. Let $\mathscr{B}$ be the graph induced by h and its neighbours, and let $\mathscr{H}_h^\dagger$ be the graph obtained from $\mathscr{H}$ by removing h and its neighbours. Note that a copy of $\mathscr{H}$ in $\mathscr{G}$ can be obtained (in a unique way) by fixing a copy of $\mathscr{N}_h$, extending it to a copy of $\mathscr{H}_h$ and also extending it to a copy of $\mathscr{B}$, where the vertex chosen for h avoids those chosen for $\mathscr{H}_h^\dagger$. The latter condition will not affect any calculations significantly, so for this sketch we will ignore it.

Fix a new constant δ_2' such that $\delta_2 \ll \delta_2' \ll \alpha$. We call a copy of $\mathscr{N}_h$ in $\mathscr{G}$ *typical* if it extends to $(1 \pm \delta_2') d_2^{d(h)} n$ copies of $\mathscr{B}$. We denote the set of all typical copies of $\mathscr{N}_h$ by Typ. An application of the extension lemma for graphs shows that all but at most $\delta_2' |\mathscr{N}_h|_{\mathscr{G}}$ copies of $\mathscr{N}_h$ are typical.

Given a copy N_h of $\mathscr{N}_h$, we denote by $|N_h \to \mathscr{H}_h|_{\mathscr{G}}$ the number of copies of $\mathscr{H}_h$ in $\mathscr{G}$ which extend N_h, and $|N_h \to \mathscr{B}|_{\mathscr{G}}$ is defined similarly. Then

$$\begin{aligned} |\mathscr{H}|_{\mathscr{G}} &\geq \sum_{N_h \in \mathrm{Typ}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} |N_h \to \mathscr{B}|_{\mathscr{G}} \\ &\geq (1-\delta_2') d_2^{d(h)} n \sum_{N_h \in \mathrm{Typ}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} \\ &= (1-\delta_2') d_2^{e(\mathscr{H})-e(\mathscr{H}_h)} n \left(|\mathscr{H}_h|_{\mathscr{G}} - \sum_{N_h \notin \mathrm{Typ}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} \right). \end{aligned}$$

So it remains to show that the sum in the last line is negligible. We can do this since there are at most $\delta_2' |\mathscr{N}_h|_{\mathscr{G}}$ atypical copies of $\mathscr{N}_h$, each extending to at most $|\mathscr{H}_h^\dagger|_{\mathscr{G}}$ copies of $\mathscr{H}_h$. By applying the induction hypothesis $|\mathscr{N}_h|$ times, we can also relate $|\mathscr{H}_h^\dagger|_{\mathscr{G}}$ to $|\mathscr{H}_h|_{\mathscr{G}}$ by

$$|\mathscr{H}_h|_{\mathscr{G}} \geq (1-\alpha)^{|\mathscr{N}_h|} d_2^{e(\mathscr{H}_h)-e(\mathscr{H}_h^\dagger)} n^{|\mathscr{N}_h|} |\mathscr{H}_h^\dagger|_{\mathscr{G}}.$$

Moreover, by the counting lemma for graphs, $|\mathscr{N}_h|_{\mathscr{G}} \leq 2d_2^{e(\mathscr{N}_h)} n^{|\mathscr{N}_h|}$. Thus the number of copies of $\mathscr{H}_h$ which come from atypical copies of $\mathscr{N}_h$ is at most

$$\begin{aligned}\delta_2' |\mathscr{N}_h|_{\mathscr{G}} |\mathscr{H}_h^\dagger|_{\mathscr{G}} &\leq 2\delta_2' d_2^{e(\mathscr{N}_h)} n^{|\mathscr{N}_h|} \frac{|\mathscr{H}_h|_{\mathscr{G}}}{(1-\alpha)^{|\mathscr{N}_h|} d_2^{e(\mathscr{H}_h)-e(\mathscr{H}_h^\dagger)} n^{|\mathscr{N}_h|}} \\ &= \frac{2\delta_2'}{(1-\alpha)^{|\mathscr{N}_h|} d_2^{e(\mathscr{H}_h)-e(\mathscr{H}_h^\dagger)-e(\mathscr{N}_h)}} |\mathscr{H}_h|_{\mathscr{G}} \leq \sqrt{\delta_2'} |\mathscr{H}_h|_{\mathscr{G}}\end{aligned}$$

where the last inequality follows since $|\mathscr{N}_h| \leq \Delta$ and $e(\mathscr{H}_h) - e(\mathscr{H}_h^\dagger) - e(\mathscr{N}_h) \leq \Delta^2$, as well as $\delta_2' \ll d_2, 1/\Delta$. Combining all the calculations proves the claim. □

4.2. **The hypergraph case.** In the hypergraph case, there will be additional problems because the number of atypical copies of $\mathscr{N}_h$ may be $\delta_3 |\mathscr{N}_h|_{\mathscr{G}}$, and it is not necessarily the case that $\delta_3 \ll d_2$, so the approximation at the end of the proof in Section 4.1 will not generalise. Thus more care will be needed.

We fix new constants β and δ_2' such that

$$\delta_2 \ll \delta_2' \ll d_2, d_3, 1/\Delta$$

and

$$c, \delta_2', \delta_3 \ll \beta \ll \alpha.$$

As mentioned earlier, we will prove Lemma 3 by induction on $|\mathscr{H}|$. We first show that we may assume that the component $\mathscr{C}_1$ of $\mathscr{H}$ containing h satisfies $|\mathscr{C}_1| > \Delta^5$. So suppose this is not the case and let $\mathscr{C}_2 := \mathscr{H} - \mathscr{C}_1$. Every copy of $\mathscr{H}$ can be obtained by first choosing a copy C_2 of $\mathscr{C}_2$ and then choosing a copy C_1 of $\mathscr{C}_1$ which is disjoint from C_2. Thus

$$|\mathscr{H}|_{\mathscr{G}} = \sum_{C_2 \subseteq \mathscr{G}} |\mathscr{C}_1|_{\mathscr{G} - C_2} \geq \sum_{C_2 \subseteq \mathscr{G}} \frac{(1-c)^{\Delta^5}(1-\beta)}{1+\beta} |\mathscr{C}_1|_{\mathscr{G}} \geq (1-3\beta)|\mathscr{C}_1|_{\mathscr{G}} |\mathscr{C}_2|_{\mathscr{G}}. \tag{1}$$

Here we applied the counting lemma (Lemma 4) in $\mathscr{G} - C_2$ and in $\mathscr{G}$ to obtain the first inequality. On the other hand, for $\mathscr{H}_h := \mathscr{H} - h$ we have

$$|\mathscr{H}_h|_{\mathscr{G}} \leq |\mathscr{C}_1 - h|_{\mathscr{G}} |\mathscr{C}_2|_{\mathscr{G}} \leq \frac{(1+\beta)|\mathscr{C}_1|_{\mathscr{G}} |\mathscr{C}_2|_{\mathscr{G}}}{(1-\beta) d_2^{e_2(\mathscr{C}_1) - e_2(\mathscr{C}_1 - h)} d_3^{e_3(\mathscr{C}_1) - e_3(\mathscr{C}_1 - h)} n}, \tag{2}$$

where the second inequality follows from the application of the counting lemma to $\mathscr{C}_1$ and $\mathscr{C}_1 - h$. Combining (1) and (2) gives the result claimed above. Note that in particular, this deals with the start of the induction. So we may assume that $|\mathscr{H}| > \Delta^5$ and that Lemma 3 holds for all complexes with fewer than $|\mathscr{H}|$ vertices. Also, the above assumption on $\mathscr{C}_1$ together with the fact that $\mathscr{H}$ has maximum degree Δ implies that the set of all those vertices of $\mathscr{H}$ which (in the underlying graph) have distance at least 4 to h is nonempty. This will be convenient later on.

For induced subcomplexes $\mathscr{H}'' \subseteq \mathscr{H}' \subseteq \mathscr{H}$ and a copy H'' of $\mathscr{H}''$ in $\mathscr{G}$, we denote by $|H'' \to \mathscr{H}'|_{\mathscr{G}}$ the number of copies of $\mathscr{H}'$ in $\mathscr{G}$ which extend H''. We set

$$\overline{|\mathscr{H}'' \to \mathscr{H}'|} := d_2^{e_2(\mathscr{H}') - e_2(\mathscr{H}'')} d_3^{e_3(\mathscr{H}') - e_3(\mathscr{H}'')} n^{|\mathscr{H}'| - |\mathscr{H}''|}.$$

Thus $\overline{|\mathscr{H}'' \to \mathscr{H}'|}$ is roughly the expected number of ways a copy of $\mathscr{H}''$ in $\mathscr{G}$ could be extended to a copy of $\mathscr{H}'$ if $\mathscr{G}$ were a random complex.

Given the vertex $h \in \mathscr{H}$ as in Lemma 3, we write $\mathscr{N}_h$ for the subcomplex of $\mathscr{H}$ induced by all the neighbours of h in $\mathscr{H}$. We write $\mathscr{B}$ for the subcomplex of $\mathscr{H}$ induced by $V(\mathscr{N}_h) \cup \{h\}$. We call a copy N_h of $\mathscr{N}_h$ in $\mathscr{G}$ *typical* if N_h can be extended to at least $(1-\beta)\overline{|\mathscr{N}_h \to \mathscr{B}|}$

copies of $\mathscr{B}$. If we knew that every copy of $\mathscr{N}_h$ in $\mathscr{G}$ were typical, then the induction step would follow immediately since this would imply that $|\mathscr{H}|_\mathscr{G}$ is roughly

$$\sum_{N_h\subseteq\mathscr{G}} |N_h \to \mathscr{H}_h|_\mathscr{G} |N_h \to \mathscr{B}|_\mathscr{G} \geq (1-\beta)\overline{|\mathscr{N}_h \to \mathscr{B}|} \sum_{N_h\subseteq\mathscr{G}} |N_h \to \mathscr{H}_h|_\mathscr{G} = (1-\beta)\overline{|\mathscr{N}_h \to \mathscr{B}|}|\mathscr{H}_h|_\mathscr{G}.$$

Indeed, this would hold since each copy of $\mathscr{H}$ in $\mathscr{G}$ can be obtained by first choosing a copy N_h of $\mathscr{N}_h$, then extending N_h to some copy of $\mathscr{H}_h$ and then extending N_h to a copy of $\mathscr{B}$. However, the extension lemma (Lemma 5) only implies that almost all copies of $\mathscr{N}_h$ are typical, which makes things more complicated. So let Typ denote the set of all typical copies of $\mathscr{N}_h$ in $\mathscr{G}$ and Atyp the set of all other copies. Lemma 5 implies that

$$|\mathrm{Typ}| \geq (1-\beta)|\mathscr{N}_h|_\mathscr{G}. \tag{3}$$

We now define an analogous set where we refer to the underlying graph P of $\mathscr{G}$ instead of $\mathscr{G}$ itself. More precisely, we call a copy N_h of $\mathscr{N}_h$ *useful* if the following holds: Let $x_1,\dots,x_\ell$ be any distinct vertices of N_h and let $x'_1,\dots,x'_\ell$ be the corresponding vertices in $\mathscr{N}_h$. If a vertex class X_i contains a common neighbour of $x'_1,\dots,x'_\ell$, then in the underlying graph P the common neighbourhood of $x_1,\dots,x_\ell$ in V_i has size $(1\pm\delta_2)^\ell d_2^\ell n$. We denote by Usef the set of all these copies of $\mathscr{N}_h$.

We will now show that almost all copies of $\mathscr{N}_h$ in $\mathscr{G}$ are useful. First recall that since $\mathscr{G}$ respects the partition of $\mathscr{H}$, the bipartite graphs $P[V_i,V_j]$ are (d_2,δ_2)-regular whenever $\mathscr{H}$ contains an edge between X_i and X_j. Together with the fact that $|\mathscr{N}_h|\leq\Delta$ this shows at most $2\Delta^2 2^\Delta\delta_2 n^{|\mathscr{N}_h|}$ of the $|\mathscr{N}_h|$-tuples of vertices in $\mathscr{G}$ do not satisfy the above neighbourhood condition in some of the relevant vertex classes V_i. Indeed, to see this first note that the graph regularity implies that each vertex class contains at most $2\delta_2 n$ vertices having degree $\neq(1\pm\delta_2)d_2|A|$ in any given sufficiently large subset A of V_i. Thus the number of ℓ-tuples $x_1,\dots,x_\ell$ of vertices in $\mathscr{G}$ whose common neighbourhood in V_i has size $\neq(1\pm\delta_2)^\ell d_2^\ell n$ is at most $2\ell\delta_2 n^\ell$. Given $x'_1,\dots,x'_\ell$, there are at most Δ choices for V_i. The bound now follows since there are at most 2^Δ choices for $\{x'_1,\dots,x'_\ell\}$.

On the other hand, Lemma 4 implies that $|\mathscr{N}_h|_\mathscr{G}\geq\frac{1}{2}(d_2d_3)^{\Delta^2}n^{|\mathscr{N}_h|}$. Altogether this shows that

$$|\mathrm{Usef}| \geq |\mathscr{N}_h|_\mathscr{G} - 2\Delta^2 2^\Delta\delta_2 n^{|\mathscr{N}_h|} \geq (1-\delta'_2)|\mathscr{N}_h|_\mathscr{G}. \tag{4}$$

Recall that each copy of $\mathscr{H}$ in $\mathscr{G}$ can be obtained by first choosing a copy N_h of $\mathscr{N}_h$, then extending N_h to some copy H_h of $\mathscr{H}_h$ and then extending N_h to a copy of $\mathscr{B}$. In the final step we have to choose a vertex $x\in\mathscr{G}$ which can play the role of h. If N_h is typical then there are at least $(1-\beta)\overline{|\mathscr{N}_h\to\mathscr{B}|}$ possible choices for x. However, we have to make sure that x does not already lie in H_h. The latter condition excludes at most $cn\leq\beta\overline{|\mathscr{N}_h\to\mathscr{B}|}$ of the possible choices for x. So altogether we have that

$$\begin{aligned}
|\mathscr{H}|_\mathscr{G} &\geq (1-2\beta)\overline{|\mathscr{N}_h\to\mathscr{B}|}\sum_{N_h\in\mathrm{Typ}}|N_h\to\mathscr{H}_h|_\mathscr{G}\\
&= (1-2\beta)\overline{|\mathscr{N}_h\to\mathscr{B}|}\left(\sum_{N_h\subseteq\mathscr{G}}|N_h\to\mathscr{H}_h|_\mathscr{G} - \sum_{N_h\in\mathrm{Atyp}}|N_h\to\mathscr{H}_h|_\mathscr{G}\right)\\
&\geq (1-2\beta)\overline{|\mathscr{N}_h\to\mathscr{B}|}\left(|\mathscr{H}_h|_\mathscr{G} - \sum_{N_h\in\mathrm{Atyp}\cap\mathrm{Usef}}|N_h\to\mathscr{H}_h|_\mathscr{G} - \sum_{N_h\notin\mathrm{Usef}}|N_h\to\mathscr{H}_h|_\mathscr{G}\right). \qquad (5)
\end{aligned}$$

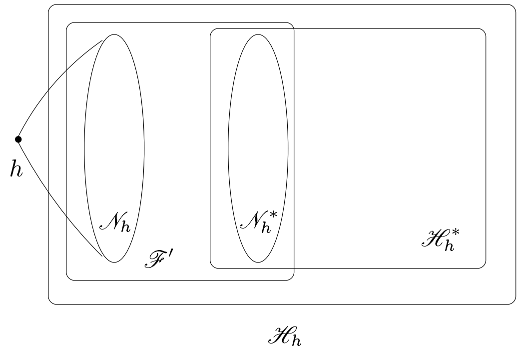

FIGURE 1. The hypergraph $\mathscr{H}$

So our aim now is to prove that each of the last two sums in (5) contributes no more than a small proportion of $|\mathscr{H}_h|_{\mathscr{G}}$. More precisely, we will show that

$$\sum_{N_h \in \mathrm{Atyp} \cap \mathrm{Usef}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} + \sum_{N_h \notin \mathrm{Usef}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq \beta^{1/2} |\mathscr{H}_h|_{\mathscr{G}}. \tag{6}$$

Since $\beta \ll \alpha$ this then proves the induction step. To prove (6), we bound both sums separately. In both cases, we bound $|N_h \to \mathscr{H}_h|_{\mathscr{G}}$ in terms of its average value

$$\frac{1}{|\mathscr{N}_h|_{\mathscr{G}}} \sum_{N_h \subseteq \mathscr{G}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} = \frac{|\mathscr{H}_h|_{\mathscr{G}}}{|\mathscr{N}_h|_{\mathscr{G}}}.$$

Our upper bound for the first sum in (6) will follow easily from the next claim.

Claim 1. *Every useful copy N_h of $\mathscr{N}_h$ in $\mathscr{G}$ satisfies*

$$|N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq \frac{12}{d_3^{2\Delta^3}} \frac{|\mathscr{H}_h|_{\mathscr{G}}}{|\mathscr{N}_h|_{\mathscr{G}}}.$$

To prove this claim, we fix any useful copy N_h of $\mathscr{N}_h$. We let $\mathscr{N}_h^*$ be the subcomplex of $\mathscr{H}$ induced by the vertices which have distance 2 to the vertex set of $\mathscr{N}_h$ in the underlying graph. Recall that our assumption at the beginning of the proof of the lemma implies that $\mathscr{N}_h^*$ is nonempty. Moreover, $|\mathscr{N}_h^*| \leq \Delta^2 |\mathscr{N}_h| \leq \Delta^3$. Let $\mathscr{F}' \subseteq \mathscr{H}$ be the subcomplex of $\mathscr{H}$ that is induced by $V(\mathscr{N}_h) \cup V(\mathscr{N}_h^*)$ and all the vertices in the first neighbourhood of $\mathscr{N}_h$ in $\mathscr{H}_h$ (see Figure 1). So $h \notin V(\mathscr{F}')$. Let $\mathscr{F}$ denote the underlying graph of $\mathscr{F}'$. Given a copy N_h^* of $\mathscr{N}_h^*$ in $\mathscr{G}$, we denote by $|N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}}$ the number of ways the underlying graphs of N_h and N_h^* can be extended into a copy of $\mathscr{F}$ (within the graph P). Similarly, we set

$$\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|} := d_2^{e_2(\mathscr{F}') - e_2(\mathscr{N}_h) - e_2(\mathscr{N}_h^*)} n^{|\mathscr{F}'| - |\mathscr{N}_h| - |\mathscr{N}_h^*|}. \tag{7}$$

Thus $\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|}$ is roughly the expected number of ways the underlying graphs of disjoint copies of $\mathscr{N}_h$ and $\mathscr{N}_h^*$ can be extended into a copy of the graph $\mathscr{F}$, if $\mathscr{G}$ were a random complex.

We define a copy N_h^* of $\mathscr{N}_h^*$ to be *useful with respect to* N_h if it is disjoint from N_h and if the following holds. Let $x_1, \dots, x_\ell$ and $y_1, \dots, y_{\ell^*}$ be any distinct vertices of N_h and N_h^* respectively. Let $x_1', \dots, x_\ell'$ and $y_1', \dots, y_{\ell^*}'$ denote the corresponding vertices in $\mathscr{N}_h$ and $\mathscr{N}_h^*$. If a vertex class X_i of $\mathscr{H}$ contains a common neighbour of $x_1', \dots, x_\ell', y_1', \dots, y_{\ell^*}'$ in $\mathscr{F} - V(\mathscr{N}_h \cup \mathscr{N}_h^*)$ then in the underlying graph P the common neighbourhood of $x_1, \dots, x_\ell, y_1, \dots, y_{\ell^*}$

in V_i has size $(1\pm\delta_2)^{\ell+\ell^*}d_2^{\ell+\ell^*}n$. We denote the set of all such copies of $\mathscr{N}_h^*$ in $\mathscr{G}$ by $\mathrm{Usef}^*(N_h)$. Using the fact that N_h is useful, similarly as in (4) one can show that

$$|\mathrm{Usef}^*(N_h)| \geq (1-\delta_2')|\mathscr{N}_h^*|_{\mathscr{G}}. \tag{8}$$

(Note that the condition that a useful copy of $\mathscr{N}_h^*$ has to be disjoint from N_h does not affect the calculation significantly.) Moreover, since all the bipartite subgraphs forming P are (d_2,δ_2)-regular or empty, we see that every $N_h^* \in \mathrm{Usef}^*(N_h)$ satisfies

$$|N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}} \leq 2\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|}. \tag{9}$$

Indeed, let $\mathscr{F}^* := \mathscr{F} - V(\mathscr{N}_h \cup \mathscr{N}_h^*)$ and let $w_1,\dots,w_p$ denote the vertices of $\mathscr{F}^*$. Let $N'(w_i)$ be the neighbourhood of w_i in $V(\mathscr{N}_h \cup \mathscr{N}_h^*)$ in the graph $\mathscr{F}$. Let W_i denote the set of candidates for w_i inside the vertex class of $\mathscr{G}$ which we aim to embed w_i into. Thus W_i consists of all those vertices in that class which are joined to all the vertices in $N_h \cup N_h^*$ corresponding to $N'(w_i)$. The usefulness of N_h and N_h^* implies that $|W_i| = ((1\pm\delta_2)d_2)^{|N'(w_i)|}n$. In particular, the subgraph of P induced by the W_i's is still regular. So the counting lemma for graphs implies that the number of copies of $\mathscr{F}^*$ induced by the W_i's is at most

$$\frac{3}{2}d_2^{e(\mathscr{F}^*)}\prod_{i=1}^{p}|W_i| \overset{(7)}{\leq} 2\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|},$$

as required.

Let $\mathscr{H}_h^*$ denote the subcomplex of $\mathscr{H}$ obtained by deleting h as well as all the vertices in $\mathscr{F}' - \mathscr{N}_h^*$. Then any copy of $\mathscr{H}_h$ extending N_h can be obtained by first choosing a copy N_h^* of $\mathscr{N}_h^*$, then extending this copy to a copy H_h^* of $\mathscr{H}_h^*$, and then extending the pair N_h, N_h^* into a copy of $\mathscr{F}'$ (which avoids H_h^*). Clearly, there are at most $|N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}}$ ways to choose an extension of N_h, N_h^* into a copy of $\mathscr{F}'$. (Using the latter bound means that we are disregarding any hyperedges of $\mathscr{H}_h$ in $E_3(\mathscr{F}') \setminus E_3(\mathscr{N}_h \cup \mathscr{N}_h^*)$. This is the reason for the error term involving d_3 in the statement of Claim 1.) Thus

$$\begin{aligned}&|N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq \\ &\sum_{N_h^* \in \mathrm{Usef}^*(N_h)} |N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}}|N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} + \sum_{N_h^* \notin \mathrm{Usef}^*(N_h)} |N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}}|N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}}.\end{aligned} \tag{10}$$

The first sum in (10) can be bounded by

$$\begin{aligned}\sum_{N_h^* \in \mathrm{Usef}^*(N_h)} |N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}}|N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} &\overset{(9)}{\leq} 2\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|} \sum_{N_h^* \in \mathrm{Usef}^*(N_h)} |N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} \\ &\leq 2\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} F|}|\mathscr{H}_h^*|_{\mathscr{G}}.\end{aligned} \tag{11}$$

To bound the second sum in (10), let $\mathscr{H}_h' := \mathscr{H}_h^* - \mathscr{N}_h^*$. Note that our assumption at the start of the proof that $|\mathscr{C}_1| > \Delta^5$ implies that $\mathscr{H}_h'$ is nonempty. Then clearly

$$|N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} \leq |\mathscr{H}_h'|_{\mathscr{G}}. \tag{12}$$

We shall estimate $|\mathscr{H}_h'|_{\mathscr{G}}$ in relation to $|\mathscr{H}_h^*|_{\mathscr{G}}$. Let $s := |\mathscr{N}_h^*| = |\mathscr{H}_h^*| - |\mathscr{H}_h'|$ and suppose that $w_1,\dots,w_s$ are the vertices in $\mathscr{H}_h^* - \mathscr{H}_h' = \mathscr{N}_h^*$. For all $i = 1,\dots,s$, we let $\mathscr{N}_i'$ be the subcomplex induced by the neighbourhood of w_i in $\mathscr{H}_h^* - \{w_{i+1},\dots,w_s\}$. Let $\mathscr{B}_i'$ be the subcomplex of $\mathscr{H}$ induced by w_i and the vertices in $\mathscr{N}_i'$. Then our induction hypothesis implies that

$$\begin{aligned} |\mathscr{H}_h^*|_{\mathscr{G}} &\geq (1-\alpha)^{|\mathscr{H}_h^*|-|\mathscr{H}_h'|} \left(\prod_{i=1}^{|\mathscr{H}_h^*|-|\mathscr{H}_h'|} \overline{|\mathscr{N}_i' \to \mathscr{B}_i'|} \right) |\mathscr{H}_h'|_{\mathscr{G}} \\ &= ((1-\alpha)n)^{|\mathscr{H}_h^*|-|\mathscr{H}_h'|} d_2^{e_2(\mathscr{H}_h^*)-e_2(\mathscr{H}_h')} d_3^{e_3(\mathscr{H}_h^*)-e_3(\mathscr{H}_h')} |\mathscr{H}_h'|_{\mathscr{G}}. \qquad (13) \end{aligned}$$

(In fact, one reason for our choice of the induction hypothesis is that it allows us to relate $|\mathscr{H}_h^*|_{\mathscr{G}}$ to $|\mathscr{H}_h'|_{\mathscr{G}}$ as in (13).) But our assumption on the maximum degree of $\mathscr{H}$ implies that $e_i(\mathscr{H}_h^*) - e_i(\mathscr{H}_h') \leq \Delta |\mathscr{N}_h^*| \leq \Delta^4$ for $i = 2, 3$. So

$$|\mathscr{H}_h^*|_{\mathscr{G}} \geq \frac{1}{2} n^{|\mathscr{H}_h^*|-|\mathscr{H}_h'|} (d_2 d_3)^{\Delta^4} |\mathscr{H}_h'|_{\mathscr{G}}$$

as $\alpha \ll 1/\Delta$. Since $|\mathscr{H}_h^*| - |\mathscr{H}_h'| = |\mathscr{N}_h^*|$ the last inequality together with (8), (12) and the fact that $\delta_2' \ll d_2, d_3, 1/\Delta$ imply that

$$\sum_{N_h^* \notin \mathrm{Usef}^*(N_h)} |N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} \leq \delta_2' |\mathscr{N}_h^*|_{\mathscr{G}} |\mathscr{H}_h'|_{\mathscr{G}} \leq |\mathscr{H}_h^*|_{\mathscr{G}} \frac{2\delta_2' |\mathscr{N}_h^*|_{\mathscr{G}}}{(d_2 d_3)^{\Delta^4} n^{|\mathscr{N}_h^*|}} \leq \sqrt{\delta_2'} |\mathscr{H}_h^*|_{\mathscr{G}}. \qquad (14)$$

In the final inequality, we also used the crude bound $|\mathscr{N}_h^*|_{\mathscr{G}} \leq n^{|\mathscr{N}_h^*|}$. We can now bound the second sum in (10) by

$$\begin{aligned} \sum_{N_h^* \notin \mathrm{Usef}^*(N_h)} |N_h, N_h^* \xrightarrow{P} \mathscr{F}|_{\mathscr{G}} |N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} &\leq n^{|\mathscr{F}|-|\mathscr{N}_h|-|\mathscr{N}_h^*|} \sum_{N_h^* \notin \mathrm{Usef}^*(N_h)} |N_h^* \to \mathscr{H}_h^*|_{\mathscr{G}} \\ &\overset{(14)}{\leq} \overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|} |\mathscr{H}_h^*|_{\mathscr{G}} \frac{\sqrt{\delta_2'} n^{|\mathscr{F}|-|\mathscr{N}_h|-|\mathscr{N}_h^*|}}{\overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|}} \\ &\overset{(7)}{\leq} \overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|} |\mathscr{H}_h^*|_{\mathscr{G}}. \qquad (15) \end{aligned}$$

Indeed, to see the last inequality recall that $\delta_2' \ll d_2, 1/\Delta$. Inequalities (10), (11) and (15) together now imply that

$$|N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq 3 \overline{|\mathscr{N}_h, \mathscr{N}_h^* \xrightarrow{P} \mathscr{F}|} |\mathscr{H}_h^*|_{\mathscr{G}}. \qquad (16)$$

Similarly as in (13) one can use the induction hypothesis to show that

$$\begin{aligned} |\mathscr{H}_h|_{\mathscr{G}} &\geq ((1-\alpha)n)^{|\mathscr{H}_h|-|\mathscr{H}_h^*|} d_2^{e_2(\mathscr{H}_h)-e_2(\mathscr{H}_h^*)} d_3^{e_3(\mathscr{H}_h)-e_3(\mathscr{H}_h^*)} |\mathscr{H}_h^*|_{\mathscr{G}} \\ &\geq \frac{1}{2} n^{|\mathscr{F}'|-|\mathscr{N}_h^*|} d_2^{e_2(\mathscr{H}_h)-e_2(\mathscr{H}_h^*)} d_3^{e_3(\mathscr{H}_h)-e_3(\mathscr{H}_h^*)} |\mathscr{H}_h^*|_{\mathscr{G}}. \end{aligned}$$

Observe that $e_i(\mathscr{H}_h) - e_i(\mathscr{H}_h^*) = e_i(\mathscr{F}') - e_i(\mathscr{N}_h^*)$ for $i = 2, 3$. Moreover, the counting lemma for graphs implies that $|\mathscr{N}_h|_{\mathscr{G}} \leq 2n^{|\mathscr{N}_h|} d_2^{e_2(\mathscr{N}_h)}$. Together with (7) and (16) this shows that

$$|N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq \frac{6 d_2^{e_2(\mathscr{F}')-e_2(\mathscr{N}_h)-e_2(\mathscr{N}_h^*)} n^{|\mathscr{F}'|-|\mathscr{N}_h|-|\mathscr{N}_h^*|} |\mathscr{N}_h|_{\mathscr{G}}}{n^{|\mathscr{F}'|-|\mathscr{N}_h^*|} d_2^{e_2(\mathscr{F}')-e_2(\mathscr{N}_h^*)} d_3^{e_3(\mathscr{F}')-e_3(\mathscr{N}_h^*)}} \cdot \frac{|\mathscr{H}_h|_{\mathscr{G}}}{|\mathscr{N}_h|_{\mathscr{G}}} \leq \frac{12 |\mathscr{H}_h|_{\mathscr{G}}}{d_3^{e_3(\mathscr{F}')-e_3(\mathscr{N}_h^*)} |\mathscr{N}_h|_{\mathscr{G}}}.$$

But $e_3(\mathscr{F}') - e_3(\mathscr{N}_h^*) \leq \Delta |\mathscr{F} - \mathscr{N}_h^*| \leq \Delta(\Delta + \Delta^2) \leq 2\Delta^3$. This completes the proof of Claim 1.

In order to give an upper bound on the second sum in (6) we will need the following claim.

Claim 2. *Every copy N_h of $\mathscr{N}_h$ satisfies*
$$|N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq \frac{2}{(d_2d_3)^{\Delta^2}}\frac{|\mathscr{H}_h|_{\mathscr{G}}}{|\mathscr{N}_h|_{\mathscr{G}}}.$$

Let $\mathscr{H}_h^- := \mathscr{H}_h - \mathscr{N}_h$. Then, very crudely,
$$|N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq |\mathscr{H}_h^-|_{\mathscr{G}}. \tag{17}$$
But similarly as in (13) we have that
$$\begin{aligned} |\mathscr{H}_h|_{\mathscr{G}} &\geq ((1-\alpha)n)^{|\mathscr{N}_h|} d_2^{e_2(\mathscr{H}_h)-e_2(\mathscr{H}_h^-)} d_3^{e_3(\mathscr{H}_h)-e_3(\mathscr{H}_h^-)} |\mathscr{H}_h^-|_{\mathscr{G}} \\ &\geq \frac{1}{2}|\mathscr{N}_h|_{\mathscr{G}} d_2^{e_2(\mathscr{H}_h)-e_2(\mathscr{H}_h^-)} d_3^{e_3(\mathscr{H}_h)-e_3(\mathscr{H}_h^-)} |\mathscr{H}_h^-|_{\mathscr{G}} \\ &\geq \frac{1}{2}|\mathscr{N}_h|_{\mathscr{G}} d_2^{\Delta^2} d_3^{\Delta^2} |\mathscr{H}_h^-|_{\mathscr{G}}. \end{aligned}$$
In the final line we used the fact that $|\mathscr{H}_h - \mathscr{H}_h^-| = |\mathscr{N}_h| \leq \Delta$. Together with (17) this implies Claim 2.

Claims 1 and 2 now immediately imply (6). Indeed, using (3) and (4) and the facts that $\delta_2' \ll d_2, d_3, 1/\Delta$ and $\delta_2' \ll \beta \ll d_3, 1/\Delta$ we see that
$$\sum_{N_h \in \mathrm{Atyp}\cap \mathrm{Usef}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} + \sum_{N_h \notin \mathrm{Usef}} |N_h \to \mathscr{H}_h|_{\mathscr{G}} \leq \left(\frac{12\beta}{d_3^{2\Delta^3}} + \frac{2\delta_2'}{(d_2d_3)^{\Delta^2}}\right)|\mathscr{H}_h|_{\mathscr{G}} \leq \beta^{1/2}|\mathscr{H}_h|_{\mathscr{G}},$$
as required. This completes the proof of Lemma 3.

5. Proof of Lemma 4

In this section, we indicate how Lemma 4 follows easily from the version of the counting lemma proved in [19] (Lemma 7 below). Full details can be found in [2]. Several related versions of the counting lemma can be found in [22]. In order to state Lemma 7, we need the following definition. Given a complex $\mathscr{H}$ with vertices $x_1, \dots, x_t$, a complex $\mathscr{G}$ is called $(d_3, \delta_3, d_2, \delta_2, r, \mathscr{H})$-regular if $\mathscr{G}$ is t-partite with vertex classes $V_1, \dots, V_t$ and satisfies the following properties:

- Let P denote the underlying graph of $\mathscr{G}$. For every edge $x_ix_j \in E_2(\mathscr{H})$ the bipartite graph $P[V_i, V_j]$ is (d_2, δ_2)-regular.
- For every hyperedge $e = x_hx_ix_j \in E_3(\mathscr{H})$ there exists $d_e \geq d_3$ such that the triad $P[V_h, V_i, V_j]$ is (d_e, δ_3, r)-regular with respect to the underlying hypergraph of $\mathscr{G}$.

In this case, we say that a labelled copy H of $\mathscr{H}$ in $\mathscr{G}$ is *partition-respecting* if for all $i \in [t]$ the vertex of H corresponding to x_i is contained in V_i.

Lemma 7. *Let t, r, n_0 be positive integers and let $\beta, d_2, d_3, \delta_2, \delta_3$ be positive constants such that*
$$1/n_0 \ll 1/r \ll \delta_2 \ll \min\{\delta_3, d_2\} \leq \delta_3 \ll \beta, d_3, 1/t.$$
Then the following holds for all integers $n \geq n_0$. Suppose that $\mathscr{H}$ is a complex with vertices $x_1, \dots, x_t$. Suppose also that $\mathscr{G}$ is a $(d_3, \delta_3, d_2, \delta_2, r, \mathscr{H})$-regular complex with vertex classes $V_1, \dots, V_t$, all of size n. Then $\mathscr{G}$ contains at least
$$(1-\beta)n^t d_2^{e_2(\mathscr{H})} \prod_{e \in E_3(\mathscr{H})} d_e$$
labelled partition-respecting copies of $\mathscr{H}$.

Note that the difference to Lemma 4 is that Lemma 7 only gives a lower bound and every vertex of $\mathscr{H}$ is to be embedded into a different vertex class of $\mathscr{G}$. On the other hand, Lemma 7 allows for different 'hypergraph densities' between the clusters. (Actually, the proof below would permit this in Lemma 4 as well, see [2].)

To derive Lemma 4, first assume that each of the vertex classes X_i of $\mathscr{H}$ contains exactly one vertex (i.e. we want to embed every vertex of $\mathscr{H}$ into a different vertex class of $\mathscr{G}$). In this case we only have to deduce the upper bound in Lemma 4 from the lower bound in Lemma 7. As the (simple) proof of this is quite similar to the proof for the complete case in [17], we just describe the main idea here. So consider any $\mathscr{D} \subseteq E_3(\mathscr{H})$. Now we construct a complex $\mathscr{G}_{\mathscr{D}}$ from $\mathscr{G}$ as follows. For any triple $hij \in \mathscr{D}$, we delete all hyperedges from $\mathscr{G}[V_h, V_i, V_j]$ and add as hyperedges all those triangles contained in the underlying graph induced by V_h, V_i and V_j which did not form a hyperedge in $\mathscr{G}[V_h, V_i, V_j]$. (Thus $\mathscr{G}_{\mathscr{D}}$ may be viewed as a 'partial' complement of $\mathscr{G}$.) Now let $|\mathscr{H}^{(2)}|_{\mathscr{G}}$ denote the number of labelled partition-respecting copies of the underlying graph of $\mathscr{H}$ in (the underlying graph of) $\mathscr{G}$. Then it is easy to see that

$$\sum_{\mathscr{D} \subseteq E_3(\mathscr{H})} |\mathscr{H}|_{\mathscr{G}_{\mathscr{D}}} = |\mathscr{H}^{(2)}|_{\mathscr{G}}.$$

(This is where we need to assume that we are considering the special case when every vertex of $\mathscr{H}$ is embedded into a different vertex class of $\mathscr{G}$.) We can use the (easy) counting lemma for graphs to estimate $|\mathscr{H}^{(2)}|_{\mathscr{G}}$. Moreover, note that we are aiming for an upper bound on the summand where $\mathscr{D}$ is empty. But we can obtain this since we can apply Lemma 7 to all the remaining summands. (This is where we need that Lemma 7 allows for different 'hypergraph densities'.) A simple calculation gives the desired result.

So it remains to deduce the general case in Lemma 4 from the special case when each of the vertex classes X_i of $\mathscr{H}$ contains exactly one vertex. To achieve this, consider the following construction. Let $\mathscr{G}_1$ be the complex obtained from $\mathscr{G}$ by taking $|X_1|$ copies of $\mathscr{G}$ and identifying them in $V(\mathscr{G}) \setminus V_1$. In other words, we blow up V_1 into $|X_1|$ copies, i.e. V_1 is replaced by classes V_{1i} with $1 \leq i \leq |X_1|$. Now let $\mathscr{G}_2$ be the hypergraph obtained from $\mathscr{G}_1$ by taking $|X_2|$ copies of $\mathscr{G}_1$ and identifying them in $V(\mathscr{G}_1) \setminus V_2$. We continue in this way to obtain an $|\mathscr{H}|$-partite hypergraph $\mathscr{G}^* := \mathscr{G}_k$ (so we have blown up each V_i into $|X_i|$ copies). Now view $\mathscr{H}$ as a $|\mathscr{H}|$-partite complex $\mathscr{H}^*$ with vertex classes X_{ji}, each consisting of a single vertex, where $1 \leq j \leq k$ and $1 \leq i \leq |X_j|$. Note that every labelled partition-respecting copy of $\mathscr{H}$ in $\mathscr{G}$ yields a distinct labelled partition-respecting copy of $\mathscr{H}^*$ in $\mathscr{G}^*$ (where in the latter case, X_{ji} is mapped to V_{ji}). So $|\mathscr{H}|_{\mathscr{G}} \leq |\mathscr{H}^*|_{\mathscr{G}^*}$. On the other hand, if a labelled partition-respecting copy of $\mathscr{H}^*$ in $\mathscr{G}^*$ does not correspond to a labelled partition-respecting copy of $\mathscr{H}$ in $\mathscr{G}$ then this means that this copy of $\mathscr{H}^*$ uses (at least) two 'twin' vertices in $\mathscr{G}^*$ which correspond to the same vertex in $\mathscr{G}$. There are at most $|\mathscr{H}|n$ possibilities for choosing the first twin vertex, at most $|\mathscr{H}|$ possibilities for the second twin vertex and at most $n^{|\mathscr{H}|-2}$ possibilities for the remaining vertices. Thus $|\mathscr{H}|_{\mathscr{G}} \geq |\mathscr{H}^*|_{\mathscr{G}^*} - |\mathscr{H}|^2 n^{|\mathscr{H}|-1}$. We can now obtain the desired upper and lower bound on $|\mathscr{H}|_{\mathscr{G}}$ from the bounds on $|\mathscr{H}^*|_{\mathscr{G}^*}$ which we already know. (Note that these bounds imply that the 'error term' $|\mathscr{H}|^2 n^{|\mathscr{H}|-1}$ is negligible compared to $|\mathscr{H}^*|_{\mathscr{G}^*}$.)

6. Proof of Lemma 5

Throughout this section, whenever we refer to copies of $\mathscr{H}$ or $\mathscr{H}'$ in $\mathscr{G}$ we mean that these copies will be labelled and partition-respecting without mentioning this explicitly. As in Section 4 we denote such copies by H and H' respectively. Thus, given any copy H of $\mathscr{H}$, we have to estimate the number of extensions of H into copies of $\mathscr{H}'$ in $\mathscr{G}$. Recall that we denote the number of all these extensions by $|H \to \mathscr{H}'|_{\mathscr{G}}$. Also, as in Section 4 we write

$$\overline{|\mathscr{H} \to \mathscr{H}'|} := n^{t'-t} d_2^{e_2(\mathscr{H}')-e_2(\mathscr{H})} d_3^{e_3(\mathscr{H}')-e_3(\mathscr{H})}.$$

We will argue similarly as in the proof of Corollary 14 in [20] or the proof of Lemma 6.6 in [8]. Namely, we use the following fact that can be deduced from the Cauchy-Schwartz inequality (see also Lemma 6.5 in [8]):

Fact 8. *For any $\beta > 0$ there exists $\delta > 0$, such that for any collection of non-negative real numbers $x_1, \dots, x_N$ satisfying*

$$\sum_{i=1}^{N} x_i = (1 \pm \delta)AN \quad \text{and} \quad \sum_{i=1}^{N} x_i^2 = (1 \pm \delta)A^2 N \tag{18}$$

for some $A \geq 0$, all except at most βN of the x_i's lie in the interval $(1 \pm \beta)A$.

In our case, the collection $\{x_i\}_{i=1}^N$ will be $\{|H \to \mathscr{H}'|_{\mathscr{G}}\}_{H \subseteq \mathscr{G}}$ (so $N := |\mathscr{H}|_{\mathscr{G}}$) and we set

$$A := \overline{|\mathscr{H} \to \mathscr{H}'|}.$$

Given β as in Lemma 5, we let $\delta = \delta(\beta)$ be as in Fact 8. We may assume that the hierarchy of constants in Lemma 5 was chosen such that $\delta_3 \ll \delta$. To prove Lemma 5 it suffices to show that

$$\sum_{H \subseteq \mathscr{G}} |H \to \mathscr{H}'|_{\mathscr{G}} = (1 \pm \delta)A|\mathscr{H}|_{\mathscr{G}}, \tag{19}$$

and

$$\sum_{H \subseteq \mathscr{G}} |H \to \mathscr{H}'|_{\mathscr{G}}^2 = (1 \pm \delta)A^2|\mathscr{H}|_{\mathscr{G}}. \tag{20}$$

The counting lemma (Lemma 4) implies that

$$|\mathscr{H}'|_{\mathscr{G}} = (1 \pm \delta/8) n^{t'} d_2^{e_2(\mathscr{H}')} d_3^{e_3(\mathscr{H}')}$$

and

$$|\mathscr{H}|_{\mathscr{G}} = (1 \pm \delta/8) n^{t} d_2^{e_2(\mathscr{H})} d_3^{e_3(\mathscr{H})}. \tag{21}$$

It follows that

$$\sum_{H \subseteq \mathscr{G}} |H \to \mathscr{H}'|_{\mathscr{G}} = |\mathscr{H}'|_{\mathscr{G}} = (1 \pm \delta)A|\mathscr{H}|_{\mathscr{G}},$$

as required in (19).

To show (20) we have to estimate $\sum_{H \subseteq \mathscr{G}} |H \to \mathscr{H}'|_{\mathscr{G}}^2$. Thus consider any copy H of $\mathscr{H}$ in $\mathscr{G}$. Then $|H \to \mathscr{H}'|_{\mathscr{G}}^2$ corresponds to the number of pairs (H_1', H_2') of copies of $\mathscr{H}'$ in $\mathscr{G}$ extending H. Now let $\tilde{\mathscr{H}}'$ denote the complex which is obtained from two disjoint copies of $\mathscr{H}'$ by identifying them in $V(\mathscr{H})$. Then the copies of $\tilde{\mathscr{H}}'$ in $\mathscr{G}$ which extend H correspond bijectively to those pairs (H_1', H_2') which meet precisely in H and are disjoint otherwise.[1] On the other hand, at most $(t'-t)^2 n^{2(t'-t)-1}$ of the pairs (H_1', H_2') meet in some vertex outside H. Thus

$$\sum_{H \subseteq \mathscr{G}} |H \to \mathscr{H}'|_{\mathscr{G}}^2 \leq \sum_{H \subseteq \mathscr{G}} \left(|H \to \tilde{\mathscr{H}}'|_{\mathscr{G}} + (t'-t)^2 n^{2(t'-t)-1} \right) \leq |\tilde{\mathscr{H}}'|_{\mathscr{G}} + (t'-t)^2 n^{2t'-t-1} \tag{22}$$

[1] Again, we only consider the partition-respecting copies of $\tilde{\mathscr{H}}'$ in $\mathscr{G}$, i.e. if a vertex $\tilde{x} \in \tilde{\mathscr{H}}'$ corresponds to a vertex $x \in \mathscr{H}'$ which lies in X_i, then $\tilde{x}$ has to be embedded into V_i.

and clearly also

$$\sum_{H\subseteq \mathscr{G}} |H \to \mathscr{H}'|^2_{\mathscr{G}} \geq |\tilde{\mathscr{H}}'|_{\mathscr{G}}. \tag{23}$$

But the counting lemma implies that

$$|\tilde{\mathscr{H}}'|_{\mathscr{G}'} = (1 \pm \delta/8) n^{2t'-t} d_2^{2e_2(\mathscr{H}')-e_2(\mathscr{H})} d_3^{2e_3(\mathscr{H}')-e_3(\mathscr{H})} \overset{(21)}{=} (1 \pm \delta/2) A^2 |\mathscr{H}|_{\mathscr{G}}.$$

In particular, $(t'-t)^2 n^{2t'-t-1} \leq \delta|\tilde{\mathscr{H}}'|_{\mathscr{G}'}/8 \leq \delta A^2|\mathscr{H}|_{\mathscr{G}}/2$. Together with (22) and (23) this implies (20) and completes the proof of Lemma 5. Note that the proof above also allows for different 'hypergraph densities' between the clusters in Lemma 4, but we have not included this to avoid making the statement more technical.

7. The regularity lemma for 3-uniform hypergraphs

7.1. **The Regularity Lemma – definitions and statement.** The main purpose of this section is to introduce the regularity lemma for 3-uniform hypergraphs due to Frankl and Rödl [6]. As in the proof of the graph analogue of Theorem 1 we shall make use of it in order to obtain the necessary regular complex $\mathscr{G}$ to which we then apply the embedding lemma (see Section 8 for the details). Before we can state it, we will collect the necessary definitions.

Definition 9 ($(\ell, t, \varepsilon_1, \varepsilon_2)$-partition)**.** Let V be a set. An *$(\ell, t, \varepsilon_1, \varepsilon_2)$-partition $\mathscr{P}$ of V* is a partition into $V_0, V_1, \dots, V_t$ together with families $(P^{ij}_\alpha)^{\ell_{ij}}_{\alpha=0}$ $(1 \leq i < j \leq t)$ of edge-disjoint bipartite graphs such that

(i) $|V_1| = \cdots = |V_t| = \lfloor |V|/t \rfloor =: n$,
(ii) $\ell_{ij} \leq \ell$ for all pairs $1 \leq i < j \leq t$,
(iii) $\bigcup_{\alpha=0}^{\ell_{ij}} P^{ij}_\alpha$ is the complete bipartite graph with vertex classes V_i and V_j (for all pairs $1 \leq i < j \leq t$),
(iv) all but at most $\varepsilon_1\binom{t}{2}n^2$ edges of the complete t-partite graph $K[V_1, \dots, V_t]$ with vertex classes $V_1, \dots, V_t$ lie in some ε_2-regular graph P^{ij}_α,
(v) for all but at most $\varepsilon_1\binom{t}{2}$ pairs V_i, V_j $(1 \leq i < j \leq t)$ we have $e(P^{ij}_0) \leq \varepsilon_1 n^2$ and
$$|d_{P^{ij}_\alpha}(V_i, V_j) - 1/\ell| \leq \varepsilon_2$$
for all $\alpha = 1, \dots, \ell_{ij}$.

Definition 10 ((δ_3, r)-regular $(\ell, t, \varepsilon_1, \varepsilon_2)$-partition)**.** Suppose that $\mathscr{G}$ is a 3-uniform hypergraph and that $V_0, V_1, \dots, V_t$ is an $(\ell, t, \varepsilon_1, \varepsilon_2)$-partition of the vertex set $V(\mathscr{G})$ of $\mathscr{G}$. Set $n := |V_1| = \cdots = |V_t|$. Recall that a triad is a 3-partite graph of the form $P = P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma$ and that $t(P)$ denotes the number of triangles in P. We say that the partition $V_0, V_1, \dots, V_t$ is *(δ_3, r)-regular* if

$$\sum_{\text{irregular}} t(P) < \delta_3 |\mathscr{G}|^3,$$

where $\sum_{\text{irregular}}$ denotes the sum over all triads P which are not (δ_3, r)-regular with respect to $\mathscr{G}$.

We can now state the regularity lemma for 3-uniform hypergraphs which was proved by Frankl and Rödl [6].

Theorem 11 (Regularity lemma for 3-uniform hypergraphs)**.** *For all δ_3 and ε_1 with $0 < \varepsilon_1 \le 2\delta_3^4$, for all $t_0, \ell_0 \in \mathbb{N}$ and for all integer-valued functions $r = r(t,\ell)$ and all decreasing functions $\varepsilon_2(\ell)$ with $0 < \varepsilon_2(\ell) \le 1/\ell$, there exist integers T_0, L_0 and N_0 such that the vertex set of any 3-uniform hypergraph $\mathscr{G}$ of order $|\mathscr{G}| \ge N_0$ admits a (δ_3, r)-regular $(\ell, t, \varepsilon_1, \varepsilon_2(\ell))$-partition for some t and ℓ satisfying $t_0 \le t \le T_0$ and $\ell_0 \le \ell \le L_0$.*

The elements $V_1, \dots, V_t$ of the $(\ell, t, \varepsilon_1, \varepsilon_2(\ell))$-partition given by Theorem 11 are called *clusters*. V_0 is the *exceptional set*.

7.2. **Definition of the reduced hypergraph.** When we apply the graph regularity lemma to a graph G, we often consider the so called reduced graph, whose vertices are the clusters V_i and whose edges correspond to those pairs of clusters which induce an ε-regular bipartite graph. Analogously, we will now define a 3-uniform reduced hypergraph.

In the proof of Theorem 1 in the next section, we will fix positive constants satisfying the following hierarchy:

$$\varepsilon_1, 1/t_0, 1/\ell_0 \ll \delta_3 \ll \varepsilon_3 \ll 1/\Delta \tag{24}$$

where $\ell_0, t_0 \in \mathbb{N}$ and we choose these constants successively from right to left as explained earlier. Next, for all $\ell \ge \ell_0$ and all $t \ge t_0$ we define functions $r(t,\ell)$ and $\varepsilon_2(\ell)$ satisfying the following properties:

$$\frac{1}{r(t,\ell)} \ll \varepsilon_2(\ell) \ll \frac{1}{\ell}, \delta_3, \varepsilon_1. \tag{25}$$

Suppose that with this choice of constants we have applied the regularity lemma to a 3-uniform hypergraph $\mathscr{G}$. In particular, this gives an integer ℓ. We then define constants d_2 and δ_2 by

$$d_2 := 1/\ell \quad \text{and} \quad \delta_2 := \sqrt{\varepsilon_2}. \tag{26}$$

In order to define the reduced hypergraph corresponding to the partition of $V(\mathscr{G})$ obtained from the regularity lemma, we need the following definitions.

Definition 12 (good pair V_iV_j)**.** We call a pair V_iV_j $(1 \le i < j \le t)$ of clusters *good* if it satisfies the following two properties:

- $e(P_0^{ij}) \le \varepsilon_1 n^2$ and $|d_{P_\alpha^{ij}}(V_i, V_j) - d_2| \le \varepsilon_2$ for all $\alpha = 1, \dots, \ell_{ij}$. (This means that V_iV_j does not belong to the at most $\varepsilon_1\binom{t}{2}$ exceptional pairs described in Definition 9(v).)
- at most $\varepsilon_3\ell/6$ of the bipartite graphs P_α^{ij} $(1 \le \alpha \le \ell_{ij})$ are not (d_2, δ_2)-regular.

Later on, we will use the fact that the first condition in Definition 12 implies that $\ell_{ij} \ge \ell/2$ since $d_2 = 1/\ell$. An observation from [15] states that almost all pairs of clusters are good, but we will not make use of this explicitly.

Definition 13 (good triple $V_iV_jV_k$)**.** We call a triple $V_iV_jV_k$ $(1 \le i < j < k \le t)$ of clusters *good* if both of the following hold:

- each of the pairs V_iV_j, V_jV_k and V_iV_k is good,
- at most $\varepsilon_3\ell^3$ of the triads induced by V_i, V_j, V_k are not (δ_3, r)-regular with respect to $\mathscr{G}$.

The next proposition, which follows immediately from Proposition 5.12 in [15], states that only a small fraction of the triples $V_iV_jV_k$ are not good.

Proposition 14. *At most $40\delta_3\binom{t}{3}/\varepsilon_3$ triples $V_iV_jV_k$ of clusters are not good.*

We are now ready to define the reduced hypergraph $\mathscr{R}$.

Definition 15 (Reduced hypergraph)**.** The vertices of the *reduced hypergraph* $\mathscr{R}$ are all the clusters $V_1, \dots, V_t$. The hyperedges of $\mathscr{R}$ are precisely the good triples $V_iV_jV_k$.

Thus, like $\mathscr{G}$, also $\mathscr{R}$ is a 3-uniform hypergraph.

8. Proof of Theorem 1

In this section, we put together all the previous tools to prove Theorem 1. We will also make use of the following well-known result (see e.g. [4]).

Lemma 16. *For all $k \in \mathbb{N}$ there exists a constant $c_0 = c_0(k) < 1$ such that if $\mathscr{R}$ is a 3-uniform hypergraph on $t \geq k$ vertices, and if $e(\mathscr{R}) \geq c_0\binom{t}{3}$, then $\mathscr{R}$ contains a copy of $K_k^{(3)}$.*

We will also use the existence of hypergraph Ramsey numbers, without needing any explicit upper bounds. Roughly speaking, the proof of Theorem 1 proceeds as follows. Consider any red/blue colouring of the hyperedges of $K_m^{(3)}$, where m is a sufficiently large integer (but m will be linear in $|\mathscr{H}|$). We apply the hypergraph regularity lemma to the red subhypergraph $\mathscr{G}_{red}$ to obtain the reduced hypergraph $\mathscr{R}$, and show that $\mathscr{R}$ satisfies the conditions of Lemma 16 with $k := R(K_{2\Delta+1}^{(3)})$. Thus $\mathscr{R}$ will contain a copy of $K_k^{(3)}$. This copy corresponds to k clusters such that for each triple of these clusters almost all the triads are regular with respect to the red hypergraph $\mathscr{G}_{red}$. We will then show that between each pair V_i, V_j of these clusters one can choose one of the bipartite graphs P_α^{ij} in such a way that any triad P_{hij} consisting of the chosen bipartite graphs is regular with respect to $\mathscr{G}_{red}$. Let P denote the k-partite graph formed by all the chosen bipartite graphs. We then consider the following red/blue colouring of $K_k^{(3)}$. We colour the hyperedge hij with red if the triad P_{hij} has density at least $1/2$ with respect to $\mathscr{G}_{red}$ and blue otherwise. Since $k = R(K_{2\Delta+1}^{(3)})$ we can find a monochromatic $K_{2\Delta+1}^{(3)}$. If it is red then we can apply the embedding lemma to the corresponding $(2\Delta+1)$-partite subhypergraph of $\mathscr{G}_{red}$ and the corresponding $(2\Delta+1)$-partite subgraph P' of P to find a red copy of $\mathscr{H}$. This can be done since the chromatic number of $\mathscr{H}$ is at most $2\Delta+1$ as $\Delta(\mathscr{H}) \leq \Delta$. If our monochromatic copy of $K_{2\Delta+1}^{(3)}$ is blue then we can apply the embedding lemma to the $(2\Delta+1)$-partite subhypergraph of the blue hypergraph $\mathscr{G}_{blue} \subseteq K_m^{(3)}$ and P'.

Proof of Theorem 1. Let $m \in \mathbb{N}$ be large enough for all subsequent calculations to hold. We will check later that we can choose m to be linear in $|\mathscr{H}|$. Consider any red/blue-colouring of the hyperedges of $K_m^{(3)}$. Let $\mathscr{G}_{red}$ be the red and $\mathscr{G}_{blue}$ be the blue subhypergraph on $V(K_m^{(3)})$. We may assume without loss of generality that $e(\mathscr{G}_{red}) \geq e(\mathscr{G}_{blue})$. We apply the hypergraph regularity lemma to $\mathscr{G}_{red}$ with parameters

$$t_0 \geq R(K_{2\Delta+1}^{(3)}) =: k$$

as well as $\ell_0, \delta_3, \varepsilon_1$ and functions $r(t,\ell)$ and $\varepsilon_2(\ell)$ satisfying the hierarchies in (24) and (25).

Thus we obtain a set of clusters $V_1, \dots, V_t$, each of size n say, together with a partition $(P_\alpha^{ij})_{\alpha=0}^{\ell_{ij}}$ of the complete bipartite graph between clusters V_i and V_j (for all $1 \leq i < j \leq t$). We define d_2 and δ_2 as in (26) and let $\mathscr{R}$ denote the reduced hypergraph. Proposition 14 implies that $\mathscr{R}$ has at least $(1-\varepsilon)\binom{t}{3}$ hyperedges, where $\varepsilon := 40\delta_3/\varepsilon_3$. Thus (24) implies that $e(\mathscr{R}) \geq (1-\varepsilon)\binom{|\mathscr{R}|}{3} > c_0\binom{|\mathscr{R}|}{3}$, where c_0 is as defined in Lemma 16. Since $|\mathscr{R}| \geq t_0 \geq k$, this means that we can apply Lemma 16 to $\mathscr{R}$ to obtain a copy of $K_k^{(3)}$ in $\mathscr{R}$. Without loss of generality we may assume that the vertices of this copy are the clusters $V_1, \dots, V_k$.

As indicated before, our next aim is to show that for each of the $\binom{k}{2}$ pairs V_iV_j (with $1 \le i < j \le k$) one can choose one of the bipartite graphs P_α^{ij} in such a way that each of them is (d_2, δ_2)-regular and such that each of the $\binom{k}{3}$ triads formed by the chosen bipartite graphs is (δ_3, r)-regular with respect to $\mathscr{G}_{red}$. We will denote the chosen bipartite graph between V_i and V_j by P_{ij} and the triad between V_h, V_i and V_j by P_{hij}.

To see that such graphs P_{ij} exist, consider selecting (for each pair i, j) one of the ℓ_{ij} bipartite graphs P_α^{ij} with $1 \le \alpha \le \ell_{ij}$ uniformly at random. By Definition 12, the probability that P_{ij} is not (d_2, δ_2)-regular is at most $(\varepsilon_3 \ell/6)/\ell_{ij} \le \varepsilon_3/3$. So the probability that all of the selected bipartite graphs are (d_2, δ_2)-regular is at least

$$1 - \binom{k}{2}\frac{\varepsilon_3}{3} \overset{(24)}{>} \frac{1}{2}. \tag{27}$$

Similarly, Definition 13 implies that the probability that the triad P_{hij} is not (δ_3, r)-regular is at most $\varepsilon_3 \ell^3/\ell_{hi}\ell_{ij}\ell_{hj} \le 8\varepsilon_3$. So the probability that all of the selected P_{hij} are (δ_3, r)-regular is at least $1 - \binom{k}{3} 8\varepsilon_3 > 1/2$. Together with (27), this shows that there is some choice of bipartite graphs P_{ij} which has the required properties.

We now use the densities of the corresponding triads P_{hij} to define a red/blue-colouring of the $K_k^{(3)}$ which we found in $\mathscr{R}$: if $d_{\mathscr{G}_{red}}(P_{hij}) \ge 1/2$, then we colour the hyperedge $V_hV_iV_j$ red, otherwise we colour it blue. Since $k = R(K_{2\Delta+1}^{(3)})$, we find a monochromatic copy K of $K_{2\Delta+1}^{(3)}$ in our $K_k^{(3)}$. We now greedily assign the vertices of $\mathscr{H}$ to the clusters that form the vertex set of K, in such a way that if three vertices of $\mathscr{H}$ form a hyperedge, then they are assigned to different clusters. (We may think of this as a $(2\Delta + 1)$-vertex-colouring of $\mathscr{H}$.) We now need to show that with this assignment we can apply the embedding lemma to find a monochromatic copy of $\mathscr{H}$ in $K_m^{(3)}$.

Assume first that K is red. We already have bipartite graphs P_{ij} between the clusters in $V(K)$ which are (d_2, δ_2)-regular and form triads P_{hij} which are (δ_3, r)-regular with respect to $\mathscr{G}_{red}$. The only technical problem is that these triads do not all have the same density with respect to $\mathscr{G}_{red}$, which was one of the conditions in the embedding lemma. We do know, however, that in each case we have $d_{\mathscr{G}_{red}}(P_{hij}) \ge 1/2$. So we choose a hypergraph $\mathscr{G}'_{red} \subseteq \mathscr{G}_{red}$ such that all the graph triads are $(1/2, 3\delta_3, r)$-regular with respect to $\mathscr{G}'_{red}$. It is easy to see that such a $\mathscr{G}'_{red}$ exists: for each triple $V_hV_iV_j$ that is a hyperedge of K, consider a random subset of the hyperedges of $\mathscr{G}_{red}$ induced by V_i, V_j, V_k such that P_{hij} has density $(1 \pm \delta_3)/2$ with respect to this subset. This observation is formalized for instance in Proposition 22 of [21], which one can apply directly to obtain the above bounds on the regularity of $\mathscr{G}'_{red}$. (Alternatively, it is easy to see that the proof of Lemma 3 generalizes to different 'hypergraph densities'.) We then apply the embedding lemma (Lemma 2) to find a copy of $\mathscr{H}$ in $\mathscr{G}'_{red}$, and therefore also in $\mathscr{G}_{red}$.

On the other hand, if K is blue, we will aim to find a copy of $\mathscr{H}$ in $\mathscr{G}_{blue}$. We certainly still have a set of bipartite graphs all of which are (d_2, δ_2)-regular, but we now also need to prove that all triads are regular with respect to $\mathscr{G}_{blue}$. So suppose $\vec{Q} = (Q(1), \ldots, Q(r))$ is an r-tuple of subtriads of one of these triads P_{hij}, satisfying $t(\vec{Q}) > \delta_3 t(P_{hij})$. Let d be such that P_{hij} is (d, δ_3, r)-regular with respect to $\mathscr{G}_{red}$. Then

$$|(1-d) - d_{\mathscr{G}_{blue}}(\vec{Q})| = |d - (1 - d_{\mathscr{G}_{blue}}(\vec{Q}))| = |d - d_{\mathscr{G}_{red}}(\vec{Q})| < \delta_3.$$

Thus P_{hij} is $(1-d, \delta_3, r)$-regular with respect to $\mathscr{G}_{blue}$ (note that $\delta_3 \ll 1/2 \le 1-d$). By the same method as in the previous case, we can apply the embedding lemma to obtain a copy of $\mathscr{H}$ in $\mathscr{G}_{blue}$.

It remains to estimate how large we needed m to be in order for all of our calculations to be valid. When we apply the embedding lemma, we know we can find any subgraph $\mathscr{H}$ of maximum degree at most Δ with $|\mathscr{H}| \le cn$, where n is the size of a cluster and c is a constant chosen to satisfy the conditions of the embedding lemma. Since $n = \lfloor m/t \rfloor \ge m/2T_0$, this means that it suffices to start with an m satisfying $m \ge 2T_0|\mathscr{H}|/c$. In order to be able to apply the embedding lemma we need that $c \ll d_2, d_3 = 1/2, 1/\Delta$. We obtain $d_2 = 1/\ell$ from the regularity lemma, given constants $\delta_3, \varepsilon_1, t_0, \ell_0$, an integer-valued function $r = r(t, \ell)$ and a decreasing function $\varepsilon_2(\ell)$, all satisfying the hierarchies (24) and (25). In all cases, we can view the constants we require purely as functions of Δ. Thus c is implicitly a function solely of Δ. This is also the case for T_0 and N_0.

Finally, in order to be able to apply the regularity lemma to $\mathscr{G}_{red}$ we needed to assume that $m \ge N_0$, and in order to be able to apply the embedding lemma we needed to assume that $n \ge n_0$ (for which it is sufficient to assume that $m \ge 2T_0 n_0$). Altogether, this shows that we can take the constant C in Theorem 1 to be $\max\{2T_0/c, N_0, 2T_0 n_0\}$. □

9. Acknowledgement

We would like to thank the referees for a careful reading of the manuscript and for their helpful comments and suggestions.

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