EDGE-DECOMPOSITIONS OF GRAPHS WITH HIGH MINIMUM DEGREE

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Abstract. A fundamental theorem of Wilson states that, for every graph F, every sufficiently large F-divisible clique has an F-decomposition. Here a graph G is F-divisible if $e(F)$ divides $e(G)$ and the greatest common divisor of the degrees of F divides the greatest common divisor of the degrees of G, and G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F. We extend this result to graphs which are allowed to be far from complete: our results imply that every sufficiently large F-divisible graph G on n vertices with minimum degree at least $(1 - (9|F|^{10})^{-1} + \varepsilon)n$ has an F-decomposition. Moreover, every sufficiently large K_3-divisible graph of minimum degree $0.956n$ has a K_3-decomposition. Our result significantly improves previous results towards the long-standing conjecture of Nash-Williams that every sufficiently large K_3-divisible graph with minimum degree $3n/4$ has a K_3-decomposition. For certain graphs, we can strengthen the general bound above. In particular, we obtain the asymptotically correct threshold of $n/2 + o(n)$ for even cycles of length at least 6. Our main contribution is a general method which turns an approximate decomposition into an exact one.

1. Introduction

Given a graph F, a graph G has an *F-decomposition* (is *F-decomposable*), if the edges of G can be covered by edge-disjoint copies of F. In this paper, we always consider decomposing a large graph G into edge-disjoint copies of some small fixed graph F. The first such result was given by Kirkman [16] in 1847, who proved that the complete graph K_n has a K_3-decomposition if and only if $n \equiv 1, 3 \mod 6$. To see that $n \equiv 1, 3 \mod 6$ is a necessary condition, note that if G has a K_3-decomposition, then the degree of each vertex of G is even and $e(G)$ is divisible by 3.

There are similar necessary conditions for the existence of an F-decomposition. For a graph G, let $\gcd(G)$ be the largest integer dividing the degree of every vertex of G. Given a graph F, we say that G is *F-divisible* if $e(G)$ is divisible by $e(F)$ and $\gcd(G)$ is divisible by $\gcd(F)$. Being F-divisible is a necessary condition for being F-decomposable. However, it is not sufficient: for example, C_6 does not have a K_3-decomposition. In this terminology, Kirkman proved that every K_3-divisible clique has a K_3-decomposition. The analogue of this for general graphs F instead

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of K_3 was an open problem for a century until it was solved by Wilson [24] in 1975. Wilson proved that, for every graph F, there exist an integer $n_0 = n_0(F)$ such that every F-divisible K_n with $n \geq n_0$ has an F-decomposition.

1.1. **Decompositions of non-complete graphs.** In contrast, it is well known that the problem of deciding whether a general graph G has an F-decomposition is NP-complete for every graph F that contains a connected component with at least three edges [4]. So a major question has been to determine the smallest minimum degree that guarantees an F-decomposition in any sufficiently large F-divisible graph G. Gustavsson [10] showed that, for every fixed graph F, there exists $\varepsilon = \varepsilon(F) > 0$ and $n_0 = n_0(F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (1-\varepsilon)n$ has an F-decomposition. (This proof has not been without criticism.) In a recent breakthrough, Keevash [14] proved a hypergraph generalisation of Gustavsson's theorem. His result actually states that every sufficiently large dense quasirandom hypergraph has a decomposition into cliques (subject to the necessary divisibility conditions). The special case for complete hypergraphs settles a question regarding the existence of designs going back to the 19th century. Yuster [25] determined the asymptotic minimum degree threshold which guarantees an F-decomposition in the case when F is a bipartite graph with $\delta(F) = 1$ (which includes trees). More recently, he [30] studied the problem of finding many edge-disjoint copies of a given graph F. For a survey regarding F-decomposition of hypergraphs, directed graphs and oriented graphs, we recommend [28].

In this paper, we substantially improve existing results when F is an arbitrary graph. For $F = K_3$, Nash-Williams [19] conjectured that every sufficiently large K_3-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a K_3-decomposition. This conjecture is still wide open. For a general K_{r+1}, the following (folklore) conjecture is a natural extension of Nash-Williams' conjecture. We describe the corresponding extremal construction in Proposition 1.6.

Conjecture 1.1. *For every $r \in \mathbb{N}$ with $r \geq 2$, there exists an $n_0 = n_0(r)$ such that every K_{r+1}-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/(r+2))n$ has a K_{r+1}-decomposition.*

The following result gives the first significant step towards the bound given by the above constructions and extends to decompositions into arbitrary graphs.

Theorem 1.2. *Let F be a graph, and let $t := \max\{9\chi(F)^{10}, 6e(F)\}$. Then for each $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/t + \varepsilon)n$ has an F-decomposition.*

Note that, for any F, we have $t \leq 9|F|^{10}$. The best previous bound in this direction is the one given by Gustavsson [10], who claimed that, if F is complete, then a minimum degree bound of $(1 - 10^{-37}|F|^{-94})n$ suffices.

For the special case of triangles we obtain the following improvement to Theorem 1.2.

Theorem 1.3. *There is an n_0 such that every K_3-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq 0.956n$ has a K_3-decomposition.*

More generally, we obtain improved bounds for some other families of graphs, including cycles (see Section 1.3).

1.2. **Approximate F-decompositions.** The main contribution of this paper is actually a result that turns an 'approximate' F-decomposition into an exact F-decomposition. Let G be a graph on n vertices. For a graph F and $\eta \geq 0$, an *η-approximate F-decomposition* $\mathcal{F}$ of G is a set of edge-disjoint copies of F covering all but at most ηn^2 edges of G. Note that a 0-approximate F-decomposition is an F-decomposition. For $n \in \mathbb{N}$, let $\delta^\eta_F(n)$ be the smallest constant δ such that every graph G on n vertices with $\delta(G) \geq \delta n$ has a η-approximate F-decomposition. Let $\delta^\eta_F := \limsup_{n\to\infty} \delta^\eta_F(n)$ be the *η-approximate F-decomposition threshold.* Clearly $\delta^{\eta'}_F \geq \delta^\eta_F$ for all $\eta' \leq \eta$. Note that there are graphs with $\lim_{\eta\to 0} \delta^\eta_F = \delta^0_F$, and graphs for which this equality does not hold (see Section 12 for a further discussion).

Our main result relates the 'decomposition threshold' to the 'approximate decomposition threshold' and an additional minimum degree condition for r-regular graphs F. The dependence on r gives the correct order of magnitude, since Proposition 1.6 shows that the term $1/3r$ cannot be replaced by anything larger than $1/(r+2)$.

Theorem 1.4. *Let F be an r-regular graph. Then for each $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon, F)$ and an $\eta = \eta(\varepsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta+\varepsilon)n$, where $\delta := \max\{\delta^\eta_F, 1-1/3r\}$, has an F-decomposition.*

To derive Theorem 1.2 from Theorem 1.4 we will use a result of Yuster [26] which guarantees a fractional F-decomposition of any graph on $n$vertices with minimum degree at least $(1 - 1/9\chi(F)^{10})n$. (An improved bound on this minimum degree was claimed by Dukes [5], but his proof contains an error.) We will also use a result by Haxell and Rödl [12] relating fractional decompositions and approximate decompositions. To derive Theorem 1.3 from Theorem 1.4 we replace the fractional decomposition result of Yuster by a result of Garaschuk [9] that (for the triangle case) circumvents the error in [5]. Our proof of Theorem 1.4 gives a polynomial time randomized algorithm which produces a decomposition with high probability (see Section 11 for more details).

1.3. **Further improvements: cycle decompositions.** In Section 11, we state a version of Theorem 1.4 which is more technical but can be applied to give better bounds for some specific choices of F (Theorem 11.1). For example, in Section 12, we apply this to prove the following result on cycle decompositions.

Theorem 1.5. *Let $\ell \in \mathbb{N}$ with $\ell \geq 3$, and let*

$$\delta := \begin{cases} 1/2 & \text{if } \ell \geq 6 \text{ is even;} \\ 2/3 & \text{if } \ell = 4; \\ 0.956 & \text{if } \ell \text{ is odd.} \end{cases}$$

Then for each $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon, \ell)$ such that every C_ℓ-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \varepsilon)n$ has a C_ℓ-decomposition.

The special case when $\ell = 4$ improves a result of Bryant and Cavenagh [3], who showed that every C_4-divisible graph G on n vertices with minimum degree at least $(31/32+o(1))n$ has a C_4-decomposition. For even cycles of length at least 6 the value of the constant δ in Theorem 1.5 is the best possible (see Proposition 12.1). For odd cycles, Theorem 11.1 and Lemma 12.2 together with the lower bound observed in Proposition 12.1 imply that the only obstacle to obtaining the best possible value of δ is finding the correct value of the approximate decomposition threshold $\delta^\eta_{C_\ell}$. It would be interesting to find other examples of graphs F for which Theorem 11.1 can be used to obtain optimal or near optimal results.

1.4. **Extremal graphs for Conjecture 1.1.** The following example from [22] shows that the minimum degree condition in Conjecture 1.1 is optimal. We include a proof for completeness.

Proposition 1.6. *For every $r \in \mathbb{N}$ with $r \geq 2$, there exist infinitely many n such that there exists a K_{r+1}-divisible graph G on n vertices with $\delta(G) = \lceil (1-1/(r+2))n \rceil - 1$ without a K_{r+1}-decomposition.*

Proof. Let $\ell, s \in \mathbb{N}$. We first consider the case when $r := 2\ell$. Let $h := (sr+1)(r+1)$. Let $K_{2\ell+2} - M$ be the subgraph of $K_{2\ell+2}$ left after removing a perfect matching. Let $G_h^{2\ell}$ be the graph constructed by blowing up each vertex of $K_{r+2} - M$ to a copy of K_h. Thus $G_h^{2\ell}$ has $n := (r+2)h$ vertices and is d-regular with $d := (h-1) + rh = (r+1)n/(r+2) - 1$. Since r divides d and $r+1$ divides h, $\binom{r+1}{2}$ divides $e(G_h^{2\ell})$, implying that $G_h^{2\ell}$ is K_{r+1}-divisible. Call an edge *internal* in $G_h^{2\ell}$ if it lies entirely within one of the copies of K_h. The number of internal edges is $I_h^{2\ell} := (r+2)\binom{h}{2}$. Since $G_h^{2\ell}$ is a blow-up of $K_{r+2} - M$, each copy of K_{r+1} in $G_h^{2\ell}$ must contain at least $r/2$ internal edges. Thus the number of edge-disjoint copies of K_{r+1} in $G_h^{2\ell}$ is at most $I_h^{2\ell}/(r/2) < e(G_h^{2\ell})/\binom{r+1}{2}$. Therefore $G_h^{2\ell}$ does not have a K_{r+1}-decomposition.

For $r := 2\ell + 1$, let $h := (s(r+1)+1)r$. Let $G_h^{2\ell+1}$ be the graph obtained from $G_h^{2\ell}$ by adding a set W of $h+1$ new vertices and joining each new vertex to each vertex in $V(G_h^{2\ell})$. Note that $G_h^{2\ell+1}$ has $n := (r+2)h + 1$ vertices and is d-regular with $d := (r+1)h = (r+1)(n-1)/(r+2) = \lceil (1 - 1/(r+2))n \rceil - 1$. Since $r(r+1)$ divides d, $\binom{r+1}{2}$ divides $e(G_h^{2\ell+1})$, implying that $G_h^{2\ell+1}$ is K_{r+1}-divisible. Let the *internal* edges of $G_h^{2\ell+1}$ be the internal edges of $G_h^{2\ell}$. Thus the number of internal edges is $I_h^{2\ell+1} := (r+1)\binom{h}{2}$. Note that each copy of K_{r+1} in $G_h^{2\ell+1}$ must contain at least $(r-1)/2$ internal edges. Moreover, if K_{r+1} contains precisely $(r-1)/2$ internal edges, then K_{r+1} must contain a vertex in W. Hence there are at most $d|W|/r = (r+1)(h+1)(s(r+1)+1)$ edge-disjoint copies of K_{r+1} in $G_h^{2\ell+1}$ that contain precisely $(r-1)/2$ internal edges. Therefore, the number of edge-disjoint copies of K_{r+1} in $G_h^{2\ell+1}$ is at most

$$
\begin{aligned}
&(r+1)(h+1)(s(r+1)+1) + \frac{I_h^{2\ell+1} - (r+1)(h+1)(s(r+1)+1)\frac{r-1}{2}}{(r+1)/2} \\
&= h(h-1) + 2(h+1)(s(r+1)+1) = (s(r+1)+1)((r+2)h - (r-2)) \\
&< (s(r+1)+1)((r+2)h+1) = \frac{e(G_h^{2\ell+1})}{\binom{r+1}{2}}.
\end{aligned}
$$

Therefore $G_s^{2\ell+1}$ does not have a K_{r+1}-decomposition. $\square$

2. Sketches of proofs

2.1. **Proof of Theorem 1.2 using Theorem 1.4.** The idea of this proof is quite natural. Given a graph F as in Theorem 1.2, we find an F-divisible regular graph R such that both the degree r of R and the η-approximate decomposition threshold δ_R^η are not too large. By removing a small number of copies of F from G, we may assume that G is also R-divisible. By Theorem 1.4, G has an R-decomposition and so an

F-decomposition, provided $\delta(G) \geq \max\{\delta_R^\eta, 1-1/3r\}$. This reduction is carried out in Section 6.

To obtain the explicit bound on $\delta(G)$, we apply a result of Yuster [26] on fractional decompositions in graphs of large minimum degree together with a result of Haxell and Rödl [12] relating fractional decompositions to approximate decompositions. We collect these tools in Section 5.

2.2. **Proof of Theorem 1.4.** The proof of Theorem 1.4 uses the 'absorbing' approach. This method was first used for finding K_3-factors (that is, a spanning union of vertex-disjoint copies of K_3) by Krivelevich [17] and for finding Hamilton cycles in hypergraphs by Rödl, Ruciński and Szemerédi [21]. An absorbing approach for finding decompositions was first used by Kühn and Osthus [18].

More precisely, the basic idea behind the proof of Theorem 1.4 can be described as follows. Let G be a graph as in Theorem 1.4. Suppose that we can find a sparse F-divisible subgraph A^* of G which is an F-absorber in the following sense: $A^* \cup H^*$ has an F-decomposition whenever H^* is a sparse F-divisible graph on $V(G)$ which is edge-disjoint from A^*. Let G' be the subgraph of G remaining after removing the edges of A^*. Since A^* is sparse, $\delta(G') \geq (\delta_F^\eta + \varepsilon/2)n$. By the definition of δ_F^η, G' has an η-approximate F-decomposition $\mathcal{F}$. Let H^* be the leftover (that is, the subgraph of G' remaining after removing all edges in $\mathcal{F}$). Note that H^* is also F-divisible. Since $A^* \cup H^*$ has an F-decomposition, so does G.

Unfortunately, this naive approach fails for the following reason: we have no control on the leftover H^*. More precisely, the natural way to obtain A^* would be to construct it as the edge-disjoint union of graphs A such that each such A has an F-decomposition and, for each possible leftover graph H^*, there is a distinct A so that $A \cup H^*$ has an F-decomposition. However, a typical leftover graph H^* has ηn^2 edges, so the number of possibilities for H^* is exponential in n. So we have no hope of finding all the required graphs A in G (and thus to construct A^*). To overcome this problem, we reduce the number of possible configurations of H^* (in turn reducing the number of graphs A required) as follows. Roughly speaking, we iteratively find approximate decompositions of the leftover so that eventually our final leftover H^* only has $O(n)$ edges whose location is very constrained—so one can view this step as finding a 'near optimal' F-decomposition.

To illustrate this, suppose that $m \in \mathbb{N}$ is bounded and n is divisible by m. Let $\mathcal{P} := \{V_1, \ldots, V_q\}$ be a partition of $V(G)$ into parts of size m (so $q = m/n$). We further suppose that H^* is a vertex-disjoint union of F-divisible graphs $H_1^*, \ldots, H_q^*$ such that $V(H_i) \subseteq V_i$ for each i. Hence to construct A^*, we only need to find one A for each possible H_i^*. (To be more precise, A^* will now consist of edge-disjoint graphs A such that each A has an F-decomposition and for each possible H_i^*, there is a distinct A so that $A \cup H_i^*$ has an F-decomposition.) For a fixed i, there are at most $2^{\binom{|V_i|}{2}} = 2^{\binom{m}{2}}$ possible configurations of H_i^*. Since m is bounded, in order to construct A^* we would only need to find $q2^{\binom{m}{2}} = 2^{\binom{m}{2}}n/m$ different A. Essentially, this is what Lemma 8.1 achieves.

We now describe in more detail the iterative approach which achieves the above setting. Recall that G' is the subgraph of G remaining after removing all the edges of A^*. Since A^* is sparse, G' has roughly the same properties as G. Our new objective is to find edge-disjoint copies of F covering all edges of G' that do not lie entirely within V_i for some i. Since each V_i has bounded size, these edge-disjoint copies of F

will cover all but at most a linear number of edges of G'. As indicated above, we use an iterative approach to achieve this. We proceed as follows. Let $k \in \mathbb{N}$. Let $\mathcal{P}_1$ be an equipartition of $V(G)$ into k parts, and let G_1 be the k-partite subgraph of G' induced by $\mathcal{P}_1$ (here k is large but bounded). Suppose that we can cover the edges of G_1 by copies of F which use only a small proportion of the edges not in G_1. Call the leftover graph H_1. Let $\mathcal{P}_2$ be an equipartition of $V(G)$ into k^2 parts obtained by dividing each $V \in \mathcal{P}_1$ into k parts. Let G_2 be the k^2-partite subgraph of H_1 induced by $\mathcal{P}_2$. Each component of G_2 will form a k-partite graph lying within some $V \in \mathcal{P}_1$. So by applying the same argument to each component of G_2 in turn and iterating $\log_k(n/m)$ times we obtain an equipartition $\mathcal{P} = \mathcal{P}_\ell$ of $V(G)$ with $|V| = m$ for each $V \in \mathcal{P}$ such that all edges of G' that do not lie entirely within some $V \in \mathcal{P}$ can be covered by edge-disjoint copies of F.

In Section 4 we prove an embedding lemma that allows us to find certain subgraphs in a dense graph. We will use this throughout the paper. The formal definition of $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_\ell$ is given in Section 7. We construct the absorber graph A^* in Section 8. The 'near optimal' decomposition result is proved in Sections 9 and 10. Finally, we prove Theorem 1.4 in Section 11.

3. Notation

Let G be a graph, and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$. We write $G[V_1]$ for the subgraph of G induced by the vertex set V_1, $G[V_1, V_2]$ for the bipartite subgraph induced by the vertex classes V_1 and V_2, and $G[\mathcal{P}] := G[V_1, \dots, V_k]$ for the k-partite subgraph of G induced by the k-partition $\mathcal{P}$. Write $V_{<i}$ for $V_1 \cup \dots \cup V_{i-1}$ and $V_{\le i}$ for $V_1 \cup \dots \cup V_i$. We say that $\mathcal{P}$ is *equitable* (or *k-equitable*) if $\big||V_i| - |V_j|\big| \le 1$ for all $1 \le i, j \le k$. For $V \subseteq V(G)$, $\mathcal{P}[V]$ denotes the restriction of $\mathcal{P}$ to V. Note that a k-equitable refinement of a k-equitable partition $\mathcal{P}$ (obtained by taking a k-equitable partition of each $V \in \mathcal{P}$) is a k^2-equitable partition of $V(G)$.

Given a graph G and disjoint $U, V \subseteq V(G)$, let $e_G(U) := e(G[U])$ and $e_G(U, V) := e(G[U, V])$. For sets $S, V \subseteq V(G)$, we write $N_G(S, V) := \{v \in V : xv \in E(G) \text{ for all } x \in S\}$, and $d_G(S, V) := |N_G(S, V)|$. If $S = \{v\}$ is a singleton, we instead write $N_G(v, V)$ and $d_G(v, V)$. We sometimes omit the subscript G if it is clear from the context.

For graphs G and H, we write $G - H$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$, and $G \setminus H$ for the subgraph of G induced by the vertex set $V(G) \setminus V(H)$. For a set of edges E, we write $G \cup E$ for the graph with vertex set $V(G) \cup V(E)$ and edge set $E(G) \cup E$.

For $r \in \mathbb{N}$, a graph G is *r-divisible* if r divides the degree $d(v)$ of v for all $v \in V(G)$.

For an integer p and a graph F, we write pF for the graph consisting of p vertex-disjoint copies of F. If G is a graph and pF is a spanning subgraph of G, then pF is an *F-factor* in G.

The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \le 1$ (where n is the order of the graph), then there is a non-decreasing function $f : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \le 1$ and all $n \in \mathbb{N}$ with $b \le f(c)$, $a \le f(b)$ and $1/n \le f(a)$. Hierarchies with more constants are defined in a similar way. We write $a = b \pm c$ to mean $a \in [b - c, b + c]$.

4. Finding subgraphs

In this section we will prove a result guaranteeing that in our given graph G we can always remove certain subgraphs that we need without significantly reducing the minimum degree of G. (These subgraphs might for example be the absorbers and parity graphs defined in Sections 8 and 9.)

Let G and H be graphs. Suppose that for each vertex of H we specify a set of vertices of G. We will seek a copy of H in G that is compatible with this specification. More formally, let $\mathcal{P}$ be a partition of $V(G)$. We say that a graph H is a *$\mathcal{P}$-labelled graph* if

- each vertex of H is labelled either $V(G)$, $\{v\}$ for some $v \in V(G)$, or V for some $V \in \mathcal{P}$;
- the vertices labelled by singletons have distinct labels and form an independent set in H.

We call the vertices labelled by singletons *root vertices*; the other vertices are *free vertices.*

An *embedding of H into G compatible with its labelling* is an injective graph homomorphism $\phi : H \to G$ such that each vertex gets mapped to an element of its label.

Given a graph H and $U \subseteq V(H)$ with $e(H[U]) = 0$, we define the *degeneracy of H rooted at U* to be the least d for which there is an ordering $v_1, \dots, v_b$ of the vertices of H such that

- there is an a such that $U = \{v_1, \dots, v_a\}$;
- for $a < j \leq b$, v_j is adjacent to at most d of the v_i with $i < j$.

Note that the requirement that the vertices in U come first means that the degeneracy of H rooted at U might be larger than the usual degeneracy of H. The *degeneracy of a $\mathcal{P}$-labelled graph H* is the degeneracy of H rooted at U, where U is the set of root vertices of H.

Lemma 4.1. *Let $n, k, d, b \in \mathbb{N}$ and let $\eta, \varepsilon > 0$ with $1/n \ll \eta \ll \varepsilon, 1/d, 1/b, 1/k$. Let G be a graph on n vertices, and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be an equitable partition of $V(G)$ such that, for each $1 \leq i \leq k$ and each $S \subseteq V(G)$ with $|S| \leq d$, $d_G(S, V_i) \geq \varepsilon|V_i|$. Let $m \leq \eta n^2$, let $s \leq \eta n$ and let $H_1, \dots, H_m$ be $\mathcal{P}$-labelled graphs such that*

(i) *for each $1 \leq i \leq m$, $|H_i| \leq b$;*
(ii) *the degeneracy of each H_i is at most d;*
(iii) *for each $v \in V(G)$, the number of indices $1 \leq i \leq m$ such that some vertex of H_i is labelled $\{v\}$ is at most s.*

Then there exist edge-disjoint embeddings $\phi(H_1), \dots, \phi(H_m)$ of $H_1, \dots, H_m$ compatible with their labellings such that the subgraph $H := \bigcup_{i=1}^m \phi(H_i)$ of G satisfies $\Delta(H) \leq \varepsilon n$.

Proof. For each $v \in V(G)$ and each $0 \leq j \leq m$, let $s(v, j)$ be the number of indices $1 \leq i \leq j$ such that some vertex of H_i is labelled $\{v\}$; so $s(v, j) \leq \eta n$.

Suppose that, for some $1 \leq j \leq m$, we have already embedded $H_1, \dots, H_{j-1}$ such that

$$d_{G_{j-1}}(v) \leq \eta^{1/2} n + (s(v, j-1) + 1)b, \tag{4.1}$$

where G_{j-1} consists of the subgraph of G used to embed $H_1, \dots, H_{j-1}$. Our next aim is to embed H_j into $G - G_{j-1}$ such that (4.1) holds with j replaced by $j+1$. By (ii), we can order the vertices of H_j such that root vertices of H_j precede free vertices of H_j and each free vertex is preceded by at most d of its neighbours. Suppose that we have already embedded some vertices of H_j one by one in this order and that the next vertex of H_j to be embedded is x.

Let $B := \{v \in V(G) : d_{G_{j-1}}(v) \geq \eta^{1/2}n\}$ be the set of vertices that are in danger of being used too many times. Since $e(G_{j-1}) \leq mdb \leq \eta dbn^2$, we have that

$$|B| \leq 2\eta dbn^2/\eta^{1/2}n \leq 2\eta^{1/2}dbn.$$

If x is a root vertex, then we can embed x at its assigned position because we have yet to embed any of its neighbours.

If x is a free vertex, then at most d of its neighbours have already been embedded. Let U be the set of images of these neighbours, and let V be the label of x. Since $|U| \leq d$, we have that $d_G(U, V) \geq \varepsilon|V|$. Thus we have that

$$\begin{aligned} d_{G-G_{j-1}}(U, V) &\geq d_G(U, V) - \textstyle\sum_{u\in U} d_{G_{j-1}}(u, V) \\ &\overset{(4.1)}{\geq} \varepsilon|V| - d(\eta^{1/2}n + (\eta n + 1)b) > |B| + |H_j|. \end{aligned}$$

So we can choose a suitable image for x outside of B.

Suppose that we have completed the embedding of H_j. We will now check that (4.1) holds with j replaced by $j+1$. Clearly (4.1) holds for every $v \in V(G)\setminus B$. But if $v \in B$, then (4.1) holds for v as well because non-root vertices of H_j were embedded outside of B and, if v is the image of a root vertex of H_j, then $s(v, j) = s(v, j-1)+1$.

Finally observe that, by (4.1), $\Delta(H) = \Delta(G_m) \leq \eta^{1/2}n + (\eta n + 1)b \leq \varepsilon n$. □

5. Fractional and approximate F-decompositions

Let F and G be graphs. Define $p_F(G)$ to be the maximum number of edges in G that can be covered by edge-disjoint copies of F. So if G has an η-approximate F-decomposition, then $e(G) - p_F(G) \leq \eta n^2$ (where G has n vertices).

Theorem 5.1 (Yuster [29]). *Let F be a graph with $\chi := \chi(F)$. For all $\eta > 0$, there exists an $n_0 = n_0(\eta, F)$ such that every graph G on $n \geq n_0$ vertices satisfies $p_F(G) \geq p_{K_\chi}(G) - \eta n^2$.*

Corollary 5.2. *Let F be a graph with $\chi := \chi(F)$. Then $\delta^\eta_F \leq \delta^{\eta/2}_{K_\chi}$ for all $\eta > 0$.*

Proof. Let $\eta > 0$ and let G be a sufficiently large graph on n vertices with $\delta(G) \geq \delta^{\eta/2}_{K_\chi}(n)n$. By the definition of $\delta^{\eta/2}_{K_\chi}(n)$ and Theorem 5.1,

$$e(G) \leq p_{K_\chi}(G) + \eta n^2/2 \leq p_F(G) + \eta n^2.$$

Therefore $\delta^\eta_F(n) \leq \delta^{\eta/2}_{K_\chi}(n)$ for all sufficiently large n, implying $\delta^\eta_F \leq \delta^{\eta/2}_{K_\chi}$. □

Write $\nu_F(G) := p_F(G)/e(F)$ for the maximum number of edge-disjoint copies of F in G. (So $\nu_F(G) = p_F(G)/e(F)$.) If G has an F-decomposition, then $\nu_F(G) = e(G)/e(F)$. We now introduce a fractional version of $\nu_F(G)$. Let $\binom{G}{F}$ denote the set of copies of F in G. A function ψ from $\binom{G}{F}$ to $[0, 1]$ is a *fractional F-packing* of G if $\sum_{F'\in\binom{G}{F}:e\in F'} \psi(F') \leq 1$ for each $e \in E(G)$. The *weight* of ψ is $|\psi| := \sum_{F'\in\binom{G}{F}} \psi(F')$. Let $\nu^*_F(G)$ be the maximum value of $|\psi|$ over all fractional F-packings ψ of G. Clearly, $\nu^*_F(G) \geq \nu_F(G)$. If $\nu^*_F(G) = e(G)/e(F)$, then we say that G has a *fractional F-decomposition.*

In fact, $\nu_F(G)$ and $\nu^*_F(G)$ are closely related. Haxell and Rödl [12] proved that any fractional packing can be converted into a genuine integer packing that covers only slightly fewer edges. (An alternative proof was given by Yuster [27].)

Theorem 5.3. [12] *Let F be a graph and let $\eta > 0$. Then there is an $n_0 = n_0(F, \eta)$ such that for every graph G on $n \geq n_0$ vertices, $\nu_F(G) \geq \nu^*_F(G) - \eta n^2$.*

For a graph F and $n \in \mathbb{N}$, let $\delta^*_F(n)$ be the smallest δ such that every graph G on n vertices with $\delta(G) \geq \delta n$ has a fractional F-decomposition. Let $\delta^*_F := \limsup_{n\to\infty} \delta^*_F(n)$ be the *fractional F-decomposition threshold*. The following corollary is an immediate consequence of Theorem 5.3 and the definitions of δ^η_F and δ^*_F.

Corollary 5.4. *For every graph F and every $\eta > 0$, we have $\delta^\eta_F \leq \delta^*_F$.*

The best known bound on $\delta^*_{K_{r+1}}$ is given by Yuster [26].

Theorem 5.5 (Yuster [26])**.** *For $r \in \mathbb{N}$ with $r \geq 2$, $\delta^*_{K_{r+1}} \leq 1 - 1/(9(r+1)^{10})$.*

Garaschuk [9] further improved the bound in the case when $r = 2$.

Theorem 5.6 (Garaschuk [9])**.** *We have that $\delta^*_{K_3} \leq \frac{95-\sqrt{185}}{104} < 0.956$.*

We can apply these results to obtain an upper bound on δ^η_F in terms of the chromatic number of F.

Lemma 5.7. *Let F be a graph with $\chi := \chi(F)$ and let $\eta > 0$. Then $\delta^\eta_F \leq 1 - 1/(9\chi^{10})$. Moreover, if $\chi = 3$, then $\delta^\eta_F < 0.956$.*

Proof. Corollary 5.2 implies that $\delta^\eta_F \leq \delta^{\eta/2}_{K_\chi}$. The result now easily follows from Corollary 5.4 and Theorems 5.5 and 5.6. □

Note that Theorem 1.3 follows immediately from Theorem 1.4 and Lemma 5.7.

6. Deriving Theorem 1.2 from Theorem 1.4

In this section we extend Theorem 1.4, which applies to regular graphs F, to Theorem 1.2, which does not require the assumption of regularity. Our approach is to combine multiple copies of F into a regular graph R and then apply Theorem 1.4 to R. We cannot do this immediately, as an F-divisible graph G need not in general also be R-divisible. We can however ensure that the extra divisibility conditions hold by removing a small number of copies of F from G.

We first prove that we can combine multiple copies of F to obtain a regular graph whose degree and chromatic number are not too large.

Lemma 6.1. *Let F be a graph. There is an F-decomposable r-regular graph R with $r = 2e(F)$ and $\chi(R) = \chi(F)$.*

We now give the main idea of the proof. Throughout the proof of the lemma, we write $[a] := \{0, 1, \dots, a-1\}$, thought of as the set of residue classes modulo a. Let $k := \chi(F)$ and fix a k-colouring of F. Let t be the size of the largest colour class. By adding isolated vertices to F if necessary, we may assume that $V(F) = [k] \times [t]$ with the k colour classes of F being $\{i\} \times [t]$ for each $i \in [k]$ (so there is no edge between (x_1, y_1) and (x_2, y_2) if $x_1 = x_2$).

For any injective function θ defined on the vertex set of a graph H, let $\theta(H)$ be the graph on the vertex set $\theta(V(H))$ for which $\theta : V(H) \to \theta(V(H))$ is an isomorphism.

Thus for $w \in [k] \times [t]$, $F+w$ is the graph obtained from F by translating each vertex by w inside $[k] \times [t]$. (To be precise, $F+w := \theta_w(F)$, where $\theta_w : (a,b) \mapsto (a+i, b+j)$ with $w = (i,j)$.) Note that $F+w$ is still k-partite with the k colour classes being $\{i\} \times [t]$ for each $i \in [k]$. Since each vertex of F is assigned to each possible position in $[k] \times [t] = V(F)$ exactly once under these translations, for each $x \in V(F)$ we have that $\sum_{w\in[k]\times[t]} d_{F+w}(x) = 2e(F)$. We would like to take R to be $\bigcup_{w\in[k]\times[t]} F+w$. However, that might produce multiple edges, so we will actually take more copies of F spread across a larger vertex set. In this way, we can achieve a similar result without producing multiple edges.

More precisely, the vertex set of R will be $V := [k] \times [t] \times [k^2t]$. (The length of the third dimension is chosen so that the multiplication maps $x \mapsto ax$ from $[kt]$ to $[k^2t]$ are injective for $a \in [k] \setminus \{0\}$.) We will embed copies of F in k^2t sets of disjoint $[k] \times [t]$ 'slices' of V. Intuitively, these sets of slices will be taken at different angles to ensure that we do not create multiple edges.

Proof of Lemma 6.1. Let $k := \chi(F)$ and fix a k-colouring of F. Let t be the size of the largest colour class. By adding isolated vertices to F if necessary, we may assume that $V(F) = [k] \times [t]$ with the k colour classes of F being $\{i\} \times [t]$ for each $i \in [k]$. Let $V := [k] \times [t] \times [k^2t]$.

For $\ell \in [k^2t]$ and $s \in [kt]$, let $\phi_{\ell,s} : [k] \times [t] \to V$ be defined by $\phi_{\ell,s}(x,y) := (x, y, \ell + xs)$. Define the *slice* $\Phi_{\ell,s}$ to be $\{\phi_{\ell,s}(x,y) : (x,y) \in [k] \times [t]\}$. Note that, for fixed s, the set $\{\Phi_{0,s}, \ldots, \Phi_{k^2t-1,s}\}$ of slices forms a partition of V.

For $w \in [k] \times [t]$, observe that $\phi_{\ell,s}(F+w)$ is k-partite with k-colourings induced by projections onto the first coordinate of V. Indeed, recall that $F+w$ has colour classes $\{0\} \times [t]$, $\{1\} \times [t]$, $\ldots$, $\{k-1\} \times [t]$ and $\phi_{\ell,s}$ preserves first coordinates. So $\phi_{\ell,s}(F+w)$ has no edges between vertices which agree in the first coordinate.

We will show that, given two points $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ with $x_1 \neq x_2$, there is at most one pair (ℓ, s) such that v_1 and v_2 are contained in $\Phi_{\ell,s}$. Indeed, suppose that v_1 and v_2 are contained in both $\Phi_{\ell,s}$ and $\Phi_{\ell',s'}$. Then $z_1 = \ell + x_1 s = \ell' + x_1 s'$ and $z_2 = \ell + x_2 s = \ell' + x_2 s'$, so $z_2 - z_1 = (x_2 - x_1)s = (x_2 - x_1)s'$. It follows that $s = s'$ since the map $u \mapsto (x_1 - x_2)u$ from $[kt]$ to $[k^2t]$ is injective, hence also that $\ell = z_1 - x_1 s = z_1 - x_1 s' = \ell'$. Recall that $\phi_{\ell,s}(F+w)$ never has an edge between two vertices which agree in the first coordinate. So for any w, w', we have that $\phi_{\ell,s}(F+w)$ and $\phi_{\ell',s'}(F+w')$ are edge-disjoint whenever $(\ell, s) \neq (\ell', s')$.

Now fix an enumeration $w_0, \ldots, w_{kt-1}$ of $[k] \times [t]$. Define $R := \bigcup_{\ell\in[k^2t], s\in[kt]} \Phi_{\ell,s}(F + w_s)$ with vertex set V. Clearly, R has an F-decomposition, is k-partite (with colour classes $\{i\} \times [t] \times [k^2t]$ for $i \in [k]$), and has no multiple edges. Since $\Phi_{0,s}, \ldots, \Phi_{k^2t-1,s}$ partition V for each $s \in [kt]$, for any vertex $v = (x, y, z) \in V$ and any $s \in [kt]$ there is precisely one $\ell \in [k^2t]$ such that v is a vertex of $\Phi_{\ell,s}(F + w_s)$. Thus

$$d_R(v) = \sum_{s\in[kt]} d_{F+w_s}((x,y)) = \sum_{u\in V(F)} d_F(u) = 2e(F).$$

Hence R is $2e(F)$-regular. □

We next show that, given a graph F, we can turn an F-divisible graph into an R-divisible graph by removing a small number of copies of F.

Lemma 6.2. *Let F be a graph and let R be an F-decomposable r-regular graph with $r = 2e(F)$. Let $\varepsilon > 0$. Then there exists an $n_0 = n_0(\varepsilon, F)$ such that, for $n \geq n_0$, the following holds. Let G be an F-divisible graph on n vertices with $\delta(G) \geq (1 - 1/r +$*

$2\varepsilon)n$. *Then there is an F-decomposable subgraph H of G such that $\Delta(H) \le \varepsilon n$ and $G - H$ is R-divisible.*

Proof. Choose $0 \le t < e(R)/e(F)$ such that $e(G) \equiv te(F) \mod e(R)$. Let $F_1, F_2, \dots, F_t$ be t vertex-disjoint copies of F in G, and let $G_0 := G - F_1 - \cdots - F_t$. Then G_0 remains F-divisible and $e(G_0)$ is divisible by $e(R)$. Note that $\delta(G_0) \ge (1 - 1/r + \varepsilon)n$.

Consider an F-decomposition $\mathcal{F}$ of R and fix an $F' \in \mathcal{F}$. Let $\mathcal{D} \subseteq \mathbb{N}$ be the set of vertex degrees of F. For each $d \in \mathcal{D}$, let v_d be a vertex of F' with $d_{F'}(v_d) = d$, and let S_d be the star consisting of v_d together with the incident edges of F'. Let R_d be the graph obtained from $R - S_d$ by adding a new vertex v'_d attached to the neighbours of v_d in F'. By construction, R_d is F-decomposable, $|R_d| = |R| + 1$, $e(R_d) = e(R)$ and every vertex of R_d has degree r except for v'_d, which has degree d, and v_d, which has degree $r - d$.

Fix an enumeration $u_1, \dots, u_n$ of $V(G)$ and, for each $1 \le i \le n-1$, choose $0 \le a_i < r$ such that $\sum_{j=1}^{i} d_{G_0}(u_j) \equiv a_i \mod r$. Since both R and G_0 are F-divisible, each a_i is divisible by $\gcd(F)$, so there exists a multiset T_i with $d \in \mathcal{D}$ for all $d \in T_i$ such that $\sum_{d \in T_i} d \equiv a_i \mod r$. Moreover, since there exist only r possible values for a_i, we may assume that there exists a $c = c(F)$ such that $|T_i| \le c$ for all i.

Let $\mathcal{P}_0 := \{V(G)\}$ be the trivial partition of $V(G)$. For each $1 \le i \le n-1$ and each $d \in T_i$, choose a $\mathcal{P}_0$-labelled copy of R_d such that the copy of v'_d is labelled $\{u_i\}$, the copy of v_d is labeled $\{u_{i+1}\}$ and all other vertices are labelled $V(G)$ (we may assume that these copies are vertex disjoint). Let $\mathcal{R}_i$ be the set of copies of R_d (one for each $d \in T_i$). Let $\mathcal{R} := \bigcup_{i=1}^{n-1} \mathcal{R}_i$. So $|\mathcal{R}_i| = |T_i| \le c$ for all i and $|\mathcal{R}| \le c(n-1)$. For each i, the number of indices such that some vertex of R_d in $\mathcal{R}$ is labelled $\{u_i\}$ is at most $|T_i| + |T_{i-1}| \le 2c$. (Here $|T_0| = |T_n| = 0$.) Recall that each copy of R_d has degeneracy at most r since $\Delta(R_d) = r$. Pick η such that $1/n \ll \eta \ll \varepsilon, 1/r, 1/f$ and apply Lemma 4.1 with G_0, 1, $|R| + 1$, r, $\varepsilon/2$, $\mathcal{P}_0$, $\mathcal{R}$ playing the roles of G, k, b, d, ε, $\mathcal{P}$, $\{H_1, \dots, H_m\}$. We obtain edge-disjoint embeddings $\phi(R_d)$ for all $R_d \in \mathcal{R}$ into G_0, which are compatible with their labelling and such that $\Delta(\bigcup_{R_d \in \mathcal{R}} \phi(R_d)) \le \varepsilon n/2$. Let $H_0 := \bigcup_{R_d \in \mathcal{R}} \phi(R_d)$; so $\Delta(H_0) \le \varepsilon n/2$.

Let $G_1 := G_0 - H_0$. Note that, for each $1 \le i \le n-1$ and each $R_d \in \mathcal{R}_i$, we have $d_{\phi(R_d)}(u_i) \equiv d \mod r$, $d_{\phi(R_d)}(u_{i+1}) \equiv -d \mod r$, and $d_{\phi(R_d)}(u_j) \equiv 0 \mod r$ for each $j \notin \{i, i+1\}$. Recall that $\sum_{d \in T_i} d \equiv a_i \equiv \sum_{j=1}^{i} d_{G_0}(u_j) \mod r$ for each $1 \le i \le n-1$. We have that

$$d_{H_0}(u_1) \equiv \sum_{R_d \in \mathcal{R}_1} d_{\phi(R_d)}(u_1) \equiv \sum_{d \in T_1} d \equiv d_{G_0}(u_1) \mod r,$$

so r divides $d_{G_1}(u_1)$. Similarly, for $2 \le i \le n-1$ we have that

$$\begin{aligned} d_{H_0}(u_i) &= \sum_{j=1}^{n-1} \sum_{R_d \in \mathcal{R}_j} d_{\phi(R_d)}(u_i) \equiv \sum_{R_d \in \mathcal{R}_{i-1}} d_{\phi(R_d)}(u_i) + \sum_{R_{d'} \in \mathcal{R}_i} d_{\phi(R_{d'})}(u_i) \mod r \\ &\equiv -\sum_{d \in T_{i-1}} d + \sum_{d' \in T_i} d' \equiv -a_{i-1} + a_i \equiv d_{G_0}(u_i) \mod r, \end{aligned}$$

so r divides $d_{G_1}(u_i)$. Recall that $e(R)$ divides $e(G_0)$ and $r = 2e(R)$, so r divides $2e(G_0)$. Finally, for $i = n$ we have that

$$d_{H_0}(u_n) \equiv \sum_{R_d \in \mathcal{R}_{n-1}} d_{\phi(R_d)}(u_n) \equiv -\sum_{d \in T_{n-1}} d \equiv -\sum_{j=1}^{n-1} d_{G_0}(u_j) \mod r,$$

so $d_{G_1}(u_n) \equiv \sum_{j=1}^{n} d_{G_0}(u_j) \equiv 2e(G_0) \equiv 0 \mod r$. Hence G_1 is r-divisible. Since G_1 was obtained from G_0 by deleting graphs with $e(R)$ edges, $e(G_1)$ is divisible by $e(R)$, so G_1 is R-divisible. Take $H := H_0 \cup F_1 \cup \cdots \cup F_t$ and observe that $\Delta(H) \leq \varepsilon n/2 + tr \leq \varepsilon n$. □

We are now ready to deduce Theorem 1.2.

Proof of Theorem 1.2. Choose $n_0 \in \mathbb{N}$ and $\eta > 0$ such that $1/n_0 \ll \eta \ll \varepsilon, 1/|F|$. Let $n \geq n_0$ and let G be an F-divisible graph on n vertices with $\delta(G) \geq (1 - 1/t + \varepsilon)n$. By Lemma 6.1, there is an F-decomposable r-regular graph R with $r = 2e(F)$ and $\chi(R) = \chi(F)$. By Lemma 6.2, there is an F-decomposable subgraph H of G such that $\Delta(H) \leq \varepsilon n/2$ and $G' := G - H$ is R-divisible.

Let $\delta := 1 - 1/t$. Lemma 5.7 implies that $\delta_R^\eta \leq 1 - (9\chi(F)^{10})^{-1} \leq \delta$. Moreover, $\delta(G') \geq \delta(G) - \Delta(H) \geq (\delta + \varepsilon/2)n$. Theorem 1.4 now implies that G' has an R-decomposition, hence also an F-decomposition. □

7. Random subgraphs and partitions

Let $m, n, N \in \mathbb{N}$ with $\max\{m, n\} < N$. Recall that the hypergeometric distribution with parameters N, n and m is the distribution of the random variable X defined as follows. Let S be a random subset of $\{1, 2, \ldots, N\}$ of size n and let $X := |S \cap \{1, 2, \ldots, m\}|$. We use the following simple form of Hoeffding's inequality, which we shall apply to both binomial and hypergeometric random variables.

Lemma 7.1 (see [13, Remark 2.5 and Theorem 2.10])**.** *Let $X \sim B(n, p)$ or let X have a hypergeometric distribution with parameters N, n, m. Then*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq 2e^{-2t^2/n}.$$

The following lemma is a simple consequence of Lemma 7.1.

Lemma 7.2. *Let $k, s \in \mathbb{N}$ and let $0 < \gamma, \rho < 1$. There is an $n_0 = n_0(k, s, \gamma)$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices and let $V_1, \ldots, V_k$ be an equitable partition of its vertex set. Let H be a graph on $V(G)$. Then there is a subgraph R of G such that, for each $1 \leq i \leq k$ and each $S \subseteq V(G)$ with $|S| \leq s$,*

$$d_R(S, V_i) = \rho^{|S|} d_G(S, V_i) \pm \gamma |V_i|$$

and for each $x, y \in V(G)$,

$$d_H(y, N_R(x, V_i)) = \rho d_H(y, N_G(x, V_i)) \pm \gamma n.$$

Proof. Let R be a random subgraph of G in which each edge is retained with probability ρ, independently from all other edges. By Lemma 7.1, for each $1 \leq i \leq k$ and each $S \subseteq V(G)$ with $|S| \leq s$,

$$\mathbb{P}(|d_R(S, V_i) - \rho^{|S|} d_G(S, V_i)| \geq \gamma |V_i|) \leq 2e^{-2(\gamma |V_i|)^2/|V_i|} \leq 2e^{-2\gamma^2 \lfloor n/k \rfloor}.$$

Similarly, for each $x, y \in V(G)$,

$$\mathbb{P}(|d_H(y, N_R(x, V_i)) - \rho d_H(y, N_G(x, V_i))| \geq \gamma n) \leq 2e^{-2\gamma^2 n}.$$

Since there are only at most $k(n+1)^s + kn^2$ conditions to check and each fails with probability exponentially small in n, some choice of R has the required properties if n is sufficiently large. □

Let G be a graph. For $k \in \mathbb{N}$ and $\delta > 0$, a (k, δ)*-partition for* G is an equitable partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of $V(G)$ such that, for each $1 \leq i \leq k$ and each $v \in V(G)$, $d_G(v, V_i) \geq \delta |V_i|$. We will often use the fact that if $\mathcal{P}$ is a $(k, \delta + \varepsilon)$-partition for G and H is a subgraph of G with $\Delta(H) \leq \varepsilon n/2k$, then $\mathcal{P}$ is a (k, δ)-partition for $G - H$.

Proposition 7.3. *Let $k \in \mathbb{N}$, and let $0 < \delta < 1$. Then there exists an $n_0 = n_0(k)$ such that any graph G on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$ has a $(k, \delta - 2n^{-1/3})$-partition.*

Proof. Consider a random equitable partition of $V(G)$ into $V_1, \dots, V_k$. For each $1 \leq i \leq k$ and for each $v \in V(G)$, by Lemma 7.1 we have that

$$\mathbb{P}(d(v, V_i) \leq \delta |V_i| - n^{2/3}/k) \leq 2e^{-2n^{1/3}/k}.$$

So for n sufficiently large we can choose an equitable partition $V_1, \dots, V_k$ such that, for each $i \leq k$ and $v \in V(G)$,

$$d(v, V_i) \geq \delta |V_i| - n^{2/3}/k \geq (\delta - 2n^{-1/3})|V_i|,$$

as required. □

Let $\mathcal{P}_1$ be a partition of $V(G)$ and for each $1 < i \leq \ell$, let $\mathcal{P}_i$ be a refinement of $\mathcal{P}_{i-1}$. We call $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ a (k, δ, m)*-partition sequence for* G if

(i) $\mathcal{P}_1$ is a (k, δ)-partition for G;
(ii) for each $2 \leq i \leq \ell$ and each $V \in \mathcal{P}_{i-1}$, $\mathcal{P}_i[V]$ is a (k, δ)-partition for $G[V]$;
(iii) for each $V \in \mathcal{P}_\ell$, $|V| = m$ or $m - 1$.

Note that (i) and (ii) imply that each $\mathcal{P}_i$ is an equitable partition of $V(G)$.

Lemma 7.4. *Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\delta, \varepsilon > 0$. There exists an $m_0 = m_0(k, \varepsilon)$ such that for all $m' \geq m_0$, any graph G on $n \geq m'$ vertices with $\delta(G) \geq \delta n$ has a $(k, \delta - \varepsilon, m)$-partition sequence for some $m' \leq m \leq km'$.*

Proof. Take $m_0 \geq \max\{n_0(k), 1000/\varepsilon^3\}$, where n_0 is the function from Proposition 7.3, and let $m' \geq m_0$. Let $\ell := \lfloor \log_k(n/m') \rfloor$. Define $\mathcal{P}_0, \dots, \mathcal{P}_\ell$ as follows. Let $\mathcal{P}_0 := \{V(G)\}$. For $j \in \mathbb{N}$, let $a_j := n^{-1/3} + (n/k)^{-1/3} + \dots + (n/k^{j-1})^{-1/3}$. Suppose that for some $1 \leq i \leq \ell$ we have already chosen $\mathcal{P}_0, \dots, \mathcal{P}_{i-1}$ such that, for each $1 \leq j \leq i - 1$ and each $V \in \mathcal{P}_{j-1}$, $\mathcal{P}_j[V]$ is a $(k, \delta - 2a_j)$-partition for $G[V]$. Since $|V| + 1 \geq n/k^{i-1} \geq n/k^{\ell-1} \geq m_0$, for each $V \in \mathcal{P}_{i-1}$ we can choose by Proposition 7.3 a $(k, \delta - 2a_i)$-partition for $G[V]$. Observing that

$$a_\ell \leq \frac{(n/k^{\ell-1})^{-1/3}}{1 - k^{-1/3}} \leq \frac{m_0^{-1/3}}{1 - 2^{-1/3}} \leq \frac{\varepsilon}{2}$$

completes the proof with $m = \lceil n/k^\ell \rceil$. □

8. Absorbers

Suppose that G is an F-divisible graph on n vertices with large minimum degree. Let $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be a (k, δ, m)-partition sequence for G given by Lemma 7.4. In our proof of Theorem 1.4, we will choose the partition so that m is bounded (i.e. each $V \in \mathcal{P}_\ell$ has bounded size). In Section 10 we will show that G can be decomposed into many copies of F and a leftover graph H^* such that $e(H^*[\mathcal{P}_\ell]) = 0$. Our aim in this section is to prove the following lemma. It guarantees the existence of an 'absorber' A^* in a dense graph G, which can absorb this leftover graph H^* (i.e. $A^* \cup H^*$ has an F-decomposition whatever the precise structure of H^*).

Lemma 8.1. *Suppose that $n, m, r, f \in \mathbb{N}$ and $\varepsilon > 0$ with $1/n \ll 1/m \ll 1/r, 1/f, \varepsilon$. Let $\delta := 1 - 1/3r + \varepsilon$, and let $q := \lceil n/m \rceil$. Suppose that F is an r-regular graph on f vertices and G is a graph on n vertices. Let $\mathcal{P} = \{V_1, \dots, V_q\}$ be an equitable partition of $V(G)$ such that, for each $1 \le i \le q$, $|V_i| = m$ or $m - 1$. Suppose that $\delta(G[\mathcal{P}]) \ge \delta n$ and $\delta(G[V_i]) \ge \delta |V_i|$ for each $1 \le i \le q$. Then G contains an F-divisible subgraph A^* such that*

(i) *$\Delta(A^*[\mathcal{P}]) \le \varepsilon^2 n$ and $\Delta(A^*[V_i]) \le r$ for each $1 \le i \le q$, and*
(ii) *if H^* is an F-divisible graph on $V(G)$ that is edge-disjoint from A^* and has $e(H^*[\mathcal{P}]) = 0$, then $A^* \cup H^*$ has an F-decomposition.*

Note that Lemma 8.1 implies that A^* itself has an F-decomposition (by taking H^* to be the empty graph). The crucial building blocks for the graph A^* in Lemma 8.1 are F-absorbers. An *F-absorber for a graph H* is a graph A such that

- A and $H \cup A$ each have F-decompositions;
- $A[V(H)]$ is empty.

Here, we sketch the proof of Lemma 8.1. The graph A^* given by Lemma 8.1 will consist of an edge-disjoint union of a set $\mathcal{A}$ of F-absorbers and a set $\mathcal{M}$ of 'edge-movers'. These graphs have low degeneracy and will be found using Lemma 4.1. The edge-movers will ensure that each $H^*[V_i]$ can be assumed to be F-divisible. Then for each $1 \le i \le q$, $\mathcal{A}$ will contain an F-absorber A_i for $H^*[V_i]$.

In the next subsection we explicitly construct an F-absorber for a given F-divisible graph H (where we may think of H as one of the possibilities for $H^*[V_i]$). We will construct this F-absorber A in a series of steps: A will consist of two 'transformers' T_1 and T_2, where T_1 will transform H into a specific graph L_h with $h := e(H)$ and T_2 will transform L_h into p vertex-disjoint copies of F, where $p := e(H)/e(F)$. This latter graph is trivially F-decomposable. Notice that if an F-absorber for H exists, then H is F-divisible. Therefore, for the rest of this section, all graphs H are assumed to be F-divisible.

8.1. An F-absorber for a given graph H.

Given an r-regular graph F and two vertex-disjoint graphs H and H', an *$(H, H')_F$-transformer* is a graph T such that

- $T \cup H$ and $T \cup H'$ each have F-decompositions;
- $V(H \cup H') \subseteq V(T)$ and $T[V(H \cup H')]$ is empty.

Thus if $\emptyset$ is an empty graph, then an $(H, \emptyset)_F$-transformer is an F-absorber for H. Write $H \sim_F H'$ if there exists an $(H, H')_F$-transformer. The relation $\sim_F$ is clearly symmetric. We now show that it is transitive on collections of vertex-disjoint graphs.

Proposition 8.2. *Let $r \in \mathbb{N}$ and let F be an r-regular graph. Suppose that H, H' and H'' are vertex-disjoint graphs. Let T_1 be an $(H, H')_F$-transformer, and let T_2 be* *an $(H', H'')_F$-transformer such that $V(T_1) \cap V(T_2) = V(H')$. Then $T := T_1 \cup H' \cup T_2$ is an $(H, H'')_F$-transformer.*

Proof. Observe that $T \cup H = (T_1 \cup H) \cup (T_2 \cup H')$ and $T \cup H'' = (T_1 \cup H') \cup (T_2 \cup H'')$ each have F-decompositions. □

We will show that in fact $H \sim_F H'$ for all vertex-disjoint F-divisible graphs H and H'. Since the empty graph is F-divisible, this in turn implies that every such H has an F-absorber. We will further show that, for each such H, we can find an F-absorber for H which has low degeneracy (rooted at $V(H)$).

We say that a graph H' is *obtained from a graph H by identifying vertices* if there is a sequence of graphs $H_0, \dots, H_s$ and vertices $x_i, y_i \in V(H_i)$ such that

(i) $H_0 = H$ and $H_s = H'$;
(ii) $(N_{H_i}(x_i) \cup \{x_i\}) \cap (N_{H_i}(y_i) \cup \{y_i\}) = \emptyset$ for all i;
(iii) for each $0 \leq i < s$, H_{i+1} is obtained from H_i by identifying the vertices x_i and y_i.

Condition (ii) ensures that the identifications do not produce multiple edges. Note that if H' can be obtained from H by identifying vertices, then there exists a graph homomorphism $\phi : H \to H'$ from H to H' that is edge-bijective. Recall that a graph H is r-divisible if r divides $d(v)$ for all $v \in V(H)$.

Fact 8.3. *Let $r \in \mathbb{N}$ and let H be an r-divisible graph. Then there is an r-regular graph H_0 such that H can be obtained from H_0 by identifying vertices.*

Proof. Split each vertex of degree sr in H into s new vertices each of degree r. □

Fact 8.3 and the next lemma together imply that, for every F-divisible graph H', there is some r-regular graph H such that $H \sim_F H'$. Recall that the degeneracy of a graph H' rooted at $U \subseteq V(H')$ was defined in Section 4.

Lemma 8.4. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Let H be an r-regular graph. Let H' be a copy of a graph obtained from H by identifying vertices. Suppose that H and H' are vertex-disjoint. Then $H \sim_F H'$. Moreover, there exists an $(H, H')_F$-transformer T such that the degeneracy of T rooted at $V(H \cup H')$ is at most $3r$ and $|T| \leq fr|H| + |H'| + fe(H)$.*

Proof. Let uv be an edge of F and let $u, v, z_1, \dots, z_{f-2}$ be the vertices of F. Let $N_F(u) = \{v, z_{a_1}, \dots, z_{a_{r-1}}\}$ and $N_F(v) = \{u, z_{b_1}, \dots, z_{b_{r-1}}\}$. (The indices a_i and b_i will be fixed throughout the rest of the proof.)

Let $\phi : H \to H'$ be a graph homomorphism from H to H' that is edge-bijective. Orient the edges of H arbitrarily. Then ϕ induces an orientation of H'. Throughout the rest of the proof, we view H and H' as oriented graphs and we write xy for the oriented edge from x to y.

For each $e \in E(H)$, let $Z^e := \{z_1^{(e)}, \dots, z_{f-2}^{(e)}\}$ be a set of $f-2$ vertices such that $V(H)$, $V(H')$, Z^e and $Z^{e'}$ are disjoint for all distinct $e, e' \in E(H)$. Define a graph T_1 as follows:

(i) $V(T_1) := V(H) \cup V(H') \cup \bigcup_{e \in E(H)} Z^e$;
(ii) $E_1 := \{xz_{a_i}^{(xy)}, yz_{b_i}^{(xy)} : 1 \leq i \leq r-1 \text{ and } xy \in E(H)\}$;
(iii) $E_2 := \{z_i^{(xy)} z_j^{(xy)} : z_i z_j \in E(F) \text{ and } xy \in E(H)\}$;
(iv) $E_3 := \{\phi(x) z_{a_i}^{(xy)}, \phi(y) z_{b_i}^{(xy)} : 1 \leq i \leq r-1 \text{ and } xy \in E(H)\}$;

(v) $E(T_1) := E_1 \cup E_2 \cup E_3$.

Note that $T_1[V(H \cup H')]$ is empty. Note also that $H \cup E_1 \cup E_2$ can be decomposed into $e(H)$ copies of F, where each copy of F has vertex set $\{x, y\} \cup Z^{(xy)}$ for some edge $xy \in E(H)$. Similarly, $H' \cup E_2 \cup E_3$ can be decomposed into $e(H)$ copies of F. In summary,

$$H \cup E_1 \cup E_2 \text{ and } H' \cup E_2 \cup E_3 \text{ each have } F\text{-decompositions.} \tag{8.1}$$

Note that every vertex $z \in V(T_1) \setminus V(H \cup H')$ satisfies

$$d_{T_1}(z) \leq \max\{r, 1 + (r-1) + 1, 2 + (r-2) + 2\} = r + 2. \tag{8.2}$$

We will now construct an additional graph T_2 such that both $T_2 \cup E_1$ and $T_2 \cup E_3$ have an F-decomposition. It will then follow that $T_1 \cup T_2$ is an $(H, H')_F$-transformer. Note that E_1 is the edge-disjoint union of $|H|$ stars $K_{1,r(r-1)}$ with centres in $V(H)$. We will obtain T_2 by viewing each star $K_{1,r(r-1)}$ as the union of $r-1$ smaller stars $K_{1,r}$, whose leaves form independent sets in T_1, and extending each of the smaller stars to a copy of F.

For each $x \in V(H)$, each neighbour y of x in H and each $1 \leq j \leq r-1$, let $u_j^{(xy)} := z_{a_j}^{(xy)}$ if the edge between x and y in H is directed toward y; otherwise let $u_j^{(xy)} := z_{b_j}^{(yx)}$. For each $x \in V(H)$ and each $1 \leq j \leq r-1$, let $N_j^x := \{u_j^{(xy)} : y \in N_H(x)\}$. The N_j^x partition $N_{T_1}(x)$ and each N_j^x forms an independent set in T_1.

For each $x \in V(H)$ and each $1 \leq j \leq r-1$, let W_j^x be a set of $f - (r+1)$ new vertices, disjoint from both $V(T_1)$ and the other $W_{j'}^{x'}$. Fix a vertex $x_0 \in V(F)$. Define a graph T_j^x on vertex set $V(T_j^x) := N_j^x \cup W_j^x$ such that T_j^x is isomorphic to $F \setminus x_0$ and the image of $N_F(x_0)$ is precisely N_j^x. Then the T_j^x are edge-disjoint and, for each $x \in V(H)$ and each $1 \leq j \leq r-1$, both $T_1[\{x\} \cup N_j^x] \cup T_j^x$ and $T_1[\{\phi(x)\} \cup N_j^x] \cup T_j^x$ are copies of F. Let $T_2 := \bigcup_{x \in V(H)} \bigcup_{j=1}^{r-1} T_j^x$ and let $T := T_1 \cup T_2$. See Figure 1 for an example with $F = C_6$.

We now claim that T is an $(H, H')_F$-transformer. Note that T_2 is edge-disjoint from T_1. Since $T_2[V(H \cup H')]$ is empty, $T[V(H \cup H')]$ is empty. Note that $T_2 \cup E_1$ has an F-decomposition into $(r-1)|H|$ copies of F, where each copy of F has vertex set $\{x\} \cup V(T_j^x)$ for some $x \in V(H)$ and some $1 \leq j \leq r-1$. Together with (8.1), this implies that $T \cup H' = (T_2 \cup E_1) \cup (E_2 \cup E_3 \cup H')$ has an F-decomposition. Similarly $T_2 \cup E_3$ has an F-decomposition into $(r-1)|H|$ copies of F, where each F has vertex set $\{\phi(x)\} \cup V(T_j^x)$ for some $x \in V(H)$ and some $1 \leq j \leq r-1$. So $T \cup H = (T_2 \cup E_3) \cup (H \cup E_1 \cup E_2)$ also has an F-decomposition. Hence T is indeed an $(H, H')_F$-transformer.

Note that each vertex in W_j^x has degree r in T. By (8.2), each vertex $z \in V(T_1) \setminus V(H \cup H')$ has degree at most $r+2+2(r-1) = 3r$ in T. Therefore, T has degeneracy at most $3r$ rooted at $V(H \cup H')$ and $|T| = |H| + |H'| + (f-2)e(H) + (f-r-1)(r-1)|H| \leq fr|H| + |H'| + fe(H)$. □

We remark that if the girth of F is large, then the degeneracy of the $(H, H')_F$-transformer constructed in the proof of Lemma 8.4, rooted at $V(H \cup H')$, is in fact smaller than $3r$. We will use this fact, captured by the following lemma, in Section 12.

Lemma 8.5. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Suppose that F contains a vertex which is not contained in any triangle in F. Let H be an*

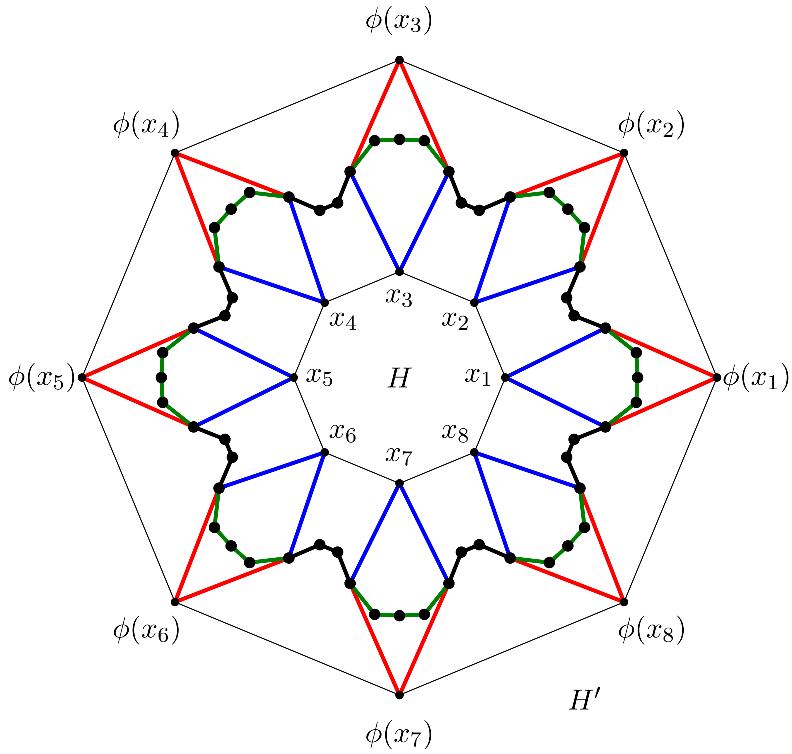

FIGURE 1. An $(H,H')_{C_6}$- transformer, where H and H' are vertex-disjoint copies of C_8.

r-regular graph. Let H' be a copy of a graph obtained from H by identifying vertices. Suppose that H and H' are vertex-disjoint. Then $H \sim_F H'$. Moreover, there exists an $(H,H')_F$-transformer T such that

(i) *the degeneracy of T rooted at $V(H \cup H')$ is at most $r+1$;*
(ii) *if F contains an edge uv that is not contained in any triangle or cycle of length 4 in F, then the degeneracy of T rooted at $V(H \cup H')$ is at most r.*

Proof. Let x_0 be a vertex of F which is not contained in any triangle in F. So $N_F(x_0)$ is an independent set in F. Also, F must contain an edge uv which is not contained in a triangle (since $r \geq 1$, we can take any edge incident to x_0). So $N_F(u)$ and $N_F(v)$ are disjoint. Moreover, if (ii) holds, then uv is the only edge in F between $N_F(u)$ and $N_F(v)$.

Let $u, v, z_1, \dots, z_{f-2}$ be the vertices of F. Let $N_F(u) = \{v, z_{a_1}, \dots, z_{a_{r-1}}\}$ and $N_F(v) = \{u, z_{b_1}, \dots, z_{b_{r-1}}\}$. Let T be the $(H,H')_F$-transformer as defined in the proof of Lemma 8.4 (with x_0 playing the role of x_0 in the proof of Lemma 8.4). To see that the degeneracy of T rooted at $V(H \cup H')$ is as desired, consider the vertices in H, H', $T_1 \setminus (H \cup H')$ and $T_2 \setminus T_1$ in that order with the vertices of $T_1 \setminus (H \cup H')$ ordered such that for each edge $xy \in E(H)$, the vertices $z^{(xy)}_{a_1}, \dots, z^{(xy)}_{a_{r-1}}, z^{(xy)}_{b_1}, \dots, z^{(xy)}_{b_{r-1}}$ come before $z^{(xy)}_j$ for $j \notin \{a_1, \dots, a_{r-1}, b_1, \dots, b_{r-1}\}$. □

Recall that the relation $\sim_F$ is transitive (on vertex-disjoint graphs) by Proposition 8.2. By Lemma 8.4, to show that $H \sim_F H'$ it suffices to show that there exists an r-regular graph H_0 (vertex-disjoint from both H and H') so that we can obtain

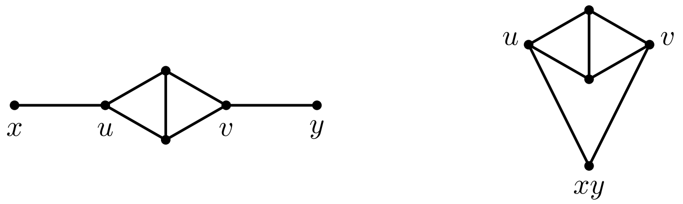

FIGURE 2. A K_4-expanded edge and a K_4-expanded loop.

both H and H' from a copy of H_0 by identifying vertices. In Lemma 8.7 we will construct such an H_0 for r-divisible graphs H and H' with the same number of edges.

Fix an edge $uv \in E(F)$. The following construction will enable us to identify vertices even if they are adjacent. Given a graph H and an edge xy of H, the *F-expansion of xy via (u,v)* is defined as follows. Consider a copy F' of F which is vertex-disjoint from H. Delete xy from H and uv from F' and join x to u and join y to v (see Figure 2).

If $x \in V(H)$, then *H with a copy of F attached to x via v* is the graph obtained from $F' \cup H$ by identifying x and v (where as before, F' is a copy of F which is vertex-disjoint from H).

Fact 8.6. *Let F be an r-regular graph and let $uv \in E(F)$. Suppose that the graph H' is obtained from a graph H by F-expanding an edge $xy \in E(H)$ via (u,v). Then the graph obtained from H' by identifying x and v is H with a copy of F attached to x via v.*

Recall that we have fixed an edge uv of F. An *F-expanded loop L* is the F-expansion of an edge xy via (u,v) with the vertices x and y identified (see Figure 2). Write L_h for h vertex-disjoint copies of L with their distinguished vertices identified. (The edge $uv \in E(F)$ used in F-expansions is always the same, so L_h is uniquely defined.)

Lemma 8.7. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Suppose that H is an r-divisible graph with $h := e(H)$, and that L_h is vertex-disjoint from H. Then $H \sim_F L_h$. Moreover, there exists an $(H, L_h)_F$-transformer T such that the degeneracy of T rooted at $V(H \cup L_h)$ is at most $3r$ and $|T| \leq |H| + |L_h| + 7f^2rh$.*

Proof. Recall that we have fixed an edge uv of F. For each edge $e \in E(H)$, attach a copy of F to one of its endpoints (chosen arbitrarily) via v; call the resulting graph H_{att}. Note that $|H_{\text{att}}| = |H| + (f-1)h$ and $e(H_{\text{att}}) = (e(F)+1)\,h$. Let H_{exp} be the graph obtained from H by F-expanding every edge in H via (u,v). By Fact 8.6, we can choose H_{exp} and H_{att} such that H_{att} can be obtained from H_{exp} by identifying vertices. By Fact 8.3, there is an r-regular graph H_0 such that H_{exp} (and so also H_{att}) can be obtained from (a copy of) H_0 by identifying vertices.

Lemma 8.4 implies that $H_0 \sim_F H_{\text{att}}$ and that there exists an $(H_0, H_{\text{att}})_F$-transformer T_1 such that the degeneracy of T_1 rooted at $V(H_0 \cup H_{\text{att}})$ is at most $3r$ and

$$|T_1| \leq fr|H_0| + |H_{\text{att}}| + fe(H_0). \tag{8.3}$$

Furthermore, we can choose T_1 such that $V(T_1) \cap V(L_h) = \emptyset$.

In H_{exp} the original vertices of H are non-adjacent with disjoint neighbourhoods, so by identifying all original vertices of H we obtain a copy of L_h from H_{exp}. Hence L_h can also be obtained from H_0 by identifying vertices, so Lemma 8.4 implies that there exists an $(H_0, L_h)_F$-transformer T_2 such that the degeneracy of T_2 rooted at $V(H_0 \cup L_h)$ is at most $3r$ and

$$|T_2| \leq fr|H_0| + |L_h| + fe(H_0). \tag{8.4}$$

Furthermore, we can choose T_2 such that $V(T_1) \cap V(T_2) = V(H_0)$. So T_1 and T_2 are edge-disjoint.

By Proposition 8.2, $T_1 \cup H_0 \cup T_2$ is an $(H_{\text{att}}, L_h)_F$-transformer. Define the graph T to be $(H_{\text{att}} - H) \cup T_1 \cup H_0 \cup T_2$. Since $H_{\text{att}} - H$ trivially has an F-decomposition, it follows that T is an $(H, L_h)_F$-transformer. To see that T has degeneracy at most $3r$ rooted at $V(H \cup L_h)$, consider the vertices in $H \cup L_h$, $H_{\text{att}} \setminus H$, H_0, $T_1 \setminus (H_{\text{att}} \cup H_0)$ and $T_2 \setminus (L_h \cup H_0)$ in that order.

Recall that $|H_{\text{att}}| = |H| + (f-1)h$ and $e(H_0) = e(H_{\text{att}}) = (e(F)+1)\,h \leq rfh$. Since H_0 is r-regular, $|H_0| = 2e(H_0)/r \leq 2fh$. By (8.3) and (8.4),

$$\begin{aligned}|T| = |T_1| + |T_2| - |H_0| &\leq |H_{\text{att}}| + |L_h| + 2fr|H_0| + 2fe(H_0)\\ &\leq |H| + |L_h| + 7f^2rh.\end{aligned}$$

This completes the proof of the lemma. □

We can now combine Lemma 8.7 and Proposition 8.2 to show that every F-divisible graph H has an F-absorber. Recall that pF consists of p vertex-disjoint copies of F.

Lemma 8.8. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Let H be an F-divisible graph. Then there is an F-absorber A for H such that the degeneracy of A rooted at $V(H)$ is at most $3r$ and $|A| \leq 9f^2r|H|^2$.*

Proof. Let $h := e(H)$ and let $p := e(H)/e(F)$. Let H, L_h and pF be vertex-disjoint. By Lemma 8.7, there exists an $(H, L_h)_F$-transformer T_1 such that the degeneracy of T_1 rooted at $V(H \cup L_h)$ is at most $3r$ and

$$|T_1| \leq |H| + |L_h| + 7f^2rh \leq |L_h| + 4f^2r|H|^2.$$

Similarly by Lemma 8.7, there exists an $(L_h, pF)_F$-transformer T_2 such that the degeneracy of T_2 rooted at $V(L_h \cup pF)$ is at most $3r$ and

$$|T_2| \leq |pF| + |L_h| + 7f^2rh = pf + hf + 1 + 7f^2rh \leq 5f^2r|H|^2.$$

Furthermore, we can choose T_1 and T_2 such that $V(T_1) \cap V(T_2) = V(L_h)$. Let $A' := T_1 \cup L_h \cup T_2$ and let $A := A' \cup pF$. By Proposition 8.2, A' is an $(H, pF)_F$-transformer. Thus A is an F-absorber for H with $|A| = |T_1| + |T_2| - |L_h| \leq 9f^2r|H|^2$. To see that the degeneracy of A rooted at $V(H)$ is at most $3r$, consider the vertices in H, L_h, pF and $T_1 \setminus (H \cup L_h)$ and $T_2 \setminus (pF \cup L_h)$ in that order (with the vertices of L_h ordered such that the distinguished vertex comes first). □

8.2. **Proof of Lemma 8.1.** Let H be an F-divisible graph and let $\mathcal{P} = \{V_1, \dots, V_q\}$ be a partition of its vertex set with $e(H[\mathcal{P}]) = 0$. (So H is the disjoint union of the $H[V_i]$.) We would like to absorb H by using Lemma 8.8 to find an F-absorber for each graph $H[V_i]$ separately. However, note that some $H[V_i]$ might not be F-divisible, as $e(H[V_i])$ might not be divisible by $e(F)$ for some $1 \leq i \leq q$. We will use 'edge-movers' to fix this problem. We first make the following simple observation, which will be used in the construction of these edge-movers.

Proposition 8.9. *Let $r \in \mathbb{N}$ and let*

$$a := a_r = \begin{cases} r & \text{if } r \text{ is odd,} \\ r/2 & \text{if } r \text{ is even.} \end{cases}$$

(i) *Let H be an r-divisible graph. Then $e(H)$ is divisible by a.*
(ii) *Let $f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. If r is odd, then let Q be an r-regular bipartite graph with each vertex class having size $f+1$. If r is even, then let Q be an r-regular graph on $2f+1$ vertices consisting of $r/2$ edge-disjoint Hamilton cycles on $V(Q)$. Then $e(Q) \equiv a \mod e(F)$.*

Proof. (i) holds since $2e(H) = \sum_{v \in V(H)} d_H(v) = pr$ for some $p \in \mathbb{N}$.

To see (ii), note that $e(F) = rf/2$. If r is odd, then $e(Q) = rf + r$; if r is even, then $e(Q) = rf + r/2$. □

Let U and V be disjoint vertex sets. Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. A *$(U, V)_F$-edge-mover* is a graph M such that

(i) M can be decomposed into Q, $\widetilde{Q}$ and A;
(ii) Q is r-regular and $V(Q) \subseteq U$;
(iii) $\widetilde{Q}$ is r-regular and $V(\widetilde{Q}) \subseteq V$;
(iv) $e(Q) \equiv a \mod e(F)$ and $e(\widetilde{Q}) \equiv -a \mod e(F)$, where a is as defined in Proposition 8.9;
(v) A is an F-absorber for $Q \cup \widetilde{Q}$.

Since A is an F-absorber for $Q \cup \widetilde{Q}$, both M and A have F-decompositions. Roughly speaking, a $(U, V)_F$-edge-mover allows us to move $a \mod e(F)$ edges from V to U (by adding Q and $\widetilde{Q}$ to the existing graph).

We are now ready to prove Lemma 8.1. In the proof, we find the copies of Q and $\widetilde{Q}$ in $G - G[\mathcal{P}]$, and the F-absorbers in $G[\mathcal{P}]$.

Proof of Lemma 8.1. Let a and Q be as defined in Proposition 8.9. Let $\widetilde{Q} := (f-1)Q$. Thus $\chi(Q) = \chi(\widetilde{Q}) \leq r+1$. Note that $\delta(G[V_i]) \geq (1 - 1/r + \varepsilon)|V_i|$ and $1/|V_i| \ll 1/r, 1/f$. So by the Erdős–Stone–Simonovits theorem [6, 23], for each $1 \leq i < q$, we can find f copies of Q in $G[V_i]$, and, for each $1 < i \leq q$, we can find f copies of $\widetilde{Q}$ in $G[V_i]$ so that all of these copies are vertex-disjoint. Call these copies $Q_1^i, \ldots, Q_f^i$ and $\widetilde{Q}_1^i, \ldots, \widetilde{Q}_f^i$ respectively.

Proposition 8.9(ii) implies that $Q_j^i \cup \widetilde{Q}_j^{i+1}$ is F-divisible for all $1 \leq i < q$ and all $1 \leq j \leq f$. Apply Lemma 8.8 to obtain an F-absorber A_j^i for $Q_j^i \cup \widetilde{Q}_j^{i+1}$ such that the degeneracy of A_j^i rooted at $V(Q_j^i \cup \widetilde{Q}_j^{i+1})$ is at most $3r$ and $|A_j^i| \leq 9f^2rm^2$ (with room to spare).

Let $H_1, \ldots, H_p$ be an enumeration of all F-divisible graphs H such that $V(H) \subseteq V_i$ for some $1 \leq i \leq q$. Since $|V_i| \leq m$ for all $1 \leq i \leq q$, for each i there are at most $2^{\binom{m}{2}}$ many $H_{j'}$ with $V(H_{j'}) \subseteq V_i$. Thus $p \leq 2^{\binom{m}{2}} q$. For each $1 \leq j' \leq p$, apply Lemma 8.8 to obtain an F-absorber $A_{j'}$ for $H_{j'}$ such that the degeneracy of $A_{j'}$ rooted at $V(H_{j'})$ is at most $3r$ and $|A_{j'}| \leq 9f^2rm^2$.

We now find the F-absorbers A_j^i and $A_{j'}$ in $G[\mathcal{P}]$ as follows. The number of F-absorbers we need to find is $(q-1)f + p$, and each of these F-absorbers has order at most $b := 9f^2rm^2$. Let $\mathcal{P}_0 := \{V(G)\}$ be the trivial partition of $V(G)$. Note that we can view each of the A_j^i and $A_{j'}$ as a $\mathcal{P}_0$-labelled graph. (For example, the

$\mathcal{P}_0$-labelled graph A^i_j is such that each $v \in V(Q^i_j \cup \widetilde{Q}^{i+1}_j)$ is labelled $\{v\}$ and every other vertex of A^i_j is labelled $V(G)$.) Note that each $v \in V(G)$ is a root for at most $s := 1 + 2^{\binom{m}{2}}$ of the A^i_j and $A_{j'}$. Since $\delta(G[\mathcal{P}]) \geq (1 - 1/3r + \varepsilon)n$, we have $d_{G[\mathcal{P}]}(S) \geq \varepsilon n$ for any $S \subseteq V(G)$ with $|S| \leq 3r$. Pick η with $1/n \ll \eta \ll 1/m$ and apply Lemma 4.1 with $G[\mathcal{P}], 1, 3r, \varepsilon^2, \mathcal{P}_0, A^1_1, A^1_2, \dots, A^{q-1}_f, A_1, \dots, A_p$ playing the roles of $G, k, d, \varepsilon, \mathcal{P}, H_1, \dots, H_m$. We obtain edge-disjoint embeddings $\phi(A^1_1)$, $\phi(A^1_2)$, $\dots$, $\phi(A^{q-1}_f)$, $\phi(A_1)$, $\dots$, $\phi(A_p)$ of A^1_1, A^1_2, $\dots$, A^{q-1}_f, A_1, $\dots$, A_p into $G[\mathcal{P}]$, which are compatible with their labellings and, moreover,

$$\Delta\Big(\bigcup_{i=1}^{q-1}\bigcup_{j=1}^{f} \phi(A^i_j) \cup \bigcup_{j'=1}^{p} \phi(A_{j'})\Big) \leq \varepsilon^2 n. \tag{8.5}$$

For each $1 \leq i < q$ and each $1 \leq j \leq f$, let $M^i_j := Q^i_j \cup \widetilde{Q}^{i+1}_j \cup \phi(A^i_j)$. Using Proposition 8.9 it is easy to check that M^i_j is a $(V_i, V_{i+1})_F$-edge-mover. Let $M := \bigcup_{i=1}^{q-1}\bigcup_{j=1}^{f} M^i_j$, and let $A^* := M \cup \bigcup_{j'=1}^{p} \phi(A_{j'})$.

We now show that A^* has the desired properties. Since A^* is an edge-disjoint union of F-absorbers and edge-movers, A^* is F-divisible. Note that $A^*[V_1] = \bigcup_{j=1}^{f} Q^1_j$, $A^*[V_q] = \bigcup_{j=1}^{f} \widetilde{Q}^q_j$ and, for each $1 < i < q$, $A^*[V_i] = \bigcup_{j=1}^{f} Q^i_j \cup \widetilde{Q}^i_j$. Thus $\Delta(A^*[V_i]) = r$ for each $1 \leq i \leq q$. Moreover, $\Delta(A^*[\mathcal{P}]) \leq \varepsilon^2 n$ by (8.5).

Let H^* be an F-divisible graph on $V(G)$ that is edge-disjoint from A^* and has $e(H^*[\mathcal{P}]) = 0$. First we show that $H^* \cup M$ can be decomposed into a graph H' and a set $\mathcal{F}$ of edge-disjoint copies of F such that $e(H'[\mathcal{P}]) = 0$ and for each $1 \leq i \leq q$, $H'[V_i]$ is F-divisible. Recall the definition of a from Proposition 8.9. Proposition 8.9(i) applied to $H^*[V_{\leq i}]$ tells us that, for each $1 \leq i \leq q$, we have $e(H^*[V_{\leq i}]) \equiv -p_i a \mod e(F)$ for some integer p_i with $0 \leq p_i < f$. Set $p_0 := 0$. For each $1 \leq i < q$, add $Q^i_1, \dots, Q^i_{p_i}, \widetilde{Q}^{i+1}_1, \dots, \widetilde{Q}^{i+1}_{p_i}$ to H^* to obtain H'. Since each $Q^i_j \cup \widetilde{Q}^{i+1}_j$ is F-divisible, so is H'. Also, for each $1 \leq i < q$,

$$\begin{aligned}
e(H'[V_i]) &= e(H^*[V_i]) + \sum_{j=1}^{p_i} e(Q^i_j) + \sum_{j'=1}^{p_{i-1}} e(\widetilde{Q}^i_{j'}) \\
&\equiv e(H^*[V_i]) + p_i a - p_{i-1} a \mod e(F) \\
&\equiv e(H^*[V_i]) - e(H^*[V_{\leq i}]) + e(H^*[V_{\leq i-1}]) \equiv 0 \mod e(F).
\end{aligned}$$

Moreover, since H^* is F-divisible,

$$\begin{aligned}
e(H'[V_q]) &= e(H^*[V_q]) + \sum_{j'=1}^{p_{q-1}} e(\widetilde{Q}^i_{j'}) \equiv e(H^*[V_q]) - p_{q-1} a \mod e(F) \\
&\equiv e(H^*[V_q]) + e(H^*[V_{<q}]) \equiv e(H^*) \equiv 0 \mod e(F).
\end{aligned}$$

Therefore $H'[V_i]$ is F-divisible for each $1 \leq i \leq q$. Note that $M - H'$ can be decomposed into $\phi(A^i_1), \dots, \phi(A^i_{p_i}), M^i_{p_i+1}, \dots, M^i_f$ for each $1 \leq i < q$, each of which has an F-decomposition. Hence $H^* \cup M$ can be decomposed into a graph H' and a set $\mathcal{F}$ of edge-disjoint copies of F such that $e(H'[\mathcal{P}]) = 0$ and for each $1 \leq i \leq q$, $H'[V_i]$ is F-divisible as claimed.

Since each $H'[V_i]$ is F-divisible, there exists a $1 \leq j'_i \leq p$ such that $A_{j'_i}$ is an F-absorber for $H'[V_i]$. Note that the indices j'_i are distinct for different $1 \leq i \leq q$.

Therefore $H' \cup \bigcup_{j'=1}^{p} \phi(A_{j'})$ has an F-decomposition $\mathcal{F}'$, so $H^* \cup A^*$ has an F-decomposition $\mathcal{F} \cup \mathcal{F}'$. This completes the proof of the lemma. □

8.3. **A strengthening of Lemma 8.1 for certain graphs F.** Let F be an r-regular graph on f vertices. Define d_F to be the smallest integer d such that for every pair of vertex-disjoint graphs H, H' such that H is r-regular and H' can be obtained from a copy of H by identifying vertices, there exists an $(H, H')_F$-transformer T such that the degeneracy of T rooted at $V(H \cup H')$ is at most d.

With this terminology, Lemma 8.4 has the following immediate corollary.

Corollary 8.10. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Then $d_F \leq 3r$.*

Our argument in this section actually gives the following stronger lemma. We omit its proof since it is virtually identical to the proof of Lemma 8.1 (with Lemma 8.4 replaced by the definition of d_F). Note that we do not have an explicit bound on the number of vertices of the $(H, H')_F$-transformer T of degeneracy d_F. However, by the definition d_F there is a function g so that $|T| \leq g(|H|)$, and such a bound is all we need to apply Lemma 4.1.

Lemma 8.11. *Suppose that $n, m, r, f \in \mathbb{N}$ and $\varepsilon > 0$ with $1/n \ll 1/m \ll 1/r, 1/f, \varepsilon$. Suppose that F is an r-regular graph on f vertices. Let $\delta := 1 - \min\{1/r, 1/d_F\} + \varepsilon$, and let $q := \lceil n/m \rceil$. Let G be a graph on n vertices. Let $\mathcal{P} = \{V_1, \dots, V_q\}$ be an equitable partition of $V(G)$ such that, for each $1 \leq i \leq q$, $|V_i| = m$ or $m-1$. Suppose that $\delta(G[\mathcal{P}]) \geq \delta n$ and $\delta(G[V_i]) \geq \delta |V_i|$ for each $1 \leq i \leq q$. Then G contains an F-divisible subgraph A^* such that*

(i) *$\Delta(A^*[\mathcal{P}]) \leq \varepsilon^2 n$ and $\Delta(A^*[V_i]) \leq r$ for each $1 \leq i \leq q$, and*
(ii) *if H^* is an F-divisible graph on $V(G)$ that is edge-disjoint from A^* and has $e(H^*[\mathcal{P}]) = 0$, then $A^* \cup H^*$ has an F-decomposition.*

9. Parity graphs

Let F be an r-regular graph, let x be a vertex of F, and let $F_x := F[N_F(x)]$. Let G be an F-divisible graph with a (k, δ)-partition $\mathcal{P} = \{V_1, \dots, V_k\}$, and suppose that $G[\mathcal{P}]$ is sparse. Our aim is to use a small number of edges from $G - G[\mathcal{P}]$ to cover all edges of $G[\mathcal{P}]$ by copies of F. We will do this by, for each $1 \leq i < j \leq k$ and each $v \in V_i$, finding an F_x-factor in $N_G(v, V_j)$. We will then extend each copy of F_x to a copy of $F - x$ using Lemma 4.1. Together with the edges incident to v, these copies of $F - x$ will form copies of F. An obvious necessary condition for this to work is that each $d_G(v, V_j)$ is divisible by r. In this section we show that we can find certain structures, which we call parity graphs, that can be used to ensure that this divisibility condition holds.

Let U and V be disjoint subsets of $V(G)$ and let $x, y \in U$. Let F be an r-regular graph. An *xy-shifter with parameters U, V, F* is a graph S with $V(S) \subseteq U \cup V$ such that $xy \notin E(S)$ and

(i) $d_S(x, V) \equiv -1 \mod r$, $d_S(y, V) \equiv 1 \mod r$ and, for all $u \in U \setminus \{x, y\}$, $d_S(u, V) \equiv 0 \mod r$;
(ii) S has an F-decomposition.

Condition (i) allows us to move excess degree (mod r) from x to y.

Let $uv \in E(F)$. For a graph H and an edge $xy \in E(H)$, *H with a copy of F glued along xy via uv* is a graph obtained from H by adding a copy F' of F that is vertex-disjoint from H and identifying u with x and v with y.

Proposition 9.1. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Let U and V be disjoint vertex sets with $|U| \geq r+2$ and $|V| \geq \binom{r+1}{2}(f-2)$, and let $x, y \in U$. Then there exists an xy-shifter S with parameters U, V, F with $r+2$ vertices in U, $\binom{r+1}{2}(f-2)$ vertices in V and degeneracy at most r rooted at $\{x, y\}$.*

Proof. Pick r distinct vertices $u_1, \ldots, u_r$ in $U \setminus \{x, y\}$. We first define a subgraph S_0 of S on vertex set $\{x, y, u_1, \ldots, u_r\} \subseteq U$. Join x to u_1, join y to $u_2, \ldots, u_r$ and join $u_1, \ldots, u_r$ completely. (So if x and y were identified we would obtain a copy of K_{r+1}.) Thus $d_{S_0}(x) = 1$, $d_{S_0}(y) = r-1$, and $d_{S_0}(u_j) = r$ for $1 \leq j \leq r$.

Let $uv \in E(F)$. Let S be the graph obtained from S_0 by gluing a copy of F along each edge of S_0 via uv such that $V(F) \setminus \{u, v\} \subseteq V$ (and these sets are disjoint for different copies). Then S has an F-decomposition, $d_S(x, V) = r-1$, $d_S(y, V) = (r-1)^2$ and $d_S(u_j, V) = r(r-1)$ for each $1 \leq j \leq r$. Ordering $V(S)$ such that x and y are the first two vertices, and all other vertices in S_0 precede those in $S \setminus S_0$, shows that the degeneracy of S is at most r. □

Let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be an equitable partition of a vertex set V. An *F-parity graph with respect to $\mathcal{P}$* is an F-decomposable graph P on V such that, for every r-divisible graph G on V that is edge-disjoint from P and there is a subgraph P' of P such that

(P1) for each $2 \leq i \leq k$ and each $x \in V_{<i}$, r divides $d_{G \cup P'}(x, V_i)$;
(P2) $P - P'$ has an F-decomposition.

Next we show that F-parity graphs exist.

Proposition 9.2. *Let $r, f, k \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be an equitable partition of a vertex set V. Let $P_2, \ldots, P_k$ be edge-disjoint graphs on V such that, for each $2 \leq i \leq k$,*

- *P_i is the edge-disjoint union of E_i and D_i;*
- *E_i is the edge-disjoint union of $r-1$ copies of F, each with 2 adjacent vertices in V_i and $f-2$ vertices in V_{i-1};*
- *D_i is the edge-disjoint union of $r-1$ u_ju_{j+1}-shifters with parameters $V_{<i}, V_i, F$ for each $1 \leq j < |V_{<i}|$, where $u_1, \ldots, u_{|V_{<j}|}$ is an enumeration of $V_{<i}$.*

Then $P := P_2 \cup \cdots \cup P_k$ is an F-parity graph with respect to $\mathcal{P}$.

Proof. The proof is by induction on k. If $k = 1$, then there is nothing to prove, so assume that $k \geq 2$. Since each D_i has an F-decomposition, so does P.

Let G be an r-divisible graph on V that is edge-disjoint from P. First we show that there is a subgraph P'_k of P such that

(i) for each $x \in V_{<k}$, r divides $d_{G \cup P'_k}(x, V_i)$;
(ii) $P_k - P'_k$ has an F-decomposition.

Suppose that $e_G(V_k) \equiv t \mod r$, where $0 < t \leq r$. Form a graph G_0 from G by adding $r-t$ of the copies of F from E_k to G. Then $2e_{G_0}(V_k) \equiv 0 \mod r$, so

$$\sum_{v \in V_{<k}} d_{G_0}(v, V_k) = e_{G_0}(V_{<k}, V_k) = \sum_{v \in V_k} d_{G_0}(v) - 2e_{G_0}(V_k) \equiv 0 \mod r. \quad (9.1)$$

Let $\ell := |V_{<k}|$ and let $u_1, \dots, u_\ell$ be the enumeration of $V_{<k}$ used in the definition of D_k.

Let $0 \le t_1 < r$ be such that $d_{G_0}(u_1, V_k) \equiv t_1 \mod r$. Add t_1 of the u_1u_2-shifters in D_k to G_0 to obtain G_1 in which $d_{G_1}(u_1, V_k) \equiv 0 \mod r$ and $d_{G_1}(u_i, V_k) \equiv d_{G_0}(u_i, V_k) \mod r$ for all $3 \le i \le \ell$.

Let $0 \le t_2 < r$ be such that $d_{G_1}(u_2, V_k) \equiv t_2 \mod r$. Add t_2 of the u_2u_3-shifters in D_k to G_1 to obtain G_2 in which $d_{G_2}(u_1, V_k) \equiv d_{G_2}(u_2, V_k) \equiv 0 \mod r$ and $d_{G_2}(u_i, V_k) \equiv d_{G_1}(u_i, V_k) \equiv d_{G_0}(u_i, V_k) \mod r$ for all $4 \le i \le \ell$.

Continuing in this way, we eventually obtain $G_{\ell-1}$ in which $d_{G_{\ell-1}}(u_i, V_k) \equiv 0 \mod r$ for each $1 \le i \le \ell - 1$. Note that

$$\begin{aligned} d_{G_{\ell-1}}(u_\ell, V_k) &\equiv d_{G_{\ell-2}}(u_\ell, V_k) + d_{G_{\ell-2}}(u_{\ell-1}, V_k) \mod r \\ &\equiv d_{G_0}(u_\ell, V_k) + d_{G_{\ell-3}}(u_{\ell-1}, V_k) + d_{G_{\ell-3}}(u_{\ell-2}, V_k) \mod r \\ &\equiv \sum_{v \in V_{<k}} d_{G_0}(v, V_k) \equiv 0 \mod r, \end{aligned}$$

where the last equality holds by (9.1). Let $P'_k := G_{\ell-1} - G$; then (i) holds. Observe also that $P_k - P'_k$ consists of some copies of F from E_k and some shifters from D_k, each of which has an F-decomposition, so (ii) holds.

Let $G^* := (G \cup P'_k)[V_{\le k-1}]$, $P^* := P_2 \cup \dots \cup P_{k-1}$ and $\mathcal{P}^* := \{V_1, \dots, V_{k-1}\}$. Note that G^* and P^* are edge-disjoint. Recall that G, P_k and $P_k - P'_k$ are r-divisible. So $G \cup P'_k$ is r-divisible. Thus (i) implies that G^* is also r-divisible. By the induction hypothesis, P^* is an F-parity graph with respect to $\mathcal{P}^*$. Therefore, there exists a subgraph P_0 of P^* such that for each $2 \le i \le k-1$ and each $x \in V_{<i}$, r divides $d_{G^* \cup P_0}(x, V_i)$ and $P^* - P_0$ has an F-decomposition. Let $P' := P_0 \cup P'_k$. Then P' satisfies (P2). Note that $(G \cup P')[V_{<k}, V_k] = (G \cup P'_k)[V_{<k}, V_k]$ and $(G \cup P')[V_{<i}, V_i] = (G^* \cup P_0)[V_{<i}, V_i]$ for all $1 \le i < k$. Thus P' satisfies (P1). Therefore P is an F-parity graph with respect to $\mathcal{P}$. □

The next lemma finds an F-parity graph P as in Proposition 9.2 within a dense graph G using Lemma 4.1.

Lemma 9.3. *Let $r, f \in \mathbb{N}$ and let F be an r-regular graph on f vertices. Let $\gamma > 0$. Then there exists an $n_0 = n_0(k, \gamma, F)$ such that the following holds. Let G be a graph on $n \ge n_0$ vertices and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a (k, δ)-partition for G with $\delta \ge 1 - 1/r + \gamma$. Then G contains an F-parity graph P with respect to $\mathcal{P}$ such that $\Delta(P) \le \gamma n$.*

Proof. It is enough to show that we can embed a graph P as described in Proposition 9.2 into G in such a way that the maximum degree of the image of the embedding is not too large. We will assign labels to the graphs making up P and then check that the conditions of Lemma 4.1 hold.

For each $2 \le i \le k$ and each $1 \le j \le r-1$, let $F'_{i,j}$ be a $\mathcal{P}$-labelled copy of F with 2 adjacent vertices labelled V_i and $f-2$ vertices labelled V_{i-1}.

For each $2 \le i \le k$, let $n_{<i} := |V_{<i}|$ and let $u^i_1, \dots, u^i_{n_{<i}}$ be an enumeration of the vertices of $V_{<i}$. For each $2 \le i \le k$ and each $1 \le j < n_{<i}$, apply Proposition 9.1 to obtain a $u^i_j u^i_{j+1}$-shifter $S_{i,j}$ with parameters $V_{<i}, V_i, F$ such that $|S_{i,j}| = r + 2 + \binom{r+1}{2}(f-2)$ and $S_{i,j}$ has degeneracy at most r rooted at $\{u^i_j, u^i_{j+1}\}$. We may view $S_{i,j}$ as a $\mathcal{P}$-labelled graph by giving u^i_j the label $\{u^i_j\}$, giving u^i_{j+1} the label $\{u^i_{j+1}\}$, giving u the label V_i for all $u \in V(S_{i,j}) \cap V_i$ and giving u' the label V_{i-1} for all

$u' \in (V(S_{i,j}) \cap V_{<i}) \setminus \{u^i_j, u^i_{j+1}\}$. Let $S'_{i,j,1}, \dots, S'_{i,j,r-1}$ be $r-1$ copies of $S_{i,j}$, and let

$$\mathcal{F} := \bigcup_{i=2}^{k} \bigcup_{\ell=1}^{r-1} \Big(\{F'_{i,\ell}\} \cup \bigcup_{j=1}^{n_{<i}-1} \{S'_{i,j,\ell}\} \Big).$$

So $\mathcal{F}$ is a family of $\mathcal{P}$-labelled graphs and $|\mathcal{F}| \leq krn$. For each $F' \in \mathcal{F}$, $|F'| \leq r + 2 + \binom{r+1}{2}(f-2)$ and F' has degeneracy at most r. Furthermore, each $v \in V(G)$ is a root vertex for at most $2rk$ members of $\mathcal{F}$. Since $\mathcal{P}$ is a (k,δ)-partition for G with $\delta \geq 1 - 1/r + \gamma$, we have that $d_G(S, V_i) \geq \gamma |V_i|$ for each $S \subseteq V(G)$ with $|S| \leq r$ and each $1 \leq i \leq k$. Therefore we can apply Lemma 4.1 to find edge-disjoint embedding $\phi(F')$ for all $F' \in \mathcal{F}$ in G in such a way that $\Delta(\bigcup_{F' \in \mathcal{F}} \phi(F')) \leq \gamma n$. Take $P := \bigcup_{F' \in \mathcal{F}} \phi(F')$. $\square$

10. Near optimal decompositions

Let G be a dense graph as defined in Theorem 1.4, and let $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be a $(k, \delta + \varepsilon, m)$-partition sequence for G. In Section 8, we constructed a graph A^* that can 'absorb' any F-divisible graph H^* satisfying $e(H^*[\mathcal{P}_\ell]) = 0$. Our aim in this section is to show that we can indeed decompose G into edge-disjoint copies of F and such a remainder H^*. More precisely, in this section, we prove the following lemma, which guarantees the existence of such a 'near optimal' F-decomposition (in particular note that, if m is bounded, then $e(H^*)$ is at most linear in n).

Lemma 10.1. *Let $r, f, m, k, \ell \in \mathbb{N}$ and let $\varepsilon, \eta > 0$ with $1/m \ll \eta \ll 1/k \ll \varepsilon, 1/r, 1/f$. Let F be an r-regular graph on f vertices and let G be an r-divisible graph. Let $\delta := \max\{\delta^\eta_F, 1 - 1/(r+1)\}$. Suppose that $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is a $(k, \delta + \varepsilon, m)$-partition sequence for G. Then there exists a subgraph H^* of $\bigcup_{V \in \mathcal{P}_\ell} G[V]$ such that $G - H^*$ has an F-decomposition. In particular, if G is F-divisible, then so is H^*.*

Recall that the definition of δ^η_F implies that G contains an η-approximate F-decomposition. We would like the remainder H^* in Lemma 10.1 to contain no edges of $G[\mathcal{P}_\ell]$, but the definition of δ^η_F does not guarantee this. The key idea of the proof of Lemma 10.1 is to proceed via an iterative process, which repeatedly invokes the definition of δ^η_F. More precisely, suppose that we are able to prove the following result:

- (†) If $\mathcal{P}$ is a (k,δ)-partition for a graph G, then $G[\mathcal{P}]$ can be covered by edge-disjoint copies of F in G which use only a small number of edges from $G - G[\mathcal{P}]$.

Suppose that we apply (†) with $\mathcal{P} = \mathcal{P}_1$. We are then left with edges in $G - G[\mathcal{P}_1] = \bigcup_{V \in \mathcal{P}_1} G[V]$. But since $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is a $(k, \delta + \varepsilon, m)$ partition sequence and we have used very few edges of $G - G[\mathcal{P}_1]$, we have for each $V \in \mathcal{P}_1$ that $\mathcal{P}_2[V]$ is a (k,δ)-partition of the remaining part of $G[V]$. So we can apply (†) to each part to cover the remaining edges of $G[\mathcal{P}_2]$ by edge-disjoint copies of F, using only a few edges from $G - G[\mathcal{P}_2]$. Continuing in this way, we eventually obtain edge-disjoint copies of F covering all edges of $G - G[\mathcal{P}_\ell]$, which implies Lemma 10.1. (To avoid our bound on the minimum degree deteriorating in each step, we actually prove a stronger version of (†) which gives us more control on the edges we use from $G - G[\mathcal{P}]$.)

The rest of this section is divided into three subsections. In Section 10.1, we show that we can find an approximate F-decomposition of $G[\mathcal{P}]$ such that the remainder has low maximum degree (at the cost of using a small number of additional edges from $G - G[\mathcal{P}]$). In Section 10.2 we show how such a remainder of low maximum degree can be covered by copies of F. In Section 10.3 we give a formal statement of (†) and perform the iteration described above.

10.1. **Bounding maximum degree of the remainder graph.** Consider an η-approximate F-decomposition $\mathcal{F}$ of $G[\mathcal{P}]$ guaranteed by the definition of δ_F^η. Let H be the remainder of $G[\mathcal{P}]$ (after removing all the edges of $\mathcal{F}$), and suppose that $d_H(x, V)$ is large for some $V \in \mathcal{P}$ and some $x \in V(G) \setminus V$. Note that x together with a copy of K_r that lies in $N_H(x, V)$ forms a copy of K_{r+1}. Using some additional vertices and edges inside V, we can then extend a spanning subgraph of this copy of K_{r+1} to a copy of F. So we can reduce $d_H(x, V)$ by finding vertex-disjoint copies of K_r lying entirely in $N_H(x, V)$, which are then extended into copies of F. This is formalised in Lemma 10.6. To find the above copies of K_r we shall use the Hajnal–Szemerédi theorem [11].

Theorem 10.2 ([11]). *Let $r \in \mathbb{N}$ with $r \geq 2$. Every graph G on n vertices with $\delta(G) \geq (1 - 1/r)n$ contains $\lfloor n/r \rfloor$ vertex-disjoint copies of K_r.*

Lemma 10.3. *Let $r, k, n \in \mathbb{N}$ and let $\gamma > 0$ with $1/n \ll \gamma, 1/k, 1/r$. Let H be a graph on n vertices. Let $U, V \subseteq V(H)$ be disjoint with $|V| \geq \lfloor n/k \rfloor$. Suppose that, for each $x \in U$ and each $y \in V$,*

(i) *r divides $d_H(x, V)$;*
(ii) *$\delta(H[N_H(x, V)]) \geq (1 - 1/r)d_H(x, V) + \gamma|V|$;*
(iii) *$d_H(y, U) \leq \gamma|V|/r$.*

Then there is a subgraph H_V of $H[V]$ such that $H[U, V] \cup H_V$ has a K_{r+1}-decomposition and $\Delta(H_V) \leq \gamma|V|$.

Proof. For each $x \in U$ in turn we will choose a K_r-factor from the unused part of $H[N_H(x, V)]$ and take H_V to be the union of these edge-disjoint K_r-factors.

We claim that we can choose these K_r-factors greedily. Indeed, suppose we seek a K_r-factor for $x \in U$. Consider any vertex $y \in N_H(x, V)$. By (iii), at most $rd_H(y, U) \leq \gamma|V|$ of the edges at y in $H[N_H(x, V)]$ have been used already. So by (i), (ii) and Theorem 10.2 there exists a K_r-factor in the unused part of $H[N_H(x, V)]$.

Since at most $rd_H(y, U) \leq \gamma|V|$ edges are used at each $y \in V$, we have that $\Delta(H_V) \leq \gamma|V|$. □

Lemma 10.4. *Let $r, f, k, n \in \mathbb{N}$ and let $\eta, \gamma > 0$ with $1/n \ll \eta \ll \gamma, 1/k, 1/r, 1/f$. Let F be an r-regular graph on f vertices and let H be a graph on n vertices. Let $U, V \subseteq V(H)$ be disjoint with $|V| \geq \lfloor n/k \rfloor$. Suppose that, for each $x \in U$ and each $y \in V$,*

(i) *r divides $d_H(x, V)$;*
(ii) *$\delta(H[N_H(x, V)]) \geq (1 - 1/r)d_H(x, V) + \gamma|V|$;*
(iii) *$d_H(y, U) \leq \eta|V|$;*
(iv) *$\delta(H[V]) \geq (1 - 1/r + 2\gamma)|V|$.*

Then there is a subgraph H'_V of $H[V]$ such that $H[U, V] \cup H'_V$ has an F-decomposition and $\Delta(H'_V) \leq 2\gamma|V|$.

Proof. Apply Lemma 10.3 to obtain a subgraph H_V of $H[V]$ such that $H[U, V] \cup H_V$ has a K_{r+1}-decomposition and $\Delta(H_V) \leq \gamma|V|$. Let $W_1, \dots, W_p$ be an enumeration of such a K_{r+1}-decomposition of $H[U, V] \cup H_V$ such that each W_j has vertex set

$\{w_j\} \cup W'_j$ with $w_j \in U$ and $W'_j \subseteq V$. Note that

$$p \leq \sum_{y \in V} d_H(y, U) \leq \eta |V|^2.$$

Let $H' := H[V] - H_V$; then $|H'| = |V|$ and $\delta(H') \geq (1 - 1/r + \gamma)|V|$ by (iv). Let $u \in V(F)$ and let $F^* := F \setminus \{u\} - F[N_F(u)]$. Note that F^* trivially has degeneracy at most r rooted at $N_F(u)$. Let $F_1^*, \ldots, F_p^*$ be copies of F^*. We now embed $F_1^*, \ldots, F_p^*$ into H' in such a way that, for each F_j^*, the image of $N_F(u)$ is precisely W'_j as follows. Let $s := \eta|V|$ and let $\mathcal{P}_0 := \{V\}$ be the trivial partition of V. We view each F_j^* as a $\mathcal{P}_0$-labelled graph such that the root vertices of F_j^* are precisely $N_F(u)$, and the union of their labels is W'_j; each other vertex of F^* is labelled V. There are at most $d_H(y, U) \leq s$ indices j with $1 \leq j \leq p$ such that some vertex of F_j^* is labelled $\{y\}$. Since $\delta(H') \geq (1 - 1/r + \gamma)|V|$, we have that $d_{H'}(S) \geq \gamma|V|$ for each $S \subseteq V$ with $|S| \leq r$. So by Lemma 4.1, with H', 1, r, f, γ, $\mathcal{P}_0$, $F_1^*, \ldots, F_p^*$ playing the roles of G, k, d, b, ε, $\mathcal{P}$, $H_1, \ldots, H_m$, there exist edge-disjoint embeddings $\phi(F_1^*)$, $\ldots$, $\phi(F_p^*)$ of F_1^*, $\ldots$, F_p^* into H' which are compatible with their labelling such that $\Delta(\bigcup_{j=1}^p \phi(F_j^*)) \leq \gamma|V|$.

Each $W_j \cup \phi(F_j^*)$ contains a copy F_j of F such that $H[U, V] \subseteq \bigcup_{j=1}^p F_j$. Let $H'_V := \bigcup F_j[V]$. Note that $H[U, V] \cup H'_V$ has an F-decomposition and $\Delta(H'_V) \leq \Delta(H_V) + \Delta(\bigcup_{j=1}^p \phi(F_j^*)) \leq 2\gamma|V|$. □

Proposition 10.5. *Let $r, k \in \mathbb{N}$ and let $\varepsilon \geq 0$. Let G be a graph and let $\mathcal{P}$ be a $(k, 1 - 1/(r + 1) + \varepsilon)$-partition for G. Let $x \in V(G)$ and let $V \in \mathcal{P}$. Then*

$$\delta(G[N_G(x, V)]) \geq (1 - 1/r)d_G(x, V) + \varepsilon|V|.$$

Proof. Let $y \in N_G(x, V)$. Since $d_G(y, V), d_G(x, V) \geq (1 - 1/(r + 1) + \varepsilon)|V|$,

$$\begin{aligned} d_G(y, N_G(x, V)) &\geq d_G(x, V) + d_G(y, V) - |V| \geq d_G(x, V) - \frac{|V|}{r + 1} + \varepsilon|V| \\ &\geq (1 - 1/r)d_G(x, V) + \varepsilon|V|. \end{aligned}$$ □

Lemma 10.6. *Let $r, f, k, n \in \mathbb{N}$ and let $\gamma, \eta > 0$ with $1/n \ll \eta \ll \gamma, 1/k, 1/r, 1/f$. Let F be an r-regular graph on f vertices and let G be a graph on n vertices. Let $\delta := \max\{\delta_F^\eta, 1 - 1/(r+1)\}$. Suppose that $\mathcal{P} = \{V_1, \ldots, V_k\}$ is a $(k, \delta + 3\gamma)$-partition for G. Then there is a subgraph H of G such that*

(a) *$G - H$ has an F-decomposition;*
(b) *$\Delta(H[\mathcal{P}]) \leq \gamma n$.*
(c) *for each $1 \leq i \leq k$, $\Delta(G[V_i] - H[V_i]) \leq 2\gamma|V_i|$.*

Proof. By the definition of δ_F^η, there exists an η-approximate F-decomposition $\mathcal{F}$ of $G[\mathcal{P}]$. Let G_0 be the subgraph of $G[\mathcal{P}]$ which consists of the uncovered edges; so $e(G_0) \leq \eta n^2$. Let $B := \{v \in V(G) : d_{G_0}(v) > \eta^{1/2}n\}$ and let $A := V(G) \setminus B$; observe that $|B| \leq 2\eta^{1/2}n$. Let H' be the union of G_0 and each of the copies of $F \in \mathcal{F}$ that contains a vertex of B. Note that $G - H'$ has an F-decomposition $\mathcal{F}_1$ and that

$$N_{H'}(v) = N_{G[\mathcal{P}]}(v) \text{ for all } v \in B. \tag{10.1}$$

For any $v \in A$, at most $|B|$ copies of F containing v were added to G_0 to form H', so $d_{H'}(v) \leq \eta^{1/2}n + 2r\eta^{1/2}n = (2r + 1)\eta^{1/2}n$.

We now find a set $\mathcal{F}_2$ of edge-disjoint copies of F that cover most of the edges incident on B in H'. To do this we will use some edges of $G - G[\mathcal{P}]$.

For each $1 \leq i \leq k$, let $B_i := B \setminus V_i$ and let $V'_i := V_i \setminus B$. Let H^*_i be the graph on vertex set $V(G)$ with $E(H'_i) := E(H'[B_i, V'_i]) \cup E(G[V'_i])$. Note that the H^*_i are edge-disjoint. By removing at most $r-1$ edges incident to each $v \in B_i$ from H^*_i, we obtaining a spanning subgraph H'_i of H^*_i which has the property that r divides $d_{H'_i}(v, V'_i)$ for all $v \in B_i$.

We aim to apply Lemma 10.4 to each H'_i with $B_i, V'_i, \eta^{1/3}$ playing the roles of U, V, η. We now check that conditions (i)–(iv) of Lemma 10.4 hold for H'_i.

Condition (i) holds by our construction. Note that for all $v \in B_i$, (10.1) implies that $d_G(v, V_i) \leq d_{H'_i}(v, V'_i) + |B| + r - 1$ (recall that we deleted at most additional $r-1$ edges at v to obtain H'_i from H^*_i). Recall that $\mathcal{P}$ is a $(k, 1-1/(r+1)+3\gamma)$-partition for G. By Proposition 10.5, for all $v \in B_i$ we have that

$$\begin{aligned}\delta(H'_i[N_{H'_i}(v, V'_i)]) = \delta(G[N_{H'_i}(v, V'_i)]) &\geq \delta(G[N_G(v, V_i)]) - |B| - (r-1)\\ &\geq (1-1/r)d_G(v, V_i) + 3\gamma|V_i| - 3\eta^{1/2}n\\ &\geq (1-1/r)d_{H'_i}(v, V'_i) + \gamma|V'_i|,\end{aligned}$$

so condition (ii) of Lemma 10.4 holds. Condition (iii) holds since $d_{H'_i}(y, B_i) \leq |B| \leq 2\eta^{1/2}n \leq \eta^{1/3}|V'_i|$ for all $y \in V'_i$. To see that (iv) holds, notice that

$$\delta(H'_i[V'_i]) \geq (1-1/(r+1)+3\gamma)|V_i| - |B| \geq (1-1/r+2\gamma)|V'_i|.$$

So by Lemma 10.4, there is a subgraph H_i of $H'_i[V'_i]$ such that $H'_i[B_i, V'_i] \cup H_i$ has an F-decomposition $\mathcal{F}'_i$ and $\Delta(H_i) \leq 2\gamma|V_i|$. Let $\mathcal{F}_2 := \bigcup_{i=1}^k \mathcal{F}'_i$.

Let $H := H' \cup (G - G[\mathcal{P}]) - \bigcup_{i=1}^k (H'_i[B_i, V'_i] \cup H_i) = G - \bigcup \mathcal{F}_1 - \bigcup \mathcal{F}_2$. Then (a) holds. To see that (b) holds note that, for each $v \in A$, $d_{H[\mathcal{P}]}(v) \leq d_{H'}(v) \leq (2r+1)\eta^{1/2}n \leq \gamma n$ and, for each $v \in B$, $d_{H[\mathcal{P}]}(v) = d_{H' - \bigcup \mathcal{F}_2}(v) \leq |B| + k(r-1) \leq 3\eta^{1/2}n \leq \gamma n$. Finally, (c) holds since $(G-H)[V_i] = H_i$. □

10.2. **Covering a pseudorandom remainder.** Lemma 10.6 gives us an approximate F-decomposition such that the remainder H has the property that $H[\mathcal{P}]$ has low maximum degree. We can also use an F-parity graph from Section 9 to ensure that, for each $2 \leq i \leq k$ and each $x \in V_{<i}$, r divides $d_H(x, V_i)$. We now cover all remaining edges of $H[\mathcal{P}]$ by using a small number of edges from $H - H[\mathcal{P}]$. We are unable to apply Lemma 10.4 directly, as the greedy algorithm used to prove Lemma 10.3 fails when H is approximately regular and U is much larger than V. However, if H is pseudorandom then we can recover an appropriate version of Lemma 10.3 by using a random greedy algorithm instead; this is because, when the codegrees of H are small, an edge used in one copy of K_r will only be contained in a small proportion of the other neighbourhoods that we consider.

Throughout this subsection H should be thought of as a random graph of density ρ. In Section 10.3 we will justify this assumption by combining the low degree remainder from Lemma 10.6 with a random subgraph of G of larger density.

Lemma 10.7. *Let $r, k, n \in \mathbb{N}$ and let $\rho > 0$ with $1/n \ll 1/r, 1/k, \rho \leq 1$. Let H be a graph on n vertices. Suppose that $U_1, \dots, U_p$ are subsets of $V(H)$ with $p \leq kn$ such that*

(i) *r divides $|U_j|$ for all $1 \leq j \leq p$;*
(ii) *$\delta(H[U_j]) \geq (1-1/r)|U_j| + 9rk\rho^{3/2}n$ for all $1 \leq j \leq p$;*
(iii) *$|U_j \cap U_{j'}| \leq 2\rho^2 n$ for distinct $1 \leq j, j' \leq p$;*
(iv) *each $v \in V(H)$ is contained in at most $2k\rho n$ of the U_j.*

Then there exist edge-disjoint subgraphs $T_1, \dots, T_p$ in H such that each T_j is a K_r-factor in $H[U_j]$.

We will use the following simple result.

Proposition 10.8 (Jain, see [20, Lemma 8])**.** *Let $X_1, \dots, X_n$ be Bernoulli random variables such that, for any $1 \le s \le n$ and any $x_1, \dots, x_{s-1} \in \{0,1\}$,*

$$\mathbb{P}(X_s = 1 \mid X_1 = x_1, \dots, X_{s-1} = x_{s-1}) \le p.$$

Let $B \sim B(n,p)$. Then $\mathbb{P}(X \ge a) \le \mathbb{P}(B \ge a)$ for any $a \ge 0$.

Proof of Lemma 10.7. Let $t := \lceil 8k\rho^{3/2}n \rceil$, and let $H_j := H[U_j]$ for all $1 \le j \le p$. We construct $T_1, \dots, T_p$ in turn using a randomised algorithm. Suppose that we have already found $T_1, \dots, T_{s-1}$ for some $1 \le s \le p$; we will find T_s as follows.

Let $G_{s-1} := \bigcup_{i=1}^{s-1} T_i$ be the subgraph of H consisting of the edges that have already been used. Let $H'_s := H_s - G_{s-1}[U_s]$. If $\Delta(G_{s-1}[U_s]) > r\rho^{3/2}n$, then let $A_1, \dots, A_t$ be empty graphs on U_s. If $\Delta(G_{s-1}[U_s]) \le r\rho^{3/2}n$, then $\delta(H'_s) \ge (1 - 1/r)|H'_s| + 8kr\rho^{3/2}n \ge (1-1/r)|H'_s| + (r-1)(t-1)$ by (ii). So by (i) and Theorem 10.2, there exist t edge-disjoint K_r-factors $A_1, \dots, A_t$ in H'_s.

In either case, we have found edge-disjoint subgraphs $A_1, \dots, A_t$ of H'_s. Pick $1 \le i \le t$ uniformly at random and set $T_s := A_i$. To prove the lemma, it suffices to show that, with positive probability,

$$\Delta(G_{s-1}[U_s]) \le r\rho^{3/2}n \text{ for all } 1 \le s \le p. \tag{10.2}$$

Consider $1 \le j \le p$ and $u \in U_j$. For $1 \le s \le p$, let $Y_s^{j,u}$ be the indicator function of the event that T_s contains an edge incident to u in H_j. Let $X^{j,u} := \sum_{s=1}^{p} Y_s^{j,u}$. Note that if $Y_s^{j,u} = 1$, then at most $r-1$ edges at u in H_j are used for T_s, so $d_{G_p}(u, U_j) \le rX^{j,u}$. Therefore to prove (10.2) it suffices to show that $X^{j,u} \le \rho^{3/2}n$ for all $1 \le j \le p$ and $u \in U_j$.

Fix $1 \le j \le p$ and $u \in U_j$. Let $J^{j,u}$ be the set of indices $s \ne j$ such that $u \in U_s$. By (iv), $|J^{j,u}| \le 2k\rho n$. Note that $Y_s^{j,u} = 0$ for all $s \notin J^{j,u} \cup \{j\}$. So

$$X^{j,u} \le 1 + \sum_{s \in J^{j,u}} Y_s^{j,u}. \tag{10.3}$$

Let $s_1, \dots, s_{|J^{j,u}|}$ be the enumeration of $J^{j,u}$ such that $s_b < s_{b+1}$ for all $1 \le b \le |J^{j,u}|$. For $b \le |J^{j,u}|$, note that $d_{H_{s_b}}(u, U_j) \le |U_j \cap U_{s_b}| \le 2\rho^2 n$ by (iii). So at most $2\rho^2 n$ of the subgraphs A_i that we picked in H'_{s_b} contain an edge incident to u in H_j. This implies that

$$\mathbb{P}(Y_{s_b}^{j,u} = 1 \mid Y_{s_1}^{j,u} = y_1, \dots, Y_{s_{b-1}}^{j,u} = y_{b-1}) \le \frac{2\rho^2 n}{t} \le \frac{\rho^{1/2}}{4k}$$

for all $y_1, \dots, y_{b-1} \in \{0,1\}$, all $1 \le b \le |J^{j,u}|$ and any values of the y_i. Let $B \sim B(|J^{j,u}|, \rho^{1/2}/4k)$. By (10.3), Proposition 10.8, Lemma 7.1 and the fact that $|J^{j,u}| \le 2k\rho n$ we have that

$$\begin{aligned}
\mathbb{P}(X^{j,u} > \rho^{3/2}n) &\le \mathbb{P}\Big(\sum_{s \in J^{j,u}} Y_s^{j,u} > 3\rho^{3/2}n/4\Big) \le \mathbb{P}(B > 3\rho^{3/2}n/4) \\
&\le \mathbb{P}(|B - \mathbb{E}(B)| > \rho^{3/2}n/4) \le 2e^{-\rho^2 n/16k}.
\end{aligned}$$

Since there are at most kn^2 pairs (j,u), there is a choice of $T_1, \dots, T_p$ such that $X^{j,u} \le \rho^{3/2}n$ for all $1 \le j \le p$ and all $u \in U_j$. □

We now use Lemma 10.7 to prove the corresponding version of Lemma 10.3.

Corollary 10.9. *Let $r, k, n \in \mathbb{N}$ and let $\rho > 0$ with $1/n \ll 1/r, 1/k, \rho \leq 1$. Let H be a graph on n vertices. Let $U, V \subseteq V(H)$ be disjoint with $|V| \geq \lfloor n/k \rfloor$. Suppose that, for all distinct $x, x' \in U$ and each $y \in V$,*

(i) *r divides $d_H(x, V)$;*
(ii) *$\delta(H[N_H(x, V)]) \geq (1 - 1/r)d_H(x, V) + 9rk\rho^{3/2}|V|$;*
(iii) *$|N_H(x, V) \cap N_H(x', V)| \leq 2\rho^2|V|$;*
(iv) *$d_H(y, U) \leq 2k\rho|V|$.*

Then there is a subgraph H_V of $H[V]$ such that $H[U, V] \cup H_V$ has a K_{r+1}-decomposition and $\Delta(H_V) \leq 2rk\rho|V|$.

Proof. Let $p := |U|$; note that $p \leq k|V|$. Let $u_1, \ldots, u_p$ be an enumeration of U. Let $U_j := N_H(u_j, V)$ for all $1 \leq j \leq p$. Apply Lemma 10.7 with $H[V], |V|$ playing the roles of H, n to obtain edge-disjoint subgraphs $T_1, \ldots, T_p$ in $H[V]$ such that each T_j is a K_r-factor in $H[U_j]$. Let $H_V := \bigcup_{j=1}^{p} T_j$. Note that $H[U, V] \cup H_V = \bigcup_{j=1}^{p}(H[\{u_j\}, U_j] \cup T_j)$ has a K_{r+1}-decomposition. Since $d_{H_V}(y) \leq rd_H(y, U) \leq 2rk\rho|V|$ for each $y \in V$ by (iv), we have $\Delta(H_V) \leq 2rk\rho|V|$. $\square$

The following lemma follows from Corollary 10.9 in the same way that Lemma 10.4 follows from Lemma 10.3, so we omit a detailed proof.

Lemma 10.10. *Let $r, k, n, f \in \mathbb{N}$ and let $\alpha, \rho > 0$ with $1/n \ll \rho \ll \alpha, 1/k, 1/r, 1/f \leq 1$. Let F be an r-regular graph on f vertices and let H be a graph on n vertices. Let $U, V \subseteq V(H)$ be disjoint with $|V| \geq \lfloor n/k \rfloor$. Suppose that, for all distinct $x, x' \in U$ and each $y \in V$,*

(i) *r divides $d_H(x, V)$;*
(ii) *$\delta(H[N_H(x, V)]) \geq (1 - 1/r)d_H(x, V) + 9rk\rho^{3/2}|V|$;*
(iii) *$|N_H(x, V) \cap N_H(x', V)| \leq 2\rho^2|V|$;*
(iv) *$d_H(y, U) \leq 2k\rho|V|$;*
(v) *$\delta(H[V]) \geq (1 - 1/r + 2\alpha)|V|$.*

Then there is a subgraph H'_V of $H[V]$ such that $H[U, V] \cup H'_V$ has an F-decomposition and $\Delta(H'_V) \leq 2\alpha|V|$.

Lemma 10.10 easily implies the following corollary.

Corollary 10.11. *Let $r, k, n, f \in \mathbb{N}$ and let $\alpha, \rho > 0$ with $1/n \ll \rho \ll \alpha, 1/k, 1/r, 1/f \leq 1$. Let F be an r-regular graph on f vertices and let H be a graph on n vertices. Let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be an equitable partition of $V(H)$. Suppose that, for each $2 \leq i \leq k$, all distinct $x, x' \in V_{<i}$ and each $y \in V_i$,*

(i) *r divides $d_H(x, V_i)$;*
(ii) *$\delta(H[N_H(x, V_i)]) \geq (1 - 1/r)d_H(x, V_i) + 9rk\rho^{3/2}|V_i|$;*
(iii) *$|N_H(x, V_i) \cap N_H(x', V_i)| \leq 2\rho^2|V_i|$;*
(iv) *$d_H(y, V_{<i}) \leq 2k\rho|V_i|$;*
(v) *$\delta(H[V_i]) \geq (1 - 1/r + 2\alpha)|V_i|$.*

Then there is a subgraph H_0 of $H - H[\mathcal{P}]$ such that $H[\mathcal{P}] \cup H_0$ has an F-decomposition and $\Delta(H_0) \leq 2\alpha n$.

Proof. For each $2 \leq i \leq k$, let $U_i := V_{<i}$, and let H_i be the graph on $V(H)$ with $E(H_i) := E(H[U_i, V_i]) \cup E(H[V_i])$. Note that $H_2, \ldots, H_k$ are pairwise edge-disjoint and $H[\mathcal{P}] \subseteq \bigcup_{i=2}^{k} H_i$. We apply Lemma 10.10 to each H_i with U_i,V_i playing the

roles of U,V to obtain a subgraph H_i' of $H_i[V_i]$ such that $H_i[U_i, V_i] \cup H_i'$ has an F-decomposition and $\Delta(H_i') \leq 2\alpha|V_i|$. Let $H_0 := \bigcup_{i=2}^k H_i'$. Note that $H[\mathcal{P}] \cup H_0 = \bigcup_{i=2}^k (H_i[U_i, V_i] \cup H_i')$ has an F-decomposition and $\Delta(H_0) = \max_{2\leq i\leq k} \Delta(H_i') \leq 2\alpha n$ since $V(H_i') \subseteq V_i$ for each i. □

10.3. **Proof of Lemma 10.1.** We now present the formal version of the statement (†) at the beginning of Section 10. Recall that if $\mathcal{P}$ is a $(k, \delta+\varepsilon)$-partition for G and H is a subgraph of G with $\Delta(H) \leq \varepsilon n/2k$, then $\mathcal{P}$ is a (k, δ)-partition for $G - H$.

Lemma 10.12. *Let $r, f, k, n \in \mathbb{N}$ and let $\eta, \varepsilon > 0$ with $1/n \ll \eta \ll 1/k, \varepsilon, 1/r, 1/f$. Let F be an r-regular graph on f vertices. Let G be an r-divisible graph on n vertices and let G_0 be a subgraph of $G - G[\mathcal{P}]$. Let $\delta := \max\{\delta_F^\eta, 1 - 1/(r+1)\}$. Suppose that $\mathcal{P} = \{V_1, \dots, V_k\}$ is a $(k, \delta + 3\varepsilon)$-partition for $G - G_0$. Then there is a subgraph H of $G - G[\mathcal{P}] - G_0$ such that $G[\mathcal{P}] \cup H$ has an F-decomposition and $\Delta(H) \leq \varepsilon n/2k^2$.*

In our application of Lemma 10.12 the graph G_0 will consist of edges which will be used in later iterations and are therefore not allowed to be used in the current one, so H needs to avoid G_0.

The proof of Lemma 10.12 uses Corollary 10.11. In order to guarantee that condition (ii) of Corollary 10.11 will hold, we first remove a sparse random graph R from $G[\mathcal{P}]$. We then add R back to the remainder graph H obtained from Lemma 10.6 so that $H[\mathcal{P}]$ essentially behaves like a random subgraph of $G[\mathcal{P}]$.

Proof of Lemma 10.12. Choose γ, ρ such that $1/n \ll \eta \ll \gamma \ll \rho \ll 1/k, \varepsilon, 1/r, 1/f$. Let $G_1 := G - G_0$, and let $G_1' := G_1 - G[\mathcal{P}]$. By Lemma 7.2, there is a subgraph R of $G_1[\mathcal{P}]$ such that, for each $1 \leq i \leq k$ and all distinct $x, y \in V(G)$,

$$d_R(x, V_i) = \rho d_{G_1[\mathcal{P}]}(x, V_i) \pm \gamma|V_i|; \tag{10.4}$$

$$d_R(\{x, y\}, V_i) \leq \rho^2 d_{G_1[\mathcal{P}]}(\{x, y\}, V_i) + \gamma|V_i| \leq (\rho^2 + \gamma)|V_i|; \tag{10.5}$$

$$d_{G_1'}(y, N_R(x, V_i)) \geq \rho d_{G_1'}(y, N_{G_1}(x, V_i)) - \gamma n. \tag{10.6}$$

For each $2 \leq i \leq k$, each $x \in V_{<i}$ and each $y \in N_{G_1}(x, V_i)$, we have $d_{G_1'}(y, N_{G_1}(x, V_i)) = d_{G_1}(y, N_{G_1}(x, V_i))$, so (10.6) and Proposition 10.5 imply that

$$\begin{aligned} d_{G_1'}(y, N_R(x, V_i)) &\geq \rho\left((1 - 1/r)d_{G_1}(x, V_i) + \varepsilon|V_i|\right) - \gamma n \\ &\geq (1 - 1/r)\rho d_{G_1}(x, V_i) + 10rk\rho^{3/2}|V_i|. \end{aligned} \tag{10.7}$$

Let $G_2 := G_1 - R$. Note that $\mathcal{P}$ is a $(k, \delta + 2\varepsilon)$-partition for G_2 since $\rho \ll \varepsilon$. By Lemma 9.3, G_2 contains an F-parity graph P with respect to $\mathcal{P}$ such that

$$\Delta(P) \leq \gamma n. \tag{10.8}$$

Let $G_3 := G_2 - P$. Note that $\mathcal{P}$ is a $(k, \delta + 3\gamma)$-partition for G_3 as $\gamma \ll \varepsilon$. Apply Lemma 10.6 to G_3 to obtain a subgraph G_4 of G_3 such that

(a) $G_3 - G_4$ has an F-decomposition $\mathcal{F}_1$;
(b) $\Delta(G_4[\mathcal{P}]) \leq \gamma n$;
(c) for each $1 \leq i \leq k$, $\Delta(\bigcup \mathcal{F}_1[V_i]) = \Delta(G_3[V_i] - G_4[V_i]) \leq 2\gamma|V_i|$.

Recall that P is an F-parity graph, so has an F-decomposition. Note that $G^* := R \cup G_4 \cup G_0 = G - P - \bigcup \mathcal{F}_1$ is obtained from G by removing a set of edge-disjoint copies of F, so G^* is r-divisible. Since P is an F-parity graph with respect to $\mathcal{P}$, there is a subgraph P' of P such that $P - P'$ has an F-decomposition $\mathcal{F}_2$ and r divides $d_{G^*\cup P'}(x, V_i)$ for each $2 \le i \le k$ and each $x \in V_{<i}$. Note that, by (10.8),

$$\Delta(\bigcup \mathcal{F}_2[V_i]) \le \Delta(P) \le \gamma n. \tag{10.9}$$

Let $G_5 := G^* \cup P' - G_0$. Note that

$$G_5 = G_1 - \bigcup \mathcal{F}_1 - \bigcup \mathcal{F}_2 = R \cup G_4 \cup P'. \tag{10.10}$$

We will now check that conditions (i)–(v) of Corollary 10.11 hold with G_5 playing the role of H. Recall that $e(G_0[\mathcal{P}]) = 0$ and that r divides $d_{G^*\cup P'}(x, V_i)$ for each $2 \le i \le k$ and each $x \in V_{<i}$. So condition (i) holds. Consider $2 \le i \le k$ and $x \in V_{<i}$. By (10.10), (10.4), (b) and (10.8), we have that

$$d_{G_5}(x, V_i) \le d_R(x, V_i) + \Delta(G_4[\mathcal{P}]) + \Delta(P) \le \rho d_{G_1[\mathcal{P}]}(x, V_i) + 3\gamma n. \tag{10.11}$$

Therefore, using (10.7), (c) and (10.9) in the second line, we have that for $y \in N_{G_5}(x, V_i) \subseteq N_{G_1}(x, V_i)$,

$$\begin{aligned} d_{G_5}(y, N_{G_5}(x, V_i)) &\overset{(10.10)}{\ge} d_{G_1'}(y, N_R(x, V_i)) - \Delta(\bigcup \mathcal{F}_1[V_i]) - \Delta(\bigcup \mathcal{F}_2[V_i]) \\ &\ge (1 - 1/r)\rho d_{G_1}(x, V_i) + 10rk\rho^{3/2}|V_i| - 2\gamma n - \gamma n \\ &\overset{(10.11)}{\ge} (1 - 1/r)(d_{G_5}(x, V_i) - 3\gamma n) + 10rk\rho^{3/2}|V_i| - 3\gamma n \\ &\ge (1 - 1/r)d_{G_5}(x, V_i) + 9rk\rho^{3/2}|V_i|. \end{aligned}$$

Thus condition (ii) of Corollary 10.11 holds.

To see that conditions (iii) and (iv) of Corollary 10.11 hold, note that, for all distinct $x, x' \in V_{<i}$ and each $2 \le i \le k$,

$$|N_{G_5}(x, V_i) \cap N_{G_5}(x', V_i)| \overset{(10.10)}{\le} d_R(\{x, x'\}, V_i) + \Delta(G_4[\mathcal{P}]) + \Delta(P) \le 2\rho^2|V_i|,$$

where the second inequality holds by (10.5), (b) and (10.8). Similarly, for each $y \in V_i$ and each $1 \le i \le k$,

$$d_{G_5}(y, V_{<i}) \overset{(10.10)}{\le} \Delta(R) + \Delta(G_4[\mathcal{P}]) + \Delta(P) \overset{(10.4),(b),(10.8)}{\le} (\rho + 3\gamma)n \le 2\rho k|V_i|.$$

Set $\alpha := \varepsilon n/4k^2$. Recall that $\mathcal{P}$ is a $(k, 1 - 1/(r + 1) + \varepsilon)$-partition for G_1. Thus, for each $1 \le i \le k$,

$$\begin{aligned} \delta(G_5[V_i]) &\overset{(10.10)}{\ge} \delta(G_1[V_i]) - \Delta(\bigcup \mathcal{F}_1[V_i]) - \Delta(\bigcup \mathcal{F}_2[V_i]) \\ &\overset{(c),(10.9)}{\ge} (1 - 1/(r + 1) + \varepsilon)|V_i| - 3\gamma n \ge (1 - 1/r + 2\alpha)|V_i|, \end{aligned}$$

implying condition (v) of Corollary 10.11. So by Corollary 10.11, there is a subgraph H of $G_5 - G_5[\mathcal{P}]$ such that $G_5[\mathcal{P}] \cup H$ has an F-decomposition $\mathcal{F}_3$ and $\Delta(H) \le \varepsilon n/2k^2$. In particular, $G[\mathcal{P}] \cup H = G_1[\mathcal{P}] \cup H$ also has an F-decomposition $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ by (10.10). □

As described at the beginning of this section, we can now iteratively apply Lemma 10.12 to a sequence of partitions to prove the following lemma, which immediately implies Lemma 10.1.

Lemma 10.13. *Let $r, f, m, k, \ell \in \mathbb{N}$ and let $\varepsilon, \eta > 0$ with $1/m \ll \eta \ll 1/k \ll \varepsilon, 1/r, 1/f$.Let F be an r-regular graph on f vertices and let G be an r-divisible graph. Let $\delta := \max\{\delta_F^\eta, 1 - 1/(r+1)\}$. Suppose that $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is a sequence of partitions of $V(G)$ such that*

(i) *$\mathcal{P}_1$ is a $(k, \delta + \varepsilon)$-partition for G;*
(ii) *for each $2 \le i \le \ell$ and each $V \in \mathcal{P}_{i-1}$, $\mathcal{P}_i[V]$ is a $(k, \delta + 2\varepsilon)$-partition for $G[V]$;*
(iii) *each $V \in \mathcal{P}_\ell$ has size $m - 1$ or m.*

Then there exists a subgraph H^* of $\bigcup_{V \in \mathcal{P}_\ell} G[V]$ such that $G - H^*$ has an F-decomposition.

Proof. Let $n := |G|$, and let $G_0 := G - G[\mathcal{P}_1] - G[\mathcal{P}_2]$. (If ℓ=1, then let G_0 be the empty graph.)

Note that $\mathcal{P}_1$ is a $(k, \delta + \varepsilon)$-partition for $G - G_0$. Apply Lemma 10.12 to obtain a subgraph H of $G - G[\mathcal{P}_1] - G_0$ such that $G[\mathcal{P}_1] \cup H$ has an F-decomposition $\mathcal{F}_0$ and $\Delta(H) \le \varepsilon n/2k^2$. This proves the case when $\ell = 1$ (by setting $H^* := G - G[\mathcal{P}_1] - H$), so we proceed by induction and assume that $\ell \ge 2$.

Obtain H as above and let $G' := G - G[\mathcal{P}_1] - H$. Consider $U \in \mathcal{P}_1$. Note that $G'[U]$ is r-divisible. Since $\Delta(H) \le \varepsilon n/2k^2$, we have that $\mathcal{P}_2[U]$ is a $(k, \delta + \varepsilon)$-partition for $G'[U]$. Since H is edge-disjoint from G_0, for each $3 \le i \le \ell$ and each $V \in \mathcal{P}_{i-1}[U]$ we have that $\mathcal{P}_i[V]$ is a $(k, \delta + 2\varepsilon)$-partition for $G'[V]$. So we can apply the induction hypothesis to $G'[U], \mathcal{P}_2[U], \dots, \mathcal{P}_\ell[U]$ to obtain a subgraph H_U^* of $\bigcup_{V \in \mathcal{P}_\ell[U]} G[V]$ such that $G'[U] - H_U^*$ has an F-decomposition $\mathcal{F}_U$.

Set $H^* := \bigcup_{U \in \mathcal{P}_1} H_U^*$. Observe that H^* is a subgraph of $\bigcup_{V \in \mathcal{P}_\ell} G[V] = G - G[\mathcal{P}_\ell]$ and $G - H^*$ has an F-decomposition $\mathcal{F}_0 \cup \bigcup_{U \in \mathcal{P}_1} \mathcal{F}_U$. □

10.4. **A strengthening of Lemma 10.1 for certain graphs F.** Suppose that F is an r-regular graph that is not a vertex-disjoint union of copies of K_{r+1}. Then Lemma 10.1 still holds if we replace $\delta := \max\{\delta_F^\eta, 1 - 1/(r+1)\}$ by $\delta := \max\{\delta_F^\eta, 1 - 1/r\}$.

Lemma 10.14. *Let $r, f, m, k, \ell \in \mathbb{N}$ and let $\varepsilon, \eta > 0$ with $1/m \ll \eta \ll 1/k \ll \varepsilon, 1/r, 1/f$. Let F be an r-regular graph on f vertices such that F is not a vertex-disjoint union of copies of K_{r+1}. Let G be an r-divisible graph. Let $\delta := \max\{\delta_F^\eta, 1 - 1/r\}$. Suppose that $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is a $(k, \delta + \varepsilon, m)$-partition sequence for G. Then there exists a subgraph H^* of $\bigcup_{V \in \mathcal{P}_\ell} G[V]$ such that $G - H^*$ has an F-decomposition. In particular, if G is F-divisible, then so is H^*.*

We now sketch a proof of Lemma 10.14 obtained by modifying the proof of Lemma 10.1. Note that the application of Theorem 10.2 (in the proof of Lemma 10.3) is the only point in the proof of Lemma 10.1 where we need that $\delta \ge 1 - 1/(r+1)$ (rather than $\delta \ge 1 - 1/r$). Since F is not a vertex-disjoint union of copies of K_{r+1}, there exists a vertex x in F such that $F_x := F[N_F(x)]$ is not complete. Note that $\chi(F_x) \le r - 1$ as $|F_x| = r$ and $F_x \ne K_r$. Suppose that H, x and V are as described at the beginning of Section 10.1. Then it suffices to find an F_x-factor in $N_H(x, V)$ (rather than a K_r-factor). So we can replace Theorem 10.2 by the following result.

Theorem 10.15 (Alon and Yuster [1])**.** *For every graph F and every $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon, F)$ such that every graph G on $n \ge n_0$ vertices with $\delta(G) \ge (1 - 1/\chi(F) + \varepsilon)n$ contains $\lfloor |G|/|F| \rfloor$ vertex-disjoint copies of F.*

The proof of Lemma 10.14 is otherwise the same as that of Lemma 10.1.

11. Proof of Theorem 1.4

We can now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Without loss of generality we may assume that $\varepsilon \ll 1/r, 1/f$. Choose $k, m', n_0 \in \mathbb{N}$ and $\eta > 0$ such that $1/n_0 \ll 1/m' \ll \eta \ll 1/k \ll \varepsilon \ll 1/r, 1/f$, and let $\varepsilon' := \varepsilon/3$. Let G be an F-divisible graph on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + 3\varepsilon')n$. By Proposition 7.4, there is a $(k, \delta + 2\varepsilon', m)$-partition sequence $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ for G such that $m' \leq m \leq km'$. Let $G_1 := G[\mathcal{P}_1]$, and let $G_{\ell+1} := G - G[\mathcal{P}_\ell]$. Note that $\delta(G_1[\mathcal{P}_\ell]) \geq \delta(G_1) \geq (\delta + 2\varepsilon')(n - \lceil n/k \rceil) \geq (\delta + \varepsilon')n$ as $1/k \ll \varepsilon' \ll 1$. Since $\delta \geq 1 - 1/3r$, we can apply Lemma 8.1 to $G_1 \cup G_{\ell+1}$ (with $\mathcal{P}_\ell, (\varepsilon'/2k)^{1/2}$ playing the roles of $\mathcal{P}, \varepsilon$) to obtain an F-divisible subgraph A^* of $G_1 \cup G_{\ell+1}$ such that

(i) $\Delta(A^*[\mathcal{P}_\ell]) \leq \varepsilon' n/2k$ and $\Delta(A^*[V]) \leq r$ for each $V \in \mathcal{P}_\ell$, and
(ii) if H^* is an F-divisible graph on $V(G)$ wthat is edge-disjoint from A^* and has $e(H^*[\mathcal{P}_\ell]) = 0$, then $A^* \cup H^*$ has an F-decomposition.

Let $G' := G - A^*$; then G' is F-divisible. Note that for each $V \in \mathcal{P}_1$ and each $v \in V(G)$, we have $d_{G'}(v, V) \geq d_G(v, V) - \Delta(A) \geq (\delta + \varepsilon')|V|$. So $\mathcal{P}_1$ is a $(k, \delta + \varepsilon')$-partition for G'. Note that $\Delta(A^* - A^*[\mathcal{P}_1]) \leq r$ by (i), so $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is a $(k, \delta + \varepsilon', m)$-partition sequence for G'.

Apply Lemma 10.1 to obtain an F-divisible subgraph H of $\bigcup_{V \in \mathcal{P}_\ell} G'[V]$ such that $G' - H$ has an F-decomposition. But now $A^* \cup H$ has an F-decomposition by (ii). $\square$

Note that all of our arguments can be carried out in polynomial time, and all probabilistic arguments give the desired structure with high probability. Haxell and Rödl's original proof of Theorem 5.3 gave a polynomial time algorithm for converting a fractional decomposition to an approximate decomposition, and Kierstead, Kostochka, Mydlarz and Szemerédi [15] found an alternative proof of Theorem 10.2 which gave a polynomial time algorithm for finding K_r-factors.

We can actually obtain a stronger version of Theorem 1.4 which can be applied to obtain better bounds for certain graphs F. It involves the parameter d_F introduced in Section 8.3 that measures the degeneracy of the most efficient transformer for F. The proof of Theorem 11.1 is the same as that of Theorem 1.4 except that we replace Lemma 8.1 with Lemma 8.11 and Lemma 10.1 with Lemma 10.14 (if F is not a vertex-disjoint union of copies of K_{r+1}).

Theorem 11.1. *Let F be an r-regular graph on f vertices. Then for all $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon, F)$ and an $\eta := \eta(\varepsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \varepsilon)n$, where*

$$\delta := \begin{cases} \max\{\delta_F^\eta, 1 - \frac{1}{d_F}, 1 - \frac{1}{r+1}\} & \text{if } F \text{ is a vertex-disjoint union of copies of } K_{r+1}, \\ \max\{\delta_F^\eta, 1 - \frac{1}{d_F}, 1 - \frac{1}{r}\} & \text{otherwise,} \end{cases}$$

has an F-decomposition.

Our proof of Theorem 11.1 can also be carried out in polynomial time, since the F-factor guaranteed in Theorem 10.15 can be obtained in polynomial time (see the discussion after [28, Theorem 2.6]).

Note that Theorem 11.1 implies Theorem 1.4 since Corollary 8.10 states that $d_F \leq 3r$ for any r-regular graph F. However for some graphs F one can obtain much better bounds on d_F, yielding improved overall bounds. We illustrate this for the case of cycles in Section 12.

12. Decompositions into cycles and bipartite graphs

In this section we consider C_ℓ-decompositions and deduce Theorem 1.5 from Theorem 11.1. For even $\ell \geq 6$, the constant $\delta = 1/2$ in Theorem 1.5 is best possible. We now describe the construction giving the lower bound. The existence of a special construction for the case $\ell = 4$ is perhaps surprising and was first observed by Winkler and Kahn (see [25]).

Proposition 12.1. *Let $\ell \in \mathbb{N}$ with $\ell \geq 3$, and let*

$$\delta := \begin{cases} 1/2 & \text{if } \ell \geq 6 \text{ is even;} \\ 3/5 & \text{if } \ell = 4; \\ \frac{\ell}{2(\ell-1)} & \text{if } \ell \text{ is odd.} \end{cases}$$

Then there are infinitely many C_ℓ-divisible graphs G with $\delta(G) \geq \delta|G| - 1$ that are not C_ℓ-decomposable.

Note that the case $\ell = 3$ describes an extremal example for the triangle decomposition conjecture of Nash-Williams.

Proof. Case $\ell \geq 6$ even. Let n be such that $n \equiv \ell + 1 \mod 2\ell$. So $n - 1$ is even and ℓ divides $n(n-1)$ but not $\binom{n}{2}$. Let G be the vertex-disjoint union of two cliques of order n. Then G is C_ℓ-divisible with $\delta(G) = |G|/2 - 1$, but neither connected component is itself C_ℓ-divisible, so G cannot be C_ℓ-decomposable.

Case $\ell = 4$. Let n be such that $n \equiv 3 \mod 8$. So 8 divides $3n - 1$. Let G be the graph obtained from C_5 by blowing up each vertex to a clique K_n of order n. The degree of each vertex is $3n - 1$, which is even, and the total number of edges is $5n(3n-1)/2$, which is divisible by 4. We have that $\delta(G) = 3n - 1 = 3|G|/5 - 1$, but each copy of C_4 in G contains an even number of edges between the copies of K_n, and the number of such edges is $5n^2$, which is odd, so G cannot be C_4-decomposable.

Case ℓ odd. Let G be the graph obtained from $K_{\ell-1,\ell-1}$ by blowing up each vertex to a clique of odd order n such that ℓ divides n. The degree of each vertex is $\ell n - 1$, which is even, and the total number of edges is $(\ell - 1)n(\ell n - 1)$, which is divisible by ℓ. We have that $\delta(G) = \ell n - 1 = \frac{\ell|G|}{2(\ell-1)} - 1$, but each copy of C_ℓ in G contains an edge of one of the copies of K_n, and the number of such edges is only $(\ell - 1)n(n - 1) < e(G)/\ell$, so G cannot be C_ℓ-decomposable. □

The construction for the lower bound for C_4-decompositions can be modified to give a lower bound of $3|G|/5$ for $K_{r,r}$-decompositions when r is even as follows. In the proof of Proposition 12.1 for the case $\ell = 4$, if we further choose n such that $2r^2$ divides $3n - 1$, then the resulting graph G is also $K_{r,r}$-divisible. Note that for r even, $K_{r,r}$ has a C_4-decomposition. Since G has no C_4-decomposition, G has no $K_{r,r}$-decomposition.

We prove Theorem 1.5 using Theorem 11.1. Recall the definition of d_{C_ℓ} in Section 8.3. We now bound d_{C_ℓ} above for $\ell \geq 3$.

Lemma 12.2. *For $\ell \in \mathbb{N}$ with $\ell \geq 3$,*

$$d_{C_\ell} \leq \begin{cases} 4 & \text{if } \ell = 3, \\ 3 & \text{if } \ell = 4, \\ 2 & \text{if } \ell \geq 5. \end{cases}$$

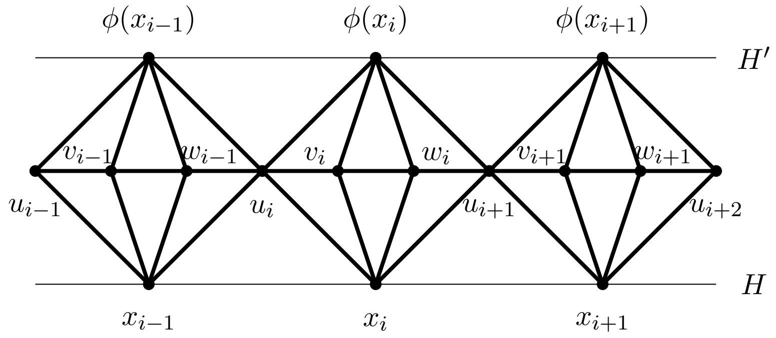

FIGURE 3. A $(H, H')_{C_3}$- transformer T.

Proof. Lemma 8.5 implies that the lemma holds for $\ell \geq 4$. So we may assume that $\ell = 3$. Let H be a 2-regular graph and let H' be obtained from H by identifying vertices. Suppose that H and H' are edge-disjoint. Recall that an $(H, H')_{C_3}$-transformer T is a graph such that

- $T \cup H$ and $T \cup H'$ each have C_3-decompositions;
- $V(H \cup H') \subseteq V(T)$ and $T[V(H \cup H')]$ is empty.

To show that $d_{C_3} \leq 4$, it suffices to show that there exists an $(H, H')_{C_3}$-transformer T such that the degeneracy of T rooted at $V(H \cup H')$ is at most 4.

Let $\phi : H \to H'$ be a graph homomorphism from H to H' that is edge-bijective. Note that H is a union of vertex-disjoint cycles $C_{s_1}, \dots, C_{s_p}$. So H' decomposes into $\phi(C_{s_1}), \dots, \phi(C_{s_p})$. Suppose that, for each $1 \leq j \leq p$, there exists a $(C_{s_j}, \phi(C_{s_j}))_{C_3}$-transformer T_j such that the degeneracy of T_j rooted at $V(C_{s_j} \cup \phi(C_{s_j}))$ is at most 4. We further choose the T_j such that $V(T_j) \cap V(H \cup H') = V(C_{s_j} \cup \phi(C_{s_j}))$ and $V(T_j) \cap V(T_{j'}) \subseteq V(H \cup H')$ for all $j \neq j'$. In particular, the T_j are edge-disjoint. Let $T := \bigcup_{1 \leq j \leq p} T_j$. Then T is an $(H, H')_{C_3}$-transformer such that the degeneracy of T rooted at $V(H \cup H')$ is at most 4. Therefore, we may assume that H is a cycle $x_1 x_2 \dots x_s x_1$.

Let $\{u_i, v_i, w_i : 1 \leq i \leq s\}$ be a set of $3s$ vertices disjoint from $V(H \cup H')$. Define a graph T as follows:

(i) $V(T_1) := V(H) \cup V(H') \cup \{u_i, v_i, w_i : 1 \leq i \leq s\}$;
(ii) $E_1 := \{x_i u_i, x_i v_i, x_i w_i, x_i u_{i+1} : 1 \leq i \leq s\}$;
(iii) $E_2 := \{v_i w_i : 1 \leq i \leq s\}$;
(iv) $E_3 := \{u_i v_i, w_i u_{i+1} : 1 \leq i \leq s\}$;
(v) $E_4 := \{\phi(x_i) u_i, \phi(x_i) v_i, \phi(x_i) w_i, \phi(x_i) u_{i+1} : 1 \leq i \leq s\}$;
(vi) $E(T_1) := E_1 \cup E_2 \cup E_3 \cup E_4$.

Here the indices are considered modulo s. Note that $T[V(H \cup H')]$ is empty.

Note also that $H \cup E_1 \cup E_2$ can be decomposed into $2s$ copies of C_3, where each C_3 has vertex set either $\{x_i, x_{i+1}, u_{i+1}\}$ or $\{x_i, v_i, w_i\}$ for some $1 \leq i \leq s$. Note also that $E_3 \cup E_4$ can be decomposed into $2s$ copies of C_3, where each C_3 has vertex set either $\{\phi(x_i), u_i, v_i\}$ or $\{\phi(x_i), w_i, u_{i+1}\}$ for some $1 \leq i \leq s$. Thus $H \cup T$ has a C_3-decomposition. Similarly, $H' \cup T$ has a C_3-decomposition. Therefore T is an $(H, H')_{C_3}$-transformer. To see that the degeneracy of T rooted at $V(H) \cup V(H')$ is at most 4, consider the vertices in H, H', $\{u_i : 1 \leq i \leq s\}$, $\{v_i, w_i : 1 \leq i \leq s\}$ in that order. This completes the proof of the lemma. $\square$

We now prove Theorem 1.5. In the special case of triangles ($\ell = 3$), if one can show that $\delta^{\eta}_{C_3} \le 3/4$ for all $\eta > 0$, then our proof immediately implies an asymptotic version of Nash–Williams' conjecture on triangle decompositions (that is, Conjecture 1.1 for $r = 2$).

Proof of Theorem 1.5. By Theorem 11.1 and Lemma 12.2, it suffices to show that $\lim_{\eta\to 0} \delta^{\eta}_{C_\ell} \le \delta$. For odd ℓ, this follows from Lemma 5.7. For even ℓ, it follows (with room to spare) from the fact that any graph on n vertices with at least $50\ell n^{1+2/\ell}$ edges contains a copy of C_ℓ (see [2]), so we can obtain an η-approximate C_ℓ-decomposition greedily. □

If F is an r-regular bipartite graph, then Theorem 11.1 implies the following result, which applies for instance to the complete bipartite graph $K_{r,r}$.

Corollary 12.3. *Let F be an r-regular bipartite graph. Then for each $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon, F)$ such that every F-divisible graph G on $n \ge n_0$ vertices with $\delta(G) \ge (1 - 1/(r + 1))n$ has an F-decomposition.*

Proof. Since F is bipartite, $\delta^{\eta}_F = 0$ for all $\eta > 0$. Indeed, it follows from the Erdős–Simonovits–Stone theorem [7, 8] that we can obtain an η-approximate F-decomposition greedily (since the Turán density of bipartite graphs is 0). Since $d_F \le r + 1$ by Lemma 8.5, the result now follows from Theorem 11.1. □

References

[1] N. Alon and R. Yuster. *H*-factors in dense graphs. *J. Combin. Theory Ser. B*, 66(2):269–282, 1996.

[2] J.A. Bondy and M. Simonovits. Cycles of even length in graphs. *J. Combin. Theory Ser. B*, 16:97–105, 1974.

[3] D. Bryant and N. Cavenagh. Decomposing graphs of high minimum degree into 4-cycles. *J. Graph Theory*, to appear.

[4] D. Dor and M. Tarsi. Graph decomposition is NPC - a complete proof of Holyer's conjecture. In *STOC '92 Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*, pages 252–263, 1992.

[5] P. Dukes. Rational decomposition of dense hypergraphs and some related eigenvalue estimates. *Linear Algebra Appl.*, 436(9):3736–3746, 2012.

[6] P. Erdős. On some new inequalities concerning extremal properties of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 77–81. Academic Press, New York, 1968.

[7] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966.

[8] P. Erdös and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[9] K. Garaschuk. *Linear methods for rational triangle decompositions.* PhD thesis, Univ. of Victoria, https://dspace.library.uvic.ca//handle/1828/5665, 2014.

[10] T. Gustavsson. *Decompositions of large graphs and digraphs with high minimum degree.* PhD thesis, Univ. of Stockholm, 1991.

[11] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623. North-Holland, Amsterdam, 1970.

[12] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, 2001.

[13] S. Janson, T. Łuczak, and A. Ruciński. *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[14] P. Keevash. The existence of designs. *arXiv preprint arXiv:1401.3665*, 2014.

[15] H.A. Kierstead, A.V. Kostochka, M. Mydlarz, and E. Szemerédi. A fast algorithm for equitable coloring. *Combinatorica*, 30(2):217–224, 2010.

[16] T.P. Kirkman. On a problem in combinatorics. *Cambridge Dublin Mathematical Journal*, 2:191–204, 1847.

[17] M. Krivelevich. Triangle factors in random graphs. *Combin. Probab. Comput.*, 6(3):337–347, 1997.

[18] D. Kühn and D. Osthus. Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments. *Adv. Math.*, 237:62–146, 2013.

[19] C.St.J.A. Nash-Williams. An unsolved problem concerning decomposition of graphs into triangles. In *Combinatorial Theory and its Applications III*, pages 1179–183. North Holland, 1970.

[20] R. Raman. The power of collision: randomized parallel algorithms for chaining and integer sorting. In *Foundations of software technology and theoretical computer science (Bangalore, 1990)*, volume 472 of *Lecture Notes in Comput. Sci.*, pages 161–175. Springer, Berlin, 1990.

[21] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):229–251, 2006.

[22] M. Schlund. *Graph Decompositions, Latin Squares, and Games.* Diploma thesis, Technische Universität München, 2011.

[23] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

[24] R.M. Wilson. Decompositions of complete graphs into subgraphs isomorphic to a given graph. In *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*, pages 647–659. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[25] R Yuster. The decomposition threshold for bipartite graphs with minimum degree one. *Random Structures Algorithms*, 21(2):121–134, 2002.

[26] R. Yuster. Asymptotically optimal K_k-packings of dense graphs via fractional K_k-decompositions. *J. Combin. Theory Ser. B*, 95(1):1–11, 2005.

[27] R. Yuster. Integer and fractional packing of families of graphs. *Random Structures Algorithms*, 26(1-2):110–118, 2005.

[28] R. Yuster. Combinatorial and computational aspects of graph packing and graph decomposition. *Computer Science Review*, 1(1):12–26, 2007.

[29] R. Yuster. H-packing of k-chromatic graphs. *Mosc. J. Comb. Number Theory*, 2(1):73–88, 2012.

[30] R. Yuster. Edge-disjoint cliques in graphs with high minimum degree. *SIAM J. Combin.*, 28(2):893–910, 2014.