

RESOLUTION OF THE OBERWOLFACH PROBLEM

STEFAN GLOCK, FELIX JOOS, JAEHOON KIM, DANIELA KÜHN, AND DERYK OSTHUS

ABSTRACT. The Oberwolfach problem, posed by Ringel in 1967, asks for a decomposition of K_{2n+1} into edge-disjoint copies of a given 2-factor. We show that this can be achieved for all large n . We actually prove a significantly more general result, which allows for decompositions into more general types of factors. In particular, this also resolves the Hamilton-Waterloo problem for large n .

1. INTRODUCTION

A central theme in Combinatorics and related areas is the decomposition of large discrete objects into simpler or smaller objects. In graph theory, this can be traced back to the 18th century, when Euler asked for which orders orthogonal Latin squares exist (which was finally answered by Bose, Shrikhande, and Parker [7]). This question can be reformulated as the existence question for resolvable triangle decompositions in the balanced complete tripartite graph. (Here a resolvable triangle decomposition is a decomposition into edge-disjoint triangle factors.) In the 19th century, Walecki proved the existence of decompositions of the complete graph K_n (with n odd) into edge-disjoint Hamilton cycles and Kirkman formulated the school girl problem. The latter triggered the question for which n the complete graph on n vertices admits a resolvable triangle decomposition, which was finally resolved in the 1970s by Ray-Chaudhuri and Wilson [37] and independently by Lu [32] (see also [40]). This topic has developed into a vast area with connections e.g. to statistical design and scheduling, Latin squares and arrays, graph labellings as well as combinatorial probability.

A far reaching generalisation of Walecki's theorem and Kirkman's school girl problem is the following problem posed by Ringel in Oberwolfach in 1967 (cf. [20]).

Problem 1.1 (Oberwolfach problem). *Let $n \in \mathbb{N}$ and let F be a 2-regular graph on n vertices. For which (odd) n and F does K_n decompose into edge-disjoint copies of F ?*

Addressing conference participants in Oberwolfach, Ringel fittingly formulated his problem as a scheduling assignment for diners: assume n people are to be seated around round tables for $\frac{n-1}{2}$ meals, where the total number of seats is equal to n , but the tables may have different sizes. Is it possible to find a seating chart such that every person sits next to any other person exactly once?

We answer this affirmatively for all sufficiently large n . We make no attempt to estimate or optimize the smallest n for which our proof works. This makes our argument significantly clearer. It is known that for $n \leq 60$, all but four instances have a solution [14, 38].

A generalisation of the Oberwolfach problem is the Hamilton-Waterloo problem (cf. [1]); here, two cycle factors are given and it is prescribed how often each of them is to be used in the decomposition. Informally, this means the conference takes place in two nearby venues (Hamilton and Waterloo), with possibly different seating arrangements. We also resolve this problem in the affirmative (for large n) via the following even more general result. We allow an arbitrary collection of types of cycle factors, as long as one type appears linearly many

Date: January 25, 2021.

The research leading to these results was partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant 306349 (S. Glock, J. Kim and D. Osthus), and the DFG, grant no. JO 1457/1-1 (F. Joos). The research was also partially supported by the EPSRC, grant no. EP/N019504/1, and by the Royal Society and the Wolfson Foundation (D. Kühn).

times. This immediately implies that the Hamilton-Waterloo problem has a solution for large n for any bounded number of given cycle factors.

Theorem 1.2. *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all odd $n \geq n_0$ the following holds. Let F_1, \dots, F_k be 2-regular graphs on n vertices and let $m_1, \dots, m_k \in \mathbb{N}$ be such that $\sum_{i \in [k]} m_i = (n-1)/2$ and $m_1 \geq \alpha n$. Then K_n admits a decomposition into graphs $H_1, \dots, H_{(n-1)/2}$ such that for exactly m_i integers j , the graph H_j is isomorphic to F_i .*

Here we say a graph G admits a *decomposition* into graphs H_1, \dots, H_t if there exist edge-disjoint copies of H_1, \dots, H_t in G such that every edge of G belongs to exactly one copy.

Several authors (see e.g. Huang, Kotzig, and Rosa [23]) considered a variant of the Oberwolfach problem for even n ; to be precise, here we ask for a decomposition of K_n minus a perfect matching into $n/2 - 1$ copies of some given n -vertex 2-regular graph F . We will deduce Theorem 1.2 from a more general result (Theorem 1.3) which also covers this case.

The Oberwolfach problem and its variants have attracted the attention of many researchers, resulting in more than 100 research papers covering a large number of partial results. Most notably, Bryant and Scharaschkin [11] proved it for infinitely many n . Traetta [39] solved the case when F consists of two cycles only, Bryant and Danziger [8] solved the variant for even n if all cycles are of even length, Alspach, Schellenberg, Stinson, and Wagner [5] solved the case when all cycles have equal length (see [22] for the analogous result for n even), and Hilton and Johnson [21] solved the case when all but one cycle have equal length. The bipartite analogue of the Oberwolfach problem (2-factorizations of $K_{n,n}$) was solved in [35]. We refer to the surveys [4, 10] for more results on the Oberwolfach problem.

A related conjecture of Alspach from 1981 stated that for all odd n the complete graph K_n can be decomposed into any collection of cycles of length at most n whose lengths sum up to $\binom{n}{2}$. This was solved by Bryant, Horsley, and Pettersson [9].

Most classical results in the area are based on algebraic approaches, often by exploiting symmetries. More recently, major progress for decomposition problems has been achieved via absorbing techniques in combination with approximate decomposition results (often also in conjunction with probabilistic ideas). This started off with decompositions into Hamilton cycles [13, 31], followed by the existence of combinatorial designs [19, 26, 27] and progress on the tree packing conjecture [25]. In this paper, at a very high level, we also pursue such an approach. As approximate decomposition results, we exploit a hypergraph matching argument due to Alon and Yuster [3] (which in turn is based on the Rödl nibble via the Pippenger-Spencer theorem [36]) and a bandwidth theorem for approximate decompositions due to Condon, Kim, Kühn, and Osthus [12]. Our absorption procedure utilizes as a key element a very special case of a recent result of Keevash on resolvable designs [27].

Whenever we only seek an approximate decomposition of a graph G , the target graphs can be significantly more general and divisibility conditions disappear. In particular, Allen, Böttcher, Hladký, and Piguet [2] considered approximate decompositions into graphs of bounded degeneracy and maximum degree $o(n/\log n)$ whenever the host graph G is sufficiently quasirandom, and Kim, Kühn, Osthus, and Tyomkyn [29] considered approximate decompositions into graphs of bounded degree in host graphs G satisfying weaker quasirandom properties (namely, ε -superregularity, see Section 3.3). Their resulting blow-up lemma for approximate decompositions was a key ingredient for [12, 25] (and thus for the current paper too). It also implies that an approximate solution to the Oberwolfach problem can always be found (this was obtained independently by Ferber, Lee, and Mousset [17]).

Our Theorem 1.2 actually follows from the following more general Theorem 1.3, which allows separable graphs. An n -vertex graph H is said to be ξ -*separable* if there exists a set S of at most ξn vertices such that every component of $H \setminus S$ has size at most ξn . Examples of separable graphs include cycles, powers of cycles, planar graphs, and F -factors. More generally, for bounded degree graphs, the notion of separability is equivalent to that of small bandwidth.

Theorem 1.3. *For given $\Delta \in \mathbb{N}$ and $\alpha > 0$, there exist $\xi_0 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $\xi < \xi_0$. Let \mathcal{F}, \mathcal{H} be collections of graphs satisfying the following:*

- \mathcal{F} is a collection of at least αn copies of F , where F is a 2-regular n -vertex graph;
- each $H \in \mathcal{H}$ is a ξ -separable n -vertex r_H -regular graph for some $r_H \leq \Delta$;
- $e(\mathcal{F} \cup \mathcal{H}) = \binom{n}{2}$.

Then K_n decomposes into $\mathcal{F} \cup \mathcal{H}$.

Clearly, Theorem 1.3 implies Theorem 1.2 and also its corresponding version if n is even and we ask for a decomposition of K_n minus a perfect matching.

While far more general than the Oberwolfach problem, Theorem 1.3 may be just the tip of the iceberg, and it seems possible that the following is true.

Conjecture 1.4. *For all $\Delta \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that the following holds for all $n \geq n_0$. Let F_1, \dots, F_t be n -vertex graphs such that F_i is r_i -regular for some $r_i \leq \Delta$ and $\sum_{i \in [t]} r_i = n - 1$. Then there is a decomposition of K_n into F_1, \dots, F_t .*

The above conjecture is implicit in the ‘meta-conjecture’ on decompositions proposed in [2].

Rather than considering decompositions of the complete graph K_n , it is also natural to consider decompositions of host graphs of large minimum degree (this has applications e.g. to the completion of partial decompositions of K_n). Indeed, a famous conjecture of Nash-Williams [34] states that every n -vertex graph G of minimum degree at least $3n/4$ has a triangle decomposition (subject to the necessary divisibility conditions). The following conjecture would (asymptotically) transfer this to arbitrary 2-regular spanning graphs.

Conjecture 1.5. *Suppose G is an n -vertex r -regular graph with even $r \geq \frac{3}{4}n + o(n)$ and F is a 2-regular graph on n vertices. Then G decomposes into copies of F .*

The (asymptotic version of the) Nash-Williams conjecture was reduced to its fractional version in [6]. In combination with [15], this shows that the Nash-Williams conjecture holds with $3n/4$ replaced by $9n/10 + o(n)$. There has also been considerable progress on decomposition problems involving such host graphs of large minimum degree into other fixed subgraphs H rather than triangles [18, 33]. It turns out that the chromatic number of H is a crucial parameter for this problem. In particular, as proved in [18], for bipartite graphs H the ‘decomposition threshold’ is always at most $\frac{2}{3}n + o(n)$.

Clearly, one can generalise Conjecture 1.5 in this direction, e.g. to determine the decomposition threshold for K_r -factors. It might also be true that the ‘ $3/4$ ’ in Conjecture 1.5 can be replaced by ‘ $2/3$ ’ if F consists only of even cycles. We are confident that the ideas from this paper will be helpful in approaching these and other related problems. As remarked in Section 5, the current method already allows us to replace K_n by any $2r$ -regular host graph G with $2r \geq (1 - \varepsilon)n$.

This paper is organised as follows. In Section 2 we provide an overview of our approach and in Section 3 we collect several embedding and decomposition results including the bandwidth theorem for approximate decompositions from [12] and a special case of a result of Keevash [27] on resolvable designs. We prove Theorem 1.3 in Section 4 and add some concluding remarks in Section 5.

2. PROOF SKETCH

For simplicity, we just sketch the argument for the setting of the Oberwolfach problem; that is, we aim to decompose K_n into $\frac{n-1}{2}$ copies of an n -vertex 2-regular graph F . The proof essentially splits into two cases. In the first case we assume that almost all vertices of F belong to ‘short’ cycles, of length at most 500. Note that there must be some cycle length, say ℓ^* , such that at least $n/600$ vertices of F lie in cycles of length ℓ^* . We will take a suitable number of random slices of the edges of K_n and then first embed, for every desired

copy of F , all cycles whose lengths are different from ℓ^* . For this, we use standard tools based on the Rödl nibble. We then complete the decomposition by embedding all the cycles of length ℓ^* . This last step uses a special case of a recent result of Keevash on the existence of resolvable designs [27].

The second case is much more involved and forms the core of the proof. We are now guaranteed that a (small) proportion of vertices of F lies in ‘long’ cycles. To motivate our approach, consider the following simplified setup. Suppose F consists only of cycles whose lengths are divisible by 3, and suppose for the moment we seek an F -decomposition of a 3-partite graph G with equitable vertex partition (V_1, V_2, V_3) (so G is a C_3 -blowup). Let ℓ_1, \dots, ℓ_t be the sequence of cycle lengths appearing in F . Now, take any permutation π on V_3 which consists of cycles of lengths $\ell_1/3, \dots, \ell_t/3$. For instance, a C_3 in F corresponds to a fixed point in π , and a C_6 in F corresponds to a transposition in π . Now, define an auxiliary graph $\pi(G)$ by ‘rewiring’ the edges between V_2 and V_3 according to π . More precisely, we ensure that $E_{\pi(G)}(V_2, V_3) = \{v_2\pi(v_3) : v_2v_3 \in E(G)\}$. Suppose that F' is a C_3 -factor in $\pi(G)$. By ‘reversing’ the rewiring, we obtain a copy of F in G . More precisely, let $\pi^{-1}(F')$ be the graph obtained from F' by replacing $F'[V_2, V_3]$ with $\{v_2\pi^{-1}(v_3) : v_2v_3 \in E(F')\}$. Clearly, $\pi^{-1}(F') \cong F$ and $\pi^{-1}(F') \subseteq G$. What is more, this rewiring is canonical in the following sense: if F' and F'' are edge-disjoint C_3 -factors in $\pi(G)$, then $\pi^{-1}(F')$ and $\pi^{-1}(F'')$ will be edge-disjoint copies of F in G . Thus, a resolvable C_3 -decomposition of $\pi(G)$ immediately translates into an F -decomposition of G .

Similarly, if all cycle lengths in F are divisible by 4, we can reduce the problem of finding an F -decomposition of a C_4 -blowup to the problem of finding a resolvable C_4 -decomposition of a suitably rewired C_4 -blowup. In order to deal with arbitrary 2-regular graphs F , we interweave such constructions for C_3 , C_4 and C_5 . In Sections 4.1 and 4.2, we will construct an ‘absorbing graph’ G which is a partite graph on 18 vertex classes such that finding an F -decomposition of G can be reduced to finding resolvable C_3, C_4, C_5 -decompositions of suitable auxiliary graphs, in a similar way as sketched above. Crucially, G has this property in a robust sense: even if we delete an arbitrary sparse graph L from G , as long as some necessary divisibility conditions hold, we are still able to find an F -decomposition of $G - L$.

The overall strategy is thus as follows: first, we remove G from K_n . Then we find an approximate decomposition of the remainder, which leaves a sparse leftover. For this, we employ the recent bandwidth theorem for approximate decompositions [12]. (The existence of an approximate decomposition of $K_n - G$ would also follow directly from the blow-up lemma for approximate decompositions [29], but this would leave a leftover whose density is larger than that of the absorbing graph G , making our approach infeasible.) We then deal with this leftover by using some edges of G , in a very careful way, such that the remainder of G is still appropriately divisible. The remainder of G then decomposes as sketched above. In order to decompose the auxiliary graphs, we again use a very special case of the main result in [27]. The fact that we are guaranteed that F has some long cycles will be helpful to construct the absorbing graph G , more precisely, to ensure that all the 18 vertex classes are of linear size. It is also essential when dealing with the leftover of the approximate decomposition.

3. PRELIMINARIES

In this section, we first introduce some notation, then a Chernoff-type concentration inequality, several graph embedding tools and notation concerning quasirandomness, as well as the framework of the result of Keevash on resolvable decompositions.

3.1. Notation. For a 2-regular graph F , let $\mathcal{C}(F)$ denote the collection of cycles in F . We also refer to a 2-regular graph as a *cycle factor*. Let G be a graph. We denote by $e(G)$ the number of edges of G , and by $|G|$ the number of vertices of G . For sets $U, U' \subseteq V(G)$, we define $e_G(U)$ as the number of edges of the graph induced by U and $e_G(U, U')$ as the number of pairs $(u, u') \in U \times U'$ such that $uu' \in E(G)$. Hence, $e_G(U, U) = 2e_G(U)$. For a vertex

$v \in V(G)$, we define $d_G(v, U) := |N_G(v) \cap U|$, where $N_G(v)$ is the neighbourhood of v in G . For a subgraph $H \subseteq G$, we write $G - H$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$. We write $G \setminus X$ for the graph obtained from G by removing the vertices of X , and $G \setminus H := G \setminus V(H)$. Given two graphs G_1 and G_2 , define $G_1 \Delta G_2$ to be the graph on $V(G_1) \cup V(G_2)$ whose edge set is $(E(G_1) \setminus E(G_2)) \cup (E(G_2) \setminus E(G_1))$.

Given graphs F and G , a function $\sigma: V(F) \rightarrow V(G)$ is a *homomorphism* if $\sigma(x)\sigma(y) \in E(G)$ for all $xy \in E(F)$. An injective homomorphism is called an *embedding*. (Note that non-edges need not be preserved, that is, the corresponding subgraphs of F in G are not required to be induced.) For a fixed graph F , an *F-factor* in a graph G is a collection \mathcal{F} of vertex-disjoint copies of F in G which cover all vertices of G . An *F-decomposition* of G is a collection \mathcal{F} of edge-disjoint copies of F in G which cover all edges of G . An *F-decomposition* \mathcal{F} is called *resolvable* if it can be partitioned into *F-factors*.

For a collection of graphs \mathcal{H} , we define $e(\mathcal{H})$ by $\sum_{H \in \mathcal{H}} e(H)$. We write $G - \mathcal{H}$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus \bigcup_{H \in \mathcal{H}} E(H)$. We also write $\Delta(\mathcal{H})$ for the maximum degree of $\bigcup_{H \in \mathcal{H}} H$. We say that $\mathcal{H} = \{H_1, \dots, H_t\}$ *packs* into a graph G if there exist edge-disjoint subgraphs H'_1, \dots, H'_t in G such that H'_i is a copy of H_i for each $i \in [t]$.

Let \mathcal{X} be a set of disjoint vertex sets and R a graph on \mathcal{X} . If G is a graph with vertex partition \mathcal{X} , where each $X \in \mathcal{X}$ is independent in G , and such that $e_G(X, X') = 0$ for all distinct $X, X' \in \mathcal{X}$ with $XX' \notin E(R)$, then we say that G has *reduced graph* R .

For a digraph D and a vertex $v \in V(D)$, we write $d_D^+(v)$ and $d_D^-(v)$ for the number of outgoing and incoming arcs at v , respectively. We say that D is *r-regular* if $d_D^+(v) = d_D^-(v) = r$ for all $v \in V(D)$. Sometimes we write \vec{D} for an oriented graph and then D is the undirected graph obtained from \vec{D} by ignoring the orientations of \vec{D} .

We write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $[n] := \{1, \dots, n\}$. For a set of objects indexed by $[t]$, we often treat indices modulo t , and define $a \pmod t$ to be the unique integer $b \in [t]$ such that $a \equiv b \pmod t$. For $a, b, c \in \mathbb{R}$, we write $a = b \pm c$ whenever $a \in [b-c, b+c]$. For $a, b, c \in (0, 1]$, we write $a \ll b \ll c$ in our statements to mean that there are increasing functions $f, g: (0, 1] \rightarrow (0, 1]$ such that whenever $a \leq f(b)$ and $b \leq g(c)$, then the subsequent result holds.

3.2. Probabilistic tools. At several stages of our proof we will apply the following standard Chernoff-type concentration inequalities.

Lemma 3.1 (see [24, Corollary 2.3, Corollary 2.4 and Theorem 2.8]). *Let X be the sum of n independent Bernoulli random variables. Then the following hold.*

- (i) *For all $0 \leq \varepsilon \leq 3/2$, we have $\mathbb{P}[|X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]] \leq 2e^{-\varepsilon^2 \mathbb{E}[X]/3}$.*
- (ii) *If $t \geq 7\mathbb{E}[X]$, then $\mathbb{P}[X \geq t] \leq e^{-t}$.*

The following follows easily from Lemma 3.1(i). An explicit derivation can be found in [19].

Lemma 3.2. *Let $1/n \ll p, \alpha, 1/B$. Let \mathcal{I} be a set of size at least αn and let $(X_i)_{i \in \mathcal{I}}$ be a family of Bernoulli random variables with $\mathbb{P}[X_i = 1] \geq p$. Suppose that \mathcal{I} can be partitioned into at most B sets $\mathcal{I}_1, \dots, \mathcal{I}_k$ such that for each $j \in [k]$, the variables $(X_i)_{i \in \mathcal{I}_j}$ are independent. Let $X := \sum_{i \in \mathcal{I}} X_i$. Then we have*

$$\mathbb{P}\left[X \neq (1 \pm n^{-1/5})\mathbb{E}[X]\right] \leq e^{-n^{1/6}}.$$

3.3. Embedding and decomposition results. Frequently in our proof we want to embed parts of a 2-regular graph into ‘random-like’ graphs. For such a task, the blow-up lemma developed by Komlós, Sárközy and Szemerédi [30] is a standard tool. Roughly speaking, it says that given a k -partite graph G that is ‘super-regular’ between any two vertex classes, and a k -partite bounded-degree graph H with a matching vertex partition, then H is a subgraph of G . The notion ‘super-regular’ is tailored towards being used after an application of Szemerédi’s regularity lemma. Since we do not use Szemerédi’s regularity lemma, but work essentially in random subgraphs of the complete graph, we can use a more convenient notion which is defined as follows.

We say that a graph G on n vertices is (ε, d) -*quasirandom* if $d_G(v) = (d \pm \varepsilon)n$ for all $v \in V(G)$ and $|N_G(v_1) \cap N_G(v_2)| = (d^2 \pm \varepsilon)n$ for all distinct $v_1, v_2 \in V(G)$.

If V_1 and V_2 are disjoint vertex sets in G , we also say that $G[V_1, V_2]$ is (ε, d) -*quasirandom* if for both $j \in [2]$, we have that $d_G(v, V_{3-j}) = (d \pm \varepsilon)|V_{3-j}|$ for all $v \in V_j$ and $|N_G(v_1) \cap N_G(v_2) \cap V_{3-j}| = (d^2 \pm \varepsilon)|V_{3-j}|$ for all distinct $v_1, v_2 \in V_j$. It is well known that these conditions imply super-regularity (see [16]), that is, in addition to the degree condition, one also knows that between any large enough sets $V'_1 \subseteq V_1, V'_2 \subseteq V_2$, the edge density is very close to d . We need the following version of the blow-up lemma of Komlós, Sárközy and Szemerédi, which we just state in the setting of quasirandom pairs.

Lemma 3.3 (Blow-up lemma, [30]). *Let $1/n \ll \varepsilon \ll \kappa, d, 1/\Delta, 1/r$. Suppose that G is an n -vertex graph with vertex partition \mathcal{X} and reduced graph R , where $|R| \leq r$. Assume that $G[X, X']$ is $(\varepsilon, d_{XX'})$ -quasirandom for some $d_{XX'} \geq d$ whenever $XX' \in E(R)$. Assume also that $\min_{X \in \mathcal{X}} |X| \geq \kappa \max_{X \in \mathcal{X}} |X|$.*

Let H be a graph with $\Delta(H) \leq \Delta$. Let $X_0 \subseteq V(H)$ be independent such that $|X_0| \leq \varepsilon n$ and no two vertices in X_0 have a common neighbour in H . Assume that $\sigma: H \rightarrow R$ is a homomorphism such that $|\sigma^{-1}(X)| = |X|$ for all $X \in \mathcal{X}$, and $\phi_0: X_0 \rightarrow V(G)$ is an injective function such that $\phi_0(x) \in \sigma(x)$ for all $x \in X_0$.

Then there exists an embedding $\phi: H \rightarrow G$ which extends ϕ_0 such that $\phi(x) \in \sigma(x)$ for all $x \in V(H)$.

The following is a straightforward consequence of the blow-up lemma.

Corollary 3.4. *Suppose $1/n \ll \varepsilon \ll d$. Let G be an (ε, d) -quasirandom n -vertex graph and suppose that H is a graph on (at most) n vertices with $\Delta(H) \leq 2$. Suppose an independent set $X_0 \subseteq V(H)$ such that $|X_0| \leq \varepsilon n$ and an injective function $\phi_0: X_0 \rightarrow V(G)$ are given. Then there exists an embedding ϕ of H into G which extends ϕ_0 .*

The following result due to Condon, Kim, Kühn and Osthus [12] is a key ingredient in our approach. It is in turn based on the blow-up lemma for approximate decompositions [29], and uses Szemerédi's regularity lemma.

Theorem 3.5 ([12]). *For all $\Delta \in \mathbb{N} \setminus \{1\}$, $0 < \nu < 1$, there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $1 - \frac{1}{200\Delta} \leq d \leq 1$ the following holds: Suppose that \mathcal{H} is a collection of n -vertex ξ -separable graphs and G is an n -vertex graph such that*

- (i) $d_G(x) = (d \pm \xi)n$ for all $x \in V(G)$;
- (ii) $\Delta(H) \leq \Delta$ for all $H \in \mathcal{H}$;
- (iii) $e(\mathcal{H}) \leq (1 - \nu)e(G)$.

Then \mathcal{H} packs into G .

3.4. Partite decompositions of typical graphs. We will make use of a recent result of Keevash [27] on partite decompositions of typical (hyper-)graphs. His result applies in a far more general setting and we only need a simple consequence thereof here (see Theorem 3.6). For simplicity, we only introduce the relevant concepts for partite graph decompositions.

We first define typicality. Roughly speaking, a graph is typical if common neighbourhoods are as large as one would expect in a random graph. We also need such a notion for partite structures. Suppose that G is a graph with vertex partition (V_1, \dots, V_t) (we do not assume that these sets are independent). Let $\tau: V(G) \rightarrow [t]$ denote the assignment function such that $v \in V_{\tau(v)}$ for all $v \in V(G)$. Given a symmetric matrix $D \in [0, 1]^{t \times t}$, $s \in \mathbb{N}$ and $\varepsilon > 0$, we say that G is (ε, s, D) -*typical* if for any set $S \subseteq V(G)$ with $|S| \leq s$ and any $i \in [t]$, we have that

$$\left| V_i \cap \bigcap_{v \in S} N_G(v) \right| = (1 \pm \varepsilon)|V_i| \prod_{v \in S} D_{\tau(v)i}. \quad (3.1)$$

Note that if G is a weighted binomial random graph where an edge between V_i and V_j is included with probability D_{ij} , then (3.1) holds with high probability. Note also that if

$D_{ij} = 0$ and (3.1) holds, then $e_G(V_i, V_j) = 0$. We will often write $D_{V_i V_j}$ instead of D_{ij} and D_{V_i} instead of $D_{V_i V_i}$. If $t = 1$, we simply write (ε, s, D_{11}) -typical instead of (ε, s, D) -typical.

Our aim is to decompose G into a given graph H , following a prescribed pattern. More precisely, let $\sigma: V(H) \rightarrow [t]$ be an assignment of the vertices of H to the vertex partition classes of G . Given an embedding $\phi: H \rightarrow G$ of H into G such that $\tau(\phi(x)) = \sigma(x)$ for all $x \in V(H)$, we say that $\phi(H)$ is a σ -copy of H in G . A collection \mathcal{H} of edge-disjoint σ -copies of H in G is called an (H, σ) -packing in G . We say that \mathcal{H} is an (H, σ) -decomposition of G if, in addition, every edge of G is covered.

Roughly speaking, Theorem 3.6 guarantees an (H, σ) -decomposition of G under two assumptions: typicality and divisibility. We have already defined typicality of G . Now we relate the density matrix D to the given assignment σ of H . Let (P_1, \dots, P_t) be the partition of $V(H)$ induced by σ , i.e. $P_i = \sigma^{-1}(i)$ for each $i \in [t]$. We let $I^{H, \sigma}$ denote the symmetric indicator $(t \times t)$ -matrix defined as $I_{ij}^{H, \sigma} := 1$ if $e_H(P_i, P_j) > 0$ and $I_{ij}^{H, \sigma} := 0$ otherwise. We will later require that $D \geq d \cdot I^{H, \sigma}$. This accounts for the fact that if some edge of H is mapped to (V_i, V_j) , then we require this pair to be sufficiently dense in G .

Finally, we say that G is (H, σ) -divisible if the following hold:

- (i) there exists $m \in \mathbb{N}_0$ such that $e_G(V_i, V_j) = m \cdot e_H(P_i, P_j)$ for all $i, j \in [t]$;
- (ii) for all $i \in [t]$ and every $v \in V_i$, there are $(a_x^v)_{x \in P_i} \in \mathbb{N}_0$ such that $d_G(v, V_j) = \sum_{x \in P_i} a_x^v \cdot d_H(x, P_j)$ for all $j \in [t]$.

It is easy to see that (H, σ) -divisibility is a necessary condition for an (H, σ) -decomposition to exist. If σ is bijective, we follow the notation in [27] and simply say that G is H -balanced if it is (H, σ) -divisible, and an (H, σ) -decomposition is simply called a *partite H -decomposition*.

Theorem 3.6 ([27, cf. Theorem 7.8]). *Let $1/n \ll \varepsilon \ll 1/s \ll d, 1/h$. Let G be (ε, s, D) -typical with vertex partition (V_1, \dots, V_t) such that $dn \leq |V_i| \leq n$ for all $i \in [t]$. Let H be a graph on h vertices and $\sigma: V(H) \rightarrow [t]$. Suppose that $D \geq d \cdot I^{H, \sigma}$ and that G is (H, σ) -divisible. Then G has an (H, σ) -decomposition.*

We now use Theorem 3.6 to deduce the two results about resolvable cycle decompositions which we will need later. The following will be used in Case 1 of our proof, where most vertices of F are contained in cycles of length at most 500.

Corollary 3.7. *Let $1/n \ll \varepsilon \ll 1/s \ll d, 1/\ell$. Let G be (ε, s, D) -typical with vertex partition (V, U) such that $dn \leq |V|, |U| \leq n$. Assume that $D_V, D_{VU} \geq d$ and $D_U = 0$. Let W_ℓ be the wheel graph with ℓ spokes and hub w , and let σ assign w to U and all other vertices to V . Assume that $d_G(v, V) = 2d_G(v, U)$ for all $v \in V$ and $\ell \mid d_G(u)$ for all $u \in U$. Then G has a (W_ℓ, σ) -decomposition.*

Proof. Note that G is (W_ℓ, σ) -divisible, so we can apply Theorem 3.6. □

Note that given such a decomposition, for every vertex $u \in U$, the collection of all cycles which together with u form a wheel in the decomposition form a C_ℓ -factor of $G[N_G(u)]$.

We will also need the following approximate version of Corollary 3.7, which is much simpler to prove and follows from standard hypergraph matching results based on the Rödl nibble. Note that we do not need to assume divisibility in this case, which makes it more convenient to apply.

Corollary 3.8. *Let $1/n \ll \varepsilon \ll \gamma, d, 1/\ell$. Let G be (ε, ℓ, D) -typical with vertex partition (V, U) such that $dn \leq |V|, |U| \leq n$. Assume that $D_V, D_{VU} \geq d$ with $D_{UV}|U| = (1 \pm \varepsilon)|D_V||V|/2$, and $D_U = 0$. Let W_ℓ be the wheel graph with ℓ spokes and hub w , and let σ assign w to U and all other vertices to V . Then G has a (W_ℓ, σ) -packing such that the leftover L satisfies $\Delta(L) \leq \gamma n$.*

Proof. Define an auxiliary 2ℓ -uniform hypergraph \mathcal{H} with vertex set $E(G)$ where the edges of \mathcal{H} correspond to σ -copies of W_ℓ in G . Using the typicality condition, we can count that every edge of G between V and U lies in $(1 \pm \varepsilon)^{\ell-1} \cdot \frac{1}{2} D_V^\ell D_{UV}^{\ell-1} |V|^{\ell-1}$ σ -copies of W_ℓ ,

and every edge contained in V lies in $(1 \pm \varepsilon)^{\ell-1} D_V^{\ell-1} D_{UV}^\ell |U||V|^{\ell-2}$ such copies. Using $D_{UV}|U| = (1 \pm \varepsilon) D_V |V|/2$, we can conclude that $d_{\mathcal{H}}(e) = (1 \pm \varepsilon)^{\ell/2} D_V^\ell D_{UV}^{\ell-1} |V|^{\ell-1}$ for all $e \in V(\mathcal{H})$, i.e. \mathcal{H} is almost regular. For $v \in V(G)$, let $F_v \subseteq V(\mathcal{H})$ be the set of edges of G which are incident to v . A result of Alon and Yuster [3, Theorem 1.2] implies that there exists a matching M in \mathcal{H} such that for each set F_v , all but at most $\gamma|F_v|/2$ vertices of F_v are covered by M . Clearly, M corresponds to a (W_ℓ, σ) -packing in G such that the leftover graph L satisfies $\Delta(L) \leq \gamma n$. \square

The next result will be used in Case 2 of our proof. It asserts the existence of resolvable partite cycle decompositions in typical partite graphs. In the proof, we add a new vertex class of size equal to the number of cycle factors required for a resolvable decomposition, and join it completely to the rest of the graph. A wheel decomposition of this auxiliary graph encodes a resolvable cycle decomposition of the original graph. A very similar reduction has also been used e.g. in [27] to derive the existence of resolvable designs. We include a short proof for completeness.

Corollary 3.9. *Let $1/n \ll \varepsilon \ll 1/s \ll d, 1/\ell$. Let G be (ε, s, D) -typical with vertex partition (V_1, \dots, V_ℓ) into sets of size n each, where $D_{i,i+1} := D_{i+1,i} := d$ for all $i \in [\ell]$ (indices modulo ℓ) and $D_{ij} = 0$ otherwise. Assume that there exists $r \in \mathbb{N}$ such that $d_G(v, V_{i-1}) = d_G(v, V_{i+1}) = r$ for all $v \in V_i$ and $i \in [\ell]$ (indices modulo ℓ). Then G has a resolvable partite C_ℓ -decomposition.*

Proof. Let W_ℓ be the wheel graph with ℓ spokes and hub w . We extend G to a graph G' by adding a new vertex class V_w of size r and joining each $v \in V_w$ to all vertices of G . Accordingly, extend D to a $((\ell+1) \times (\ell+1))$ -matrix D' by defining $D'_{V_w} := 0$ and $D'_{V_w V_i} := D'_{V_i V_w} := 1$ for all $i \in [\ell]$. Noting that $V_i \subseteq N_{G'}(v)$ for all $i \in [\ell]$ and $v \in V_w$, it is easy to see that the typicality of G directly implies that G' is (ε, s, D') -typical.

It is also straightforward to check that G' is W_ℓ -balanced. Thus, applying Theorem 3.6 to G' yields a partite W_ℓ -decomposition \mathcal{W} of G' . This gives us a resolvable partite C_ℓ -decomposition \mathcal{C} of G as follows. For each vertex $v \in V_w$, let \mathcal{W}_v be the set of all copies of W_ℓ in \mathcal{W} which contain v . Let \mathcal{C}_v be obtained from \mathcal{W}_v by removing v from each element. Then \mathcal{C}_v is a C_ℓ -factor of G , and $\mathcal{C} := \bigcup_{v \in V_w} \mathcal{C}_v$ is a resolvable partite C_ℓ -decomposition of G . \square

The following proposition states that random slices of typical graphs are again typical. This is an easy consequence of Lemma 3.1(i).

Proposition 3.10. *Let $1/n \ll \varepsilon, 1/s, d, p$. Let G be (ε, s, D) -typical with vertex partition (V_1, \dots, V_t) such that $dn \leq |V_i| \leq n$ for all $i \in [t]$ and $D \geq dI$ for some indicator $(t \times t)$ -matrix I . Suppose we choose a random subgraph G' of G by including each edge independently with probability p . Then G' is $(1.1\varepsilon, s, pD)$ -typical with probability at least $1 - e^{-\sqrt{n}}$.*

We conclude this subsection with the following simple fact about the robustness of the typicality property.

Proposition 3.11. *Let G be (ε, s, D) -typical with vertex partition (V_1, \dots, V_t) such that $dn \leq |V_i| \leq n$ for all $i \in [t]$ and $D \geq dI$ for some indicator $(t \times t)$ -matrix I . Suppose that L is a graph on $V(G)$ such that whenever $L[V_i, V_j]$ is non-empty, then neither is $G[V_i, V_j]$. Suppose that $\Delta(L) \leq \gamma n$. Then $G \Delta L$ is still $(\varepsilon + s\gamma d^{-s-1}, s, D)$ -typical.*

Proof. Consider a set S of at most s vertices of G and some $i \in [t]$. If $D_{\tau(v)i} = 0$ for some $v \in S$, then $d_{G \Delta L}(v, V_i) = 0$ and hence there is nothing to show. So we can assume that $D_{\tau(v)i} \geq d$ for all $v \in S$. We then have $|V_i \cap \bigcap_{v \in S} N_{G \Delta L}(v)| = |V_i \cap \bigcap_{v \in S} N_G(v)| \pm s\gamma n$ and $s\gamma n \leq s\gamma d^{-s-1} |V_i| \prod_{v \in S} D_{\tau(v)i}$. \square

4. MAIN PROOF

In this section, we prove Theorem 1.3. The proof divides into two cases. The first case assumes that the given 2-regular graph F has very few vertices in long cycles. In the second case, we are guaranteed that linearly many vertices of F lie in long cycles. The latter case is much more involved than the first case and needs some additional preliminary work. In the following subsections, we will develop the necessary tools for Case 2.

The following definition will be used throughout the section. Let \mathcal{X} be a set of disjoint vertex sets and R an oriented graph on \mathcal{X} . Assume that G is a graph with vertex partition \mathcal{X} and reduced graph R (where each $X \in \mathcal{X}$ is independent in G). Let G_R be the oriented graph obtained from G by orienting every edge $e \in E(G)$ with the same direction as the reduced edge of R corresponding to e .

We say that G is r -balanced with respect to (\mathcal{X}, R) if G_R is r -regular, that is, $d_{G_R}^+(v) = d_{G_R}^-(v) = r$ for all $v \in V(G)$. We simply say that G is *balanced* if it is r -balanced for some $r \in \mathbb{N}$.

4.1. Cyclic partitions. As sketched in Section 2, we reduce the problem of finding an F -decomposition of a graph to finding resolvable C_3, C_4, C_5 -decompositions of suitable auxiliary graphs.

For $\ell \in \mathbb{N}$, we say that (a_1, \dots, a_t) is a *cyclic partition of ℓ* if $a_i \in \mathbb{N}$ for all $i \in [t]$ and $\sum_{i \in [t]} a_i = \ell$. We identify (a_1, \dots, a_t) with $(a_i, \dots, a_t, a_1, \dots, a_{i-1})$ for each $i \in [t]$ and treat indices modulo t . Moreover, for $S \subseteq \mathbb{N}$, we say that (a_1, \dots, a_t) is a *cyclic S -partition of ℓ* if $a_i \in S$ for all $i \in [t]$.

For a cyclic partition $\mathbf{a} = (a_1, \dots, a_t)$ and a sequence $a' = (a'_1, \dots, a'_{t'})$, we let $\mathbf{c}^{\mathbf{a}}(a')$ denote the number of appearances of a' in \mathbf{a} , that is,

$$\mathbf{c}^{\mathbf{a}}(a') := |\{i \in [t] : a_{i+j \bmod t} = a'_j \text{ for all } j \in [t']\}|.$$

Note that if e.g. $\mathbf{a} = (a)$ and $a' = (a, a)$, we have $\mathbf{c}^{\mathbf{a}}(a') = 1$. To improve readability, we write $\mathbf{c}^{\mathbf{a}}(a)$ instead of $\mathbf{c}^{\mathbf{a}}((a))$ and $\mathbf{c}^{\mathbf{a}}(a, b)$ instead of $\mathbf{c}^{\mathbf{a}}((a, b))$. For a sequence a , we let a^m denote the sequence which is the concatenation of m copies of a .

In this paper, we will only consider cyclic $\{3, 4, 5\}$ -partitions \mathbf{a} where $\mathbf{c}^{\mathbf{a}}(a, b) = \mathbf{c}^{\mathbf{a}}(b, a)$ for all $a, b \in \{3, 4, 5\}$. For brevity, we will simply call these the *admissible* partitions. Let

$$\mathcal{I} := \{3, 4, 5\} \times \{3, 4, 5\}.$$

We now make some easy observations regarding admissible partitions.

Proposition 4.1. *For each $\ell \in \mathbb{N}$ with $\ell \geq 3$, there exists an admissible partition of ℓ . Moreover, if $\ell \geq 500$, then there exists an admissible partition \mathbf{a} of ℓ such that $\mathbf{c}^{\mathbf{a}}((a, b)^6) \geq \ell/200$ for all $(a, b) \in \mathcal{I}$.*

Proof. For the first part, write $\ell = 3x + 4y$ with $x, y \in \mathbb{Z}$. If $\ell \geq 12$, we can clearly assume $x, y \geq 0$, and this can also easily be checked if $\ell < 12$ and $\ell \neq 5$. Thus, unless $\ell = 5$, we obtain a partition \mathbf{a} consisting of x 3's and y 4's (in any order). Since $\mathbf{c}^{\mathbf{a}}(3, 4) = \mathbf{c}^{\mathbf{a}}(4, 3)$, the partition \mathbf{a} is also admissible. If $\ell = 5$, we simply take (5). (This is in fact the reason why we allow 5's in admissible partitions, otherwise 3 and 4 would suffice.)

Now, assume that $\ell \geq 500$. Choose $\ell - 61 < \ell' \leq \ell - 13$ such that ℓ' is divisible by 48. Then there is a partition of ℓ' such that each number 3, 4, 5 appears exactly $\ell'/12$ times. Moreover, similarly as in the first part, there is an admissible partition \mathbf{a}' of $\ell - \ell' \geq 13$ that consists of a positive number of 4's followed by a positive number of 3's. We now construct \mathbf{a} by concatenating $a^{\ell'/48}$ for each $a \in \{(3, 3), (3, 4), (4, 4), (4, 5), (5, 5), (3, 5)\}$ (in this order), and finally attaching \mathbf{a}' . It is easy to check that \mathbf{a} is an admissible partition of ℓ . Moreover, by construction, we have $\mathbf{c}^{\mathbf{a}}((a, b)^6) \geq \ell'/48 - 6 \geq \ell/200$ for all $(a, b) \in \mathcal{I}$. \square

Fact 4.2. *For any admissible partition \mathbf{a} of $\ell \geq 3$ the following hold:*

- (i) $\sum_{a \in \{3,4,5\}} a \cdot \mathbf{c}^{\mathbf{a}}(a) = \ell$;
- (ii) *for each $a \in \{3,4,5\}$, we have $\mathbf{c}^{\mathbf{a}}(a) = \sum_{b \in \{3,4,5\}} \mathbf{c}^{\mathbf{a}}(a,b)$.*

For every $\ell \in \mathbb{N}$ with $\ell \geq 3$, let \mathbf{a}^ℓ be an admissible partition of ℓ . Moreover, if $\ell \geq 500$, we also assume that $\mathbf{c}^{\mathbf{a}^\ell}((a,b)^6) \geq \ell/200$ for all $(a,b) \in \mathcal{I}$. We fix \mathbf{a}^ℓ throughout the remainder of the paper. Moreover, whenever C is a cycle of length ℓ , we let $\mathbf{a}^C := \mathbf{a}^\ell$ and $\mathbf{c}^C(\cdot) := \mathbf{c}^{\mathbf{a}^\ell}(\cdot)$. Hence, given any 2-regular graph F , we obtain a collection

$$\mathbf{a}^F := \{\mathbf{a}^C : C \in \mathcal{C}(F)\}$$

of cyclic partitions. Since we fix our choice of \mathbf{a}^ℓ throughout, this collection is unique for every 2-regular graph F . We define the counting function

$$\mathbf{c}^F(\cdot) := \sum_{C \in \mathcal{C}(F)} \mathbf{c}^C(\cdot). \quad (4.1)$$

Proposition 4.3. *Let F be a 2-regular graph on n vertices. Then the following hold:*

- (i) $\mathbf{c}^F(a,b) = \mathbf{c}^F(b,a)$ for all $(a,b) \in \mathcal{I}$;
- (ii) $\sum_{a \in \{3,4,5\}} a \cdot \mathbf{c}^F(a) = n$;
- (iii) *for each $a \in \{3,4,5\}$, we have $\mathbf{c}^F(a) = \sum_{b \in \{3,4,5\}} \mathbf{c}^F(a,b)$;*
- (iv) *if at least ηn vertices of F lie in cycles of length at least 500, then $\mathbf{c}^F((a,b)^6) \geq \eta n/200$ for all $(a,b) \in \mathcal{I}$.*

Proof. Items (i), (ii) and (iii) follow directly from Fact 4.2 and (4.1). (iv) follows by assumption on \mathbf{a}^ℓ and (4.1). \square

We now describe the structure of the absorbing graph, which we call an F -partition. Roughly speaking, an F -partition (\mathcal{X}, R) consists of a suitable partition \mathcal{X} of an n -set V into 18 sets and an (oriented) reduced graph R on \mathcal{X} . Depending on the different parts of the proof, we will endow such a partition with additional structure. For instance, the sizes of the vertex classes in \mathcal{X} are chosen such that there is a natural embedding of F into V with reduced graph R (cf. Definition 4.5), hence the name F -partition. Another crucial ingredient is a ‘rewiring’ permutation π on a special subset $Y \subseteq V$ (cf. Definition 4.7 and Figure 1). The absorbing graph will be a graph G with vertex partition \mathcal{X} and reduced graph R (cf. Section 4.2).

Definition 4.4 (F -partition). Let F be a 2-regular graph and assume that V is a (vertex) set of size $|F|$. An F -partition (\mathcal{X}, R) of V is defined as follows:

- For all $a \in \{3,4,5\}$ and $i \in [a]$, let X_i^a be a subset of V of size $\mathbf{c}^F(a)$ such that

$$\hat{\mathcal{X}} := (X_i^a)_{a \in \{3,4,5\}, i \in [a]}$$

is a partition of V . This is possible by Proposition 4.3(ii). Moreover, for each $a \in \{3,4,5\}$, we partition X_1^a further using Proposition 4.3(iii), that is, let

$$X_1^a = X_1^{a,3} \cup X_1^{a,4} \cup X_1^{a,5} \quad (4.2)$$

be a partition into sets of size $\mathbf{c}^F(a,3)$, $\mathbf{c}^F(a,4)$ and $\mathbf{c}^F(a,5)$, respectively. Let

$$\mathcal{X} := \bigcup_{a \in \{3,4,5\}} \{X_1^{a,3}, X_1^{a,4}, X_1^{a,5}, X_2^a, \dots, X_a^a\}$$

- We define two (oriented) reduced graphs, one for the partition $\hat{\mathcal{X}}$ and one for the refined partition \mathcal{X} . Define \hat{R} on $\hat{\mathcal{X}}$ as the union of the (oriented) cycles $X_1^a X_2^a \dots X_a^a X_1^a$ for each $a \in \{3,4,5\}$. Define R on \mathcal{X} as the union of an (oriented) path $X_2^a \dots X_a^a$ for each $a \in \{3,4,5\}$ as well as the (oriented) paths $X_a^a X_1^{a,b} X_2^b$ for all $(a,b) \in \mathcal{I}$ (cf. Figure 1).

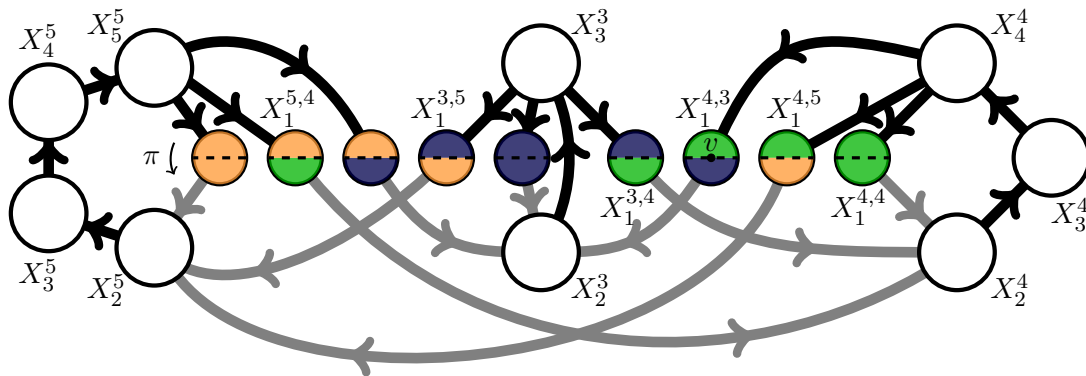


FIGURE 1. An F -partition with rewiring permutation π . The permutation π acts on the vertices in the coloured clusters (denoted by Y). Every vertex has an upper and a lower colour (the vertex v has upper colour green and lower colour blue). The lower colour of v coincides with the upper colour of $\pi(v)$; in the figure, $\pi(v)$ lies in $X_1^{3,3} \cup X_1^{3,4} \cup X_1^{3,5}$ (cf. (4.3),(4.4)).

- Note that $\hat{\mathcal{X}}$ and \hat{R} are uniquely determined by \mathcal{X} and R . Moreover, we let

$$Y := \bigcup_{(a,b) \in \mathcal{I}} X_1^{a,b} = \bigcup_{a \in \{3,4,5\}} X_1^a.$$

Note that if at least $\eta|F|$ vertices of F lie in cycles of length at least 500, then Proposition 4.3(iv) implies that $|X| \geq \eta|F|/200$ for all $X \in \mathcal{X}$.

Definition 4.5 (F -homomorphism). Let (\mathcal{X}, R) be an F -partition. An F -homomorphism $\sigma: F \rightarrow R$ for (\mathcal{X}, R) is defined as follows: Consider $C \in \mathcal{C}(F)$. We define a homomorphism $\sigma_C: C \rightarrow R$ simply by walking in R as indicated by \mathbf{a}^C . More precisely, let $\mathbf{a}^C = (a_1, \dots, a_t)$ and write

$$C = v_{1,1}v_{1,2} \dots v_{1,a_1}v_{2,1} \dots v_{2,a_2} \dots v_{t,1} \dots v_{t,a_t}v_{1,1}.$$

(Recall that $\sum_{i \in [t]} a_t = |C|$.) For $i \in [t]$ and $j \in [a_i] \setminus \{1\}$, define $\sigma_C(v_{i,1}) := X_1^{a_i-1, a_i}$ (where $a_0 := a_t$) and $\sigma_C(v_{i,j}) := X_j^{a_i}$. Let $\sigma := \bigcup_{C \in \mathcal{C}(F)} \sigma_C$. Clearly, $\sigma: F \rightarrow R$ is a homomorphism.

We record some easy properties of such a homomorphism.

Fact 4.6. Let (\mathcal{X}, R) be an F -partition and $\sigma: F \rightarrow R$ an F -homomorphism for (\mathcal{X}, R) . Let \vec{F} denote the orientation of F obtained by orienting $xy \in E(F)$ with the orientation of $\sigma(x)\sigma(y)$ in R . Then $|\sigma^{-1}(X)| = |X|$ for all $X \in \mathcal{X}$, and \vec{F} is 1-regular.

Proof. Let σ be defined as in Definition 4.5. Consider $C \in \mathcal{C}(F)$. Note that we have $|\sigma_C^{-1}(X_i^a)| = \mathbf{c}^C(a)$ for all $a \in \{3, 4, 5\}$ and $i \in [a] \setminus \{1\}$, and $|\sigma_C^{-1}(X_1^{a,b})| = \mathbf{c}^C(a, b)$ for all $(a, b) \in \mathcal{I}$. Moreover, if \vec{C} denotes the orientation of C obtained by orienting $xy \in E(C)$ with the orientation of $\sigma_C(x)\sigma_C(y)$ in R , then \vec{C} is 1-regular. Hence, \vec{F} is clearly 1-regular. Moreover, for all $a \in \{3, 4, 5\}$ and $i \in [a] \setminus \{1\}$, we have

$$|\sigma^{-1}(X_i^a)| = \sum_{C \in \mathcal{C}(F)} |\sigma_C^{-1}(X_i^a)| = \sum_{C \in \mathcal{C}(F)} \mathbf{c}^C(a) \stackrel{(4.1)}{=} \mathbf{c}^F(a) = |X_i^a|,$$

and similarly, for all $(a, b) \in \mathcal{I}$, we have $|\sigma^{-1}(X_1^{a,b})| = \mathbf{c}^F(a, b) = |X_1^{a,b}|$. \square

Another property of an F -homomorphism σ , which will be crucial in Section 4.4, is that there are many subpaths of F whose σ -image winds at least 5 times around a given cycle in R . This follows from Proposition 4.3(iv).

4.2. The robust decomposition lemma. In this subsection, we show that given an F -partition (\mathcal{X}, R) , we can find a graph G with vertex partition \mathcal{X} and reduced graph R , such that for any sparse subgraph $L \subseteq G$, the remainder $G - L$ has an F -decomposition, subject to some divisibility conditions (cf. Lemma 4.11). This is arguably the core of our proof.

The construction of G is based on a ‘rewiring permutation’. This permutation controls how we ‘merge’ resolvable C_3, C_4, C_5 -decompositions of suitable auxiliary graphs into an F -decomposition.

Definition 4.7 (F -rewiring). Let V be a set and assume that (\mathcal{X}, R) is an F -partition of V . Let $\hat{\mathcal{X}}, \hat{R}, Y$ be as in Definition 4.4. An F -rewiring (π, π^*) for (\mathcal{X}, R) is defined as follows:

- We define a permutation π on Y that is the disjoint union of permutations $\{\pi_C\}_{C \in \mathcal{C}(F)}$. To this end, for each $C \in \mathcal{C}(F)$, let $Y_C \subseteq Y$ be a set which consists of $\mathfrak{c}^C(a, b)$ vertices from $X_1^{a,b}$ for each $(a, b) \in \mathcal{I}$ such that $(Y_C)_{C \in \mathcal{C}(F)}$ is a partition of Y . This is possible since $|X_1^{a,b}| = \mathfrak{c}^F(a, b) \stackrel{(4.1)}{=} \sum_{C \in \mathcal{C}(F)} \mathfrak{c}^C(a, b)$ for each $(a, b) \in \mathcal{I}$.

Consider a cycle $C \in \mathcal{C}(F)$ and let $\mathbf{a}^C = (a_1, \dots, a_t)$. For each $i \in [t]$, choose a vertex $v_i \in Y_C \cap X_1^{a_i, a_{i+1}}$ (where $a_{t+1} := a_1$) such that the vertices v_1, \dots, v_t are distinct. This is possible since $|Y_C \cap X_1^{a,b}| = \mathfrak{c}^C(a, b)$ for each $(a, b) \in \mathcal{I}$. In particular, $Y_C = \{v_1, \dots, v_t\}$. Now, define $\pi_C(v_i) := v_{i+1}$ for each $i \in [t-1]$, and $\pi_C(v_t) := v_1$. (In particular, if $t = 1$, then v_1 becomes a fixed point of π_C .) Finally, let $\pi: Y \rightarrow Y$ be the permutation which consists of all the cycles $(\pi_C)_{C \in \mathcal{C}(F)}$.

Clearly, we have

$$\pi(X_1^{a,b}) \subseteq X_1^{b,3} \cup X_1^{b,4} \cup X_1^{b,5} = X_1^b, \quad (4.3)$$

$$\pi^{-1}(X_1^{a,b}) \subseteq X_1^{3,a} \cup X_1^{4,a} \cup X_1^{5,a} \quad (4.4)$$

for all $(a, b) \in \mathcal{I}$ (cf. Figure 1).

- Using the permutation π , we define a bijection π^* which ‘rewires’ edges between Y and $X_2^3 \cup X_2^4 \cup X_2^5$. Let E^∞ be the set of all edges vv' such that $v \in X$, $v' \in X'$ with $XX' \in E(R)$. Similarly, let $E^{\circ\circ}$ be the set of all edges vv' such that $v \in X$, $v' \in X'$ with $XX' \in E(\hat{R})$. For an edge $e = yv \in E^\infty$ with $y \in Y$ and $v \in X_2^a$ for some $a \in \{3, 4, 5\}$, we define

$$\pi^*(e) := \pi(y)v \quad (4.5)$$

and $\pi^*(e) := e$ otherwise. By (4.3), we have $\pi^*(e) \in E^{\circ\circ}$, thus $\pi^*: E^\infty \rightarrow E^{\circ\circ}$. Clearly, π^* is bijective, with $\pi^{*-1}(e) = \pi^{-1}(y)v$ if $e = yv$ with $y \in Y$ and $v \in X_2^a$ for some $a \in \{3, 4, 5\}$, and $\pi^{*-1}(e) = e$ otherwise.

We use π^* to switch between graphs with reduced graphs R and \hat{R} , respectively. Let \mathcal{G}^∞ be the set of all graphs G on V with $E(G) \subseteq E^\infty$, and let $\mathcal{G}^{\circ\circ}$ be the set of all graphs G on V with $E(G) \subseteq E^{\circ\circ}$. Clearly, π^* induces a bijection between \mathcal{G}^∞ and $\mathcal{G}^{\circ\circ}$, which we call π^* again.

The following are the crucial properties of our rewiring procedure.

Lemma 4.8. *Let $V, \mathcal{X}, R, \hat{\mathcal{X}}, \hat{R}, Y, \pi, \pi^*, \mathcal{G}^\infty, \mathcal{G}^{\circ\circ}$ be as in Definitions 4.4 and 4.7. Then the following hold:*

- (i) $G \in \mathcal{G}^\infty$ is r -balanced with respect to (\mathcal{X}, R) if and only if $\pi^*(G) \in \mathcal{G}^{\circ\circ}$ is r -balanced with respect to $(\hat{\mathcal{X}}, \hat{R})$.
- (ii) If $H \in \mathcal{G}^{\circ\circ}$ consists of an a -partite C_a -factor on (X_1^a, \dots, X_a^a) for each $a \in \{3, 4, 5\}$, then $\pi^{*-1}(H) \cong F$.

Proof. (i) This follows easily from the definitions.

(ii) Suppose that $H \in \mathcal{G}^{\circ\circ}$ consists of an a -partite C_a -factor on (X_1^a, \dots, X_a^a) for each $a \in \{3, 4, 5\}$. For a vertex $y \in X_1^a$, let $C^y = yx_2^{(y)} \dots x_a^{(y)}y$ be the copy of C_a which contains y , where $x_i^{(y)} \in X_i^a$ for all $i \in [a] \setminus \{1\}$. Recall that $\pi: Y \rightarrow Y$ is the composition of all cycles

$(\pi_C)_{C \in \mathcal{C}(F)}$, where π_C is a permutation on Y_C and $(Y_C)_{C \in \mathcal{C}(F)}$ is a partition of Y . Clearly,

$$E(H) = \bigcup_{y \in Y} E(C^y) = \bigcup_{C \in \mathcal{C}(F)} \bigcup_{y \in Y_C} E(C^y). \quad (4.6)$$

The crucial observation is that π^{*-1} merges the cycles $(C^y)_{y \in Y_C}$ to one copy of C .

Claim 1: For each $C \in \mathcal{C}(F)$, let H^C be the graph with vertex set $\bigcup_{y \in Y_C} V(C^y)$ and edge set $\pi^{*-1}(\bigcup_{y \in Y_C} E(C^y))$. Then $H^C \cong C$.

Proof of claim: Fix some $C \in \mathcal{C}(F)$. Note first that for every $y \in Y_C$, the set of vertices spanned by the edges of $\pi^{*-1}(E(C^y))$ is $V(C^y) \cup \{\pi^{-1}(y)\}$. Indeed, since $\pi(Y_C) = Y_C$, we have $\pi^{-1}(y) \in V(C^{y'})$ for some $y' \in Y_C$. Thus, H^C is well-defined.

Now, let $\mathbf{a}^C = (a_1, \dots, a_t)$ and write $Y_C = \{y_1, \dots, y_t\}$, where $\pi(y_i) = y_{i+1}$ and $y_i \in X_1^{a_i, a_{i+1}}$ for all $i \in [t]$ (where $y_{t+1} := y_1$ and $a_{t+1} := a_1$). Consider $i \in [t]$. Recall that $x_2^{(y_i)}$ is the neighbour of y_i on C^{y_i} in $X_2^{a_i}$. By definition of π^* , we have that

$$\pi^{*-1}(E(C^{y_i})) = (E(C^{y_i}) - \{y_i x_2^{(y_i)}\}) \cup \{\pi^{-1}(y_i) x_2^{(y_i)}\}.$$

Thus, $P_i := (V(C^{y_i}) \cup \{y_{i-1}\}, \pi^{*-1}(E(C^{y_i})))$ is a path from y_i to y_{i-1} of length a_i . Since the internal vertices of the paths P_1, \dots, P_t are mutually disjoint, we conclude that the union H^C of these t paths is a cycle of length $a_1 + \dots + a_t = |C|$. \square

Since $Y_C \cap Y_{C'} = \emptyset$ for distinct $C, C' \in \mathcal{C}(F)$ and $V(C^y) \cap V(C^{y'}) = \emptyset$ for distinct $y, y' \in Y$, the graphs $(H^C)_{C \in \mathcal{C}(F)}$ are pairwise vertex-disjoint. Therefore,

$$\pi^{*-1}(H) \stackrel{(4.6)}{=} \bigcup_{C \in \mathcal{C}(F)} H^C \cong F.$$

\square

The following lemma will be used to define the absorbing graph.

Lemma 4.9. *Suppose $1/n \ll \varepsilon \ll 1/s, \beta, d, 1/t$. Let V_1, \dots, V_t be sets of size n each (indices modulo t). Let \mathcal{U} be a collection of at most $e^{n^{1/10}}$ sets such that for every $U \in \mathcal{U}$, we have $|U| \geq \beta n$ and $U \subseteq V_i$ for some $i \in [t]$. Then there exists a graph G with vertex partition (V_1, \dots, V_t) satisfying the following:*

- (i) *for all $i \in [t]$, all $S \subseteq V_{i-1} \cup V_{i+1}$ with $|S| \leq s$ and each $U \in \mathcal{U}$ with $U \subseteq V_i$, we have $|U \cap \bigcap_{v \in S} N_G(v)| = (1 \pm \varepsilon) d^{|S|} |U|$;*
- (ii) *$G[V_i, V_{i+1}]$ is $\lfloor dn \rfloor$ -regular for all $i \in [t]$.*

Proof. For each $i \in [t]$, let \mathcal{M}_i be a decomposition of the complete bipartite graph between V_i and V_{i+1} into perfect matchings. Equivalently, consider an n -edge-colouring. Let G be the (random) graph obtained by activating each $M \in \bigcup_{i \in [t]} \mathcal{M}_i$ independently with probability d . For each $i \in [t]$, let $\mathcal{M}'_i \subseteq \mathcal{M}_i$ be the (random) set of activated matchings. Clearly, $G[V_i, V_{i+1}]$ is $|\mathcal{M}'_i|$ -regular.

Using Lemma 3.2 and a union bound, it is easy to see that G satisfies (i) with probability at least $1 - e^{-n^{1/7}}$. Moreover, standard properties of the binomial distribution yield that G satisfies (ii) with probability at least $\Omega_d(n^{-t/2})$. Thus, there exists a graph G with the desired properties. \square

Later on, we will also need the following related result.

Proposition 4.10. *Suppose $1/n \ll \varepsilon \ll d$. Let V be a vertex set of size n . There exists a $2\lfloor dn/2 \rfloor$ -regular (ε, d) -quasirandom graph on V .*

Proof. Let $r := 2\lfloor dn/2 \rfloor$. If n is even, K_n has a 1-factorization and we can generate the desired graph as the union of random perfect matchings as above. If n is odd, we take a quasirandom r -regular graph G' on $n-1$ vertices first. Then choose a random subset $N \subseteq V(G')$ of size r . It is well known that $G'[N]$ has with high probability a perfect matching M (since r is even). Then removing M and adding a new vertex with neighbourhood N yields the desired graph. \square

We are now ready to prove the robust decomposition lemma. It guarantees the existence of an absorbing graph G which not only has an F -decomposition itself, but even after removing a sparse balanced subgraph, it is still F -decomposable.

Lemma 4.11. *Suppose $1/n \ll \varepsilon \ll \eta, \alpha$. Let F be a 2-regular graph on n vertices and assume that at least ηn vertices lie in cycles of length at least 500. Let V be a vertex set of size n and let (\mathcal{X}, R) be an F -partition of V . There exists a graph G with vertex partition \mathcal{X} and reduced graph R which satisfies the following properties:*

- (i) G is $2r$ -regular for some $r \leq \alpha n$;
- (ii) for all $XX' \in E(R)$, the pair $G[X, X']$ is $(\varepsilon, d_{XX'})$ -quasirandom for some $\alpha\eta/200 \leq d_{XX'} \leq \alpha$;
- (iii) for every subgraph $L \subseteq G$ which is r_L -balanced with respect to (\mathcal{X}, R) for some $r_L \leq \sqrt{\varepsilon}n$, the remainder $G - L$ has an F -decomposition.

Proof. We choose a new constant $s \in \mathbb{N}$ such that $1/n \ll \varepsilon \ll 1/s \ll \eta, \alpha$. Let $c := \min\{\mathbf{c}^F(3), \mathbf{c}^F(4), \mathbf{c}^F(5)\}$. By Proposition 4.3(iv), we have $c \geq \eta n/200$. Define $d_a := \frac{\alpha c}{\mathbf{c}^F(a)}$ for each $a \in \{3, 4, 5\}$. Thus, $\alpha\eta/200 \leq d_3, d_4, d_5 \leq \alpha$. Let $r := \lfloor \alpha c \rfloor$. Clearly, we have $\lfloor d_a \mathbf{c}^F(a) \rfloor = r$ for each $a \in \{3, 4, 5\}$. Let $\hat{\mathcal{X}}, \hat{R}$ be as in Definition 4.4 and let (π, π^*) be an F -rewiring for (\mathcal{X}, R) . For $a \in \{3, 4, 5\}$, define

$$\mathcal{U}_a := \{X_1^a, \dots, X_a^a, X_1^{a,3}, X_1^{a,4}, X_1^{a,5}, \pi(X_1^{3,a}), \pi(X_1^{4,a}), \pi(X_1^{5,a})\}.$$

Recall from (4.2), (4.3) and Proposition 4.3(iv) that $X_1^{a,3}, X_1^{a,4}, X_1^{a,5}, \pi(X_1^{3,a}), \pi(X_1^{4,a}), \pi(X_1^{5,a})$ are all subsets of X_1^a of size at least $\eta n/200 \geq \frac{\eta}{200} \mathbf{c}^F(a)$.

We now apply Lemma 4.9 for each $a \in \{3, 4, 5\}$ (with $\mathbf{c}^F(a), d_a, \eta/200, \mathcal{U}_a$ playing the roles of n, d, β, \mathcal{U}) to obtain a graph G_a with vertex partition (X_1^a, \dots, X_a^a) whose reduced graph is the cycle $X_1^a X_2^a \dots X_a^a X_1^a$ and which satisfies the following (indices modulo a):

- (a) for all $i \in [a]$, all $S \subseteq X_{i-1}^a \cup X_{i+1}^a$ with $|S| \leq s$ and each $U \in \mathcal{U}_a$ with $U \subseteq X_i^a$, we have $|U \cap \bigcap_{v \in S} N_{G_a}(v)| = (1 \pm \varepsilon) d_a^{|S|} |U|$;
- (b) $G_a[X_i^a, X_{i+1}^a]$ is r -regular for all $i \in [a]$.

For each $a \in \{3, 4, 5\}$, define the density $(a \times a)$ -matrix D^a as $D_{i,i+1}^a := D_{i+1,i}^a := d_a$ for all $i \in [a]$ and $D_{i,i'}^a := 0$ otherwise. It follows immediately from (a) that

$$G_a \text{ is } (\varepsilon, s, D^a)\text{-typical.} \tag{4.7}$$

Now, define

$$G := \pi^{*-1}(G_3 \cup G_4 \cup G_5).$$

We claim that G is the desired graph.

First, observe that $G_3 \cup G_4 \cup G_5$ is r -balanced with respect to $(\hat{\mathcal{X}}, \hat{R})$, and thus G is r -balanced with respect to (\mathcal{X}, R) by Lemma 4.8(i). In particular, G is $2r$ -regular, proving (i).

Next, from (a) and our choice of \mathcal{U}_a , we can also deduce that for each $a \in \{3, 4, 5\}$, the following pairs are (ε, d_a) -quasirandom:

$$\begin{aligned} G_a[X_i^a, X_{i+1}^a] & \text{ for each } i \in \{2, \dots, a-1\}; \\ G_a[X_a^a, X_1^{a,b}] & \text{ for each } b \in \{3, 4, 5\}; \\ G_a[\pi(X_1^{b,a}), X_2^a] & \text{ for each } b \in \{3, 4, 5\}. \end{aligned}$$

Note that for all $(a, b) \in \mathcal{I}$, we have $G_a[\pi(X_1^{b,a}), X_2^a] \cong G[X_1^{b,a}, X_2^a]$. This proves (ii).

It remains to prove the essential claim, which is (iii). To this end, let $L \subseteq G$ be any subgraph which is r_L -balanced with respect to (\mathcal{X}, R) for some $r_L \leq \sqrt{\varepsilon n}$. Let $r' := r - r_L$. Clearly, $G - L$ is then r' -balanced with respect to (\mathcal{X}, R) . Lemma 4.8(i) implies that $G' := \pi^*(G - L)$ is r' -balanced with respect to $(\hat{\mathcal{X}}, \hat{R})$. For $a \in \{3, 4, 5\}$, let

$$G'_a := G'[X_1^a \cup \dots \cup X_a^a].$$

Hence,

$$G' = G'_3 \cup G'_4 \cup G'_5. \quad (4.8)$$

The balancedness of G' implies that $d_{G'_a}(v, X_{i-1}^a) = d_{G'_a}(v, X_{i+1}^a) = r'$ for all $v \in X_i^a$ and $i \in [a]$ (indices modulo a).

Moreover, since $G'_a = G_a - \pi^*(L)$ and $\Delta(\pi^*(L)) \leq 2\sqrt{\varepsilon n} \leq \varepsilon^{1/3}|V(G_a)|$, Proposition 3.11 and (4.7) imply that G'_a is still $(\varepsilon^{1/4}, s, D^a)$ -typical. Therefore, by Corollary 3.9, G'_a has a resolvable partite C_a -decomposition. Let $F_1^a, \dots, F_{r'}^a$ be the C_a -factors in such a decomposition.

Now, for each $t \in [r']$, define

$$F_t := F_t^3 \cup F_t^4 \cup F_t^5. \quad (4.9)$$

By (4.8), we have that $\{F_t : t \in [r']\}$ is a decomposition of G' . Thus, $\mathcal{F} := \{\pi^{*-1}(F_t) : t \in [r']\}$ is a decomposition of $\pi^{*-1}(G') = G - L$. Crucially, by Lemma 4.8(ii), $\pi^{*-1}(F_t) \cong F$ for all $t \in [r']$. We conclude that \mathcal{F} is an F -decomposition of $G - L$, completing the proof. \square

4.3. Crossing edge absorption. In this subsection, we prove the main tool for the first step of our absorption process (Lemma 4.12). Roughly speaking, it states that given a graph G with a vertex partition (V_1, \dots, V_t) which is quasirandom inside the clusters and sufficiently sparse between the clusters, we can utilize copies of F to cover all edges between clusters.

Lemma 4.12. *Suppose $1/n \ll \varepsilon \ll \eta, 1/t$. Let F be an n -vertex 2-regular graph with at least ηn vertices belonging to cycles of length at least $15t$. Let (V_1, \dots, V_t) be a vertex partition of an n -vertex graph G with vertex set V such that $|V_i| \geq \eta n$ and $G[V_i]$ is (ε, d_i) -quasirandom for some $d_i \geq \eta$ for all $i \in [t]$. Let L be a graph with vertex set V such that the following hold:*

- (i) $\Delta(L) \leq \varepsilon n$;
- (ii) $e_L(V_i, V_j) \geq \sqrt{\varepsilon n}$ for all distinct $i, j \in [t]$ and $e_L(V_i) = 0$ for all $i \in [t]$;
- (iii) $e_L(V_i, V \setminus V_i)$ is even for all $i \in [t]$.

Then there exists a subgraph G' of G such that $G' \cup L$ has an F -decomposition.

Note that condition (iii) is clearly necessary. Moreover, we need at least a few edges between clusters (e.g. if F is a Hamilton cycle) and it turns out that the condition in (ii) suffices.

In the proof of Lemma 4.12 we rely on Lemma 4.14. The statement of Lemma 4.14 is similar to the statement of Lemma 4.12. However, in Lemma 4.14 L is a matching which we cover with a single copy of F . For the proof of Lemma 4.14 we use Lemma 4.13.

Lemma 4.13. *Let $n, t \in \mathbb{N}$. Let F be an n -vertex 2-regular graph where all vertices belong to cycles of length at least 30. Let $n_1, \dots, n_t \in \mathbb{N}$ be such that $\sum_{i \in [t]} n_i = n$ and $n_i \geq 50$ for all $i \in [t]$. Then there exists an assignment $f : V(F) \rightarrow [t]$ such that the following hold:*

- (i) $|f^{-1}(i)| = n_i$ for all $i \in [t]$;
- (ii) $|f(x) - f(y)| \leq 1$ for all $xy \in E(F)$;
- (iii) the set E of all edges $xy \in E(F)$ with $|\{f(x), f(y)\}| = 2$ is an induced matching in F ;
- (iv) for all $i \in [t-1]$, there are exactly four edges $xy \in E(F)$ with $\{f(x), f(y)\} = \{i, i+1\}$.

Proof. The proof consists of three steps. In the first step, we define a very simple assignment $f_1 : V(F) \rightarrow [t]$. In the second step, we modify f_1 slightly to obtain a different assignment f_2 . Both assignments essentially ignore the edges of F and only make sure that (i) holds with f replaced by f_1 and f_2 , respectively. In the final step, it is then easy to obtain an assignment f from f_2 which satisfies (i)–(iv).

We write C^1, \dots, C^s for the cycles in F . Let x_1, \dots, x_n be an ordering of the vertices of F such that all vertices in C^i precede all vertices in C^j for all $i < j$. For $i \in [t]$, let $I_i := \{1 + \sum_{j < i} n_j, 2 + \sum_{j < i} n_j, \dots, \sum_{j < i} n_j\}$. Clearly, I_1, \dots, I_t is a partition of $[n]$. This also gives rise to our first assignment f_1 by defining $f_1(x_k) := i$ whenever $k \in I_i$.

Recall that every cycle of F has length at least 30. Hence for all $i \in [t]$, there are (disjoint) intervals $I_i^-, I_i^+ \subseteq I_i$ of size 6 such that

- (a) for all $k, k' \in I_i^-$, the vertices $x_k, x_{k'}$ belong to the same cycle of F and analogously for I_i^+ ;
- (b) $\max I_i^- \leq \min I_i + 10$ and $\min I_i^+ \geq \max I_i - 10$.

Next, we slightly modify f_1 . Roughly speaking, f_2 arises from f_1 by interchanging I_i^+ and I_{i+1}^- for all $i \in [t-1]$. To be precise, for all $k \in [n]$, let $f_2(x_k) := i$ whenever $k \in (I_i \setminus (I_i^- \cup I_i^+)) \cup I_{i-1}^+ \cup I_{i+1}^-$ where $I_0^+ := I_1^-$ and $I_{t+1}^- := I_t^+$.

For all $i \in [t]$ and $j \in [s]$, let $n(i, j) := |f_2^{-1}(i) \cap V(C^j)|$. The following statements are easy observations that follow directly from our construction:

- (A) for all $j \in [s]$, in the sequence $n(1, j), n(2, j), \dots, n(t, j)$ all non-zero elements appear consecutively and if $n(i, j) > 0$, then $n(i, j) \geq 6$;
- (B) for all $i \in [t-1]$, there are either one or two $j \in [s]$ such that both $n(i, j) > 0$ and $n(i+1, j) > 0$.

It is easy to find an assignment $f : V(C^j) \rightarrow [t]$ of the vertices of a single cycle C^j such that

- $|f^{-1}(i) \cap V(C^j)| = |f_2^{-1}(i) \cap V(C^j)|$ for all $i \in [t]$;
- $|f(x) - f(y)| \leq 1$ for all $xy \in E(C^j)$;
- the set E of all edges $xy \in E(C^j)$ with $|\{f(x), f(y)\}| = 2$ is an induced matching in F ;
- and the number of edges $xy \in E(C^j)$ with $\{f(x), f(y)\} = \{i, i+1\}$ equals 4 if j is the unique integer such that both $n(i, j), n(i+1, j)$ are positive, 2 if $n(i, j), n(i+1, j), n(i, j^*), n(i+1, j^*)$ are positive for some $j^* \in [s] \setminus \{j\}$ and otherwise 0.

Combining these assignments for all cycles yields an assignment f satisfying (i)–(iv). \square

Lemma 4.14. *Suppose $1/n \ll \varepsilon \ll \eta, 1/t$. Let F be an n -vertex 2-regular graph with at least ηn vertices belonging to cycles of length at least $15t$. Let (V_1, \dots, V_t) be a vertex partition of an n -vertex graph G with vertex set V such that $|V_i| \geq \eta n$ and $G[V_i]$ is (ε, d_i) -quasirandom for some $d_i \geq \eta$ for all $i \in [t]$. Let M be a matching of size at most εn such that $e_M(V_i) = 0$ for all $i \in [t]$ and $e_M(V_i, V_{i+1}) \geq 4$ for all $i \in [t-1]$. Moreover, suppose that $e_M(V_i, V \setminus V_i)$ is even for all $i \in [t]$. Then there exists an embedding $\phi : F \rightarrow G \cup M$ such that $M \subseteq \phi(F)$.*

Proof. For $i \in [t]$, define $n_i := |V_i|$. We first allocate all short cycles of F to the clusters V_i . To this end, let $F^{<}$ be the subgraph of F which consists of all cycles of length less than $15t$, and let F^{\geq} be the subgraph of F which consists of all cycles of length at least $15t$. Let $\sigma' : V(F^{<}) \rightarrow [t]$ be such that $n'_i := n_i - |\sigma'^{-1}(i)| \geq \eta n / 2t$ for all $i \in [t]$ and $\sigma'(x) = \sigma'(y)$ for all $xy \in E(F^{<})$. Clearly, σ' exists.

Our next goal is to find a function $\sigma'' : V(F^{\geq}) \rightarrow [t]$ such that the following hold:

- (a) $|\sigma''^{-1}(i)| = n'_i$ for all $i \in [t]$;
- (b) the set E of all edges $xy \in E(F^{\geq})$ with $|\{\sigma''(x), \sigma''(y)\}| = 2$ is an induced matching in F ;
- (c) for all distinct $i, j \in [t]$, there are exactly $e_M(V_i, V_j)$ edges $xy \in E(F^{\geq})$ with $\{\sigma''(x), \sigma''(y)\} = \{i, j\}$.

We will find σ'' with the help of Lemma 4.13. Before this, we show how to complete the proof based on (a)–(c). Assume that σ'' satisfies (a)–(c). Obviously, we will embed E onto M . Let $\phi_0: V(E) \rightarrow V(M)$ be such that $\phi_0(E) = M$, and $\phi_0(x) \in V_{\sigma''(x)}$ for all $x \in V(E)$. Such an embedding exists by (c). For $i \in [t]$, let $F_i := F[\sigma'^{-1}(i) \cup \sigma''^{-1}(i)]$. Clearly, $F = \bigcup_{i \in [t]} F_i \cup E$. Now, consider $i \in [t]$. By (a), we have $|F_i| = n_i$. Moreover, by (b), $V(E) \cap \sigma''^{-1}(i)$ is independent in F_i . Thus, by Corollary 3.4, there exists an embedding $\phi_i: F_i \rightarrow G[V_i]$ such that $\phi_i(x) = \phi_0(x)$ for all $x \in V(E) \cap \sigma''^{-1}(i)$. It is easy to see that $\phi := \bigcup_{i \in [t]} \phi_i$ is the desired embedding, where $\phi(E) = M$.

It remains to find σ'' . Clearly, we may assume that $t \geq 2$. First, let R be the graph on $[t]$ where $ij \in E(R)$ if and only if $e_M(V_i, V_j)$ is odd. Clearly, R is Eulerian. Hence, R has a decomposition \mathcal{C} into cycles. For $C \in \mathcal{C}$ and $i \in [t]$, we define

$$g(C, i) := \begin{cases} 0 & \text{if } i \notin V(C); \\ |C| - 1 & \text{if } i = \min V(C); \\ -1 & \text{otherwise.} \end{cases}$$

Note that $\sum_{i \in [t]} g(C, i) = 0$ for all $C \in \mathcal{C}$. For all $i, j \in [t]$ with $|j - i| > 1$, we define $\text{def}(i, j) := \lfloor e_M(V_i, V_j)/2 \rfloor$. Moreover, for all $i \in [t - 1]$, we define $\text{def}(i, i + 1) := \lfloor e_M(V_i, V_{i+1})/2 \rfloor - 2$. By assumption, we have $\text{def}(i, j) \geq 0$ for all distinct $i, j \in [t]$.

Now, for all $i \in [t]$, we define

$$n_i^* := n'_i - \sum_{j=1}^{i-1} 3\text{def}(i, j) + \sum_{j=i+1}^t 3\text{def}(i, j) + \sum_{C \in \mathcal{C}} 3g(C, i). \quad (4.10)$$

Observe that $\sum_{i \in [t]} n_i^* = \sum_{i \in [t]} n'_i = |V(F^{\geq})|$. Moreover, for each $i \in [t]$, we have

$$|n_i^* - n'_i| \leq \sum_{j \in [t] \setminus \{i\}} 3\text{def}(i, j) + \sum_{C \in \mathcal{C}} 3|g(C, i)| \leq 3\epsilon n/2 + 3t^2 \leq 2\epsilon n. \quad (4.11)$$

Thus, we can apply Lemma 4.13 to find a function $f: V(F^{\geq}) \rightarrow [t]$ such that the following hold:

- (i) $|f^{-1}(i)| = n_i^*$ for all $i \in [t]$;
- (ii) $|f(x) - f(y)| \leq 1$ for all $xy \in E(F^{\geq})$;
- (iii) the set E^* of all edges $xy \in E(F^{\geq})$ with $|\{f(x), f(y)\}| = 2$ is an induced matching in F^{\geq} ;
- (iv) for all $i \in [t - 1]$, there are exactly four edges $xy \in E(F^{\geq})$ with $\{f(x), f(y)\} = \{i, i + 1\}$.

We will obtain σ'' from f by changing the image of a few vertices. Roughly speaking, we repeatedly take a subpath of F which is currently embedded into V_i and then move three consecutive vertices into another cluster V_j . Apart from moving three vertices from V_i to V_j , this also produces two crossing edges (which do not share an endpoint).

More precisely, for $i \in [t]$, an i -target is a subpath $P \subseteq F^{\geq} \setminus f^{-1}(V(E^*))$ of length $3t$ such that $f(V(P)) = \{i\}$. Clearly, for each $i \in [t]$, there are at least $2\epsilon n$ vertex-disjoint i -targets.

For all $i \in [t]$, let \mathcal{P}_i be a set of $\sum_{j=i+1}^t \text{def}(i, j)$ vertex-disjoint i -targets, and let $\{\mathcal{P}_{i,j} : j = i + 1, \dots, t\}$ be a partition of \mathcal{P}_i such that $|\mathcal{P}_{i,j}| = \text{def}(i, j)$. For every $C \in \mathcal{C}$ with $i = \min V(C)$, let P_C be an i -target which is vertex-disjoint from all the previously chosen targets (as there are $2\epsilon n$ vertex-disjoint i -targets, P_C exists by (4.11)). Let $\mathcal{P} := \bigcup_{i \in [t]} \mathcal{P}_i \cup \bigcup_{C \in \mathcal{C}} P_C$ be the set of all these targets.

We now define σ'' . For each target $P \in \mathcal{P}$, write $P = x_1^{(P)} \dots x_{3t+1}^{(P)}$. For every vertex $x \in V(F^{\geq})$ which is not contained in any $P \in \mathcal{P}$, we let $\sigma''(x) := f(x)$. For all $i, j \in [t]$ with

$i < j$ and all $P \in \mathcal{P}_{i,j}$, we define, for all $k \in [3t+1]$,

$$\sigma''(x_k^{(P)}) := \begin{cases} j & \text{if } k \in \{3, 4, 5\}; \\ i & \text{otherwise.} \end{cases}$$

Now consider $C \in \mathcal{C}$. Write $C = i_1 i_2 \dots i_\ell i_1$ such that $i_1 = \min V(C)$. We define $\sigma''(x_{3(k-1)+s}^{(P_C)}) := i_k$ for all $k \in [\ell] \setminus \{1\}$ and $s \in \{0, 1, 2\}$, and $\sigma''(x_s^{(P_C)}) := i_1$ for $s \in \{1, 2\} \cup \{3\ell, \dots, 3t+1\}$.

We claim that σ'' satisfies (a)–(c). Clearly, for all $i \in [t]$, we have

$$|\sigma''^{-1}(i)| = |f^{-1}(i)| + \sum_{j=1}^{i-1} 3|\mathcal{P}_{j,i}| - \sum_{j=i+1}^t 3|\mathcal{P}_{i,j}| - \sum_{C \in \mathcal{C}} 3g(C, i) \stackrel{(4.10),(i)}{=} n'_i,$$

thus σ'' satisfies (a). Condition (b) clearly holds for the restriction $\sigma''|_P$ for every $P \in \mathcal{P}$. Thus, (b) follows from (iii) and the fact that all the paths in \mathcal{P} are vertex-disjoint and for every i -target P , we have $\sigma''(x_s^{(P)}) = i$ for $s \in \{1, 2, 3t, 3t+1\}$.

Finally, let $i, j \in [t]$ with $i < j$. The union of all $P \in \mathcal{P}_{i,j}$ gives rise to exactly $2\text{def}(i, j)$ edges $xy \in E(F^\geq)$ with $\{\sigma''(x), \sigma''(y)\} = \{i, j\}$. Moreover, if $e_M(V_i, V_j)$ is even, then there is no cycle $C \in \mathcal{C}$ with $ij \in E(C)$. If $e_M(V_i, V_j)$ is odd, then there is exactly one cycle $C \in \mathcal{C}$ with $ij \in E(C)$, and P_C gives rise to exactly one edge $xy \in E(F^\geq)$ with $\{\sigma''(x), \sigma''(y)\} = \{i, j\}$. Together with (ii), (iv) and the definition of $\text{def}(i, j)$, this implies (c). \square

We can now prove Lemma 4.12. Essentially, we need to decompose L into suitable matchings and then apply Lemma 4.14 to each matching.

Proof of Lemma 4.12. We call a matching $M \subseteq L$ *good* if $e(M) \leq 2\varepsilon^{1/4}n$, $e_M(V_i, V_j) \geq 4$ for all distinct $i, j \in [t]$, and $e_M(V_i, V \setminus V_i)$ is even for all $i \in [t]$. The main part of the proof is to partition $E(L)$ into good matchings. We achieve this in three steps. In the first step, we find a good matching M_0 such that for $L' := L - M_0$ the number of edges between two clusters is always even. In the second step, we partition $L'[V_i, V_j]$ into matchings of size 2, and in the final step we combine these matchings to obtain a decomposition of L' into good matchings.

To this end, let $p_{ij} := 1$ if $e_L(V_i, V_j)$ is odd and $p_{ij} := 0$ otherwise, for all distinct $i, j \in [t]$. Let $M_0 \subseteq L$ be a matching such that $M_0[V_i, V_j]$ consists of $4 + p_{ij}$ edges for all distinct $i, j \in [t]$. Clearly, M_0 exists. Note that for each $i \in [t]$, we have

$$e_{M_0}(V_i, V \setminus V_i) \equiv \sum_{j \in [t] \setminus \{i\}} p_{ij} \equiv \sum_{j \in [t] \setminus \{i\}} e_L(V_i, V_j) = e_L(V_i, V \setminus V_i) \equiv 0 \pmod{2}.$$

Thus, M_0 is a good matching. Let $L' := L - M_0$. Obviously, $e_{L'}(V_i, V_j)$ is even for all distinct $i, j \in [t]$.

Next, for all distinct $i, j \in [t]$, we partition the edges of $L'[V_i, V_j]$ into matchings of size 2. Since the maximum degree of the line graph of $L'(V_i, V_j)$ is at most $2\Delta(L') \leq e_{L'}(V_i, V_j)/2$, the complement of the line graph of $L'(V_i, V_j)$ has a perfect matching, which yields the desired partition. We call such a matching of size 2 an *edge pair*.

An *admissible* colouring is a proper edge-colouring of L' where two edges that form an edge pair receive the same colour.

We now admissibly colour L' with $s := \lceil \varepsilon^{3/4}n \rceil$ colours. First, since $e_{L'}(V_i, V_j) - 4s - 8t\Delta(L) \geq 2$, it is easy to find a partial admissible colouring such that for every pair $ij \in \binom{[t]}{2}$ and each colour $k \in [s]$, exactly two edge pairs in $L'[V_i, V_j]$ are coloured k . In a second phase, we order the remaining edge pairs arbitrarily and then colour them successively, each time picking an available colour that appears least often in the current colouring. Clearly, since at each step, there are at least $s - 4\Delta(L') > \varepsilon^{3/4}n/2$ colours available for the considered edge pair, no colour class will have size more than $2\varepsilon^{1/4}n$ in the completed colouring.

It is easy to see that each colour class of this colouring is a good matching. To conclude, we obtained a decomposition \mathcal{M} of L into $s+1$ good matchings. Finally, we apply Lemma 4.14

$s + 1$ times in turn to cover all these matchings. More precisely, we embed one copy of F into the union of $M \in \mathcal{M}$ and the subgraph of G induced by all edges that are not covered by earlier applications of Lemma 4.14. Clearly, at any stage of the procedure the uncovered edges in $G[V_i]$ induce an $(\varepsilon^{1/2}, d_i)$ -quasirandom graph for every $i \in [t]$. \square

4.4. Atom absorption. This subsection is devoted to the proof of the following lemma which states that if we are given a graph G as in Lemma 4.11(ii), and a regular ‘leftover’ inside each partition class, then we can absorb this leftover by using a few edges of G . Moreover, the subgraph A which we use from G will be balanced with respect to (\mathcal{X}, R) , thus ensuring that $G - A$ still has an F -decomposition (cf. Lemma 4.11(iii)).

Lemma 4.15. *Suppose $1/n \ll \varepsilon \ll \eta$. Let F be a 2-regular graph on n vertices and assume that at least ηn vertices lie in cycles of length at least 500. Let V be a vertex set of size n and assume that (\mathcal{X}, R) is an F -partition. Assume that G is a graph with vertex partition \mathcal{X} and reduced graph R such that for all $XX' \in E(R)$, the pair $G[X, X']$ is $(\varepsilon, d_{XX'})$ -quasirandom for some $d_{XX'} \geq \eta$. Suppose that $r \leq \varepsilon n$ and that L_X is a $2r$ -regular graph on X for each $X \in \mathcal{X}$. Let $L := \bigcup_{X \in \mathcal{X}} L_X$. Then there exists a subgraph $A \subseteq G$ which is r_A -balanced with respect to (\mathcal{X}, R) for some $r_A \leq \sqrt{\varepsilon} n$ such that $A \cup L$ has an F -decomposition.*

As in the previous subsection, we will prove Lemma 4.15 by splitting L into suitable matchings and then employing the blow-up lemma to extend each such matching into a copy of F . How the latter can be done for a single matching is proved in Lemma 4.17. Recall that since (\mathcal{X}, R) is an F -partition, Definition 4.5 yields a natural homomorphism $\sigma: F \rightarrow R$ for which we could straightforwardly apply the blow-up lemma to find a copy of F in G . However, such a copy of F would not contain any edges of L . In order to extend a given matching into a copy of F , we will locally modify σ . Because of divisibility issues, we perform this surgery not for single matching edges, but group them into smallest balanced edge sets, which we refer to as atoms. For each atom, we will perform a slight surgery on σ to make sure that the copy of F found with the blow-up lemma will cover this particular atom.

For $a \in \{3, 4, 5\}$, an $\{a\}$ -atom is a matching of size a , consisting of one edge inside $X_1^{a,a}$ and one edge inside each vertex class X_i^a , for each $i \in [a] \setminus \{1\}$. For distinct $a, b \in \{3, 4, 5\}$, an $\{a, b\}$ -atom is a matching of size $a + b$, consisting of one edge inside each of $X_1^{a,b}$ and $X_1^{b,a}$, one edge inside each vertex class X_i^a , for each $i \in [a] \setminus \{1\}$, and one edge inside each vertex class X_i^b , for each $i \in [b] \setminus \{1\}$. Let \mathcal{I}' be the set of all subsets S of $\{3, 4, 5\}$ with $|S| \in \{1, 2\}$. An atom is an S -atom for some $S \in \mathcal{I}'$. Clearly, if O is an S -atom, then $e(O) = \sum_{s \in S} s$. We say that a graph is *internally balanced* if it is the union of edge-disjoint atoms.

Fact 4.16. *A graph H is internally balanced if and only if the following conditions hold:*

$$e_H(X_1^a) = \dots = e_H(X_a^a) \text{ for all } a \in \{3, 4, 5\}; \quad (4.12)$$

$$e_H(X_1^{a,b}) = e_H(X_1^{b,a}) \text{ for all } (a, b) \in \mathcal{I}. \quad (4.13)$$

Proof. Observe first that every atom satisfies (4.12) and (4.13). Thus, if H is the edge-disjoint union of atoms, it also satisfies (4.12) and (4.13). For the converse, repeatedly remove atoms from H , until this is no longer possible. Using (4.12) and (4.13), it is easy to see that at the end of this procedure, there can be no edge left. \square

Recall that for a subgraph A of G , where G is as in the statements of Lemmas 4.15 and 4.17, we write A_R for the oriented graph obtained from A by orienting the edges according to the orientation of R .

Lemma 4.17. *Suppose $1/n \ll \varepsilon \ll \eta$. Let F be a 2-regular graph on n vertices and assume that at least ηn vertices lie in cycles of length at least 500. Let V be a vertex set of size n and assume that (\mathcal{X}, R) is an F -partition. Assume that G is a graph with vertex partition \mathcal{X} and reduced graph R such that for all $XX' \in E(R)$, the pair $G[X, X']$ is $(\varepsilon, d_{XX'})$ -quasirandom*

for some $d_{XX'} \geq \eta$. Suppose that \vec{M} is an oriented internally balanced matching of size at most εn . Then there exists a subgraph $A \subseteq G$ such that $A \cup M \cong F$ and for every $v \in V$, we have $d_{A \cup \vec{M}}^+(v) = d_{A \cup \vec{M}}^-(v) = 1$.

Proof. Let \mathcal{A} be a decomposition of M into atoms. Let $\sigma: F \rightarrow R$ be an F -homomorphism for (\mathcal{X}, R) (cf. Definition 4.5) and let \vec{F} denote the orientation of F obtained by orienting $xy \in E(F)$ with the orientation of $\sigma(x)\sigma(y)$. By Fact 4.6, we have

$$|\sigma^{-1}(X)| = |X| \text{ for all } X \in \mathcal{X} \quad (4.14)$$

and \vec{F} is 1-regular.

Observe that R has a unique oriented ℓ -cycle D_ℓ for each $\ell \in \{3, 4, 5, 7, 8, 9\}$, namely, for distinct $a, b \in \{3, 4, 5\}$, we have

$$\begin{aligned} D_a &= X_1^{a,a} X_2^a \dots X_a^a X_1^{a,a}; \\ D_{a+b} &= X_1^{b,a} X_2^a \dots X_a^a X_1^{a,b} X_2^b \dots X_b^b X_1^{b,a}. \end{aligned}$$

Note that if O is an atom, then O consists of exactly one edge inside each vertex class $Z \in V(D_{e(O)})$. For $\ell \in \{3, 4, 5, 7, 8, 9\}$, we say that $P \subseteq F$ is an ℓ -target if P is a subpath of some cycle $C \subseteq \mathcal{C}(F)$ such that $\sigma(V(P)) \subseteq \bigcup_{X \in V(D_\ell)} X$ and P has length 5ℓ . (In other words, the σ -image of P is a closed walk winding five times around the cycle D_ℓ .) Our strategy is as follows. We modify σ (and obtain σ') in such a way that the σ' -image of P only winds four times around D_ℓ but ‘repeats’ each cluster of D_ℓ at some point exactly once; that is, two consecutive vertices of P are assigned to the same cluster. This ensures that we can cover the edges of O (which are inside the clusters) and at the same time exactly the same number of vertices are assigned to every particular cluster by σ and σ' .

First, we assign to each atom $O \in \mathcal{A}$ an $e(O)$ -target P_O such that all those targets are vertex-disjoint. This is possible by Proposition 4.3(iv). Indeed, for every $a \in \{3, 4, 5\}$, every appearance of $(a)^6$ in \mathbf{a}^C for some $C \in \mathcal{C}(F)$ yields an a -target, and for all distinct $a, b \in \{3, 4, 5\}$, every appearance of $(a, b)^6$ in \mathbf{a}^C for some $C \in \mathcal{C}(F)$ yields an $(a+b)$ -target. Hence, for every $\ell \in \{3, 4, 5, 7, 8, 9\}$, there are at least $\eta n/200$ distinct ℓ -targets. Since $|\mathcal{A}| \leq |M| \leq \varepsilon n$, we can (greedily) choose an $e(O)$ -target P_O for each $O \in \mathcal{A}$ such that all the paths P_O are vertex-disjoint.

For each atom $O \in \mathcal{A}$, we will now partially embed P_O to cover the edges of O . Crucially, for this we use exactly as many vertices from each cluster as indicated by σ . Thus, we will be able to apply the blow-up lemma to complete the embedding.

We now describe how the homomorphism σ can be modified on $V(P_O)$ to absorb O . Consider some $O \in \mathcal{A}$ and let $\ell = e(O)$. Write $P_O = x_1 x_2 \dots x_{5\ell+1}$. We may assume that $D_\ell = Z_1 Z_2 \dots Z_\ell Z_1$ is oriented as in R and $\sigma(x_1) = Z_1$. Hence, $\sigma(x_i) = Z_{i \bmod \ell}$ for all $i \in [5\ell+1]$. Thus, $|\sigma^{-1}(Z_i) \cap V(P_O)| = 5$ for $i \in [\ell] \setminus \{1\}$ and $|\sigma^{-1}(Z_1) \cap V(P_O)| = 6$. Recall that O consists of exactly one edge inside each vertex class Z_i . Hence, we can write $E(O) = \{v_i w_i : i \in [\ell]\}$, where $v_i, w_i \in Z_i$ and $v_i w_i$ is oriented towards w_i by \vec{M} .

We now mark ℓ edges of P_O which will later be embedded onto O . More precisely, let $g := 4$ if $\ell \in \{4, 5, 7, 8\}$ and $g := 5$ if $\ell \in \{3, 9\}$. Let

$$E_O := \{x_{2+g(i-1)} x_{3+g(i-1)} : i \in [\ell]\}.$$

Clearly, $E_O \subseteq E(P_O)$, and

$$\text{no endvertex of } P_O \text{ is incident to an edge of } E_O. \quad (4.15)$$

We now define a new function $\sigma_O: V(P_O) \rightarrow V(R)$ as follows: Let $\sigma_O(x_1) := Z_1 = \sigma(x_1)$. For $i \in [5\ell]$, let $\sigma_O(x_{i+1}) := \sigma_O(x_i)$ if $x_i x_{i+1} \in E_O$, and let $\sigma_O(x_{i+1})$ be the outneighbour of $\sigma_O(x_i)$ on D_ℓ otherwise. For $i \in [\ell]$, let $j_i \in [\ell]$ be such that $\sigma_O(x_{2+g(i-1)}) = Z_{j_i}$. By construction, we have $j_{i+1} \equiv j_i + (g-1) \pmod{\ell}$. By the choice of g , we have that $g-1$ and ℓ are coprime. Thus, $\{j_1, \dots, j_\ell\} = [\ell]$. In other words, the σ_O -image of P may be viewed

as a walk in D_ℓ which ‘stops’ in each $Z \in V(D_\ell)$ exactly once. This implies that for all $Z \in V(D_\ell)$, we have

$$|\sigma_O^{-1}(Z)| = |\sigma^{-1}(Z) \cap V(P_O)|. \quad (4.16)$$

Note that σ_O is not a homomorphism of P_O into R . However,

$$\sigma_O: P_O - E_O \rightarrow R \text{ is a homomorphism.} \quad (4.17)$$

Moreover, it is compatible with σ in the following sense:

$$\sigma_O(x_1) = \sigma(x_1) = \sigma_O(x_{5\ell+1}) = \sigma(x_{5\ell+1}). \quad (4.18)$$

Define $X_O^+ := \{x_{2+g(i-1)} : i \in [\ell]\}$ and $X_O^- := \{x_{3+g(i-1)} : i \in [\ell]\}$. Moreover, define $\phi_O: X_O^- \cup X_O^+ \rightarrow V$ by

$$\begin{aligned} \phi_O(x_{2+g(i-1)}) &:= v_{j_i}, \\ \phi_O(x_{3+g(i-1)}) &:= w_{j_i} \end{aligned}$$

for all $i \in [\ell]$. Hence $\phi_O(X_O^+)$ is the set of tails of O , and $\phi_O(X_O^-)$ is the set of heads of O .

Having done this for all $O \in \mathcal{A}$, we define

$$\vec{F}' := \vec{F} - \bigcup_{O \in \mathcal{A}} E_O; \quad X^\pm := \bigcup_{O \in \mathcal{A}} X_O^\pm; \quad \phi' := \bigcup_{O \in \mathcal{A}} \phi_O.$$

Observe that

$$d_{\vec{F}'}^\pm(x) = \begin{cases} 0 & \text{if } x \in X^\pm; \\ 1 & \text{otherwise.} \end{cases} \quad (4.19)$$

Note that since $g \geq 4$ and using (4.15), no two vertices in $X^+ \cup X^-$ have a common neighbour in F . Observe that $\phi'(\vec{F}'[X^+ \cup X^-]) = \vec{M}$. Define $\sigma': V(F) \rightarrow V(R)$ by

$$\sigma'(x) := \begin{cases} \sigma_O(x) & \text{if } x \in V(P_O) \text{ for some } O \in \mathcal{A}; \\ \sigma(x) & \text{otherwise.} \end{cases}$$

By (4.17) and (4.18), we have that σ' is a homomorphism from F' into R . Moreover, from (4.14) and (4.16) we can deduce that $|\sigma'^{-1}(X)| = |X|$ for all $X \in \mathcal{X}$.

Finally, we can apply the blow-up lemma (Lemma 3.3) to obtain an embedding $\phi: F' \rightarrow G$ which extends ϕ' such that $\phi(x) \in \sigma'(x)$ for all $x \in V(F)$. Then $A := \phi(F')$ is the desired graph. Indeed, we clearly have $A \cup M \cong F$. Moreover, from (4.19) and the definition of σ' , it is evident that

$$d_{A_R}^\pm(v) = \begin{cases} 0 & \text{if } v \in \phi(X^\pm); \\ 1 & \text{otherwise.} \end{cases}$$

Hence, $A_R \cup \vec{M}$ is 1-regular, as desired. \square

We will now deduce Lemma 4.15 from Lemma 4.17. In order to partition L into internally balanced matchings, it is convenient for us to exploit the following simple fact on the matching sequencibility of graphs.

Fact 4.18. *Let G be a regular graph on n vertices. Then there is an ordering of the edges of G such that any $n/12$ consecutive edges form a matching.*

Proof. Suppose G is r -regular. As the edges of G can be properly coloured with $r + 1$ colours, there exist matchings M_1, \dots, M_{r+1} in G which partition $E(G)$ (suppose $|M_1| \leq \dots \leq |M_{r+1}|$). Observe that $|M_2| \geq n/4$. Now, assume that for some $i \in [r]$, the edges of M_1, \dots, M_i are ordered such that any $n/12$ consecutive edges form a matching. We call an edge in M_{i+1} *blocked* if it shares a vertex with some edge in M_i that belongs to the last $n/12$ edges in the current ordering. Clearly, at most $n/6$ edges of M_{i+1} are blocked. Thus, we can

extend the ordering by putting $n/12$ unblocked edges of M_{i+1} first and then the remaining ones. \square

We remark that the graph A in Lemma 4.17 is not balanced with respect to (\mathcal{X}, R) . However, the ‘inbalancedness’ of A is encoded in the orientation of the matching M . By decomposing L into suitable oriented matchings, it is not too difficult to ensure that the union of all graphs A over all these matchings will indeed be balanced.

Proof of Lemma 4.15. Note that since L is $2r$ -regular, there exists an orientation \vec{L} of L such that \vec{L} is r -regular, that is, $d_{\vec{L}}^+(v) = d_{\vec{L}}^-(v) = r$ for all $v \in V$.

We first partition L into small internally balanced matchings. For each $X \in \mathcal{X}$, apply Fact 4.18 to obtain an ordering of the edges of L_X such that any $\eta n/10^4$ consecutive edges form a matching (recall that $|X| \geq \eta n/200$). We view each such ordering as a stack from which we repeatedly choose the first edge which has not been chosen before. We successively choose a set of edges M as follows: If for some $S \in \mathcal{I}'$, there exists an S -atom of unchosen edges, then add those edges to M (taking the first unchosen edges from the appropriate stacks). Repeat this until $|M| \geq 2\sqrt{\varepsilon}n$ or no atom can be found. Clearly, $|M| \leq 2\sqrt{\varepsilon}n + 9 \leq \eta n/10^4$ and thus M is a matching. Moreover, since M is the edge-disjoint union of atoms, it is internally balanced. By repeating this procedure, we obtain a collection \mathcal{M} of edge-disjoint matchings. Note that for all $a \in \{3, 4, 5\}$ and $i \in [a]$, we have $e_L(X_i^a) = r|X_i^a| = rc^F(a)$. Similarly, for $(a, b) \in \mathcal{I}$, using Proposition 4.3(i), we have that $e_L(X_1^{a,b}) = r|X_1^{a,b}| = rc^F(a, b) = rc^F(b, a) = r|X_1^{b,a}| = e_L(X_1^{b,a})$. Clearly, since in each step we removed an atom from L , Fact 4.16 implies that the set of unchosen edges is the edge-disjoint union of atoms. In particular, we can find an atom as above until the very last edge of L is chosen. We conclude that \mathcal{M} is a decomposition of L , and that all matchings in \mathcal{M} have size at least $2\sqrt{\varepsilon}n$, except possibly for the last one. Thus,

$$|\mathcal{M}| \leq \frac{e(L)}{2\sqrt{\varepsilon}n} + 1 = \frac{rn}{2\sqrt{\varepsilon}n} + 1 \leq \sqrt{\varepsilon}n. \quad (4.20)$$

From now on, we view \mathcal{M} as a set of oriented internally balanced matchings, where every edge simply inherits its orientation from \vec{L} .

We now apply Lemma 4.17 successively for each $\vec{M} \in \mathcal{M}$, to find a graph $A^M \subseteq G$ such that the following hold:

- (i) $A^M \cup M \cong F$;
- (ii) for every $v \in V$, we have $d_{A^M \cup M}^+(v) = d_{A^M \cup M}^-(v) = 1$;
- (iii) all the graphs in $\{A^M\}_{\vec{M} \in \mathcal{M}}$ are pairwise edge-disjoint.

Suppose that for some subset $\mathcal{M}' \subseteq \mathcal{M}$ and all $\vec{M}' \in \mathcal{M}'$ we have already found $A^{M'}$ satisfying (i)–(iii). Now we need to be able to find A^M . Let $A' := \bigcup_{\vec{M}' \in \mathcal{M}'} A^{M'}$ and $G' := G - A'$. Clearly, $\Delta(A') \leq 2|\mathcal{M}'| \leq 2\sqrt{\varepsilon}n$, and thus $G[X, X']$ is still $(\varepsilon^{1/3}, d_{XX'})$ -quasirandom whenever $XX' \in E(R)$. Thus, by Lemma 4.17, there exists $A^M \subseteq G'$ which satisfies (i) and (ii). Clearly, A^M is edge-disjoint from A' , thus (iii) holds as well. Hence, we can find A^M for every $\vec{M} \in \mathcal{M}$.

Let $A := \bigcup_{\vec{M} \in \mathcal{M}} A^M$. Clearly, $A \subseteq G$ and $A \cup L$ has an F -decomposition by (i) and (iii). Moreover, for every $v \in V$, we have

$$d_{A \cup L}^+(v) = \sum_{\vec{M} \in \mathcal{M}} d_{A^M \cup M}^+(v) \stackrel{(ii)}{=} |\mathcal{M}| - \sum_{\vec{M} \in \mathcal{M}} d_{\vec{M}}^+(v) = |\mathcal{M}| - d_L^+(v) = |\mathcal{M}| - r. \quad (4.21)$$

Similarly, $d_{A \cup L}^-(v) = |\mathcal{M}| - r$ for every $v \in V$. Thus, A is r_A -balanced with respect to (\mathcal{X}, R) , where $r_A := |\mathcal{M}| - r$. Clearly, $r_A \leq |\mathcal{M}| \leq \sqrt{\varepsilon}n$. \square

4.5. Proof of Theorem 1.3. We are now ready to complete the proof of Theorem 1.3. The essential work has been done in the previous subsections. We now combine those results to finish the proof.

Proof of Theorem 1.3. Let F be as in the statement of Theorem 1.3. For $\ell \in [n]$, let n_ℓ be the number of vertices in F in cycles of length ℓ . Of course,

$$\sum_{\ell=3}^n n_\ell = n. \quad (4.22)$$

We may clearly assume that $\Delta \geq 2$ and that $\alpha \leq \frac{1}{400\Delta}$. Further, we choose new constants $\xi, \varepsilon, \mu, \eta > 0$ and $s \in \mathbb{N}$ such that

$$1/n \ll \xi \ll \varepsilon \ll \mu \ll \eta \ll 1/s \ll \alpha, 1/\Delta.$$

Let $V := V(K_n)$.

Case 1: $\sum_{\ell=3}^{500} n_\ell \geq (1 - \eta)n$.

For the first part of the proof we assume that at least $(1 - \eta)n$ vertices of F are contained in cycles of length at most 500. Since F is obviously ξ -separable, by moving a suitable number of copies of F from \mathcal{F} to \mathcal{H} , we may also assume that $|\mathcal{F}| = \lceil \alpha n \rceil = (1 \pm \xi)\alpha n$.

We will first use Theorem 3.5 to embed all graphs from \mathcal{H} . Let $d := 1 - 2\alpha + \varepsilon$. We partition the edges of K_n into two graphs G', G'' such that the following hold:

$$G' \text{ is } (\xi, s, d)\text{-typical and } G'' \text{ is } (\xi, s, 1 - d)\text{-typical.} \quad (4.23)$$

That such a partition exists can be seen easily via a probabilistic argument: for every edge independently, include it in G' with probability d and in G'' otherwise. Clearly, K_n is $(\xi/2, s, 1)$ -typical. Thus, Proposition 3.10 implies that G' and G'' satisfy (4.23) with high probability.

Now we use Theorem 3.5 to pack \mathcal{H} into G' . Note that $d \geq 1 - \frac{1}{200\Delta}$. By (4.23), we have that $d_{G'}(x) = (1 \pm \xi)dn$ for all $x \in V(G')$. Moreover, $e(\mathcal{H}) = \binom{n}{2} - |\mathcal{F}|n \leq (1 - 2\alpha + 2\alpha\xi)\binom{n}{2}$, whereas $e(G') \geq (1 - \xi)d\binom{n}{2} \geq (1 - 2\alpha + \varepsilon/2)\binom{n}{2}$, implying that $e(\mathcal{H}) \leq (1 - \varepsilon/3)e(G')$. Thus, by Theorem 3.5, \mathcal{H} packs into G' . Let L_0 be the leftover of this packing in G' . Note that since $e(\mathcal{H}) = \binom{n}{2} - |\mathcal{F}|n \geq (1 - 2\alpha - 2\xi)n^2/2$ and all graphs in \mathcal{H} are regular, at least $(1 - 2\alpha - 2\xi)n$ edges are covered at every vertex. Thus, we have $\Delta(L_0) \leq (1 + \xi)dn - (1 - 2\alpha - 2\xi)n \leq 2\varepsilon n$. We now add this leftover back to G'' . That is, define $G := G'' \cup L_0$.

It remains to show that G has an F -decomposition. Using Proposition 3.11, we can see that G is $(\varepsilon^{3/4}, s, 2\alpha - \varepsilon)$ -typical, say, and therefore $(\sqrt{\varepsilon}, s, 2\alpha)$ -typical. Note that G is automatically $2|\mathcal{F}|$ -regular. Before starting to decompose G , we first split F into three subgraphs according to the cycle lengths appearing in F .

Let $\ell^* \in \{3, \dots, 500\}$ be such that $n_{\ell^*} \geq n/600$. Clearly, ℓ^* exists since $\sum_{\ell=3}^{500} n_\ell \geq (1 - \eta)n$. Moreover, define

$$I := \{\ell \in [500] \setminus \{1, 2, \ell^*\} : n_\ell \geq \eta n\}.$$

That is, I consists of ‘significant’ cycle lengths appearing in F (other than ℓ^*). Note that I might be empty. Define F_1 to be the disjoint union of n_{ℓ^*}/ℓ^* cycles of length ℓ^* . Let F_2 be the disjoint union of n_ℓ/ℓ cycles of length ℓ , for each $\ell \in I$. Finally, define F_3 as the disjoint union of n_ℓ/ℓ cycles of length ℓ , for each $\ell \in [n] \setminus (\{1, 2, \ell^*\} \cup I)$. Thus, F is the disjoint union of F_1, F_2 and F_3 . Observe that

$$|F_3| \leq \eta n + 500\eta n \leq \eta^{1/2}n. \quad (4.24)$$

We will find the desired F -decomposition of G in three steps. First, we embed all the copies of F_2 , then those of F_3 , and finally we complete the decomposition by embedding all copies of F_1 . In order to keep track of the vertices which we have already used to embed some part of some copy of F , we introduce a set of new vertices.

Consider a vertex set U of size exactly $|\mathcal{F}|$ disjoint from V . Let \tilde{G} be the graph on vertex set $V \cup U$ with all edges from G and all edges between V and U . Each vertex in U will represent one copy of F in the required decomposition of G . Let W_F be the graph obtained from F by adding a universal vertex. We will decompose \tilde{G} into copies of W_F such that the universal vertices lie in U . Clearly, this corresponds to an F -decomposition of G .

Define the (2×2) -density matrix D by $D_V := 2\alpha$, $D_{UV} := 1$ and $D_U := 0$. Clearly, the following hold:

- (a) \tilde{G} is $(\sqrt{\varepsilon}, s, D)$ -typical;
- (b) $d_{\tilde{G}}(v, V) = d_G(v) = 2d_{\tilde{G}}(v, U) = 2|\mathcal{F}|$ for all $v \in V$;
- (c) $d_{\tilde{G}}(u) = n$ for all $u \in U$;
- (d) $e_{\tilde{G}}(V) = e_{\tilde{G}}(V, U) = |\mathcal{F}|n$.

Here, (a) holds since G is $(\sqrt{\varepsilon}, s, 2\alpha)$ -typical.

We commence with the embedding of the copies of F_2 . To this end, partition the edge set of \tilde{G} (randomly) into graphs $\{\tilde{G}_\ell\}_{\ell \in I \cup \{\ell^*\}}$, by including, for each $\ell \in I$, each edge independently into \tilde{G}_ℓ with probability $p_\ell := \frac{n_\ell}{n} + \mu$, and into \tilde{G}_{ℓ^*} with the remaining probability $p_{\ell^*} := 1 - \sum_{\ell \in I} p_\ell$. Note that since $\ell^* \notin I$, we have

$$\sum_{\ell \in I} p_\ell = \sum_{\ell \in I} \frac{n_\ell}{n} + |I|\mu \stackrel{(4.22)}{\leq} 1 - \frac{1}{600} + 500\mu \leq 1 - \frac{1}{700}.$$

Thus, $p_{\ell^*} \geq \frac{1}{700}$. Since each graph \tilde{G}_ℓ is a random subgraph of \tilde{G} , Proposition 3.10 implies that for each $\ell \in I \cup \{\ell^*\}$, with probability at least $1 - 1/n$, we have that

$$\tilde{G}_\ell \text{ is } (2\sqrt{\varepsilon}, s, p_\ell D)\text{-typical.} \quad (4.25)$$

In particular, a decomposition of \tilde{G} into graphs $\{\tilde{G}_\ell\}_{\ell \in I \cup \{\ell^*\}}$ with these properties exists. From now on, fix such a decomposition.

For $\ell \in \mathbb{N} \setminus \{1, 2\}$, let W_ℓ be the wheel graph with ℓ spokes and hub w , and let $\sigma_\ell: V(W_\ell) \rightarrow \{V, U\}$ assign w to U and all other vertices of $V(W_\ell)$ to V .

Recall that $p_\ell D_{UV}|U| = p_\ell(1 \pm \xi)\alpha n = (1 \pm \xi)p_\ell D_V|V|/2$. Thus, for each $\ell \in I$ separately, we can apply Corollary 3.8 to find a (W_ℓ, σ_ℓ) -packing \mathcal{W}_ℓ in \tilde{G}_ℓ such that the leftover L_ℓ satisfies $\Delta(L_\ell) \leq \mu^2 n$.

Claim 1: For every $\ell \in I$, there exists $\mathcal{W}'_\ell \subseteq \mathcal{W}_\ell$ such that every $u \in U$ is the hub of exactly n_ℓ/ℓ wheels in \mathcal{W}'_ℓ , and the new leftover $L'_\ell := \tilde{G}_\ell - \mathcal{W}'_\ell$ satisfies $\Delta(L'_\ell) \leq 25\sqrt{\mu}n$.

Proof of claim: Fix some $\ell \in I$. Consider $u \in U$. Let s'_u be the number of wheels in \mathcal{W}_ℓ which contain u . Clearly, $d_{L_\ell}(u) = d_{\tilde{G}_\ell}(u) - s'_u \cdot \ell$. The typicality of \tilde{G}_ℓ implies that $d_{\tilde{G}_\ell}(u) = (1 \pm 2\sqrt{\varepsilon})p_\ell n = n_\ell + \mu n \pm 2\sqrt{\varepsilon}n$. Since $\Delta(L_\ell) \leq \mu^2 n$, we conclude that $s'_u = (d_{\tilde{G}_\ell}(u) - d_{L_\ell}(u))/\ell = \frac{n_\ell}{\ell} + \frac{\mu n}{\ell} \pm 2\mu^2 n$. In particular, $s'_u \geq \frac{n_\ell}{\ell}$. Let $s_u := s'_u - \frac{n_\ell}{\ell}$. Note that $s_u \leq 2\mu n/\ell$.

For each $u \in U$, we want to discard s_u wheels from \mathcal{W}_ℓ which contain u . We can do this independently for each $u \in U$, however we need to be careful not to delete too many wheels which contain a particular vertex $v \in V$. Thus, for each $u \in U$, we pick s_u wheels from \mathcal{W}_ℓ which contain u uniformly at random, independently of the choices for other vertices $u' \in U$. Clearly, this yields $\mathcal{W}'_\ell \subseteq \mathcal{W}_\ell$ such that every $u \in U$ is the hub of exactly n_ℓ/ℓ wheels in \mathcal{W}'_ℓ . Let L'_ℓ be the new leftover. Clearly, $d_{L'_\ell}(u) \leq \Delta(L_\ell) + \ell \cdot s_u \leq 3\mu n$ for all $u \in U$. Now, consider $v \in V$. If $W \in \mathcal{W}_\ell$ contains v and u , then the probability that W is discarded is s_u/s'_u . Since $\ell \in I$, we have $s'_u \geq n_\ell/\ell \geq \eta n/\ell$ and thus $s_u/s'_u \leq \sqrt{\mu}$. With Lemma 3.1(ii), we conclude that with high probability, for all $v \in V$, at most $7\sqrt{\mu}|U|$ wheels containing v are discarded from \mathcal{W}_ℓ , implying that $d_{L'_\ell}(v) \leq \Delta(L_\ell) + 7\sqrt{\mu}|U| \cdot 3 \leq 25\sqrt{\mu}n$. Thus, there exists a choice of \mathcal{W}'_ℓ for which $\Delta(L'_\ell) \leq 25\sqrt{\mu}n$. –

Recall that each $u \in U$ encodes a copy F_u of F in the desired decomposition of G . At this stage, if we consider all wheels in $\bigcup_{\ell \in I} \mathcal{W}'_\ell$ which contain u , this yields a copy $F_{2,u}$ of F_2 . This concludes the first step.

To continue, we combine all the leftover graphs L'_ℓ with \tilde{G}_{ℓ^*} , that is, we define

$$G^* := \tilde{G}_{\ell^*} \cup \bigcup_{\ell \in I} L'_\ell.$$

By Proposition 3.11 and (4.25), G^* is still $(\mu^{1/3}, s, p_{\ell^*}D)$ -typical. In the next step, we embed all copies of F_3 . For each $u \in U$, we want to embed a copy $F_{3,u}$ of F_3 into $G^*[N_{G^*}(u)]$. To ensure that these embeddings are edge-disjoint, we proceed sequentially. Suppose that for some subset $U' \subseteq U$, we have already successfully embedded such copies $F_{3,u'}$ edge-disjointly whilst ensuring that

$$\Delta(F_{3,U'}) \leq 2\eta^{1/4}n, \quad (4.26)$$

where $F_{3,U'} := \bigcup_{u' \in U'} F_{3,u'}$. Now we want to find $F_{3,u}$ such that the above holds with U' replaced by $U' \cup \{u\}$. Let $B := \{v \in V : d_{F_{3,U'}}(v) \geq \eta^{1/4}n\}$. Since $e(F_{3,U'}) \leq |U'| \cdot |F_3| \leq \eta^{1/2}n^2$ by (4.24), we deduce that $|B| \leq 2\eta^{1/4}n$. We exclude the vertices of B and the edges of $F_{3,U'}$ when finding $F_{3,u}$. Let $G_u := G^*[N_{G^*}(u) \setminus B] - F_{3,U'}$.

Claim 2: Every pair of vertices in G_u has at least $p_{\ell^}^3 \alpha^2 n$ common neighbours (in G_u).*

Proof of claim: Let v, v' be two vertices in G_u . Since G^* is $(\mu^{1/3}, s, p_{\ell^*}D)$ -typical, we have that v, v', u have $(1 \pm \mu^{1/3})p_{\ell^*}^3 D_{UV} D_V^2 n$ common neighbours in G^* . In particular, v and v' have at least $(1 - \mu^{1/3})4\alpha^2 p_{\ell^*}^3 n$ common neighbours in $G^*[N_{G^*}(u)]$. At most $|B| + 2\Delta(F_{3,U'}) \leq 6\eta^{1/4}n$ have to be discarded. \square

By Claim 2 and since $|F_3| \leq \eta^{1/2}n$, we can greedily find a copy $F_{3,u}$ of F_3 in G_u by choosing one vertex after the other. Clearly, (4.26) still holds with U' replaced by $U' \cup \{u\}$. Thus, we can carry out this embedding for all $u \in U$, which completes the second step.

It remains to embed all copies of F_1 . For each $u \in U$, let $W_{3,u}$ be the graph obtained from $F_{3,u}$ by adding all edges from $V(F_{3,u})$ to u . Moreover, let $\mathcal{W}^3 := \{W_{3,u} : u \in U\}$. Since $|F_3| \leq \eta^{1/2}n$ and using (4.26), we have that $\Delta(\mathcal{W}^3) \leq 3\eta^{1/4}n$. Finally, define

$$\hat{G} := G^* - \mathcal{W}^3.$$

By Proposition 3.11, \hat{G} is still $(\eta^{1/5}, s, p_{\ell^*}D)$ -typical. Recall that $p_{\ell^*} \geq 1/700$ and $\eta \ll 1$. Moreover, \hat{G} is obtained from \tilde{G} by removing edge-disjoint wheel graphs with hubs in U . Thus, from (b), we have $d_{\hat{G}}(v, V) = 2d_{\tilde{G}}(v, U)$ for all $v \in V$. We also have $d_{\hat{G}}(u) = n - |F_2| - |F_3| = |F_1| = n_{\ell^*}$ for all $u \in U$. Hence, by Corollary 3.7, \hat{G} has a $(W_{\ell^*}, \sigma_{\ell^*})$ -decomposition \mathcal{W}_{ℓ^*} . For $u \in U$, let $F_{1,u}$ be the collection of all ℓ^* -cycles which together with u form a wheel in \mathcal{W}_{ℓ^*} . Clearly, $F_{1,u}$ is a copy of F_1 .

Therefore, for every $u \in U$, the graph $F_{1,u} \cup F_{2,u} \cup F_{3,u}$ is a copy of F in G , and all these copies are edge-disjoint. Hence, G has an F -decomposition. This completes the proof of Case 1.

Case 2: $\sum_{\ell=500}^n n_\ell \geq \eta n$.

We assume now that at least an η -fraction of vertices in F lie in cycles of length at least 500. Our strategy is as follows. Let (\mathcal{X}, R) be an F -partition of V . We select the following three edge-disjoint graphs. Given (\mathcal{X}, R) , we set aside an absorbing graph G^{abs} as in Lemma 4.11 and also reserve a regular graph $G_{\mathcal{X}}$ which is the union of quasirandom graphs G_X on X for each $X \in \mathcal{X}$. We also set aside a random edge slice \tilde{G} of the remaining graph. The graph \tilde{G} is much sparser than $G_{\mathcal{X}}$, and $G_{\mathcal{X}}$ is much sparser than G^{abs} . Then we apply the bandwidth theorem for approximate decompositions (Theorem 3.5) to find a packing of \mathcal{H}

in the remainder. This yields a very sparse uncovered leftover. Afterwards, we add \tilde{G} to the leftover from this packing to make it sufficiently well behaved. By Lemma 4.12, we can cover this new leftover with a few copies of F by using (additionally) only edges of $G_{\mathcal{X}}$. In a further step, we utilize Lemma 4.15 to cover the remaining edges of $G_{\mathcal{X}}$ by using very few edges of G^{abs} . In particular, by Lemma 4.11 the remaining subgraph of G^{abs} will still admit an F -decomposition.

Now we turn to the details. Let (\mathcal{X}, R) be an F -partition of V . First, we apply Lemma 4.11 (with η playing the roles of α and η , and μ playing the role of ε) to find a graph G^{abs} with vertex partition \mathcal{X} and reduced graph R which satisfies the following properties:

- (i) G^{abs} is $2r$ -regular for some $r \leq \eta n$;
- (ii) for all $XX' \in E(R)$, the pair $G^{abs}[X, X']$ is $(\mu, d_{XX'})$ -quasirandom for some $\eta^2/200 \leq d_{XX'} \leq \eta$;
- (iii) for every subgraph $A \subseteq G^{abs}$ that is r_A -balanced with respect to (\mathcal{X}, R) for some $r_A \leq \sqrt{\mu n}$, the remainder $G^{abs} - A$ has an F -decomposition.

Let $c := \min_{(a,b) \in \mathcal{I}} c^F(a, b)$. By Proposition 4.3(iv), we have $c \geq \eta n/200$. Define $d_{X_i^a} := \frac{\mu c}{c^F(a)}$ for each $a \in \{3, 4, 5\}$ and $i \in [a] \setminus \{1\}$, and set $d_{X_1^{a,b}} := \frac{\mu c}{c^F(a,b)}$ for all $(a, b) \in \mathcal{I}$. Thus, $\mu \eta/200 \leq d_X \leq \mu$ for all $X \in \mathcal{X}$. Let $r^\circ := \lfloor \mu c/2 \rfloor$. Clearly, we have $\lfloor d_X |X|/2 \rfloor = r^\circ$ for all $X \in \mathcal{X}$. Now, for each $X \in \mathcal{X}$, we apply Proposition 4.10 to find a graph G_X on X which is $2r^\circ$ -regular and (ε, d_X) -quasirandom. Let

$$G_{\mathcal{X}} := \bigcup_{X \in \mathcal{X}} G_X.$$

Observe that $K_n - G^{abs} - G_{\mathcal{X}}$ is an $(n - 1 - 2r - 2r^\circ)$ -regular graph. Next, we select in $K_n - G^{abs} - G_{\mathcal{X}}$ every edge independently with probability $\varepsilon \cdot \frac{n-1}{n-1-2r-2r^\circ}$. Using Lemma 3.1(i), with probability at least $1/2$, this yields a graph \tilde{G} where $d_{\tilde{G}}(v) = \varepsilon n \pm \xi n$ for all $v \in V$ and $e_{\tilde{G}}(X, X') \geq \varepsilon^2 n^2 \geq \sqrt{2\varepsilon} n$ for all distinct $X, X' \in \mathcal{X}$. Let \tilde{G} be some graph with these properties. Now, let

$$G' := K_n - G^{abs} - G_{\mathcal{X}} - \tilde{G}.$$

We will first use Theorem 3.5 to pack \mathcal{H} into G' . By moving some copies of F from \mathcal{F} to \mathcal{H} , we may assume that $|\mathcal{F}| = r + r^\circ + \lfloor \varepsilon n \rfloor$. Let $d := 1 - \frac{2r+2r^\circ}{n} - \varepsilon$. Note that $d \geq 1 - \frac{1}{200\Delta}$ by (i). Clearly, $d_{G'}(v) = (d \pm 2\xi)n$ for all $v \in V$. Observe that $e(\mathcal{H}) = \binom{n}{2} - |\mathcal{F}|n \leq (d - 0.9\varepsilon)n^2/2$ and hence $e(\mathcal{H}) \leq (1 - \varepsilon/2)e(G')$. Thus, by Theorem 3.5, \mathcal{H} packs into G' . Let L_1 be the leftover of this packing in G' , and let $L'_1 := L_1 \cup \tilde{G}$. It remains to show that $G^{abs} \cup G_{\mathcal{X}} \cup L'_1$ has an F -decomposition.

First note that because all graphs in \mathcal{H} are regular, G^{abs} is $2r$ -regular, and $G_{\mathcal{X}}$ is $2r^\circ$ -regular, we can conclude that

$$d_{L'_1}(v) = 2|\mathcal{F}| - 2r - 2r^\circ = 2\lfloor \varepsilon n \rfloor \text{ for all } v \in V. \quad (4.27)$$

Let $L''_1 := L'_1 - \bigcup_{X \in \mathcal{X}} L'_1[X]$ be the subgraph of L'_1 which consists of all the ‘crossing’ edges.

Clearly, $\Delta(L''_1) \leq \Delta(L'_1) \leq 2\varepsilon n$ by (4.27). Moreover, for all distinct $X, X' \in \mathcal{X}$, we have $e_{L''_1}(X, X') \geq e_{\tilde{G}}(X, X') \geq \sqrt{2\varepsilon} n$. Crucially, for all $X \in \mathcal{X}$, we have that

$$e_{L''_1}(X, V \setminus X) = e_{L'_1}(X, V \setminus X) = \sum_{v \in X} d_{L'_1}(v) - 2e_{L'_1}(X) \stackrel{(4.27)}{\equiv} 0 \pmod{2}.$$

Hence, we can employ Lemma 4.12 (with $2\varepsilon, \mu \eta/200, 18$ playing the roles of ε, η, t) to obtain a subgraph $G'_{\mathcal{X}}$ of $G_{\mathcal{X}}$ such that $G'_{\mathcal{X}} \cup L''_1$ has an F -decomposition.

We now define

$$L_2 := (G_{\mathcal{X}} - G'_{\mathcal{X}}) \cup \bigcup_{X \in \mathcal{X}} L'_1[X] = (G_{\mathcal{X}} \cup L'_1) - (G'_{\mathcal{X}} \cup L''_1).$$

Note that it remains to show that $G^{abs} \cup L_2 = (G^{abs} \cup G_{\mathcal{X}} \cup L'_1) - (G'_{\mathcal{X}} \cup L''_1)$ has an F -decomposition. Clearly, both endpoints of any edge in L_2 lie in the same part $X \in \mathcal{X}$. Moreover, as $d_{L_2}(v) = 2r^\circ + 2\lfloor \varepsilon n \rfloor - d_{G'_{\mathcal{X}} \cup L''_1}(v)$ for all $v \in V$ and $G'_{\mathcal{X}} \cup L''_1$ is even-regular, we deduce that L_2 is $2s$ -regular for some $s \in \mathbb{N}$ with $s \leq \mu n/2 + \varepsilon n \leq \mu n$. Hence, by (ii), we can apply Lemma 4.15 (with μ playing the role of ε) to find a subgraph $A \subseteq G^{abs}$ such that $A \cup L_2$ has an F -decomposition and A is r_A -balanced with respect to (\mathcal{X}, R) for some $r_A \leq \sqrt{\mu n}$. Finally, by (iii), $G^{abs} - A$ also has an F -decomposition, which completes the proof. \square

5. CONCLUDING REMARKS

Note that our main result (Theorem 1.3) is formulated in terms of decompositions of the complete graph K_n . Essentially the same argument allows us to replace K_n by any n -vertex regular host graph G which is almost complete in the sense that the degrees are $n - o(n)$. This allows us to find an ‘Oberwolfach factorization’ where the first $o(n)$ F -factors in K_n can be chosen arbitrarily.

Theorem 5.1. *For given $\Delta \in \mathbb{N}$ and $\alpha > 0$, there exist $\xi_0 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $\xi < \xi_0$. Let G be an r -regular n -vertex graph with $r \geq (1 - \xi)n$ and let \mathcal{F}, \mathcal{H} be collections of graphs satisfying the following:*

- \mathcal{F} is a collection of at least αn copies of F , where F is a 2-regular n -vertex graph;
- each $H \in \mathcal{H}$ is a ξ -separable n -vertex r_H -regular graph for some $r_H \leq \Delta$;
- $e(\mathcal{F} \cup \mathcal{H}) = e(G)$.

Then G decomposes into $\mathcal{F} \cup \mathcal{H}$.

The proof of Theorem 5.1 is essentially the same as that of Theorem 1.3. The only significant difference is the construction of the absorbing graph G in the proof of Lemma 4.11. In Lemma 4.11, it was sufficient to prove simply the existence of such a graph. We achieved this by first proving the existence of suitable ‘permuted blow-ups’ G_a for $a \in \{3, 4, 5\}$. The existence of the G_a in turn followed from Lemma 4.9. We now need to find G in an r -regular graph G^* of degree $r \geq (1 - \xi)n$. For this, we argue as in the proofs of Lemmas 4.9 and 4.11: We first choose an F -partition (\mathcal{X}, R) of $V = V(G^*)$ and let G^{**} denote the spanning subgraph of G^* containing all those edges which correspond to edges of R . Then for each $a \in \{3, 4, 5\}$, the graph $\pi^*(G^{**})[X_1^a \cup \dots \cup X_a^a]$ is a blown-up cycle. Our aim now is to construct a graph $G_a \subseteq \pi^*(G^{**})[X_1^a \cup \dots \cup X_a^a]$ satisfying (a) and (b) in the proof of Lemma 4.11. This can be done similarly as before: since for each $i \in [a]$ the pair $\pi^*(G^{**})[X_i^a, X_{i+1}^a]$ has minimum degree at least $(1 - \sqrt{\xi})|X_i^a|$, it contains at least $(1 - \xi^{1/3})|X_i^a|$ edge-disjoint perfect matchings. Choosing a suitable number of these at random as in the proof of Lemma 4.9 again gives the desired graph G_a .

Finally, we remark that Keevash and Staden [28] have recently given a new proof of our result. Their approach is also based on [27] and uses probabilistic techniques and the absorption method. They do not require the condition $m_1 \geq \alpha n$ in Theorem 1.2, that is, they solve the problem when $F_1, \dots, F_{(n-1)/2}$ can be arbitrary 2-factors.

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(S. Glock) INSTITUTE FOR THEORETICAL STUDIES, ETH ZÜRICH, SWITZERLAND

Email address: `stefan.glock@eth-its.ethz.ch`

(F. Joos) INSTITUT FÜR INFORMATIK, UNIVERSITÄT HEIDELBERG, GERMANY

Email address: `joos@informatik.uni-heidelberg.de`

(J. Kim) DEPARTMENT OF MATHEMATICAL SCIENCES, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, SOUTH KOREA

Email address: `jaehoon.kim@kaist.ac.kr`

(D. Kühn) SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, UNITED KINGDOM

Email address: `d.kuhn@bham.ac.uk`

(D. Osthus) SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, UNITED KINGDOM

Email address: `d.osthus@bham.ac.uk`