HYPERGRAPH F-DESIGNS FOR ARBITRARY F

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Abstract. We solve the existence problem for F-designs for arbitrary r-uniform hypergraphs F. In particular, this shows that, given any r-uniform hypergraph F, the trivially necessary divisibility conditions are sufficient to guarantee a decomposition of any sufficiently large complete r-uniform hypergraph $G = K_n^{(r)}$ into edge-disjoint copies of F, which answers a question asked e.g. by Keevash. The graph case $r = 2$ forms one of the cornerstones of design theory and was proved by Wilson in 1975. The case when F is complete corresponds to the existence of block designs, a problem going back to the 19th century, which was first settled by Keevash.

More generally, our results extend to F-designs of quasi-random hypergraphs G and of hypergraphs G of suitably large minimum degree. Our approach builds on results and methods we recently introduced in our new proof of the existence conjecture for block designs.

1. Introduction

1.1. **Background.** A *hypergraph* G is a pair (V, E), where $V = V(G)$ is the vertex set of G and the edge set E is a set of subsets of V. We often identify G with E, in particular, we let $|G| := |E|$. We say that G is an r-*graph* if every edge has size r. We let $K_n^{(r)}$ denote the complete r-graph on n vertices.

Let G and F be r-graphs. An F-*decomposition of* G is a collection $\mathcal{F}$ of copies of F in G such that every edge of G is contained in exactly one of these copies. (Throughout the paper, we always assume that F is non-empty without mentioning this explicitly.) More generally, an (F, λ)-*design of* G is a collection $\mathcal{F}$ of distinct copies of F in G such that every edge of G is contained in exactly λ of these copies. Such a design can only exist if G satisfies certain divisibility conditions (e.g. if F is a graph triangle and $\lambda = 1$, then G must have even vertex degrees and the number of edges must be a multiple of three). If F is complete, such designs are also referred to as block designs.

The question of the existence of such designs goes back to the 19th century. The first general result was due to Kirkman [16], who proved the existence of Steiner triple systems (i.e. triangle decompositions of complete graphs) under the appropriate divisibility conditions. In a groundbreaking series of papers which transformed the area, Wilson [30, 31, 32, 33] solved the existence problem in the graph setting (i.e. when $r = 2$) by showing that the trivially necessary divisibility conditions imply the existence of (F, λ)-designs in $K_n^{(2)}$ for sufficiently large n. More generally, the existence conjecture postulated that the necessary divisibility conditions are also sufficient to ensure the existence of block designs with given parameters in $K_n^{(r)}$.

Answering a question of Erdős and Hanani [10], Rödl [25] was able to give an approximate solution to the existence conjecture by constructing near optimal packings of edge-disjoint copies of $K_f^{(r)}$ in $K_n^{(r)}$, i.e. packings which cover almost all the edges of $K_n^{(r)}$. (For this, he introduced his now famous Rödl nibble method, which has since had a major impact in many areas.) More recently, Kuperberg, Lovett and Peled [21] were able to prove probabilistically the existence of non-trivial designs for a large range of parameters (but their result requires that λ is comparatively large). Apart from this, progress for $r \geq 3$ was mainly limited to explicit constructions for rather restrictive parameters (see e.g. [7, 29]).

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In a recent breakthrough, Keevash [14] proved the existence of $(K_f^{(r)},\lambda)$-designs in $K_n^{(r)}$ for arbitrary (but fixed) r, f and λ, provided n is sufficiently large. In particular, his result implies the existence of Steiner systems for any admissible range of parameters as long as n is sufficiently large compared to f (Steiner systems are block designs with $\lambda = 1$). The approach in [14] involved randomised algebraic constructions and yielded a far-reaching generalisation to block designs in quasirandom r-graphs. This in turn was extended in [12], where we developed a non-algebraic approach based on iterative absorption, which additionally yielded resilience versions and the existence of block designs in hypergraphs of large minimum degree. This naturally raises the question of whether F-designs also exist for arbitrary r-graphs F. Here, we answer this affirmatively, by building on methods and results from [12].

1.2. **F-designs in quasirandom hypergraphs.** We now describe the degree conditions which are trivially necessary for the existence of an F-design in an r-graph G. For a set $S \subseteq V(G)$ with $0 \le |S| \le r$, the $(r-|S|)$-graph $G(S)$ has vertex set $V(G)\setminus S$ and contains all $(r-|S|)$-subsets of $V(G)\setminus S$ that together with S form an edge in G. ($G(S)$ is often called the *link graph of* S.) Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum $(r-1)$-degree of an r-graph G, respectively, that is, the minimum/maximum value of $|G(S)|$ over all $S \subseteq V(G)$ of size $r-1$. For a (non-empty) r-graph F, we define the *divisibility vector of* F as $Deg(F) := (d_0,\dots,d_{r-1}) \in \mathbb{N}^r$, where $d_i := \gcd\{|F(S)| \,:\, S \in \binom{V(F)}{i}\}$, and we set $Deg(F)_i := d_i$ for $0 \le i \le r-1$. Note that $d_0 = |F|$. So if F is the Fano plane, we have $Deg(F) = (7,3,1)$.

Given r-graphs F and G, G is called (F,λ)-*divisible* if $Deg(F)_i \mid \lambda |G(S)|$ for all $0 \le i \le r-1$ and all $S \in \binom{V(G)}{i}$. Note that G must be (F,λ)-divisible in order to admit an (F,λ)-design. For simplicity, we say that G is F-*divisible* if G is $(F,1)$-divisible. Thus F-divisibility of G is necessary for the existence of an F-decomposition of G.

As a special case, the following result implies that (F,λ)-divisibility is sufficient to guarantee the existence of an (F,λ)-design when G is complete and λ is not too large. This answers a question asked e.g. by Keevash [14].

In fact, rather than requiring G to be complete, it suffices that G is quasirandom in the following sense. An r-graph G on n vertices is called (c,h,p)-*typical* if for any set A of $(r-1)$-subsets of $V(G)$ with $|A| \le h$ we have $|\bigcap_{S\in A} G(S)| = (1\pm c)p^{|A|}n$. Note that this is what one would expect in a random r-graph with edge probability p.

Theorem 1.1 (F-designs in typical hypergraphs)**.** *For all $f, r \in \mathbb{N}$ with $f > r$ and all $c, p \in (0,1]$ with*

$$c \le 0.9(p/2)^h/(q^r 4^q), \ \text{ where } q := 2f\cdot f! \text{ and } h := 2^r\binom{q+r}{r},$$

there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds for all $n \ge n_0$. Let F be any r-graph on f vertices and let $\lambda \in \mathbb{N}$ with $\lambda \le \gamma n$. Suppose that G is a (c,h,p)-typical r-graph on n vertices. Then G has an (F,λ)-design if it is (F,λ)-divisible.

The main result in [14] is also stated in the setting of typical r-graphs, but additionally requires that $c \ll 1/h \ll p, 1/f$ and that $\lambda = \mathcal{O}(1)$ and F is complete. The case when F is complete and λ is bounded is also a special case of our recent result on designs in supercomplexes (see Theorem 1.4 in [12]). Previous results in the case when $r \ge 3$ and F is not complete are very sporadic – for instance Hanani [13] settled the problem if F is an octahedron (viewed as a 3-uniform hypergraph) and G is complete.

As a very special case, Theorem 1.1 resolves a conjecture of Archdeacon on self-dual embeddings of random graphs in orientable surfaces: as proved in [2], a graph has such an embedding if it has a decomposition into $K := K_4^{(2)}$ and $K' := K_5^{(2)}$. Suppose G is a (c,h,p)-typical 2-graph on n vertices with an even number of edges and $1/n \ll c \ll 1/h \ll p$ (which almost surely holds for the binomial random graph $G_{n,p}$ if we remove at most one edge). Now remove a suitable number of copies of K from G to ensure that the leftover G' satisfies $16 \mid |G'|$. Let F be the vertex-disjoint union of K and K'. Since $Deg(F)_1 = 1$, G' is F-divisible. Thus we can apply

Theorem 1.1 to obtain an F-decomposition of G'. If the number of edges is odd, a similar argument yields self-dual embeddings in non-orientable surfaces.

In Section 8, we will deduce Theorem 1.1 from a more general result on F-decompositions in supercomplexes G (Theorem 3.8). (The condition of G being a supercomplex is considerably less restrictive than typicality.) Moreover, the F-designs we obtain will have the additional property that $|V(F') \cap V(F'')| \leq r$ for all distinct F', F'' which are included in the design. It is easy to see that with this additional property the bound on λ in Theorem 1.1 is best possible up to the value of γ.

We can also deduce the following result which yields 'near-optimal' F-packings in typical r-graphs which are not divisible. (An F-packing in G is a collection of edge-disjoint copies of F in G.)

Theorem 1.2. *For all $f, r \in \mathbb{N}$ with $f > r$ and all $c, p \in (0, 1]$ with*

$$c \leq 0.9p^h/(q^r 4^q), \text{ where } q := 2f \cdot f! \text{ and } h := 2^r \binom{q+r}{r},$$

there exist $n_0, C \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let F be any r-graph on f vertices. Suppose that G is a (c, h, p)-typical r-graph on n vertices. Then G has an F-packing $\mathcal{F}$ such that the leftover L consisting of all uncovered edges satisfies $\Delta(L) \leq C$.

1.3. **F-designs in hypergraphs of large minimum degree.** Once the existence question is settled, a next natural step is to seek F-designs and F-decompositions in r-graphs of large minimum degree. Our next result gives a bound on the minimum degree which ensures an F-decomposition for 'weakly regular' r-graphs F. These are defined as follows.

Definition 1.3 (weakly regular)**.** Let F be an r-graph. We say that F is *weakly $(s_0, \ldots, s_{r-1})$-regular* if for all $0 \leq i \leq r-1$ and all $S \in \binom{V(F)}{i}$, we have $|F(S)| \in \{0, s_i\}$. We simply say that F is *weakly regular* if it is weakly $(s_0, \ldots, s_{r-1})$-regular for suitable s_i's.

So for example, cliques, the Fano plane and the octahedron are all weakly regular but a 3-uniform tight or loose cycle is not.

Theorem 1.4 (F-decompositions in hypergraphs of large minimum degree)**.** *Let F be a weakly regular r-graph on f vertices. Let*

$$c^{\diamond}_F := \frac{r!}{3 \cdot 14^r f^{2r}}.$$

There exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that G is an r-graph on n vertices with $\delta(G) \geq (1 - c^{\diamond}_F)n$. Then G has an F-decomposition if it is F-divisible.

Note that Theorem 1.4 implies that every packing of edge-disjoint copies of F into $K^{(r)}_n$ with overall maximum degree at most $c^{\diamond}_F n$ can be extended into an F-decomposition of $K^{(r)}_n$ (provided $K^{(r)}_n$ is F-divisible).

An analogous (but significantly worse) constant $c^{\diamond}_F$ for r-graphs F which are not weakly regular immediately follows from the case $p = 1$ of Theorem 1.1. These results lead to the concept of the 'decomposition threshold' δ_F of a given r-graph F.

Definition 1.5 (Decomposition threshold)**.** *Given an r-graph F, let δ_F be the infimum of all $\delta \in [0, 1]$ with the following property: There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, every F-divisible r-graph G on n vertices with $\delta(G) \geq \delta n$ has an F-decomposition.*

By Theorem 1.4, we have $\delta_F \leq 1 - c^{\diamond}_F$ whenever F is weakly regular. As noted in [12], for all $r, f, n_0 \in \mathbb{N}$, there exists an r-graph G_n on $n \geq n_0$ vertices with $\delta(G_n) \geq (1 - b_r \frac{\log f}{f^{r-1}})n$ such that G_n does not contain a single copy of $K^{(r)}_f$, where $b_r > 0$ only depends on r. (This can be seen by adapting a construction from [18] which is based on a result from [27].)

Previously, the only positive result for the hypergraph case $r \geq 3$ was due to Yuster [34], who showed that if T is a linear r-uniform hypertree, then every T-divisible r-graph G on n

vertices with minimum vertex degree at least $(\frac{1}{2^{r-1}} + o(1))\binom{n}{r-1}$ has a T-decomposition. This is asymptotically best possible for nontrivial T. Moreover, the result implies that $\delta_T \leq 1/2^{r-1}$.

For the graph case $r = 2$, much more is known about the decomposition threshold: the results in [4, 11] establish a close connection between δ_F and the fractional decomposition threshold δ^*_F (which is defined as in Definition 1.5, but with an F-decomposition replaced by a fractional F-decomposition). In particular, the results in [4, 11] imply that $\delta_F \leq \max\{\delta^*_F, 1-1/(\chi(F)+1)\}$ and that $\delta_F = \delta^*_F$ if F is a complete graph.

Together with recent results on the fractional decomposition threshold for cliques in [3, 8], this gives the best current bounds on δ_F for general F. It would be very interesting to establish a similar connection in the hypergraph case.

Also, for bipartite graphs the decomposition threshold was completely determined in [11]. It would be interesting to see if this can be generalised to r-partite r-graphs. On the other hand, even the decomposition threshold of a graph triangle is still unknown (a beautiful conjecture of Nash-Williams [23] would imply that the value is 3/4).

1.4. **Counting.** An approximate F-decomposition of $K^{(r)}_n$ is a set of edge-disjoint copies of F in $K^{(r)}_n$ which together cover almost all edges of $K^{(r)}_n$. Given good bounds on the number of approximate F-decompositions of $K^{(r)}_n$ whose set of leftover edges forms a typical r-graph, one can apply Theorem 1.1 to obtain corresponding bounds on the number of F-decompositions in $K^{(r)}_n$ (see [14, 15] for the clique case). Such bounds on the number of approximate F-decompositions can be achieved by considering either a random greedy F-removal process or an associated F-nibble removal process.

1.5. **Outline of the paper.** As mentioned earlier, our main result (Theorem 3.8) actually concerns F-decompositions in so-called supercomplexes. We will define supercomplexes in Section 3 and derive Theorems 1.1, 1.2 and 1.4 from Theorem 3.8 in Section 8. The definition of a supercomplex G involves mainly the distribution of cliques of size f in G (where $f = |V(F)|$). The notion is weaker than usual notions of quasirandomness. This has two main advantages: firstly, our proof is by induction on r, and working with this weaker notion is essential to make the induction proof work. Secondly, this allows us to deduce Theorems 1.1, 1.2 and 1.4 from a single statement.

However, Theorem 3.8 applies only to F-decompositions of a supercomplex G for weakly regular r-graphs F (which allows us to deduce Theorem 1.4 but not Theorem 1.1). To deal with this, in Section 8 we first provide an explicit construction which shows that every r-graph F can be 'perfectly' packed into a suitable weakly regular r-graph F^*. In particular, F^* has an F-decomposition. The idea is then to apply Theorem 3.8 to find an F^*-decomposition in G. Unfortunately, G may not be F^*-divisible. To overcome this, in Section 9 we show that we can remove a small set of copies of F from G to achieve that the leftover G' of G is now F^*-divisible (see Lemma 8.4 for the statement). This now implies Theorem 1.1 for F-decompositions, i.e. for $\lambda = 1$. However, by repeatedly applying Theorem 3.8 in a suitable way, we can actually allow λ to be as large as required in Theorem 1.1.

It thus remains to prove Theorem 3.8 itself. We achieve this via the iterative absorption method. The idea is to iteratively extend a packing of edge-disjoint copies of F until the set H of uncovered edges is very small. This final set can then be 'absorbed' into an r-graph A we set aside at the beginning of the proof (in the sense that $A \cup H$ has an F-decomposition). This iterative approach to decompositions was first introduced in [17, 20] in the context of Hamilton decompositions of graphs. (Absorption itself was pioneered earlier for spanning structures e.g. in [19, 26], but as remarked e.g. in [14], such direct approaches are not feasible in the decomposition setting.)

This approach relies on being able to find a suitable approximate F-decomposition in each iteration, whose existence we derive in Section 5. The iteration process is underpinned by a so-called 'vortex', which consists of an appropriate nested sequence of vertex subsets of G (after each iteration, the current set of uncovered edges is constrained to the next vertex subset in the sequence). These vortices are discussed in Section 6. The final absorption step is described in Section 7.

As mentioned earlier, the current proof builds on the framework introduced in [12]. In fact, several parts of the argument in [12] can either be used directly or can be straightforwardly adapted to the current setting, so we do not repeat them here. In particular, this applies to the Cover down lemma (Lemma 6.9), which is the key result that allows the iteration to work. Thus in the current paper we concentrate on the parts which involve significant new ideas (e.g. the absorption process). For details of the parts which can be straightforwardly adapted, we refer to the appendix of the arxiv version of the current paper. Altogether, this illustrates the versatility of our framework and we thus believe that it can be developed in further settings.

As a byproduct of the construction of the weakly regular r-graph F^* outlined above, we prove the existence of resolvable clique decompositions in complete partite r-graphs G (see Theorem 8.1). The construction is explicit and exploits the property that all square submatrices of so-called Cauchy matrices over finite fields are invertible. We believe this construction to be of independent interest. A natural question leading on from the current work would be to obtain such resolvable decompositions also in the general (non-partite) case. For decompositions of $K_n^{(2)}$ into $K_f^{(2)}$, this is due to Ray-Chaudhuri and Wilson [24]. For recent progress see [9, 22].

2. Notation

2.1. **Basic terminology.** We let $[n]$ denote the set $\{1,\dots,n\}$, where $[0]:=\emptyset$. Moreover, let $[n]_0:=[n]\cup\{0\}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. As usual, $\binom{n}{i}$ denotes the binomial coefficient, where we set $\binom{n}{i}:=0$ if $i>n$ or $i<0$. Moreover, given a set X and $i\in\mathbb{N}_0$, we write $\binom{X}{i}$ for the collection of all i-subsets of X. Hence, $\binom{X}{i}=\emptyset$ if $i>|X|$. If F is a collection of sets, we define $\bigcup F:=\bigcup_{f\in F}f$. We write $A\mathbin{\dot\cup}B$ for the union of A and B if we want to emphasise that A and B are disjoint.

We write $X\sim B(n,p)$ if X has binomial distribution with parameters n,p, and we write $bin(n,p,i):=\binom{n}{i}p^i(1-p)^{n-i}$. So by the above convention, $bin(n,p,i)=0$ if $i>n$ or $i<0$.

We say that an event holds *with high probability (whp)* if the probability that it holds tends to 1 as $n\to\infty$ (where n usually denotes the number of vertices).

We write $x\ll y$ to mean that for any $y\in(0,1]$ there exists an $x_0\in(0,1)$ such that for all $x\le x_0$ the subsequent statement holds. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. We will always assume that the constants in our hierarchies are reals in $(0,1]$. Moreover, if $1/x$ appears in a hierarchy, this implicitly means that x is a natural number. More precisely, $1/x\ll y$ means that for any $y\in(0,1]$ there exists an $x_0\in\mathbb{N}$ such that for all $x\in\mathbb{N}$ with $x\ge x_0$ the subsequent statement holds.

We write $a=b\pm c$ if $b-c\le a\le b+c$. Equations containing $\pm$ are always to be interpreted from left to right, e.g. $b_1\pm c_1=b_2\pm c_2$ means that $b_1-c_1\ge b_2-c_2$ and $b_1+c_1\le b_2+c_2$.

When dealing with multisets, we treat multiple appearances of the same element as distinct elements. In particular, two subsets A,B of a multiset can be disjoint even if they both contain a copy of the same element, and if A and B are disjoint, then the multiplicity of an element in the union $A\cup B$ is obtained by adding the multiplicities of this element in A and B (rather than just taking the maximum).

2.2. **Hypergraphs and complexes.** Let G be an r-graph. Note that $G(\emptyset)=G$. For a set $S\subseteq V(G)$ with $|S|\le r$ and $L\subseteq G(S)$, let $S\uplus L:=\{S\cup e\,:\,e\in L\}$. Clearly, there is a natural bijection between L and $S\uplus L$.

For $i\in[r-1]_0$, we define $\delta_i(G)$ and $\Delta_i(G)$ as the minimum and maximum value of $|G(S)|$ over all i-subsets S of $V(G)$, respectively. As before, we let $\delta(G):=\delta_{r-1}(G)$ and $\Delta(G):=\Delta_{r-1}(G)$. Note that $\delta_0(G)=\Delta_0(G)=|G(\emptyset)|=|G|$.

For two r-graphs G and G', we let $G-G'$ denote the r-graph obtained from G by deleting all edges of G'. We write G_1+G_2 to mean the vertex-disjoint union of G_1 and G_2, and $t\cdot G$ to mean the vertex-disjoint union of t copies of G.

Let F and G be r-graphs. An *F-packing in G* is a set $\mathcal{F}$ of edge-disjoint copies of F in G. We let $\mathcal{F}^{(r)}$ denote the r-graph consisting of all covered edges of G, i.e. $\mathcal{F}^{(r)} = \bigcup_{F' \in \mathcal{F}} F'$.

A *multi-r-graph* G consists of a set of vertices $V(G)$ and a multiset of edges $E(G)$, where each $e \in E(G)$ is a subset of $V(G)$ of size r. We will often identify a multi-r-graph with its edge set. For $S \subseteq V(G)$, let $|G(S)|$ denote the number of edges of G that contain S (counted with multiplicities). If $|S| = r$, then $|G(S)|$ is called the *multiplicity of S in G*. We say that G is *F-divisible* if $Deg(F)_{|S|}$ divides $|G(S)|$ for all $S \subseteq V(G)$ with $|S| \leq r-1$. An F-decomposition of G is a collection $\mathcal{F}$ of copies of F in G such that every edge $e \in G$ is covered precisely once. (Thus if $S \subseteq V(G)$ has size r, then there are precisely $|G(S)|$ copies of F in $\mathcal{F}$ in which S forms an edge.)

Definition 2.1. A *complex* G is a hypergraph which is closed under inclusion, that is, whenever $e' \subseteq e \in G$ we have $e' \in G$. If G is a complex and $i \in \mathbb{N}_0$, we write $G^{(i)}$ for the i-graph on $V(G)$ consisting of all $e \in G$ with $|e| = i$. We say that a complex is empty if $\emptyset \notin G^{(0)}$, that is, if G does not contain any edges.

Suppose G is a complex and $e \subseteq V(G)$. Define $G(e)$ as the complex on vertex set $V(G) \setminus e$ containing all sets $e' \subseteq V(G) \setminus e$ such that $e \cup e' \in G$. Clearly, if $e \notin G$, then $G(e)$ is empty. Observe that if $|e| = i$ and $r \geq i$, then $G^{(r)}(e) = G(e)^{(r-i)}$. We say that G' is a *subcomplex* of G if G' is a complex and a subhypergraph of G.

For a set U, define $G[U]$ as the complex on $U \cap V(G)$ containing all $e \in G$ with $e \subseteq U$. Moreover, for an r-graph H, let $G[H]$ be the complex on $V(G)$ with edge set

$$G[H] := \{e \in G : \binom{e}{r} \subseteq H\},$$

and define $G - H := G[G^{(r)} - H]$. So for $i \in [r-1]$, $G[H]^{(i)} = G^{(i)}$. For $i > r$, we might have $G[H]^{(i)} \subsetneq G^{(i)}$. Moreover, if $H \subseteq G^{(r)}$, then $G[H]^{(r)} = H$. Note that for an r_1-graph H_1 and an r_2-graph H_2, we have $(G[H_1])[H_2] = (G[H_2])[H_1]$. Also, $(G - H_1) - H_2 = (G - H_2) - H_1$, so we may write this as $G - H_1 - H_2$.

If G_1 and G_2 are complexes, we define $G_1 \cap G_2$ as the complex on vertex set $V(G_1) \cap V(G_2)$ containing all sets e with $e \in G_1$ and $e \in G_2$. We say that G_1 and G_2 are *i-disjoint* if $G_1^{(i)} \cap G_2^{(i)}$ is empty.

For any hypergraph H, let $H^{\leq}$ be the complex on $V(H)$ *generated by* H, that is,

$$H^{\leq} := \{e \subseteq V(H) : \exists e' \in H \text{ such that } e \subseteq e'\}.$$

For an r-graph H, we let $H^{\leftrightarrow}$ denote the complex on $V(H)$ that is *induced by* H, that is,

$$H^{\leftrightarrow} := \{e \subseteq V(H) : \binom{e}{r} \subseteq H\}.$$

Note that $H^{\leftrightarrow(r)} = H$ and for each $i \in [r-1]_0$, $H^{\leftrightarrow(i)}$ is the complete i-graph on $V(H)$. We let K_n denote the the complete complex on n vertices.

3. Decompositions of supercomplexes

3.1. **Supercomplexes.** We prove our main decomposition theorem for so-called 'supercomplexes', which were introduced in [12]. The crucial property appearing in the definition is that of 'regularity', which means that every r-set of a given complex G is contained in roughly the same number of f-sets (where $f = |V(F)|$). If we view G as a complex which is induced by some r-graph, this means that every edge lies in roughly the same number of cliques of size f. It turns out that this set of conditions is appropriate even when F is not a clique.

A key advantage of the notion of a supercomplex is that the conditions are very flexible, which will enable us to 'boost' their parameters (see Lemma 3.5 below). The following definitions are the same as in [12].

Definition 3.1. Let G be a complex on n vertices, $f \in \mathbb{N}$ and $r \in [f-1]_0$, $0 \leq \varepsilon, d, \xi \leq 1$. We say that G is

(i) (ε, d, f, r)-*regular*, if for all $e \in G^{(r)}$ we have

$$|G^{(f)}(e)| = (d \pm \varepsilon)n^{f-r};$$

(ii) (ξ, f, r)-*dense*, if for all $e \in G^{(r)}$, we have

$$|G^{(f)}(e)| \geq \xi n^{f-r};$$

(iii) (ξ, f, r)-*extendable*, if $G^{(r)}$ is empty or there exists a subset $X \subseteq V(G)$ with $|X| \geq \xi n$ such that for all $e \in \binom{X}{r}$, there are at least ξn^{f-r} $(f-r)$-sets $Q \subseteq V(G) \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G^{(r)}$.

We say that G is a *full* (ε, ξ, f, r)-*complex* if G is

- (ε, d, f, r)-regular for some $d \geq \xi$,
- $(\xi, f+r, r)$-dense,
- (ξ, f, r)-extendable.

We say that G is an (ε, ξ, f, r)-*complex* if there exists an f-graph Y on $V(G)$ such that $G[Y]$ is a full (ε, ξ, f, r)-complex. Note that $G[Y]^{(r)} = G^{(r)}$ (recall that $r < f$).

Definition 3.2. (supercomplex) Let G be a complex. We say that G is an (ε, ξ, f, r)-*supercomplex* if for every $i \in [r]_0$ and every set $B \subseteq G^{(i)}$ with $1 \leq |B| \leq 2^i$, we have that $\bigcap_{b \in B} G(b)$ is an $(\varepsilon, \xi, f-i, r-i)$-complex.

In particular, taking $i = 0$ and $B = \{\emptyset\}$ implies that every (ε, ξ, f, r)-supercomplex is also an (ε, ξ, f, r)-complex. Moreover, the above definition ensures that if G is a supercomplex and $b, b' \in G^{(i)}$, then $G(b) \cap G(b')$ is also a supercomplex (cf. Proposition 4.4).

The following examples from [12] demonstrate that the definition of supercomplexes generalises the notion of typicality.

Example 3.3. Let $1/n \ll 1/f$ and $r \in [f-1]$. Then the complete complex K_n is a $(0, 0.99/f!, f, r)$-supercomplex.

Example 3.4. Suppose that $1/n \ll c, p, 1/f$, that $r \in [f-1]$ and that G is a $(c, 2^r\binom{f+r}{r}, p)$-typical r-graph on n vertices. Then $G^{\leftrightarrow}$ is an (ε, ξ, f, r)-supercomplex, where

$$\varepsilon := 2^{f-r+1}c/(f-r)! \quad \text{and} \quad \xi := (1 - 2^{f+1}c)p^{2^r\binom{f+r}{r}}/f!.$$

As mentioned above, the following lemma allows us to 'boost' the regularity parameters (and thus deduce results with 'effective' bounds). It is an easy consequence of our Boost lemma (Lemma 5.2). The key to the proof is that we can (probabilistically) choose some $Y \subseteq G^{(f)}$ so that the parameters of $G[Y]$ in Definition 3.1(i) are better than those of G, i.e. the resulting distribution of f-sets is more uniform.

Lemma 3.5 ([12])**.** *Let $1/n \ll \varepsilon, \xi, 1/f$ and $r \in [f-1]$ with $2(2\sqrt{\mathrm{e}})^r\varepsilon \leq \xi$. Let $\xi' := 0.9(1/4)^{\binom{f+r}{f}}\xi$. If G is an (ε, ξ, f, r)-complex on n vertices, then G is an $(n^{-1/3}, \xi', f, r)$-complex. In particular, if G is an (ε, ξ, f, r)-supercomplex, then it is a $(2n^{-1/3}, \xi', f, r)$-supercomplex.*

3.2. **The main complex decomposition theorem.** The statement of our main complex decomposition theorem involves the concept of 'well separated' decompositions. This did not appear in [12], but is crucial for our inductive proof to work in the context of F-decompositions.

Definition 3.6 (well separated)**.** Let F be an r-graph and let $\mathcal{F}$ be an F-packing (in some r-graph G). We say that $\mathcal{F}$ is κ-*well separated* if the following hold:

(WS1) for all distinct $F', F'' \in \mathcal{F}$, we have $|V(F') \cap V(F'')| \leq r$.
(WS2) for every r-set e, the number of $F' \in \mathcal{F}$ with $e \subseteq V(F')$ is at most κ.

We simply say that $\mathcal{F}$ is *well separated* if (WS1) holds.

For instance, any $K_f^{(r)}$-packing is automatically 1-well separated. Moreover, if an F-packing $\mathcal{F}$ is 1-well separated, then for all distinct $F', F'' \in \mathcal{F}$, we have $|V(F') \cap V(F'')| < r$. On the other hand, if F is not complete, we cannot require $|V(F') \cap V(F'')| < r$ in (WS1): this would make it impossible to find an F-decomposition of $K_n^{(r)}$. The notion of being well-separated is a natural relaxation of this requirement, we discuss this in more detail after stating Theorem 3.8.

We now define F-divisibility and F-decompositions for complexes G (rather than r-graphs G).

Definition 3.7. Let F be an r-graph and $f := |V(F)|$. A complex G is *F-divisible* if $G^{(r)}$ is F-divisible. An *F-packing in* G is an F-packing $\mathcal{F}$ in $G^{(r)}$ such that $V(F') \in G^{(f)}$ for all $F' \in \mathcal{F}$. Similarly, we say that $\mathcal{F}$ is an *F-decomposition of* G if $\mathcal{F}$ is an F-packing in G and $\mathcal{F}^{(r)} = G^{(r)}$.

Note that this implies that every copy F' of F used in an F-packing in G is 'supported' by a clique, i.e. $G^{(r)}[V(F')] \cong K_f^{(r)}$.

We can now state our main complex decomposition theorem.

Theorem 3.8 (Main complex decomposition theorem)**.** *For all $r \in \mathbb{N}$, the following is true.*

$(*)_r$ *Let $1/n \ll 1/\kappa, \varepsilon \ll \xi, 1/f$ and $f > r$. Let F be a weakly regular r-graph on f vertices and let G be an F-divisible (ε, ξ, f, r)-supercomplex on n vertices. Then G has a κ-well separated F-decomposition.*

We will prove $(*)_r$ by induction on r in Section 8. We do not make any attempt to optimise the values that we obtain for κ.

We now motivate Definitions 3.6 and 3.7. This involves the following additional concepts, which are also convenient later.

Definition 3.9. Let $f := |V(F)|$ and suppose that $\mathcal{F}$ is a well separated F-packing. We let $\mathcal{F}^{\leq}$ denote the complex generated by the f-graph $\{V(F') : F' \in \mathcal{F}\}$. We say that well separated F-packings $\mathcal{F}_1, \mathcal{F}_2$ are *i-disjoint* if $\mathcal{F}_1^{\leq}, \mathcal{F}_2^{\leq}$ are i-disjoint (or equivalently, if $|V(F') \cap V(F'')| < i$ for all $F' \in \mathcal{F}_1$ and $F'' \in \mathcal{F}_2$).

Note that if F is a well-separated F-packing, then the f-graph $\{V(F') : F' \in \mathcal{F}\}$ is simple. Moreover, observe that (WS2) is equivalent to the condition $\Delta_r(\mathcal{F}^{\leq(f)}) \leq \kappa$. Furthermore, if $\mathcal{F}$ is a well separated F-packing in a complex G, then $\mathcal{F}^{\leq}$ is a subcomplex of G by Definition 3.7. Clearly, we have $\mathcal{F}^{(r)} \subseteq \mathcal{F}^{\leq(r)}$, but in general equality does not hold. On the other hand, if $\mathcal{F}$ is an F-decomposition of G, then $\mathcal{F}^{(r)} = G^{(r)}$ which implies $\mathcal{F}^{(r)} = \mathcal{F}^{\leq(r)}$.

We now discuss (WS1). During our proof, we will need to find an F-packing which covers a given set of edges. This gives rise to the following task of 'covering down locally'.

$(\star)$ *Given a set $S \subseteq V(G)$ of size $1 \leq i \leq r-1$, find an F-packing $\mathcal{F}$ which covers all edges of G that contain S.*

(This is crucial in the proof of Lemma 6.9. Moreover, a two-sided version of this involving sets S, S' is needed to construct parts of our absorbers, see Section 7.1.)

A natural approach to achieve $(\star)$ is as follows: Let $T \in \binom{V(F)}{i}$. Suppose that by using the main theorem inductively, we can find an $F(T)$-decomposition $\mathcal{F}'$ of $G(S)$. We now wish to obtain $\mathcal{F}$ by 'extending' $\mathcal{F}'$ as follows: For each copy F' of $F(T)$ in $\mathcal{F}'$, we define a copy $F'_\triangleleft$ of F by 'adding S back', that is, $F'_\triangleleft$ has vertex set $V(F') \cup S$ and S plays the role of T in $F'_\triangleleft$. Then $F'_\triangleleft$ covers all edges e with $S \subseteq e$ and $e \setminus S \in F'$. Since $\mathcal{F}'$ is an $F(T)$-decomposition of $G(S)$, the union of all $F'_\triangleleft$ would indeed cover all edges of G that contain S, as desired. There are two issues with this 'extension' though. Firstly, it is not clear that $F'_\triangleleft$ is a subgraph of G. Secondly, for distinct $F', F'' \in \mathcal{F}'$, it is not clear that $F'_\triangleleft$ and $F''_\triangleleft$ are edge-disjoint. Definition 3.7 (and the succeeding remark) allows us to resolve the first issue. Indeed, if $\mathcal{F}'$ is an $F(T)$-decomposition of the complex $G(S)$, then from $V(F') \in G(S)^{(f-i)}$, we can deduce $V(F'_\triangleleft) \in G^{(f)}$ and thus that $F'_\triangleleft$ is a subgraph of $G^{(r)}$.

We now consider the second issue. This does not arise if F is a clique. Indeed, in that case $F(T)$ is a copy of $K_{f-i}^{(r-i)}$, and thus for distinct $F', F'' \in \mathcal{F}'$ we have $|V(F') \cap V(F'')| < r - i$. Hence $|V(F'_\triangleleft) \cap V(F''_\triangleleft)| < r - i + |S| = r$, i.e. $F'_\triangleleft$ and $F''_\triangleleft$ are edge-disjoint. If however F is not a clique, then $F', F'' \in \mathcal{F}'$ can overlap in $r - i$ or more vertices (they could in fact have the same vertex set), and the above argument does not work. We will show that under the assumption that $\mathcal{F}'$ is well separated, we can overcome this issue and still carry out the above 'extension'. (Moreover, the resulting F-packing $\mathcal{F}$ will in fact be well separated itself, see Definition 6.10 and Proposition 6.11). For this it is useful to note that $F(T)$ is an $(r-i)$-graph, and thus we already have $|V(F') \cap V(F'')| \leq r-i$ if $\mathcal{F}'$ is well separated.

The reason why we also include (WS2) in Definition 3.6 is as follows. Suppose we have already found a well separated F-packing $\mathcal{F}_1$ in G and now want to find another well separated F-packing $\mathcal{F}_2$ such that we can combine $\mathcal{F}_1$ and $\mathcal{F}_2$. If we find $\mathcal{F}_2$ in $G - \mathcal{F}_1^{(r)}$, then $\mathcal{F}_1^{(r)}$ and $\mathcal{F}_2^{(r)}$ are edge-disjoint and thus $\mathcal{F}_1 \cup \mathcal{F}_2$ will be an F-packing in G, but it is not necessarily well separated. We therefore find $\mathcal{F}_2$ in $G - \mathcal{F}_1^{(r)} - \mathcal{F}_1^{\leq(r+1)}$. This ensures that $\mathcal{F}_1$ and $\mathcal{F}_2$ are $(r+1)$-disjoint, which in turn implies that $\mathcal{F}_1 \cup \mathcal{F}_2$ is indeed well separated, as required. But in order to be able to construct $\mathcal{F}_2$, we need to ensure that $G - \mathcal{F}_1^{(r)} - \mathcal{F}_1^{\leq(r+1)}$ is still a supercomplex, which is true if $\Delta(\mathcal{F}_1^{(r)})$ and $\Delta(\mathcal{F}_1^{\leq(r+1)})$ are small (cf. Proposition 4.5). The latter in turn is ensured by (WS2) via Fact 4.3.

Finally, we discuss why we prove Theorem 3.8 for weakly regular r-graphs F. Most importantly, the 'regularity' of the degrees will be crucial for the construction of our absorbers (most notably in Lemma 7.22). Beyond that, weakly regular graphs also have useful closure properties (cf. Proposition 4.2): they are closed under taking link graphs and divisibility is inherited by link graphs in a natural way.

We prove Theorem 3.8 in Sections 5–7 and 8.1. As described in Section 1.5, we generalise this to arbitrary F via Lemma 8.2 (proved in Section 8.2) and Lemma 8.4 (proved in Section 9): Lemma 8.2 shows that for every given r-graph F, there is a weakly regular r-graph F^* which has an F-decomposition. Lemma 8.4 then complements this by showing that every F-divisible r-graph G can be transformed into an F^*-divisible r-graph G' by removing a sparse F-decomposable subgraph of G.

4. Tools

4.1. **Basic tools.** We will often use the following 'handshaking lemma' for r-graphs: Let G be an r-graph and $0 \leq i \leq k \leq r-1$. Then for every $S \in \binom{V(G)}{i}$ we have

$$|G(S)| = \binom{r-i}{r-k}^{-1} \sum_{T \in \binom{V(G)}{k} :\, S \subseteq T} |G(T)|. \tag{4.1}$$

Proposition 4.1. *Let F be an r-graph. Then there exist infinitely many $n \in \mathbb{N}$ such that $K_n^{(r)}$ is F-divisible.*

Proof. Let $p := \prod_{i=0}^{r-1} Deg(F)_i$. We will show that for every $a \in \mathbb{N}$, if we let $n = r!ap + r - 1$ then $K_n^{(r)}$ is F-divisible. Clearly, this implies the claim. In order to see that $K_n^{(r)}$ is F-divisible, it is sufficient to show that $p \mid \binom{n-i}{r-i}$ for all $i \in [r-1]_0$. It is easy to see that this holds for the above choice of n. □

The following proposition shows that the class of weakly regular uniform hypergraphs is closed under taking link graphs.

Proposition 4.2. *Let F be a weakly regular r-graph and let $i \in [r-1]$. Suppose that $S \in \binom{V(F)}{i}$ and that $F(S)$ is non-empty. Then $F(S)$ is a weakly regular $(r-i)$-graph and $Deg(F(S))_j = Deg(F)_{i+j}$ for all $j \in [r-i-1]_0$.*

Proof. Let $s_0, \ldots, s_{r-1}$ be such that F is weakly $(s_0, \ldots, s_{r-1})$-regular. Note that since F is non-empty, we have $s_j > 0$ for all $j \in [r-1]_0$ (and the s_i's are unique). Consider $j \in [r-i-1]_0$. For all $T \in \binom{V(F(S))}{j}$, we have $|F(S)(T)| = |F(S \cup T)| \in \{0, s_{i+j}\}$. Hence, $F(S)$ is weakly $(s_i, \ldots, s_{r-1})$-regular. Since F is non-empty, we have $Deg(F) = (s_0, \ldots, s_{r-1})$, and since $F(S)$ is non-empty too by assumption, we have $Deg(F(S)) = (s_i, \ldots, s_{r-1})$. Therefore, $Deg(F(S))_j = Deg(F)_{i+j}$ for all $j \in [r-i-1]_0$. □

We now list some useful properties of well separated F-packings.

Fact 4.3. *Let G be a complex and F an r-graph on $f > r$ vertices. Suppose that $\mathcal{F}$ is a κ-well separated F-packing (in G) and $\mathcal{F}'$ is a κ'-well separated F-packing (in G). Then the following hold.*

(i) $\Delta(\mathcal{F}^{\leq(r+1)}) \leq \kappa(f-r)$.
(ii) *If $\mathcal{F}^{(r)}$ and $\mathcal{F}'^{(r)}$ are edge-disjoint and $\mathcal{F}$ and $\mathcal{F}'$ are $(r+1)$-disjoint, then $\mathcal{F} \cup \mathcal{F}'$ is a $(\kappa+\kappa')$-well separated F-packing (in G).*
(iii) *If $\mathcal{F}$ and $\mathcal{F}'$ are r-disjoint, then $\mathcal{F} \cup \mathcal{F}'$ is a $\max\{\kappa,\kappa'\}$-well separated F-packing (in G).*

4.2. **Some properties of supercomplexes.** The following properties of supercomplexes were proved in [12].

Proposition 4.4 ([12])**.** *Let G be an (ε,ξ,f,r)-supercomplex and let $B \subseteq G^{(i)}$ with $1 \leq |B| \leq 2^i$ for some $i \in [r]_0$. Then $\bigcap_{b\in B} G(b)$ is an $(\varepsilon,\xi,f-i,r-i)$-supercomplex.*

Proposition 4.5 ([12])**.** *Let $f,r' \in \mathbb{N}$ and $r \in \mathbb{N}_0$ with $f > r$ and $r' \geq r$. Let G be a complex on $n \geq r2^{r+1}$ vertices and let H be an r'-graph on $V(G)$ with $\Delta(H) \leq \gamma n$. Then the following hold:*

(i) *If G is (ε,d,f,r)-regular, then $G-H$ is $(\varepsilon+2^r\gamma,d,f,r)$-regular.*
(ii) *If G is (ξ,f,r)-dense, then $G-H$ is $(\xi-2^r\gamma,f,r)$-dense.*
(iii) *If G is (ξ,f,r)-extendable, then $G-H$ is $(\xi-2^r\gamma,f,r)$-extendable.*
(iv) *If G is an (ε,ξ,f,r)-complex, then $G-H$ is an $(\varepsilon+2^r\gamma,\xi-2^r\gamma,f,r)$-complex.*
(v) *If G is an (ε,ξ,f,r)-supercomplex, then $G-H$ is an $(\varepsilon+2^{2r+1}\gamma,\xi-2^{2r+1}\gamma,f,r)$-supercomplex.*

Corollary 4.6 ([12])**.** *Let $1/n \ll \varepsilon,\gamma,\xi,p,1/f$ and $r \in [f-1]$. Let*

$$\xi' := 0.95\xi p^{2^r\binom{f+r}{r}} \geq 0.95\xi p^{(8^f)} \text{ and } \gamma' := 1.1\cdot 2^r\frac{\binom{f+r}{r}}{(f-r)!}\gamma.$$

Suppose that G is an (ε,ξ,f,r)-supercomplex on n vertices and that $H \subseteq G^{(r)}$ is a random subgraph obtained by including every edge of $G^{(r)}$ independently with probability p. Then whp the following holds: for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \gamma n$, $G[H \triangle L]$ is a $(3\varepsilon+\gamma',\xi'-\gamma',f,r)$-supercomplex.

4.3. **Rooted Embeddings.** We now prove a result (Lemma 4.7) which allows us to find edge-disjoint embeddings of graphs with a prescribed 'root embedding'. Let T be an r-graph and suppose that $X \subseteq V(T)$ is such that $T[X]$ is empty. A *root of* (T,X) is a set $S \subseteq X$ with $|S| \in [r-1]$ and $|T(S)| > 0$.

For an r-graph G, we say that $\Lambda\colon X \to V(G)$ is a *G-labelling of* (T,X) if Λ is injective. Our aim is to embed T into G such that the roots of (T,X) are embedded at their assigned position. More precisely, given a G-labelling Λ of (T,X), we say that ϕ is a *Λ-faithful embedding of* (T,X) *into* G if ϕ is an injective homomorphism from T to G with $\phi\restriction_X = \Lambda$. Moreover, for a set $S \subseteq V(G)$ with $|S| \in [r-1]$, we say that Λ *roots* S if $S \subseteq \mathrm{Im}(\Lambda)$ and $|T(\Lambda^{-1}(S))| > 0$, i.e. if $\Lambda^{-1}(S)$ is a root of (T,X).

The *degeneracy of T rooted at X* is the smallest D such that there exists an ordering $v_1,\dots,v_k$ of the vertices of $V(T)\setminus X$ such that for every $\ell \in [k]$, we have

$$|T[X \cup \{v_1,\dots,v_\ell\}](v_\ell)| \leq D,$$

i.e. every vertex is contained in at most D edges which lie to the left of that vertex in the ordering.

We need to be able to embed many copies of (T,X) simultaneously (with different labellings) into a given host graph G such that the different embeddings are edge-disjoint. In fact, we need a slightly stronger disjointness criterion. Ideally, we would like to have that two distinct embeddings intersect in less than r vertices. However, this is in general not possible because of the desired rooting. We therefore introduce the following concept of a *hull*. We will ensure that the hulls are edge-disjoint, which will be sufficient for our purposes. Given (T,X) as above, the *hull of* (T,X) is the r-graph T' on $V(T)$ with $e \in T'$ if and only if $e \cap X = \emptyset$ or $e \cap X$ is a root of (T,X). Note that $T \subseteq T' \subseteq K^{(r)}_{V(T)} - K^{(r)}_X$, where $K^{(r)}_Z$ denotes the complete r-graph with vertex set Z. Moreover, the roots of (T',X) are precisely the roots of (T,X).

Lemma 4.7. *Let $1/n \ll \gamma \ll \xi, 1/t, 1/D$ and $r \in [t]$. Suppose that $\alpha \in (0,1]$ is an arbitrary scalar (which might depend on n) and let $m \leq \alpha\gamma n^r$ be an integer. For every $j \in [m]$, let T_j be an r-graph on at most t vertices and $X_j \subseteq V(T_j)$ such that $T_j[X_j]$ is empty and T_j has degeneracy at most D rooted at X_j. Let G be an r-graph on n vertices such that for all $A \subseteq \binom{V(G)}{r-1}$ with $|A| \leq D$, we have $|\bigcap_{S\in A} G(S)| \geq \xi n$. Let O be an $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq \gamma n$. For every $j \in [m]$, let Λ_j be a G-labelling of (T_j, X_j). Suppose that for all $S \subseteq V(G)$ with $|S| \in [r-1]$, we have that*

$$|\{j \in [m] \,:\, \Lambda_j \text{ roots } S\}| \leq \alpha\gamma n^{r-|S|} - 1. \tag{4.2}$$

Then for every $j \in [m]$, there exists a Λ_j-faithful embedding ϕ_j of (T_j, X_j) into G such that the following hold:

(i) *for all distinct $j, j' \in [m]$, the hulls of $(\phi_j(T_j), \mathrm{Im}(\Lambda_j))$ and $(\phi_{j'}(T_{j'}), \mathrm{Im}(\Lambda_{j'}))$ are edge-disjoint;*

(ii) *for all $j \in [m]$ and $e \in O$ with $e \subseteq \mathrm{Im}(\phi_j)$, we have $e \subseteq \mathrm{Im}(\Lambda_j)$;*

(iii) $\Delta(\bigcup_{j\in[m]} \phi_j(T_j)) \leq \alpha\gamma^{(2^{-r})} n$.

Note that (i) implies that $\phi_1(T_1), \dots, \phi_m(T_m)$ are edge-disjoint. We also remark that the T_j do not have to be distinct; in fact, they could all be copies of a single r-graph T.

Proof. For $j \in [m]$ and a set $S \subseteq V(G)$ with $|S| \in [r-1]$, let

$$root(S,j) := |\{j' \in [j] \,:\, \Lambda_{j'} \text{ roots } S\}|.$$

We will define $\phi_1, \dots, \phi_m$ successively. Once ϕ_j is defined, we let K_j denote the hull of $(\phi_j(T_j), \mathrm{Im}(\Lambda_j))$. Note that $\phi_j(T_j) \subseteq K_j$ and that K_j is not necessarily a subgraph of G.

Suppose that for some $j \in [m]$, we have already defined $\phi_1, \dots, \phi_{j-1}$ such that $K_1, \dots, K_{j-1}$ are edge-disjoint, (ii) holds for all $j' \in [j-1]$, and the following holds for $G_j := \bigcup_{j'\in[j-1]} K_{j'}$, all $i \in [r-1]$ and all $S \in \binom{V(G)}{i}$:

$$|G_j(S)| \leq \alpha\gamma^{(2^{-i})} n^{r-i} + (root(S, j-1) + 1)2^t. \tag{4.3}$$

Note that (4.3) together with (4.2) implies that for all $i \in [r-1]$ and all $S \in \binom{V(G)}{i}$, we have

$$|G_j(S)| \leq 2\alpha\gamma^{(2^{-i})} n^{r-i}. \tag{4.4}$$

We will now define a Λ_j-faithful embedding ϕ_j of (T_j, X_j) into G such that K_j is edge-disjoint from G_j, (ii) holds for j, and (4.3) holds with j replaced by $j+1$. For $i \in [r-1]$, define $BAD_i := \{S \in \binom{V(G)}{i} \,:\, |G_j(S)| \geq \alpha\gamma^{(2^{-i})} n^{r-i}\}$. We view BAD_i as an i-graph. We claim that for all $i \in [r-1]$,

$$\Delta(BAD_i) \leq \gamma^{(2^{-r})} n. \tag{4.5}$$

Consider $i \in [r-1]$ and suppose that there exists some $S \in \binom{V(G)}{i-1}$ such that $|BAD_i(S)| > \gamma^{(2^{-r})} n$. We then have that

$$\begin{aligned} |G_j(S)| &= \frac{1}{r-i+1} \sum_{v\in V(G)\setminus S} |G_j(S \cup \{v\})| \geq r^{-1} \sum_{v\in BAD_i(S)} |G_j(S\cup\{v\})| \\ &\geq r^{-1}|BAD_i(S)|\alpha\gamma^{(2^{-i})} n^{r-i} \geq r^{-1}\gamma^{(2^{-r})} n \alpha\gamma^{(2^{-i})} n^{r-i} = r^{-1}\alpha\gamma^{(2^{-r}+2^{-i})} n^{r-(i-1)}. \end{aligned}$$

This contradicts (4.4) if $i-1 > 0$ since $2^{-r} + 2^{-i} < 2^{-(i-1)}$. If $i = 1$, then $S = \emptyset$ and we have $|G_j| \geq r^{-1}\alpha\gamma^{(2^{-r}+2^{-1})} n^r$, which is also a contradiction since $|G_j| \leq m\binom{t}{r} \leq \binom{t}{r}\alpha\gamma n^r$ and $2^{-r} + 2^{-1} < 1$ (as $r \geq 2$ if $i \in [r-1]$). This proves (4.5).

We now embed the vertices of T_j such that the obtained embedding ϕ_j is Λ_j-faithful. First, embed every vertex from X_j at its assigned position. Since T_j has degeneracy at most D rooted at X_j, there exists an ordering $v_1, \dots, v_k$ of the vertices of $V(T_j) \setminus X_j$ such that for every $\ell \in [k]$, we have

$$|T_j[X_j \cup \{v_1, \dots, v_\ell\}](v_\ell)| \le D. \tag{4.6}$$

Suppose that for some $\ell \in [k]$, we have already embedded $v_1, \dots, v_{\ell-1}$. We now want to define $\phi_j(v_\ell)$. Let $U := \{\phi_j(v) : v \in X_j \cup \{v_1, \dots, v_{\ell-1}\}\}$ be the set of vertices which have already been used as images for ϕ_j. Let A contain all $(r-1)$-subsets S of U such that $\phi_j^{-1}(S) \cup \{v_\ell\} \in T_j$. We need to choose $\phi_j(v_\ell)$ from the set $(\bigcap_{S \in A} G(S)) \setminus U$ in order to complete ϕ_j to an injective homomorphism from T_j to G. By (4.6), we have $|A| \le D$. Thus, by assumption, $|\bigcap_{S \in A} G(S)| \ge \xi n$.

For $i \in [r-1]$, let O_i consist of all vertices $x \in V(G)$ such that there exists some $S \in \binom{U}{i-1}$ such that $S \cup \{x\} \in BAD_i$ (so $BAD_1 = \binom{O_1}{1}$). We have

$$|O_i| \le \binom{|U|}{i-1} \Delta(BAD_i) \stackrel{(4.5)}{\le} \binom{t}{i-1} \gamma^{(2^{-r})} n.$$

Let O_r consist of all vertices $x \in V(G)$ such that $S \cup \{x\} \in G_j$ for some $S \in \binom{U}{r-1}$. By (4.4), we have that $|O_r| \le \binom{|U|}{r-1} \Delta(G_j) \le \binom{t}{r-1} 2\alpha\gamma^{(2^{-(r-1)})} n \le \binom{t}{r-1} \gamma^{(2^{-r})} n$. Finally, let O_{r+1} be the set of all vertices $x \in V(G)$ such that there exists some $S \in \binom{U}{r}$ such that $S \cup \{x\} \in O$. By assumption, we have $|O_{r+1}| \le \binom{|U|}{r} \Delta(O) \le \binom{t}{r} \gamma n$.

Crucially, we have

$$|\bigcap_{S \in A} G(S)| - |U| - \sum_{i=1}^{r+1} |O_i| \ge \xi n - t - 2^t \gamma^{(2^{-r})} n > 0.$$

Thus, there exists a vertex $x \in V(G)$ such that $x \notin U \cup O_1 \cup \dots \cup O_{r+1}$ and $S \cup \{x\} \in G$ for all $S \in A$. Define $\phi_j(v_\ell) := x$.

Continuing in this way until ϕ_j is defined for every $v \in V(T_j)$ yields an injective homomorphism from T_j to G. By definition of O_{r+1}, (ii) holds for j. Moreover, by definition of O_r, K_j is edge-disjoint from G_j. It remains to show that (4.3) holds with j replaced by $j+1$. Let $i \in [r-1]$ and $S \in \binom{V(G)}{i}$. If $S \notin BAD_i$, then we have $|G_{j+1}(S)| \le |G_j(S)| + \binom{t-i}{r-i} \le \alpha\gamma^{(2^{-i})} n^{r-i} + 2^t$, so (4.3) holds. Now, assume that $S \in BAD_i$. If $S \subseteq \mathrm{Im}(\Lambda_j)$ and $|T_j(\Lambda_j^{-1}(S))| > 0$, then $root(S, j) = root(S, j-1) + 1$ and thus $|G_{j+1}(S)| \le |G_j(S)| + \binom{t-i}{r-i} \le \alpha\gamma^{(2^{-i})} n^{r-i} + (root(S, j-1)+1)2^t + \binom{t-i}{r-i} \le \alpha\gamma^{(2^{-i})} n^{r-i} + (root(S, j)+1)2^t$ and (4.3) holds. Suppose next that $S \nsubseteq \mathrm{Im}(\Lambda_j)$. We claim that $S \nsubseteq V(\phi_j(T_j))$. Suppose, for a contradiction, that $S \subseteq V(\phi_j(T_j))$. Let $\ell := \max\{\ell' \in [k] : \phi_j(v_{\ell'}) \in S\}$. (Note that the maximum exists since $(S \cap V(\phi_j(T_j))) \setminus \mathrm{Im}(\Lambda_j)$ is not empty.) Hence, $x := \phi_j(v_\ell) \in S$. Recall that when we defined $\phi_j(v_\ell)$, $\phi_j(v)$ had already been defined for all $v \in X_j \cup \{v_1, \dots, v_{\ell-1}\}$ and hence $S \setminus \{x\} \subseteq U$. But since $S \in BAD_i$, we have $x \in O_i$, in contradiction to $x = \phi_j(v_\ell)$. Thus, $S \nsubseteq V(\phi_j(T_j)) = V(K_j)$, which clearly implies that $|G_{j+1}(S)| = |G_j(S)|$ and (4.3) holds. The last remaining case is if $S \subseteq \mathrm{Im}(\Lambda_j)$ but $|T_j(\Lambda_j^{-1}(S))| = 0$. But then S is not a root of $(\phi_j(T_j), \mathrm{Im}(\Lambda_j))$ and thus not a root of $(K_j, \mathrm{Im}(\Lambda_j))$. Hence $|K_j(S)| = 0$ and therefore $|G_{j+1}(S)| = |G_j(S)|$ as well.

Finally, if $j = m$, then the fact that (4.3) holds with j replaced by $j+1$ together with (4.2) implies that $\Delta(\bigcup_{j \in [m]} \phi_j(T_j)) \le 2\alpha\gamma^{(2^{-(r-1)})} n \le \alpha\gamma^{(2^{-r})} n$. □

5. Approximate F-decompositions

The majority of the edges which are covered during our iterative absorption procedure are covered by 'approximate' F-decompositions of certain parts of G, i.e. F-packings which cover almost all the edges in these parts. For cliques, the existence of such packings was first proved by Rödl [25], introducing what is now called the 'nibble' technique. Here, we derive a result on approximate F-decompositions which is suitable for our needs (Lemma 5.3).

We will derive the F-nibble lemma (Lemma 5.3) from the special case when F is a clique. This in turn was derived in [12] from a result in [1] which allows us to assume that the leftover of an approximate clique decomposition has appropriately bounded maximum degree.

Lemma 5.1 (Boosted nibble lemma, [12])**.** *Let $1/n \ll \gamma, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let G be a complex on n vertices such that G is (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense for some $d \geq \xi$. Then G contains a $K_f^{(r)}$-packing $\mathcal{K}$ such that $\Delta(G^{(r)} - \mathcal{K}^{(r)}) \leq \gamma n$.*

Crucially, we do not need to assume that $\varepsilon \ll \gamma$ in Lemma 5.1. The reason for this is the so-called Boost lemma from [12], which allows us to 'boost' the regularity parameters of a suitable complex and which is an important ingredient in the proof of both Lemma 5.1 and Lemma 5.3.

Lemma 5.2 (Boost lemma, [12])**.** *Let $1/n \ll \varepsilon, \xi, 1/f$ and $r \in [f-1]$ such that $2(2\sqrt{\mathrm{e}})^r \varepsilon \leq \xi$. Let $\xi' := 0.9(1/4)^{\binom{f+r}{f}}\xi$. Suppose that G is a complex on n vertices and that G is (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense for some $d \geq \xi$. Then there exists $Y \subseteq G^{(f)}$ such that $G[Y]$ is $(n^{-(f-r)/2.01}, d/2, f, r)$-regular and $(\xi', f+r, r)$-dense.*

We now prove an F-nibble lemma which allows us to find κ-well separated approximate F-decompositions in supercomplexes.

Lemma 5.3 (F-nibble lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f vertices. Let G be a complex on n vertices such that G is (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense for some $d \geq \xi$. Then G contains a κ-well separated F-packing $\mathcal{F}$ such that $\Delta(G^{(r)} - \mathcal{F}^{(r)}) \leq \gamma n$.*

Let F be an r-graph on f vertices. Given a collection $\mathcal{K}$ of edge-disjoint copies of $K_f^{(r)}$, we define the *$\mathcal{K}$-random F-packing* $\mathcal{F}$ as follows: For every $K \in \mathcal{K}$, choose a random bijection from $V(F)$ to $V(K)$ and let F_K be a copy of F on $V(K)$ embedded by this bijection. Let $\mathcal{F} := \{F_K \,:\, K \in \mathcal{K}\}$.

Clearly, if $\mathcal{K}$ is a $K_f^{(r)}$-decomposition of a complex G, then the $\mathcal{K}$-random F-packing $\mathcal{F}$ is a 1-well separated F-packing in G. Moreover, writing $p := 1 - |F|/\binom{f}{r}$, we have $|\mathcal{F}^{(r)}| = |F||\mathcal{K}| = |F||G^{(r)}|/\binom{f}{r} = (1-p)|G^{(r)}|$, and for every $e \in G^{(r)}$, we have $\mathbb{P}(e \in G^{(r)} - \mathcal{F}^{(r)}) = p$. As turns out, the leftover $G^{(r)} - \mathcal{F}^{(r)}$ behaves essentially like a p-random subgraph of $G^{(r)}$ (cf. Lemma 5.4). Our strategy to prove Lemma 5.3 is thus as follows: We apply Lemma 5.1 to G to obtain a $K_f^{(r)}$-packing $\mathcal{K}_1$ such that $\Delta(G^{(r)} - \mathcal{K}_1^{(r)}) \leq \gamma n$. The leftover here is negligible, so assume for the moment that $\mathcal{K}_1$ is a $K_f^{(r)}$-decomposition. We then choose a $\mathcal{K}_1$-random F-packing $\mathcal{F}_1$ in G and continue the process with $G - \mathcal{F}_1^{(r)}$. In each step, the leftover decreases by a factor of p. Thus after $\log_p \gamma$ steps, the leftover will have maximum degree at most γn.

Lemma 5.4. *Let $1/n \ll \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f-vertices with $p := 1 - |F|/\binom{f}{r} \in (0,1)$. Let G be an (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense complex on n vertices for some $d \geq \xi$. Suppose that $\mathcal{K}$ is a $K_f^{(r)}$-decomposition of G. Let $\mathcal{F}$ be the $\mathcal{K}$-random F-packing in G. Then whp the following hold for $G' := G - \mathcal{K}^{\leq(r+1)} - \mathcal{F}^{(r)}$.*

(i) *G' is $(2\varepsilon, p^{\binom{f}{r}-1}d, f, r)$-regular;*
(ii) *G' is $(0.9p^{\binom{f+r}{r}-1}\xi, f+r, r)$-dense;*
(iii) *$\Delta(G'^{(r)}) \leq 1.1p\Delta(G^{(r)})$.*

Since the assertions follow easily from the definitions, we omit the proof here.

Proof of Lemma 5.3. Let $p := 1 - |F|/\binom{f}{r}$. If $F = K_f^{(r)}$, then we are done by Lemma 5.1. We may thus assume that $p \in (0,1)$. Choose $\varepsilon' > 0$ such that $1/n \ll \varepsilon' \ll 1/\kappa \ll \gamma, \varepsilon \ll p, 1-p, \xi, 1/f$. We will now repeatedly apply Lemma 5.1. More precisely, let $\xi_0 := 0.9(1/4)^{\binom{f+r}{f}}\xi$ and define $\xi_j := (0.5p)^{j\binom{f+r}{r}}\xi_0$ for $j \geq 1$. For every $j \in [\kappa]_0$, we will find $\mathcal{F}_j$ and G_j such that the following hold:

(a)$_j$ $\mathcal{F}_j$ is a j-well separated F-packing in G and $G_j \subseteq G - \mathcal{F}_j^{(r)}$;
(b)$_j$ $\Delta(L_j) \le j\varepsilon' n$, where $L_j := G^{(r)} - \mathcal{F}_j^{(r)} - G_j^{(r)}$;
(c)$_j$ G_j is $(2^{(r+1)j}\varepsilon', d_j, f, r)$-regular and $(\xi_j, f+r, r)$-dense for some $d_j \ge \xi_j$;
(d)$_j$ $\mathcal{F}_j^{\le}$ and G_j are $(r+1)$-disjoint;
(e)$_j$ $\Delta(G_j^{(r)}) \le (1.1p)^j n$.

First, apply Lemma 5.2 to G in order to find $Y \subseteq G^{(f)}$ such that $G_0 := G[Y]$ is $(\varepsilon', d/2, f, r)$-regular and $(\xi_0, f+r, r)$-dense. Hence, (a)$_0$–(e)$_0$ hold with $\mathcal{F}_0 := \emptyset$. Also note that $\mathcal{F}_\kappa$ will be a κ-well separated F-packing in G and $\Delta(G^{(r)} - \mathcal{F}_\kappa^{(r)}) \le \Delta(L_\kappa) + \Delta(G_\kappa^{(r)}) \le \kappa\varepsilon' n + (1.1p)^\kappa n \le \gamma n$, so we can take $\mathcal{F} := \mathcal{F}_\kappa$.

Now, assume that for some $j \in [\kappa]$, we have found $\mathcal{F}_{j-1}$ and G_{j-1} and now need to find $\mathcal{F}_j$ and G_j. By (c)$_{j-1}$, G_{j-1} is $(\sqrt{\varepsilon'}, d_{j-1}, f, r)$-regular and $(\xi_{j-1}, f+r, r)$-dense for some $d_{j-1} \ge \xi_{j-1}$. Thus, we can apply Lemma 5.1 to obtain a $K_f^{(r)}$-packing $\mathcal{K}_j$ in G_{j-1} such that $\Delta(L_j') \le \varepsilon' n$, where $L_j' := G_{j-1}^{(r)} - \mathcal{K}_j^{(r)}$. Let $G_j' := G_{j-1} - L_j'$. Clearly, $\mathcal{K}_j$ is a $K_f^{(r)}$-decomposition of G_j'. Moreover, by (c)$_{j-1}$ and Proposition 4.5 we have that G_j' is $(2^{(r+1)(j-1)+r}\varepsilon', d_{j-1}, f, r)$-regular and $(0.9\xi_{j-1}, f+r, r)$-dense. By Lemma 5.4, there exists a 1-well separated F-packing $\mathcal{F}_j'$ in G_j' such that the following hold for $G_j := G_j' - \mathcal{F}_j'^{(r)} - \mathcal{K}_j^{\le(r+1)} = G_j' - \mathcal{F}_j'^{(r)} - \mathcal{F}_j'^{\le(r+1)}$:

(i) G_j is $(2^{(r+1)(j-1)+r+1}\varepsilon', p^{\binom{f}{r}-1}d_{j-1}, f, r)$-regular;
(ii) G_j is $(0.81p^{\binom{f+r}{r}-1}\xi_{j-1}, f+r, r)$-dense;
(iii) $\Delta(G_j^{(r)}) \le 1.1p\Delta(G_j'^{(r)})$.

Let $\mathcal{F}_j := \mathcal{F}_{j-1} \cup \mathcal{F}_j'$ and $L_j := G^{(r)} - \mathcal{F}_j^{(r)} - G_j^{(r)}$. Note that $\mathcal{F}_{j-1}^{(r)} \cap \mathcal{F}_j'^{(r)} = \emptyset$ by (a)$_{j-1}$. Moreover, $\mathcal{F}_{j-1}$ and $\mathcal{F}_j'$ are $(r+1)$-disjoint by (d)$_{j-1}$. Thus, $\mathcal{F}_j$ is $(j-1+1)$-well separated by Fact 4.3(ii). Moreover, using (a)$_{j-1}$, we have

$$G_j \subseteq G_{j-1} - \mathcal{F}_j'^{(r)} \subseteq G - \mathcal{F}_{j-1}^{(r)} - \mathcal{F}_j'^{(r)},$$

thus (a)$_j$ holds. Observe that $L_j \setminus L_{j-1} \subseteq L_j'$. Thus, we clearly have $\Delta(L_j) \le \Delta(L_{j-1}) + \Delta(L_j') \le j\varepsilon' n$, so (b)$_j$ holds. Moreover, (c)$_j$ follows directly from (i) and (ii), and (e)$_j$ follows from (e)$_{j-1}$ and (iii). To see (d)$_j$, observe that $\mathcal{F}_{j-1}^{\le}$ and G_j are $(r+1)$-disjoint by (d)$_{j-1}$ and since $G_j \subseteq G_{j-1}$, and $\mathcal{F}_j'^{\le}$ and G_j are $(r+1)$-disjoint by definition of G_j. Thus, (a)$_j$–(e)$_j$ hold and the proof is completed. □

6. Vortices

A vortex is best thought of as a sequence of nested 'random-like' subsets of the vertex set of a supercomplex G. In our approach, the final set of the vortex has bounded size.

The main results of this section are Lemmas 6.3 and 6.4, where the first one shows that vortices exist, and the latter one shows that given a vortex, we can find an F-packing covering all edges which do not lie inside the final vortex set. We now give the formal definition of what it means to be a 'random-like' subset.

Definition 6.1. Let G be a complex on n vertices. We say that U is $(\varepsilon, \mu, \xi, f, r)$-*random in* G if there exists an f-graph Y on $V(G)$ such that the following hold:

(R1) $U \subseteq V(G)$ with $|U| = \mu n \pm n^{2/3}$;
(R2) there exists $d \ge \xi$ such that for all $x \in [f-r]_0$ and all $e \in G^{(r)}$, we have that

$$|\{Q \in G[Y]^{(f)}(e) \,:\, |Q \cap U| = x\}| = (1 \pm \varepsilon) bin(f-r, \mu, x) d n^{f-r};$$

(R3) for all $e \in G^{(r)}$ we have $|G[Y]^{(f+r)}(e)[U]| \ge \xi(\mu n)^f$;
(R4) for all $h \in [r]_0$ and all $B \subseteq G^{(h)}$ with $1 \le |B| \le 2^h$ we have that $\bigcap_{b \in B} G(b)[U]$ is an $(\varepsilon, \xi, f-h, r-h)$-complex.

Having defined what it means to be a 'random-like' subset, we can now define what a vortex is.

Definition 6.2 (Vortex)**.** Let G be a complex. An $(\varepsilon,\mu,\xi,f,r,m)$-*vortex in* G is a sequence $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ such that

(V1) $U_0 = V(G)$;
(V2) $|U_i| = \lfloor \mu |U_{i-1}| \rfloor$ for all $i \in [\ell]$;
(V3) $|U_\ell| = m$;
(V4) for all $i \in [\ell]$, U_i is $(\varepsilon,\mu,\xi,f,r)$-random in $G[U_{i-1}]$;
(V5) for all $i \in [\ell-1]$, $U_i \setminus U_{i+1}$ is $(\varepsilon,\mu(1-\mu),\xi,f,r)$-random in $G[U_{i-1}]$.

As shown in [12], a vortex can be found in a supercomplex by repeatedly taking random subsets.

Lemma 6.3 ([12])**.** *Let* $1/m' \ll \varepsilon \ll \mu,\xi,1/f$ *such that* $\mu \le 1/2$ *and* $r \in [f-1]$. *Let* G *be an* (ε,ξ,f,r)*-supercomplex on* $n \ge m'$ *vertices. Then there exists a* $(2\sqrt{\varepsilon},\mu,\xi-\varepsilon,f,r,m)$*-vortex in* G *for some* $\mu m' \le m \le m'$.

The following is the main lemma of this section. Given a vortex in a supercomplex G, it allows us to cover all edges of $G^{(r)}$ except possibly some from inside the final vortex set (see Lemma 7.13 in [12] for the corresponding result in the case when F is a clique).

Lemma 6.4. *Let* $1/m \ll 1/\kappa \ll \varepsilon \ll \mu \ll \xi, 1/f$ *and* $r \in [f-1]$. *Assume that* $(*)_k$ *is true for all* $k \in [r-1]$. *Let* F *be a weakly regular* r*-graph on* f *vertices. Let* G *be an* F*-divisible* (ε,ξ,f,r)*-supercomplex and* $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ *an* $(\varepsilon,\mu,\xi,f,r,m)$*-vortex in* G. *Then there exists a* 4κ*-well separated* F*-packing* $\mathcal{F}$ *in* G *which covers all edges of* $G^{(r)}$ *except possibly some inside* U_ℓ.

The proof of Lemma 6.4 consists of an 'iterative absorption' procedure, where the key ingredient is the Cover down lemma (Lemma 6.9). Roughly speaking, given a supercomplex G and a 'random-like' subset $U \subseteq V(G)$, the Cover down lemma allows us to find a 'partial absorber' $H \subseteq G^{(r)}$ such that for any sparse $L \subseteq G^{(r)}$, $H \cup L$ has an F-packing which covers all edges of $H \cup L$ except possibly some inside U. Together with the F-nibble lemma (Lemma 5.3), this allows us to cover all edges of G except possibly some inside U whilst using only few edges inside U. Indeed, set aside H as above, which is reasonably sparse. Then apply the Lemma 5.3 to $G - G^{(r)}[U] - H$ to obtain an F-packing $\mathcal{F}_{nibble}$ with a very sparse leftover L. Combine H and L to find an F-packing $\mathcal{F}_{clean}$ whose leftover lies inside U.

Now, if $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ is a vortex, then U_1 is 'random-like' in G and thus we can cover all edges which are not inside U_1 by using only few edges inside U_1 (and in this step we forbid edges inside U_2 from being used.) Then U_2 is still 'random-like' in the remainder of $G[U_1]$, and hence we can iterate until we have covered all edges of G except possibly some inside U_ℓ. The proof of Lemma 6.4 is very similar to that of Lemma 7.13 in [12], thus we omit it here.

We record the following easy tools from [12] for later use.

Fact 6.5 ([12])**.** *The following hold.*

(i) *If* G *is an* (ε,ξ,f,r)*-supercomplex, then* $V(G)$ *is* $(\varepsilon/\xi,1,\xi,f,r)$*-random in* G.
(ii) *If* U *is* $(\varepsilon,\mu,\xi,f,r)$*-random in* G, *then* $G[U]$ *is an* (ε,ξ,f,r)*-supercomplex.*

Proposition 6.6 ([12])**.** *Let* $1/n \ll \varepsilon \ll \mu_1,\mu_2,1-\mu_2,\xi,1/f$ *and* $r \in [f-1]$. *Let* G *be a complex on* n *vertices and let* $U \subseteq V(G)$ *be of size* $\lfloor \mu_1 n \rfloor$ *and* $(\varepsilon,\mu_1,\xi,f,r)$*-random in* G. *Then there exists* $\tilde{U} \subseteq U$ *of size* $\lfloor \mu_2 |U| \rfloor$ *such that*

(i) $\tilde{U}$ *is* $(\varepsilon + |U|^{-1/6},\mu_2,\xi - |U|^{1/6},f,r)$*-random in* $G[U]$ *and*
(ii) $U \setminus \tilde{U}$ *is* $(\varepsilon + |U|^{-1/6},\mu_1(1-\mu_2),\xi - |U|^{1/6},f,r)$*-random in* G.

Proposition 6.7 ([12])**.** *Let* $1/n \ll \varepsilon \ll \mu,\xi,1/f$ *such that* $\mu \le 1/2$ *and* $r \in [f-1]$. *Suppose that* G *is a complex on* n *vertices and* U *is* $(\varepsilon,\mu,\xi,f,r)$*-random in* G. *Suppose that* $L \subseteq G^{(r)}$ *and* $O \subseteq G^{(r+1)}$ *satisfy* $\Delta(L) \le \varepsilon n$ *and* $\Delta(O) \le \varepsilon n$. *Then* U *is still* $(\sqrt{\varepsilon},\mu,\xi-\sqrt{\varepsilon},f,r)$*-random in* $G - L - O$.

6.1. **The Cover down lemma.** Recall that the Cover down lemma allows us to replace a given leftover L with a new leftover which is restricted to some small set of vertices U. We now provide the formal statement.

Definition 6.8. Let G be a complex on n vertices and $H \subseteq G^{(r)}$. We say that G is (ξ, f, r)-*dense with respect to* H if for all $e \in G^{(r)}$, we have $|G[H \cup \{e\}]^{(f)}(e)| \geq \xi n^{f-r}$.

Lemma 6.9 (Cover down lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/f$ and $r \in [f-1]$ with $\mu \leq 1/2$. Assume that $(*)_i$ is true for all $i \in [r-1]$ and that F is a weakly regular r-graph on f vertices. Let G be a complex on n vertices and suppose that U is $(\varepsilon, \mu, \xi, f, r)$-random in G. Let $\tilde{G}$ be a complex on $V(G)$ with $G \subseteq \tilde{G}$ such that $\tilde{G}$ is (ε, f, r)-dense with respect to $G^{(r)} - G^{(r)}[\bar{U}]$, where $\bar{U} := V(G) \setminus U$.*

Then there exists a subgraph $H^* \subseteq G^{(r)} - G^{(r)}[\bar{U}]$ with $\Delta(H^*) \leq \nu n$ such that for any $L \subseteq \tilde{G}^{(r)}$ with $\Delta(L) \leq \gamma n$ and $H^* \cup L$ being F-divisible and any $(r+1)$-graph O on $V(G)$ with $\Delta(O) \leq \gamma n$, there exists a κ-well separated F-packing in $\tilde{G}[H^* \cup L] - O$ which covers all edges of $H^* \cup L$ except possibly some inside U.

Roughly speaking, the proof of the Cover down lemma proceeds as follows. Suppose that we have already chosen H^* and that L is any sparse (leftover) r-graph. For an edge $e \in H^* \cup L$, we refer to $|e \cap U|$ as its type. Since L is very sparse, we can greedily cover all edges of L in a first step. In particular, this covers all type-0-edges. We will now continue and cover all type-1-edges. Note that every type-1-edge contains a unique $S \in \binom{V(G)\setminus U}{r-1}$. For a given set $S \in \binom{V(G)\setminus U}{r-1}$, we would like to cover all remaining edges of H^* that contain S simultaneously. Assuming a suitable choice of H^*, this can be achieved as follows. Let L_S be the link graph of S after the first step. Let $T \in \binom{V(F)}{r-1}$ be such that $F(T)$ is non-empty. By Proposition 4.2, L_S will be $F(T)$-divisible. Thus, by $(*)_1$, L_S has a κ-well separated $F(T)$-decomposition $\mathcal{F}'_S$. Proposition 6.11 below implies that we can extend $\mathcal{F}'_S$ to a κ-well separated F-packing $\mathcal{F}_S$ which covers all edges that contain S.

However, in order to cover all type-1-edges, we need to obtain such a packing $\mathcal{F}_S$ for every $S \in \binom{V(G)\setminus U}{r-1}$, and these packings are to be r-disjoint for their union to be a κ-well separated F-packing again. The real difficulty thus lies in choosing H^* in such a way that the link graphs L_S do not interfere too much with each other, and then to choose the decompositions $\mathcal{F}'_S$ sequentially. We would then continue to cover all type-2-edges using $(*)_2$, etc., until we finally cover all type-$(r-1)$-edges using $(*)_{r-1}$. The only remaining edges are then type-r-edges, which are contained in U, as desired.

Proving the Cover down lemma for cliques presented one of the main challenges in [12]. However, with Proposition 6.11 in hand, the proof carries over to general (weakly regular) F without significant modifications, and is thus omitted here.

We now show how the notion of well separated F-packings allows us to 'extend' a decomposition of a link complex to a packing which covers all edges that contain a given set S (cf. the discussion in Section 3.2).

Definition 6.10. Let F be an r-graph, $i \in [r-1]$ and assume that $T \in \binom{V(F)}{i}$ is such that $F(T)$ is non-empty. Let G be a complex and $S \in \binom{V(G)}{i}$. Suppose that $\mathcal{F}'$ is a well separated $F(T)$-packing in $G(S)$. We then define $S \triangleleft \mathcal{F}'$ as follows: For each $F' \in \mathcal{F}'$, let $F'_{\triangleleft}$ be an (arbitrary) copy of F on vertex set $S \cup V(F')$ such that $F'_{\triangleleft}(S) = F'$. Let

$$S \triangleleft \mathcal{F}' := \{F'_{\triangleleft} : F' \in \mathcal{F}'\}.$$

The following proposition is crucial and guarantees that the above extension yields a packing which covers the desired set of edges. It replaces Fact 10.1 of [12] in the proof of the Cover down lemma. It is also used in the construction of so-called 'transformers' (see Section 7.1).

Proposition 6.11. *Let F, r, i, T, G, S be as in Definition 6.10. Let $L \subseteq G(S)^{(r-i)}$. Suppose that $\mathcal{F}'$ is a κ-well separated $F(T)$-decomposition of $G(S)[L]$. Then $\mathcal{F} := S \triangleleft \mathcal{F}'$ is a κ-well separated F-packing in G and $\{e \in \mathcal{F}^{(r)} : S \subseteq e\} = S \uplus L$.*

In particular, if $L = G(S)^{(r-i)}$, i.e. if $\mathcal{F}'$ is a κ-well separated $F(T)$-decomposition of $G(S)$, then $\mathcal{F}$ is a κ-well separated F-packing in G which covers all r-edges of G that contain S.

Proof. We first check that $\mathcal{F}$ is an F-packing in G. Let $f := |V(F)|$. For each $F' \in \mathcal{F}'$, we have $V(F') \in G(S)[L]^{(f-i)} \subseteq G(S)^{(f-i)}$. Hence, $V(F'_\triangleleft) \in G^{(f)}$. In particular, $G^{(r)}[V(F'_\triangleleft)]$ is a clique and thus $F'_\triangleleft$ is a subgraph of $G^{(r)}$. Suppose, for a contradiction, that for distinct $F', F'' \in \mathcal{F}'$, $F'_\triangleleft$ and $F''_\triangleleft$ both contain $e \in G^{(r)}$. By (WS1) we have that $|V(F') \cap V(F'')| \le r - i$, and thus we must have $e = S \cup (V(F') \cap V(F''))$. Since $V(F') \cap V(F'') \in G(S)[L]$, we have $e \setminus S \in G(S)[L]^{(r-i)}$, and thus $e \setminus S$ belongs to at most one of F' and F''. Without loss of generality, assume that $e \setminus S \notin F'$. Then we have $e \setminus S \notin F'_\triangleleft(S)$ and thus $e \notin F'_\triangleleft$, a contradiction. Thus, $\mathcal{F}$ is an F-packing in G.

We next show that $\mathcal{F}$ is κ-well separated. Clearly, for distinct $F', F'' \in \mathcal{F}'$, we have $|V(F'_\triangleleft) \cap V(F''_\triangleleft)| \le r - i + |S| = r$, so (WS1) holds. To check (WS2), consider $e \in \binom{V(G)}{r}$. Let e' be an $(r-i)$-subset of $e \setminus S$. By definition of $\mathcal{F}$, we have that the number of $F'_\triangleleft \in \mathcal{F}$ with $e \subseteq V(F'_\triangleleft)$ is at most the number of $F' \in \mathcal{F}'$ with $e' \subseteq V(F')$, where the latter is at most κ since $\mathcal{F}'$ is κ-well separated.

Finally, we check that $\{e \in \mathcal{F}^{(r)} : S \subseteq e\} = S \uplus L$. Let e be any r-set with $S \subseteq e$. By Definition 6.10, we have $e \in \mathcal{F}^{(r)}$ if and only if $e \setminus S \in \mathcal{F}'^{(r-i)}$. Since $\mathcal{F}'$ is an $F(T)$-decomposition of $G(S)[L]^{(r-i)} = L$, we have $e \setminus S \in \mathcal{F}'^{(r-i)}$ if and only if $e \setminus S \in L$. Thus, $e \in \mathcal{F}^{(r)}$ if and only if $e \in S \uplus L$. □

7. Absorbers

In this section we show that for any (divisible) r-graph H in a supercomplex G, we can find an 'exclusive' absorber r-graph A (as discussed in Section 1.5, one may think of H as a potential leftover from an approximate F-decomposition and A will be set aside earlier to absorb H into an F-decomposition). The following definition makes this precise. The main result of this section is Lemma 7.2, which constructs an absorber provided that F is weakly regular.

Definition 7.1 (Absorber)**.** Let F, H and A be r-graphs. We say that A is an F-*absorber for* H if A and H are edge-disjoint and both A and $A \cup H$ have an F-decomposition. More generally, if G is a complex and $H \subseteq G^{(r)}$, then $A \subseteq G^{(r)}$ is a κ-*well separated* F-*absorber for* H *in* G if A and H are edge-disjoint and there exist κ-well separated F-packings $\mathcal{F}_\circ$ and $\mathcal{F}_\bullet$ in G such that $\mathcal{F}_\circ^{(r)} = A$ and $\mathcal{F}_\bullet^{(r)} = A \cup H$.

Lemma 7.2 (Absorbing lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma, 1/h, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let F be a weakly regular r-graph on f vertices, let G be an (ε, ξ, f, r)-supercomplex on n vertices and let H be an F-divisible subgraph of $G^{(r)}$ with $|H| \le h$. Then there exists a κ-well separated F-absorber A for H in G with $\Delta(A) \le \gamma n$.*

We now briefly discuss the case $r = 1$. For the case $F = K_f^{(1)}$, a construction of an F-absorber for any F-divisible r-graph H in a supercomplex G is given in [12]. It is easy to see that this absorber is 1-well separated. Essentially the same construction also works if F contains some isolated vertices. Thus, for the remainder of this section, we will assume that $r \ge 2$.

The building blocks of our absorbers will be so-called 'transformers', first introduced in [4]. Roughly speaking, a transformer T can be viewed as transforming a given leftover graph H into a new leftover H' (where we set aside T and H' earlier).

Definition 7.3 (Transformer)**.** Let F be an r-graph, G a complex and assume that $H, H' \subseteq G^{(r)}$. A subgraph $T \subseteq G^{(r)}$ is a κ-*well separated* $(H, H'; F)$-*transformer in* G if T is edge-disjoint from both H and H' and there exist κ-well separated F-packings $\mathcal{F}$ and $\mathcal{F}'$ in G such that $\mathcal{F}^{(r)} = T \cup H$ and $\mathcal{F}'^{(r)} = T \cup H'$.

Our 'Transforming lemma' (Lemma 7.5) guarantees the existence of a transformer for H and H' if H' is *obtained from* H *by identifying vertices* (modulo deleting some isolated vertices from H'). To make this more precise, given a multi-r-graph H and $x, x' \in V(H)$, we say that x and x'

are *identifiable* if $|H(\{x,x'\})| = 0$, that is, if identifying x and x' does not create an edge of size less than r. For multi-r-graphs H and H', we write $H \leftrightsquigarrow H'$ if there is a sequence $H_0,\dots,H_t$ of multi-r-graphs such that $H_0 \cong H$, H_t is obtained from H' by deleting isolated vertices, and for every $i \in [t]$, there are two identifiable vertices $x,x' \in V(H_{i-1})$ such that H_i is obtained from H_{i-1} by identifying x and x'.

If H and H' are (simple) r-graphs and $H \leftrightsquigarrow H'$, we just write $H \rightsquigarrow H'$ to indicate the fact that during the identification steps, only vertices $x,x' \in V(H_{i-1})$ with $H_{i-1}(\{x\}) \cap H_{i-1}(\{x'\}) = \emptyset$ were identified (i.e. if we did not create multiple edges).

Clearly, $\leftrightsquigarrow$ is a reflexive and transitive relation on the class of multi-r-graphs, and $\rightsquigarrow$ is a reflexive and transitive relation on the class of r-graphs.

It is easy to see that $H \rightsquigarrow H'$ if and only if there is an *edge-bijective homomorphism* from H to H' (see Proposition 7.4(i)). Given r-graphs H, H', a *homomorphism from H to H'* is a map $\phi\colon V(H) \to V(H')$ such that $\phi(e) \in H'$ for all $e \in H$. Note that this implies that $\phi\restriction_e$ is injective for all $e \in H$. We let $\phi(H)$ denote the subgraph of H' with vertex set $\phi(V(H))$ and edge set $\{\phi(e) : e \in H\}$. We say that ϕ is *edge-bijective* if $|H| = |\phi(H)| = |H'|$. For two r-graphs H and H', we write $H \overset{\phi}{\rightsquigarrow} H'$ if ϕ is an edge-bijective homomorphism from H to H'.

We now record a few simple observations about the relation $\rightsquigarrow$ for future reference.

Proposition 7.4. *The following hold.*

(i) *$H \rightsquigarrow H'$ if and only if there exists ϕ such that $H \overset{\phi}{\rightsquigarrow} H'$.*

(ii) *Let $H_1, H_1', \dots, H_t, H_t'$ be r-graphs such that $H_1,\dots,H_t$ are vertex-disjoint and $H_1',\dots,H_t'$ are edge-disjoint and $H_i \cong H_i'$ for all $i \in [t]$. Then*

$$H_1 + \dots + H_t \rightsquigarrow H_1' \cup \dots \cup H_t'.$$

(iii) *If $H \rightsquigarrow H'$ and H is F-divisible, then H' is F-divisible.*

7.1. **Transformers.** The following lemma guarantees the existence of a transformer from H to H' if F is weakly regular and $H \rightsquigarrow H'$. The proof relies inductively on the assertion of the main complex decomposition theorem (Theorem 3.8).

Lemma 7.5 (Transforming lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma, 1/h, \varepsilon \ll \xi, 1/f$ and $2 \le r < f$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let F be a weakly regular r-graph on f vertices, let G be an (ε,ξ,f,r)-supercomplex on n vertices and let H,H' be vertex-disjoint F-divisible subgraphs of $G^{(r)}$ of order at most h and such that $H \rightsquigarrow H'$. Then there exists a κ-well separated $(H,H';F)$-transformer T in G with $\Delta(T) \le \gamma n$.*

A key operation in the proof of Lemma 7.5 is the ability to find 'localised transformers'. Let $i \in [r-1]$ and let $S \subseteq V(H)$, $S' \subseteq V(H')$ and $S^* \subseteq V(F)$ be sets of size i. For an $(r-i)$-graph L in the link graph of both S and S', we can view an $F(S^*)$-decomposition $\mathcal{F}_L$ of L (which exists by $(*)_{r-i}$) as a localised transformer between $S \uplus L$ and $S' \uplus L$. Indeed, similarly to the situation described in Sections 3.2 and 6.1, we can extend $\mathcal{F}_L$ 'by adding S back' to obtain an F-packing $\mathcal{F}$ which covers all edges of $S \uplus L$. By 'mirroring' this extension, we can also obtain an F-packing $\mathcal{F}'$ which covers all edges of $S' \uplus L$ (see Definition 7.8 and Proposition 7.9). To make this more precise, we introduce the following notation.

Definition 7.6. Let V be a set and let V_1, V_2 be disjoint subsets of V having equal size. Let $\phi\colon V_1 \to V_2$ be a bijection. For a set $S \subseteq V \setminus V_2$, define $\phi(S) := (S \setminus V_1) \cup \phi(S \cap V_1)$. Moreover, for an r-graph R with $V(R) \subseteq V \setminus V_2$, we let $\phi(R)$ be the r-graph on $\phi(V(R))$ with edge set $\{\phi(e) : e \in R\}$.

The following facts are easy to see.

Fact 7.7. *Suppose that V, V_1, V_2 and ϕ are as above. Then the following hold for every r-graph R with $V(R) \subseteq V \setminus V_2$:*

(i) *$\phi(R) \cong R$;*

(ii) *if* $R = R_1 \cup \ldots \cup R_k$, *then* $\phi(R) = \phi(R_1) \cup \ldots \cup \phi(R_k)$ *and thus* $\phi(R_1) = \phi(R) - \phi(R_2) - \cdots - \phi(R_k)$.

The following definition is a two-sided version of Definition 6.10.

Definition 7.8. Let F be an r-graph, $i \in [r-1]$ and assume that $S^* \in \binom{V(F)}{i}$ is such that $F(S^*)$ is non-empty. Let G be a complex and assume that $S_1, S_2 \in \binom{V(G)}{i}$ are disjoint and that a bijection $\phi\colon S_1 \to S_2$ is given. Suppose that $\mathcal{F}'$ is a well separated $F(S^*)$-packing in $G(S_1) \cap G(S_2)$. We then define $S_1 \triangleleft \mathcal{F}' \triangleright S_2$ as follows: For each $F' \in \mathcal{F}'$ and $j \in \{1,2\}$, let F'_j be a copy of F on vertex set $S_j \cup V(F')$ such that $F'_j(S_j) = F'$ and such that $\phi(F'_1) = F'_2$. Let

$$\mathcal{F}_1 := \{F'_1 \,:\, F' \in \mathcal{F}'\};$$
$$\mathcal{F}_2 := \{F'_2 \,:\, F' \in \mathcal{F}'\};$$
$$S_1 \triangleleft \mathcal{F}' \triangleright S_2 := (\mathcal{F}_1, \mathcal{F}_2).$$

The next proposition is proved using its one-sided counterpart, Proposition 6.11. As in Proposition 6.11, the notion of well separatedness (Definition 3.6) is crucial here.

Proposition 7.9. *Let F, r, i, S^*, G, S_1, S_2 and ϕ be as in Definition 7.8. Suppose that $L \subseteq G(S_1)^{(r-i)} \cap G(S_2)^{(r-i)}$ and that $\mathcal{F}'$ is a κ-well separated $F(S^*)$-decomposition of $(G(S_1) \cap G(S_2))[L]$. Then the following holds for $(\mathcal{F}_1, \mathcal{F}_2) = S_1 \triangleleft \mathcal{F}' \triangleright S_2$:*

(i) *for $j \in [2]$, $\mathcal{F}_j$ is a κ-well separated F-packing in G with $\{e \in \mathcal{F}_j^{(r)} \,:\, S_j \subseteq e\} = S_j \uplus L$;*

(ii) *$V(\mathcal{F}_1^{(r)}) \subseteq V(G) \setminus S_2$ and $\phi(\mathcal{F}_1^{(r)}) = \mathcal{F}_2^{(r)}$.*

Proof. Let $j \in [2]$. Since $(G(S_1) \cap G(S_2))[L] \subseteq G(S_j)$, we can view $\mathcal{F}_j$ as $S_j \triangleleft \mathcal{F}'$ (cf. Definition 6.10). Moreover, since $(G(S_1) \cap G(S_2))[L]^{(r-i)} = L = G(S_j)[L]^{(r-i)}$, we can conclude that $\mathcal{F}'$ is a κ-well separated $F(S^*)$-decomposition of $G(S_j)[L]$. Thus, by Proposition 6.11, $\mathcal{F}_j$ is a κ-well separated F-packing in G with $\{e \in \mathcal{F}_j^{(r)} \,:\, S_j \subseteq e\} = S_j \uplus L$.

Moreover, we have $V(\mathcal{F}_1^{(r)}) \subseteq \bigcup_{F' \in \mathcal{F}'} V(F'_1) \subseteq V(G) \setminus S_2$ and by Fact 7.7(ii)

$$\phi(\mathcal{F}_1^{(r)}) = \phi(\dot{\bigcup_{F' \in \mathcal{F}'}} F'_1) = \dot{\bigcup_{F' \in \mathcal{F}'}} \phi(F'_1) = \dot{\bigcup_{F' \in \mathcal{F}'}} F'_2 = \mathcal{F}_2^{(r)}.$$

□

We now sketch the proof of Lemma 7.5. Given Proposition 7.9, the details are very similar to the proof of Lemma 8.5 in [12] and thus omitted here.

Suppose for simplicity that H' is simply a copy of H, i.e. $H' = \phi(H)$ where ϕ is an isomorphism from H to H'. We aim to construct an $(H, H'; F)$-transformer. In a first step, for every edge $e \in H$, we introduce a set X_e of $|V(F)| - r$ new vertices and let F_e be a copy of F such that $V(F_e) = e \cup X_e$ and $e \in F_e$. Let $T_1 := \bigcup_{e \in H} F_e[X_e]$ and $R_1 := \bigcup_{e \in H} F_e - T_1 - H$. Clearly, $\{F_e \,:\, e \in H\}$ is an F-decomposition of $H \cup R_1 \cup T_1$. By Fact 7.7(ii), we also have that $\{\phi(F_e) \,:\, e \in H\}$ is an F-decomposition of $H' \cup \phi(R_1) \cup T_1$. Hence, T_1 is an $(H \cup R_1, H' \cup \phi(R_1); F)$-transformer. Note that at this stage, it would suffice to find an $(R_1, \phi(R_1); F)$-transformer T'_1, as then $T_1 \cup T'_1 \cup R_1 \cup \phi(R_1)$ would be an $(H, H'; F)$-transformer. The crucial difference now to the original problem is that every edge of R_1 contains at most $r-1$ vertices from $V(H)$. On the other hand, every edge in R_1 contains at least one vertex in $V(H)$ as otherwise it would belong to T_1. We view this as Step 1 and will now proceed inductively. After Step i, we will have an r-graph R_i and an $(H \cup R_i, H' \cup \phi(R_i); F)$-transformer T_i such that every edge $e \in R_i$ satisfies $1 \leq |e \cap V(H)| \leq r - i$. Thus, after Step r we can terminate the process as R_r must be empty and thus T_r is an $(H, H'; F)$-transformer.

In Step $i+1$, where $i \in [r-1]$, we use $(*)_i$ inductively as follows. Let R'_i consist of all edges of R_i which intersect $V(H)$ in $r-i$ vertices. We decompose R'_i into 'local' parts. For every edge $e \in R'_i$, there exists a unique set $S \in \binom{V(H)}{r-i}$ such that $S \subseteq e$. For each $S \in \binom{V(H)}{r-i}$, let $L_S := R'_i(S)$. Note that the 'local' parts $S \uplus L_S$ form a decomposition of R'_i. The problem of finding R_{i+1} and T_{i+1} can be reduced to finding a 'localised transformer' between $S \uplus L_S$

and $\phi(S) \uplus L_S$ for every S, as described above. At this stage, by Proposition 4.2, L_S will automatically be $F(S^*)$-divisible, where $S^* \in \binom{V(F)}{r-i}$ is such that $F(S^*)$ is non-empty. If we were given an $F(S^*)$-decomposition $\mathcal{F}'_S$ of L_S, we could use Proposition 7.9 to extend $\mathcal{F}'_S$ to an F-packing $\mathcal{F}_S$ which covers all edges of $S \uplus L_S$, and all new edges created by this extension intersect S (and $V(H)$) in at most $r-i-1$ vertices, as desired. It is possible to combine these localised transformers with T_i and R_i in such a way that we obtain T_{i+1} and R_{i+1}.

Unfortunately, $(G(S) \cap G(\phi(S)))[L_S]$ might not be a supercomplex (one can think of L_S as some leftover from previous steps) and so $\mathcal{F}'_S$ may not exist. However, by Proposition 4.4, we have that $G(S) \cap G(\phi(S))$ is a supercomplex. Thus we can (randomly) choose a suitable i-subgraph A_S of $(G(S) \cap G(\phi(S)))^{(i)}$ such that A_S is $F(S^*)$-divisible and edge-disjoint from L_S. Instead of building a localised transformer for L_S directly, we will now build one for A_S and one for $A_S \cup L_S$, using $(*)_i$ both times to find the desired $F(S^*)$-decomposition. These can then be combined into a localised transformer for L_S.

7.2. **Canonical multi-r-graphs.** Roughly speaking, the aim of this section is to show that any F-divisible r-graph H can be transformed into a canonical multigraph M_h which does not depend on the structure of H. However, it turns out that for this we need to move to a 'dual' setting, where we consider ∇H which is obtained from H by applying an F-extension operator ∇. This operator allows us to switch between multi-r-graphs (which arise naturally in the construction but are not present in the complex G we are decomposing) and (simple) r-graphs (see e.g. Fact 7.15).

Given a multi-r-graph H and a set X of size r, we say that ψ is an *X-orientation of H* if ψ is a collection of bijective maps $\psi_e \colon X \to e$, one for each $e \in H$. (For $r = 2$ and $X = \{1, 2\}$, say, this coincides with the notion of an oriented multigraph, e.g. by viewing $\psi_e(1)$ as the tail and $\psi_e(2)$ as the head of e, where parallel edges can be oriented in opposite directions.)

Given an r-graph F and a distinguished edge $e_0 \in F$, we introduce the following 'extension' operators $\tilde{\nabla}_{(F,e_0)}$ and $\nabla_{(F,e_0)}$.

Definition 7.10 (Extension operators $\tilde{\nabla}$ and ∇)**.** Given a (multi-)r-graph H with an e_0-orientation ψ, let $\tilde{\nabla}_{(F,e_0)}(H,\psi)$ be obtained from H by extending every edge of H into a copy of F, with e_0 being the rooted edge. More precisely, let Z_e be vertex sets of size $|V(F) \setminus e_0|$ such that $Z_e \cap Z_{e'} = \emptyset$ for all distinct (but possibly parallel) $e, e' \in H$ and $V(H) \cap Z_e = \emptyset$ for all $e \in H$. For each $e \in H$, let F_e be a copy of F on vertex set $e \cup Z_e$ such that $\psi_e(v)$ plays the role of v for all $v \in e_0$ and Z_e plays the role of $V(F) \setminus e_0$. Then $\tilde{\nabla}_{(F,e_0)}(H,\psi) := \bigcup_{e \in H} F_e$. Let $\nabla_{(F,e_0)}(H,\psi) := \tilde{\nabla}_{(F,e_0)}(H,\psi) - H$.

Note that $\nabla_{(F,e_0)}(H,\psi)$ is a (simple) r-graph even if H is a multi-r-graph. If F, e_0 and ψ are clear from the context, or if we only want to motivate an argument before giving the formal proof, we just write $\tilde{\nabla} H$ and ∇H.

Fact 7.11. *Let F be an r-graph and $e_0 \in F$. Let H be a multi-r-graph and let ψ be any e_0-orientation of H. Then the following hold:*

(i) *$\tilde{\nabla}_{(F,e_0)}(H,\psi)$ is F-decomposable;*
(ii) *$\nabla_{(F,e_0)}(H,\psi)$ is F-divisible if and only if H is F-divisible.*

The goal of this subsection is to show that for every $h \in \mathbb{N}$, there is a multi-r-graph M_h such that for any F-divisible r-graph H on at most h vertices, we have

$$\nabla(\nabla(H + t \cdot F) + s \cdot F) \rightsquigarrow \nabla M_h \tag{7.1}$$

for suitable $s, t \in \mathbb{N}$. The multigraph M_h is *canonical* in the sense that it does not depend on H, but only on h. The benefit is, very roughly speaking, that it allows us to transform any given leftover r-graph H into the empty r-graph, which is trivially decomposable, and this will enable us to construct an absorber for H. Indeed, to see that (7.1) allows us to transform H into the empty r-graph, let

$$H' := \nabla(\nabla(H + t \cdot F) + s \cdot F) = \nabla\nabla H + t \cdot \nabla\nabla F + s \cdot \nabla F$$

and observe that the r-graph $T := \nabla H + t \cdot \tilde{\nabla} F + s \cdot F$ 'between' H and H' can be chosen in such a way that

$$T \cup H = \tilde{\nabla} H + t \cdot \tilde{\nabla} F + s \cdot F,$$
$$T \cup H' = \tilde{\nabla}(\nabla H) + t \cdot (\tilde{\nabla}(\nabla F) \cup F) + s \cdot \tilde{\nabla} F,$$

i.e. T is an $(H, H'; F)$-transformer (cf. Fact 7.11(i)). Hence, together with (7.1) and Lemma 7.5, this means that we can transform H into ∇M_h. Since M_h does not depend on H, we can also transform the empty r-graph into ∇M_h, and by transitivity we can transform H into the empty graph, which amounts to an absorber for H (the detailed proof of this can be found in Section 7.3).

We now give the rigorous statement of (7.1), which is the main lemma of this subsection.

Lemma 7.12. *Let $r \geq 2$ and assume that $(*)_i$ is true for all $i \in [r-1]$. Let F be a weakly regular r-graph and $e_0 \in F$. Then for all $h \in \mathbb{N}$, there exists a multi-r-graph M_h such that for any F-divisible r-graph H on at most h vertices, we have*

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F, \psi_3) \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi_2)$$

for suitable $s, t \in \mathbb{N}$, where ψ_1 and ψ_2 can be arbitrary e_0-orientations of $H + t \cdot F$ and M_h, respectively, and ψ_3 is an e_0-orientation depending on these.

The above graphs $\nabla(\nabla(H + t \cdot F) + s \cdot F)$ and ∇M_h will be part of our F-absorber for H. We therefore need to make sure that we can actually find them in a supercomplex G. This requirement is formalised by the following definition.

Definition 7.13. Let G be a complex, $X \subseteq V(G)$, F an r-graph with $f := |V(F)|$ and $e_0 \in F$. Suppose that $H \subseteq G^{(r)}$ and that ψ is an e_0-orientation of H. By *extending H with a copy of $\nabla_{(F,e_0)}(H, \psi)$ in G (whilst avoiding X)* we mean the following: for each $e \in H$, let $Z_e \in G^{(f)}(e)$ be such that $Z_e \cap (V(H) \cup X) = \emptyset$ for every $e \in H$ and $Z_e \cap Z_{e'} = \emptyset$ for all distinct $e, e' \in H$. For each $e \in H$, let F_e be a copy of F on vertex set $e \cup Z_e$ (so $F_e \subseteq G^{(r)}$) such that $\psi_e(v)$ plays the role of v for all $v \in e_0$ and Z_e plays the role of $V(F) \setminus e_0$. Let $H^\nabla := \bigcup_{e \in H} F_e - H$ and $\mathcal{F} := \{F_e : e \in H\}$ be the output of this.

For our purposes, the set $|V(H) \cup X|$ will have a small bounded size compared to $|V(G)|$. Thus, if the $G^{(f)}(e)$ are large enough (which is the case e.g. in an (ε, ξ, f, r)-supercomplex), then the above extension can be carried out simply by picking the sets Z_e one by one.

Fact 7.14. *Let $(H^\nabla, \mathcal{F})$ be obtained by extending $H \subseteq G^{(r)}$ with a copy of $\nabla_{(F,e_0)}(H, \psi)$ in G. Then $H^\nabla \subseteq G^{(r)}$ is a copy of $\nabla_{(F,e_0)}(H, \psi)$ and $\mathcal{F}$ is a 1-well separated F-packing in G with $\mathcal{F}^{(r)} = H \cup H^\nabla$ such that for all $F' \in \mathcal{F}$, $|V(F') \cap V(H)| \leq r$.*

For a partition $\mathcal{P} = \{V_x\}_{x \in X}$ whose classes are indexed by a set X, we define $V_Y := \bigcup_{x \in Y} V_x$ for every subset $Y \subseteq X$. Recall that for a multi-r-graph H and $e \in \binom{V(H)}{r}$, $|H(e)|$ denotes the multiplicity of e in H. For multi-r-graphs H, H', we write $H \overset{\mathcal{P}}{\leftrightsquigarrow} H'$ if $\mathcal{P} = \{V_{x'}\}_{x' \in V(H')}$ is a partition of $V(H)$ such that

(I1) for all $x' \in V(H')$ and $e \in H$, $|V_{x'} \cap e| \leq 1$;
(I2) for all $e' \in \binom{V(H')}{r}$, $\sum_{e \in \binom{V_{e'}}{r}} |H(e)| = |H'(e')|$.

Given $\mathcal{P}$, define $\phi_\mathcal{P} \colon V(H) \to V(H')$ as $\phi_\mathcal{P}(x) := x'$ where x' is the unique $x' \in V(H')$ such that $x \in V_{x'}$. Note that by (I1), we have $|\{\phi_\mathcal{P}(x) : x \in e\}| = r$ for all $e \in H$. Further, by (I2), there exists a bijection $\Phi_\mathcal{P} \colon H \to H'$ between the multi-edge-sets of H and H' such that for every edge $e \in H$, the image $\Phi_\mathcal{P}(e)$ is an edge consisting of the vertices $\phi_\mathcal{P}(x)$ for all $x \in e$. It is easy to see that $H \leftrightsquigarrow H'$ if and only if there is some $\mathcal{P}$ such that $H \overset{\mathcal{P}}{\leftrightsquigarrow} H'$.

The extension operator ∇ is well behaved with respect to the identification relation $\leftrightsquigarrow$ in the following sense: if $H \leftrightsquigarrow H'$, then $\nabla H \rightsquigarrow \nabla H'$. More precisely, let H and H' be multi-r-graphs and suppose that $H \overset{\mathcal{P}}{\leftrightsquigarrow} H'$. Let $\phi_\mathcal{P}$ and $\Phi_\mathcal{P}$ be defined as above. Let F be an r-graph and

$e_0 \in F$. For any e_0-orientation ψ' of H', we define an e_0-orientation ψ of H *induced by* ψ' as follows: for every $e \in H$, let $e' := \Phi_{\mathcal{P}}(e)$ be the image of e with respect to $\overset{\mathcal{P}}{\rightsquigarrow}$. We have that $\phi_{\mathcal{P}}\restriction_e : e \to e'$ is a bijection. We now define the bijection $\psi_e : e_0 \to e$ as $\psi_e := \phi_{\mathcal{P}}\restriction_e^{-1} \circ \psi'_{e'}$, where $\psi'_{e'} : e_0 \to e'$. Thus, the collection ψ of all ψ_e, $e \in H$, is an e_0-orientation of H. It is easy to see that ψ satisfies the following.

Fact 7.15. *Let F be an r-graph and $e_0 \in F$. Let H, H' be multi-r-graphs and suppose that $H \rightsquigarrow H'$. Then for any e_0-orientation ψ' of H', we have $\nabla_{(F,e_0)}(H,\psi) \rightsquigarrow \nabla_{(F,e_0)}(H',\psi')$, where ψ is induced by ψ'.*

We now define the multi-r-graphs which will serve as the canonical multi-r-graphs M_h in (7.1). For $r \in \mathbb{N}$, let $\mathcal{M}_r$ contain all pairs $(k,m) \in \mathbb{N}_0^2$ such that $\frac{m}{r-i}\binom{k-i}{r-1-i}$ is an integer for all $i \in [r-1]_0$.

Definition 7.16 (Canonical multi-r-graph)**.** Let F^* be an r-graph and $e^* \in F^*$. Let $V' := V(F^*) \setminus e^*$. If $(k,m) \in \mathcal{M}_r$, define the multi-r-graph $M_{k,m}^{(F^*,e^*)}$ on vertex set $[k] \mathbin{\dot\cup} V'$ such that for every $e \in \binom{[k]\cup V'}{r}$, the multiplicity of e is

$$|M_{k,m}^{(F^*,e^*)}(e)| = \begin{cases} 0 & \text{if } e \subseteq [k]; \\ \frac{m}{r-|e\cap[k]|}\binom{k-|e\cap[k]|}{r-1-|e\cap[k]|} & \text{if } |e \cap [k]| > 0, |e \cap V'| > 0; \\ 0 & \text{if } e \subseteq V', e \notin F^*; \\ \frac{m}{r}\binom{k}{r-1} & \text{if } e \subseteq V', e \in F^*. \end{cases}$$

We will require the graph F^* in Definition 7.16 to have a certain symmetry property with respect to e^*, which we now define. We will prove the existence of a suitable (F-decomposable) symmetric r-extender in Lemma 7.23.

Definition 7.17 (symmetric r-extender)**.** We say that (F^*,e^*) is a *symmetric r-extender* if F^* is an r-graph, $e^* \in F^*$ and the following holds:

(SE) for all $e' \in \binom{V(F^*)}{r}$ with $e' \cap e^* \neq \emptyset$, we have $e' \in F^*$.

Note that if (F^*,e^*) is a symmetric r-extender, then the operators $\tilde{\nabla}_{(F^*,e^*)}, \nabla_{(F^*,e^*)}$ are labelling-invariant, i.e. $\tilde{\nabla}_{(F^*,e^*)}(H,\psi_1) \cong \tilde{\nabla}_{(F^*,e^*)}(H,\psi_2)$ and $\nabla_{(F^*,e^*)}(H,\psi_1) \cong \nabla_{(F^*,e^*)}(H,\psi_2)$ for all e^*-orientations ψ_1,ψ_2 of a multi-r-graph H. We therefore simply write $\tilde{\nabla}_{(F^*,e^*)}H$ and $\nabla_{(F^*,e^*)}H$ in this case.

To prove Lemma 7.12 we introduce so called strong colourings. Let H be an r-graph and C a set. A map $c\colon V(H) \to C$ is a *strong C-colouring of H* if for all distinct $x,y \in V(H)$ with $|H(\{x,y\})| > 0$, we have $c(x) \neq c(y)$, that is, no colour appears twice in one edge. For $\alpha \in C$, we let $c^{-1}(\alpha)$ denote the set of all vertices coloured α. For a set $C' \subseteq C$, we let $c^{\subseteq}(C') := \{e \in H : C' \subseteq c(e)\}$. We say that c is *m-regular* if $|c^{\subseteq}(C')| = m$ for all $C' \in \binom{C}{r-1}$. For example, an r-partite r-graph H trivially has a strong $|H|$-regular $[r]$-colouring.

Fact 7.18. *Let H be an r-graph and let c be a strong m-regular $[k]$-colouring of H. Then $|c^{\subseteq}(C')| = \frac{m}{r-i}\binom{k-i}{r-1-i}$ for all $i \in [r-1]_0$ and all $C' \in \binom{[k]}{i}$.*

Lemma 7.19. *Let (F^*,e^*) be a symmetric r-extender. Suppose that H is an r-graph and suppose that c is a strong m-regular $[k]$-colouring of H. Then $(k,m) \in \mathcal{M}_r$ and*

$$\nabla_{(F^*,e^*)}H \rightsquigarrow M_{k,m}^{(F^*,e^*)}.$$

Proof. By Fact 7.18, $(k,m) \in \mathcal{M}_r$, thus $M_{k,m}^{(F^*,e^*)}$ is defined. Recall that $M_{k,m}^{(F^*,e^*)}$ has vertex set $[k] \cup V'$, where $V' := V(F^*) \setminus e^*$. Let $V(H) \cup \bigcup_{e\in H} Z_e$ be the vertex set of $\nabla_{(F^*,e^*)}H$ as in Definition 7.10, with $Z_e = \{z_{e,v} : v \in V'\}$. We define a partition $\mathcal{P}$ of $V(H) \cup \bigcup_{e\in H} Z_e$ as follows: for all $i \in [k]$, let $V_i := c^{-1}(i)$. For all $v \in V'$, let $V_v := \{z_{e,v} : e \in H\}$. We now claim that $\nabla_{(F^*,e^*)}H \overset{\mathcal{P}}{\rightsquigarrow} M_{k,m}^{(F^*,e^*)}$.

Clearly, $\mathcal{P}$ satisfies (I1) because c is a strong colouring of H. For a set $e' \in \binom{[k]\cup V'}{r}$, define

$$S_{e'} := \{e'' \in \nabla_{(F^*,e^*)}H \,:\, e'' \subseteq V_{e'}\}.$$

Since $\nabla_{(F^*,e^*)}H$ is simple, in order to check (I2), it is enough to show that for all $e' \in \binom{[k]\cup V'}{r}$, we have $|S_{e'}| = |M_{k,m}^{(F^*,e^*)}(e')|$. We distinguish three cases.

Case 1: $e' \subseteq [k]$

In this case, $|M_{k,m}^{(F^*,e^*)}(e')| = 0$. Since $V_{e'} \subseteq V(H)$ and $(\nabla_{(F^*,e^*)}H)[V(H)]$ is empty, we have $S_{e'} = \emptyset$, as desired.

Case 2: $e' \subseteq V'$

In this case, $S_{e'}$ consists of all edges of $\nabla_{(F^*,e^*)}H$ which play the role of e' in F_e^* for some $e \in H$. Hence, if $e' \notin F^*$, then $|S_{e'}| = 0$, and if $e' \in F^*$, then $|S_{e'}| = |H|$. Fact 7.18 applied with $i = 0$ yields $|H| = \frac{m}{r}\binom{k}{r-1}$, as desired.

Case 3: $|e' \cap [k]| > 0$ and $|e' \cap V'| > 0$

We claim that $|S_{e'}| = |c^{\subseteq}(e' \cap [k])|$. In order to see this, we define a bijection $\pi\colon c^{\subseteq}(e' \cap [k]) \to S_{e'}$ as follows: for every $e \in H$ with $e' \cap [k] \subseteq c(e)$, define

$$\pi(e) := (e \cap c^{-1}(e' \cap [k])) \cup \{z_{e,v} \,:\, v \in e' \cap V'\}.$$

We first show that $\pi(e) \in S_{e'}$. Note that $e \cap c^{-1}(e' \cap [k])$ is a subset of e of size $|e' \cap [k]|$ and $\{z_{e,v} \,:\, v \in e' \cap V'\}$ is a subset of Z_e of size $|e' \cap V'|$. Hence, $\pi(e) \in \binom{V(F_e^*)}{r}$ and $|\pi(e) \cap e| = |e' \cap [k]| > 0$. Thus, by (SE), we have $\pi(e) \in F_e^* \subseteq \nabla_{(F^*,e^*)}H$. (This is in fact the crucial point where we need (SE).) Moreover,

$$\pi(e) \subseteq c^{-1}(e' \cap [k]) \cup \{z_{e,v} \,:\, v \in e' \cap V'\} \subseteq V_{e'\cap[k]} \cup V_{e'\cap V'} = V_{e'}.$$

Therefore, $\pi(e) \in S_{e'}$. It is straightforward to see that π is injective. Finally, for every $e'' \in S_{e'}$, we have $e'' = \pi(e)$, where $e \in H$ is the unique edge of H with $e'' \in F_e^*$. This establishes our claim that π is bijective and hence $|S_{e'}| = |c^{\subseteq}(e' \cap [k])|$. Since $1 \leq |e' \cap [k]| \leq r-1$, Fact 7.18 implies that

$$|S_{e'}| = |c^{\subseteq}(e' \cap [k])| = \frac{m}{r - |e' \cap [k]|}\binom{k - |e' \cap [k]|}{r - 1 - |e' \cap [k]|} = |M_{k,m}^{(F^*,e^*)}(e')|,$$

as required. □

Next, we establish the existence of suitable strong regular colourings. As a tool we need the following result about decompositions of very dense multi-r-graphs (which we will apply with $r-1$ playing the role of r). We omit the proof as it is essentially the same as that of Corollary 8.16 in [12].

Lemma 7.20. *Let $r \in \mathbb{N}$ and assume that $(*)_r$ is true. Let $1/n \ll 1/h, 1/f$ with $f > r$, let F be a weakly regular r-graph on f vertices and assume that $K_n^{(r)}$ is F-divisible. Let $m \in \mathbb{N}$. Suppose that H is an F-divisible multi-r-graph on $[h]$ with multiplicity at most $m-1$ and let K be the complete multi-r-graph on $[n]$ with multiplicity m. Then $K - H$ has an F-decomposition.*

The next lemma guarantees the existence of a suitable strong regular colouring. For this, we apply Lemma 7.20 to the shadow of F. For an r-graph F, define the *shadow F^{sh} of F* to be the $(r-1)$-graph on $V(F)$ where an $(r-1)$-set S is an edge if and only if $|F(S)| > 0$. We need the following fact.

Fact 7.21. *If F is a weakly $(s_0, \dots, s_{r-1})$-regular r-graph, then F^{sh} is a weakly $(s_0', \dots, s_{r-2}')$-regular $(r-1)$-graph, where $s_i' := \frac{r-i}{s_{r-1}} s_i$ for all $i \in [r-2]_0$.*

Proof. Let $i \in [r-2]_0$. For every $T \in \binom{V(F)}{i}$, we have $|F^{sh}(T)| = \frac{r-i}{s_{r-1}}|F(T)|$ since every edge of F which contains T contains $r-i$ edges of F^{sh} which contain T, but each such edge of F^{sh} is contained in s_{r-1} such edges of F. This implies the claim. □

Lemma 7.22. *Let $r \geq 2$ and assume that $(*)_{r-1}$ holds. Let F be a weakly regular r-graph. Then for all $h \in \mathbb{N}$, there exist $k, m \in \mathbb{N}$ such that for any F-divisible r-graph H on at most h vertices, there exists $t \in \mathbb{N}$ such that $H + t \cdot F$ has a strong m-regular $[k]$-colouring.*

Proof. Let $f := |V(F)|$ and suppose that F is weakly $(s_0, \dots, s_{r-1})$-regular. Thus, for every $S \in \binom{V(F)}{r-1}$, we have

$$|F(S)| = \begin{cases} s_{r-1} & \text{if } S \in F^{sh}; \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

By Proposition 4.1, we can choose $k \in \mathbb{N}$ such that $1/k \ll 1/h, 1/f$ and such that $K_k^{(r-1)}$ is F^{sh}-divisible. Let G be the complete multi-$(r-1)$-graph on $[k]$ with multiplicity $m' := h+1$ and let $m := s_{r-1}m'$.

Let H be any F-divisible r-graph on at most h vertices. By adding isolated vertices to H if necessary, we may assume that $V(H) = [h]$. We first define a multi-$(r-1)$-graph H' on $[h]$ as follows: For each $S \in \binom{[h]}{r-1}$, let the multiplicity of S in H' be $|H'(S)| := |H(S)|$. Clearly, H' has multiplicity at most h. Observe that for each $S \subseteq [h]$ with $|S| \leq r-1$, we have

$$|H'(S)| = (r - |S|)|H(S)|. \tag{7.3}$$

Note that since H is F-divisible, we have that $s_{r-1} \mid |H(S)|$ for all $S \in \binom{[h]}{r-1}$. Thus, the multiplicity of each $S \in \binom{[h]}{r-1}$ in H' is divisible by s_{r-1}. Let H'' be the multi-$(r-1)$-graph on $[h]$ obtained from H' by dividing the multiplicity of each $S \in \binom{[h]}{r-1}$ by s_{r-1}. Hence, by (7.3), for all $S \subseteq [h]$ with $|S| \leq r-1$, we have

$$|H''(S)| = \frac{|H'(S)|}{s_{r-1}} = \frac{r-|S|}{s_{r-1}}|H(S)|. \tag{7.4}$$

For each $S \in \binom{[k]}{r-1}$ with $S \not\subseteq [h]$, we set $|H''(S)| := |H(S)| := 0$. Then (7.4) still holds.

We claim that H'' is F^{sh}-divisible. Recall that by Fact 7.21,

$$F^{sh} \text{ is weakly } (\frac{r}{s_{r-1}}s_0, \dots, \frac{r-i}{s_{r-1}}s_i, \dots, \frac{2}{s_{r-1}}s_{r-2})\text{-regular.}$$

Let $i \in [r-2]_0$ and let $S \in \binom{[h]}{i}$. We need to show that $|H''(S)| \equiv 0 \mod Deg(F^{sh})_i$, where $Deg(F^{sh})_i = \frac{r-i}{s_{r-1}}s_i$. Since H is F-divisible, we have $|H(S)| \equiv 0 \mod s_i$. Together with (7.4), we deduce that $|H''(S)| \equiv 0 \mod \frac{r-i}{s_{r-1}}s_i$. Hence, H'' is F^{sh}-divisible. Therefore, by Lemma 7.20 (with $k, m', r-1, F^{sh}$ playing the roles of n, m, r, F) and our choice of k, $G - H''$ has an F^{sh}-decomposition $\mathcal{F}$ into t edge-disjoint copies $F_1', \dots, F_t'$ of F^{sh}.

We will show that t is as required in Lemma 7.22. To do this, let $F_1, \dots, F_t$ be vertex-disjoint copies of F which are also vertex-disjoint from H. We will now define a strong m-regular $[k]$-colouring c of

$$H^+ := H \cup \bigcup_{j \in [t]} F_j.$$

Let c_0 be the identity map on $V(H) = [h]$, and for each $j \in [t]$, let

$$c_j \colon V(F_j) \to V(F_j') \text{ be an isomorphism from } F_j^{sh} \text{ to } F_j' \tag{7.5}$$

(recall that $V(F_j^{sh}) = V(F_j)$). Since $H, F_1, \dots, F_t$ are vertex-disjoint and $V(H) \cup \bigcup_{j \in [t]} V(F_j') \subseteq [k]$, we can combine $c_0, c_1, \dots, c_t$ to a map

$$c \colon V(H^+) \to [k],$$

i.e. for $x \in V(H^+)$, we let $c(x) := c_j(x)$, where either j is the unique index for which $x \in V(F_j)$ or $j = 0$ if $x \in V(H)$. For every edge $e \in H^+$, we have $e \subseteq V(H)$ or $e \subseteq V(F_j)$ for some $j \in [t]$, thus $c\restriction_e$ is injective. Therefore, c is a strong $[k]$-colouring of H^+.

It remains to check that c is m-regular. Let $C \in \binom{[k]}{r-1}$. Clearly, $|c^{\subseteq}(C)| = \sum_{j=0}^{t} |c_j^{\subseteq}(C)|$. Since every c_j is a bijection, we have

$$|c_0^{\subseteq}(C)| = |\{e \in H \,:\, c_0^{-1}(C) \subseteq e\}| = |H(c_0^{-1}(C))| = |H(C)| \quad \text{and}$$
$$|c_j^{\subseteq}(C)| = |F_j(c_j^{-1}(C))| \overset{(7.2)}{=} \begin{cases} s_{r-1} & \text{if } c_j^{-1}(C) \in F_j^{sh} \overset{(7.5)}{\Leftrightarrow} C \in F_j'; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have $|c^{\subseteq}(C)| = |H(C)| + s_{r-1}|J(C)|$, where

$$J(C) := \{j \in [t] \,:\, C \in F_j'\}.$$

Now crucially, since $\mathcal{F}$ is an F^{sh}-decomposition of $G - H''$, we have that $|J(C)|$ is equal to the multiplicity of C in $G - H''$, i.e. $|J(C)| = m' - |H''(C)|$. Thus,

$$|c^{\subseteq}(C)| = |H(C)| + s_{r-1}|J(C)| \overset{(7.4)}{=} s_{r-1}(|H''(C)| + |J(C)|) = s_{r-1}m' = m,$$

completing the proof. □

Before we can prove Lemma 7.12, we need to show the existence of a symmetric r-extender F^* which is F-decomposable. For some F we could actually take $F^* = F$ (e.g. if F is a clique). For general (weakly regular) r-graphs F, we will use the Cover down lemma (Lemma 6.9) to find F^*. At first sight, appealing to the Cover down lemma may seem rather heavy handed, but a direct construction seems to be quite difficult.

Lemma 7.23. *Let F be a weakly regular r-graph, $e_0 \in F$ and assume that $(*)_i$ is true for all $i \in [r-1]$. There exists a symmetric r-extender (F^*, e^*) such that F^* has an F-decomposition $\mathcal{F}$ with $e^* \in F' \in \mathcal{F}$ and e^* plays the role of e_0 in F'.*

Proof. Let $f := |V(F)|$. By Proposition 4.1, we can choose $n \in \mathbb{N}$ and $\gamma, \varepsilon, \nu, \mu > 0$ such that $1/n \ll \gamma \ll \varepsilon \ll \nu \ll \mu \ll 1/f$ and such that $K_n^{(r)}$ is F-divisible. By Example 3.3, K_n is a $(0, 0.99/f!, f, r)$-supercomplex. By Fact 6.5(i) and Proposition 6.6, there exists $U \subseteq V(K_n)$ of size $\lfloor \mu n \rfloor$ which is $(\varepsilon, \mu, 0.9/f!, f, r)$-random in K_n. Let $\bar{U} := V(K_n) \setminus U$. Using (R2) of Definition 6.1, it is easy to see that K_n is (ε, f, r)-dense with respect to $K_n^{(r)} - K_n^{(r)}[\bar{U}]$ (see Definition 6.8). Thus, by the Cover down lemma (Lemma 6.9), there exists a subgraph H^* of $K_n^{(r)} - K_n^{(r)}[\bar{U}]$ with $\Delta(H^*) \le \nu n$ and the following property: for all $L \subseteq K_n^{(r)}$ such that $\Delta(L) \le \gamma n$ and $H^* \cup L$ is F-divisible, $H^* \cup L$ has an F-packing which covers all edges except possibly some inside U.

Let F' be a copy of F with $V(F') \subseteq \bar{U}$. Let $G_{nibble} := K_n - H^* - F'$. By Proposition 4.5(v), G_{nibble} is a $(2^{2r+2}\nu, 0.8/f!, f, r)$-supercomplex. Thus, by Lemma 5.3, there exists an F-packing $\mathcal{F}_{nibble}$ in $G_{nibble}^{(r)}$ such that $\Delta(L) \le \gamma n$, where $L := G_{nibble}^{(r)} - \mathcal{F}_{nibble}^{(r)}$. Clearly, $H^* \cup L = K_n^{(r)} - \mathcal{F}_{nibble}^{(r)} - F'$ is F-divisible. Thus, there exists an F-packing $\mathcal{F}^*$ in $H^* \cup L$ which covers all edges of $H^* \cup L$ except possibly some inside U. Let $\mathcal{F} := \{F'\} \cup \mathcal{F}_{nibble} \cup \mathcal{F}^*$. Let $F^* := \mathcal{F}^{(r)}$ and let e^* be the edge in F' which plays the role of e_0.

Clearly, $\mathcal{F}$ is an F-decomposition of F^* with $e^* \in F' \in \mathcal{F}$ and e^* plays the role of e_0 in F'. It remains to check (SE). Let $e' \in \binom{V(K_n^{(r)})}{r}$ with $e' \cap e^* \neq \emptyset$. Since $e^* \subseteq \bar{U}$, e' cannot be inside U. Thus, e' is covered by $\mathcal{F}$ and we have $e' \in F^*$. □

Note that $|V(F^*)|$ is quite large here, in particular $1/|V(F^*)| \ll 1/f$ for $f = |V(F)|$. This means that G being an (ε, ξ, f, r)-supercomplex does not necessarily allow us to extend a given subgraph H of $G^{(r)}$ to a copy of $\nabla_{(F^*, e^*)} H$ as described in Definition 7.13. Fortunately, this will in fact not be necessary, as F^* will only serve as an abstract auxiliary graph and will not appear as a subgraph of the absorber. (This is crucial since otherwise we would not be able to prove our main theorems with explicit bounds, let alone the bounds given in Theorem 1.4.)

We are now ready to prove Lemma 7.12.

Proof of Lemma 7.12. Given F and e_0, we first apply Lemma 7.23 to obtain a symmetric r-extender (F^*, e^*) such that F^* has an F-decomposition $\mathcal{F}$ with $e^* \in F' \in \mathcal{F}$ and e^* plays the role of e_0 in F'. For given $h \in \mathbb{N}$, let $k, m \in \mathbb{N}$ be as in Lemma 7.22. Clearly, we may assume that there exists an F-divisible r-graph on at most h vertices. Together with Lemma 7.19, this implies that $(k, m) \in \mathcal{M}_r$. Define

$$M_h := M_{k,m}^{(F^*,e^*)}.$$

Now, let H be any F-divisible r-graph on at most h vertices. By Lemma 7.22, there exists $t \in \mathbb{N}$ such that $H + t \cdot F$ has a strong m-regular $[k]$-colouring. By Lemma 7.19, we have

$$\nabla_{(F^*,e^*)}(H + t \cdot F) \rightsquigarrow\!\!\!\rightsquigarrow M_h.$$

Let ψ_1 be any e_0-orientation of $H + t \cdot F$. Observe that since e^* plays the role of e_0 in F', $\nabla_{(F^*,e^*)}(H + t \cdot F)$ can be decomposed into a copy of $\nabla_{(F,e_0)}(H + t \cdot F, \psi_1)$ and s copies of F (where $s = |H + t \cdot F| \cdot |\mathcal{F} \setminus \{F'\}|$). Hence, we have

$$\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F \rightsquigarrow \nabla_{(F^*,e^*)}(H + t \cdot F)$$

by Proposition 7.4(ii). Thus, $\nabla_{(F,e_0)}(H+t\cdot F, \psi_1)+s\cdot F \rightsquigarrow\!\!\!\rightsquigarrow M_h$ by transitivity of $\rightsquigarrow\!\!\!\rightsquigarrow$. Finally, let ψ_2 be any e_0-orientation of M_h. By Fact 7.15, there exists an e_0-orientation ψ_3 of $\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F$ such that

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F, \psi_3) \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi_2).$$

□

7.3. **Proof of the Absorbing lemma.** As discussed at the beginning of Section 7.2, we can now combine Lemma 7.5 and Lemma 7.12 to construct the desired absorber by concatenating transformers between certain auxiliary r-graphs, in particular the extension ∇M_h of the canonical multi-r-graph M_h. It is relatively straightforward to find these auxiliary r-graphs within a given supercomplex G. The step when we need to find ∇M_h is the reason why the definition of a supercomplex includes the notion of extendability.

Proof of Lemma 7.2. If H is empty, then we can take A to be empty, so let us assume that H is not empty. In particular, $G^{(r)}$ is not empty. Recall also that we assume $r \geq 2$. Let $e_0 \in F$ and let M_h be as in Lemma 7.12. Fix any e_0-orientation ψ of M_h. By Lemma 7.12, there exist $t_1, t_2, s_1, s_2, \psi_1, \psi_2, \psi_1', \psi_2'$ such that

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H + t_1 \cdot F, \psi_1) + s_1 \cdot F, \psi_1') \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi); \tag{7.6}$$

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(t_2 \cdot F, \psi_2) + s_2 \cdot F, \psi_2') \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi). \tag{7.7}$$

We can assume that $1/n \ll 1/\ell$ where $\ell := \max\{|V(M_h)|, t_1, t_2, s_1, s_2\}$.

Since G is $(\xi, f + r, r)$-dense, there exist disjoint $Q_{1,1}, \dots, Q_{1,t_1}, Q_{2,1}, \dots, Q_{2,t_2} \in G^{(f)}$ which are also disjoint from $V(H)$. For $i \in [2]$ and $j \in [t_i]$, let $F_{i,j}$ be a copy of F with $V(F_{i,j}) = Q_{i,j}$. Let $H_1 := H \cup \bigcup_{j\in[t_1]} F_{1,j}$ and $H_2 := \bigcup_{j\in[t_2]} F_{2,j}$ and for $i \in [2]$, define

$$\mathcal{F}_i := \{F_{i,j} \,:\, j \in [t_i]\}.$$

So H_1 is a copy of $H + t_1 \cdot F$ and H_2 is a copy of $t_2 \cdot F$. In fact, we will from now on assume (by redefining ψ_i and ψ_i') that for $i \in [2]$, we have

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H_i, \psi_i) + s_i \cdot F, \psi_i') \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi). \tag{7.8}$$

For $i \in [2]$, let $(H_i', \mathcal{F}_i')$ be obtained by extending H_i with a copy of $\nabla_{(F,e_0)}(H_i, \psi_i)$ in G (cf. Definition 7.13). We can assume that H_1' and H_2' are vertex-disjoint by first choosing H_1' whilst avoiding $V(H_2)$ and subsequently choosing H_2' whilst avoiding $V(H_1')$. (To see that this is possible we can e.g. use the fact that G is (ε, d, f, r)-regular for some $d \geq \xi$.)

There exist disjoint $Q_{1,1}', \dots, Q_{1,s_1}', Q_{2,1}', \dots, Q_{2,s_2}' \in G^{(f)}$ which are also disjoint from $V(H_1') \cup V(H_2')$. For $i \in [2]$ and $j \in [s_i]$, let $F_{i,j}'$ be a copy of F with $V(F_{i,j}') = Q_{i,j}'$. For $i \in [2]$, let

$$H_i'' := H_i' \cup \bigcup_{j\in[s_i]} F_{i,j}';$$

$$\mathcal{F}_i'' := \{F_{i,j}' \,:\, j \in [s_i]\}.$$

Since H_i'' is a copy of $\nabla_{(F,e_0)}(H_i,\psi_i)+s_i\cdot F$, we can assume (by redefining ψ_i') that

$$\nabla_{(F,e_0)}(H_i'',\psi_i') \rightsquigarrow \nabla_{(F,e_0)}(M_h,\psi). \tag{7.9}$$

For $i\in[2]$, let $(H_i''',\mathcal{F}_i''')$ be obtained by extending H_i'' with a copy of $\nabla_{(F,e_0)}(H_i'',\psi_i')$ in G (cf. Definition 7.13). We can assume that H_1''' and H_2''' are vertex-disjoint.

Since G is (ξ,f,r)-extendable, it is straightforward to find a copy M' of $\nabla_{(F,e_0)}(M_h,\psi)$ in $G^{(r)}$ which is vertex-disjoint from H_1''' and H_2'''.

Since H_i''' is a copy of $\nabla_{(F,e_0)}(H_i'',\psi_i')$, by (7.9) we have $H_i'''\rightsquigarrow M'$ for $i\in[2]$. Using Fact 7.11(ii) repeatedly, we can see that both H_1''' and H_2''' are F-divisible. Together with Proposition 7.4(iii), this implies that M' is F-divisible as well.

Let $T_1:=(H_1-H)\cup H_1''$ and $T_2:=H_2\cup H_2''$. For $i\in[2]$, let

$$\mathcal{F}_{i,1}:=\mathcal{F}_i'\cup\mathcal{F}_i'' \text{ and } \mathcal{F}_{i,2}:=\mathcal{F}_i\cup\mathcal{F}_i'''.$$

We claim that $\mathcal{F}_{1,1},\mathcal{F}_{1,2},\mathcal{F}_{2,1},\mathcal{F}_{2,2}$ are 2-well separated F-packings in G such that

$$\mathcal{F}_{1,1}^{(r)}=T_1\cup H,\quad \mathcal{F}_{1,2}^{(r)}=T_1\cup H_1''',\quad \mathcal{F}_{2,2}^{(r)}=T_2\cup H_2''' \quad\text{and}\quad \mathcal{F}_{2,1}^{(r)}=T_2. \tag{7.10}$$

(In particular, T_1 is a 2-well separated $(H,H_1''';F)$-transformer in G and T_2 is a 2-well separated $(H_2''',\emptyset;F)$-transformer in G.) Indeed, we clearly have that $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_1'',\mathcal{F}_2''$ are 1-well separated F-packings in G, where $\mathcal{F}_1^{(r)}=H_1-H$, $\mathcal{F}_2^{(r)}=H_2$, and for $i\in[2]$, $\mathcal{F}_i''^{(r)}=H_i''-H_i'$. Moreover, by Fact 7.14, for $i\in[2]$, $\mathcal{F}_i'$ and $\mathcal{F}_i'''$ are 1-well separated F-packings in G with $\mathcal{F}_i'^{(r)}=H_i\cup H_i'$ and $\mathcal{F}_i'''^{(r)}=H_i''\cup H_i'''$. Note that

$$\begin{aligned} T_1\cup H &= H_1\cup H_1''=(H_1\cup H_1')\mathbin{\dot\cup}(H_1''-H_1')=\mathcal{F}_1'^{(r)}\mathbin{\dot\cup}\mathcal{F}_1''^{(r)}=\mathcal{F}_{1,1}^{(r)};\\ T_1\cup H_1''' &= (H_1-H)\mathbin{\dot\cup}(H_1''\cup H_1''')=\mathcal{F}_1^{(r)}\mathbin{\dot\cup}\mathcal{F}_1'''^{(r)}=\mathcal{F}_{1,2}^{(r)};\\ T_2\cup H_2''' &= H_2\mathbin{\dot\cup}(H_2''\cup H_2''')=\mathcal{F}_2^{(r)}\mathbin{\dot\cup}\mathcal{F}_2'''^{(r)}=\mathcal{F}_{2,2}^{(r)};\\ T_2 &= H_2\cup H_2''=(H_2\cup H_2')\mathbin{\dot\cup}(H_2''-H_2')=\mathcal{F}_2'^{(r)}\mathbin{\dot\cup}\mathcal{F}_2''^{(r)}=\mathcal{F}_{2,1}^{(r)}. \end{aligned}$$

To check that $\mathcal{F}_{1,1}$, $\mathcal{F}_{1,2}$, $\mathcal{F}_{2,1}$ and $\mathcal{F}_{2,2}$ are 2-well separated F-packings, by Fact 4.3(ii) it is now enough to show for $i\in[2]$ that $\mathcal{F}_i'$ and $\mathcal{F}_i''$ are $(r+1)$-disjoint and that $\mathcal{F}_i$ and $\mathcal{F}_i'''$ are $(r+1)$-disjoint. Note that for all $F'\in\mathcal{F}_i'$ and $F''\in\mathcal{F}_i''$, we have $V(F')\subseteq V(H_i')$ and $V(F'')\cap V(H_i')=\emptyset$, thus $V(F')\cap V(F'')=\emptyset$. For all $F'\in\mathcal{F}_i$ and $F''\in\mathcal{F}_i'''$, we have $V(F')\subseteq V(H_i)$ and $|V(F'')\cap V(H_i)|\le|V(F'')\cap V(H_i'')|\le r$ by Fact 7.14, thus $|V(F')\cap V(F'')|\le r$. This completes the proof of (7.10).

Let

$$\begin{aligned} O_r &:= H_1\cup H_1''\cup H_2\cup H_2'';\\ O_{r+1,3} &:= \mathcal{F}_{1,1}^{\le(r+1)}\cup\mathcal{F}_{1,2}^{\le(r+1)}\cup\mathcal{F}_{2,1}^{\le(r+1)}\cup\mathcal{F}_{2,2}^{\le(r+1)}. \end{aligned}$$

By Fact 4.3(i), $\Delta(O_{r+1,3})\le 8(f-r)$. Note that $H_1''',M'\subseteq G^{(r)}-(O_r\cup H_2''')$. Thus, by Proposition 4.5(v) and Lemma 7.5, there exists a $(\kappa/3)$-well separated $(H_1''',M';F)$-transformer T_3 in $G-(O_r\cup H_2''')-O_{r+1,3}$ with $\Delta(T_3)\le\gamma n/3$. Let $\mathcal{F}_{3,1}$ and $\mathcal{F}_{3,2}$ be $(\kappa/3)$-well separated F-packings in $G-(O_r\cup H_2''')-O_{r+1,3}$ such that $\mathcal{F}_{3,1}^{(r)}=T_3\cup H_1'''$ and $\mathcal{F}_{3,2}^{(r)}=T_3\cup M'$.

Similarly, let $O_{r+1,4}:=O_{r+1,3}\cup\mathcal{F}_{3,1}^{\le(r+1)}\cup\mathcal{F}_{3,2}^{\le(r+1)}$. By Fact 4.3(i), $\Delta(O_{r+1,4})\le(8+2\kappa/3)(f-r)$. Note that $H_2''',M'\subseteq G^{(r)}-(O_r\cup H_1'''\cup T_3)$. Using Proposition 4.5(v) and Lemma 7.5 again, we can find a $(\kappa/3)$-well separated $(H_2''',M';F)$-transformer T_4 in $G-(O_r\cup H_1'''\cup T_3)-O_{r+1,4}$ with $\Delta(T_4)\le\gamma n/3$. Let $\mathcal{F}_{4,1}$ and $\mathcal{F}_{4,2}$ be $(\kappa/3)$-well separated F-packings in $G-(O_r\cup H_1'''\cup T_3)-O_{r+1,4}$ such that of $\mathcal{F}_{4,1}^{(r)}=T_4\cup H_2'''$ and $\mathcal{F}_{4,2}^{(r)}=T_4\cup M'$.

Let

$$\begin{aligned} A &:= T_1\mathbin{\dot\cup}H_1'''\mathbin{\dot\cup}T_3\mathbin{\dot\cup}M'\mathbin{\dot\cup}T_4\mathbin{\dot\cup}H_2'''\mathbin{\dot\cup}T_2;\\ \mathcal{F}_\circ &:= \mathcal{F}_{1,2}\cup\mathcal{F}_{3,2}\cup\mathcal{F}_{4,1}\cup\mathcal{F}_{2,1};\\ \mathcal{F}_\bullet &:= \mathcal{F}_{1,1}\cup\mathcal{F}_{3,1}\cup\mathcal{F}_{4,2}\cup\mathcal{F}_{2,2}. \end{aligned}$$

Clearly, $A \subseteq G^{(r)}$, and $\Delta(A) \leq \gamma n$. Moreover, A and H are edge-disjoint. Using (7.10), we can check that

$$\mathcal{F}_{\circ}^{(r)} = \mathcal{F}_{1,2}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{3,2}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{4,1}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{2,1}^{(r)} = (T_1 \cup H_1''') \mathbin{\dot\cup} (T_3 \cup M') \mathbin{\dot\cup} (T_4 \cup H_2''') \mathbin{\dot\cup} T_2 = A;$$

$$\mathcal{F}_{\bullet}^{(r)} = \mathcal{F}_{1,1}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{3,1}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{4,2}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{2,2}^{(r)} = (H \cup T_1) \mathbin{\dot\cup} (H_1''' \cup T_3) \mathbin{\dot\cup} (M' \cup T_4) \mathbin{\dot\cup} (H_2''' \cup T_2) = A \cup H.$$

By definition of $O_{r+1,3}$ and $O_{r+1,4}$, we have that $\mathcal{F}_{1,2}, \mathcal{F}_{3,2}, \mathcal{F}_{4,1}, \mathcal{F}_{2,1}$ are $(r+1)$-disjoint. Thus, $\mathcal{F}_{\circ}$ is a $(2\cdot\kappa/3+4)$-well separated F-packing in G by Fact 4.3(ii). Similarly, $\mathcal{F}_{\bullet}$ is a $(2\cdot\kappa/3+4)$-well separated F-packing in G. So A is indeed a κ-well separated F-absorber for H in G. □

8. Proof of the main theorems

8.1. Main complex decomposition theorem. We can now deduce our main decomposition result for supercomplexes. The main ingredients for the proof of Theorem 3.8 are Lemma 6.3 (to find a vortex), Lemma 7.2 (to find absorbers for the possible leftovers in the final vortex set), and Lemma 6.4 (to cover all edges outside the final vortex set).

Proof of Theorem 3.8. We proceed by induction on r. The case $r = 1$ forms the base case of the induction and in this case we do not rely on any inductive assumption. Suppose that $r \in \mathbb{N}$ and that $(*)_i$ is true for all $i \in [r-1]$.

We may assume that $1/n \ll 1/\kappa \ll \varepsilon$. Choose new constants $\kappa', m' \in \mathbb{N}$ and $\gamma, \mu > 0$ such that

$$1/n \ll 1/\kappa \ll \gamma \ll 1/m' \ll 1/\kappa' \ll \varepsilon \ll \mu \ll \xi, 1/f$$

and suppose that F is a weakly regular r-graph on $f > r$ vertices.

Let G be an F-divisible (ε, ξ, f, r)-supercomplex on n vertices. We are to show the existence of a κ-well separated F-decomposition of G. By Lemma 6.3, there exists a $(2\sqrt{\varepsilon}, \mu, \xi-\varepsilon, f, r, m)$-vortex $U_0, U_1, \dots, U_\ell$ in G for some $\mu m' \leq m \leq m'$. Let $H_1, \dots, H_s$ be an enumeration of all spanning F-divisible subgraphs of $G[U_\ell]^{(r)}$. Clearly, $s \leq 2^{\binom{m}{r}}$. We will now find edge-disjoint subgraphs $A_1, \dots, A_s$ of $G^{(r)}$ and $\sqrt{\kappa}$-well separated F-packings $\mathcal{F}_{1,\circ}, \mathcal{F}_{1,\bullet}, \dots, \mathcal{F}_{s,\circ}, \mathcal{F}_{s,\bullet}$ in G such that for all $i \in [s]$ we have that

- (A1) $\mathcal{F}_{i,\circ}^{(r)} = A_i$ and $\mathcal{F}_{i,\bullet}^{(r)} = A_i \cup H_i$;
- (A2) $\Delta(A_i) \leq \gamma n$;
- (A3) $A_i[U_1]$ is empty;
- (A4) $\mathcal{F}_{i,\bullet}^{\leq}, G[U_1], \mathcal{F}_{1,\circ}^{\leq}, \dots, \mathcal{F}_{i-1,\circ}^{\leq}, \mathcal{F}_{i+1,\circ}^{\leq}, \dots, \mathcal{F}_{s,\circ}^{\leq}$ are $(r+1)$-disjoint.

Suppose that for some $t \in [s]$, we have already found edge-disjoint $A_1, \dots, A_{t-1}$ together with $\mathcal{F}_{1,\circ}, \mathcal{F}_{1,\bullet}, \dots, \mathcal{F}_{t-1,\circ}, \mathcal{F}_{t-1,\bullet}$ that satisfy (A1)–(A4) (with $t-1$ playing the role of s). Let

$$T_t := (G^{(r)}[U_1] - H_t) \cup \bigcup_{i \in [t-1]} A_i;$$

$$T_t' := G^{(r+1)}[U_1] \cup \bigcup_{i \in [t-1]} (\mathcal{F}_{i,\circ}^{\leq(r+1)} \cup \mathcal{F}_{i,\bullet}^{\leq(r+1)}).$$

Clearly, $\Delta(T_t) \leq \mu n + s\gamma n \leq 2\mu n$ by (V2) and (A2). Also, $\Delta(T_t') \leq \mu n + 2s\sqrt{\kappa}(f-r) \leq 2\mu n$ by (V2) and Fact 4.3(i). Thus, applying Proposition 4.5(v) twice we see that $G_{abs,t} := G - T_t - T_t'$ is still a $(\sqrt{\mu}, \xi/2, f, r)$-supercomplex. Moreover, $H_t \subseteq G_{abs,t}^{(r)}$ by (A3). Hence, by Lemma 7.2, there exists a $\sqrt{\kappa}$-well separated F-absorber A_t for H_t in $G_{abs,t}$ with $\Delta(A_t) \leq \gamma n$. Let $\mathcal{F}_{t,\circ}$ and $\mathcal{F}_{t,\bullet}$ be $\sqrt{\kappa}$-well separated F-packings in $G_{abs,t} \subseteq G$ such that $\mathcal{F}_{t,\circ}^{(r)} = A_t$ and $\mathcal{F}_{t,\bullet}^{(r)} = A_t \cup H_t$. Clearly, A_t is edge-disjoint from $A_1, \dots, A_{t-1}$. Moreover, (A3) holds since $G_{abs,t}^{(r)}[U_1] = H_t$ and A_t is edge-disjoint from H_t, and (A4) holds with t playing the role of s due to the definition of T_t'.

Let $A^* := A_1 \cup \dots \cup A_s$ and $T^* := \bigcup_{i \in [s]} (\mathcal{F}_{i,\circ}^{\leq(r+1)} \cup \mathcal{F}_{i,\bullet}^{\leq(r+1)})$. We claim that the following hold:

(A1′) for every F-divisible subgraph H^* of $G[U_\ell]^{(r)}$, $A^* \cup H^*$ has an $s\sqrt{\kappa}$-well separated F-decomposition $\mathcal{F}^*$ with $\mathcal{F}^{*\leq} \subseteq G[T^*]$;
(A2′) $\Delta(A^*) \leq \varepsilon n$ and $\Delta(T^*) \leq 2s\sqrt{\kappa}(f-r) \leq \varepsilon n$;
(A3′) $A^*[U_1]$ and $T^*[U_1]$ are empty.

For (A1′), we have that $H^* = H_t$ for some $t \in [s]$. Then $\mathcal{F}^* := \mathcal{F}_{t,\bullet} \cup \bigcup_{i\in[s]\setminus\{t\}} \mathcal{F}_{i,\circ}$ is an F-decomposition of $A^* \cup H^* = (A_t \cup H_t) \cup \bigcup_{i\in[s]\setminus\{t\}} A_i$ by (A1) and since $H_t, A_1, \dots, A_s$ are pairwise edge-disjoint. By (A4) and Fact 4.3(ii), $\mathcal{F}^*$ is $s\sqrt{\kappa}$-well separated. We clearly have $\mathcal{F}^{*\leq} \subseteq G$ and $\mathcal{F}^{*\leq(r+1)} \subseteq T^*$. Thus $\mathcal{F}^{*\leq} \subseteq G[T^*]$ and so (A1′) holds. It is straightforward to check that (A2′) follows from (A2) and Fact 4.3(i), and that (A3′) follows from (A3) and (A4).

Let $G_{almost} := G - A^* - T^*$. By (A2′) and Proposition 4.5(v), G_{almost} is an $(\sqrt{\varepsilon}, \xi/2, f, r)$-supercomplex. Moreover, since A^* must be F-divisible, we have that G_{almost} is F-divisible. By (A3′), $U_1, \dots, U_\ell$ is a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, f, r, m)$-vortex in $G_{almost}[U_1]$. Moreover, (A2′) and Proposition 6.7 imply that U_1 is $(\varepsilon^{1/5}, \mu, \xi/2, f, r)$-random in G_{almost} and $U_1 \setminus U_2$ is $(\varepsilon^{1/5}, \mu(1-\mu), \xi/2, f, r)$-random in G_{almost}. Hence, $U_0, U_1, \dots, U_\ell$ is still an $(\varepsilon^{1/5}, \mu, \xi/2, f, r, m)$-vortex in G_{almost}. Thus, by Lemma 6.4, there exists a $4\kappa'$-well separated F-packing $\mathcal{F}_{almost}$ in G_{almost} which covers all edges of $G^{(r)}_{almost}$ except possibly some inside U_ℓ. Let $H^* := (G^{(r)}_{almost} - \mathcal{F}^{(r)}_{almost})[U_\ell]$. Since H^* is F-divisible, $A^* \cup H^*$ has an $s\sqrt{\kappa}$-well separated F-decomposition $\mathcal{F}^*$ with $\mathcal{F}^{*\leq} \subseteq G[T^*]$ by (A1′). Clearly,

$$G^{(r)} = G^{(r)}_{almost} \mathbin{\dot\cup} A^* = \mathcal{F}^{(r)}_{almost} \mathbin{\dot\cup} H^* \mathbin{\dot\cup} A^* = \mathcal{F}^{(r)}_{almost} \mathbin{\dot\cup} \mathcal{F}^{*(r)},$$

and $\mathcal{F}_{almost}$ and $\mathcal{F}^*$ are $(r+1)$-disjoint. Thus, by Fact 4.3(ii), $\mathcal{F}_{almost} \cup \mathcal{F}^*$ is a $(4\kappa' + s\sqrt{\kappa})$-well separated F-decomposition of G, completing the proof. □

8.2. **Resolvable partite designs.** Perhaps surprisingly, it is much easier to obtain decompositions of complete partite r-graphs than of complete (non-partite) r-graphs. In fact, we can obtain (explicit) resolvable decompositions (sometimes referred to as *Kirkman systems* or *large sets of designs*) in the partite setting using basic linear algebra. We believe that this result and the corresponding construction are of independent interest. Here, we will use this result to show that for every r-graph F, there is a weakly regular r-graph F^* which is F-decomposable (see Lemma 8.2).

Let G be a complex. We say that a $K^{(r)}_f$-decomposition $\mathcal{K}$ of G is *resolvable* if $\mathcal{K}$ can be partitioned into $K^{(r-1)}_f$-decompositions of G, that is, $\mathcal{K}^{\leq(f)}$ can be partitioned into sets $Y_1, \dots, Y_t$ such that for each $i \in [t]$, $\mathcal{K}_i := \{G^{(r-1)}[Q] : Q \in Y_i\}$ is a $K^{(r-1)}_f$-decomposition of G. Clearly, $\mathcal{K}_1, \dots, \mathcal{K}_t$ are r-disjoint.

Let $K_{n\times k}$ be the complete k-partite complex with each vertex class having size n. More precisely, $K_{n\times k}$ has vertex set $V_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_k$ such that $|V_i| = n$ for all $i \in [k]$ and $e \in K_{n\times k}$ if and only if e is *crossing*, that is, intersects with each V_i in at most one vertex. Since every subset of a crossing set is crossing, this defines a complex.

Theorem 8.1. *Let q be a prime power and $2f \leq q$. Then for every $r \in [f-1]$, $K_{q\times f}$ has a resolvable $K^{(r)}_f$-decomposition.*

Let us first motivate the proof of Theorem 8.1. Let $\mathbb{F}$ be the finite field of order q. Assume that each class of $K_{q\times f}$ is a copy of $\mathbb{F}$. Suppose further that we are given a matrix $A \in \mathbb{F}^{(f-r)\times f}$ with the property that every $(f-r)\times(f-r)$-submatrix is invertible. Identifying $K^{(f)}_{q\times f}$ with $\mathbb{F}^f$ in the obvious way, we let $\mathcal{K}$ be the set of all $Q \in K^{(f)}_{q\times f}$ with $AQ = 0$. Fixing the entries of r coordinates of Q (which can be viewed as fixing an r-set) transforms this into an equation $A'Q' = b'$, where A' is an $(f-r)\times(f-r)$-submatrix of A. Thus, there exists a unique solution, which will translate into the fact that every r-set of $K_{q\times f}$ is contained in exactly one f-set of $\mathcal{K}$, i.e. we have a $K^{(r)}_f$-decomposition.

There are several known classes of matrices over finite fields which have the desired property that every square submatrix is invertible. We use so-called Cauchy matrices, introduced by

Cauchy [6], which are very convenient for our purposes. For an application of Cauchy matrices to coding theory, see e.g. [5].

Let $\mathbb{F}$ be a field and let $x_1,\dots,x_m,y_1,\dots,y_n$ be distinct elements of $\mathbb{F}$. The *Cauchy matrix generated by* $(x_i)_{i\in[m]}$ *and* $(y_j)_{j\in[n]}$ is the $m\times n$-matrix $A\in\mathbb{F}^{m\times n}$ defined by $a_{i,j}:=(x_i-y_j)^{-1}$. Obviously, every submatrix of a Cauchy matrix is itself a Cauchy matrix. For $m=n$, it is well known that the Cauchy determinant is given by the following formula (cf. [28]):

$$\det(A)=\frac{\prod_{1\le i<j\le n}(x_j-x_i)(y_i-y_j)}{\prod_{1\le i,j\le n}(x_i-y_j)}.$$

In particular, every square Cauchy matrix is invertible.

Proof of Theorem 8.1. Let $\mathbb{F}$ be the finite field of order q. Since $2f\le q$, there exists a Cauchy matrix $A\in\mathbb{F}^{(f-r+1)\times f}$. Let $\mathbf{\hat{a}}$ be the final row of A and let $A'\in\mathbb{F}^{(f-r)\times f}$ be obtained from A by deleting $\mathbf{\hat{a}}$.

We assume that the vertex set of $K_{q\times f}$ is $\mathbb{F}\times[f]$. Hence, for every $e\in K_{q\times f}$, there are unique $1\le i_1<\dots<i_{|e|}\le f$ and $x_1,\dots,x_{|e|}\in\mathbb{F}$ such that $e=\{(x_j,i_j)\,:\,j\in[|e|]\}$. Let

$$I_e:=\{i_1,\dots,i_{|e|}\}\subseteq[f]\quad\text{and}\quad \mathbf{x_e}:=\begin{pmatrix}x_1\\ \vdots\\ x_{|e|}\end{pmatrix}\in\mathbb{F}^{|e|}.$$

Clearly, $Q\in K^{(f)}_{q\times f}$ is uniquely determined by $\mathbf{x_Q}$.

Define $Y\subseteq K^{(f)}_{q\times f}$ as the set of all $Q\in K^{(f)}_{q\times f}$ which satisfy $A'\cdot\mathbf{x_Q}=\mathbf{0}$. Moreover, for each $x^*\in\mathbb{F}$, define $Y_{x^*}\subseteq Y$ as the set of all $Q\in Y$ which satisfy $\mathbf{\hat{a}}\cdot\mathbf{x_Q}=x^*$. Clearly, $\{Y_{x^*}\,:\,x^*\in\mathbb{F}\}$ is a partition of Y. Let $\mathcal{K}:=\{K^{(r)}_{q\times f}[Q]\,:\,Q\in Y\}$ and $\mathcal{K}_{x^*}:=\{K^{(r-1)}_{q\times f}[Q]\,:\,Q\in Y_{x^*}\}$ for each $x^*\in\mathbb{F}$. We claim that $\mathcal{K}$ is a $K^{(r)}_f$-decomposition of $K_{q\times f}$ and that $\mathcal{K}_{x^*}$ is a $K^{(r-1)}_f$-decomposition of $K_{q\times f}$ for each $x^*\in\mathbb{F}$.

For $I\subseteq[f]$, let A_I be the $(f-r+1)\times|I|$-submatrix of A obtained by deleting the columns which are indexed by $[f]\setminus I$. Similarly, for $I\subseteq[f]$, let A'_I be the $(f-r)\times|I|$-submatrix of A' obtained by deleting the columns which are indexed by $[f]\setminus I$. Finally, for a vector $\mathbf{x}\in\mathbb{F}^f$ and $I\subseteq[f]$, let $\mathbf{x}_I\in\mathbb{F}^{|I|}$ be the vector obtained from $\mathbf{x}$ by deleting the coordinates not in I.

Observe that for all $e\in K_{q\times f}$ and $Q\in K^{(f)}_{q\times f}$, we have

$$e\subseteq Q\text{ if and only if }\mathbf{x_Q}_{I_e}=\mathbf{x_e}.\tag{8.1}$$

Consider $e\in K^{(r)}_{q\times f}$. By (8.1), the number of $Q\in Y$ containing e is equal to the number of $\mathbf{x}\in\mathbb{F}^f$ such that $A'\cdot\mathbf{x}=\mathbf{0}$ and $\mathbf{x}_{I_e}=\mathbf{x_e}$, or equivalently, the number of $\mathbf{x'}\in\mathbb{F}^{f-r}$ satisfying $A'_{I_e}\cdot\mathbf{x_e}+A'_{[f]\setminus I_e}\cdot\mathbf{x'}=\mathbf{0}$. Since $A'_{[f]\setminus I_e}$ is an $(f-r)\times(f-r)$-Cauchy matrix, the equation $A'_{[f]\setminus I_e}\cdot\mathbf{x'}=-A'_{I_e}\cdot\mathbf{x_e}$ has a unique solution $\mathbf{x'}\in\mathbb{F}^{f-r}$, i.e. there is exactly one $Q\in Y$ which contains e. Thus, $\mathcal{K}$ is a $K^{(r)}_f$-decomposition of $K_{q\times f}$.

Now, fix $x^*\in\mathbb{F}$ and $e\in K^{(r-1)}_{q\times f}$. By (8.1), the number of $Q\in Y_{x^*}$ containing e is equal to the number of $\mathbf{x}\in\mathbb{F}^f$ such that $A'\cdot\mathbf{x}=\mathbf{0}$, $\mathbf{\hat{a}}\cdot\mathbf{x}=x^*$ and $\mathbf{x}_{I_e}=\mathbf{x_e}$, or equivalently, the number of $\mathbf{x'}\in\mathbb{F}^{f-(r-1)}$ satisfying $A_{I_e}\cdot\mathbf{x_e}+A_{[f]\setminus I_e}\cdot\mathbf{x'}=\begin{pmatrix}\mathbf{0}\\ x^*\end{pmatrix}$. Since $A_{[f]\setminus I_e}$ is an $(f-r+1)\times(f-r+1)$-Cauchy matrix, this equation has a unique solution $\mathbf{x'}\in\mathbb{F}^{f-r+1}$, i.e. there is exactly one $Q\in Y_{x^*}$ which contains e. Hence, $\mathcal{K}_{x^*}$ is a $K^{(r-1)}_f$-decomposition of $K_{q\times f}$. □

Our application of Theorem 8.1 is as follows.

Lemma 8.2. *Let $2\le r<f$. Let F be any r-graph on f vertices. There exists a weakly regular r-graph F^* on at most $2f\cdot f!$ vertices which has a 1-well separated F-decomposition.*

Proof. Choose a prime power q with $f!\le q\le 2f!$. Let $V(F)=\{v_1,\dots,v_f\}$. By Theorem 8.1, there exists a resolvable $K^{(r)}_f$-decomposition $\mathcal{K}$ of $K_{q\times f}$. Let the vertex classes of $K_{q\times f}$ be

$V_1,\dots,V_f$. Let $\mathcal{K}_1,\dots,\mathcal{K}_q$ be a partition of $\mathcal{K}$ into $K_f^{(r-1)}$-decompositions of $K_{q\times f}$. (We will only need $\mathcal{K}_1,\dots,\mathcal{K}_{f!}$.) We now construct F^* with vertex set $V(K_{q\times f})$ as follows: Let $\pi_1,\dots,\pi_{f!}$ be an enumeration of all permutations on $[f]$. For every $i\in[f!]$ and $Q\in\mathcal{K}_i^{\le(f)}$, let $F_{i,Q}$ be a copy of F with $V(F)=Q$ such that for every $j\in[f]$, the unique vertex in $Q\cap V_{\pi_i(j)}$ plays the role of v_j. Let

$$F^* := \bigcup_{i\in[f!],Q\in\mathcal{K}_i^{\le(f)}} F_{i,Q};$$

$$\mathcal{F} := \{F_{i,Q} \,:\, i\in[f!], Q\in\mathcal{K}_i^{\le(f)}\}.$$

Since $\mathcal{K}_1,\dots,\mathcal{K}_{f!}$ are r-disjoint, we have $|V(F')\cap V(F'')|<r$ for all distinct $F',F''\in\mathcal{F}$. Thus, $\mathcal{F}$ is a 1-well separated F-decomposition of F^*.

We now show that F^* is weakly regular. Let $i\in[r-1]_0$ and $S\in\binom{V(F^*)}{i}$. If S is not crossing, then $|F^*(S)|=0$, so assume that S is crossing. If $i=r-1$, then S plays the role of every $(r-1)$-subset of $V(F)$ exactly k times, where k is the number of permutations on $[f]$ that map $[r-1]$ to $[r-1]$. Hence,

$$|F^*(S)| = |F|rk = |F|\cdot r!(f-r+1)! =: s_{r-1}.$$

If $i<r-1$, then S is contained in exactly $c_i := \binom{f-i}{r-1-i}q^{r-1-i}$ crossing $(r-1)$-sets. Thus,

$$|F^*(S)| = \frac{s_{r-1}c_i}{r-i} =: s_i.$$

Therefore, F^* is weakly $(s_0,\dots,s_{r-1})$-regular. □

8.3. **Proofs of Theorems 1.1, 1.2 and 1.4.** We now prove our main theorems which guarantee F-decompositions in r-graphs of high minimum degree (for weakly regular r-graphs F, see Theorem 1.4), and F-designs in typical r-graphs (for arbitrary r-graphs F, see Theorem 1.1). We will also derive Theorem 1.2.

We first prove the minimum degree version (for weakly regular r-graphs F). Instead of directly proving Theorem 1.4 we actually prove a stronger 'local resilience version'. Let $\mathcal{H}_r(n,p)$ denote the random binomial r-graph on $[n]$ whose edges appear independently with probability p.

Theorem 8.3 (Resilience version)**.** *Let $p\in(0,1]$ and $f,r\in\mathbb{N}$ with $f>r$ and let*

$$c(f,r,p) := \frac{r!p^{2^r\binom{f+r}{r}}}{3\cdot 14^r f^{2r}}.$$

Then the following holds whp for $H\sim\mathcal{H}_r(n,p)$. For every weakly regular r-graph F on f vertices and any r-graph L on $[n]$ with $\Delta(L)\le c(f,r,p)n$, $H\triangle L$ has an F-decomposition whenever it is F-divisible.

The case $p=1$ immediately implies Theorem 1.4.

Proof. Choose $n_0\in\mathbb{N}$ and $\varepsilon>0$ such that $1/n_0\ll\varepsilon\ll p,1/f$ and let $n\ge n_0$,

$$c' := \frac{1.1\cdot 2^r\binom{f+r}{r}}{(f-r)!}c(f,r,p),\quad \xi := 0.99/f!,\quad \xi' := 0.95\xi p^{2^r\binom{f+r}{r}},\quad \xi'' := 0.9(1/4)^{\binom{f+r}{f}}(\xi'-c').$$

Recall that the complete complex K_n is an (ε,ξ,f,r)-supercomplex (cf. Example 3.3). Let $H\sim\mathcal{H}_r(n,p)$. We can view H as a random subgraph of $K_n^{(r)}$. By Corollary 4.6, the following holds whp for all $L\subseteq K_n^{(r)}$ with $\Delta(L)\le c(f,r,p)n$:

$$K_n[H\triangle L] \text{ is a } (3\varepsilon+c',\xi'-c',f,r)\text{-supercomplex.}$$

Note that $c'\le\frac{p^{2^r\binom{f+r}{r}}}{2.7(2\sqrt{e})^r f!}$. Thus, $2(2\sqrt{e})^r\cdot(3\varepsilon+c')\le\xi'-c'$. Lemma 3.5 now implies that $K_n[H\triangle L]$ is an (ε,ξ'',f,r)-supercomplex. Hence, if $H\triangle L$ is F-divisible, it has an F-decomposition by Theorem 3.8. □

Next, we derive Theorem 1.1. As indicated previously, we cannot apply Theorem 3.8 directly, but have to carry out two reductions. As shown in Lemma 8.2, we can 'perfectly' pack any given r-graph F into a weakly regular r-graph F^*. We also need the following lemma, which we will prove later in Section 9. It allows us to remove a sparse F-decomposable subgraph L from an F-divisible r-graph G to achieve that $G-L$ is F^*-divisible. Note that we do not need to assume that F^* is weakly regular.

Lemma 8.4. *Let $1/n \ll \gamma \ll \xi, 1/f^*$ and $r \in [f^*-1]$. Let F be an r-graph. Let F^* be an r-graph on f^* vertices which has a 1-well separated F-decomposition. Let G be an r-graph on n vertices such that for all $A \subseteq \binom{V(G)}{r-1}$ with $|A| \leq \binom{f^*-1}{r-1}$, we have $|\bigcap_{S\in A} G(S)| \geq \xi n$. Let O be an $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq \gamma n$. Then there exists an F-divisible subgraph $D \subseteq G$ with $\Delta(D) \leq \gamma^{-2}$ such that the following holds: for every F-divisible r-graph H on $V(G)$ which is edge-disjoint from D, there exists a subgraph $D^* \subseteq D$ such that $H \cup D^*$ is F^*-divisible and $D - D^*$ has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint.*

In particular, we will apply this lemma when G is F-divisible and thus $H := G-D$ is F-divisible. Then $L := D - D^*$ is a subgraph of G with $\Delta(L) \leq \gamma^{-2}$ and has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint. Moreover, $G - L = H \cup D^*$ is F^*-divisible.

We can deduce the following corollary from the case $F = K_r^{(r)}$ of Lemma 8.4.

Corollary 8.5. *Let $1/n \ll \gamma \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f vertices. Let G be an r-graph on n vertices such that for all $A \subseteq \binom{V(G)}{r-1}$ with $|A| \leq \binom{f-1}{r-1}$, we have $|\bigcap_{S\in A} G(S)| \geq \xi n$. Then there exists a subgraph $D \subseteq G$ with $\Delta(D) \leq \gamma^{-2}$ such that the following holds: for any r-graph H on $V(G)$ which is edge-disjoint from D, there exists a subgraph $D^* \subseteq D$ such that $H \cup D^*$ is F-divisible.*

In particular, using $H := G - D$, there exists a subgraph $L := D - D^* \subseteq G$ with $\Delta(L) \leq \gamma^{-2}$ such that $G - L = H \cup D^*$ is F-divisible.

Proof. Apply Lemma 8.4 with $F, K_r^{(r)}$ playing the roles of F^*, F. □

We now prove the following theorem, which immediately implies the case $\lambda = 1$ of Theorem 1.1.

Theorem 8.6. *Let $1/n \ll \gamma, 1/\kappa \ll c, p, 1/f$ and $r \in [f-1]$, and*

$$c \leq p^h/(q^r 4^q), \text{ where } h := 2^r\binom{q+r}{r} \text{ and } q := 2f \cdot f!. \tag{8.2}$$

Let F be any r-graph on f vertices. Suppose that G is a (c,h,p)-typical F-divisible r-graph on n vertices. Let O be an $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq \gamma n$. Then G has a κ-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint.

Proof. By Lemma 8.2, there exists a weakly regular r-graph F^* on $f^* \leq q$ vertices which has a 1-well separated F-decomposition.

By Lemma 8.4 (with $0.5p^{\binom{f^*-1}{r-1}}$ playing the role of ξ), there exists a subgraph $L \subseteq G$ with $\Delta(L) \leq \gamma^{-2}$ such that $G-L$ is F^*-divisible and L has a 1-well separated F-decomposition $\mathcal{F}_{div}$ such that $\mathcal{F}_{div}^{\leq(r+1)}$ and O are edge-disjoint. By Fact 4.3(i), $\Delta(\mathcal{F}_{div}^{\leq(r+1)}) \leq f-r$. Let

$$G' := G^{\leftrightarrow} - L - \mathcal{F}_{div}^{\leq(r+1)} - O.$$

By Example 3.4, $G^{\leftrightarrow}$ is an $(\varepsilon, \xi, f^*, r)$-supercomplex, where $\varepsilon := 2^{f^*-r+1}c/(f^*-r)!$ and $\xi := (1-2^{f^*+1}c)p^{2^r\binom{f^*+r}{r}}/f^*!$. Observe that assumption (8.2) now guarantees that $2(2\sqrt{\mathrm{e}})^r\varepsilon \leq \xi$. Thus, by Lemma 3.5, $G^{\leftrightarrow}$ is a (γ, ξ', f^*, r)-supercomplex, where $\xi' := 0.9(1/4)^{\binom{f^*+r}{r}}\xi$. By Proposition 4.5(v), we have that G' is a $(\sqrt{\gamma}, \xi'/2, f^*, r)$-supercomplex. Moreover, G' is F^*-divisible. Thus, by Theorem 3.8, G' has a $(\kappa-1)$-well separated F^*-decomposition $\mathcal{F}^*$. Since F^* has a 1-well separated F-decomposition, we can conclude that G' has a $(\kappa-1)$-well separated

F-decomposition $\mathcal{F}_{complex}$. Let $\mathcal{F} := \mathcal{F}_{div} \cup \mathcal{F}_{complex}$. By Fact 4.3(ii), $\mathcal{F}$ is a κ-well separated F-decomposition of G. Moreover, $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint. □

It remains to derive Theorem 1.1 from Theorem 8.6 and Corollary 8.5.

Proof of Theorem 1.1. Choose a new constant $\kappa \in \mathbb{N}$ such that

$$1/n \ll \gamma \ll 1/\kappa \ll c, p, 1/f.$$

Suppose that G is a (c, h, p)-typical (F, λ)-divisible r-graph on n vertices. Split G into two subgraphs G'_1 and G'_2 which are both $(c + \gamma, h, p/2)$-typical (a standard Chernoff-type bound shows that whp a random splitting of G yields the desired property).

By Corollary 8.5 (applied with $G'_2, 0.5(p/2)^{\binom{f-1}{r-1}}$ playing the roles of G, ξ), there exists a subgraph $L^* \subseteq G'_2$ with $\Delta(L^*) \leq \kappa$ such that $G_2 := G'_2 - L^*$ is F-divisible. Let $G_1 := G'_1 \cup L^* = G - G_2$. Clearly, G_1 is still (F, λ)-divisible. By repeated applications of Corollary 8.5, we can find edge-disjoint subgraphs $L_1, \dots, L_\lambda$ of G_1 such that $R_i := G_1 - L_i$ is F-divisible and $\Delta(L_i) \leq \kappa$ for all $i \in [\lambda]$. Indeed, suppose that we have already found $L_1, \dots, L_{i-1}$. Then $\Delta(L_1 \cup \dots \cup L_{i-1}) \leq \lambda\kappa \leq \gamma^{1/2} n$ (recall that $\lambda \leq \gamma n$). Thus, by Corollary 8.5, there exists a subgraph $L_i \subseteq G'_1 - (L_1 \cup \dots \cup L_{i-1})$ with $\Delta(L_i) \leq \kappa$ such that $G_1 - L_i$ is F-divisible.

Let $G''_2 := G_2 \cup L_1 \cup \dots \cup L_\lambda$. We claim that G''_2 is F-divisible. Indeed, let $S \subseteq V(G)$ with $|S| \leq r-1$. We then have that $|G''_2(S)| = |G_2(S)| + \sum_{i\in[\lambda]} |(G_1 - R_i)(S)| = |G_2(S)| + \lambda |G_1(S)| - \sum_{i\in[\lambda]} |R_i(S)| \equiv 0 \mod Deg(F)_{|S|}$.

Since G'_1 and G'_2 are both $(c + \gamma, h, p/2)$-typical and $\Delta(L^* \cup L_1 \cup \dots \cup L_\lambda) \leq 2\gamma^{1/2} n$, we have that each of G_2, G''_2, $R_1, \dots, R_\lambda$ is $(c + \gamma^{1/3}, h, p/2)$-typical (and they are F-divisible by construction).

Using Theorem 8.6 repeatedly, we can thus find κ-well separated F-decompositions $\mathcal{F}_1, \dots, \mathcal{F}_{\lambda-1}$ of G_2, a κ-well separated F-decomposition $\mathcal{F}^*$ of G''_2, and for each $i \in [\lambda]$, a κ-well separated F-decomposition $\mathcal{F}'_i$ of R_i. Moreover, we can assume that all these decompositions are pairwise $(r+1)$-disjoint. Indeed, this can be achieved by choosing them successively: Let O consist of the $(r+1)$-sets which are covered by the decompositions we have already found. Then by Fact 4.3(i) we have that $\Delta(O) \leq 2\lambda \cdot \kappa(f - r) \leq \gamma^{1/2} n$. Hence, using Theorem 8.6, we can find the next κ-well separated F-decomposition which is $(r + 1)$-disjoint from the previously chosen ones.

Then $\mathcal{F} := \mathcal{F}^* \cup \bigcup_{i\in[\lambda-1]} \mathcal{F}_i \cup \bigcup_{i\in[\lambda]} \mathcal{F}'_i$ is the desired (F, λ)-design. Indeed, every edge of $G_1 - (L_1 \cup \dots \cup L_\lambda)$ is covered by each of $\mathcal{F}'_1, \dots, \mathcal{F}'_\lambda$. For each $i \in [\lambda]$, every edge of L_i is covered by $\mathcal{F}^*$ and each of $\mathcal{F}'_1, \dots, \mathcal{F}'_{i-1}, \mathcal{F}'_{i+1}, \dots, \mathcal{F}'_\lambda$. Finally, every edge of G_2 is covered by each of $\mathcal{F}_1, \dots, \mathcal{F}_{\lambda-1}$ and $\mathcal{F}^*$. □

Finally, we also prove Theorem 1.2, which is an immediate consequence of Theorem 8.6 and Corollary 8.5.

Proof of Theorem 1.2. Apply Corollary 8.5 (with $G, 0.5p^{\binom{f-1}{r-1}}$ playing the roles of G, ξ) to find a subgraph $L \subseteq G$ with $\Delta(L) \leq C$ such that $G - L$ is F-divisible. It is easy to see that $G - L$ is $(1.1c, h, p)$-typical. Thus, we can apply Theorem 8.6 to obtain an F-decomposition $\mathcal{F}$ of $G - L$. □

9. Achieving divisibility

It remains to show that we can turn every F-divisible r-graph G into an F^*-divisible r-graph G' by removing a sparse F-decomposable subgraph of G, that is, to prove Lemma 8.4. Note that in Lemma 8.4, we do not need to assume that F^* is weakly regular. On the other hand, our argument heavily relies on the assumption that F^* is F-decomposable.

We first sketch the argument. Let F^* be F-decomposable, let $b_k := Deg(F^*)_k$ and $h_k := Deg(F)_k$. Clearly, we have $h_k \mid b_k$. First, consider the case $k = 0$. Then $b_0 = |F^*|$ and $h_0 = |F|$. We know that $|G|$ is divisible by h_0. Let $0 \leq x < b_0$ be such that $|G| \equiv x \mod b_0$. Since h_0 divides $|G|$ and b_0, it follows that $x = ah_0$ for some $0 \leq a < b_0/h_0$. Thus, removing a

edge-disjoint copies of F from G yields an r-graph G' such that $|G'| = |G| - ah_0 \equiv 0 \mod b_0$, as desired. This will in fact be the first step of our argument.

We then proceed by achieving $Deg(G')_1 \equiv 0 \mod b_1$. Suppose that the vertices of G' are ordered $v_1, \dots, v_n$. We will construct a *degree shifter* which will fix the degree of v_1 by allowing the degree of v_2 to change, whereas all other degrees are unaffected (modulo b_1). Step by step, we will fix all the degrees from $v_1, \dots, v_{n-1}$. Fortunately, the degree of v_n will then automatically be divisible by b_1. For $k > 1$, we will proceed similarly, but the procedure becomes more intricate. It is in general impossible to shift degree from one k-set to another one without affecting the degrees of any other k-set. Roughly speaking, the degree shifter will contain a set of $2k$ special 'root vertices', and the degrees of precisely 2^k k-subsets of this root set change, whereas all other k-degrees are unaffected (modulo b_k). This will allow us to fix all the degrees of k-sets in G' except the ones inside some final $(2k-1)$-set, where we use induction on k as well. Fortunately, the remaining k-sets will again automatically satisfy the desired divisibility condition (cf. Lemma 9.5).

The proof of Lemma 8.4 divides into three parts. In the first subsection, we will construct the degree shifters. In the second subsection, we show on a very abstract level (without considering a particular host graph) how the shifting has to proceed in order to achieve overall divisibility. Finally, we will prove Lemma 8.4 by embedding our constructed shifters (using Lemma 4.7) according to the given shifting procedure.

9.1. **Degree shifters.** The aim of this subsection is to show the existence of certain r-graphs which we call degree shifters. They allow us to locally 'shift' degree among the k-sets of some host graph G.

Definition 9.1 ($\mathbf{x}$-shifter)**.** Let $1 \le k < r$ and let F, F^* be r-graphs. Given an r-graph T_k and distinct vertices $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1$ of T_k, we say that T_k is an $(x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1)$*-shifter with respect to* F, F^* if the following hold:

(SH1) T_k has a 1-well separated F-decomposition $\mathcal{F}$ such that for all $F' \in \mathcal{F}$ and all $i \in [k]$, $|V(F') \cap \{x_i^0, x_i^1\}| \le 1$;

(SH2) $|T_k(S)| \equiv 0 \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_k)$ with $|S| < k$;

(SH3) for all $S \in \binom{V(T_k)}{k}$,
$$|T_k(S)| \equiv \begin{cases} (-1)^{\sum_{i\in[k]} z_i} Deg(F)_k \mod Deg(F^*)_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod Deg(F^*)_k & \text{otherwise.} \end{cases}$$

We will now show that such shifters exist. Ultimately, we seek to find them as rooted subgraphs in some host graph G. Therefore, we impose additional conditions which will allow us to apply Lemma 4.7.

Lemma 9.2. *Let $1 \le k < r$, let F, F^* be r-graphs and suppose that F^* has a 1-well separated F-decomposition $\mathcal{F}$. Let $f^* := |V(F^*)|$. There exists an $(x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1)$-shifter T_k with respect to F, F^* such that $T_k[X]$ is empty and T_k has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X, where $X := \{x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1\}$.*

In order to prove Lemma 9.2, we will first prove a multigraph version (Lemma 9.4), which is more convenient for our construction. We will then recover the desired (simple) r-graph by applying an operation similar to the extension operator $\nabla_{(F,e_0)}$ defined in Section 7.2. The difference is that instead of extending every edge to a copy of F, we will consider an F-decomposition of the multigraph shifter and then extend every copy of F in this decomposition to a copy of F^* (and then delete the original multigraph).

For a word $w = w_1 \dots w_k \in \{0,1\}^k$, let $|w|_0$ denote the number of 0's in w and let $|w|_1$ denote the number of 1's in w. Let $W_e(k)$ be the set of words $w \in \{0,1\}^k$ with $|w|_1$ being even, and let $W_o(k)$ be the set of words $w \in \{0,1\}^k$ with $|w|_1$ being odd.

Fact 9.3. *For every $k \ge 1$, $|W_e(k)| = |W_o(k)| = 2^{k-1}$.*

Lemma 9.4. *Let $1 \leq k < r$ and let F, F^* be r-graphs such that F^* is F-decomposable. Let $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1$ be distinct vertices. There exists a multi-r-graph T_k^* which satisfies (SH1)–(SH3), except that $\mathcal{F}$ does not need to be 1-well separated.*

Proof. Let $\mathcal{S}_k := \binom{V(F)}{k}$. For every $S^* \in \mathcal{S}_k$, we will construct a multi-r-graph T_{k,S^*} such that $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1 \in V(T_{k,S^*})$ and

(sh1) T_{k,S^*} has an F-decomposition $\mathcal{F}$ such that for all $F' \in \mathcal{F}$ and all $i \in [k]$, $|V(F') \cap \{x_i^0, x_i^1\}| \leq 1$;
(sh2) $|T_{k,S^*}(S)| \equiv 0 \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_{k,S^*})$ with $|S| < k$;
(sh3) for all $S \in \binom{V(T_{k,S^*})}{k}$,

$$|T_{k,S^*}(S)| \equiv \begin{cases} (-1)^{\sum_{i\in[k]} z_i}|F(S^*)| \mod Deg(F^*)_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod Deg(F^*)_k & \text{otherwise.} \end{cases}$$

Following from this, it easy to construct T_k^* by overlaying the above multi-r-graphs T_{k,S^*}. Indeed, there are integers $(a'_{S^*})_{S^*\in\mathcal{S}_k}$ such that $\sum_{S^*\in\mathcal{S}_k} a'_{S^*}|F(S^*)| = Deg(F)_k$. Hence, there are positive integers $(a_{S^*})_{S^*\in\mathcal{S}_k}$ such that

$$\sum_{S^*\in\mathcal{S}_k} a_{S^*}|F(S^*)| \equiv Deg(F)_k \mod Deg(F^*)_k. \tag{9.1}$$

Therefore, we take T_k^* to be the union of a_{S^*} copies of T_{k,S^*} for each $S^* \in \mathcal{S}_k$. Then T_k^* has the desired properties.

Let $S^* \in \mathcal{S}_k$. It remains to construct T_{k,S^*}. Let $X_0 := \{x_1^0, \dots, x_k^0\}$ and $X_1 := \{x_1^1, \dots, x_k^1\}$. We may assume that $V(F^*) \cap (X_0 \cup X_1) = \emptyset$. Let $\mathcal{F}^*$ be an F-decomposition of F^* and $F' \in \mathcal{F}^*$. Let $X = \{x_1, \dots, x_k\} \subseteq V(F')$ be the k-set which plays the role of S^* in F', in particular $|F'(X)| = |F(S^*)|$. We first define an auxiliary r-graph T_{1,x_k} as follows: Let F'' be obtained from F' by replacing x_k with a new vertex $\hat{x}_k$. Then let

$$T_{1,x_k} := (F^* - F') \cup F''.$$

Clearly, $(\mathcal{F}^* \setminus \{F'\}) \cup \{F''\}$ is an F-decomposition of T_{1,x_k}. Moreover, observe that for every set $S \subseteq V(T_{1,x_k})$ with $|S| < r$, we have

$$|T_{1,x_k}(S)| = \begin{cases} 0 & \text{if } \{x_k, \hat{x}_k\} \subseteq S; \\ |F^*(S)| & \text{if } \{x_k, \hat{x}_k\} \cap S = \emptyset; \\ |F^*(S)| - |F'(S)| & \text{if } x_k \in S, \hat{x}_k \notin S; \\ |F''(S)| = |F'((S \setminus \{\hat{x}_k\}) \cup \{x_k\})| & \text{if } x_k \notin S, \hat{x}_k \in S. \end{cases} \tag{9.2}$$

We now overlay copies of T_{1,x_k} in a suitable way in order to obtain the multi-r-graph T_{k,S^*}. The vertex set of T_{k,S^*} will be

$$V(T_{k,S^*}) = (V(F^*) \setminus X) \,\dot\cup\, X_0 \,\dot\cup\, X_1.$$

For every word $w = w_1 \dots w_{k-1} \in \{0,1\}^{k-1}$, let T_w be a copy of T_{1,x_k}, where

(a) for each $i \in [k-1]$, $x_i^{w_i}$ plays the role of x_i (and $x_i^{1-w_i} \notin V(T_w)$);
(b) if $|w|_1$ is odd, then x_k^0 plays the role of x_k and x_k^1 plays the role of $\hat{x}_k$, whereas if $|w|_1$ is even, then x_k^0 plays the role of $\hat{x}_k$ and x_k^1 plays the role of x_k;
(c) the vertices in $V(T_{1,x_k}) \setminus \{x_1, \dots, x_{k-1}, x_k, \hat{x}_k\}$ keep their role.

Let

$$T_{k,S^*} := \bigcup_{w\in\{0,1\}^{k-1}} T_w.$$

(Note that if $k = 1$, then T_{k,S^*} is just a copy of T_{1,x_k}, where x_1^0 plays the role of $\hat{x}_1$ and x_1^1 plays the role of x_1.) We claim that T_{k,S^*} satisfies (sh1)–(sh3). Clearly, (sh1) is satisfied because each T_w is a copy of T_{1,x_k} which is F-decomposable, and for all $w \in \{0,1\}^{k-1}$ and all $i \in [k-1]$, $|V(T_w) \cap \{x_i^0, x_i^1\}| = 1$, and since $x_k \notin V(F'')$.

We will now use (9.2) in order to determine an expression for $|T_{k,S^*}(S)|$ (see (9.3)) which will imply (sh2) and (sh3). Call $S \subseteq V(T_{k,S^*})$ *degenerate* if $\{x_i^0, x_i^1\} \subseteq S$ for some $i \in [k]$. Clearly, if S is degenerate, then $|T_w(S)| = 0$ for all $w \in \{0,1\}^{k-1}$. If $S \subseteq V(T_{k,S^*})$ is non-degenerate, define $I(S)$ as the set of all indices $i \in [k]$ such that $|S \cap \{x_i^0, x_i^1\}| = 1$, and define the 'projection'

$$\pi(S) := (S \setminus (X_0 \cup X_1)) \cup \{x_i \,:\, i \in I(S)\}.$$

Clearly, $\pi(S) \subseteq V(F^*)$ and $|\pi(S)| = |S|$. Note that if $S \subseteq V(T_w)$ and $k \notin I(S)$, then S plays the role of $\pi(S) \subseteq V(T_{1,x_k})$ in T_w by (a). For $i \in I(S)$, let $z_i(S) \in \{0,1\}$ be such that $S \cap \{x_i^0, x_i^1\} = \{x_i^{z_i(S)}\}$, and let $z(S) := \sum_{i \in I(S)} z_i(S)$. We claim that the following holds:

$$|T_{k,S^*}(S)| \equiv \begin{cases} (-1)^{z(S)}|F'(\pi(S))| \mod \mathit{Deg}(F^*)_{|S|} & \text{if } S \text{ is non-degenerate} \\ & \text{and } |I(S)| = k; \\ 0 \mod \mathit{Deg}(F^*)_{|S|} & \text{otherwise.} \end{cases} \tag{9.3}$$

As seen above, if S is degenerate, then we have $|T_{k,S^*}(S)| = 0$. From now on, we assume that S is non-degenerate. Let $W(S)$ be the set of words $w = w_1 \dots w_{k-1} \in \{0,1\}^{k-1}$ such that $w_i = z_i(S)$ for all $i \in I(S) \setminus \{k\}$. Clearly, if $w \in \{0,1\}^{k-1} \setminus W(S)$, then $|T_w(S)| = 0$ by (a). Suppose that $w \in W(S)$. If $k \notin I(S)$, then S plays the role of $\pi(S)$ in T_w and hence we have $|T_w(S)| = |T_{1,x_k}(\pi(S))| = |F^*(\pi(S))|$ by (9.2). It follows that $|T_{k,S^*}(S)| \equiv 0 \mod \mathit{Deg}(F^*)_{|S|}$, as required.

From now on, suppose that $k \in I(S)$. Let

$$W_e(S) := \{w \in W(S) \,:\, |w|_1 + z_k(S) \text{ is even}\};$$
$$W_o(S) := \{w \in W(S) \,:\, |w|_1 + z_k(S) \text{ is odd}\}.$$

By (b), we know that $x_k^{z_k(S)}$ plays the role of x_k in T_w if $w \in W_o(S)$ and the role of $\hat{x}_k$ if $w \in W_e(S)$. Hence, if $w \in W_o(S)$ then S plays the role of $\pi(S)$ in T_w, and if $w \in W_e(S)$, then S plays the role of $(\pi(S) \setminus \{x_k\}) \cup \{\hat{x}_k\}$ in T_w. Thus, we have

$$|T_w(S)| = \begin{cases} |T_{1,x_k}(\pi(S))| \overset{(9.2)}{=} |F^*(\pi(S))| - |F'(\pi(S))| & \text{if } w \in W_o(S); \\ |T_{1,x_k}((\pi(S) \setminus \{x_k\}) \cup \{\hat{x}_k\})| \overset{(9.2)}{=} |F'(\pi(S))| & \text{if } w \in W_e(S); \\ 0 & \text{if } w \notin W(S). \end{cases}$$

It follows that

$$|T_{k,S^*}(S)| = \sum_{w \in \{0,1\}^{k-1}} |T_w(S)| \equiv (|W_e(S)| - |W_o(S)|)|F'(\pi(S))| \mod \mathit{Deg}(F^*)_{|S|}.$$

Observe that

$$|W_e(S)| = |\{w' \in \{0,1\}^{k-|I(S)|} \,:\, |w'|_1 + z(S) \text{ is even}\}|;$$
$$|W_o(S)| = |\{w' \in \{0,1\}^{k-|I(S)|} \,:\, |w'|_1 + z(S) \text{ is odd}\}|.$$

Hence, if $|I(S)| < k$, then by Fact 9.3 we have $|W_e(S)| = |W_o(S)| = 2^{k-|I(S)|-1}$. If $|I(S)| = k$, then $|W_e(S)| = 1$ if $z(S)$ is even and $|W_e(S)| = 0$ if $z(S)$ is odd, and for $W_o(S)$, the reverse holds. Altogether, this implies (9.3).

It remains to show that (9.3) implies (sh2) and (sh3). Clearly, (sh2) holds. Indeed, if $|S| < k$, then S is degenerate or we have $|I(S)| < k$, and (9.3) implies that $|T_{k,S^*}(S)| \equiv 0 \mod \mathit{Deg}(F^*)_{|S|}$.

Finally, consider $S \in \binom{V(T_{k,S^*})}{k}$. If S does not have the form $\{x_i^{z_i} \,:\, i \in [k]\}$ for suitable $z_1, \dots, z_k \in \{0,1\}$, then S is degenerate or $|I(S)| < k$ and (9.3) implies that $|T_{k,S^*}(S)| \equiv 0 \mod \mathit{Deg}(F^*)_k$, as required. Assume now that $S = \{x_i^{z_i} : i \in [k]\}$ for suitable $z_1, \dots, z_k \in \{0,1\}$. Then S is not degenerate, $I(S) = [k]$, $z(S) = \sum_{i \in [k]} z_i$ and $\pi(S) = \{x_1, \dots, x_k\} = X$, in which case (9.3) implies that

$$|T_{k,S^*}(S)| \equiv (-1)^{z(S)}|F'(X)| = (-1)^{z(S)}|F(S^*)| \mod \mathit{Deg}(F^*)_k,$$

as required for (sh3). □

Proof of Lemma 9.2. By applying Lemma 9.4 (with x_k^0 and x_k^1 swapping their roles), we can see that there exists a multi-r-graph T_k^* with $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1 \in V(T_k^*)$ such that the following properties hold:

- T_k^* has an F-decomposition $\{F_1, \dots, F_m\}$ such that for all $j \in [m]$ and all $i \in [k]$, we have $|V(F_j) \cap \{x_i^0, x_i^1\}| \leq 1$;
- $|T_k^*(S)| \equiv 0 \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_k^*)$ with $|S| < k$;
- for all $S \in \binom{V(T_k^*)}{k}$,

$$|T_k^*(S)| \equiv \begin{cases} (-1)^{\sum_{i \in [k-1]} z_i + (1-z_k)} Deg(F)_k \mod Deg(F^*)_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod Deg(F^*)_k & \text{otherwise.} \end{cases}$$

Let $f := |V(F)|$. For every $j \in [m]$, let Z_j be a set of $f^* - f$ new vertices, such that $Z_j \cap Z_{j'} = \emptyset$ for all distinct $j, j' \in [m]$ and $Z_j \cap V(T_k^*) = \emptyset$ for all $j \in [m]$. Now, for every $j \in [m]$, let F_j^* be a copy of F^* on vertex set $V(F_j) \cup Z_j$ such that $\mathcal{F}_j \cup \{F_j\}$ is a 1-well separated F-decomposition of F_j^*. In particular, we have that

(a) $(F_j^* - F_j)[V(F_j)]$ is empty;
(b) $\mathcal{F}_j$ is a 1-well separated F-decomposition of $F_j^* - F_j$ such that for all $F' \in \mathcal{F}_j$, $|V(F') \cap V(F_j)| \leq r - 1$.

Let

$$T_k := \dot{\bigcup_{j \in [m]}} (F_j^* - F_j).$$

We claim that T_k is the desired shifter. First, observe that T_k is a (simple) r-graph since $(F_j^* - F_j)[V(F_j)]$ is empty for every $j \in [m]$ by (a). Moreover, since $\mathcal{F}_1, \dots, \mathcal{F}_m$ are r-disjoint by (b), Fact 4.3(iii) implies that $\mathcal{F} := \mathcal{F}_1 \cup \dots \cup \mathcal{F}_m$ is a 1-well separated F-decomposition of T_k, and for each $j \in [m]$, all $F' \in \mathcal{F}_j$ and all $i \in [k]$, we have $|V(F') \cap \{x_i^0, x_i^1\}| \leq |V(F_j) \cap \{x_i^0, x_i^1\}| \leq 1$. Thus, (SH1) holds.

Moreover, note that for every $j \in [m]$, we have $|(F_j^* - F_j)(S)| \equiv -|F_j(S)| \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_k)$ with $|S| \leq r - 1$. Thus,

$$|T_k(S)| \equiv \sum_{j \in [m]} -|F_j(S)| = -|T_k^*(S)| \mod Deg(F^*)_{|S|}$$

for all $S \subseteq V(T_k)$ with $|S| \leq r - 1$. Hence, (SH2) clearly holds. If $S = \{x_i^{z_i} : i \in [k]\}$ for suitable $z_1, \dots, z_k \in \{0, 1\}$, then

$$|T_k(S)| \equiv -|T_k^*(S)| \equiv (-1)^{\sum_{i \in [k]} z_i} Deg(F)_k \mod Deg(F^*)_k$$

and (SH3) holds. Thus, T_k is indeed an $(x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1)$-shifter with respect to F, F^*.

Finally, to see that T_k has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X, consider the vertices of $V(T_k) \setminus X$ in an ordering where the vertices of $V(T_k^*) \setminus X$ precede all the vertices in sets Z_j, for $j \in [m]$. Note that $T_k[V(T_k^*)]$ is empty by (a), i.e. a vertex in $V(T_k^*) \setminus X$ has no 'backward' edges. Moreover, if $z \in Z_j$ for some $j \in [m]$, then $|T_k(\{z\})| = |F_j^*(\{z\})| \leq \binom{f^*-1}{r-1}$. □

9.2. **Shifting procedure.** In the previous section, we constructed degree shifters which allow us to locally change the degrees of k-sets in some host graph. We will now show how to combine these local shifts in order to transform any given F-divisible r-graph G into an F^*-divisible r-graph. It turns out to be more convenient to consider the shifting for 'r-set functions' rather than r-graphs. We will then recover the graph theoretical statement by considering a graph as an indicator set function (see below).

Let $\phi : \binom{V}{r} \to \mathbb{Z}$. (Think of ϕ as the multiplicity function of a multi-r-graph.) We extend ϕ to $\phi : \bigcup_{k\in[r]_0} \binom{V}{k} \to \mathbb{Z}$ by defining for all $S \subseteq V$ with $|S| = k \le r$,

$$\phi(S) := \sum_{S'\in\binom{V}{r}:S\subseteq S'} \phi(S'). \tag{9.4}$$

Thus for all $0 \le i \le k \le r$ and all $S \in \binom{V}{i}$,

$$\binom{r-i}{k-i}\phi(S) = \sum_{S'\in\binom{V}{k}:S\subseteq S'} \phi(S'). \tag{9.5}$$

For $k \in [r-1]_0$ and $b_0, \dots, b_k \in \mathbb{N}$, we say that ϕ is $(b_0, \dots, b_k)$-*divisible* if $b_{|S|} \mid \phi(S)$ for all $S \subseteq V$ with $|S| \le k$.

If G is an r-graph with $V(G) \subseteq V$, we define $\mathbb{1}_G \colon \binom{V}{r} \to \mathbb{Z}$ as

$$\mathbb{1}_G(S) := \begin{cases} 1 & \text{if } S \in G; \\ 0 & \text{if } S \notin G. \end{cases}$$

and extend $\mathbb{1}_G$ as in (9.4). Hence, for a set $S \subseteq V$ with $|S| < r$, we have $\mathbb{1}_G(S) = |G(S)|$. Thus, (9.5) corresponds to the handshaking lemma for r-graphs (cf. (4.1)). Clearly, if G and G' are edge-disjoint, then we have $\mathbb{1}_G + \mathbb{1}_{G'} = \mathbb{1}_{G\cup G'}$. Moreover, for an r-graph F, G is F-divisible if and only if $\mathbb{1}_G$ is $(Deg(F)_0, \dots, Deg(F)_{r-1})$-divisible.

As mentioned before, our strategy is to successively fix the degrees of k-sets until we have fixed the degrees of all k-sets except possibly the degrees of those k-sets contained in some final vertex set K which is too small as to continue with the shifting. However, as the following lemma shows, divisibility is then automatically satisfied for all the k-sets lying inside K. For this to work it is essential that the degrees of all i-sets for $i < k$ are already fixed.

Lemma 9.5. *Let $1 \le k < r$ and $b_0, \dots, b_k \in \mathbb{N}$ be such that $\binom{r-i}{k-i}b_i \equiv 0 \mod b_k$ for all $i \in [k]_0$. Let $\phi : \binom{V}{r} \to \mathbb{Z}$ be a $(b_0, \dots, b_{k-1})$-divisible function. Suppose that there exists a subset $K \subseteq V$ of size $2k-1$ such that if $S \in \binom{V}{k}$ with $\phi(S) \not\equiv 0 \mod b_k$, then $S \subseteq K$. Then ϕ is $(b_0, \dots, b_k)$-divisible.*

Proof. Let $\mathcal{K}$ be the set of all subsets T'' of K of size less than k. We first claim that for all $T'' \in \mathcal{K}$, we have

$$\sum_{T'\in\binom{K}{k}\colon T''\subseteq T'} \phi(T') \equiv 0 \mod b_k. \tag{9.6}$$

Indeed, suppose that $|T''| = i < k$, then we have

$$\sum_{T'\in\binom{K}{k}\colon T''\subseteq T'} \phi(T') \equiv \sum_{T'\in\binom{V}{k}\colon T''\subseteq T'} \phi(T') \overset{(9.5)}{=} \binom{r-i}{k-i}\phi(T'') \mod b_k.$$

Since ϕ is $(b_0, \dots, b_{k-1})$-divisible, we have $\phi(T'') \equiv 0 \mod b_i$, and since $\binom{r-i}{k-i}b_i \equiv 0 \mod b_k$, the claim follows.

Let $T \in \binom{K}{k}$. We need to show that $\phi(T) \equiv 0 \mod b_k$. To this end, define the function $f \colon \mathcal{K} \to \mathbb{Z}$ as

$$f(T'') := \begin{cases} (-1)^{|T''|} & \text{if } T'' \subseteq K \setminus T; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for all $T' \in \binom{K}{k}$, we have

$$\sum_{T''\subsetneq T'} f(T'') = \begin{cases} 1 & \text{if } T' = T; \\ 0 & \text{otherwise.} \end{cases} \tag{9.7}$$

Indeed, let $T' \in \binom{K}{k}$, and set $t := |T' \setminus T|$. We then check that (using $|K| < 2k$ in the first equality)

$$\sum_{T'' \subsetneq T'} f(T'') = \sum_{T'' \subseteq (K \setminus T) \cap T'} (-1)^{|T''|} = \sum_{j=0}^{t} (-1)^j \binom{t}{j} = \begin{cases} 1 & \text{if } t = 0; \\ 0 & \text{if } t > 0. \end{cases}$$

We can now conclude that

$$\phi(T) \overset{(9.7)}{=} \sum_{T' \in \binom{K}{k}} \phi(T') \sum_{T'' \subsetneq T'} f(T'') = \sum_{T'' \in \mathcal{K}} f(T'') \left(\sum_{T' \in \binom{K}{k} \colon T'' \subseteq T'} \phi(T') \right) \overset{(9.6)}{\equiv} 0 \mod b_k,$$

as desired. □

We now define a more abstract version of degree shifters, which we call adapters. They represent the effect of shifters and will finally be replaced by shifters again.

Definition 9.6 ($\mathbf{x}$-adapter). Let V be a vertex set and $k, r, b_0, \ldots, b_k, h_k \in \mathbb{N}$ be such that $k < r$ and $h_k \mid b_k$. For distinct vertices $x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1$ in V, we say that $\tau \colon \binom{V}{r} \to \mathbb{Z}$ is an $(x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1)$*-adapter with respect to* $(b_0, \ldots, b_k; h_k)$ if τ is $(b_0, \ldots, b_{k-1})$-divisible and for all $S \in \binom{V}{k}$,

$$\tau(S) \equiv \begin{cases} (-1)^{\sum_{i \in [k]} z_i} h_k \mod b_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod b_k & \text{otherwise.} \end{cases}$$

Note that such an adapter τ is $(b_0, \ldots, b_{k-1}, h_k)$-divisible.

Fact 9.7. *If T is an $\mathbf{x}$-shifter with respect to F, F^*, then $\mathbb{1}_T$ is an $\mathbf{x}$-adapter with respect to $(Deg(F^*)_0, \ldots, Deg(F^*)_k; Deg(F)_k)$.*

The following definition is crucial for the shifting procedure. Given some function ϕ, we intend to add adapters in order to obtain a divisible function. Every adapter is characterised by a tuple $\mathbf{x}$ consisting of $2k$ distinct vertices, which tells us where to apply the adapter. All these tuples are contained within a multiset Ω, which we call a balancer. Ω is capable of dealing with any input function ϕ in the sense that there is a multisubset of Ω which tells us where to apply the adapters in order to make ϕ divisible. Moreover, as we finally want to replace the adapters by shifters (and thus embed them into some host graph), there must not be too many of them.

Definition 9.8 (balancer). Let $r, k, b_0, \ldots, b_k \in \mathbb{N}$ with $k < r$ and let U, V be sets with $U \subseteq V$. Let Ω_k be a multiset containing ordered tuples $\mathbf{x} = (x_1, \ldots, x_{2k})$, where $x_1, \ldots, x_{2k} \in U$ are distinct. We say that Ω_k is a $(b_0, \ldots, b_k)$*-balancer for V with uniformity r acting on U* if for any $h_k \in \mathbb{N}$ with $h_k \mid b_k$, the following holds: let $\phi \colon \binom{V}{r} \to \mathbb{Z}$ be any $(b_0, \ldots, b_{k-1}, h_k)$-divisible function such that $S \subseteq U$ whenever $S \in \binom{V}{k}$ and $\phi(S) \not\equiv 0 \mod b_k$. There exists a multisubset Ω' of Ω_k such that $\phi + \tau_{\Omega'}$ is $(b_0, \ldots, b_k)$-divisible, where $\tau_{\Omega'} := \sum_{\mathbf{x} \in \Omega'} \tau_{\mathbf{x}}$ and $\tau_{\mathbf{x}}$ is any $\mathbf{x}$-adapter with respect to $(b_0, \ldots, b_k; h_k)$.

For a set $S \in \binom{V}{k}$, let $\deg_{\Omega_k}(S)$ be the number of $\mathbf{x} = (x_1, \ldots, x_{2k}) \in \Omega_k$ such that $|S \cap \{x_i, x_{i+k}\}| = 1$ for all $i \in [k]$. Furthermore, we denote $\Delta(\Omega_k)$ to be the maximum value of $\deg_{\Omega_k}(S)$ over all $S \in \binom{V}{k}$.

The following lemma shows that these balancers exist, i.e. that the local shifts performed by the degree shifters guaranteed by Lemma 9.2 are sufficient to obtain global divisibility (for which we apply Lemma 9.5).

Lemma 9.9. *Let $1 \leq k < r$. Let $b_0, \ldots, b_k \in \mathbb{N}$ be such that $\binom{r-s}{k-s} b_s \equiv 0 \mod b_k$ for all $s \in [k]_0$. Let U be a set of $n \geq 2k$ vertices and $U \subseteq V$. Then there exists a $(b_0, \ldots, b_k)$-balancer Ω_k for V with uniformity r acting on U such that $\Delta(\Omega_k) \leq 2^k (k!)^2 b_k$.*

Proof. We will proceed by induction on k. First, consider the case when $k = 1$. Write $U = \{v_1, \ldots, v_n\}$. Define Ω_1 to be the multiset containing precisely $b_1 - 1$ copies of (v_j, v_{j+1}) for all $j \in [n-1]$. Note that $\Delta(\Omega_1) \leq 2b_1$.

We now show that Ω_1 is a (b_0, b_1)-balancer for V with uniformity r acting on U. Let $\phi : \binom{V}{r} \to \mathbb{Z}$ be (b_0, h_1)-divisible for some $h_1 \in \mathbb{N}$ with $h_1 \mid b_1$, such that $v \in U$ whenever $v \in V$ and $\phi(\{v\}) \not\equiv 0 \mod b_1$. Let $m_0 := 0$. For each $j \in [n-1]$, let $0 \leq m_j < b_1$ be such that $(m_{j-1} - m_j)h_1 \equiv \phi(\{v_j\}) \mod b_1$. Let $\Omega' \subseteq \Omega_1$ consist of precisely m_j copies of (v_j, v_{j+1}) for all $j \in [n-1]$. Let $\tau := \sum_{\mathbf{x} \in \Omega'} \tau_{\mathbf{x}}$, where $\tau_{\mathbf{x}}$ is an $\mathbf{x}$-adapter with respect to $(b_0, b_1; h_1)$, and let $\phi' := \phi + \tau$. Clearly, ϕ' is (b_0)-divisible. Note that, for all $j \in [n-1]$,

$$\begin{aligned} \tau(\{v_j\}) &\equiv m_{j-1}\tau_{(v_{j-1},v_j)}(\{v_j\}) + m_j \tau_{(v_j,v_{j+1})}(\{v_j\}) \mod b_1 \\ &\equiv (-m_{j-1} + m_j)h_1 \equiv -\phi(\{v_j\}) \mod b_1, \end{aligned} \tag{9.8}$$

implying that $\phi'(\{v_j\}) \equiv 0 \mod b_1$ for all $j \in [n-1]$. Moreover, for all $v \in V \setminus U$, we have $\phi(\{v\}) \equiv 0 \mod b_1$ by assumption and $\tau(\{v\}) \equiv 0 \mod b_1$ since no element of Ω_1 contains v. Thus, by Lemma 9.5 (with $\{v_n\}$ playing the role of K), ϕ' is (b_0, b_1)-divisible, as required.

We now assume that $k > 1$ and that the statement holds for smaller k. Again, write $U = \{v_1, \dots, v_n\}$. For every $\ell \in [n]$, let $U_\ell := \{v_j : j \in [\ell]\}$. We construct Ω_k inductively. For each $\ell \in \{2k, \dots, n\}$, we define a multiset $\Omega_{k,\ell}$ as follows. Let $\Omega_{k-1,\ell-1}$ be a $(b_1, \dots, b_k)$-balancer for $V \setminus \{v_\ell\}$ with uniformity $r-1$ acting on $U_{\ell-1}$ and

$$\Delta(\Omega_{k-1,\ell-1}) \leq 2^{k-1}(k-1)!^2 b_k.$$

(Indeed, $\Omega_{k-1,\ell-1}$ exists by our induction hypothesis with $r-1, k-1, b_1, \dots, b_k, U_{\ell-1}, V \setminus \{v_\ell\}$ playing the roles of $r, k, b_0, \dots, b_k, U, V$.) For each $\mathbf{v} = (v_{j_1}, \dots, v_{j_{2k-2}}) \in \Omega_{k-1,\ell-1}$, let

$$\mathbf{v}' := (v_\ell, v_{j_1}, \dots, v_{j_{k-1}}, v_{j_{\mathbf{v}}}, v_{j_k}, \dots, v_{j_{2k-2}}) \in U_\ell \times U_{\ell-1}^{2k-1}, \tag{9.9}$$

such that $j_{\mathbf{v}} \in \{\ell - 2k + 1, \dots, \ell\} \setminus \{\ell, j_1, \dots, j_{2k-2}\}$ (which exists since $\ell \geq 2k$). We let $\Omega_{k,\ell} := \{\mathbf{v}' : \mathbf{v} \in \Omega_{k-1,\ell-1}\}$. Now, define

$$\Omega_k := \bigcup_{\ell=2k}^{n} \Omega_{k,\ell}.$$

Claim 1: $\Delta(\Omega_k) \leq 2^k(k!)^2 b_k$

Proof of claim: Consider any $S \in \binom{V}{k}$. Clearly, if $S \not\subseteq U$, then $\deg_{\Omega_k}(S) = 0$, so assume that $S \subseteq U$. Let i_0 be the largest $i \in [n]$ such that $v_i \in S$.

First note that for all $\ell \in \{2k, \dots, n\}$, we have

$$\deg_{\Omega_{k,\ell}}(S) \leq \sum_{v \in S} \deg_{\Omega_{k-1,\ell-1}}(S \setminus \{v\}) \leq k\Delta(\Omega_{k-1,\ell-1}).$$

On the other hand, we claim that if $\ell < i_0$ or $\ell \geq i_0 + 2k$, then $\deg_{\Omega_{k,\ell}}(S) = 0$. Indeed, in the first case, we have $S \not\subseteq U_\ell$ which clearly implies that $\deg_{\Omega_{k,\ell}}(S) = 0$. In the latter case, for any $\mathbf{v} \in \Omega_{k-1,\ell-1}$, we have $j_{\mathbf{v}} \geq \ell - 2k + 1 > i_0$ and thus $|S \cap \{v_\ell, v_{j_{\mathbf{v}}}\}| = 0$, which also implies $\deg_{\Omega_{k,\ell}}(S) = 0$. Therefore,

$$\deg_{\Omega_k}(S) = \sum_{\ell=2k}^{n} \deg_{\Omega_{k,\ell}}(S) \leq 2k^2 \Delta(\Omega_{k-1,\ell-1}) \leq 2^k(k!)^2 b_k,$$

as required. –

We now show that Ω_k is indeed a $(b_0, \dots, b_k)$-balancer on V with uniformity r acting on U. The key to this is the following claim, which we will apply repeatedly.

Claim 2: Let $2k \leq \ell \leq n$. Let $\phi_\ell \colon \binom{V}{r} \to \mathbb{Z}$ be any $(b_0, \dots, b_{k-1}, h_k)$-divisible function for some $h_k \in \mathbb{N}$ with $h_k \mid b_k$. Suppose that if $\phi_\ell(S) \not\equiv 0 \mod b_k$ for some $S \in \binom{V}{k}$, then $S \subseteq U_\ell$. Then there exists $\Omega'_{k,\ell} \subseteq \Omega_{k,\ell}$ such that $\phi_{\ell-1} := \phi_\ell + \tau_{\Omega'_{k,\ell}}$ is $(b_0, \dots, b_{k-1}, h_k)$-divisible and if $\phi_{\ell-1}(S) \not\equiv 0 \mod b_k$ for some $S \in \binom{V}{k}$, then $S \subseteq U_{\ell-1}$.

(Here, $\tau_{\Omega'_{k,\ell}}$ is as in Definition 9.8, i.e. $\tau_{\Omega'_{k,\ell}} := \sum_{\mathbf{v}' \in \Omega'_{k,\ell}} \tau_{\mathbf{v}'}$ and $\tau_{\mathbf{v}'}$ is an arbitrary $\mathbf{v}'$-adapter with respect to $(b_0, \dots, b_k; h_k)$.)

Proof of claim: Define $\rho : \binom{V \setminus \{v_\ell\}}{r-1} \to \mathbb{Z}$ such that for all $S \in \binom{V \setminus \{v_\ell\}}{r-1}$,

$$\rho(S) := \phi_\ell(S \cup \{v_\ell\}).$$

It is easy to check that this identity transfers to smaller sets S, that is, for all $S \subseteq V \setminus \{v_\ell\}$, with $|S| \leq r-1$, we have $\rho(S) = \phi_\ell(S \cup \{v_\ell\})$, where $\rho(S)$ and $\phi_\ell(S \cup \{v_\ell\})$ are as defined in (9.4).

Hence, since ϕ_ℓ is $(b_0, \dots, b_{k-1}, h_k)$-divisible, ρ is $(b_1, \dots, b_{k-1}, h_k)$-divisible. Moreover, for all $S \in \binom{V \setminus \{v_\ell\}}{k-1}$ with $\rho(S) \not\equiv 0 \mod b_k$, we have $S \subseteq U_{\ell-1}$.

Recall that $\Omega_{k-1,\ell-1}$ is a $(b_1, \dots, b_k)$-balancer for $V \setminus \{v_\ell\}$ with uniformity $r-1$ acting on $U_{\ell-1}$. Thus, there exists a multiset $\Omega' \subseteq \Omega_{k-1,\ell-1}$ such that

$$\rho + \tau_{\Omega'} \text{ is } (b_1, \dots, b_k)\text{-divisible.} \tag{9.10}$$

Let $\Omega'_{k,\ell} \subseteq \Omega_{k,\ell}$ be induced by Ω', that is, $\Omega'_{k,\ell} := \{\mathbf{v}' : \mathbf{v} \in \Omega'\}$ (see (9.9)). Let $\mathbf{v}' \in \Omega'_{k,\ell}$ and let $\tau_{\mathbf{v}'}$ be any $\mathbf{v}'$-adapter with respect to $(b_0, \dots, b_k; h_k)$. As noted after Definition 9.6, $\tau_{\mathbf{v}'}$ is $(b_0, \dots, b_{k-1}, h_k)$-divisible. Crucially, if $S \in \binom{V}{k}$ and $v_\ell \in S$, then $\tau_{\mathbf{v}'}(S) \equiv \tau_{\mathbf{v}}(S \setminus \{v_\ell\})$ mod b_k. Indeed, let $x_1^0, \dots, x_{k-1}^0, x_1^1, \dots, x_{k-1}^1$ be such that $\mathbf{v} = (x_1^0, \dots, x_{k-1}^0, x_1^1, \dots, x_{k-1}^1)$ and thus $\mathbf{v}' = (v_\ell, x_1^0, \dots, x_{k-1}^0, v_{j_\mathbf{v}}, x_1^1, \dots, x_{k-1}^1)$. Then by Definition 9.6, as $v_\ell \in S$, we have

$$\begin{aligned}\tau_{\mathbf{v}'}(S) &\equiv \begin{cases} (-1)^{0+\sum_{i \in [k-1]} z_i} h_k \mod b_k & \text{if } S \setminus \{v_\ell\} = \{x_i^{z_i} : i \in [k-1]\}, \\ 0 \mod b_k & \text{otherwise,} \end{cases} \\ &\equiv \tau_{\mathbf{v}}(S \setminus \{v_\ell\}) \mod b_k.\end{aligned}$$

Let $\tau_{\Omega'_{k,\ell}} := \sum_{\mathbf{v}' \in \Omega'_{k,\ell}} \tau_{\mathbf{v}'}$ and $\phi_{\ell-1} := \phi_\ell + \tau_{\Omega'_{k,\ell}}$. Note that for all $S \not\subseteq U_\ell$, we have $\tau_{\Omega'_{k,\ell}}(S) = 0$ by (9.9). Moreover, if $S \in \binom{V}{k}$ and $v_\ell \in S$, then $\tau_{\Omega'_{k,\ell}}(S) \equiv \tau_{\Omega'}(S \setminus \{v_\ell\}) \mod b_k$ by the above.

Clearly, $\phi_{\ell-1}$ is $(b_0, \dots, b_{k-1}, h_k)$-divisible. Now, consider any $S \in \binom{V}{k}$ with $S \not\subseteq U_{\ell-1}$. If $S \not\subseteq U_\ell$, then

$$\phi_{\ell-1}(S) = \phi_\ell(S) + \tau_{\Omega'_{k,\ell}}(S) \equiv 0 + 0 \equiv 0 \mod b_k.$$

If $S \subseteq U_\ell$, then since $S \not\subseteq U_{\ell-1}$ we must have $v_\ell \in S$, and so

$$\phi_{\ell-1}(S) = \phi_\ell(S) + \tau_{\Omega'_{k,\ell}}(S) \equiv \rho(S \setminus \{v_\ell\}) + \tau_{\Omega'}(S \setminus \{v_\ell\}) \overset{(9.10)}{\equiv} 0 \mod b_k.$$

This completes the proof of the claim. –

Now, let $h_k \in \mathbb{N}$ with $h_k \mid b_k$ and let $\phi\colon \binom{V}{r} \to \mathbb{Z}$ be any $(b_0, \dots, b_{k-1}, h_k)$-divisible function such that $S \subseteq U$ whenever $S \in \binom{V}{k}$ and $\phi(S) \not\equiv 0 \mod b_k$. Let $\phi_n := \phi$ and note that $U = U_n$. Thus, by Claim 2, there exists $\Omega'_{k,n} \subseteq \Omega_{k,n}$ such that $\phi_{n-1} := \phi_n + \tau_{\Omega'_{k,n}}$ is $(b_0, \dots, b_{k-1}, h_k)$-divisible and if $\phi_{n-1}(S) \not\equiv 0 \mod b_k$ for some $S \in \binom{V}{k}$, then $S \subseteq U_{n-1}$. Repeating this step finally yields some $\Omega'_k \subseteq \Omega_k$ such that $\phi^* := \phi + \tau_{\Omega'_k}$ is $(b_0, \dots, b_{k-1}, h_k)$-divisible and such that $S \subseteq U_{2k-1}$ whenever $S \in \binom{V}{k}$ and $\phi(S) \not\equiv 0 \mod b_k$. By Lemma 9.5 (with U_{2k-1} playing the role of K), ϕ^* is then $(b_0, \dots, b_k)$-divisible. Thus Ω_k is indeed a $(b_0, \dots, b_k)$-balancer. □

9.3. **Proof of Lemma 8.4.** We now prove Lemma 8.4. For this, we consider the balancers Ω_k guaranteed by Lemma 9.9. Recall that these consist of suitable adapters, and that Lemma 9.2 guarantees the existence of shifters corresponding to these adapters. It remains to embed these shifters in a suitable way, which is achieved via Lemma 4.7. The following fact will help us to verify the conditions of Lemma 9.9.

Fact 9.10. *Let F be an r-graph. Then for all $0 \leq i \leq k < r$, we have $\binom{r-i}{k-i} Deg(F)_i \equiv 0$ mod $Deg(F)_k$.*

Proof. Let S be any i-set in $V(F)$. By (4.1), we have that

$$\binom{r-i}{k-i}|F(S)| = \sum_{T\in\binom{V(F)}{k}\colon S\subseteq T} |F(T)| \equiv 0 \mod Deg(F)_k,$$

and this implies the claim. □

Proof of Lemma 8.4. Let $x_1^0,\dots,x_{r-1}^0,x_1^1,\dots,x_{r-1}^1$ be distinct vertices (not in $V(G)$). For $k\in[r-1]$, let $X_k := \{x_1^0,\dots,x_k^0,x_1^1,\dots,x_k^1\}$. By Lemma 9.2, for every $k\in[r-1]$, there exists an $(x_1^0,\dots,x_k^0,x_1^1,\dots,x_k^1)$-shifter T_k with respect to F,F^* such that $T_k[X_k]$ is empty and T_k has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X_k. Note that (SH1) implies that

$$|T_k(\{x_i^0,x_i^1\})| = 0 \text{ for all } i\in[k]. \tag{9.11}$$

We may assume that there exists $t\geq \max_{k\in[r-1]}|V(T_k)|$ such that $1/n \ll \gamma \ll 1/t \ll \xi, 1/f^*$. Let $Deg(F)=(h_0,h_1,\dots,h_{r-1})$ and let $Deg(F^*)=(b_0,b_1,\dots,b_{r-1})$. Since F^* is F-decomposable and thus F-divisible, we have $h_k\mid b_k$ for all $k\in[r-1]_0$.

By Fact 9.10, we have $\binom{r-i}{k-i}b_i\equiv 0 \mod b_k$ for all $0\leq i\leq k<r$. For each $k\in[r-1]$ with $h_k<b_k$, we apply Lemma 9.9 to obtain a $(b_0,\dots,b_k)$-balancer Ω_k for $V(G)$ with uniformity r acting on $V(G)$ such that $\Delta(\Omega_k)\leq 2^k(k!)^2b_k$. For values of k for which we have $h_k=b_k$, we let $\Omega_k:=\emptyset$. For every $k\in[r-1]$ and every $\mathbf{v}=(v_1,\dots,v_{2k})\in\Omega_k$, define the labelling $\Lambda_{\mathbf{v}}\colon X_k\to V(G)$ by setting $\Lambda_{\mathbf{v}}(x_i^0):=v_i$ and $\Lambda_{\mathbf{v}}(x_i^1):=v_{i+k}$ for all $i\in[k]$.

For technical reasons, let T_0 be a copy of F and let $X_0:=\emptyset$. Let Ω_0 be the multiset containing b_0/h_0 copies of $\emptyset$, and for every $\mathbf{v}\in\Omega_0$, let $\Lambda_{\mathbf{v}}\colon X_0\to V(G)$ be the trivial G-labelling of (T_0,X_0). Note that T_0 has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X_0. Note also that $\Lambda_{\mathbf{v}}$ does not root any set $S\subseteq V(G)$ with $|S|\in[r-1]$.

We will apply Lemma 4.7 in order to find faithful embeddings of the T_k into G. Let $\Omega := \bigcup_{k=0}^{r-1}\Omega_k$. Let $\alpha:=\gamma^{-2}/n$.

Claim 1: For every $k\in[r-1]$ and every $S\subseteq V(G)$ with $|S|\in[r-1]$, we have $|\{\mathbf{v}\in\Omega_k : \Lambda_{\mathbf{v}}$ roots $S\}|\leq r^{-1}\alpha\gamma n^{r-|S|}$. Moreover, $|\Omega_k|\leq r^{-1}\alpha\gamma n^r$.

Proof of claim: Let $k\in[r-1]$ and $S\subseteq V(G)$ with $|S|\in[r-1]$. Consider any $\mathbf{v}=(v_1,\dots,v_{2k})\in\Omega_k$ and suppose that $\Lambda_{\mathbf{v}}$ roots S, i.e. $S\subseteq\{v_1,\dots,v_{2k}\}$ and $|T_k(\Lambda_{\mathbf{v}}^{-1}(S))|>0$. Note that if we had $\{x_i^0,x_i^1\}\subseteq\Lambda_{\mathbf{v}}^{-1}(S)$ for some $i\in[k]$ then $|T_k(\Lambda_{\mathbf{v}}^{-1}(S))|=0$ by (9.11), a contradiction. We deduce that $|S\cap\{v_i,v_{i+k}\}|\leq 1$ for all $i\in[k]$, in particular $|S|\leq k$. Thus there exists $S'\supseteq S$ with $|S'|=k$ and such that $|S'\cap\{v_i,v_{i+k}\}|=1$ for all $i\in[k]$. However, there are at most $n^{k-|S|}$ sets S' with $|S'|=k$ and $S'\supseteq S$, and for each such S', the number of $\mathbf{v}=(v_1,\dots,v_{2k})\in\Omega_k$ with $|S'\cap\{v_i,v_{i+k}\}|=1$ for all $i\in[k]$ is at most $\Delta(\Omega_k)$. Thus, $|\{\mathbf{v}\in\Omega_k : \Lambda_{\mathbf{v}} \text{ roots } S\}|\leq n^{k-|S|}\Delta(\Omega_k)\leq n^{r-1-|S|}2^k(k!)^2b_k\leq r^{-1}\alpha\gamma n^{r-|S|}$. Similarly, we have $|\Omega_k|\leq n^k\Delta(\Omega_k)\leq r^{-1}\alpha\gamma n^r$. –

Claim 1 implies that for every $S\subseteq V(G)$ with $|S|\in[r-1]$, we have

$$|\{\mathbf{v}\in\Omega : \Lambda_{\mathbf{v}} \text{ roots } S\}|\leq \alpha\gamma n^{r-|S|}-1,$$

and we have $|\Omega|\leq b_0/h_0+\sum_{k=1}^{r-1}|\Omega_k|\leq\alpha\gamma n^r$. Therefore, by Lemma 4.7, for every $k\in[r-1]_0$ and every $\mathbf{v}\in\Omega_k$, there exists a $\Lambda_{\mathbf{v}}$-faithful embedding $\phi_{\mathbf{v}}$ of (T_k,X_k) into G, such that, letting $T_{\mathbf{v}}:=\phi_{\mathbf{v}}(T_k)$, the following hold:

(a) for all distinct $\mathbf{v}_1,\mathbf{v}_2\in\Omega$, the hulls of $(T_{\mathbf{v}_1},\mathrm{Im}(\Lambda_{\mathbf{v}_1}))$ and $(T_{\mathbf{v}_2},\mathrm{Im}(\Lambda_{\mathbf{v}_2}))$ are edge-disjoint;
(b) for all $\mathbf{v}\in\Omega$ and $e\in O$ with $e\subseteq V(T_{\mathbf{v}})$, we have $e\subseteq\mathrm{Im}(\Lambda_{\mathbf{v}})$;
(c) $\Delta(\bigcup_{\mathbf{v}\in\Omega}T_{\mathbf{v}})\leq\alpha\gamma^{(2^{-r})}n$.

Note that by (a), all the graphs $T_{\mathbf{v}}$ are edge-disjoint. Let

$$D:=\bigcup_{\mathbf{v}\in\Omega}T_{\mathbf{v}}.$$

By (c), we have $\Delta(D) \le \gamma^{-2}$. We will now show that D is as desired.

For every $k \in [r-1]$ and $\mathbf{v} \in \Omega_k$, we have that $T_{\mathbf{v}}$ is a $\mathbf{v}$-shifter with respect to F, F^* by definition of $\Lambda_{\mathbf{v}}$ and since $\phi_{\mathbf{v}}$ is $\Lambda_{\mathbf{v}}$-faithful. Thus, by Fact 9.7,

$$\mathbb{1}_{T_{\mathbf{v}}} \text{ is a } \mathbf{v}\text{-adapter with respect to } (b_0, \dots, b_k; h_k). \tag{9.12}$$

Claim 2: For every $\Omega' \subseteq \Omega$, $\bigcup_{\mathbf{v}\in\Omega'} T_{\mathbf{v}}$ has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\le(r+1)}$ and O are edge-disjoint.

Proof of claim: Clearly, for every $\mathbf{v} \in \Omega_0$, $T_{\mathbf{v}}$ is a copy of F and thus has a 1-well separated F-decomposition $\mathcal{F}_{\mathbf{v}} = \{T_{\mathbf{v}}\}$. Moreover, for each $k \in [r-1]$ and all $\mathbf{v} = (v_1, \dots, v_{2k}) \in \Omega_k$, $T_{\mathbf{v}}$ has a 1-well separated F-decomposition $\mathcal{F}_{\mathbf{v}}$ by (SH1) such that for all $F' \in \mathcal{F}_{\mathbf{v}}$ and all $i \in [k]$, $|V(F') \cap \{v_i, v_{i+k}\}| \le 1$.

In order to prove the claim, it is thus sufficient to show that for all distinct $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$, $\mathcal{F}_{\mathbf{v}_1}$ and $\mathcal{F}_{\mathbf{v}_2}$ are r-disjoint (implying that $\mathcal{F} := \bigcup_{\mathbf{v}\in\Omega'} \mathcal{F}_{\mathbf{v}}$ is 1-well separated by Fact 4.3(iii)) and that for every $\mathbf{v} \in \Omega$, $\mathcal{F}_{\mathbf{v}}^{\le(r+1)}$ and O are edge-disjoint.

To this end, we first show that for every $\mathbf{v} \in \Omega$ and $F' \in \mathcal{F}_{\mathbf{v}}$, we have that $|V(F') \cap \mathrm{Im}(\Lambda_{\mathbf{v}})| < r$ and every $e \in \binom{V(F')}{r}$ belongs to the hull of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$. If $\mathbf{v} \in \Omega_0$, this is clear since $\mathrm{Im}(\Lambda_{\mathbf{v}}) = \emptyset$ and $F' = T_{\mathbf{v}}$, so suppose that $\mathbf{v} = (v_1, \dots, v_{2k}) \in \Omega_k$ for some $k \in [r-1]$. (In particular, $h_k < b_k$.) By the above, we have $|V(F') \cap \{v_i, v_{i+k}\}| \le 1$ for all $i \in [k]$. In particular, $|V(F') \cap \mathrm{Im}(\Lambda_{\mathbf{v}})| \le k < r$, as desired. Moreover, suppose that $e \in \binom{V(F')}{r}$. If $e \cap \mathrm{Im}(\Lambda_{\mathbf{v}}) = \emptyset$, then e belongs to the hull of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$, so suppose further that $S := e \cap \mathrm{Im}(\Lambda_{\mathbf{v}})$ is not empty. Clearly, $|S \cap \{v_i, v_{i+k}\}| \le |V(F') \cap \{v_i, v_{i+k}\}| \le 1$ for all $i \in [k]$. Thus, there exists $S' \supseteq S$ with $|S'| = k$ and $|S' \cap \{v_i, v_{i+k}\}| = 1$ for all $i \in [k]$. By (SH3) (and since $h_k < b_k$), we have that $|T_{\mathbf{v}}(S')| > 0$, which clearly implies that $|T_{\mathbf{v}}(S)| > 0$. Thus, $e \cap \mathrm{Im}(\Lambda_{\mathbf{v}}) = S$ is a root of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$ and therefore e belongs to the hull of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$.

Now, consider distinct $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$ and suppose, for a contradiction, that there is $e \in \binom{V(G)}{r}$ such that $e \subseteq V(F') \cap V(F'')$ for some $F' \in \mathcal{F}_{\mathbf{v}_1}$ and $F'' \in \mathcal{F}_{\mathbf{v}_2}$. But by the above, e belongs to the hulls of both $(T_{\mathbf{v}_1}, \mathrm{Im}(\Lambda_{\mathbf{v}_1}))$ and $(T_{\mathbf{v}_2}, \mathrm{Im}(\Lambda_{\mathbf{v}_2}))$, a contradiction to (a).

Finally, consider $\mathbf{v} \in \Omega$ and $e \in O$. We claim that $e \notin \mathcal{F}_{\mathbf{v}}^{\le(r+1)}$. Let $F' \in \mathcal{F}_{\mathbf{v}}$ and suppose, for a contradiction, that $e \subseteq V(F')$. By (b), we have $e \subseteq \mathrm{Im}(\Lambda_{\mathbf{v}})$. On the other hand, by the above, we have $|V(F') \cap \mathrm{Im}(\Lambda_{\mathbf{v}})| < r$, a contradiction. —

Clearly, D is F-divisible by Claim 2. We will now show that for every F-divisible r-graph H on $V(G)$ which is edge-disjoint from D, there exists a subgraph $D^* \subseteq D$ such that $H \cup D^*$ is F^*-divisible and $D - D^*$ has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\le(r+1)}$ and O are edge-disjoint.

Let H be any F-divisible r-graph on $V(G)$ which is edge-disjoint from D. We will inductively prove that the following holds for all $k \in [r-1]_0$:

SHIFT$_k$ there exists $\Omega_k^* \subseteq \Omega_0 \cup \dots \cup \Omega_k$ such that $\mathbb{1}_{H \cup D_k^*}$ is $(b_0, \dots, b_k)$-divisible, where $D_k^* := \bigcup_{\mathbf{v}\in\Omega_k^*} T_{\mathbf{v}}$.

We first establish SHIFT$_0$. Since H is F-divisible, we have $|H| \equiv 0 \mod h_0$. Since $h_0 \mid b_0$, there exists some $0 \le a < b_0/h_0$ such that $|H| \equiv ah_0 \mod b_0$. Let Ω_0^* be the multisubset of Ω_0 consisting of $b_0/h_0 - a$ copies of $\emptyset$. Let $D_0^* := \bigcup_{\mathbf{v}\in\Omega_0^*} T_{\mathbf{v}}$. Hence, D_0^* is the edge-disjoint union of $b_0/h_0 - a$ copies of F. We thus have $|H \cup D_0^*| \equiv ah_0 + |F|(b_0/h_0 - a) \equiv ah_0 + b_0 - ah_0 \equiv 0 \mod b_0$. Therefore, $\mathbb{1}_{H\cup D_0^*}$ is (b_0)-divisible, as required.

Suppose now that SHIFT$_{k-1}$ holds for some $k \in [r-1]$, that is, there is $\Omega_{k-1}^* \subseteq \Omega_0 \cup \dots \cup \Omega_{k-1}$ such that $\mathbb{1}_{H\cup D_{k-1}^*}$ is $(b_0, \dots, b_{k-1})$-divisible, where $D_{k-1}^* := \bigcup_{\mathbf{v}\in\Omega_{k-1}^*} T_{\mathbf{v}}$. Note that D_{k-1}^* is F-divisible by Claim 2. Thus, since both H and D_{k-1}^* are F-divisible, we have $\mathbb{1}_{H\cup D_{k-1}^*}(S) = |(H \cup D_{k-1}^*)(S)| \equiv 0 \mod h_k$ for all $S \in \binom{V(G)}{k}$. Hence, $\mathbb{1}_{H\cup D_{k-1}^*}$ is in fact $(b_0, \dots, b_{k-1}, h_k)$-divisible. Thus, if $h_k = b_k$, then $\mathbb{1}_{H\cup D_{k-1}^*}$ is $(b_0, \dots, b_k)$-divisible and we let $\Omega_k' := \emptyset$. Now, assume that $h_k < b_k$. Recall that Ω_k is a $(b_0, \dots, b_k)$-balancer and that $h_k \mid b_k$. Thus, there exists a multisubset Ω_k' of Ω_k such that the function $\mathbb{1}_{H\cup D_{k-1}^*} + \sum_{\mathbf{v}\in\Omega_k'} \tau_{\mathbf{v}}$ is $(b_0, \dots, b_k)$-divisible, where $\tau_{\mathbf{v}}$ is any $\mathbf{v}$-adapter with respect to $(b_0, \dots, b_k; h_k)$. Recall that by (9.12) we can take $\tau_{\mathbf{v}} = \mathbb{1}_{T_{\mathbf{v}}}$. In both cases, let

$$\Omega_k^* := \Omega_{k-1}^* \cup \Omega_k' \subseteq \Omega_0 \cup \cdots \cup \Omega_k;$$
$$D_k' := \bigcup_{\mathbf{v} \in \Omega_k'} T_{\mathbf{v}};$$
$$D_k^* := \bigcup_{\mathbf{v} \in \Omega_k^*} T_{\mathbf{v}} = D_{k-1}^* \cup D_k'.$$

Thus, $\sum_{\mathbf{v} \in \Omega_k'} \tau_{\mathbf{v}} = \mathbb{1}_{D_k'}$ and hence $\mathbb{1}_{H \cup D_k^*} = \mathbb{1}_{H \cup D_{k-1}^*} + \mathbb{1}_{D_k'}$ is $(b_0, \dots, b_k)$-divisible, as required.

Finally, SHIFT$_{r-1}$ implies that there exists $\Omega_{r-1}^* \subseteq \Omega$ such that $\mathbb{1}_{H \cup D^*}$ is $(b_0, \dots, b_{r-1})$-divisible, where $D^* := \bigcup_{\mathbf{v} \in \Omega_{r-1}^*} T_{\mathbf{v}}$. Clearly, $D^* \subseteq D$, and we have that $H \cup D^*$ is F^*-divisible. Finally, by Claim 2,

$$D - D^* = \bigcup_{\mathbf{v} \in \Omega \setminus \Omega_{r-1}^*} T_{\mathbf{v}}$$

has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq (r+1)}$ and O are edge-disjoint, completing the proof. □

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