A TOURNAMENT VERSION OF THE LOVÁSZ PATH REMOVAL CONJECTURE

JAEHOON KIM, DANIELA KÜHN, AND DERYK OSTHUS

Abstract. We prove the following tournament version of the Lovász path removal conjecture: for each $k \in \mathbb{N}$ there exists $g(k) \in \mathbb{N}$ so that every strongly $g(k)$-connected tournament T has the property that for any ordered pair x, y of vertices there is a path P from x to y in T for which $T \setminus V(P)$ is strongly k-connected. In fact, we prove the following stronger statement, which provides a tournament analogue of a long-standing conjecture of Thomassen: suppose that T is a strongly 10^7k^6m-connected tournament. Then for every set M of m vertices in T, there is a partition V_1, V_2 of $V(T)$ such that (i) $M \subseteq V_1$, (ii) for $i = 1, 2$ the subtournament $T[V_i]$ is strongly k-connected, and (iii) every vertex in V_1 has at least k out-neighbours and at least k in-neighbours in V_2.

1. Introduction

The famous Lovász path removal conjecture states that for every $k \in \mathbb{N}$ there exists $g(k) \in \mathbb{N}$ such that for every pair x, y of vertices in a $g(k)$-connected graph G we can find an induced path P joining x and y in G for which $G \setminus V(P)$ is k-connected. It is not hard to show that $g(1) = 3$. Chen, Gould and Yu [1] as well as Kriesell [4] independently showed that $g(2) = 5$. In general, the conjecture is still wide open (a version for edge-connectivity was proved in [3]). More generally, one can also ask for the existence of a non-separating subdivision of a graph H with prescribed branch vertices such that the paths joining the branch vertices are induced (the path removal conjecture then corresponds to the special case when H consists of a single edge).

We prove such a result for tournaments. The natural tournament analogue of an induced path is a backwards transitive path: here a directed path $P = x_1 \dots x_t$ in a tournament T is *backwards-transitive* if x_ix_j is an edge of T whenever $i \geq j + 2$.

Theorem 1.1. *Let $k, m \in \mathbb{N}$. Suppose that T is a strongly $10^{23}k^6m^{13}$-connected tournament, that M is a set of m vertices in T, that H is a digraph on m vertices and that ϕ is a bijection from $V(H)$ to M. Then T contains a subdivision H^* of H such that*
- (i) *for each $h \in V(H)$ the branch vertex of H^* corresponding to h is $\phi(h)$,*
- (ii) *$T \setminus V(H^*)$ is k-connected,*
- (iii) *for every edge e of H, the path P_e of H^* corresponding to e is backwards-transitive.*

Our proof shows that in the case when H is an edge it suffices for T to be strongly $2 \cdot 10^7k^6$-connected. We will derive Theorem 1.1 from the following result on tournament partitions:

Theorem 1.2. *Let T be a tournament and $k, m \in \mathbb{N}$. If T is strongly 10^7k^6m-connected then for any set $M \subseteq V(T)$ with $|M| = m$, there exists a partition V_1, V_2 of $V(T)$ such that $M \subseteq V_1$, $T[V_1]$*

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and $T[V_2]$ are both strongly k-connected, and every vertex in V_1 has at least k out-neighbours and at least k in-neighbours in V_2.

We have made no attempt to optimize the bound on the connectivity in Theorem 1.2. (It would be straightforward to obtain minor improvements at the expense of more careful calculations.) On the other hand, it would be interesting to obtain the correct order of magnitude (in terms of m and k) for the connectivity bound.

Kühn, Osthus and Townsend [7] earlier proved the weaker result that every strongly $10^8k^6\log(4k)$-connected tournament T has a vertex partition V_1, V_2 such that $T[V_1]$ and $T[V_2]$ are both strongly k-connected (with some control over the sizes of V_1 and V_2). This proved a conjecture of Thomassen. [7] raised the question whether this can be extended to digraphs. A graph version of this was already proved much earlier by Hajnal [2] and Thomassen [10].

As described later, our proof of Theorem 1.2 develops ideas in [7]. These in turn are based on the concept of robust linkage structures which were introduced in [5] to prove a conjecture of Thomassen on edge-disjoint Hamilton cycles in highly connected tournaments. Further (asymptotically optimal) results leading on from these approaches were obtained by Pokrovskiy [8, 9].

Theorem 1.2 is a tournament analogue of the following long-standing conjecture of Thomassen [11].

Conjecture 1.3. *For every $k \in \mathbb{N}$ there exists $f(k) \in \mathbb{N}$ such that if G is a $f(k)$-connected graph and $M \subseteq V(G)$ consists of k vertices then there exists a partition V_1, V_2 of $V(G)$ such that $M \subseteq V_1$, both $G[V_1]$ and $G[V_2]$ are k-connected, and each vertex in V_1 has at least k neighbours in V_2.*

The case $|M| = 2$ would already imply the path removal conjecture. The case $M = \emptyset$ was proved in [6]. It implies the existence of non-separating subdivisions (without prescribed branch vertices) in highly connected graphs.

2. Notation and tools

Given $k \in \mathbb{N}$, we let $[k] := \{1, \ldots, k\}$ and $\log k := \log_2 k$. We write $V(G)$ and $E(G)$ for the set of vertices and the set of edges in a digraph G. We let $|G| := |V(G)|$. If $u, v \in V(G)$ we write uv for the directed edge from u to v. We write $d^-_G(v)$ and $d^+_G(v)$ for the in-degree and the out-degree of a vertex v in G. We write $\delta^-(G)$ and $\delta^+(G)$ for the minimum in-degree and the minimum out-degree of G and let $\delta^0(G) := \min\{\delta^-(G), \delta^+(G)\}$. A set $A \subseteq V(G)$ *in-dominates* a set $B \subseteq V(G)$ if for every vertex $b \in B$ there exists a vertex $a \in A$ such that $ba \in E(G)$. Similarly, we say that A *out-dominates* B if for every vertex $b \in B$ there exists a vertex $a \in A$ such that $ab \in E(G)$. We say that a tournament T is *transitive* if we may enumerate its vertices $v_1, \ldots, v_m$ such that $v_iv_j \in E(T)$ if and only if $i < j$. In this case we call v_1 the *source* of T and v_m the *sink* of T. When referring to subpaths of tournaments, we always mean that these paths are directed (i.e. consistently oriented). The *length* of a path is the number of its edges. We say that two paths are disjoint if they are vertex-disjoint. A tournament T is *strongly k-connected* if $|T| > k$ and for every set $F \subseteq V(T)$ with $|F| < k$ and every ordered pair x, y of vertices in $V(T) \setminus F$ there exists a path from x to y in $T - F$. A tournament T is called *k-linked* if $|T| \geq 2k$ and whenever $x_1, \ldots, x_k, y_1, \ldots, y_k$ are $2k$ distinct vertices of T there exist disjoint paths $P_1, \ldots, P_k$ such that P_i is a directed path from x_i to y_i for each $i \in [k]$.

We now collect the tools which we need in our proof of Theorem 1.2. We will use the following well known fact.

Proposition 2.1. *Let $k \in \mathbb{N}$ and let T be a tournament. Then T contains less than $2k$ vertices of out-degree less than k, and T contains less than $2k$ vertices of in-degree less than k.*

The following proposition is a straightforward consequence of the definition of linkedness.

Proposition 2.2. *Let $k \in \mathbb{N}$. Then a tournament T is k-linked if and only if $|T| \geq 2k$ and whenever $(x_1, y_1), \dots, (x_k, y_k)$ are ordered pairs of (not necessarily distinct) vertices of T, there exist distinct internally disjoint paths $P_1, \dots, P_k$ such that for all $i \in [k]$ we have that P_i is a directed path from x_i to y_i and that $\{x_1, \dots, x_k, y_1, \dots, y_k\} \cap V(P_i) = \{x_i, y_i\}$.*

We will also use the following bound from [5] on the connectivity which forces a tournament to be highly linked.

Theorem 2.3. *For each $k \in \mathbb{N}$ every strongly $452k$-connected tournament is k-linked.*

The following two lemmas from [7] guarantee that every tournament contains almost out-dominating and almost in-dominating sets which are not too large.

Lemma 2.4. *Let T be a tournament, let $v \in V(T)$ and $c \in \mathbb{N}$. Then there exist disjoint sets $A, E \subseteq V(T)$ such that the following properties hold:*

(i) $1 \leq |A| \leq c$ *and $T[A]$ is a transitive tournament with sink v,*
(ii) A *out-dominates $V(T) \setminus (A \cup E)$,*
(iii) $|E| \leq (1/2)^{c-1} d^-_T(v)$.

The next lemma follows immediately from Lemma 2.4 by reversing the orientations of all edges.

Lemma 2.5. *Let T be a tournament, let $v \in V(T)$ and $c \in \mathbb{N}$. Then there exist disjoint sets $B, E \subseteq V(T)$ such that the following properties hold:*

(i) $1 \leq |B| \leq c$ *and $T[B]$ is a transitive tournament with source v,*
(ii) B *in-dominates $V(T) \setminus (B \cup E)$,*
(iii) $|E| \leq (1/2)^{c-1} d^+_T(v)$.

We will also need the following observation, which guarantees a small set Z of vertices in a tournament such that every vertex outside Z has many out- and in-neighbours in Z.

Proposition 2.6. *Let $k, n \in \mathbb{N}$ and let T be a tournament on $n \geq 16$ vertices. Then there is a set $Z \subseteq V(T)$ of size $|Z| \leq 3k \log n$ such that each vertex in $V(T) \setminus Z$ has at least k out-neighbours and at least k in-neighbours in Z.*

Proof. We may assume that $n \geq 3k \log n$. Let $c := \lceil \log n \rceil + 1 \leq (3 \log n)/2$. Note that Lemma 2.5 implies that T contains an in-dominating set V_1 of size at most c. Apply Lemma 2.5 again to $T \setminus V_1$ to find an in-dominating set V_2 of $T \setminus V_1$ with size at most c. Continue in this way to obtain disjoint sets $V_1, \dots, V_k$. Now apply Lemma 2.4 repeatedly to obtain disjoint sets $U_1, \dots, U_k$, each of size at most c, such that each U_i is an out-dominating set in $T \setminus (U_1 \cup \dots \cup U_{i-1})$. We can take $Z := V_1 \cup \dots \cup V_k \cup U_1 \cdots \cup U_k$. □

Recall that a subpath $Q = q_1 \dots q_{|Q|}$ of a tournament T is backwards-transitive if $q_i q_j \in E(T)$ whenever $i \geq j + 2$. The following lemma is a slight strengthening of Lemma 2.7 in [7]. The proof is identical to that in [7], so we omit it here.

Lemma 2.7. *Let $k, \ell \in \mathbb{N}$, let T be a tournament and let $Q_1, \dots, Q_\ell$ be disjoint backwards-transitive paths in T such that $|Q_j| \geq k+1$ for all $j \in [\ell]$ and $V(T) = V(Q_1 \cup \dots \cup Q_\ell)$. Let U' be the set consisting of the first $k+1$ vertices in Q_j for all $j \in [\ell]$ and let W' be the set consisting of the last $k+1$ vertices in Q_j for all $j \in [\ell]$. Then there exist sets U, W satisfying the following properties:*

- *$U \subseteq U' \subseteq V(T)$ and $W \subseteq W' \subseteq V(T)$,*
- *$|U|, |W| \leq 2k(k+1)$,*
- *for any set $F \subseteq V(T)$ of size at most $k-1$, and for every vertex v in $V(T) \setminus F$, there exists a directed path (possibly of length 0) in $T[(U' \cup \{v\}) \setminus F]$ from v to a vertex in U and a directed path in $T[(W' \cup \{v\}) \setminus F]$ from a vertex in W to v.*

Note that U' and W' may not be disjoint, and $|U'| = |W'| = \ell(k+1)$.

3. Proofs of Theorems 1.1 and 1.2

Before we prove Theorem 1.2, we will show how it can be used to derive Theorem 1.1.

Proof of Theorem 1.1. Apply Theorem 1.2 to obtain a partition V_1, V_2 of $V(T)$ such that $M \subseteq V_1$, $T[V_1]$ and $T[V_2]$ are both strongly $452km^2$-connected, and every vertex in V_1 has at least k out-neighbours and at least k in-neighbours in V_2. Theorem 2.3 now implies that $T[V_1]$ is m^2-linked. Together with Proposition 2.2 this in turn implies that $T[V_1]$ contains a subdivision H^* of H such that for each $h \in V(H)$ the branch vertex of H^* corresponding to h is $\phi(h)$. By shortening the paths between the branch vertices if necessary, we may assume that they are backwards-transitive. Since every vertex in V_1 has at least k out-neighbours and at least k in-neighbours in V_2 it follows that $T[V_2 \cup (V_1 \setminus V(H^*))]$ is strongly k-connected, as desired. $\square$

We now give a brief idea of the argument in the proof of Theorem 1.2 under the much stronger assumptions that $k \gg \log n$ and $|M| = 1$. In this case we can find $2k$ disjoint sets $A_1, \dots, A_{2k} \subseteq V(T)$ of size $o(k)$ which are out-dominating. We can also find $2k$ sets $B_1, \dots, B_{2k} \subseteq V(T)$ of size $o(k)$ which are in-dominating such that all the B_i are disjoint from each other and from $A_1, \dots, A_{2k}$. Moreover, we can choose these sets in such a way that each A_i and each B_i induces a transitive subtournament of T. We now use the fact that T is $(10^7k^6/452)$-linked to find, for each $i \in [2k]$, a path P_i from the sink of B_i to the source of A_i such that all the P_i are pairwise disjoint. We now assign $A_i \cup B_i \cup V(P_i)$ to V_1 for all $i \leq k$ and to V_2 for all $i > k$. We assign the remaining vertices arbitrarily. By relabeling V_1 and V_2 if necessary, we may assume that $M \subseteq V_1$.

It is easy to see that both $T[V_1]$ and $T[V_2]$ are strongly k-connected. Indeed, consider some $F \subseteq V_1$ with $|F| < k$. So there exists $i \in [k]$ such that F avoids $A_i \cup B_i \cup V(P_i)$. Consider any $x, y \in V_1 \setminus F$. Since B_i is in-dominating, there is an edge from x to some $x' \in B_i$. Similarly, since A_i is out-dominating, there is an edge from some $y' \in A_i$ to y. Then P_i, xx', $y'y$ together with the edge from x' to the sink of B_i and the edge from the source of A_i to y' form a path in $T[V_1 \setminus F]$ from x to y, as required. A similar argument shows that $T[V_2]$ is k-connected too. Moreover, each $x \in V_1$ has k in-neighbours and k out-neighbours in V_2 since x receives an edge from A_i and sends an edge to B_i for all $i > k$.

In general, the problem with this approach is that we cannot guarantee such (small) dominating sets when k is bounded. However, we can still find small sets which dominate a large proportion of $V(T)$. With some new ideas one can use these to ensure strong k-connectivity of both $T[V_1]$ and $T[V_2]$ as well as high in- and outdegree of the vertices in V_1 from and to V_2. Significant additional difficulties arise when $|M| > 1$.

Proof of Theorem 1.2. Let $X := \{x_1, x_2, \dots, x_{20k}\} \subseteq V(T) \setminus M$ consist of $20k$ vertices whose in-degree in T is as small as possible, and let $Y := \{y_1, y_2, \dots, y_{20k}\}$ be a set of $20k$ vertices in $V(T) \setminus (M \cup X)$ whose out-degree in T is as small as possible. Define

$$\hat{\delta}^-(T) := \min_{v \in V(T) \setminus (M \cup X)} d^-_T(v) \qquad \text{and} \qquad \hat{\delta}^+(T) := \min_{v \in V(T) \setminus (M \cup Y)} d^+_T(v).$$

Let $c := \lceil \log(80k) \rceil + 1 \leq 9k$. Apply Lemmas 2.4 and 2.5 with parameter c repeatedly (removing M and the dominating sets each time) to obtain disjoint sets of vertices $A_1, \dots, A_{20k}, B_1, \dots, B_{20k}$ and sets of vertices $E_{A_1}, \dots, E_{A_{20k}}, E_{B_1}, \dots, E_{B_{20k}}$ satisfying the following properties for all $i \in [20k]$, where we write $D := \bigcup_{i=1}^{20k}(A_i \cup B_i)$, $D_1 := \bigcup_{i=1}^{19k}(A_i \cup B_i)$ and $D_2 := \bigcup_{i=19k+1}^{20k}(A_i \cup B_i)$:

- (D1) $1 \leq |A_i| \leq c$ and $T[A_i]$ is a transitive tournament with sink x_i,
- (D2) $1 \leq |B_i| \leq c$ and $T[B_i]$ is a transitive tournament with source y_i,
- (D3) A_i out-dominates $V(T) \setminus (M \cup D \cup E_{A_i})$ in T,
- (D4) B_i in-dominates $V(T) \setminus (M \cup D \cup E_{B_i})$ in T,
- (D5) $|E_{A_i}| \leq (1/2)^{c-1}\hat{\delta}^-(T)$,
- (D6) $|E_{B_i}| \leq (1/2)^{c-1}\hat{\delta}^+(T)$.

Let $E_A := E_{A_1} \cup \dots \cup E_{A_{20k}}$, $E_B := E_{B_1} \cup \dots \cup E_{B_{20k}}$, $E'_A := E_{A_{19k+1}} \cup \dots \cup E_{A_{20k}}$, $E'_B := E_{B_{19k+1}} \cup \dots \cup E_{B_{20k}}$, $E := E_A \cup E_B$ and $E' := E'_A \cup E'_B$. Note that

$$|E'_A| \leq |E_A| \leq 20k \left(\frac{1}{2}\right)^{c-1} \hat{\delta}^-(T) \leq \frac{\hat{\delta}^-(T)}{4} \quad \text{and} \quad |E'_B| \leq |E_B| \leq \frac{\hat{\delta}^+(T)}{4} \tag{3.1}$$

by our choice of c. Moreover, we may assume that $|E_A| \leq |E_B|$. (The case $|E_A| > |E_B|$ follows by a symmetric argument.) In particular, this implies that

$$|E'| \leq |E| \leq |E_A| + |E_B| \leq 2|E_B| \leq \frac{\hat{\delta}^+(T)}{2}. \tag{3.2}$$

Our aim is to use the almost-dominating sets A_i, B_i in order to construct the desired partition V_1, V_2 of $V(T)$. More precisely, we will iteratively colour the vertices of T with colours α and β, and at each step V_α will consist of all vertices of colour α and V_β is defined similarly. At the end of our argument, every vertex of T will be coloured either with α or with β, i.e. V_α, V_β will form a partition of $V(T)$. Our aim is to colour the vertices in such a way that we can take $V_1 := V_\alpha$ and $V_2 := V_\beta$. We start with no vertices of T coloured, and we then colour the vertices in $M \cup D_1 = M \cup \bigcup_{i=1}^{19k}(A_i \cup B_i)$ by α and the vertices in $D_2 = \bigcup_{i=19k+1}^{20k}(A_i \cup B_i)$ by β.

At each step and for each $\gamma \in \{\alpha, \beta\}$, we call a vertex $v \in V_\gamma$ *forwards-safe* if for any set $F \not\ni v$ of at most $k-1$ vertices, there is a directed path (possibly of length 0) in $T[V_\gamma \setminus F]$ from v to $V_\gamma \setminus (M \cup D \cup E_B \cup F)$. Similarly, we say that $v \in V_\gamma$ is *backwards-safe* if for any set $F \not\ni v$ of at most $k-1$ vertices, there is a directed path (possibly of length 0) in $T[V_\gamma \setminus F]$ from $V_\gamma \setminus (M \cup D \cup E_A \cup F)$ to v. We will call a vertex $v \in V_\gamma$ *partition-safe* if either $v \notin M \cup D \cup E$ or $\gamma = \beta$ or v has at least k out-neighbours and k in-neighbours of colour β. Finally, we call a vertex *safe* if it is forwards-safe, backwards-safe and partition-safe. Note that the following properties are satisfied at every step (for each $\gamma \in \{\alpha, \beta\}$):

- (S1) all coloured vertices in $V(T) \setminus (M \cup D \cup E)$ are safe,
- (S2) all coloured vertices in $V(T) \setminus (M \cup D \cup E_B)$ are forwards-safe and all coloured vertices in $V(T) \setminus (M \cup D \cup E_A)$ are backwards-safe,
- (S3) if $v \in V_\gamma$ has at least k forwards-safe out-neighbours of colour γ then v itself is forwards-safe; the analogue holds if v has at least k backwards-safe in-neighbours of colour γ,

(S4) all coloured vertices in $V_\alpha \setminus (M \cup D \cup E'_A)$ which have at least k out-neighbours in V_β are partition-safe,

(S5) if $v \in V_\gamma$ is safe and in the next step we enlarge V_γ by colouring some more (previously uncoloured) vertices then v is still safe.

Indeed, to check (S4) note that (D3) implies that every vertex in $V(T) \setminus (M \cup D \cup E'_A)$ has an in-neighbour in A_s for each $19k < s \leq 20k$ and that all vertices in these A_s are coloured β.

In what follows, by a (partial) colouring of the vertices of T we always mean a colouring with colours α and β in which all the vertices in $M \cup D_1$ are coloured α and all the vertices in D_2 are coloured β.

Claim 1: *Suppose that there are distinct indices $i_1, \dots, i_k \in [19k]$, distinct indices $i'_1, \dots, i'_k \in [19k]$ and subpaths $Q_1, \dots, Q_k$ and $P_{19k+1}, \dots, P_{20k}$ of T satisfying the following properties:*

- *for each $s \in [k]$ the path Q_s joins the sink of $B_{i'_s}$ to the source of A_{i_s},*
- *for each $19k < s \leq 20k$ the path P_s joins the sink of B_s to the source of A_s,*
- *the paths $Q_1, \dots, Q_k$ and $P_{19k+1}, \dots, P_{20k}$ are disjoint from each other and meet $D \cup M$ only in their endvertices.*

Suppose that we have coloured all vertices of T such that

- *every vertex in $M \cup D_1 \cup V(Q_1) \cup \dots \cup V(Q_k)$ is coloured α,*
- *every vertex in $D_2 \cup V(P_{19k+1}) \cup \dots \cup V(P_{20k})$ is coloured β,*
- *every vertex is safe.*

Then the sets $V_1 := V_\alpha$ and $V_2 := V_\beta$ form a partition of $V(T)$ as required in Theorem 1.2.

To prove Claim 1, we first show that $T[V_\alpha]$ is strongly k-connected. So consider any set F of at most $k-1$ vertices and any two vertices $x, y \in V_\alpha \setminus F$. We need to check that $T[V_\alpha \setminus F]$ contains a path from x to y. Since x is forwards-safe there exists a path P_x in $T[V_\alpha \setminus F]$ from x to some vertex $x' \in V_\alpha \setminus (M \cup D \cup E_B \cup F)$. Similarly, since y is backwards-safe there exists a path P_y in $T[V_\alpha \setminus F]$ from some vertex $y' \in V_\alpha \setminus (M \cup D \cup E_A \cup F)$ to y. Let $s \in [k]$ be such that F avoids $A_{i_s} \cup V(Q_s) \cup B_{i'_s}$. Since $x' \notin M \cup D \cup E_B$, (D4) implies that x' sends an edge to $B_{i'_s}$. Similarly, since $y' \notin M \cup D \cup E_A$, (D3) implies that y' receives an edge from A_{i_s}. Altogether this implies that $T[V(P_x) \cup V(P_y) \cup A_{i_s} \cup V(Q_s) \cup B_{i'_s}] \subseteq T[V_\alpha \setminus F]$ contains path from x to y, as desired.

A similar argument shows that V_β is strongly k-connected too. It remains to show that any vertex $x \in V_\alpha$ has k in-neighbours and k out-neighbours in V_β. Since x is partition-safe this is clear if $x \in M \cup D \cup E$. If $x \notin M \cup D \cup E$ then (D3) and (D4) together imply that, for every $19k < s \leq 20k$, x sends an edge to $B_s \subseteq V_\beta$ and receives an edge from $A_s \subseteq V_\beta$. This completes the proof of Claim 1.

Claim 2: *Consider a partial colouring of $V(T)$ and let C denote the set of previously coloured vertices. (So $M \cup D \subseteq C$.) Let $Z \subseteq V(T) \setminus (M \cup X \cup Y)$ and $N \subseteq V(T) \setminus Z$ and suppose that $9k^2|Z| + |C \cup N| \leq 5 \cdot 10^6 k^6 m$. Then for every colouring of the vertices in $Z \setminus C$ there is a set $Z' \subseteq V(T) \setminus (Z \cup N \cup C)$ and a colouring of the vertices in Z' such that every vertex in $Z \cup Z'$ is safe and $|Z \cup Z'| \leq 9k^2|Z|$.*

To prove Claim 2, note that the strong 10^7k^6m-connectivity of T implies that $\delta^0(T) \geq 10^7k^6m$. Hence

$$\hat{\delta}^-(T) - |E_A| \overset{(3.1)}{\geq} \frac{\hat{\delta}^-(T)}{2} \geq \frac{\delta^0(T)}{2} \geq 5 \cdot 10^6 k^6 m, \tag{3.3}$$

and similarly

$$\hat{\delta}^+(T) - |E| \overset{(3.2)}{\geq} \frac{\hat{\delta}^+(T)}{2} \geq 5 \cdot 10^6 k^6 m. \tag{3.4}$$

Consider any colouring of $Z \setminus C$. Let Z_α be the vertices in Z coloured with α and define Z_β similarly. For each vertex $z \in Z_\beta$ in turn we greedily choose k uncoloured in-neighbours outside $N \cup E_A$, and colour them β. Then for each vertex $z \in Z_\alpha$ in turn we greedily choose $2k$ uncoloured in-neighbours outside $N \cup E_A$, and colour k of them α and k of them β. (We do not modify C in this process.) To see that we can choose all these vertices to be distinct from each other, note that the total number of vertices we wish to choose is $2k|Z_\alpha| + k|Z_\beta| \leq 2k|Z|$ and

$$|C \cup N \cup Z| + 2k|Z| \leq 5 \cdot 10^6 k^6 m \overset{(3.3)}{\leq} \hat{\delta}^-(T) - |E_A|.$$

For each vertex outside $C \setminus Z$ of colour β in turn we greedily choose k uncoloured out-neighbours outside $N \cup E$, and colour them by β. Now for each vertex outside $C \setminus Z$ of colour α in turn we greedily choose $2k$ uncoloured out-neighbours not in $N \cup E$ and colour k of them by α and k of them by β. To see that we can choose such vertices to be distinct from each other, note that the total number of vertices we wish to choose is at most $2k(1 + 2k)|Z|$ and

$$|C \cup N \cup Z| + 2k|Z| + 2k(1 + 2k)|Z| \leq |C \cup N| + 9k^2|Z| \leq 5 \cdot 10^6 k^6 m \overset{(3.4)}{\leq} \hat{\delta}^-(T) - |E|.$$

Let Z' be the set of vertices outside $C \cup Z$ that we coloured. Then $Z' \cap N = \emptyset$. Moreover, using (S1)–(S4) it is easy to check that every vertex in $Z \cup Z'$ is safe. This completes the proof of Claim 2.

$\circ = \alpha, \ \bullet = \beta$

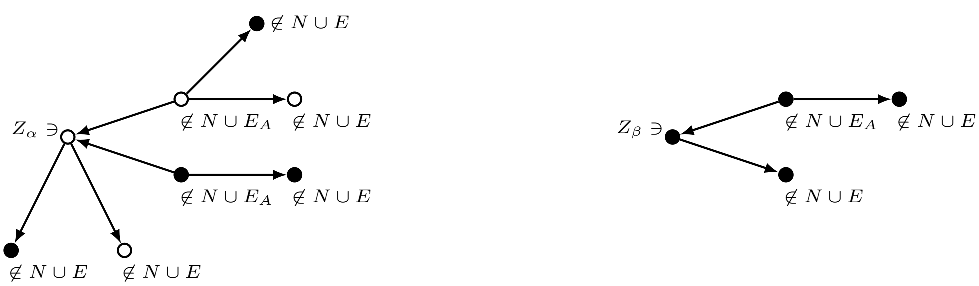

Figure 1 : The vertices chosen in the case when $k = 1$ in order to make one vertex in Z_α safe (left) and one vertex in Z_β safe (right).

Recall that we have already coloured all the vertices in $M \cup D_1$ by α and all the vertices in D_2 by β. Step by step, we will now colour further vertices of T. Our final aim is to arrive at a colouring of $V(T)$ which is as described in Claim 1. The first step is to colour some more vertices in order to achieve that all the coloured vertices are safe. In what follows, when saying that we colour some additional vertices we always mean that these vertices are uncoloured so far.

Claim 3: *We can colour some additional vertices of T in such a way that every coloured vertex is safe and the set C_1 consisting of all vertices coloured so far satisfies $|C_1| \leq 5000k^4 m$.*

To prove Claim 3, for every $v \in \{x_1, \ldots, x_{19k}, y_1, \ldots, y_{19k}\} \cup M$ in turn we greedily choose $2k$ uncoloured in-neighbours and $2k$ uncoloured out-neighbours, all distinct from each other, and colour k out-neighbours and k in-neighbours by α and the other k out- and in-neighbours by β. Similarly, for every $v \in \{x_{19k+1}, \dots, x_{20k}, y_{19k+1}, \dots, y_{20k}\}$ in turn we greedily choose k uncoloured in-neighbours and k uncolored out-neighbours, all distinct from each other and colour them β. Let Z^* denote the set of $4k(38k+m)+4k^2 \leq 160k^2m$ new vertices we just coloured and let $Z := Z^* \cup (D \setminus (X \cup Y))$. Then $|Z| \leq |Z^*| + |D| \leq 160k^2m + c \cdot 40k \leq 520k^2m$. Apply Claim 2 with $N := \emptyset$ to find a set Z' of uncoloured vertices and a colouring of these vertices such that all the vertices in $Z \cup Z'$ are safe and $|Z \cup Z'| \leq 9k^2 \cdot |Z| \leq 5000k^4m$. Our choice of Z^* and (S3) together now imply that the vertices in $X \cup Y \cup M$ are safe as well. This completes the proof of Claim 3.

Claim 4: *There are distinct indices $i_1, \dots, i_k \in [19k]$, distinct indices $i'_1, \dots, i'_k \in [19k]$ and subpaths $Q_1, \dots, Q_k$ and $P_{19k+1}, \dots, P_{20k}$ of T satisfying the following properties:*

(i) *for each $s \in [k]$ the path Q_s joins the sink of $B_{i'_s}$ to the source of A_{i_s},*
(ii) *for each $19k < s \leq 20k$ the path P_s joins the sink of B_s to the source of A_s,*
(iii) *the paths $Q_1, \dots, Q_k$ and $P_{19k+1}, \dots, P_{20k}$ are disjoint from each other and meet $C_1 \supseteq D \cup M$ only in their endvertices,*
(iv) *we can colour the internal vertices of $Q_1, \dots, Q_k$ by α, the internal vertices of $P_{19k+1}, \dots, P_{20k}$ by β and can colour some additional vertices such that the set C_4 of all coloured vertices satisfies the following properties:*
  (α) *all vertices in C_4 are safe,*
  (β) *there is a set $C_\alpha \subseteq C_4$ such that every vertex in C_α is coloured α and the number of vertices of colour α outside C_α is at most 10^6k^6m,*
  (γ) *every vertex outside C_4 which has an in-neighbour in C_α has at least k in-neighbours coloured β, and every vertex outside C_4 which has an out-neighbour in C_α has at least k out-neighbours coloured β.*

We will prove Claim 4 via a sequence of subclaims. For $i \in [20k]$ we define an *i-path* to be a directed path from the sink of B_i to the source of A_i whose interior vertices lie outside C_1. Ideally, we would like to find disjoint i-paths P_i (one for each $i \in [20k]$) such that the following properties hold:

(a) for $19k < i \leq 20k$ all interior vertices of P_i can be coloured β,
(b) there are at least k indices i with $i \in [19k]$ such that all interior vertices of P_i can be coloured α,
(c) by colouring some additional vertices we can achieve that all coloured vertices are safe.

However, we are not able to satisfy (b) (and (c)) directly. So instead, for each of the paths Q_s in Claim 4, there will be three paths P_{i_1}, P_{i_2} and P_{i_3} with $i_1, i_2, i_3 \in [19k]$ such that each P_{i_j} is an i_j-path and Q_s consists of an initial segment of P_{i_1}, a middle segment of P_{i_2}, a final segment of P_{i_3} as well as two edges joining these three segments.

More precisely, our strategy is to proceed as follows. For each $i \in [20k]$ we will first try to find a short i-path P_i such that all these i-paths are disjoint. We will then colour the vertices on these short i-paths as well as some additional vertices such that (a)–(c) are satisfied for the set I_{short} of all indices i for which we have been able to choose a short i-path (see Claim 4.1). This provides some of the paths required in Claim 4. To find the remaining paths, for all $i \notin I_{short}$ we will choose $1000k^4$ i-paths $Q_{i,1}, \dots, Q_{i,1000k^4}$ such that all these paths are internally disjoint from each other. We will then show that for each $i \notin I_{short}$ with $i > 19k$ we can take the P_i required in Claim 4 to be some $Q_{i,j}$, whereas each remaining path Q_s still required in Claim 4 will consist of one segment from each of three different paths Q_{i_1,j_1}, Q_{i_2,j_2}, Q_{i_3,j_3} with $i_1, i_2, i_3 \in [19k] \setminus I_{short}$, as described before. The reason why we start with $19k$ indices to choose the k paths Q_s in Claim 4 and why we choose many i-paths for each $i \notin I_{short}$ is that we need some extra flexibility in order to be able to satisfy part (vi) of Claim 4.

We will now choose the short i-paths. So let $\mathcal{P}_{short}$ be a collection of paths consisting of at most one i-path for each $i \in [20k]$ such that all these paths are disjoint from each other, each path has length at most $6k+10$ and, subject to this, $|\mathcal{P}_{short}|$ is as large as possible. Let I_{short} be the set of all those indices $i \in [20k]$ for which $\mathcal{P}_{short}$ contains an i-path, let $I_{short,\alpha} := I_{short} \cap [19k]$ and $I_{short,\beta} := I_{short} \setminus I_{short,\alpha}$. Moreover, set $I_{long} := [20k] \setminus I_{short}$, $I_{long,\alpha} := I_{long} \cap [19k]$ and $I_{long,\beta} := I_{long} \setminus I_{long,\alpha}$. For each $i \in I_{short}$ let P_i denote the i-path contained in $\mathcal{P}_{short}$. We will call all these i-paths short. Let V_{short} be the set of all internal vertices of P_i for all $i \in I_{short}$. Recall that the definition of an i-path implies that all the vertices in V_{short} are uncoloured so far (i.e. $V_{short} \cap C_1 = \emptyset$).

Claim 4.1: *We may colour all vertices in V_{short} as well as some additional vertices of T such that the following properties hold:*

(i) *for each $i \in I_{short,\alpha}$ all the vertices on P_i are coloured α,*
(ii) *for each $i \in I_{short,\beta}$ all the vertices on P_i are coloured β,*
(iii) *the set C_2 consisting of all vertices coloured so far has size $|C_2| \le 8000k^4m$ and all vertices in C_2 are safe.*

Note that $|V_{short}| \le 20k(6k+9) \le 300k^2$. Together with Claim 2 (applied with $N := \emptyset$ and $Z := V_{short}$) and Claim 3 this implies Claim 4.1.

Claim 4.2: *We may assume that $|\mathcal{I}_{short,\alpha}| < k$*, and hence $|\mathcal{I}_{long,\alpha}| > 18k$.

To prove Claim 4.2, suppose that $|\mathcal{I}_{short,\alpha}| \ge k$. Colour all uncoloured vertices by β. Then $|V_\alpha| \le 8000k^4m$ by Claim 4.1(iii). Since T is strongly 10^7k^6m-connected and $10^7k^6m - |V_\alpha| \ge 10^7k^6m - 8000k^4m > k$, it follows that $T[V_\beta]$ is strongly k-connected and that every vertex in V_α has at least k in-neighbours and k out-neighbours in V_β. Using the facts that $T[V_\alpha]$ contains D_1 as well as disjoint i-paths for all $i \in \mathcal{I}_{short,\alpha}$ and that all the vertices in V_α are safe, a similar argument as in the proof of Claim 1 shows that $T[V_\alpha]$ is strongly k-connected too. So the partition V_α, V_β is as desired in Theorem 1.2. This completes the proof of Claim 4.2.

Recall from Claim 4.1(iii) that the set C_2 of coloured vertices has size at most $8000k^4m$. So all uncoloured vertices together with the sinks of the B_i and the sources of the A_i for all $i \in I_{long}$ induce a strongly $(904 \cdot 10^4k^5)$-connected subtournament T' of T (with some room to spare). Theorem 2.3 implies that T' is $2 \cdot 10^4k^5$-linked. Together with Proposition 2.2 this implies that for each $i \in I_{long}$ we can find $1000k^4$ i-paths in T' such that all these $1000k^4|I_{long}|$ paths are internally disjoint and the internal vertices on all these paths lie outside C_2. We choose such a collection of paths which minimizes the size of the set V_{long} consisting of all the internal vertices on these paths. Let $Q_{i,j}$ denote the jth i-path we chose (for all $i \in I_{long}$ and all $j \in [1000k^4]$). Note that each $Q_{i,j}$ must have length at least $6k+11$ since $i \in I_{long}$. Write $Q_{i,j} = q^0_{i,j}q^1_{i,j}\dots q^{|Q_{i,j}|}_{i,j}$. So $q^0_{i,j}$ is the the sink of B_i and $q^{|Q_{i,j}|}_{i,j}$ is the source of A_i. Observe that the minimality of $|V_{long}|$ implies the following:

(Q1) each $Q_{i,j}$ induces a backwards-transitive path,
(Q2) if $v \in V(T) \setminus (C_2 \cup V_{long})$ is an out-neighbour of $q^s_{i,j}$, then v is also an out-neighbour of $q^{s'}_{i,j}$ for all $s' \ge s+3$,

(Q3) if $v \in V(T) \setminus (C_2 \cup V_{long})$ is an in-neighbour of $q^s_{i,j}$, then v is also an in-neighbour of $q^{s'}_{i,j}$ for all $s' \leq s-3$.

For all $i \in I_{long}$ and all $j \in [1000k^4]$ we let $\mathrm{int}(Q_{i,j}) := q^1_{i,j} \dots q^{|Q_{i,j}|-1}_{i,j}$ denote the interior of $Q_{i,j}$. Let $Q^1_{i,j}, \dots, Q^7_{i,j}$ be disjoint segments of $\mathrm{int}(Q_{i,j})$ such that $\mathrm{int}(Q_{i,j}) = Q^1_{i,j} \dots Q^7_{i,j}$, $|Q^1_{i,j}| = |Q^7_{i,j}| = k+1$, $|Q^2_{i,j}| = |Q^6_{i,j}| = k$ and $|Q^3_{i,j}| = |Q^5_{i,j}| = k+2$. So $q^{3k+3}_{i,j}$ is the final vertex of $Q^3_{i,j}$ and $q^{|Q_{i,j}|-3k-3}_{i,j}$ is the initial vertex of $Q^5_{i,j}$. We let

$$Q^0_{i,j} := Q^1_{i,j} \cup Q^2_{i,j} \cup Q^3_{i,j} \cup Q^5_{i,j} \cup Q^6_{i,j} \cup Q^7_{i,j}$$

and write V^0_{long} for the set of all those vertices which lie in $Q^0_{i,j}$ for some $i \in I_{long}$ and some $j \in [1000k^4]$. Thus $V^0_{long} \subseteq V_{long}$ and

$$|V^0_{long}| \leq (3k+3) \cdot 40k \cdot 1000k^4 \leq 3 \cdot 10^5 k^6.$$

Claim 4.3: *There exists an index set $I_R \subseteq I_{long,\alpha} \times [1000k^4]$ such that, writing*

$$R := \bigcup_{(i,j) \in I_R} V(Q^0_{i,j}) \quad \textit{and} \quad I_S := (I_{long,\alpha} \times [1000k^4]) \setminus I_R,$$

for every $(i,j) \in I_S$ every vertex in $Q^0_{i,j}$ has at least k in-neighbours and at least k out-neighbours in R, and such that $|I_R| \leq 700k^3$.

To prove Claim 4.3, for each $\ell \in [3k+3]$ we consider $U^\ell := \{q^\ell_{i,j} : i \in I_{long,\alpha}, j \in [1000k^4]\}$ and $V^\ell := \{q^{|Q_{i,j}|-\ell}_{i,j} : i \in I_{long,\alpha}, j \in [1000k^4]\}$. By Proposition 2.6 applied to $T[U^\ell]$, there exists a set $Z^\ell_U \subseteq U^\ell$ with $|Z^\ell_U| \leq 3k \log |U^\ell|$ and such that every vertex in $U^\ell \setminus Z^\ell_U$ has at least k out-neighbours and k in-neighbours in Z^ℓ_U. Similarly, there exists a set $Z^\ell_V \subseteq V^\ell$ with $|Z^\ell_V| \leq 3k \log |V^\ell|$ and such that every vertex in $V^\ell \setminus Z^\ell_V$ has at least k out-neighbours and k in-neighbours in Z^ℓ_V. We let $Z := \bigcup_{\ell \in [3k+3]} (Z^\ell_U \cup Z^\ell_V)$ and write I_R for the set of all those indices (i,j) for which Z contains some vertex in $Q^0_{i,j}$. Let R and I_S be as defined in the statement of Claim 4.3. Then $Z \subseteq R$ and for every $(i,j) \in I_S$ every vertex in $Q^0_{i,j}$ has at least k in-neighbours and at least k out-neighbours in $Z \subseteq R$. Moreover,

$$|I_R| \leq |Z| \leq (6k+6) \cdot 3k \log(2 \cdot 10^4 k^5) \leq 700k^3,$$

as required in Claim 4.3.

Let

$$S := \bigcup_{(i,j) \in I_S} V(Q^0_{i,j}) \quad \text{and} \quad B := \bigcup_{(i,j) \in I_{long,\beta} \times [1000k^4]} V(Q^0_{i,j}).$$

Moreover, let

$$S^{1,7} := \bigcup_{(i,j) \in I_S} V(Q^1_{i,j} \cup Q^7_{i,j}) \quad \text{and} \quad R^{1,7} := \bigcup_{(i,j) \in I_R} V(Q^1_{i,j} \cup Q^7_{i,j}),$$

and define $B^{1,7}$ similarly. Note that by Claim 4.3 every vertex in S has least k in-neighbours and at least k out-neighbours in R.

Claim 4.4: *We may colour all vertices in $S^{1,7} \cup R \cup B$ as well as some additional vertices lying outside V^0_{long} such that*

(i) *all vertices in $S^{1,7}$ are coloured α and all vertices in $R \cup B$ are coloured β,*
(ii) *all coloured vertices are safe,*

(iii) *the set* C_3 *consisting of all vertices coloured so far has size* $|C_3| \leq 5 \cdot 10^5 k^6 m$ *and* $|C_3 \setminus (C_2 \cup S^{1,7} \cup R \cup B)| \leq 220k^4$.

To prove Claim 4.4, we first colour all vertices in $S^{1,7}$ with α and all vertices in $R \cup B$ with β. Recall from (Q1) that $\{\mathrm{int}(Q_{i,j}) : (i,j) \in I_R\}$ is a collection of backwards-transitive paths with $|\mathrm{int}(Q_{i,j})| \geq k+1$. So we may apply the Lemma 2.7 to obtain sets U_R and W_R such that

(a) $U_R, W_R \subseteq R^{1,7}$,
(b) $|U_R|, |W_R| \leq 2k(k+1)$,
(c) for any set $F \subseteq V(T)$ of size at most $k-1$, and for every vertex $v \in V(T) \setminus F$ which lies on some path in $\{\mathrm{int}(Q_{i,j}) : (i,j) \in I_R\}$ there exists a directed path (possibly of length 0) in $T[(R^{1,7} \cup \{v\}) \setminus F]$ from v to a vertex in U_R and a directed path in $T[(R^{1,7} \cup \{v\}) \setminus F]$ from a vertex in W_R to v.

We next apply Lemma 2.7 to the collection of backwards-transitive paths $\{\mathrm{int}(Q_{i,j}) : (i,j) \in I_S\}$ to obtain sets $U_S, W_S \subseteq S^{1,7}$. Finally, we apply Lemma 2.7 to $\{\mathrm{int}(Q_{i,j}) : (i,j) \in I_{long,\beta} \times [1000k^4]\}$ to obtain sets $U_B, W_B \subseteq B^{1,7}$. Let $U := U_R \cup U_S \cup U_B$ and define W similarly. Apply Claim 2 with C_2, $U \cup W$, V^0_{long} playing the roles of C, Z, N to obtain a set $Z' \subseteq V(T) \setminus (V^0_{long} \cup C_2)$ and a colouring of the vertices in Z' such that every vertex in $U \cup W \cup Z'$ is safe and

$$|U \cup W \cup Z'| \leq 9k^2 |U \cup W| \leq 9k^2 \cdot 12k(k+1) \leq 220k^4.$$

So the set C_3 consisting of all vertices coloured so far satisfies $|C_3 \setminus (C_2 \cup S^{1,7} \cup R \cup B)| \leq 220k^4$ and $|C_3| \leq 220k^4 + |V^0_{long}| + |C_2| \leq 220k^4 + 3 \cdot 10^5 k^6 + 8000k^4 m \leq 5 \cdot 10^5 k^6 m$. Using (c) (and its analogue for U_S, W_S and U_B, W_B) it is now straightforward to check that (ii) holds. (To check that the vertices in $S^{1,7}$ are partition-safe we use that every vertex in S has least k in-neighbours and at least k out-neighbours in R and that all vertices in R are coloured β.) This completes the proof of Claim 4.4.

Claim 4.5: *For each* $s \in [k]$ *there are indices* $(i^\ell_s, j^\ell_s), (i^m_s, j^m_s), (i^r_s, j^r_s) \in I_S$ *such that*

(i) *the set* $\bigcup_{s \in [k]} \{i^\ell_s, i^m_s, i^r_s\}$ *has size* $3k$ *(i.e. all these indices are different from each other),*
(ii) *for each* $s \in [k]$ *and each* $2 \leq a \leq 6$ *no vertex in* $V(Q^a_{i^\ell_s, j^\ell_s} \cup Q^a_{i^m_s, j^m_s} \cup Q^a_{i^r_s, j^r_s})$ *is coloured,*
(iii) *for each* $s \in [k]$ *there is a directed edge* e^1_s *from the initial vertex of* $Q^3_{i^\ell_s, j^\ell_s}$ *to the initial vertex of* $Q^3_{i^m_s, j^m_s}$*, and a directed edge* e^2_s *from the final vertex of* $Q^5_{i^m_s, j^m_s}$ *to the final vertex of* $Q^5_{i^r_s, j^r_s}$.

Note that Claim 4.3 implies that for each $s \in I_{long,\alpha}$ there are at least $1000k^4 - |I_R| \geq 300k^4$ indices $j \in [1000k^4]$ for which $(s,j) \in I_S$. Since $|C_3 \setminus (C_2 \cup S^{1,7} \cup R \cup B)| \leq 220k^4$ by Claim 4.4(iii) and $C_2 \cap V_{long} = \emptyset$, we can pick an index $j = j(s)$ with $(s, j(s)) \in I_S$ and such that the coloured vertices on $\mathrm{int}(Q_{s,j(s)})$ are precisely those in $Q^1_{s,j(s)} \cup Q^7_{s,j(s)}$. Let $u(s)$ denote the initial vertex of $Q^3_{s,j(s)}$ (so $u(s) = q^{2k+2}_{s,j(s)}$) and let $v(s)$ denote the final vertex of $Q^5_{s,j(s)}$ (so $v(s) = q^{|Q_{s,j(s)}|-2k-2}_{s,j(s)}$).

Now consider the subtournament T_1 of T which is induced by all the vertices $v(s)$ for all $s \in I_{long,\alpha}$. Thus $|T_1| = |I_{long,\alpha}| \geq 18k$ by Claim 4.2. Together with Proposition 2.1 this implies that there is a set $I_1 \subseteq I_{long,\alpha}$ such that $|I_1| \geq 12k$ and such that for every $s \in I_1$ the vertex $v(s)$ has at least $3k$ out-neighbours in T_1. We now consider the subtournament T_2 of T which is induced by all the vertices $u(s)$ for all $s \in I_1$. By Proposition 2.1 applied to T_2 there is a set $I_2 \subseteq I_1$ such that $|I_2| \geq 6k$ and such that for every $s \in I_2$ the vertex $u(s)$ has at least $3k$ in-neighbours in T_2.

Now let $i_1^m, \dots, i_k^m$ be k distinct indices in I_2. For each $s \in [k]$ choose an index $i_s^\ell \in I_1$ such that $u(i_s^\ell)$ is an in-neighbour of $u(i_s^m)$ and such that the $2k$ indices $i_1^m, \dots, i_k^m, i_1^\ell, \dots, i_k^\ell$ are distinct. Finally, for each $s \in [k]$ choose an index $i_s^r \in I_{long,\alpha}$ such that $v(i_s^r)$ is an out-neighbour of $v(i_s^m)$ and such that the indices $i_1^r, \dots, i_k^r$ are distinct from each other and from $i_1^m, \dots, i_k^m, i_1^\ell, \dots, i_k^\ell$. This completes the proof of Claim 4.5.

We are now ready to prove Claim 4. For each $s \in [k]$ let Q_s denote the path formed by

$$Q^1_{i_s^\ell, j_s^\ell} \cup Q^2_{i_s^\ell, j_s^\ell} \cup Q^3_{i_s^m, j_s^m} \cup Q^4_{i_s^m, j_s^m} \cup Q^5_{i_s^m, j_s^m} \cup Q^6_{i_s^r, j_s^r} \cup Q^7_{i_s^r, j_s^r},$$

the initial vertices of both $Q_{i_s^\ell, j_s^\ell}$ and $Q^3_{i_s^\ell, j_s^\ell}$, the final vertices of both $Q_{i_s^r, j_s^r}$ and $Q^5_{i_s^r, j_s^r}$ as well as the edges e_s^1 and e_s^2 guaranteed by Claim 4.5(iii). Let $i'_s := i_s^\ell$ and $i_s := i_s^r$. Then Q_s joins the sink of $B_{i'_s}$ to the source of A_{i_s}, i.e. Claim 4(i) holds.

Recall that all the vertices in $Q^1_{i_s^\ell, j_s^\ell} \cup Q^7_{i_s^r, j_s^r}$ as well as the two endvertices of Q_s are coloured α, and all other vertices of Q_s are uncoloured (i.e. lie outside C_3). Colour all the (so far uncoloured) vertices of Q_s with α (for all $s \in [k]$) and then all other vertices in V_{long} which are still uncoloured with β. Let C_4 be the set of coloured vertices obtained in this way.

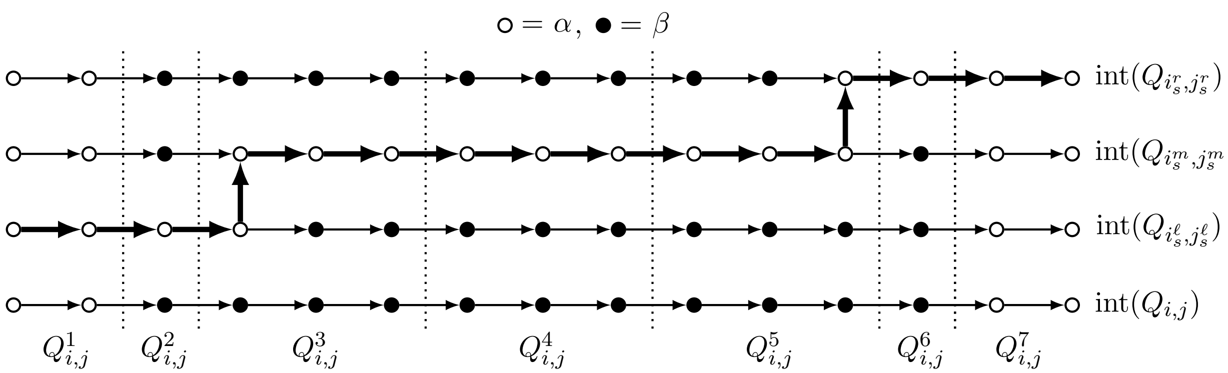

Figure 2 : Colour patterns of the paths $\mathrm{int}(Q_{i,j})$ with $(i,j) \in I_S$ in the case when $k = 1$. The thick arrows indicate $\mathrm{int}(Q_s)$.

Since $|C_3 \setminus (C_2 \cup S^{1,7} \cup R \cup B)| \le 220k^4$ by Claim 4.4(iii) and $C_2 \cap V_{long} = \emptyset$, for each $s \in I_{long,\beta}$ there is at least one index $j' = j'(s)$ such that $V(\mathrm{int}(Q_{s,j'(s)})) \cap C_3 = V(Q^0_{s,j'(s)})$. Moreover, since $V(Q^0_{s,j'(s)}) \subseteq B$, all the vertices in $V(Q^0_{s,j'(s)})$ are coloured β by Claim 4.4(i). Altogether this shows that all vertices on $Q_{s,j'(s)}$ are coloured β. For each $s \in I_{long,\beta}$ let $P_s := Q_{s,j'(s)}$. Together with the short paths P_s for all $s \in I_{short,\beta}$ this gives k paths satisfying Claim 4(ii). Our choice of the paths Q_s and P_s implies that Claim 4(iii) holds too.

Let us now check that all vertices in $C_4 \setminus C_3$ are safe. First consider any $v \in C_4 \setminus C_3$ which is coloured α. Then one of the following holds:

(a) $v \in S \setminus S^{1,7}$,
(b) $v \in V(Q^4_{i_s^m, j_s^m})$ for some $s \in [k]$.

Suppose first that (a) holds. So there exists $(i,j) \in I_S$ such that $v \in Q^2_{i,j} \cup Q^3_{i,j} \cup Q^5_{i,j} \cup Q^6_{i,j}$. Since $Q_{i,j}$ is a backwards-transitive path by (Q1), it follows that every vertex in $Q^1_{i,j}$ (except possibly its final vertex) is an out-neighbour of v and every vertex in $Q^7_{i,j}$ (except possibly its initial vertex) is an in-neighbour of v. Since all vertices in $S^{1,7} \supseteq Q^1_{i,j} \cup Q^7_{i,j}$ are coloured α and are safe, it follows that v has at least k safe in-neighbours and at least k safe out-neighbours of colour α. So by (S3) v is forwards- and backwards-safe. Since by Claim 4.3 v has at least k in-neighbours and k out-neighbours in R (and all vertices in R are coloured β) it follows that v is partition-safe. So v is safe.

Now suppose that (b) holds. As in (a) one can show that v is forwards- and backwards-safe. Moreover, by (Q1) every vertex in $Q^2_{i^m_s,j^m_s}$ is an out-neighbour of v and every vertex in $Q^6_{i^m_s,j^m_s}$ is an in-neighbour of v. But all the vertices in $Q^2_{i^m_s,j^m_s} \cup Q^6_{i^m_s,j^m_s}$ are coloured β, so v is partition-safe and thus safe.

Now consider any $v \in C_4 \setminus C_3$ which is coloured β. Then one of the following holds:

(c) $v \in S \setminus S^{1,7}$,
(d) $v \in V(Q^4_{i,j})$ for some $i \in I_{long}$ and $j \in [1000k^4]$ such that $(i,j) \notin \{(i^m_s, j^m_s) : s \in [k]\}$.

If (c) holds then v has at least k in-neighbours and k out-neighbours in R. Since all vertices in R are coloured β and are safe, this implies that v is safe. Moreover, together with Claim 4.4(ii) and the safety of the vertices in $C_4 \setminus C_3$ which are coloured α, this implies that all vertices in V^0_{long} are safe.

Now suppose that (d) holds. Since $(i,j) \notin \{(i^m_s, j^m_s) : s \in [k]\}$ all vertices in $Q^3_{i,j}$ (except possibly its initial vertex) and all vertices in $Q^5_{i,j}$ (except possibly its final vertex) are coloured β. Moreover, all these vertices are safe since they lie in V^0_{long}. By (Q1) every vertex in $\mathrm{int}(Q^3_{i,j})$ is an out-neighbour of v and every vertex in $\mathrm{int}(Q^5_{i,j})$ is an in-neighbour of v. So v is safe. This completes the proof that all vertices in $C_4 \setminus C_3$ (and thus also all coloured vertices) are safe, i.e. Claim 4(iv)(α) holds.

Let C_α be the union of $V(Q^4_{i^m_s,j^m_s})$ over all $s \in [k]$. Thus the number of vertices of colour α outside C_α is at most $|C_3| + |V^0_{long}| \leq 10^6k^6m$, i.e. Claim 4(iv)($\beta$) holds. Moreover, if $v \in V(T) \setminus C_4$ and v has an in-neighbour in some $V(Q^4_{i^m_s,j^m_s})$ then by (Q3) all vertices in $Q^6_{i^m_s,j^m_s}$ are also in-neighbours of v. But all vertices in $Q^6_{i^m_s,j^m_s}$ are coloured β. So v has at least k in-neighbours of colour β. Similarly, if v has an out-neighbour in $V(Q^4_{i^m_s,j^m_s})$ then by (Q2) all vertices in $Q^2_{i^m_s,j^m_s}$ are also out-neighbours of v. But all vertices in $Q^2_{i^m_s,j^m_s}$ are coloured β. So v has at least k out-neighbours of colour β. This shows that Claim 4(iv)(γ) holds and thus completes the proof of Claim 4.

The next claim shows that by colouring every uncoloured vertex with β, all vertices will become safe. Together with Claim 1 this then implies that the partition consisting of the colour classes V_α, V_β is as required in Theorem 1.2.

Claim 5: *We can colour all uncoloured vertices with β. Then every vertex is safe.*

Colour all uncoloured vertices (i.e. all vertices in $V(T) \setminus C_4$) with β. Consider any vertex $v \in V(T) \setminus C_4$. If $v \notin E'$ then by (D3) and (D4) v has an in-neighbour in A_s and an out-neighbour in B_s for every $19k < s \leq 20k$. Since the vertices in all these sets A_s and B_s are coloured β and are safe, this implies that v is safe.

Suppose next that $v \in E'_B \setminus E'_A$. As above it follows that v has k safe in-neighbours of colour β. If v has k out-neighbours of colour β which are lying outside E', then these out-neighbours are safe and so v is safe. So suppose that v has less than k out-neighbours of colour β which are lying outside E'. Recall from Claim 4(iv)(β) that at most 10^6k^6m vertices of colour α lie outside the set C_α. Together with the fact that $\hat{\delta}^+(T) - |E'| \geq 5 \cdot 10^6k^6m \geq k + 10^6k^6m$ by (3.4), this implies that v has an out-neighbour in C_α. But now Claim 4(iv)(γ) implies that v has k out-neighbours of colour β in C_4. Since all the vertices in C_4 are safe, this shows that v is safe.

Finally, suppose that $v \in E'_A$. As in the previous case one can show that v has k safe out-neighbours of colour β. If v has k in-neighbours of colour β which are lying outside E'_A, then these in-neighbours are safe and so v is safe. So suppose that v has less than k in-neighbours of colour β which are lying outside E'_A. Together with the fact that $\hat{\delta}^-(T) - |E'_A| \geq 5 \cdot 10^6 k^6 m \geq k + 10^6 k^6 m$ by (3.3), this implies that v has an in-neighbour in C_α. Thus Claim 4(iv)(γ) implies that v has k in-neighbours of colour β in C_4. Since all the vertices in C_4 are safe, this shows that v is safe. This completes the proof of Claim 5 and thus of Theorem 1.2. □

References

[1] G. Chen, R. Gould and X. Yu, Graph connectivity after path removal, *Combinatorica* **23** (2003) 185–203.

[2] P. Hajnal, Partition of graphs with condition on the connectivity and minimum degree, *Combinatorica* **3** (1983), 95–99.

[3] K. Kawarabayashi, O. Lee, B. Reed, P. Wollan, A weaker version of Lovász path removal conjecture. *J. Combinatorial Theory B* **98** (2008), 972-979.

[4] M. Kriesell, Induced paths in 5-connected graphs, *J. Graph Theory* **36** (2001), 52–58.

[5] D. Kühn, J. Lapinskas, D. Osthus and V. Patel, Proof of a conjecture of Thomassen on Hamilton cycles in highly connected tournaments, *Proc. London Math. Soc.*, to appear.

[6] D. Kühn and D. Osthus, Partitions of graphs with high minimum degree or connectivity, *J. Combinatorial Theory B* **88** (2003), 29–43.

[7] D. Kühn, D. Osthus and T. Townsend, Proof of a tournament partition conjecture and an application to 1-factors with prescribed cycle lengths, *Combinatorica*, to appear.

[8] A. Pokrovskiy, Highly linked tournaments, preprint, arxiv:1406.7552.

[9] A. Pokrovskiy, Edge disjoint Hamiltonian cycles in highly connected tournaments, preprint, arxiv:1406.7556.

[10] C. Thomassen, Graph decomposition with constraints on the connectivity and minimum degree, *J. Graph Theory* **7** (1983), 165–167.

[11] C. Thomassen, Graph decomposition with applications to subdivisions and path systems modulo k, *J. Graph Theory* **7** (1983), 261–271.

Jaehoon Kim, Daniela Kühn, Deryk Osthus
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {kimJS, d.kuhn, d.osthus}@bham.ac.uk