A PROOF OF THE ERDŐS–FABER–LOVÁSZ CONJECTURE

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Abstract. The Erdős-Faber-Lovász conjecture (posed in 1972) states that the chromatic index of any linear hypergraph on n vertices is at most n. In this paper, we prove this conjecture for every large n. We also provide stability versions of this result, which confirm a prediction of Kahn.

1. Introduction

Graph and hypergraph colouring problems are central to combinatorics, with applications and connections to many other areas, such as geometry, algorithm design, and information theory. As one illustrative example, the fundamental Ajtai-Komlós-Pintz-Spencer-Szemerédi (AKPSS) theorem [2] shows that locally sparse uniform hypergraphs have large independent sets. This was initially designed to disprove the famous Heilbronn conjecture in combinatorial geometry but has found numerous further applications e.g., in coding theory. The AKPSS theorem was later strengthened by Frieze and Mubayi [18] to show that linear k-uniform hypergraphs with $k \geq 3$ have small chromatic number. Here a hypergraph $\mathcal{H}$ is *linear* if every two distinct edges of $\mathcal{H}$ intersect in at most one vertex.

1.1. **The Erdős-Faber-Lovász conjecture.** In 1972, Erdős, Faber, and Lovász conjectured (see [15]) the following equivalent statements. Let $n \in \mathbb{N}$.

(i) If $A_1, \dots, A_n$ are sets of size n such that every pair of them shares at most one element, then the elements of $\bigcup_{i=1}^n A_i$ can be coloured by n colours so that all colours appear in each A_i.
(ii) If G is a graph that is the union of n cliques, each having at most n vertices, such that every pair of cliques shares at most one vertex, then the chromatic number of G is at most n.
(iii) If $\mathcal{H}$ is a linear hypergraph with n vertices, then the chromatic index of $\mathcal{H}$ is at most n.

Here the *chromatic index* $\chi'(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the smallest number of colours needed to colour the edges of $\mathcal{H}$ so that any two edges that share a vertex have different colours. The formulation (iii) is the one that we will consider throughout the paper. For simplicity, we will refer to this conjecture as the EFL conjecture.

Erdős considered this to be 'one of his three most favorite combinatorial problems' (see e.g., [29]). The simplicity and elegance of its formulation initially led the authors to believe it to be easily solved (see e.g., the discussion in [10] and [15]). It was initially designed as a simple test case for a more general theory of hypergraph colourings. However, as the difficulty became apparent Erdős offered successively increasing rewards for a proof of the conjecture, which eventually reached \$500.

Previous progress towards the conjecture includes the following results. Seymour [42] proved that every n-vertex linear hypergraph $\mathcal{H}$ has a matching of size at least $e(\mathcal{H})/n$, where $e(\mathcal{H})$ is the number of edges in $\mathcal{H}$. (Note that this immediately follows from the validity of the EFL conjecture, but it is already difficult to prove.) Kahn and Seymour [30] proved that every n-vertex linear hypergraph has fractional chromatic index at most n. Chang and Lawler [9] showed that every n-vertex linear hypergraph has chromatic index at most $\lceil 3n/2 - 2 \rceil$. Finally, a breakthrough of Kahn [25] yielded an approximate version of the conjecture, by showing that every n-vertex linear hypergraph has chromatic index at most $n + o(n)$. (His surveys [27, 29] discuss many related results and open problems.)

1.2. **Main results.** In this paper we prove the EFL conjecture for every large n.

Theorem 1.1. *For every sufficiently large n, every linear hypergraph $\mathcal{H}$ on n vertices has chromatic index at most n.*

There are three constructions for which Theorem 1.1 is known to be tight: a complete graph K_n for any odd integer n (and minor modifications thereof), a finite projective plane of order k on $n = k^2 + k + 1$ points,

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and a degenerate plane $\{\{1,2\},\ldots,\{1,n\},\{2,\ldots,n\}\}$. Note that the first example has bounded edge size (two), while the other two examples have unbounded edge size as n tends to infinity.

Kahn's proof [25] is based on a powerful method known as the Rödl nibble. Roughly speaking, this method builds a large matching using an iterative probabilistic procedure. It was originally developed by Rödl [40] to prove the Erdős–Hanani conjecture [14] on combinatorial designs. Another famous result based on this method is the Pippenger–Spencer theorem [39], which implies that the chromatic index of any uniform hypergraph $\mathcal{H}$ of maximum degree D and codegree $o(D)$ is $D+o(D)$. (Note that this in turn implies that the EFL conjecture holds for all large r-uniform linear hypergraphs of bounded uniformity $r \geq 3$.) In a seminal paper, Kahn [28] later developed the approach further to show that the same bound $D+o(D)$ even holds for the list chromatic index (an intermediate result in this direction, which also strengthens the Pippenger–Spencer theorem, was the main ingredient of his proof in [25]). The best bound on the $o(D)$ error term for the list chromatic index of such hypergraphs was obtained by Molloy and Reed [37], and for the chromatic index, the best bound was proved in [31]. Our proof will also rely on certain properties of the Rödl nibble.

In addition, our proof makes use of powerful colouring results for locally sparse graphs (Theorems 6.4 and 6.6). This line of research goes back to Ajtai, Komlós, and Szemerédi [3] who (independently of Rödl [40]) developed a very similar semi-random nibble approach to give an upper bound $O(k^2/\log k)$ on the Ramsey number $R(3,k)$ by finding large independent sets in triangle-free graphs (the matching lower bound $R(3,k)=\Omega(k^2/\log k)$ was later established by Kim [33], also using a semi-random approach). Inspired by an earlier result of Kim [32], the above Ramsey bound by Ajtai, Komlós and Szemerédi was subsequently strengthened by a highly influential result of Johansson [24], who showed that triangle-free graphs of maximum degree Δ have chromatic number $O(\Delta/\log\Delta)$. The result of Frieze and Mubayi [18] mentioned at the start of Section 1 is one of several analogues and generalizations of Johansson's Theorem. It also turns out that the condition of being triangle-free can be relaxed (in various ways) to being 'locally sparse' [1, 4, 12, 44]. We will be able to apply such results to suitable parts of the line graph of our given linear hypergraph $\mathcal{H}$.

One step in our proof involves what may be considered a 'vertex absorption' argument; here certain vertices not covered by a matching produced by the Rödl nibble are 'absorbed' into the matching to form a colour class. (Vertex) absorption as a systematic approach was introduced by Rödl, Ruciński, and Szemerédi [41] to find spanning structures in hypergraphs (with precursors including [16, 34]). Absorption ideas were first used for edge decomposition problems in [35] to solve Kelly's conjecture on tournament decompositions. We will make use of an application of the main result of [35] to the overfull subgraph conjecture (which was derived in [20]).

Kahn [27] predicted that the bound in the EFL conjecture can be improved if $\mathcal{H}$ is far from being one of the extremal examples mentioned above. We confirm his prediction by proving a 'linear' and a 'sublinear' stability result as follows.

Theorem 1.2 (linear stability)**.** *For every $\delta > 0$, there exist $n_0, \sigma > 0$ such that the following holds. For any $n \geq n_0$, if $\mathcal{H}$ is an n-vertex linear hypergraph with maximum degree at most $(1-\delta)n$ such that the number of edges of size $(1\pm\delta)\sqrt{n}$ in $\mathcal{H}$ is at most $(1-3\delta)n$, then the chromatic index of $\mathcal{H}$ is at most $(1-\sigma)n$.*

Theorem 1.3 (sublinear stability)**.** *For every $\varepsilon > 0$, there exist $n_0, \eta > 0$ such that the following holds. For any $n \geq n_0$, if $\mathcal{H}$ is an n-vertex linear hypergraph with maximum degree at most ηn and no edge $e \in \mathcal{H}$ such that $\eta\sqrt{n} < |e| < \sqrt{n}/\eta$, then the chromatic index of $\mathcal{H}$ is at most εn.*

1.3. **Related results and open problems.** Formulation (ii) of the EFL conjecture can be viewed as a statement implying that a local restriction on the local density of a graph has a strong influence on its global structure. A famous example where this is not the case is the construction by Erdős of graphs of high girth and high chromatic number. Another well known instance where this fails is a bipartite version of the EFL conjecture due to Alon, Saks, and Seymour (see Kahn [26]); they conjectured that if a graph G can be decomposed into k edge-disjoint bipartite graphs, then the chromatic number of G is at most $k+1$. This conjecture was a generalisation of the Graham–Pollak theorem [21] on edge decompositions of complete graphs into bipartite graphs, which has applications to communication complexity. However, this conjecture was disproved by Huang and Sudakov [22] in a strong form, i.e., it is not even close to being true.

A natural generalization of the EFL conjecture was suggested by Berge [6] and Füredi [19]; if $\mathcal{H}$ is a linear hypergraph with vertex set V, then the chromatic index of $\mathcal{H}$ is at most $\max_{v\in V}|\bigcup_{e\ni v} e|$. This would be a direct generalization of Vizing's theorem on the chromatic index of graphs. Finally, another beautiful question leading on from Theorem 1.1 is whether it can be extended to list colourings.

2. Overview

In this section, we provide an overview of the proof of Theorem 1.1.

2.1. **Colouring linear hypergraphs with bounded edge sizes.** Here, we discuss the proof of Theorem 1.1 in the special case when all edges of $\mathcal{H}$ have bounded size. In this subsection, we fix constants satisfying the hierarchy
$$0 < 1/n_0 \ll \xi \ll 1/r \ll \gamma \ll \varepsilon \ll \rho \ll 1,$$
we let $n \geq n_0$, and we let $\mathcal{H}$ be an n-vertex linear hypergraph such that every $e \in \mathcal{H}$ satisfies $2 \leq |e| \leq r$. We first describe the ideas which already lead to the near-optimal bound $\chi'(\mathcal{H}) \leq n+1$.

Let G be the graph with $V(G) \coloneqq V(\mathcal{H})$ and $E(G) \coloneqq \{e \in \mathcal{H} : |e| = 2\}$. The first step of the proof is to include every edge of G in a 'reservoir' R independently with probability 1/2 that we will use for 'absorption'. With high probability, each $v \in V(\mathcal{H})$ satisfies $d_R(v) = d_G(v)/2 \pm \xi n$. Since $\mathcal{H}$ is linear, this easily implies that $\Delta(\mathcal{H} \setminus R) \leq (1/2 + \xi)n$. So by the Pippenger-Spencer theorem [39], we obtain the nearly optimal bound $\chi'(\mathcal{H} \setminus R) \leq (1/2 + \gamma)n$. Now using R as a 'vertex-absorber', we would like to extend the colour classes of $\mathcal{H} \setminus R$ to cover as many vertices of U as possible, where $U \coloneqq \{u \in V(\mathcal{H}) : d_G(u) \geq (1-\varepsilon)n\}$. This would allow us to control the maximum degree in the hypergraph consisting of uncoloured edges, so that it can then be coloured with few colours. To that end, we need the following important definition.

Definition 2.1 (Perfect and nearly-perfect coverage)**.** Let $\mathcal{H}$ be a linear multi-hypergraph, let $\mathcal{N}$ be a set of edge-disjoint matchings in $\mathcal{H}$, and let $S \subseteq U \subseteq V(\mathcal{H})$.

- We say $\mathcal{N}$ has *perfect coverage* of U if each $N \in \mathcal{N}$ covers U.
- We say $\mathcal{N}$ has *nearly-perfect coverage of U with defects in S* if
  - (i) each $u \in U$ is covered by at least $|\mathcal{N}| - 1$ matchings in $\mathcal{N}$ and
  - (ii) each $N \in \mathcal{N}$ covers all but at most one vertex in U such that $U \setminus V(N) \subseteq S$.

We will construct some $\mathcal{H}' \subseteq \mathcal{H}$ and a proper edge-colouring $\psi : \mathcal{H}' \to C$ such that $\mathcal{H}' \supseteq \mathcal{H} \setminus R$, $|C| = (1/2 + \gamma)n$ and the set of colour classes $\{\psi^{-1}(c) : c \in C\}$ has nearly-perfect coverage of U (with defects in U). Crucially, this means that $\mathcal{H} \setminus \mathcal{H}'$ is a graph and satisfies $\Delta(\mathcal{H} \setminus \mathcal{H}') \leq n - |C|$. (Indeed, every vertex $u \in U$ satisfies $d_{\mathcal{H}}(u) \leq n-1$ and is covered by all but at most one of the colour classes of ψ, and every vertex $v \notin U$ satisfies $d_{\mathcal{H}\setminus\mathcal{H}'}(v) \leq d_R(v) \leq ((1-\varepsilon)/2 + \xi)n < n - |C|$.) Therefore, Vizing's theorem [43] implies that $\chi'(\mathcal{H} \setminus \mathcal{H}') \leq \Delta(\mathcal{H} \setminus \mathcal{H}') + 1 \leq n - |C| + 1$, so altogether we have $\chi'(\mathcal{H}) \leq \chi'(\mathcal{H}') + \chi'(\mathcal{H} \setminus \mathcal{H}') \leq n+1$, as claimed.

To construct $\mathcal{H}'$ and ψ we iteratively apply the Rödl nibble to (the leftover of) $\mathcal{H} \setminus R$ to successively construct large matchings N_i which are then removed from $\mathcal{H} \setminus R$ and form part of the colour classes of ψ. (The Rödl nibble is applied implicitly via Corollary 4.3, which guarantees a large matching in a suitable hypergraph.) Crucially, each matching N_i exhibits pseudorandom properties, which allow us to use some edges of R to extend N_i into a matching M_i (which will form a colour class of ψ) with nearly-perfect coverage of U, as desired. (This is why we apply the Rödl nibble in our proof rather than the Pippenger-Spencer theorem.) Thus, R acts as a 'vertex-absorber' for $U \setminus V(N_i)$ and the final edge decomposition of the unused edges of R into matchings is achieved by Vizing's theorem. (Actually, this only works if $\mathcal{H} \setminus R$ is nearly regular, which is not necessarily the case. Thus, we first embed $\mathcal{H} \setminus R$ in a suitable nearly regular hypergraph $\mathcal{H}^*$ and prove that the respective matchings in $\mathcal{H}^*$ have nearly-perfect coverage of U, which suffices for our purposes.)

Let us now discuss how to improve the bound $\chi'(\mathcal{H}) \leq n+1$ to $\chi'(\mathcal{H}) \leq n$. Let $S \coloneqq \{u \in U : d_G(u) < n-1\}$, and note that if $\{\psi^{-1}(c) : c \in C\}$ has either perfect coverage of U, or nearly-perfect coverage of U with defects in S, then $\Delta(\mathcal{H} \setminus \mathcal{H}') \leq n - 1 - |C|$. In this case, we may use the same argument as before with Vizing's theorem to obtain $\chi'(\mathcal{H}) \leq n$. However, it is not always possible to find such a colouring. For example, if $\mathcal{H}$ is a complete graph K_n for odd n (which is one of the extremal examples for Theorem 1.1), then $U = V(\mathcal{H})$ and $S = \varnothing$, so it is not possible for even a single colour class to have nearly-perfect coverage of U with defects in S. However, we can adapt the above nibble-absorption-Vizing approach to work whenever $\mathcal{H}$ is not 'close' to K_n in the following sense.

Definition 2.2 ((ρ, ε)-full)**.** Let $\mathcal{H}$ be an n-vertex linear hypergraph, and let G be the graph with $V(G) \coloneqq V(\mathcal{H})$ and $E(G) \coloneqq \{e \in \mathcal{H} : |e| = 2\}$. For $\varepsilon, \rho \in (0,1)$, $\mathcal{H}$ is (ρ, ε)-*full* if

- $|\{u \in V(\mathcal{H}) : d_G(u) \geq (1-\varepsilon)n\}| \geq (1 - 10\varepsilon)n$, and
- $|\{v \in V(\mathcal{H}) : d_G(v) = n-1\}| \geq (\rho - 15\varepsilon)n$.

As mentioned above, when $\mathcal{H}$ is not (ρ, ε)-full we can adapt the nibble-absorption-Vizing approach to show that $\chi'(\mathcal{H}) \leq n$ (with a reservoir of density ρ rather than 1/2). If $\mathcal{H}$ is (ρ, ε)-full then we will ensure that the leftover $\mathcal{H} \setminus \mathcal{H}' \subseteq R$ is a quasirandom almost regular graph (which involves a more careful choice of R – again it will have density close to ρ rather than $1/2$ but now it consists of a 'random' part and a 'regularising' part). This allows us to apply a result [20] on the overfull subgraph conjecture (see Corollary 9.6) which implies that $\chi'(\mathcal{H} \setminus \mathcal{H}') \leq \Delta(\mathcal{H} \setminus \mathcal{H}')$. (The result in [20] is obtained as a straightforward consequence of the result in [35] that robustly expanding regular graphs have a Hamilton decomposition, and thus, a 1-factorisation if they have even order.)

2.2. **Colouring linear hypergraphs where all edges are large.** Now we discuss how to prove Theorems 1.1 and 1.2 when all edges of $\mathcal{H}$ have size at least some large constant. In this step it is often very useful to consider the line graph $L(\mathcal{H})$ of $\mathcal{H}$ and use the fact that $\chi(L(\mathcal{H})) = \chi'(\mathcal{H})$. In this subsection, we fix constants satisfying the hierarchy

$$0 < 1/n_0 \ll 1/r \ll \sigma \ll \delta,$$

we let $n \geq n_0$, and we let $\mathcal{H}$ be an n-vertex linear hypergraph such that every $e \in \mathcal{H}$ satisfies $|e| > r$. Now we sketch a proof that $\chi'(\mathcal{H}) \leq n$ for such $\mathcal{H}$. If $\mathcal{H}$ is a finite projective plane of order k, where $k^2 + k + 1 = n$, then the line graph $L(\mathcal{H})$ is a clique K_n. Thus, $\chi'(\mathcal{H}) = \chi(L(\mathcal{H})) = n$, so the bound $\chi'(\mathcal{H}) \leq n$ is best possible. Thus, we refer to the case where $\mathcal{H}$ has approximately n edges of size $(1 \pm \delta)\sqrt{n}$ as the 'FPP-extremal' case. We also sketch how to prove the improved bound $\chi'(\mathcal{H}) \leq (1-\sigma)n$ if $\mathcal{H}$ is not in the FPP-extremal case. As we discuss in the next subsection, we will need this result in the proof of Theorem 1.1.

Consider an ordering $\preceq$ of the edges $e_1, e_2, \ldots, e_m$ of $\mathcal{H}$ according to their size, i.e., $e_i \preceq e_j$ if $|e_i| > |e_j|$ for every $i, j \in [m]$. For an edge $e \in \mathcal{H}$, let $d^{\preceq}_{\mathcal{H}}(e)$ denote the number of edges in $\mathcal{H}$ which intersect e and precede e in $\preceq$. Clearly, a greedy colouring following this size-monotone ordering achieves a bound of $\chi'(\mathcal{H}) \leq \max_i d^{\preceq}_{\mathcal{H}}(e_i) + 1$ (this bound was also used in [9, 25]). Moreover, it is easy to see that if this greedy colouring algorithm fails to produce a colouring with at most $(1-\sigma)n$ colours, i.e., if an edge e satisfies $d^{\preceq}_{\mathcal{H}}(e) \geq (1-\sigma)n$, then almost all of the corresponding edges that intersect e and precede e must have size close to $|e|$.

Surprisingly, if one allows some flexibility in the ordering (in particular, if we allow it to be size-monotone only up to some edge e^* such that $d^{\preceq}_{\mathcal{H}}(e^*) \geq (1-\sigma)n$ while every edge f with $e^* \preceq f$ satisfies $d^{\preceq}_{\mathcal{H}}(f) < (1-\sigma)n$), then one can show much more: Either we can modify the ordering to reduce the number of edges which come before e^*, or there is a set $W \subseteq \mathcal{H}$ (where e^* is the last edge of W) such that

(W1) $|e^*| \approx |e|$ for every $e \in W$, and
(W2) the edges of W cover almost all pairs of vertices of $\mathcal{H}$.

If $|e^*| \leq (1-\delta)\sqrt{n}$, then one can show that $L(W)$ induces a 'locally sparse' graph (as $\mathcal{H}$ is linear). Moreover, (W1) implies that the maximum degree of $L(W)$ is not too large, and thus one can show that $\chi(L(W))$ is much smaller than $(1-\sigma)n$ (leaving enough room to colour the edges preceding W with a new set of colours). This together with (W2) allows us to extend the colouring of W to all of $\mathcal{H}$ using a suitable modification of the above greedy colouring procedure for the remaining edges in $\mathcal{H}$ to obtain that $\chi'(\mathcal{H}) \leq (1-\sigma)n$, as desired.

If $|e^*| \geq (1-\delta)\sqrt{n}$, then we first colour the edges of size at least $(1-\delta)\sqrt{n}$ (in particular, the edges of W) as follows. Let $\mathcal{H}' \subseteq \mathcal{H}$ be the hypergraph consisting of these edges. If $e(\mathcal{H}') \leq n$, then, of course, we may colour the edges of $\mathcal{H}'$ with different colours. Otherwise, if $t := e(\mathcal{H}') - n > 0$, the main idea is to find a matching of size t in the complement of $L(\mathcal{H}')$ (where $L(\mathcal{H}')$ will be close to being a clique of order not much more than n). By assigning the same colour to the edges of $\mathcal{H}'$ that are adjacent in this matching, we obtain $\chi'(\mathcal{H}') = \chi(L(\mathcal{H}')) \leq n$. Now we extend the colouring to all of $\mathcal{H}$ using a suitable modification of the above greedy colouring procedure again to obtain that $\chi'(\mathcal{H}) \leq n$, as desired.

2.3. **Combining colourings of the large and small edges.** We now describe how one can prove Theorem 1.1 by building on the ideas described in Sections 2.1 and 2.2. In this subsection and throughout the rest of the paper we work with constants satisfying the following hierarchy:

$$0 < 1/n_0 \ll 1/r_0 \ll \xi \ll 1/r_1 \ll \beta \ll \kappa \ll \gamma_1 \ll \varepsilon_1 \ll \rho_1 \ll \sigma \ll \delta \ll \gamma_2 \ll \rho_2 \ll \varepsilon_2 \ll 1. \tag{2.1}$$

Some of these constants are used to characterize the edges of a hypergraph by their size, as follows.

Definition 2.3 (Edge sizes)**.** Let $\mathcal{H}$ be an n-vertex linear hypergraph with $n \geq n_0$.

- Let $\mathcal{H}_{\text{small}} := \{e \in \mathcal{H} : |e| \leq r_1\}$. An edge $e \in \mathcal{H}$ is *small* if $e \in \mathcal{H}_{\text{small}}$.
- Let $\mathcal{H}_{\text{med}} := \{e \in \mathcal{H} : r_1 < |e| \leq r_0\}$. An edge $e \in \mathcal{H}$ is *medium* if $e \in \mathcal{H}_{\text{med}}$.
- Let $\mathcal{H}_{\text{large}} := \{e \in \mathcal{H} : |e| > r_0\}$. An edge $e \in \mathcal{H}$ is *large* if $e \in \mathcal{H}_{\text{large}}$.
- Let $\mathcal{H}_{\text{ex}} := \{e \in \mathcal{H} : |e| = (1 \pm \delta)\sqrt{n}\}$. An edge $e \in \mathcal{H}$ is *FPP-extremal* if $e \in \mathcal{H}_{\text{ex}}$.

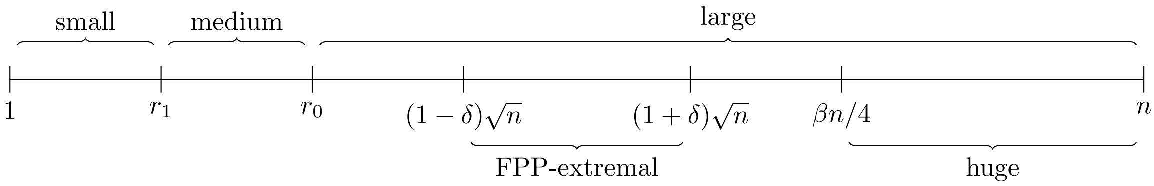

Figure 1. Types of edges based on their size

- Let $\mathcal{H}_{\mathrm{huge}} := \{e \in \mathcal{H} : |e| \geq \beta n/4\}$. An edge $e \in \mathcal{H}$ is *huge* if $e \in \mathcal{H}_{\mathrm{huge}}$.

Note that $\mathcal{H}_{\mathrm{small}}, \mathcal{H}_{\mathrm{med}}, \mathcal{H}_{\mathrm{large}}$ form a partition of the edges of $\mathcal{H}$ (see Figure 1). Also note that if $\mathcal{H}$ is an n-vertex linear hypergraph and $1/n \ll \alpha < 1$, then

$$|\{e \in \mathcal{H} : |e| \geq \alpha n\}| \leq 2/\alpha. \tag{2.2}$$

In the proof of Theorem 1.1, given an n-vertex linear hypergraph $\mathcal{H}$ with $n \geq n_0$ (where we assume $\mathcal{H}$ has no singleton edges), we first find a proper edge-colouring $\psi_1 : \mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}} \to C_1$ as discussed in Section 2.2, and then we extend it to a proper n-edge-colouring of $\mathcal{H}_{\mathrm{small}}$ by adapting the argument presented in Section 2.1. The proof proceeds slightly differently depending on whether we are in the FPP-extremal case. As discussed in the previous subsection, in the non-FPP-extremal case, $\chi'(\mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}}) \leq (1-\sigma)n$, so we may assume $|C_1| = (1-\sigma)n$. In this case, we let $\gamma := \gamma_1$, $\varepsilon := \varepsilon_1$, and $\rho := \rho_1$; in the FPP-extremal case, we let $\gamma := \gamma_2$, $\rho := \rho_2$, and $\varepsilon := \varepsilon_2$. We define G and U as in Section 2.1, and we define a suitable 'defect' set $S \subseteq U$ (whose choice now depends on the structure of $\mathcal{H}$). In order to extend the colouring ψ_1 of $\mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}}$ to $\mathcal{H}$, we need it to satisfy a few additional properties, which are provided by Theorem 6.1. Roughly, we need that

(1) each colour class of ψ_1 covers at most βn vertices, with exceptions for colour classes containing huge or medium edges, and
(2) at most γn colours are assigned by ψ_1 to colour medium edges.

We choose a 'reservoir' R from $E(G)$; how we choose it depends on whether we are in the FPP-extremal case. In the non-FPP-extremal case, we choose it as described in Section 2.1, and in the FPP-extremal case, we include every edge of G incident to a vertex of U to be in R independently with probability ρ.

Let $C_{\mathrm{hm}} \subseteq C_1$ be the set of colours assigned to a huge or medium edge by ψ_1. Note that $e(\mathcal{H}_{\mathrm{huge}}) \leq 8/\beta$ by (2.2), so consequently, by (2), $|C_{\mathrm{hm}}| \leq 3\gamma n/2$. For each $c \in C_{\mathrm{hm}}$, we use Lemma 7.11 to extend $\psi_1^{-1}(c)$ (in the sense of Section 2.1) using edges of R, so that $\{\psi_1^{-1}(c) : c \in C_{\mathrm{hm}}\}$ has nearly perfect coverage of U with defects in S. There is possibly an exceptional colour class, which we call *difficult* (see Definition 7.10), that we need to consider in this step. This situation arises if $\mathcal{H}$ is close to being a degenerate plane. If $\mathcal{H}$ is the degenerate plane, then there is a huge edge e of size $n-1$, and U consists of a single vertex of degree $n-1$. Even though $\mathcal{H}$ is not (ρ, ε)-full, if c is assigned to the edge e, it is clearly impossible to extend $\psi^{-1}(c)$ to have perfect coverage of U, which would be necessary in order to finish the colouring with Vizing's theorem in the final step. However, if there is a difficult colour class that we cannot absorb, then we show that we can colour $\mathcal{H}$ directly (see Lemma 7.12).

We now construct some $\mathcal{H}'$ with $\mathcal{H}_{\mathrm{small}} \setminus R \subseteq \mathcal{H}' \subseteq \mathcal{H}_{\mathrm{small}}$ and a proper edge-colouring $\psi_2 : \mathcal{H}' \to C_2$ such that ψ_2 is compatible with ψ_1, $|C_2|$ is slightly larger than $(1-\rho+\gamma)n$, $C_2 \cap C_{\mathrm{hm}} = \varnothing$, and $\{\psi_1^{-1}(c) \cup \psi_2^{-1}(c) : c \in C_{\mathrm{hm}} \cup C_2\}$ has nearly-perfect coverage of U with defects in S. (Actually, as in Section 2.1 we obtain this coverage property only for a suitable auxiliary hypergraph $\mathcal{H}^* \supseteq \mathcal{H}'$, but we again ignore this here for simplicity.) In the non-FPP-extremal case, since $\rho = \rho_1 \ll \sigma$, this means we can reserve a set C_{final} of colours (of size close to ρn) which are used neither by ψ_1 nor by ψ_2. Then in the final step of the proof, we can colour the leftover graph $\mathcal{H}_{\mathrm{small}} \setminus \mathcal{H}' \subseteq R$ (with colours from C_{final}) as described in Section 2.1. In the FPP-extremal case, we may have $|C_1| = n$, so we need to find a proper edge-colouring of $\mathcal{H}_{\mathrm{small}} \setminus \mathcal{H}'$ using colours from $C_1 \setminus C_2$ while avoiding conflicts with ψ_1. But in this case most pairs of vertices are contained in an edge of $\mathcal{H}_{\mathrm{ex}}$, which implies that $|U|$ is small. Moreover, every edge of the leftover graph $\mathcal{H}_{\mathrm{small}} \setminus \mathcal{H}' \subseteq R$ is incident to a vertex of U. These two properties allow us to colour the leftover graph $\mathcal{H}_{\mathrm{small}} \setminus \mathcal{H}'$ with $\Delta(\mathcal{H}_{\mathrm{small}} \setminus \mathcal{H}')$ colours while using (1) and (2) to avoid conflicts with ψ_1, as desired.

We conclude by discussing how to construct $\mathcal{H}'$ and ψ_2. Using the colours in C_2, we colour all of the edges of $\mathcal{H}_{\mathrm{small}} \setminus R$ and some of the remaining uncoloured (by ψ_1) edges of R based on the nibble and the absorption strategy outlined in Section 2.1. For this, the following properties are crucial (which follow from (1) and the definition of $\mathcal{H}_{\mathrm{med}}$ respectively).

(a) $\psi_1^{-1}(c)$ covers at most βn vertices for each $c \in C_2$, and
(b) every vertex $v \in V(\mathcal{H})$ is contained in at most $n/(r_0 - 1)$ edges that are assigned a colour in C_2 by ψ_1 (since for any $c \in C_2$, either $\psi_1^{-1}(c)$ is empty or all the edges in $\psi_1^{-1}(c)$ are large). Thus, each edge in $\mathcal{H}_{\text{small}}$ still has slightly more than $(1-\rho)n$ colours available in C_2 that do not conflict with ψ_1 (since any edge of $\mathcal{H}_{\text{small}}$ intersects at most $r_1 n/(r_0-1)$ large edges and $r_1/(r_0-1) \ll \gamma$).

We will use (a) and (b) to show that the effect of the previously coloured edges (by ψ_1) on the Rödl nibble argument is negligible, i.e., we can adapt the arguments of Section 2.1, so that the colouring ψ_2 of $\mathcal{H}'$ is compatible with ψ_1.

2.4. **Organisation of the paper.** In Section 3, we introduce some notation that we use throughout the paper, and in Section 4 we collect some tools that we use in the proof. In Section 5 we prove Theorem 1.1 for hypergraphs where every edge has size at least $(1-\delta)\sqrt{n}$, and in Section 6, we prove Theorem 6.1, which we use to colour the large and medium edges of our hypergraph. In Section 6, we also prove Theorems 1.2 and 1.3. (In particular, Theorems 1.2 and 1.3 do not rely on the subsequent sections.) In Section 7, we prove several lemmas that we use for vertex absorption, and in Section 8, we show how to combine the results of Section 7 with hypergraph matching results (based on the Rödl nibble) to colour the small edges of our hypergraph not in the reservoir. In Section 9, we prove Lemma 9.2 and introduce Corollary 9.6, both of which are used in the final step of the proof to colour the uncoloured reservoir edges. In Section 10 we show how we select the reservoir edges, and finally in Section 11, we prove Theorem 1.1.

3. Notation

For $n \in \mathbb{N}$, we write $[n] := \{k \in \mathbb{N} : 1 \leq k \leq n\}$. We write $c = a \pm b$ if $a - b \leq c \leq a + b$. We use the '$\ll$' notation to state our results. Whenever we write a hierarchy of constants, they have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < a \ll b \leq 1$, then this means that there exists a non-decreasing function $f : (0,1] \mapsto (0,1]$ such that the result holds for all $0 < a, b \leq 1$ with $a \leq f(b)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way.

A *hypergraph* $\mathcal{H}$ is an ordered pair $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is called the vertex set and $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$ is called the edge set. If $E(\mathcal{H})$ is a multiset, we refer to $\mathcal{H}$ as a multi-hypergraph. Throughout the paper we usually write $\mathcal{H}$ instead of $E(\mathcal{H})$. We say that a (multi-)hypergraph $\mathcal{H}$ is r-uniform if for every $e \in E(\mathcal{H})$ we have $|e| = r$. In particular, 2-uniform hypergraphs are simply called *graphs*.

Given any multi-hypergraph $\mathcal{H}$, let $v(\mathcal{H})$ denote the number of vertices in $\mathcal{H}$ and let $e(\mathcal{H})$ denote the number of edges in $\mathcal{H}$. Let $\mathcal{H}^{(i)} := \{e \in \mathcal{H} : |e| = i\}$. Throughout the paper, we usually denote $\mathcal{H}^{(2)}$ by G. For any subset $S \subseteq V(\mathcal{H})$, let $\mathcal{H}|_S$ be the multi-hypergraph with the vertex set $V(\mathcal{H}|_S) := S$ and edge set $\mathcal{H}|_S := \{e \cap S : e \in \mathcal{H} \text{ and } e \cap S \neq \varnothing\}$. For any vertex $v \in V(\mathcal{H})$, let $E_{\mathcal{H}}(v) := \{e \in \mathcal{H} : v \in e\}$. We define the *degree* of v by $d_{\mathcal{H}}(v) := |E_{\mathcal{H}}(v)|$. More generally, for any given multiset $R \subseteq \mathcal{H}$, let $E_R(v)$ denote the multiset of edges incident to v in R, and $d_R(v) := |E_R(v)|$. We denote the minimum and maximum degrees of the vertices in $\mathcal{H}$ by $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$, respectively. Let $V^{(d)}(\mathcal{H}) := \{v \in \mathcal{H} : d_{\mathcal{H}}(v) = d\}$. Moreover, if $d \in \mathbb{N}$, then let $V_+^{(d)}(\mathcal{H}) := \{v \in \mathcal{H} : d_{\mathcal{H}}(v) \geq d\}$ and if $x \in (0,1)$, then let $V_+^{(x)}(\mathcal{H}) := \{v \in \mathcal{H} : d_{\mathcal{H}}(v) \geq xn\}$, where $n := v(\mathcal{H})$. For any edge $e \in \mathcal{H}$, the set $N_{\mathcal{H}}(e)$ denotes the multiset of edges $f \in E(\mathcal{H}) \setminus \{e\}$ that intersect e. The subscript $\mathcal{H}$ from $N_{\mathcal{H}}(e)$ may be omitted if it is clear from the context. If $\mathcal{H}' \subseteq \mathcal{H}$ and $e \in \mathcal{H}$, then we denote $N_{\mathcal{H}}(e) \cap \mathcal{H}'$ by $N_{\mathcal{H}'}(e)$, even if $e \notin \mathcal{H}'$.

The *line graph* $L(\mathcal{H})$ of a (multi-)hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is the graph whose vertex set is $E(\mathcal{H})$, where two vertices in $L(\mathcal{H})$ are adjacent if the corresponding edges in $E(\mathcal{H})$ have a non-empty intersection. A *matching* M in $\mathcal{H}$ is a subset of pairwise disjoint edges of $\mathcal{H}$. We often regard M as a hypergraph with $V(M) := \bigcup_{e \in M} e$. For any vertex $u \in V(\mathcal{H})$, we say u is *covered* by a matching M if $u \in e$ for some $e \in M$. For any $X \subseteq V(\mathcal{H})$, we say that a matching M *covers* X if M covers every vertex in X. For any integer $k \geq 0$ and a (multi-)hypergraph $\mathcal{H}$, a map $\phi : \mathcal{H} \to [k]$ is a *proper edge-colouring* of $\mathcal{H}$ if $\phi(e) \neq \phi(f)$ for any pair of distinct edges $e, f \in \mathcal{H}$ such that $e \cap f \neq \emptyset$. For any integer i and a proper edge-colouring $\phi : \mathcal{H} \to [k]$, let $\phi^{-1}(i)$ be the set of edges $e \in \mathcal{H}$ with $\phi(e) = i$. (Note that $\phi^{-1}(i)$ is a matching, for any i.)

A (multi-)hypergraph $\mathcal{H}$ is *linear* if for any distinct $e, f \in \mathcal{H}$, $|e \cap f| \leq 1$. A linear hypergraph may contain singleton edges (but no edge is repeated). A linear multi-hypergraph may contain multiple singleton edges incident to the same vertex but any edge of size at least two cannot be repeated (as that would contradict linearity). Given any linear multi-hypergraph $\mathcal{H}$ on n vertices, and any $W \subseteq \mathcal{H}$, the *normalised volume* of W is defined as $\mathrm{vol}_{\mathcal{H}}(W) := \sum_{e \in W} \binom{|e|}{2} / \binom{n}{2}$. We sometimes omit the subscript and write $\mathrm{vol}(W)$ instead of $\mathrm{vol}_{\mathcal{H}}(W)$ when it is clear from the context. Note that since $\mathcal{H}$ is linear, $\mathrm{vol}_{\mathcal{H}}(W) \leq 1$ for any $W \subseteq \mathcal{H}$.

Given any graph G, for any subset of vertices $V' \subseteq V(G)$, we denote the subgraph of G induced by V' as $G[V'] \coloneqq (V', E')$, where $E' \coloneqq \{e \in E(G) : e \subseteq V'\}$. We write $G - V' \coloneqq G[V \setminus V']$. If $V' = \{v\}$, then we simply write $G - v$ instead of $G - \{v\}$. For any disjoint pair of subsets $S, T \subseteq V(G)$, let $E_G(S,T) := \{st \in E(G) : s \in S, t \in T\}$, and let $e_G(S,T) \coloneqq |E_G(S,T)|$. Let $\overline{G} \coloneqq (V(G), \overline{E(G)})$ denote the complement of a graph G. For any non-negative integer functions $g, f : V(G) \to \mathbb{Z}$, a subset $F \subseteq E(G)$ is a *(g,f)-factor* in G if $g(w) \leq d_F(w) \leq f(w)$ for each $w \in V(G)$.

4. Preliminaries

We often use the following weighted version of Chernoff's inequality.

Theorem 4.1 (Weighted Chernoff's inequality [11])**.** *Let $c_1, \dots, c_m > 0$ be real numbers, let $X_1, \dots, X_m$ be independent random variables taking values 0 or 1, let $X \coloneqq \sum_{i=1}^m c_i X_i$ and let $C \coloneqq \max_{i \in [m]} c_i$. Then,*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq 2e^{\frac{-t^2}{2C(\mathbb{E}[X]+t/3)}}.$$

4.1. **Pseudorandom hypergraph matchings.** Now we state a special case of a recent result of Ehard, Glock, and Joos [13] that provides a matching covering almost all vertices of every set in a given collection of sets. This result will be used in the proof of Lemma 8.1. A similar result, but with weaker bounds, was proved earlier by Alon and Yuster [5]. The proof in [13] is derived via an averaging argument from a result on the chromatic index of hypergraphs by Molloy and Reed [37], which in turn relies on the Rödl nibble.

Theorem 4.2 (Ehard, Glock, and Joos [13])**.** *Let $r \geq 2$ be an integer, and let $\varepsilon \coloneqq 1/(1500r^2)$. There exists Δ_0 such that the following holds for all $\Delta \geq \Delta_0$. Let $\mathcal{H}$ be an r-uniform linear hypergraph with $\Delta(\mathcal{H}) \leq \Delta$ and $e(\mathcal{H}) \leq \exp(\Delta^{\varepsilon^2})$. Let $\mathcal{F}^*$ be a set of subsets of $V(\mathcal{H})$ such that $|\mathcal{F}^*| \leq \exp(\Delta^{\varepsilon^2})$ and $\sum_{v \in S} d_{\mathcal{H}}(v) \geq \Delta^{26/25}$ for any $S \in \mathcal{F}^*$. Then, there exists a matching M_0 in $\mathcal{H}$ such that for any $S \in \mathcal{F}^*$, we have $|S \cap V(M_0)| = (1 \pm \Delta^{-\varepsilon^2}) \sum_{v \in S} d_{\mathcal{H}}(v)/\Delta$.*

We remark that Theorem 4.2 is a direct application of [13, Theorem 1.2] by setting $\delta \coloneqq 1/30$ and the weight functions $w_S(e) := |S \cap e|$ for $S \in \mathcal{F}^*$, where $w_S(\mathcal{H}) \geq \max_{e \in \mathcal{H}} w_S(e)\Delta^{1+\delta}$ follows by the assumption $\sum_{v \in S} d_{\mathcal{H}}(v) \geq \Delta^{26/25}$, since $w_S(\mathcal{H}) = \sum_{v \in S} d_{\mathcal{H}}(v)$ and $w_S(e) \leq r$ for any $e \in \mathcal{H}$.

Our vertex absorption arguments will actually require that the number of uncovered vertices in S is small but not too small. So we need the following 'sparsified' version of Theorem 4.2, which allows us to have better control on the number of uncovered vertices. To deduce Corollary 4.3 from Theorem 4.2, one simply applies Theorem 4.2 to obtain a matching M_0 (in $\mathcal{H}$) and then we randomly remove each edge of M_0 with probability γ to obtain a matching M which satisfies the assertion of Corollary 4.3 with positive probability. We remark that one could also derive Corollary 4.3 via a direct application of the Rödl nibble (see [31] for a proof of a stronger result based on stronger assumptions).

Corollary 4.3. *Let $0 < 1/n_0 \ll 1/r, \kappa, \gamma < 1$. For any integer $n \geq n_0$, let $\mathcal{H}$ be an r-uniform linear n-vertex hypergraph such that every vertex has degree $(1 \pm \kappa)D$, where $D \geq n^{1/100}$. Let $\mathcal{F}$ be a set of subsets of $V(\mathcal{H})$ such that $|\mathcal{F}| \leq n^{2\log n}$. Then there exists a matching M of $\mathcal{H}$ such that for any $S \in \mathcal{F}$ with $|S| \geq D^{1/20}$, we have $|S \setminus V(M)| = (\gamma \pm 4\kappa)|S|$.*

4.2. **Embedding lemma.** The following lemma allows us to embed any linear hypergraph $\mathcal{H}$ with maximum degree D into an almost regular, uniform, linear hypergraph $\mathcal{H}_{\text{unif}}$ with maximum degree D, satisfying some additional properties.

Lemma 4.4. *Let $0 < 1/N_0, 1/D_0, 1/C_0 \ll 1/r \leq 1/3$, where $r \in \mathbb{N}$. Let $N \geq N_0$, let $C \geq C_0$, let $D \geq D_0$, and let $\mathcal{H}$ be an N-vertex linear multi-hypergraph with $\Delta(\mathcal{H}) \leq D$. If every $e \in \mathcal{H}$ satisfies $|e| \leq r$, then there exists an r-uniform linear hypergraph $\mathcal{H}_{\text{unif}}$ such that the following hold.*

(4.4.1) $\mathcal{H} \subseteq \mathcal{H}_{\text{unif}}|_{V(\mathcal{H})}$ and $\mathcal{H}_{\text{unif}}|_{V(\mathcal{H})} \setminus \mathcal{H}$ only contains singleton edges.

(4.4.2) For any $v \in V(\mathcal{H}_{\text{unif}})$, $D - C \leq d_{\mathcal{H}_{\text{unif}}}(v) \leq D$. Moreover, if $d_{\mathcal{H}}(v) \geq D - C$ for $v \in V(\mathcal{H})$, then $d_{\mathcal{H}_{\text{unif}}}(v) = d_{\mathcal{H}}(v)$.

(4.4.3) $v(\mathcal{H}_{\text{unif}}) \leq r(r-1)^2 D^3 N$.

The proof of Lemma 4.4 is a straightforward modification of the proof of [31, Lemma 8.1]. Here we briefly sketch the proof. First, let $\mathcal{H}^*$ be an r-uniform linear hypergraph obtained from $\mathcal{H}$ by adding $r - |e|$ new vertices to each $e \in \mathcal{H}$. Let $T \coloneqq (r-1)^2 D^2$. For every (sufficiently large) integer $d \leq D$, by considering Steiner systems, one can easily construct a simple T-vertex r-uniform hypergraph $\mathcal{H}_d$ such that every vertex of $\mathcal{H}_d$ has degree between $d - c_r$ and d for some constant c_r depending on r. We define our desired multi-hypergraph $\mathcal{H}_{\text{unif}}$ by taking the union of T vertex-disjoint copies of $\mathcal{H}^*$, where the first copy is identified with

$\mathcal{H}^*$. Then, for each $v \in V(\mathcal{H}^*)$ with $d_{\mathcal{H}^*}(v) < D - C$, let $v^1, \dots, v^T$ be the T clone vertices of $v \in V(\mathcal{H}^*)$ in $\mathcal{H}_{\text{unif}}$, and extend $\mathcal{H}_{\text{unif}}$ by making $\mathcal{H}_{\text{unif}}[\{v^1, \dots, v^T\}]$ induce a copy of $\mathcal{H}_{D-d_{\mathcal{H}^*}(v)}$, which implies that $D - c_r \leq d_{\mathcal{H}_{\text{unif}}}(v^i) \leq D$ for $1 \leq i \leq T$.

4.3. **Some colouring results.** We use Vizing's theorem [43] in the final step of our proof when $\mathcal{H}$ is not (ρ, ε)-full in the non-FPP-extremal case. We also use the following stronger form in Lemma 7.12.

Theorem 4.5 (Vizing [43])**.** *Every graph G with $\Delta(G) \leq D$ satisfies $\chi'(G) \leq D + 1$. Moreover, if G contains at most two vertices of degree D, then $\chi'(G) \leq D$.*

The following theorem is used as one of the ingredients to prove Theorems 1.2 and 1.3 in Section 6, as well as in Section 8 to colour a small 'leftover' part in Lemma 8.3.

Theorem 4.6 (Kahn [28])**.** *Let $0 < 1/D_0 \ll \alpha, 1/r < 1$, and let $D \geq D_0$. Let $\mathcal{H}$ be a linear hypergraph such that $\Delta(\mathcal{H}) \leq D$, and every $e \in \mathcal{H}$ satisfies $|e| \leq r$. Let C be a set with $|C| \geq (1 + \alpha)D$, and for each $e \in \mathcal{H}$, let $C(e) \subseteq C$ and $|C(e)| \leq \alpha D/2$. Then there exists a proper edge-colouring $\phi : \mathcal{H} \to C$ such that $\phi(e) \notin C(e)$ for each $e \in \mathcal{H}$.*

Theorem 4.6 immediately follows from [28, Theorem 1.1] by defining a list $S(e) \coloneqq C \setminus C(e)$ of available colours for each $e \in \mathcal{H}$.

5. Colouring FPP-extremal and larger edges

In this section we prove Theorem 1.1 when all edges are FPP-extremal or larger, as follows.

Lemma 5.1. *Let $0 < 1/n_0 \ll \delta \ll 1$, and let $n \geq n_0$. If $\mathcal{H}$ is an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $|e| \geq (1 - \delta)\sqrt{n}$, then $\mathcal{H}$ has a proper edge-colouring with n colours, where each colour is assigned to at most two edges.*

First we need the following simple observations. For a hypergraph $\mathcal{H}$, recall that $L(\mathcal{H})$ denotes the line graph of $\mathcal{H}$ and $\overline{L(\mathcal{H})}$ denotes its complement.

Observation 5.2. *Let $\mathcal{H}$ be an n-vertex linear hypergraph. If there is a matching N in $\overline{L(\mathcal{H})}$ of size $e(\mathcal{H}) - n$, then $\mathcal{H}$ has a proper edge-colouring with n colours, where each colour is assigned to at most two edges.* □

A pair $\{e, f\} \subseteq \mathcal{H}$ in an n-vertex hypergraph $\mathcal{H}$ is *useful* if $e \neq f$, $e \cap f \neq \varnothing$, and $|N(e) \cap N(f)| \leq n$.

Proposition 5.3. *Let $\mathcal{H}$ be an n-vertex linear hypergraph, and let $t \coloneqq e(\mathcal{H}) - n$. If $e_1, \dots, e_{2t} \in \mathcal{H}$ are distinct pairwise intersecting edges such that $\{e_{2i-1}, e_{2i}\}$ is a useful pair for each $i \in [t]$, then $\mathcal{H}$ has a proper edge-colouring with n colours, where each colour is assigned to at most two edges.*

Proof. We may assume that $e(\mathcal{H}) > n$. We will show that there exists a matching N in $\overline{L(\mathcal{H})}$ of size t.

For $i \in [t]$, suppose we have chosen distinct edges $z_1, \dots, z_{i-1} \in \mathcal{H} \setminus \{e_1, \dots, e_{2t}\}$ where z_j is non-adjacent to at least one of e_{2j-1} or e_{2j} in $L(\mathcal{H})$ for $1 \leq j \leq i - 1$. We claim that one can choose $z_i \in \mathcal{H} \setminus \{e_1, \dots, e_{2t}\}$ distinct from $z_1, \dots, z_{i-1}$ such that z_i is non-adjacent to either e_{2i-1} or e_{2i} in $L(\mathcal{H})$. Indeed, since $|N(e_{2i-1}) \cap N(e_{2i})| \leq n$, letting $S \coloneqq \mathcal{H} \setminus (N(e_{2i-1}) \cap N(e_{2i}))$, we have $|S| \geq e(\mathcal{H}) - n$ and every $e \in S$ is non-adjacent to at least one of e_{2i-1} or e_{2i}. Since $i - 1 \leq e(\mathcal{H}) - n - 1$ we can choose $z_i \in S$ distinct from $z_1, \dots, z_{i-1}$. Moreover, since $S \cap \{e_1, \dots, e_{2t}\} = \emptyset$, we have $z_i \in \mathcal{H} \setminus \{e_1, \dots, e_{2t}\}$, as desired.

Let $z_1, z_2, \dots, z_t \in \mathcal{H} \setminus \{e_1, \dots, e_{2t}\}$ be chosen using the above procedure. Then since z_j is non-adjacent to either e_{2j-1} or e_{2j} for each $j \in [t]$, we have a matching N in $\overline{L(\mathcal{H})}$ of size $t = e(\mathcal{H}) - n$. Now applying Observation 5.2, the proof is complete. □

Proposition 5.4. *Let $\mathcal{H}$ be an n-vertex linear hypergraph, and let $\{A, B\}$ be a partition of $\mathcal{H}$ such that $|A| + |B| - n \leq |A|/4$. If for every distinct intersecting $e, f \in A$, the pair $\{e, f\}$ is useful, then $\mathcal{H}$ has a proper edge-colouring with n colours, where each colour is assigned to at most two edges.*

Proof. We may assume that $|A| + |B| \geq n + 1$. Let N be a matching of maximum size in $\overline{L(\mathcal{H})}$. If $|N| \geq e(\mathcal{H}) - n = |A| + |B| - n$, then by Observation 5.2, we have a proper colouring of $\mathcal{H}$ with the desired properties. Thus we may assume that $|N| \leq |A| + |B| - n$. Then we have

$$|A \setminus V(N)| - 2(|A| + |B| - n) \geq |A| - 4(|A| + |B| - n) \geq 0$$

since we assumed $|A| + |B| - n \leq |A|/4$. By the maximality of N, all pairs $e, f \in A \setminus V(N)$ are adjacent. Thus we may choose $2t \coloneqq 2(|A| + |B| - n)$ distinct $a_1, \dots, a_{2t} \in A \setminus V(N)$ which are pairwise adjacent such that $|N(a_{2i-1}) \cap N(a_{2i})| \leq n$ holds for each $i \in [t]$. Thus we can apply Proposition 5.3 to $\{a_1, \dots, a_{2t}\}$ to complete the proof. □

Proposition 5.5. *Let $0 < 1/n_0 \ll \delta \ll 1$, and let $n \geq n_0$ satisfy $k^2 + k + 1 \geq n \geq (k-1)^2 + k + 1$ where $n, k \in \mathbb{N}$. Let $\mathcal{H}$ be an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $|e| \geq (1-\delta)\sqrt{n}$, and let $e, f \in \mathcal{H}$ be distinct intersecting edges of size at most k. Let $w \in e \cap f$, and let m be the number edges of size at most $k-1$ containing w. If at least one of e or f has size at most $k-1$, or if $m \leq 1/(3\delta)$, then $\{e, f\}$ is a useful pair.*

Proof. If at least one of e or f has size at most $k-1$, then

$$\begin{aligned}|N(e) \cap N(f)| &\leq (|e|-1)(|f|-1) + d(w) - 2 \leq (k-1)(k-2) + \frac{n-1}{(1-\delta)\sqrt{n}-1} \\ &\leq (k^2 - k + 2) - 2k + (1+2\delta)\sqrt{n} \leq n,\end{aligned}$$

as desired.

Now we may assume $|e| = |f| = k$ and $m \leq 1/(3\delta) \leq \frac{1-2/(k-1)}{2\delta}$. Let us consider the number of vertices in $V(\mathcal{H}) \setminus (e \cup f)$ sharing an edge with w. By the definition of m and the linearity of $\mathcal{H}$, we have

$$m((1-\delta)\sqrt{n}-1) + (d(w) - 2 - m)(k-1) \leq n - |e \cup f| = n - 2k + 1 \leq k(k-1) + 2,$$

which implies

$$d(w) - 2 \leq \frac{k(k-1)+2}{k-1} + m\left(1 - \frac{(1-\delta)\sqrt{n}-1}{k-1}\right) \leq \frac{k(k-1)+2}{k-1} + 2\delta m. \tag{5.1}$$

Thus,

$$|N(e) \cap N(f)| \leq (|e|-1)(|f|-1) + d(w) - 2 \overset{(5.1)}{\leq} k^2 - k + 1 + \frac{2}{k-1} + 2\delta m \leq n,$$

so $\{e, f\}$ is a useful pair, as desired. $\square$

Now we prove Lemma 5.1.

Proof of Lemma 5.1. First of all, we may assume that $e(\mathcal{H}) > n$. Let k be a positive integer such that

$$k^2 - k + 2 = (k-1)^2 + k + 1 \leq n \leq k^2 + k + 1, \tag{5.2}$$

let $A^- := \{e \in \mathcal{H} : |e| \leq k-1\}$, let $A^+ := \{e \in \mathcal{H} : |e| = k\}$, let $A := A^- \cup A^+$, and let $B := \{e \in \mathcal{H} : |e| \geq k+1\}$. Note that

$$\mathrm{vol}_{\mathcal{H}}(B) \geq |B|\frac{k(k+1)}{n(n-1)} \overset{(5.2)}{\geq} \frac{|B|}{n}.$$

Since $|e| \geq (1-\delta)\sqrt{n}$ for all $e \in \mathcal{H}$, we have

$$\mathrm{vol}_{\mathcal{H}}(A) \geq |A|\binom{(1-\delta)\sqrt{n}}{2}\binom{n}{2}^{-1} \geq |A|\frac{1-2\delta}{n}. \tag{5.3}$$

Combining the above two inequalities with $\mathrm{vol}_{\mathcal{H}}(A) + \mathrm{vol}_{\mathcal{H}}(B) \leq 1$, we have

$$|A| + |B| - n \leq 2\delta|A|. \tag{5.4}$$

If $|A^-| \leq 300$, then by Proposition 5.5, for any distinct intersecting $e, f \in A$, we have that $\{e, f\}$ is useful. Moreover, (5.4) implies that $|A| + |B| - n \leq |A|/4$. Thus we can apply Proposition 5.4 to obtain a proper edge-colouring of $\mathcal{H}$ with the desired properties, proving the lemma in this case. Hence, we may assume that

$$|A^-| > 300. \tag{5.5}$$

Note that

$$\mathrm{vol}_{\mathcal{H}}(A^+ \cup B) \geq |B|\frac{k(k+1)}{n(n-1)} + |A^+|\frac{k(k-1)}{n(n-1)} \overset{(5.2)}{\geq} \frac{|B|}{n} + \frac{|A^+|}{n}\left(1 - \frac{3}{\sqrt{n}}\right).$$

Similarly as in (5.3), we have $\mathrm{vol}_{\mathcal{H}}(A^-) \geq |A^-|(1-2\delta)/n$. Combining the previous two inequalities with the fact that $\mathrm{vol}_{\mathcal{H}}(A^+ \cup B) + \mathrm{vol}_{\mathcal{H}}(A^-) \leq 1$, we obtain

$$e(\mathcal{H}) - n = |A^-| + |A^+ \cup B| - n \leq \frac{3|A^+|}{\sqrt{n}} + 2\delta|A^-|. \tag{5.6}$$

Thus if $|A^+| \leq \sqrt{n}|A^-|/15$, then we have $|A^-| + |A^+ \cup B| - n \leq |A^-|/4$. Using this inequality and Proposition 5.5, we can apply Proposition 5.4 with A^- and $A^+ \cup B$ playing the roles of A and B, respectively, to obtain an edge-colouring of $\mathcal{H}$ with the desired properties, proving the lemma in this case. Thus we can assume that

$$|A^+| > \frac{\sqrt{n}|A^-|}{15} \overset{(5.5)}{\geq} 20\sqrt{n}. \tag{5.7}$$

Now let $t := e(\mathcal{H}) - n$, let $L := L(\mathcal{H})$ be the line graph of $\mathcal{H}$, and let N be a maximal matching in $\overline{L}$. We assume $|N| < t$, as otherwise by Observation 5.2, we obtain the desired proper edge-colouring of $\mathcal{H}$, proving the lemma. Combining this inequality with (5.6) and (5.7), we have

$$|N| < t \leq \frac{3|A^+|}{\sqrt{n}} + 2\delta|A^-| < \frac{5|A^+|}{\sqrt{n}}. \tag{5.8}$$

Most of the remainder of the proof is devoted to the following claim.

Claim 1. *There are $2t$ distinct $e_1, \ldots, e_{2t} \in A^+$ such that*
(i) $e_1, \ldots, e_{2t}$ are pairwise intersecting and
(ii) $\{e_{2i-1}, e_{2i}\}$ is useful for $i \in [t]$.

Proof of claim: Since N is maximal, $\mathcal{H} \setminus V(N)$ is a clique in L. We choose $e_1, \ldots, e_{2t}$ in $A^+ \setminus V(N)$, which will ensure that (i) holds.

For each $x \in V(\mathcal{H})$, let $A_x := \{e \in A^- : x \in e\}$, and let $V_{\text{bad}} := \{x \in V(\mathcal{H}) : |A_x| \geq (4\delta)^{-1}\}$. Thus, if $e, f \in \mathcal{H}$ are distinct and intersecting such that $w \notin V_{\text{bad}}$ where $w \in e \cap f$, then $\{e, f\}$ is useful by Proposition 5.5. We choose $e_1, \ldots, e_{2t}$ such that $w \in e_{2i-1} \cap e_{2i}$ satisfies $w \notin V_{\text{bad}}$, which will ensure that (ii) holds. Let $\mathcal{P} := \{(w, e) : w \in V_{\text{bad}} \text{ and } w \in e \in A^-\}$, and note that $|V_{\text{bad}}| \cdot (4\delta)^{-1} \leq |\mathcal{P}| \leq (k-1)|A^-| \leq 2\sqrt{n}|A^-|$. Thus,

$$|V_{\text{bad}}| \leq 8\delta|A^-|\sqrt{n}. \tag{5.9}$$

Now let $A^* := \{e \in A^+ : |e \cap V_{\text{bad}}| \geq \sqrt{\delta n}\}$, and note that $|A^*|\sqrt{\delta n} \leq |V_{\text{bad}}|\frac{n-1}{k-1} \overset{(5.9)}{\leq} 16\delta n|A^-|$. Therefore

$$|A^*| \leq 16\sqrt{\delta n}|A^-| \overset{(5.7)}{\leq} \frac{|A^+|}{20}. \tag{5.10}$$

Thus

$$|A^+ \setminus (A^* \cup V(N))| \geq |A^+| - |A^*| - |V(N)| \overset{(5.8),(5.10)}{\geq} \frac{9|A^+|}{10} \overset{(5.7)}{\geq} 18\sqrt{n}. \tag{5.11}$$

Now we iterate the following procedure for $i \in [t']$, where $t' := \lceil |A^+|/4 \rceil$. Suppose we have chosen distinct $e_1, \ldots, e_{2(i-1)} \in A^+ \setminus (A^* \cup V(N))$ such that $\{e_{2j-1}, e_{2j}\}$ is useful for each $j \in [i-1]$. Consider the set $S_i := A^+ \setminus \big(A^* \cup V(N) \cup \{e_1, \ldots, e_{2(i-1)}\}\big)$, which has size

$$|S_i| \geq |A^+ \setminus (A^* \cup V(N))| - 2(t'-1) \overset{(5.11)}{\geq} \frac{9|A^+|}{10} - \frac{|A^+|}{2} \overset{(5.7)}{\geq} 5\sqrt{n}. \tag{5.12}$$

We first show that there exists a useful pair $\{e_{2i-1}, e_{2i}\} \subseteq S_i$. For any $e \in S_i$, we have $|e \cap V_{\text{bad}}| \leq \sqrt{\delta n}$ since $S_i \subseteq A^+ \setminus A^*$. Therefore, letting $\mathcal{P}_i := \{(w, e) : e \in S_i,\ w \in e \setminus V_{\text{bad}}\}$, we have

$$\begin{aligned}\frac{1}{|V(\mathcal{H}) \setminus V_{\text{bad}}|} \sum_{w \in V(\mathcal{H}) \setminus V_{\text{bad}}} d_{S_i}(w) &= \frac{|\mathcal{P}_i|}{|V(\mathcal{H}) \setminus V_{\text{bad}}|} \geq \frac{|\mathcal{P}_i|}{n} = \frac{1}{n}\sum_{e \in S_i} |e \setminus V_{\text{bad}}| \\ &\geq \frac{|S_i|(k - \sqrt{\delta n})}{n} \overset{(5.12),(5.2)}{\geq} \frac{5\sqrt{n} \cdot \sqrt{n}/2}{n} > 2.\end{aligned}$$

Thus there exists a vertex $w \in V(\mathcal{H}) \setminus V_{\text{bad}}$ with $d_{S_i}(w) \geq 2$, which implies that there is a useful pair $\{e_{2i-1}, e_{2i}\} \subseteq S_i$ such that $w \in e_{2i-1} \cap e_{2i}$.

The above procedure constructs a useful pair $\{e_{2i-1}, e_{2i}\} \subseteq A^+ \setminus (A^* \cup V(N))$ for each $i \in [t']$. Recall that since N is maximal, the elements of $A^+ \setminus V(N)$ are pairwise intersecting. Since $t' \geq |A^+|/4 > t = e(\mathcal{H}) - n$ by (5.8), $e_1, \ldots, e_{2t}$ satisfy (i) and (ii), as claimed. ♦

Now by combining Claim 1 and Proposition 5.3, there is a proper edge-colouring of $\mathcal{H}$ with n colours such that each colour is assigned to at most two edges, which completes the proof of Lemma 5.1. □

6. Colouring large and medium edges

The main result of this section is the following, which we use in the proof of Theorem 1.1 to colour large and medium edges.

Theorem 6.1. *Let $0 < 1/n_0 \ll 1/r_0 \ll 1/r_1, \beta \ll \gamma_1 \ll \sigma \ll \delta \ll \gamma_2 \ll 1$, and let $n \geq n_0$. If $\mathcal{H}$ is an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $|e| > r_1$, then at least one of the following holds:*

(6.1:a) There exists a proper edge-colouring of $\mathcal{H}$ using at most $(1-\sigma)n$ colours such that
- *(i) every colour assigned to a huge edge is assigned to no other edge,*
- *(ii) every medium edge is assigned a colour from a set C_{med} of size at most $\gamma_1 n$ such that for every $c \in C_{\mathrm{med}}$, at most $\gamma_1 n$ vertices are incident to an edge coloured c, and*
- *(iii) for every colour $c \notin C_{\mathrm{med}}$ not assigned to a huge edge, at most βn vertices are incident to an edge coloured c.*

(6.1:b) There exists a set of FPP-extremal edges of volume at least $1-\delta$ and a proper edge-colouring of $\mathcal{H}$ using at most n colours such that
- *(i) for every colour c assigned to a huge edge, at most δn vertices are incident to an edge coloured c,*
- *(ii) every medium edge is assigned a colour from a set C_{med} of size at most $\gamma_2 n$ such that for every $c \in C_{\mathrm{med}}$, at most $\gamma_1 n$ vertices are incident to an edge coloured c, and*
- *(iii) for every colour $c \notin C_{\mathrm{med}}$ not assigned to a huge edge, at most βn vertices are incident to an edge coloured c.*

Note that every linear hypergraph $\mathcal{H}$ satisfies $\mathrm{vol}_{\mathcal{H}}(\mathcal{H}) \leq 1$, so in (6.1:b), the FPP-extremal edges contain almost all of the pairs of vertices.

We now prove Theorem 1.2 by combining Theorem 4.6 and Theorem 6.1.

Proof of Theorem 1.2. Without loss of generality, we may assume that δ is sufficiently small. Let $0 < 1/n_0 \ll 1/r_0 \ll 1/r_1, \beta \ll \gamma \ll \sigma \ll \delta \ll 1$, and recall $\mathcal{H}_{\mathrm{small}}$, $\mathcal{H}_{\mathrm{med}}$, $\mathcal{H}_{\mathrm{large}}$, and $\mathcal{H}_{\mathrm{ex}}$ were defined in Definition 2.3. By assumption, $e(\mathcal{H}_{\mathrm{ex}}) \leq (1-3\delta)n$, so

$$\mathrm{vol}_{\mathcal{H}}(\mathcal{H}_{\mathrm{ex}}) \leq (1-3\delta)n \cdot \binom{(1+\delta)\sqrt{n}}{2}\binom{n}{2}^{-1} \leq \frac{(1-3\delta)(1+\delta)^2 n}{n-1} < 1-\delta.$$

Hence, applying Theorem 6.1 with $\mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}}$, γ, σ, and δ playing the roles of $\mathcal{H}$, γ_1, σ, and δ, respectively, we obtain a proper edge-colouring $\phi' : \mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}} \to [(1-\sigma)n]$ and $C_{\mathrm{med}} \subseteq [(1-\sigma)n]$ such that every $e \in \mathcal{H}_{\mathrm{med}}$ satisfies $\phi'(e) \in C_{\mathrm{med}}$ and $|C_{\mathrm{med}}| \leq \gamma n$.

For every $e \in \mathcal{H}_{\mathrm{small}}$, let $C(e) := \{\phi'(f) : f \in \mathcal{H}_{\mathrm{large}}, e \cap f \neq \varnothing\}$. Note that for each vertex $w \in V(\mathcal{H})$, there are at most $2r_0^{-1}n$ edges of $\mathcal{H}_{\mathrm{large}}$ incident to w. Therefore, for every $e \in \mathcal{H}_{\mathrm{small}}$, there are at most $2r_1 r_0^{-1}n$ edges $f \in \mathcal{H}_{\mathrm{large}}$ such that $e \cap f \neq \varnothing$. Hence, $|C(e)| \leq \beta n$. So applying Theorem 4.6 with $\mathcal{H}_{\mathrm{small}}$, 3β, r_1, $\lfloor (1-\delta)n \rfloor$, $[(1-\sigma)n] \setminus C_{\mathrm{med}}$ playing the roles of $\mathcal{H}$, α, r, D, C, respectively, we obtain a proper edge-colouring $\phi'' : \mathcal{H}_{\mathrm{small}} \to [(1-\sigma)n] \setminus C_{\mathrm{med}}$ such that for every $e \in \mathcal{H}_{\mathrm{small}}$, $\phi''(e) \notin C(e)$, which implies that $\phi''(e) \neq \phi'(f)$ for every $f \in \mathcal{H}_{\mathrm{large}}$ with $e \cap f \neq \varnothing$. Hence $\phi := \phi' \cup \phi'' : \mathcal{H} \to [(1-\sigma)n]$ is a proper edge-colouring, as desired. □

6.1. Reordering.

If $\preceq$ is a linear ordering of the edges of a hypergraph $\mathcal{H}$, for each $e \in \mathcal{H}$, we define $N^{\preceq}_{\mathcal{H}}(e) := \{f \in N_{\mathcal{H}}(e) : f \preceq e\}$ and $d^{\preceq}_{\mathcal{H}}(e) := |N^{\preceq}_{\mathcal{H}}(e)|$. We omit the subscript $\mathcal{H}$ when it is clear from the context. For each $e \in \mathcal{H}$, we also let $\mathcal{H}^{\preceq e} := \{f \in \mathcal{H} : f \preceq e\}$. The main result of this subsection is the following key lemma, which we use to find the ordering of the edges of $\mathcal{H}$ mentioned in Section 2.2.

Lemma 6.2 (Reordering lemma)**.** *Let $0 < 1/r_1 \ll \tau, 1/K$ where $\tau < 1$ and $K \geq 1$. If $\mathcal{H}$ is an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $|e| \geq r_1$, then there exists a linear ordering $\preceq$ of the edges of $\mathcal{H}$ such that at least one of the following holds.*

(6.2:a) Every $e \in \mathcal{H}$ satisfies $d^{\preceq}(e) \leq (1-\tau)n$.

(6.2:b) There is a set $W \subseteq \mathcal{H}$ such that
- *(W1) $\max_{e \in W} |e| \leq (1 + 3\tau^{1/4}K^4) \min_{e \in W} |e|$ and*
- *(W2) $\mathrm{vol}_{\mathcal{H}}(W) \geq \frac{(1-\tau-7\tau^{1/4}/K)^2}{1+3\tau^{1/4}K^4}$.*

Moreover, if e^* is the last edge of W, then
- *(O1) for all $f \in \mathcal{H}$ such that $e^* \preceq f$ and $f \neq e^*$, we have $d^{\preceq}(f) \leq (1-\tau)n$ and*
- *(O2) for all $e, f \in \mathcal{H}$ such that $f \preceq e \preceq e^*$, we have $|f| \geq |e|$.*

Proposition 6.3. *Let $\alpha_1, \alpha_2, \tau \geq 0$. Let $\mathcal{H}$ be an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $|e| \geq 1 + \alpha_2$. Let $e \in \mathcal{H}$, let $r := |e|$, let $m_1 := |\{f \in N(e) : |f| \geq (1+\alpha_1)r\}|$, and let $m_2 := |\{f \in N(e) : (1+\alpha_1)r > |f| \geq r/(1+\alpha_2)\}|$. If $r > 1 + \alpha_2$, then*

(i) $$(1+\alpha_1)m_1 + \frac{m_2}{1+\alpha_2} \leq n + \frac{(1+\alpha_2)n}{r-1-\alpha_2}.$$

Moreover, if $m_1 + m_2 \geq (1-\tau)n$ and $\alpha_1 > 0$, then

(ii) $$m_1 \leq \left(\tau + \frac{(1+\alpha_2)(1+\alpha_2 r)}{r-1-\alpha_2}\right)\frac{n}{\alpha_1}.$$

If $\preceq$ is an ordering of the edges of an n-vertex linear hypergraph $\mathcal{H}$ satisfying $e \preceq f$ if $|e| > |f|$, then Proposition 6.3(i) with $\alpha_1, \alpha_2, \tau = 0$ implies that every $e \in \mathcal{H}$ with $|e| \geq 2$ satisfies $d^{\preceq}(e) \leq (1+1/(|e|-1))n$. This well-known fact immediately implies that every linear n-vertex hypergraph $\mathcal{H}$ satisfies $\chi'(\mathcal{H}) \leq 2n+1$, and if all of its edges have size at least $r_1 \geq 2$, then $\chi'(\mathcal{H}) \leq (1 + 1/(r_1-1))n + 1$.

Proof of Proposition 6.3. There are exactly $r(n-r)$ pairs of vertices $\{u, v\}$ of $\mathcal{H}$ where $u \notin e$ and $v \in e$. Thus, since $\mathcal{H}$ is linear, $\sum_{f \in N(e)}(|f|-1) \leq r(n-r)$. In particular,

$$\sum_{f \in N(e) : |f| \geq (1+\alpha_1)r} ((1+\alpha_1)r - 1) + \sum_{f \in N(e):(1+\alpha_1)r > |f| \geq r/(1+\alpha_2)} (r/(1+\alpha_2) - 1) \leq r(n-r).$$

Dividing both sides of this inequality by r and rearranging terms, we obtain

$$(1+\alpha_1)m_1 + m_2/(1+\alpha_2) \leq n - r + (m_1+m_2)/r. \tag{6.1}$$

Similarly, we have

$$m_1 + m_2 \leq \frac{r(n-r)}{r/(1+\alpha_2) - 1} \leq \frac{(1+\alpha_2)rn}{r-1-\alpha_2}. \tag{6.2}$$

Substituting this inequality in the right side of (6.1), we obtain (i), as desired.

Now suppose additionally $m_1 + m_2 \geq (1-\tau)n$ and $\alpha_1 > 0$. By combining the former inequality with (i), we obtain $\alpha_1 m_1 \leq \tau n + n(1+\alpha_2)/(r-1-\alpha_2) + (1 - 1/(1+\alpha_2))m_2$. Since $1 - 1/(1+\alpha_2) \leq \alpha_2$, we have $\alpha_1 m_1 \leq \tau n + n(1+\alpha_2)/(r-1-\alpha_2) + \alpha_2 m_2$, and by combining this inequality with the bound on m_2 from (6.2), we obtain (ii), as desired. □

Proof of Lemma 6.2. We consider an ordering $\preceq$ of the edges of $\mathcal{H}$ satisfying (O1) and (O2) for some $e^* \in \mathcal{H}$ such that $e(\mathcal{H}^{\preceq e^*})$ is minimum. Note that such an ordering exists – in particular, any ordering where $f \preceq e$ whenever $|f| \geq |e|$ satisfies (O1) and (O2) for e^*, where e^* is the last edge in the ordering.

If $e(\mathcal{H}^{\preceq e^*}) = 1$, then $\preceq$ satisfies (6.2:a), as desired, so we assume we do not have this case. Now we have $d^{\preceq}(e^*) > (1-\tau)n$, or else the predecessor of e^* also satisfies (O1) and (O2), contradicting the choice of $e(\mathcal{H}^{\preceq e^*})$ to be minimum.

Let $r := |e^*|$, and let $W := \{f \preceq e^* : |f| \leq (1 + 3\tau^{1/4}K^4)r\}$. We claim that W satisfies (6.2:b). By the choice of $\preceq$, every $e \in N^{\preceq}(e^*)$ satisfies

$$|N(e) \cap \mathcal{H}^{\preceq e^*}| > (1-\tau)n, \tag{6.3}$$

or else we can make e the successor of e^*. Let $X := \{e \in N^{\preceq}(e^*) : |e| \leq (1 + K\sqrt{\tau})r\}$. By (O2), we may apply Proposition 6.3(ii) with $K\sqrt{\tau}$ and 0 playing the roles of α_1 and α_2, respectively, to obtain

$$|X| \geq (1-\tau)n - |N^{\preceq}(e^*) \setminus X| \geq \left(1 - \tau - \frac{\tau + 2/r}{K\sqrt{\tau}}\right)n \geq \left(1 - \tau - \frac{2\sqrt{\tau}}{K}\right)n. \tag{6.4}$$

Consider $e \in X$. Note that by (O2) and the definition of W, every $f \in \mathcal{H}^{\preceq e^*} \setminus W$ satisfies $|f| \geq (1 + 3\tau^{1/4}K^4)r \geq (1 + K^3\tau^{1/4})|e|$. We now aim to apply Proposition 6.3(ii) to e with $K^3\tau^{1/4}$ and $K\sqrt{\tau}$ playing the roles of α_1 and α_2, respectively. Let m_1 and m_2 be defined as in Proposition 6.3. Then $|(N(e) \cap \mathcal{H}^{\preceq e^*}) \setminus W)| \leq m_1$. Moreover, since $e \in X$, (6.3) implies that $m_1 + m_2 \geq (1-\tau)n$. Thus, we can apply Proposition 6.3 to deduce that for every $e \in X$ we have

$$|(N(e) \cap \mathcal{H}^{\preceq e^*}) \setminus W| \leq \left(\tau + \frac{1 + K\sqrt{\tau} + K\sqrt{\tau}|e| + K^2\tau|e|}{|e|/2}\right)\frac{n}{K^3\tau^{1/4}} \leq \frac{6K^2\sqrt{\tau}}{K^3\tau^{1/4}}n = \frac{6\tau^{1/4}}{K}n. \tag{6.5}$$

(In the second inequality, we used that $|e| \geq r$ by (O2).)

Now we use these inequalities to lower bound the size of W. First we claim that every $e \in X$ satisfies

$$|N(e) \cap (W \setminus N(e^*))| \geq (1 - \tau - 7\tau^{1/4}/K)n - (1 + K\sqrt{\tau})r^2. \tag{6.6}$$

To that end, we bound $|N(e) \cap N^{\preceq}(e^*)|$, as follows. Let $N_1 := \{f \in N(e) \cap N^{\preceq}(e^*) : f \cap e^* = e \cap e^*\}$ and $N_2 := (N(e) \cap N^{\preceq}(e^*)) \setminus N_1$. Since $\mathcal{H}$ is linear and every edge in $N^{\preceq}(e^*)$ has size at least r, we have $|N_1| \leq n/(r-1)$ and $|N_2| \leq (|e|-1)(|e^*|-1) \leq (1+K\sqrt{\tau})r^2$. Thus,

$$|N(e) \cap N^{\preceq}(e^*)| \leq \frac{n}{r-1} + (1+K\sqrt{\tau})r^2. \tag{6.7}$$

On the other hand, since $W \subseteq \mathcal{H}^{\preceq e^*}$, $|N(e) \cap (W \setminus N(e^*))| = |(N(e) \cap \mathcal{H}^{\preceq e^*}) \cap (W \setminus N(e^*))|$. Moreover, we have

$$|(N(e) \cap \mathcal{H}^{\preceq e^*}) \cap (W \setminus N(e^*))| \geq |N(e) \cap \mathcal{H}^{\preceq e^*}| - |(N(e) \cap \mathcal{H}^{\preceq e^*}) \setminus W| - |N(e) \cap N^{\preceq}(e^*)|.$$

Combining this inequality with (6.3), (6.5), (6.7), one can see that (6.6) follows, as claimed.

For every $f \in W \setminus N(e^*)$, we also have

$$|N(f) \cap X| \leq |N(f) \cap N(e^*)| \leq |f||e^*| \leq (1 + 3\tau^{1/4}K^4)r^2. \tag{6.8}$$

Since $\sum_{e \in X} |N(e) \cap (W \setminus N(e^*))| = |\{(e,f) : e \in X,\ f \in W \setminus N(e^*),\ e \in N(f)\}| = \sum_{f \in W \setminus N(e^*)} |N(f) \cap X|$, by combining (6.6) and (6.8), we have

$$|W \setminus N(e^*)| \geq |X| \left(\frac{(1-\tau-7\tau^{1/4}/K)n}{(1+3\tau^{1/4}K^4)r^2} - \frac{1+K\sqrt{\tau}}{1+3\tau^{1/4}K^4} \right),$$

and since $X \subseteq N(e^*) \cap W$, this inequality implies

$$\begin{aligned} |W| &\geq |X| \left(\frac{(1-\tau-7\tau^{1/4}/K)n}{(1+3\tau^{1/4}K^4)r^2} + 1 - \frac{1+K\sqrt{\tau}}{1+3\tau^{1/4}K^4} \right) \\ &\geq \left(1 - \tau - \frac{2\sqrt{\tau}}{K}\right) \left(\frac{1-\tau-7\tau^{1/4}/K}{1+3\tau^{1/4}K^4} \right) \frac{n^2}{r^2}, \end{aligned}$$

where the second inequality follows from (6.4) and the fact that $1 \geq \frac{1+K\sqrt{\tau}}{1+3\tau^{1/4}K^4}$. Thus, $\mathrm{vol}_{\mathcal{H}}(W) \geq |W|\binom{r}{2}/\binom{n}{2} \geq |W|\frac{r^2}{n^2}(1-1/r) \geq \frac{(1-\tau-7\tau^{1/4}/K)^2}{1+3\tau^{1/4}K^4}$, so W satisfies (6.2:b), as claimed, and moreover, $\preceq$ satisfies (O1) and (O2), as required. □

6.2. **Colouring locally sparse graphs.** To prove Theorem 6.1 we use the following theorem [38, Theorem 10.5], which has been improved in [8, 7, 23].

Theorem 6.4 (Molloy and Reed [38]). *Let $0 < 1/\Delta \ll \zeta \leq 1$. Let G be a graph with $\Delta(G) \leq \Delta$. If every $v \in V(G)$ satisfies $e(G[N(v)]) \leq (1-\zeta)\binom{\Delta}{2}$, then $\chi(G) \leq (1-\zeta/e^6)\Delta$.*

Corollary 6.5. *Let $0 < 1/n_0, 1/r \ll \alpha \ll \zeta < 1$, let $n \geq n_0$, and suppose $r \leq (1-\zeta)\sqrt{n}$. If $\mathcal{H}$ is an n-vertex linear hypergraph such that every $e \in \mathcal{H}$ satisfies $|e| \in [r, (1+\alpha)r]$, then $\chi'(\mathcal{H}) \leq (1-\zeta/500)n$.*

Proof. Let $\Delta := (1+\alpha)r(n-r)/(r-1)$, and let $L := L(\mathcal{H})$. For every edge $e \in \mathcal{H}$, there are at most $(1+\alpha)r(n-r)$ pairs of vertices $\{u,v\}$ of $\mathcal{H}$ where $u \notin e$ and $v \in e$. Thus, since $\mathcal{H}$ is linear and every edge has size at least r, we have $\Delta(L) \leq \Delta$. Similarly, if $e, f \in \mathcal{H}$ share a vertex, then $|N_L(e) \cap N_L(f)| \leq n/(r-1) + (1+\alpha)^2 r^2 \leq (1-5\zeta/6)n$. Thus, every $v \in V(L)$ satisfies $e(L[N(v)]) \leq \Delta(1-5\zeta/6)n/2 \leq (1-5\zeta/6)\binom{\Delta}{2}$. Therefore by Theorem 6.4, $\chi'(\mathcal{H}) = \chi(L) \leq (1-5\zeta/(6e^6))\Delta \leq (1-\zeta/500)n$, as desired. □

In the proof of Theorem 1.3, we use the following theorem, which has been further improved in [1, 4, 12, 44].

Theorem 6.6 (Alon, Krivelevich, and Sudakov [4]). *Let $0 < \zeta, 1/K_{6.6} \ll 1$. Let G be a graph with $\Delta(G) \leq \Delta$. If every $v \in V(G)$ satisfies $e(G[N(v)]) \leq \zeta\Delta^2$, then $\chi(G) \leq K_{6.6}\Delta/\log(1/\zeta)$.*

We need the following corollary of Theorem 6.6. The proof is nearly identical to the proof of Corollary 6.5, with Theorem 6.4 replaced by Theorem 6.6, so we omit it.

Corollary 6.7. *Let $0 < 1/n_0 \ll \eta \ll \alpha, \varepsilon < 1$, and let $n \geq n_0$. If $\mathcal{H}$ is an n-vertex linear hypergraph such that every $e \in \mathcal{H}$ satisfies $1/\eta \leq |e| \leq \eta\sqrt{n}$ and $\min_{e \in \mathcal{H}} |e| \geq \alpha \max_{e \in \mathcal{H}} |e|$, then $\chi'(\mathcal{H}) \leq \varepsilon n$.*

We remark that the proof of Corollary 6.7 is also similar to that of [17, Theorem 1.1], where a similar statement was proved for uniform regular linear hypergraphs (which implies that the EFL conjecture holds for all r-uniform regular linear n-vertex hypergraphs satisfying $c \leq r \leq \sqrt{n}/c$ for some constant $c > 0$).

6.3. **Proof of Theorems 1.3 and 6.1.** Let ϕ be a proper edge-colouring of an n-vertex hypergraph $\mathcal{H}$. For $\alpha \in (0,1)$, we say ϕ is *α-bounded* if every colour c satisfies at least one of the following: c is assigned to at most one $e \in \mathcal{H}$, or $\phi^{-1}(c)$ covers at most αn vertices of $\mathcal{H}$.

Proposition 6.8. *Let $\alpha > 0$, and let $\mathcal{H}$ be an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $|e| \geq r+1$.*

(i) If $M_1, \dots, M_t$ are pairwise edge-disjoint matchings in $\mathcal{H}$ that each cover at least αn vertices, then $t \leq n/(\alpha r)$.
(ii) There is an α-bounded proper edge-colouring of $\mathcal{H}$ using at most $\chi'(\mathcal{H}) + 2n/(\alpha^2 r)$ colours.

Proof. Since $\mathcal{H}$ is linear, we have $1 \geq \mathrm{vol}_{\mathcal{H}}\left(\bigcup_{i=1}^t M_i\right) = \sum_{i=1}^t \mathrm{vol}_{\mathcal{H}}(M_i)$. Moreover, for each $i \in [t]$, $\mathrm{vol}_{\mathcal{H}}(M_i) \geq \frac{\alpha n}{r+1}\binom{r+1}{2}\Big/\binom{n}{2} \geq \alpha r/n$. Combining these inequalities, we have $t \leq n/(\alpha r)$, as desired for (i).

Now let ϕ be a proper edge-colouring of $\mathcal{H}$ using a set C of $\chi'(\mathcal{H})$ colours. For each $c \in C$, let $M_c := \phi^{-1}(c)$, and let $C' := \{c \in C : |V(M_c)| > \alpha n\}$. By (i), we have $|C'| \leq n/(\alpha r)$. For each $c \in C'$, there is a partition of M_c into a set $\mathcal{M}_c$ of pairwise disjoint matchings such that every $M \in \mathcal{M}_c$ covers at least $\alpha n/2$ vertices and satisfies at least one of the following: $|M| = 1$, or M covers at most αn vertices of $\mathcal{H}$. Note that $|\mathcal{M}_c| \leq 2/\alpha$. Now for each $c \in C'$, we choose a distinct set of $|\mathcal{M}_c|$ colours C_c disjoint from C, and we define a proper edge-colouring ϕ' of $\mathcal{H}$ as follows. For each $c \in C \setminus C'$ and $e \in M_c$, we let $\phi'(e) := \phi(e)$. For each $c \in C'$ and $e \in M_c$, we let $\phi'(e) \in C_c$ such that for every $M \in \mathcal{M}_c$, every edge of M is assigned the same colour. By the choice of $\mathcal{M}_c$, every colour is either assigned to at most one $e \in \mathcal{H}$ by ϕ', or there are at most αn vertices of $\mathcal{H}$ that are incident to an edge assigned that colour, so ϕ' is α-bounded, as desired. Moreover, by the bounds on $|\mathcal{M}_c|$ and $|C'|$, the colouring ϕ' uses at most $|C| + 2|C'|/\alpha \leq \chi'(\mathcal{H}) + 2n/(\alpha^2 r)$ colours, as desired for (ii). □

Proposition 6.9. *Let $0 < 1/n_0 \ll 1/r \ll \alpha_1, \alpha_2 < 1$, and let $n \geq n_0$. Let $\preceq$ be a linear ordering of the edges of an n-vertex linear hypergraph $\mathcal{H}$ where every $e \in \mathcal{H}$ satisfies $|e| \geq r$. If C is a list-assignment for the line graph of $\mathcal{H}$ such that every $e \in \mathcal{H}$ satisfies $|C(e)| \geq d^{\preceq}(e) + \alpha_1 n$, then there is an α_2-bounded proper edge-colouring ϕ of $\mathcal{H}$ such that $\phi(e) \in C(e)$ for every $e \in \mathcal{H}$.*

Proof. Let $\mathcal{H}_{\mathrm{big}} := \{e \in \mathcal{H} : |e| \geq \alpha_2 n/2\}$, and note that $e(\mathcal{H}_{\mathrm{big}}) \leq 4/\alpha_2$ by (2.2). By possibly reordering $\preceq$ and replacing α_1 with $\alpha_1/2$, we may assume without loss of generality that every $e \in \mathcal{H}_{\mathrm{big}}$ satisfies $e \preceq f$ for $f \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{big}}$.

Choose an edge $e^* \in \mathcal{H}$ and an α_2-bounded proper edge-colouring ϕ of $\mathcal{H}^{\preceq e^*}$ satisfying $\phi(e) \in C(e)$ for every $e \in \mathcal{H}^{\preceq e^*}$ such that $e(\mathcal{H}^{\preceq e^*})$ is maximum. Note that such a choice indeed exists, for example when e^* is the first edge in $\preceq$. We claim that e^* is the last edge of $\mathcal{H}$ in $\preceq$, in which case ϕ is the desired colouring. Suppose to the contrary, and let f be the successor of e^*. We have $f \notin \mathcal{H}_{\mathrm{big}}$, or else assigning f a colour in $C(f) \setminus \{\phi(e) : e \preceq f\}$ would yield an α_2-bounded colouring of $\mathcal{H}^{\preceq f}$, contradicting the choice of e^*.

Now let $C_1 := \bigcup_{e \in N^{\preceq}(f)} \phi(e)$, and let C_2 be the set of colours c for which there are at least $\alpha_2 n/2$ vertices of $\mathcal{H}$ incident to an edge assigned the colour c. If there is a colour $c \in C(f) \setminus (C_1 \cup C_2)$, then assigning $\phi(f) := c$ would yield an α_2-bounded colouring of $\mathcal{H}^{\preceq f}$, contradicting the choice of e^*. Therefore $|C_1| + |C_2| \geq |C(f)|$, and since $|C_1| \leq d^{\preceq}(f)$, we have $|C_2| \geq \alpha_1 n/2$. However, by Proposition 6.8(i), $|C_2| \leq 2n/(\alpha_2(r-1)) < \alpha_1 n/2$, a contradiction. □

Before we prove Theorem 6.1, we prove Theorem 1.3 using Theorem 4.6, Lemma 6.2, Corollary 6.7, and Proposition 6.9. The proof of Theorem 6.1 uses similar ideas, with Corollary 6.5 instead of Corollary 6.7.

Proof of Theorem 1.3. We may assume that $\varepsilon \ll 1$. Choose n_0, η to satisfy $0 < 1/n_0, \eta \ll \varepsilon$. First we decompose $\mathcal{H}$ into three spanning subhypergraphs, as follows. Let $\mathcal{H}_1 := \{e \in \mathcal{H} : 1/\eta < |e| < \eta\sqrt{n}\}$, let $\mathcal{H}_2 := \{e \in \mathcal{H} : |e| \leq 1/\eta\}$, and let $\mathcal{H}_3 := \{e \in \mathcal{H} : |e| \geq \sqrt{n}/\eta\}$. Since $\Delta(\mathcal{H}) \leq \eta n$, by Theorem 4.6 applied to $\mathcal{H}_2$ with ηn, $1/2$, and $1/\eta$ playing the roles of D, α, and r, respectively, we have $\chi'(\mathcal{H}_2) \leq 3\eta n/2 \leq \varepsilon n/4$. Since $\mathcal{H}$ is linear and $\mathrm{vol}_{\mathcal{H}}(\mathcal{H}_3) \leq 1$, we have $e(\mathcal{H}_3) \leq 2\eta n$, and thus, $\chi'(\mathcal{H}_3) \leq 2\eta n \leq \varepsilon n/4$. Therefore it suffices to show that $\chi'(\mathcal{H}_1) \leq \varepsilon n/2$.

Without loss of generality, let us assume $\mathcal{H}_1 \neq \varnothing$. Let $\mathcal{H}_0^{\mathrm{left}} := \mathcal{H}_1$, and for every positive integer i, define spanning subhypergraphs $\mathcal{H}_i^{\mathrm{left}}, \mathcal{H}_i^{\mathrm{good}}, W_i$ of $\mathcal{H}_{i-1}^{\mathrm{left}}$ as follows. If $\mathcal{H}_{i-1}^{\mathrm{left}} = \varnothing$, then let $\mathcal{H}_i^{\mathrm{left}}, \mathcal{H}_i^{\mathrm{good}}, W_i := \varnothing$. Otherwise, apply Lemma 6.2 to $\mathcal{H}_{i-1}^{\mathrm{left}}$ with $1 - \varepsilon/6$ and ε^{-2} playing the roles of τ and K, respectively, to obtain an ordering $\preceq_i$. If $\preceq_i$ satisfies (6.2:a), then let $\mathcal{H}_i^{\mathrm{good}} := \mathcal{H}_{i-1}^{\mathrm{left}}$, and let $\mathcal{H}_i^{\mathrm{left}}, W_i := \varnothing$. Otherwise, let W_i be the set W obtained from (6.2:b), let e_i^* be the edge of W_i which comes last in $\preceq_i$, let $\mathcal{H}_i^{\mathrm{good}} := \mathcal{H}_{i-1}^{\mathrm{left}} \setminus (\mathcal{H}_{i-1}^{\mathrm{left}})^{\preceq e_i^*}$, let f_i^* be the edge of W_i which comes first in $\preceq_i$, and let $\mathcal{H}_i^{\mathrm{left}} := \mathcal{H}_{i-1}^{\mathrm{left}} \setminus \{e \in \mathcal{H}_{i-1}^{\mathrm{left}} : f_i^* \preceq_i e\}$.

By (O2), we may assume without loss of generality that every $e \in \mathcal{H}_{i-1}^{\mathrm{left}}$ satisfying $f_i^* \preceq_i e \preceq_i e_i^*$ is in W_i. By the choices of τ and K, if $\preceq_i$ and W_i satisfy (6.2:b), then

(W_i1) $\max_{e \in W_i} |e| \leq \varepsilon^{-10}|e_i^*|$ and

(W_i2) $\mathrm{vol}_{\mathcal{H}}(W_i) \geq \varepsilon^{20}$.

For any $i \geq 1$, note that the sets $W_1, \dots, W_i$ are pairwise disjoint. Moreover, if $W_i = \varnothing$ then $\mathcal{H}_i^{\mathrm{left}} = \varnothing$, and if $W_i \neq \varnothing$ then $W_1, \dots, W_{i-1}$ are also nonempty. Thus, we have $\sum_i \mathrm{vol}_{\mathcal{H}}(W_i) \leq 1$, and if $W_i \neq \varnothing$ then $i \leq \varepsilon^{-20}$ by (W_i2), so there is some integer $k \geq 1$ such that $W_k = \varnothing$, and $k \leq \varepsilon^{-20} + 1$. Hence,

$$\text{(6.9)} \qquad \mathcal{H}_1 \text{ is partitioned into } W_1, \dots, W_{k-1} \text{ and } \mathcal{H}_1^{\mathrm{good}}, \dots, \mathcal{H}_k^{\mathrm{good}},$$

where $W_1, \dots, W_{k-1}$ are nonempty, and $\mathcal{H}_1^{\mathrm{good}}, \dots, \mathcal{H}_k^{\mathrm{good}}$ could be empty.

Combine $\preceq_1, \dots, \preceq_k$ to obtain an ordering $\preceq$ of $\mathcal{H}_1$ where if $f \in \mathcal{H}_i^{\mathrm{good}} \cup W_i$, then $e \preceq f$ for every $e \in (\mathcal{H}_{i-1}^{\mathrm{left}})^{\preceq_i f}$. Let $\mathcal{H}^{\mathrm{good}} \coloneqq \bigcup_{i=1}^k \mathcal{H}_i^{\mathrm{good}}$, and note that by (6.2:a) and (O1),

$$\text{(6.10)} \qquad \text{if } e \in \mathcal{H}^{\mathrm{good}}, \text{ then } d_{\mathcal{H}_1}^{\preceq}(e) \leq \varepsilon n/6.$$

By (W_i1), and since every $e \in \mathcal{H}_1$ satisfies $1/\eta < |e| < \eta\sqrt{n}$, for each $i \in [k-1]$ we can apply Corollary 6.7 to W_i with η, ε^{10}, and $\varepsilon/(6k)$ playing the roles of η, α, and ε, respectively, to obtain a proper edge-colouring $\phi_i : W_i \to C_i$, where $|C_i| \leq \varepsilon n/(6k)$. By (6.10), we can apply Proposition 6.9 to $\mathcal{H}^{\mathrm{good}}$ with $1/\eta$, $\varepsilon/6$, and $1/2$ playing the roles of r, α_1, and α_2, respectively, to obtain a proper edge-colouring $\phi' : \mathcal{H}^{\mathrm{good}} \to C'$, where $|C'| \leq \varepsilon n/3$. We may assume without loss of generality that C', $C_1, \dots, C_{k-1}$ are pairwise disjoint. Therefore by (6.9), we can combine $\phi', \phi_1, \dots, \phi_{k-1}$ to obtain a proper edge-colouring $\phi : \mathcal{H}_1 \to C' \cup \bigcup_{i=1}^{k-1} C_i$. Since $|C'| + \sum_{i=1}^{k-1} |C_i'| \leq \varepsilon n/2$, we have $\chi'(\mathcal{H}_1) \leq \varepsilon n/2$, as desired. □

Proposition 6.10. *Let $0 < 1/n_0 \ll 1/r_0 \ll 1/r_1 \ll \gamma < 1$, and let $n \geq n_0$. If $\mathcal{H}$ is an n-vertex linear hypergraph where every $e \in \mathcal{H}$ satisfies $r_1 \leq |e| \leq r_0$, then there is a γ-bounded proper edge-colouring of $\mathcal{H}$ using at most γn colours.*

Proof. By Proposition 6.8(ii) applied with γ and $r_1 - 1$ playing the roles of α and r, respectively, there is a γ-bounded proper edge-colouring ϕ of $\mathcal{H}$ using at most $\chi'(\mathcal{H}) + 2n/(\gamma^2(r_1-1)) \leq \chi'(\mathcal{H}) + \gamma n/2$ colours. Since $\mathcal{H}$ is linear and every $e \in \mathcal{H}$ satisfies $|e| \geq r_1$, we have $\Delta(\mathcal{H}) \leq n/(r_1 - 1)$. Thus, by Theorem 4.6, $\chi'(\mathcal{H}) \leq 2n/r_1 \leq \gamma n/2$, so ϕ uses at most γn colours, as desired. □

Proof of Theorem 6.1. Recall that $\mathcal{H}_{\mathrm{med}} = \{e \in \mathcal{H} : r_1 < |e| \leq r_0\}$, $\mathcal{H}_{\mathrm{large}} = \{e \in \mathcal{H} : |e| > r_0\}$, and $\mathcal{H}_{\mathrm{huge}} = \{e \in \mathcal{H} : |e| \geq \beta n/4\}$. Let $\mathcal{H}' \coloneqq \mathcal{H} \setminus \mathcal{H}_{\mathrm{huge}}$. By Proposition 6.10, there is a γ_1-bounded proper edge-colouring ϕ_{med} of $\mathcal{H}_{\mathrm{med}}$ using a set C_{med} of at most $\gamma_1 n$ colours. We use ϕ_{med} in all cases when we prove (6.1:a), and we define C_{med} differently when we prove (6.1:b) (see Case 2.2 below).

Now we apply Lemma 6.2 several times and combine the resulting orderings to obtain an ordering $\preceq$ of $\mathcal{H}$. We also define several subhypergraphs of $\mathcal{H}$, which we assume are all spanning. First, apply Lemma 6.2 to $\mathcal{H}'$ with $1 - \gamma_2/3$, and γ_2^{-2} playing the roles of τ and K, respectively, to obtain an ordering $\preceq_1$. We define e_1^*, W_1, and $\mathcal{H}_1^{\mathrm{good}}$, as follows. If $\preceq_1$ satisfies (6.2:a), then let e_1^* be the first edge of $\mathcal{H}'$, let $W_1 \coloneqq \varnothing$, and let $\mathcal{H}_1^{\mathrm{good}} \coloneqq \mathcal{H}'$. Otherwise, let W_1 be the set W obtained from (6.2:b), let e_1^* be the last edge of W_1, and let $\mathcal{H}_1^{\mathrm{good}} \coloneqq \mathcal{H}' \setminus (\mathcal{H}')^{\preceq_1 e_1^*}$. In both cases, let $\mathcal{H}_1^{\mathrm{left}} \coloneqq \mathcal{H}' \setminus \mathcal{H}_1^{\mathrm{good}}$, and let $r_2 \coloneqq |e_1^*|$. By the choices of τ and K, and since $\delta \ll \gamma_2 \ll 1$, if $\preceq_1$ and W_1 satisfy (6.2:b), then

(W_11) $\max_{e \in W_1} |e| \leq r_2/\gamma_2^{10}$ and

(W_12) $\mathrm{vol}_{\mathcal{H}}(W_1) \geq \gamma_2^{20} > \delta$.

If $\mathcal{H}_1^{\mathrm{left}} = \varnothing$, then let $e_2^* \coloneqq e_1^*$ and $W_2, \mathcal{H}_2^{\mathrm{good}} \coloneqq \varnothing$. If $\mathcal{H}_1^{\mathrm{left}} \neq \varnothing$, we apply Lemma 6.2 to $\mathcal{H}_1^{\mathrm{left}}$ with 3σ and 1 playing the roles of τ and K, respectively, to obtain an ordering $\preceq_2$, and we define e_2^*, W_2, and $\mathcal{H}_2^{\mathrm{good}}$, as follows. If $\preceq_2$ satisfies (6.2:a), then let e_2^* be the first edge of $\mathcal{H}_1^{\mathrm{left}}$ in $\preceq_2$, let $W_2 \coloneqq \varnothing$, and let $\mathcal{H}_2^{\mathrm{good}} \coloneqq \mathcal{H}_1^{\mathrm{left}}$. Otherwise, let W_2 be the set W obtained from (6.2:b), let e_2^* be the last edge of W_2 in $\preceq_2$, and let $\mathcal{H}_2^{\mathrm{good}} \coloneqq \mathcal{H}_1^{\mathrm{left}} \setminus (\mathcal{H}_1^{\mathrm{left}})^{\preceq_2 e_2^*}$. In all cases, let $\mathcal{H}_2^{\mathrm{left}} \coloneqq (\mathcal{H}_1^{\mathrm{left}} \cup \mathcal{H}_{\mathrm{huge}}) \setminus \mathcal{H}_2^{\mathrm{good}}$, and let $r_3 \coloneqq |e_2^*|$. By the choices of τ and K, and since $\sigma \ll \delta \ll 1$, if $\preceq_2$ and W_2 satisfy (6.2:b), then

(W_21) $\max_{e \in W_2} |e| \leq (1 + 4\sigma^{1/4})r_3$ and

(W_22) $\mathrm{vol}_{\mathcal{H}}(W_2) \geq (1 - \sigma^{1/5})^3 \geq 1 - \delta^3$.

Finally, if $W_2 \neq \varnothing$, then let f^* be the edge of W_2 which comes first in $\preceq_2$, and let $\mathcal{H}_3 \coloneqq \mathcal{H}_2^{\mathrm{left}} \setminus \{e \in \mathcal{H}_1^{\mathrm{left}} : f^* \preceq_2 e\}$. Otherwise, let $\mathcal{H}_3 \coloneqq \mathcal{H}_{\mathrm{huge}}$. Thus in both cases, $\mathcal{H}_{\mathrm{huge}} \subseteq \mathcal{H}_3$. Apply Lemma 6.2 with $\mathcal{H}_3$, $1 - 1/2000$, and 2000^2 playing the roles of $\mathcal{H}$, τ, and K, respectively, to obtain an ordering $\preceq_3$. Since $W_2 \cap \mathcal{H}_3 = \varnothing$, we have $\mathrm{vol}_{\mathcal{H}}(W_2) + \mathrm{vol}_{\mathcal{H}}(\mathcal{H}_3) \leq 1$. Thus, $\preceq_3$ satisfies (6.2:a), because (6.2:b) would imply there is a set $W' \subseteq \mathcal{H}_3$ disjoint from W_2 with $\mathrm{vol}_{\mathcal{H}}(W') \geq \delta$, contradicting ($\mathrm{W}_22$).

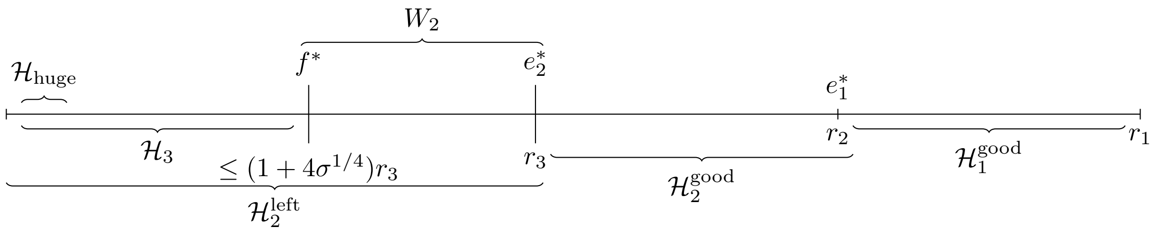

FIGURE 2. Combining three applications of the Reordering Lemma in the proof of Theorem 6.1: the ordering $\preceq$ is increasing from left to right. ($\mathcal{H}_{\text{huge}} \subseteq \mathcal{H}_3$, but $\mathcal{H}_{\text{huge}}$ need not form an initial segment.)

By (O2) of Lemma 6.2, we may assume without loss of generality that every $e \in \mathcal{H}_1^{\text{left}}$ satisfying $f^* \preceq_2 e \preceq_2 e_2^*$ is in W_2, so

$$\mathcal{H} \text{ is partitioned into } \mathcal{H}_2^{\text{left}}, \mathcal{H}_2^{\text{good}}, \text{ and } \mathcal{H}_1^{\text{good}}, \text{ and } \mathcal{H}_2^{\text{left}} \text{ is partitioned into } \mathcal{H}_3 \text{ and } W_2. \tag{6.11}$$

Combine $\preceq_1$, $\preceq_2$, and $\preceq_3$ to obtain an ordering $\preceq$ of $\mathcal{H}$ where

- if $f \in \mathcal{H}_1^{\text{good}}$, then $e \preceq f$ for every $e \in \mathcal{H}_{\text{huge}} \cup (\mathcal{H}')^{\preceq_1 f}$,
- if $f \in \mathcal{H}_2^{\text{good}} \cup W_2$, then $e \preceq f$ for every $e \in \mathcal{H}_{\text{huge}} \cup (\mathcal{H}_1^{\text{left}})^{\preceq_2 f}$, and
- if $f \in \mathcal{H}_3$, then $e \preceq f$ for every $e \in \mathcal{H}_3^{\preceq_3 f}$.

See Figure 2. Note the following.

(a) $\mathcal{H}_{\text{huge}} \subseteq \mathcal{H}_3$, and $e(\mathcal{H}_{\text{huge}}) \leq 8/\beta$ by (2.2).
(b) If $e \in \mathcal{H}_1^{\text{good}}$, then $d_{\mathcal{H}}^{\preceq}(e) \leq \gamma_2 n/2$.
(c) If $e \in \mathcal{H}_2^{\text{good}}$, then $d_{\mathcal{H}}^{\preceq}(e) \leq (1-2\sigma)n$ and $d_{\mathcal{H}_1^{\text{left}}}^{\preceq_2}(e) \leq (1-3\sigma)n$.
(d) If $e \in \mathcal{H}_3$, then $d_{\mathcal{H}}^{\preceq}(e) \leq n/2000$.
(e) If $e \in \mathcal{H}_1^{\text{left}}$, then $|e| \geq r_2 = |e_1^*|$.
(f) If $e \in \mathcal{H}_2^{\text{left}}$, then $|e| \geq r_3 = |e_2^*|$.

We consider two cases: $\preceq_2$ satisfies (6.2:a), or both $\preceq_1$ and $\preceq_2$ satisfy (6.2:b). Note that if $\preceq_1$ satisfies (6.2:a), then $\preceq_2$ vacuously satisfies (6.2:a) (that is, the former case applies).

Case 1: $\preceq_2$ satisfies (6.2:a).

In this case, we prove (6.1:a). If $\preceq_2$ satisfies (6.2:a), then $\mathcal{H}_3 \cup \mathcal{H}_2^{\text{good}} \cup \mathcal{H}_1^{\text{good}} = \mathcal{H}$ by (6.11), so every $e \in \mathcal{H}$ satisfies $d_{\mathcal{H}}^{\preceq}(e) \leq (1-2\sigma)n$ by (b), (c), and (d). Thus, by Proposition 6.9 applied to $\mathcal{H}_{\text{large}}$ with r_0, $\sigma/2$, and $\beta/5$ playing the roles of r, α_1, and α_2, respectively, we obtain a $\beta/5$-bounded proper edge-colouring ϕ_{large} of $\mathcal{H}_{\text{large}}$ using colours from a set C_{large} of size at most $(1-3\sigma/2)n$ disjoint from C_{med}. We combine ϕ_{large} and ϕ_{med} to obtain a proper edge-colouring ϕ of $\mathcal{H}$ satisfying (6.1:a), as follows. For each $e \in \mathcal{H}_{\text{large}}$, let $\phi(e) := \phi_{\text{large}}(e)$, and for each $e \in \mathcal{H}_{\text{med}}$, let $\phi(e) := \phi_{\text{med}}(e)$. Since $C_{\text{large}} \cap C_{\text{med}} = \varnothing$ and $|C_{\text{large}} \cup C_{\text{med}}| \leq (1-\sigma)n$, the colouring ϕ is proper and uses at most $(1-\sigma)n$ colours, as required. Since ϕ_{large} is $\beta/5$-bounded, ϕ satisfies (i) and (iii), and since ϕ_{med} is γ_1-bounded, ϕ satisfies (ii), as desired.

Case 2: Both $\preceq_1$ and $\preceq_2$ satisfy (6.2:b).

We assume

$$r_3 \leq \sqrt{n/(1-4\sigma)}, \tag{6.12}$$

as otherwise $\text{vol}_{\mathcal{H}}(\mathcal{H}_2^{\text{left}}) \leq 1$ and (f) would imply $e(\mathcal{H}_2^{\text{left}}) \leq (1-3\sigma)n$. Together with (c), this fact would imply that $\preceq_2$ satisfies (6.2:a).

We now consider two additional cases: $r_3 < (1-\delta)\sqrt{n}$, and $r_3 \geq (1-\delta)\sqrt{n}$. In the former case, we prove (6.1:a), and in the latter case we prove (6.1:b).

Case 2.1: $r_3 < (1-\delta)\sqrt{n}$.

Let $\zeta := 1 - r_3/\sqrt{n}$. Since $r_3 < (1-\delta)\sqrt{n}$, we have $\zeta > \delta$. First we show how to colour $W_2 \setminus \mathcal{H}_{\text{med}}$ in the following claim.

Claim 1. *There is a $\beta/5$-bounded proper edge-colouring ϕ' of $W_2 \setminus \mathcal{H}_{\text{med}}$ using at most $(1-\zeta/1000)n$ colours.*

Proof of claim: By ($W_2$1), we can apply Corollary 6.5 with r_3 and $4\sigma^{1/4}$ playing the roles of r and α, respectively, so $\chi'(W_2 \setminus \mathcal{H}_{\text{med}}) \leq (1-\zeta/500)n$. Thus, the claim follows from Proposition 6.8(ii) with $\beta/5$ and r_0 playing the roles of α and r, respectively, since $\zeta > \delta$ and $2n/((\beta/5)^2 r_0) \leq \delta^2 n \leq \zeta n/1000$. ♦

We will colour $\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}}$ with a set of colours disjoint from those that we assign to $W_2 \setminus \mathcal{H}_{\mathrm{med}}$ using the following claim.

Claim 2. *There is a $\beta/5$-bounded proper edge-colouring ϕ'' of $\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}}$ using at most $(\zeta/1000 - 2\sigma)n$ colours.*

Proof of claim: Let $k := e(\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}})$. If $k \leq (\zeta/1000 - 2\sigma)n$, then we can simply assign each edge of $\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}}$ a distinct colour and the claim holds, so we assume $k > (\zeta/1000 - 2\sigma)n$. Since $\zeta > \delta$, we have $k > 2\delta^2 n$. By (f), every edge of $\mathcal{H}_3$ has size at least r_3, so we have $\mathrm{vol}_{\mathcal{H}}(\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}}) \geq k(r_3-1)^2/n^2$. On the other hand, by ($\mathrm{W}_2$2), and since $\mathcal{H}_3 \cap W_2 = \varnothing$ by (6.11), we have $\mathrm{vol}_{\mathcal{H}}(\mathcal{H}_3) \leq \delta^3$. Thus, $2\delta^2 n < k \leq \delta^3 n^2/(r_3-1)^2$, so $r_3 < \delta^{1/4}\sqrt{n}$.

Therefore, $\zeta > 1000/1001$. Now by (d) and Proposition 6.9 applied with $\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}}$, $\preceq_3$, r_0, $1/6000$, and $\beta/5$ playing the roles of $\mathcal{H}$, $\preceq$, r, α_1, and α_2, respectively, we obtain a $\beta/5$-bounded proper edge-colouring of $\mathcal{H}_3 \setminus \mathcal{H}_{\mathrm{med}}$ using a set of at most $n/1500 \leq (\zeta/1000 - 2\sigma)n$ colours, as desired. ♦

We may assume that ϕ' and ϕ'' use disjoint sets of colours. By Claims 1 and 2, we can combine ϕ' and ϕ'' to obtain a $\beta/5$-bounded proper edge-colouring ϕ_1 of $\mathcal{H}_2^{\mathrm{left}} \setminus \mathcal{H}_{\mathrm{med}}$ using a set C_1 of at most $(1-2\sigma)n$ colours. We may assume that $C_1 \cap C_{\mathrm{med}} = \varnothing$. Let C_2 be a set of $(1-3\sigma/2)n - |C_1|$ colours disjoint from C_1 and C_{med}, and let $C_{\mathrm{huge}} \subseteq C_1$ where $c \in C_{\mathrm{huge}}$ if $\phi_1(f) = c$ for some $f \in \mathcal{H}_{\mathrm{huge}}$. By (a), $|(C_1 \cup C_2) \setminus C_{\mathrm{huge}}| \geq (1 - 7\sigma/4)n$. We extend ϕ_1 to a $\beta/5$-bounded proper edge-colouring of $\mathcal{H}_{\mathrm{large}}$ using the following claim.

Claim 3. *There is a $\beta/5$-bounded proper edge-colouring ϕ_2 of $(\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}}$ such that every $e \in (\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}}$ satisfies*
- *$\phi_2(e) \in (C_1 \cup C_2) \setminus C_{\mathrm{huge}}$ and*
- *$\phi_2(e) \neq \phi_1(f)$ for every $f \in N_{\mathcal{H}_2^{\mathrm{left}} \setminus \mathcal{H}_{\mathrm{med}}}(e)$.*

Proof of claim: We apply Proposition 6.9 to $(\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}}$, as follows. We let r_0, $\sigma/4$ and $\beta/5$ play the roles of r, α_1 and α_2, respectively, and we define the list-assignment for $L((\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}})$ as $C(e) := ((C_1 \cup C_2) \setminus C_{\mathrm{huge}}) \setminus \bigcup_{f \in N_{\mathcal{H}_2^{\mathrm{left}} \setminus \mathcal{H}_{\mathrm{med}}}(e)} \phi_1(f)$. Since every $e \in \mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}$ satisfies $|N_{\mathcal{H}_2^{\mathrm{left}}}(e)| + d^{\preceq}_{\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}}(e) = d^{\preceq}_{\mathcal{H}}(e) \leq (1-2\sigma)n$ by (b) and (c), we have $|C(e)| \geq (1-7\sigma/4)n - |N_{\mathcal{H}_2^{\mathrm{left}}}| \geq d^{\preceq}_{\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}}(e) + \sigma n/4$ as required. Therefore by Proposition 6.9, there is a $\beta/5$-bounded proper edge-colouring ϕ_2 of $(\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}}$ such that $\phi_2(e) \in C(e)$ for every $e \in (\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}}$, and the choice of $C(e)$ ensures that ϕ_2 satisfies the claim. ♦

Now we combine ϕ_1, ϕ_2, and ϕ_{med} to obtain a proper edge-colouring ϕ, and we show that ϕ satisfies (6.1:a). Indeed, by (6.11), every edge of $\mathcal{H}$ is assigned a colour by ϕ, and since $|C_1| + |C_2| + |C_{\mathrm{med}}| \leq (1-3\sigma/2) + \gamma_1 n \leq (1-\sigma)n$, the colouring ϕ uses at most $(1-\sigma)n$ colours, as required. Since $\phi_2(e) \notin C_{\mathrm{huge}}$ for each $e \in (\mathcal{H}_2^{\mathrm{good}} \cup \mathcal{H}_1^{\mathrm{good}}) \setminus \mathcal{H}_{\mathrm{med}}$, since (a) holds, and since ϕ'' is $\beta/5$-bounded, ϕ satisfies (i). Since ϕ_1 and ϕ_2 are $\beta/5$-bounded, ϕ satisfies (iii), and since ϕ_{med} is γ_1-bounded, ϕ satisfies (ii), as desired.

Case 2.2: $r_3 \geq (1-\delta)\sqrt{n}$.

By (f) and Lemma 5.1, there is a proper edge-colouring ϕ_1 of $\mathcal{H}_2^{\mathrm{left}}$ using a set C of at most n colours such that every colour is assigned to at most two edges. By ($\mathrm{W}_2$1), ($\mathrm{W}_2$2), and (6.12), there are no edges of size at least $\delta n/2$ in $\mathcal{H}$. Hence, ϕ_1 satisfies (i) of (6.1:b), and $\phi_1|_{\mathcal{H}_2^{\mathrm{left}} \setminus \mathcal{H}_{\mathrm{huge}}}$ is $\beta/2$-bounded. Let $C_{\mathrm{huge}} \subseteq C$ where $c \in C_{\mathrm{huge}}$ if $\phi_1(f) \in \mathcal{H}$ for some $f \in \mathcal{H}_{\mathrm{huge}}$, and let $C_{\mathrm{med}} \subseteq C \setminus C_{\mathrm{huge}}$ have size $\gamma_2 n$ (such a set exists by (a)).

By ($\mathrm{W}_1$2) and ($\mathrm{W}_2$2), $W_1 \cap W_2 \neq \varnothing$, so by (f), there is an edge $e \in W_1$ such that $|e| \geq r_3$. Therefore by ($\mathrm{W}_1$1), $r_2 \geq \gamma_2^{10} r_3$. Also, $r_3 \geq (1-\delta)\sqrt{n} \geq 2r_0/\gamma_2^{10}$, so $r_2 \geq 2r_0$. Thus, by (e), we have $\mathcal{H}_{\mathrm{med}} \subseteq \mathcal{H}_1^{\mathrm{good}}$.

We use the following two claims to colour $\mathcal{H}_2^{\mathrm{good}}$ and $\mathcal{H}_1^{\mathrm{good}}$. The proofs are similar to the proof of Claim 3, so we omit them.

Claim 4. *There is a $\beta/5$-bounded proper edge-colouring ϕ_2 of $\mathcal{H}_2^{\mathrm{good}}$ such that every $e \in \mathcal{H}_2^{\mathrm{good}}$ satisfies*
- *$\phi_2(e) \in C \setminus C_{\mathrm{huge}}$ and*
- *$\phi_2(e) \neq \phi_1(f)$ for every $f \in N_{\mathcal{H}_2^{\mathrm{left}}}(e)$.* ♦

Claim 5. *There is a $\gamma_1/2$-bounded proper edge-colouring ϕ_3 of $\mathcal{H}_1^{\mathrm{good}}$ such that every $e \in \mathcal{H}_1^{\mathrm{good}}$ satisfies*
- *$\phi_3(e) \in C_{\mathrm{med}}$ and*
- *$\phi_3(e) \neq \phi_1(f)$ for every $f \in N_{\mathcal{H}_2^{\mathrm{left}}}(e)$ and $\phi_3(e) \neq \phi_2(f)$ for every $f \in \cap N_{\mathcal{H}_2^{\mathrm{good}}}(e)$.* ♦

Now we combine ϕ_1, ϕ_2, and ϕ_3 to obtain a proper edge-colouring ϕ, and we show that ϕ satisfies (6.1:b). Indeed, since $|C| \leq n$, the colouring ϕ uses at most n colours, as required. Since $r_3 \geq (1-\delta)\sqrt{n}$, by (f), ($W_2$1), and (6.12), the edges in W_2 are FPP-extremal, and by ($W_2$2), $\mathrm{vol}_{\mathcal{H}}(W_2) \geq 1-\delta$, as required. Since $\phi_i(e) \notin C_{\mathrm{huge}}$ for each $e \in \mathcal{H}^{\mathrm{good}}_{4-i}$ for $i \in \{2,3\}$ and (a) holds, and since ϕ_1 satisfies (i) of (6.1:b), ϕ satisfies (i). Since $\phi_1|_{\mathcal{H}_2^{\mathrm{left}} \setminus \mathcal{H}_{\mathrm{huge}}}$ is $\beta/2$-bounded and ϕ_2 is $\beta/5$-bounded, ϕ satisfies (iii), and since moreover, $C_{\mathrm{med}} \cap C_{\mathrm{huge}} = \varnothing$, $\mathcal{H}_{\mathrm{med}} \subseteq \mathcal{H}^{\mathrm{good}}_1$, and ϕ_3 is $\gamma_1/2$-bounded, ϕ satisfies (ii), as desired. □

7. Vertex absorption to extend colour classes

In this section, we will define properties that that we need our absorbers to satisfy in order to carry out the vertex absorption step outlined in Section 2.1 (These absorbers will form part of the reservoirs R, which will be constructed in Section 10.) We also formalise various properties that allow a matching to be extended using vertex absorption.

7.1. Quasirandom properties for absorption.

Definition 7.1 (Typicality and upper regularity)**.** Let $\gamma, \rho, \xi \in (0,1)$, let G be an n-vertex graph, let $\mathcal{V}$ be a set of subsets of $V(G)$, and let $R \subseteq E(G)$. We say R is

- (ρ,γ,G)-*typical with respect to* $\mathcal{V}$ if for every $X \in \mathcal{V}$, every vertex $v \in V(G)$ satisfies $|N_R(v) \cap X| = \rho|N_G(v) \cap X| \pm \gamma n$ and $|N_R(v) \setminus X| = \rho|N_G(v) \setminus X| \pm \gamma n$, and
- *upper* (ρ,ξ,G)*-regular* if for every pair of disjoint sets $S,T \subseteq V(G)$ with $|S|,|T| \geq \xi n$, we have $|E_G(S,T) \cap R| \leq \rho e_G(S,T) + \xi|S||T|$.

We also say a graph H is upper (ρ,ξ)-regular if $E(H)$ is upper $(\rho,\xi,K_{v(H)})$-regular.

Note that if H is a subgraph of G such that $\xi v(G) < v(H)$, and $E(H)$ is upper (ρ,ξ,G)-regular, then H is upper $(\rho, \xi v(G)/v(H))$-regular.

Definition 7.2 (Absorbers)**.** Let $\xi,\gamma,\rho,\varepsilon \in (0,1)$. Let $\mathcal{H}$ be a linear multi-hypergraph, let $G := \mathcal{H}^{(2)}$, let G' be the spanning subgraph of G consisting of those edges with at least one vertex in $V_+^{(1-\varepsilon)}(G)$, and let $\mathcal{V}$ be a set of subsets of $V(\mathcal{H})$. We say R_{abs} is a $(\rho,\gamma,\xi,\varepsilon)$-*absorber for* $\mathcal{V}$ if it satisfies the following properties:

(i) $R_{\mathrm{abs}} \subseteq E(G')$,
(ii) R_{abs} is (ρ,γ,G')-typical with respect to $\mathcal{V}$, and
(iii) R_{abs} is upper (ρ,ξ,G')-regular.

Observation 7.3 (Robustness of absorbers)**.** *Let $\xi,\gamma,\rho,\varepsilon \in (0,1)$. Let $\mathcal{H}$ be a linear multi-hypergraph, and let $\mathcal{V}$ be a set of subsets of $V(\mathcal{H})$. The following hold.*

- *A $(\rho,\gamma,\xi,\varepsilon)$-absorber for $\mathcal{V}$ is also a $(\rho,\gamma',\xi,\varepsilon)$-absorber for $\mathcal{V}$, if $\gamma < \gamma' < 1$.*
- *For any $(\rho,\gamma,\xi,\varepsilon)$-absorber R for $\mathcal{V}$, if $R' \subseteq R$ and $\Delta(R-R') \leq \alpha n$, then R' is also a $(\rho,\gamma+\alpha,\xi,\varepsilon)$-absorber for $\mathcal{V}$.*

Definition 7.4 (Pseudorandom matchings)**.** Let $n \in \mathbb{N}$, $\gamma,\kappa \in (0,1)$, and let $\mathcal{H}$ be an n-vertex multi-hypergraph. For a family $\mathcal{F}$ of subsets of $V(\mathcal{H})$, a matching M in $\mathcal{H}$ is (γ,κ)-*pseudorandom* with respect to $\mathcal{F}$ if every $S \in \mathcal{F}$ satisfies $|S \setminus V(M)| = \gamma|S| \pm \kappa n$.

Definition 7.5 (Absorbable matchings)**.** Let $\xi,\kappa,\gamma,\rho,\varepsilon \in (0,1)$. Let $\mathcal{H}$ be an n-vertex linear multi-hypergraph, let $G := \mathcal{H}^{(2)}$, let $U := V_+^{(1-\varepsilon)}(G)$, and let $S \subseteq U$. Let $R \subseteq E(G)$, and let M be a matching in $\mathcal{H}$. We say $(\mathcal{H},M,R,S)$ is $(\rho,\varepsilon,\gamma,\kappa,\xi)$-*absorbable* if

(AB1) R is a $(\rho,10\gamma,\xi,\varepsilon)$-absorber for some $\mathcal{V}$ such that $U, V(\mathcal{H}) \in \mathcal{V}$,
(AB2) $M \subseteq \mathcal{H} \setminus R$, and
(AB3) at least one of the following holds:
 (i) M is (γ,κ)-pseudorandom with respect to $\mathcal{F}(R) \cup \{U,S\}$, where $\mathcal{F}(R) := \{N_R(u) \cap U : u \in U\} \cup \{N_R(u) \setminus U : u \in U\}$,
 (ii) $v(M) \leq \gamma n$, or
 (iii) $|V(M) \cap U| \leq \varepsilon n$ and $U \cup V(M), U \setminus V(M) \in \mathcal{V}$.

We say $(\mathcal{H},M,R,S)$ is

- $(\rho,\varepsilon,\gamma,\kappa,\xi)$-absorbable *by pseudorandomness of* M if (i) holds,
- $(\rho,\varepsilon,\gamma,\kappa,\xi)$-absorbable *by smallness of* M if (ii) holds, and
- $(\rho,\varepsilon,\gamma,\kappa,\xi)$-absorbable *by typicality of* R if (iii) holds.

We simply say $(\mathcal{H},M,R,S)$ is absorbable if it is $(\rho,\varepsilon,\gamma,\kappa,\xi)$-absorbable and ρ, ε, γ, κ, and ξ are clear from the context.

If $\mathcal{H}$ is an n-vertex linear hypergraph and $\phi : \mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}} \to C$ is obtained from Theorem 6.1, then we will choose an absorber R such that for each $c \in C$, $(\mathcal{H}, \phi^{-1}(c), R, S)$ is absorbable by smallness of $\phi^{-1}(c)$ if $c \in C_{\mathrm{med}}$ and by typicality of R if ϕ assigns c to a huge edge. For essentially every other $c \in C$, we will find a matching $M_c \supseteq \phi^{-1}(c)$ in Section 8 such that $(\mathcal{H}, M_c, R, S)$ is absorbable by pseudorandomness of M_c.

In Sections 7.2 and 7.3, we will show that if $(\mathcal{H}, M, R, S)$ is absorbable, then we can extend M to cover all but at most one vertex of U using the edges of R.

7.2. **Absorption for pseudorandom matchings.** We will use the following two lemmas (depending on whether $|U|$ is small or not) to extend matchings N for which $(\mathcal{H}, N, R, S)$ is absorbable by pseudorandomness of N.

Lemma 7.6. *Let $0 < 1/n_0 \ll \xi \ll \kappa \ll \gamma \ll \rho, \varepsilon \ll 1$, let $n \geq n_0$ and let $k \leq \kappa n$. Let $\mathcal{H}$ be an n-vertex linear multi-hypergraph, let $G := \mathcal{H}^{(2)}$, let $R \subseteq E(G)$, and let $U := V_+^{(1-\varepsilon)}(G)$. Let $\mathcal{N} := \{N_1, \ldots, N_k\}$ be a set of edge-disjoint matchings in $\mathcal{H}$ such that $(\mathcal{H}, N_i, R, \varnothing)$ is absorbable by pseudorandomness of N_i for each $i \in [k]$.*

If $|U| \leq n/100$, then there is a set of edge-disjoint matchings $\mathcal{N}' := \{N'_1, \ldots, N'_k\}$ in $\mathcal{H}$ such that for all $i \in [k]$,
- *$N'_i \supseteq N_i$ and $N'_i \setminus N_i \subseteq R$, and*
- *$\mathcal{N}'$ has perfect coverage of U.*

Lemma 7.7. *Let $0 < 1/n_0 \ll \xi \ll \kappa \ll \gamma \ll \rho, \varepsilon \ll 1$, let $n \geq n_0$ and let $k \leq \kappa n$. Let $\mathcal{H}$ be an n-vertex linear multi-hypergraph, let $G := \mathcal{H}^{(2)}$, let $R \subseteq E(G)$, let $U := V_+^{(1-\varepsilon)}(G)$, and let $S \subseteq U$ satisfy $|S| \geq \gamma n$ if $|U| > (1 - 2\varepsilon)n$. Let $\mathcal{N} := \{N_1, \ldots, N_k\}$ be a set of edge-disjoint matchings in $\mathcal{H}$ such that $(\mathcal{H}, N_i, R, S)$ is absorbable by pseudorandomness of N_i for each $i \in [k]$.*

If $|U| \geq n/100$, then there is a set of edge-disjoint matchings $\mathcal{N}' := \{N'_1, \ldots, N'_k\}$ in $\mathcal{H}$ such that for all $i \in [k]$,
- *$N'_i \supseteq N_i$ and $N'_i \setminus N_i \subseteq R$, and*
- *if $|U| \leq (1 - 2\varepsilon)n$, then $\mathcal{N}'$ has perfect coverage of U. Otherwise, $\mathcal{N}'$ has nearly-perfect coverage of U with defects in S.*

To prove the above lemmas we will need the following simple observations, which follow easily from Hall's theorem.

Observation 7.8. *Let $0 < \xi \ll \rho \leq 1$. If H is an upper (ρ, ξ)-regular bipartite graph with bipartition (A, B) such that $v(H) \leq \rho|A|/\xi$ and every $v \in A$ satisfies $d_H(v) \geq 2\rho|A|$, then H has a matching covering A.*

Observation 7.9. *Let $0 < 1/m \ll \xi \ll \rho \leq 1$. If G is an m-vertex, upper (ρ, ξ)-regular graph such that every $v \in V(G)$ satisfies $d_G(v) \geq 3\rho m/4$, and m is even, then G has a perfect matching.*

Proof of Lemma 7.6. For each $i \in [k]$, let H_i be the bipartite graph consisting of edges in R with the bipartition (A_i, B_i), where $A_i := U \setminus V(N_i)$ and $B_i := (V(\mathcal{H}) \setminus U) \setminus V(N_i)$.

We claim that there exist pairwise edge-disjoint matchings N_i^{abs} in H_i covering A_i for each $i \in [k]$. We find these matchings one-by-one using Observation 7.8, if $|A_i| \geq \xi n/\rho$. Otherwise, we find them greedily. To this end, we assume that for some $\ell \leq k$, we have found such matchings N_i^{abs} for $i \in [\ell - 1]$, and we show that there exists such a matching N_ℓ^{abs}, which proves the claim. Let $H'_\ell := H_\ell \setminus \bigcup_{i \in [\ell-1]} N_i^{\mathrm{abs}}$.

We first show that every vertex $u \in A_\ell$ satisfies $d_{H'_\ell}(u) \geq 2\rho|A_\ell|$. Since $|U| \leq n/100$ and R is a $(\rho, 10\gamma, \xi, \varepsilon)$-absorber for $\{V(\mathcal{H}), U\}$ by (AB1), every $u \in A_\ell$ satisfies

$$|N_R(u) \setminus U| \geq (\rho(99/100 - \varepsilon) - 10\gamma)n \geq 98\rho n/100. \tag{7.1}$$

Note that each N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F} := \mathcal{F}(R) \cup \{U\}$ by (AB3)(i). Together with (7.1), this implies that every $u \in A_\ell$ satisfies $d_{H_\ell}(u) \geq \gamma|N_R(u) \setminus U| - \kappa n \geq 97\gamma\rho n/100$. Since $\ell \leq k \leq \kappa n$, we have

$$d_{H'_\ell}(u) \geq d_{H_\ell}(u) - \kappa n \geq 96\gamma\rho n/100. \tag{7.2}$$

Since N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F} \ni U$ and $|U| \leq n/100$, we also have

$$|A_\ell| \leq \gamma|U| + \kappa n \leq \gamma n/50. \tag{7.3}$$

Combining (7.2) and (7.3), we have $d_{H'_\ell}(u) \geq 2\rho|A_\ell|$, as desired.

Note that the graph $H_\ell^* := (V(\mathcal{H}), E(H'_\ell))$ is upper (ρ, ξ)-regular since $H_\ell^* \subseteq R$, and H_ℓ^* is bipartite with the bipartition $(A_\ell, V(\mathcal{H}) \setminus A_\ell)$ where $d_{H_\ell^*}(u) \geq 2\rho|A_\ell|$, for every $u \in A_\ell$. Therefore, if $|A_\ell| \geq \xi n/\rho$, then Observation 7.8 implies that H^*_ℓ has a matching N^{abs}_ℓ covering A_ℓ, so N^{abs}_ℓ is also a matching in H'_ℓ covering A_ℓ, as claimed. Otherwise, for any $u \in A_\ell$,

$$(7.4) \qquad |A_\ell| < \xi n/\rho \le 96\gamma\rho n/100 \overset{(7.2)}{\le} d_{H'_\ell}(u),$$

so we can find such a matching greedily.

Therefore we have pairwise edge-disjoint matchings N^{abs}_i in H_i covering A_i for $i \in [k]$, as claimed, and $N^{\mathrm{abs}}_1, \dots, N^{\mathrm{abs}}_k$ are edge-disjoint from $N_1, \dots, N_k$ by (AB2). For each $i \in [k]$, let $N'_i \coloneqq N_i \cup N^{\mathrm{abs}}_i$, and let $\mathcal{N}' = \{N'_1, \dots, N'_k\}$. Hence each matching in $\mathcal{N}'$ covers U, so $\mathcal{N}'$ has perfect coverage of U, as desired. □

Proof of Lemma 7.7. Let $\mathcal{F} \coloneqq \mathcal{F}(R) \cup \{U, S\}$, where $\mathcal{F}(R) \coloneqq \{N_R(u) \cap U : u \in U\} \cup \{N_R(u) \setminus U : u \in U\}$. For each $i \in [k]$, let G_i be the graph with $V(G_i) \coloneqq V(\mathcal{H}) \setminus V(N_i)$ and $E(G_i) \coloneqq \{e \in R : e \subseteq V(\mathcal{H}) \setminus V(N_i)\}$, and let $U_i \coloneqq U \setminus V(N_i)$. Since N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F} \ni U$, we have

$$(7.5) \qquad |U_i| = \gamma|U| \pm \kappa n, \text{ so } |U_i| \ge \gamma n/200.$$

We claim that for each $i \in [k]$ there exists $u_i \in U_i$ and a matching N^{abs}_i in G_i such that the following holds. The vertices $u_1, \dots, u_k$ are distinct, the matchings $N^{\mathrm{abs}}_1, N^{\mathrm{abs}}_2, \dots, N^{\mathrm{abs}}_k$ are pairwise edge-disjoint, and N^{abs}_i covers every vertex of $U_i \setminus \{u_i\}$ for each $i \in [k]$. Moreover, if $|U| \le (1-2\varepsilon)n$, then N^{abs}_i covers every vertex of U_i for each $i \in [k]$, and otherwise $u_i \in S$.

To that end, we choose distinct $u_i \in U_i$ for each $i \in [k]$, as follows.

- If $|U| \le (1-2\varepsilon)n$, then since R is a $(\rho, 10\gamma, \xi, \varepsilon)$-absorber for $\{V(\mathcal{H}), U\}$ by (AB1), every $u \in U_i$ satisfies $|N_R(u) \setminus U| \ge (\rho\varepsilon - 10\gamma)n$. By (AB3)(i), since N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F} \supseteq \mathcal{F}(R)$ for each $i \in [k]$, this inequality implies that every $u \in U_i$ satisfies $|N_{G_i}(u) \setminus U| \ge \gamma\rho\varepsilon n/2$. Since $k \le \kappa n$ and $\kappa \ll \gamma, \rho, \varepsilon$, and (7.5) holds, we can choose $u_i \in U_i$ one-by-one such that there is a matching $\{u_i v_i : i \in [k]\}$ where $v_i \in N_{G_i}(u_i) \setminus U$ for each $i \in [k]$.
- Otherwise, $|S| \ge \gamma n$, and since N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F} \ni S$, by (AB3)(i), we have $|S \setminus V(N_i)| \ge \gamma|S| - \kappa n \ge \gamma^2 n/2 > \kappa n$ for each $i \in [k]$, so we can choose $u_i \in U_i \cap S = S \setminus V(N_i)$ one-by-one such that they are distinct, as required.

Now let $U'_i \coloneqq U_i \setminus \{u_i\}$ if $|U_i|$ is odd. Otherwise, let $U'_i \coloneqq U_i$. By the choice of the vertices u_i, it suffices to find pairwise edge-disjoint perfect matchings N'^{abs}_i in $G_i[U'_i]$ for each $i \in [k]$. Indeed if $|U| \le (1-2\varepsilon)n$ and $|U_i|$ is odd, then $N^{\mathrm{abs}}_i \coloneqq N'^{\mathrm{abs}}_i \cup \{u_i v_i\}$ satisfies the claim, and otherwise $N^{\mathrm{abs}}_i \coloneqq N'^{\mathrm{abs}}_i$ satisfies the claim.

We find these matchings one-by-one using Observation 7.9. To this end, we assume that for some $\ell \le k$, we have found such matchings N'^{abs}_i for $i \in [\ell-1]$, and we show that there exists such a matching N'^{abs}_ℓ, which proves the claim. Let $G'_\ell \coloneqq G_\ell[U'_\ell] \setminus \bigcup_{i \in [\ell-1]} N'^{\mathrm{abs}}_i$. Since $|U| \ge n/100$ and R is a $(\rho, 10\gamma, \xi, \varepsilon)$-absorber for $\{V(\mathcal{H}), U\}$ by (AB1), every $u \in U$ satisfies

$$(7.6) \qquad |N_R(u) \cap U| \ge \rho(|U| - \varepsilon n) - 10\gamma n \ge 99\rho|U|/100.$$

Note that N_ℓ is (γ, κ)-pseudorandom with respect to $\mathcal{F} \supseteq \mathcal{F}(R) \cup \{U\}$ by (AB3)(i). Together with (7.6), this implies that every $u \in U'_\ell$ satisfies $d_{G_\ell[U'_\ell]}(u) \ge \gamma|N_R(u) \cap U| - \kappa n - 1 \ge 98\gamma\rho|U|/100$. Since $\ell \le k \le \kappa n$, we have

$$(7.7) \qquad d_{G'_\ell}(u) \ge d_{G_\ell[U'_i]}(u) - \kappa n \ge 97\gamma\rho|U|/100.$$

We also have

$$(7.8) \qquad |U'_\ell| \pm 1 = |U_\ell| \overset{(7.5)}{=} \gamma|U| \pm \kappa n, \text{ so } |U'_\ell| \le 5\gamma|U|/4.$$

Since R is upper (ρ, ξ, G')-regular and $|U'_\ell| \overset{(7.8)}{\ge} \gamma n/100 - \kappa n - 1 \ge \gamma n/200$, G'_ℓ is upper $(\rho, 200\xi/\gamma)$-regular. Moreover, combining (7.7) and (7.8), we have $d_{G'_\ell}(u) \ge 3\rho|U'_\ell|/4$. So by Observation 7.9, G'_ℓ has a perfect matching N'^{abs}_ℓ, as desired.

Therefore we have pairwise edge-disjoint matchings N^{abs}_i in G_i, as claimed, which by (AB2) are edge-disjoint from $N_1, \dots, N_k$. For each $i \in [k]$, let $N'_i \coloneqq N_i \cup N^{\mathrm{abs}}_i$, and let $\mathcal{N}' = \{N'_1, \dots, N'_k\}$. Now $N'_i \supseteq N_i$ and $N'_i \setminus N_i \subseteq R$ for each $i \in [k]$, and $\mathcal{N}'$ has nearly-perfect coverage of U with defects in S, as desired. Moreover, if $|U| \le (1-2\varepsilon)n$, then $\mathcal{N}'$ has perfect coverage of U, as desired. □

7.3. **Absorption for matchings having huge edges or having few vertices.**

Definition 7.10 (Difficult matching)**.** Let $\mathcal{H}$ be an n-vertex hypergraph, let $G := \mathcal{H}^{(2)}$, and let $U := V_+^{(1-\varepsilon)}(G)$. A matching M in $\mathcal{H}$ is *difficult* if it covers at least $3|V(\mathcal{H}) \setminus U|/4$ of the vertices in $V(\mathcal{H}) \setminus U$ and $|V(\mathcal{H}) \setminus U| \geq 2$. If the matching M is difficult and consists of a single edge e, then we also say that e is *difficult*.

We will use the following lemma to extend matchings N for which $(\mathcal{H}, N, R, S)$ is either absorbable by smallness of N or by typicality of R (provided N is not difficult).

Lemma 7.11. *Let $0 < 1/n_0 \ll \xi \ll \kappa \ll \gamma \ll \rho, \varepsilon \ll 1$, and let $n \geq n_0$. Suppose $k \leq \gamma n$. Let $\mathcal{H}$ be an n-vertex linear hypergraph, let $G := \mathcal{H}^{(2)}$, let $R \subseteq E(G)$, let $U := V_+^{(1-\varepsilon)}(G)$, and let $S \subseteq U$ satisfy $|S| > (\gamma + \varepsilon)n$ if $|U| > (1 - 10\varepsilon)n$. Let $\mathcal{N} := \{N_1, \ldots, N_k\}$ be a set of edge-disjoint matchings in $\mathcal{H}$ such that for each $i \in [k]$, either*

(a) *$(\mathcal{H}, N_i, R, S)$ is $(\rho, \varepsilon, \gamma, \kappa, \xi)$-absorbable by smallness of N_i, or*

(b) *$(\mathcal{H}, N_i, R, S)$ is $(\rho, \varepsilon, \gamma, \kappa, \xi)$-absorbable by typicality of R and N_i is not difficult.*

Then there is a set $\mathcal{N}' := \{N'_1, \ldots, N'_k\}$ of pairwise edge-disjoint matchings such that for $i \in [k]$,

- *$N'_i \supseteq N_i$ and $N'_i \setminus N_i \subseteq R$, and*
- *if $|U| \leq (1 - 10\varepsilon)n$, then $\mathcal{N}'$ has perfect coverage of U. Otherwise, $\mathcal{N}'$ has nearly-perfect coverage of U with defects in S.*

Proof. Let G' be the spanning subgraph of G consisting only of the edges incident to a vertex in U. Without loss of generality, we may assume that there is an integer s such that for each $i \in [s]$, we have $|U \setminus V(N_i)| \leq n/100$, and for each $i \in \{s+1, \ldots, k\}$, we have $|U \setminus V(N_i)| > n/100$.

For each $i \in [k]$, let $U_i := U \setminus V(N_i)$, let $V_i := (V(\mathcal{H}) \setminus U) \setminus V(N_i)$, let H_i be the bipartite graph with the bipartition (U_i, V_i) and $E(H_i) := \{e \in R : |e \cap U_i| = |e \cap V_i| = 1\}$, and let G_i be the graph with $V(G_i) := V(\mathcal{H}) \setminus V(N_i)$ and $E(G_i) := \{e \in R : e \subseteq V(\mathcal{H}) \setminus V(N_i)\}$.

We first claim that there exist pairwise edge-disjoint matchings $N_1^{\text{abs}}, \ldots, N_s^{\text{abs}}$ such that for each $i \in [s]$, N_i^{abs} is a matching in H_i covering all of the vertices in U_i. We find these matchings one-by-one using Observation 7.8. To this end, we assume that for some $\ell \leq s$, we have found such matchings N_i^{abs} for $i \in [\ell - 1]$, we let $H'_\ell := H_\ell \setminus \bigcup_{t=1}^{\ell-1} N_t^{\text{abs}}$, and we show that there exists such a matching N_ℓ^{abs} in H'_ℓ, which proves the claim. It suffices to show that every $u \in U_\ell$ satisfies

$$d_{H'_\ell}(u) \geq \rho n/7 \geq 2\rho|U_\ell|. \tag{7.9}$$

Indeed, the graph $H_\ell^* := (V(\mathcal{H}), E(H'_\ell))$ is upper (ρ, ξ)-regular since $E(H_\ell^*) \subseteq R$, and H_ℓ^* is bipartite with the bipartition $(U_\ell, V(\mathcal{H}) \setminus U_\ell)$ where $d_{H_\ell^*}(u) = d_{H'_\ell}(u) \geq 2\rho|U_\ell|$ for every $u \in U_\ell$. Therefore, if $|U_\ell| \geq \xi n/\rho$, then we have a matching N_ℓ^{abs} in H_ℓ^* (and so in H'_ℓ) covering U_ℓ by Observation 7.8, as desired. Otherwise, by (7.9), $|U_\ell| < \xi n/\rho < \rho n/7 \leq d_{H'_\ell}(u)$ for every $u \in U_\ell$, so we can find the desired matching N_ℓ^{abs} covering U_ℓ greedily.

To prove (7.9), first suppose (b) holds. By (AB3)(iii), we have $v(N_\ell) \leq (3/4 + \varepsilon)n \leq 4n/5$. Thus, $|V_\ell| \geq n - v(N_\ell) - |U \setminus V(N_\ell)| \geq n/5 - n/100 \geq n/6$, and since R is $(\rho, 10\gamma, G')$-typical with respect to $\mathcal{V} \ni U \cup V(N_\ell)$ by (AB3)(iii), every $u \in U_\ell$ satisfies $d_{H'_\ell}(u) \geq (\rho(1/6 - \varepsilon) - 10\gamma - \gamma)n \geq \rho n/7 \geq 2\rho|U_\ell|$, as desired. Therefore we assume $(\mathcal{H}, N_\ell, R, S)$ is absorbable by smallness of N_ℓ, so by (AB3)(ii), we have $v(N_\ell) \leq \gamma n$. Thus, $|U| \leq |U \setminus V(N_\ell)| + v(N_\ell) \leq n/100 + \gamma n \leq n/50$, and since R is $(\rho, 10\gamma, G')$-typical with respect to $\mathcal{V} \ni U$ by (AB1), every $u \in U_\ell$ satisfies $d_{H'_\ell}(u) \geq (\rho(49/50 - \varepsilon) - 10\gamma - \gamma - \gamma)n \geq \rho n/7 \geq 2\rho|U_\ell|$, as desired. Therefore (7.9) holds in both cases, so we have the matchings $N_1^{\text{abs}}, \ldots, N_s^{\text{abs}}$, as claimed.

Claim 1. *There exist matchings $N_{s+1}^{\text{abs}}, \ldots, N_k^{\text{abs}}$ and distinct vertices $u_{s+1}, \ldots, u_k$ such that for each $i \in \{s+1, \ldots, k\}$, $u_i \in U_i$, N_i^{abs} is a matching in G_i covering all the vertices of $U_i \setminus \{u_i\}$, and the matchings $N_1^{\text{abs}}, \ldots, N_k^{\text{abs}}$ are pairwise edge-disjoint. Moreover, for each $i \in \{s+1, \ldots, k\}$, if $|U| \leq (1 - 10\varepsilon)n$, then N_i^{abs} covers all vertices in U_i, and otherwise $u_i \in S$.*

Proof of claim: We choose distinct $u_i \in U_i$ for $s + 1 \leq i \leq k$ as follows. Let $G'_i := G_i \setminus \bigcup_{t=1}^{s} N_t^{\text{abs}}$ for $s + 1 \leq i \leq k$.

- If $|U| \leq (1 - 10\varepsilon)n$, then every $u \in U$ satisfies $|N_G(u) \setminus U| \geq 9\varepsilon n$, and moreover if N_i is not difficult, then $|V(\mathcal{H}) \setminus (U \cup V(N_i))| \geq 2\varepsilon n$, which implies that every $u \in U$ satisfies $|N_G(u) \setminus (U \cup V(N_i))| \geq \varepsilon n$. If (b) holds, then R is $(\rho, 10\gamma, G')$-typical with respect to $\mathcal{V} \ni U \cup V(N_i)$ by (AB3)(iii), so we have $|N_{G'_i}(u) \setminus (U \cup V(N_i))| \geq \rho|N_G(u) \setminus (U \cup V(N_i))| - 10\gamma n - k \geq \rho\varepsilon n/2 > \gamma n \geq k$ for every $u \in U_i$. If (a) holds, then $v(N_i) \leq \gamma n$ and R is $(\rho, 10\gamma, G')$-typical with respect to $\mathcal{V} \ni U$ by (AB1), so we have $|N_{G'_i}(u) \setminus (U \cup V(N_i))| \geq \rho|N_G(u) \setminus U| - 10\gamma n - k - v(N_i) > 8\rho\varepsilon n > k$ for every $u \in U_i$. Thus,

$|N_{G'_i}(u)\setminus(U\cup V(N_i))|>k$ for each $i\in\{s+1,\dots,k\}$ and for all $u\in U_i$, and since $|U_i|>n/100>k$, we can choose $u_i\in U_i$ one-by-one such that there is a matching $\{u_iv_i : i\in\{s+1,\dots,k\}\}$ where $v_i\in N_{G'_i}(u_i)\setminus(U\cup V(N_i))$ for each $i\in\{s+1,\dots,k\}$.

- Otherwise, we have $|S|>(\gamma+\varepsilon)n$. Using (AB3)(ii) or (AB3)(iii), and $\gamma\ll\varepsilon$, we have $|V(N_i)\cap U|\le\varepsilon n$. Since $S\subseteq U$ it then follows that $|S\setminus V(N_i)|\ge|S|-|V(N_i)\cap U|>\gamma n$ for each $i\in\{s+1,\dots,k\}$. Since $k\le\gamma n$, we can choose $u_i\in S\setminus V(N_i)$ one-by-one such that they are distinct, as required.

Now let $U'_i:=U_i\setminus\{u_i\}$ if $|U_i|$ is odd. Otherwise, let $U'_i:=U_i$. By the choice of the vertices u_i, it suffices to find pairwise edge-disjoint perfect matchings ${N'}_i^{\rm abs}$ in $G'_i[U'_i]$ for each $i\in\{s+1,\dots,k\}$. Indeed, if $|U|\le(1-10\varepsilon)n$ and $|U_i|$ is odd, then $N_i^{\rm abs}:={N'}_i^{\rm abs}\cup\{u_iv_i\}$ satisfies the claim, and otherwise $N_i^{\rm abs}:={N'}_i^{\rm abs}$ satisfies the claim.

We find these matchings (${N'}_i^{\rm abs}$ for $i\in\{s+1,\dots,k\}$) one-by-one using Observation 7.9. To this end, we assume that for some $s+1\le\ell\le k$, we have found such matchings ${N'}_i^{\rm abs}$ for all $s+1\le i\le\ell-1$, we let $G''_\ell:=G'_\ell[U'_\ell]\setminus\bigcup_{s+1\le i\le\ell-1}{N'}_i^{\rm abs}$, and we show that there exists such a matching ${N'}_\ell^{\rm abs}$ in G''_ℓ, which proves the claim. Note that

$$|U'_\ell|\ge|U_\ell|-1\ge n/100-1\ge n/200. \tag{7.10}$$

Since R is upper (ρ,ξ,G)-regular and (7.10) holds, G''_ℓ is upper $(\rho,200\xi)$-regular. So by Observation 7.9, it suffices to show that every $u\in U'_\ell$ satisfies

$$d_{G''_\ell}(u)\ge3\rho|U'_\ell|/4. \tag{7.11}$$

To prove (7.11), first suppose (b) holds. Since R is a $(\rho,10\gamma,\xi,\varepsilon)$-absorber for $\mathcal{V}\ni U_\ell$ by (AB3)(iii), every $u\in U'_\ell$ satisfies

$$d_{G''_\ell}(u)\ge\rho(|U_\ell|-\varepsilon n)-10\gamma n-\gamma n-1\ge\rho(|U'_\ell|-\varepsilon n)-12\gamma n\overset{(7.10)}{\ge}3\rho|U'_\ell|/4,$$

as desired. Therefore, we assume (a) holds. Since R is $(\rho,10\gamma,G')$-typical with respect to $\mathcal{V}\ni U$ by (AB1), every $u\in U'_\ell$ satisfies

$$d_{G''_\ell}(u)\ge\rho(|U|-\varepsilon n)-10\gamma n-\gamma n-\gamma n-1\ge\rho(|U'_\ell|-\varepsilon n)-13\gamma n\overset{(7.10)}{\ge}3\rho|U'_\ell|/4,$$

as desired. Therefore (7.11) holds in both cases, so we have the matchings $N_{s+1}^{\rm abs},\dots,N_k^{\rm abs}$, which proves Claim 1. ♦

Now, letting $N'_i:=N_i\cup N_i^{\rm abs}$ for each $i\in[k]$, we have $N'_i\supseteq N_i$ and $N'_i\setminus N_i\subseteq R$, and $\mathcal{N}':=\{N'_1,\dots,N'_k\}$ has nearly-perfect coverage of U with defects in S, as desired. Moreover, if $|U|\le(1-10\varepsilon)n$, then $\mathcal{N}'$ has perfect coverage of U, as desired. □

We will use the following lemma to extend a difficult matching.

Lemma 7.12. *Let $0<1/n_0\ll\beta\ll1$, and let $n\ge n_0$. If $\mathcal{H}$ is an n-vertex linear hypergraph with no singleton edge, $G:=\mathcal{H}^{(2)}$, and $M:=\{e\}$ is a difficult matching where e is huge, then at least one of the following holds:*

(7.12:a) There is a matching M' such that $M\subseteq M'$, $M'\setminus M\subseteq E(G)$, and M' covers every vertex of $V^{(n-1)}(\mathcal{H})$ and all but at most five vertices of $V^{(n-2)}(\mathcal{H})$, or

(7.12:b) $\chi'(\mathcal{H})\le n$.

Proof. Let $U_1:=V^{(n-1)}(\mathcal{H})$, let $U_2:=V^{(n-2)}(\mathcal{H})$, let $X:=V(\mathcal{H})\setminus(e\cup U_1\cup U_2)$, and let $m:=|U_1\cup U_2|$. Since $\mathcal{H}$ is linear and e is huge, $e\cap U_i=\varnothing$ for $i\in\{1,2\}$.

First, suppose $U_2=\varnothing$. If $|U_1|$ is even, then we can find a perfect matching M_1 in $G[U_1]$, and $M':=M\cup M_1$ satisfies (7.12:a), so we assume $|U_1|$ is odd. If $X\ne\varnothing$, then there is an edge $uv\in E(G)$ such that $u\in U_1$ and $v\in X$, and there is a perfect matching M_1 in $G[U_1\setminus\{u\}]$. Now $M':=M\cup M_1\cup\{uv\}$ satisfies (7.12:a), so we assume $X=\varnothing$, and we show $\chi'(\mathcal{H})\le n$. Note that if $X,U_2=\varnothing$, then the only edge in $\mathcal{H}\setminus E(G)$ is e. Let $w\in U_1$, let M_1 be a perfect matching in $G[U_1\setminus\{w\}]$, and let $G':=\mathcal{H}\setminus(M_1\cup e)$. Now G' is a graph with exactly one vertex of degree $n-1$ (namely w), so by Theorem 4.5, $\chi'(G')\le n-1$. By combining a proper $(n-1)$-edge-colouring of G' with the colour class consisting of $M_1\cup e$, we have $\chi'(\mathcal{H})\le n$, as desired.

Therefore we assume $U_2\ne\varnothing$ and let $u\in U_2$. Let $G':=G[U_1\cup U_2]-u$ if m is odd and let $G':=G[U_1\cup U_2]$ otherwise. If G' has a perfect matching, then (7.12:a) holds, so we assume otherwise. Thus, by the Tutte-Berge formula, there is a set S such that $G'-S$ has at least $|S|+2$ odd components (since G' has an even number of vertices, $G'-S$ cannot have $|S|+1$ odd components). Note that if a vertex has degree at least $n-2$ in $\mathcal{H}$ then it has degree at least $n-3$ in G. Thus $\delta(G')\ge v(G')-3$, which implies that $|S|\le1$.

Moreover, if $|S| = 1$, then $v(G') = 4$ and the vertices in $V(G') \setminus S$ form a hyperedge of $\mathcal{H}$. In this case G' is a star on four vertices, so $|U_2| \leq |V(G') \cup \{u\}| \leq 5$ and $|U_1| \leq 1$. If $U_1 = \varnothing$, then $M' := M$ satisfies (7.12:a), and otherwise, $M' := M \cup \{uv\}$ where $v \in U_1$, satisfies (7.12:a), as desired. If $S = \varnothing$, then G' is a graph with two vertices and no edges. In this case, $U_1 = \varnothing$, so $M' := M$ satisfies (7.12:a). □

8. Colouring small edges that are not in the reservoir

In this section, we prove three lemmas which will be applied to colour all of the small edges that are not in the reservoir (where the reservoir is constructed in Section 11). Since we may need to reuse the colours already used for large edges and medium edges (given by Theorem 6.1), we need to formulate the lemmas to colour the small edges (that are not in the reservoir) by extending the colour classes given by Theorem 6.1.

The lemma below is used repeatedly in the proof of Lemma 8.2 to colour most of the non-reserved small edges in such a way that every colour class exhibits some pseudorandom properties.

Lemma 8.1 (Nibble Lemma)**.** *Let $0 < 1/n_0 \ll 1/r, \beta \ll \kappa, \gamma \ll 1$, let $n \geq n_0$, and let $D \in [n^{1/2}, n]$. Let $\mathcal{H}$ be an n-vertex linear multi-hypergraph.*

- *Let $\mathcal{H}' \subseteq \mathcal{H}$ be a linear multi-hypergraph such that $V(\mathcal{H}) = V(\mathcal{H}')$, every $e \in \mathcal{H}'$ satisfies $|e| \leq r$, and for every $w \in V(\mathcal{H}')$ we have $d_{\mathcal{H}'}(w) = (1 \pm \beta)D$,*
- *let $\mathcal{F}_V$ and $\mathcal{F}_E$ be a family of subsets in $V(\mathcal{H}')$ and $E(\mathcal{H}')$, respectively, such that $|\mathcal{F}_V|, |\mathcal{F}_E| \leq n^{\log n}$, and*
- *let $M_1, \dots, M_D \subseteq \mathcal{H} \setminus \mathcal{H}'$ be pairwise edge-disjoint matchings such that for every $i \in [D]$, $|V(M_i)| \leq \beta D$, and for every edge $e \in \mathcal{H}'$, we have $|\{i \in [D] : e \cap V(M_i) \neq \varnothing\}| \leq \beta D$.*

Then there exist pairwise edge-disjoint matchings $N_1, \dots, N_D$ in $\mathcal{H}$ such that for any $i \in [D]$,

(8.1.1) $N_i \supseteq M_i$ and $N_i \setminus M_i \subseteq \mathcal{H}'$,

(8.1.2) N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F}_V$, and

(8.1.3) $|F \setminus \bigcup_{j=1}^{D} N_j| \leq \gamma|F| + \kappa \max(|F|, D)$ for each $F \in \mathcal{F}_E$.

Note that if we let $E(\mathcal{H}') \in \mathcal{F}_E$, then (8.1.3) implies that $\bigcup_{j=1}^{D} N_j$ contains almost all of the edges of $\mathcal{H}'$. The matchings $M_1, \dots, M_D$ will play the role of some of the colour classes given by Theorem 6.1.

The overall idea of the proof of Lemma 8.1 is as follows. First we embed $\mathcal{H}'$ into an r-uniform linear hypergraph $\mathcal{H}_{\text{unif}}$ using Lemma 4.4. We then embed $\mathcal{H}_{\text{unif}}$ into an $(r+1)$-uniform auxiliary hypergraph $\mathcal{H}_{\text{aux}}$, and we find a pseudorandom matching N^* in $\mathcal{H}_{\text{aux}}$ using Corollary 4.3, which yields D edge-disjoint pseudorandom matchings $N'_1, \dots, N'_D$ in $\mathcal{H}'$. Then we will show that the matchings $N_i := N'_i \cup M_i$ for $i \in [D]$ satisfy the desired properties.

Proof. We apply Lemma 4.4 to $\mathcal{H}'$ with $(1+\beta)D$, $2\beta D$, and n playing the roles of D, C, and N, respectively, to obtain an r-uniform linear hypergraph $\mathcal{H}_{\text{unif}}$ such that

(a) $V(\mathcal{H}_{\text{unif}}) \supseteq V(\mathcal{H}')$ and $v(\mathcal{H}_{\text{unif}}) \leq n^5$, and

(b) every vertex $w \in \mathcal{H}_{\text{unif}}$ satisfies $d_{\mathcal{H}_{\text{unif}}}(w) = (1 \pm \beta)D$. Moreover, $d_{\mathcal{H}'}(w) = d_{\mathcal{H}_{\text{unif}}}(w)$ for any $w \in V(\mathcal{H}')$.

Note that (b) and (4.4.1) imply that

(c) $\mathcal{H}' = \mathcal{H}_{\text{unif}}|_{V(\mathcal{H}')}$.

Let $E_{\text{meet}} := \{e \in \mathcal{H}_{\text{unif}} : e \cap V(\mathcal{H}') \neq \varnothing\}$. By (c), we have a bijective map

$$\psi : E_{\text{meet}} \to \mathcal{H}' \text{ such that } e^* \mapsto e^* \cap V(\mathcal{H}').$$

Thus, for any $w \in V(\mathcal{H}')$, we have $E_{\mathcal{H}'}(w) = \{\psi(e^*) : w \in e^* \in \mathcal{H}_{\text{unif}}\}$. Note that the assumption $|\{i \in [D] : e \cap V(M_i) \neq \varnothing\}| \leq \beta D$ for any $e \in \mathcal{H}'$, implies that

$$\text{for every } e^* \in \mathcal{H}_{\text{unif}}, \text{ we have } |\{i \in [D] : e^* \cap V(M_i) \neq \varnothing\}| \leq \beta D. \tag{8.1}$$

We construct an $(r+1)$-uniform linear hypergraph $\mathcal{H}_{\text{aux}}$ based on $\mathcal{H}_{\text{unif}}$ and the sets $V(M_1), \dots, V(M_D)$, as follows.

- For any $i \in [D]$, let $V_i^* := \{w^i : w \in V(\mathcal{H}_{\text{unif}})\}$, where for any distinct $i_1, i_2 \in [D]$, we have $V_{i_1}^* \cap V_{i_2}^* = \varnothing$. Now let us define a map $\varphi : [D] \times V(\mathcal{H}_{\text{unif}}) \to \bigcup_{i=1}^{D} V_i^*$ such that $\varphi(i, w) := w^i$ for any $(i, w) \in [D] \times V(\mathcal{H}_{\text{unif}})$.
- For any $i \in [D]$, let $V_i := V_i^* \setminus \varphi(i, V(M_i))$.
- Let $V(\mathcal{H}_{\text{aux}}) := \mathcal{H}_{\text{unif}} \cup \bigcup_{i \in [D]} V_i$, where $\mathcal{H}_{\text{unif}} \cap V_i = \varnothing$ for $i \in [D]$.
- Let $\mathcal{H}_{\text{aux}} := \{\{f, v_1^i, \dots, v_r^i\} : f = \{v_1, \dots, v_r\} \in \mathcal{H}_{\text{unif}}, \{v_1^i, \dots, v_r^i\} \subseteq V_i, i \in [D]\}$.

Now for every $w \in V(\mathcal{H}_{\text{unif}})$ and $i \in [D]$ such that $w^i \in V_i$, since $\mathcal{H}_{\text{aux}}$ is linear, $d_{\mathcal{H}_{\text{unif}}}(w) - |V(M_i)| \leq d_{\mathcal{H}_{\text{aux}}}(w^i) \leq d_{\mathcal{H}_{\text{unif}}}(w)$, since V_i is obtained from V_i^* by deleting $|V(M_i)|$ vertices. Since $|V(M_i)| \leq \beta D$

and (b) holds, this implies $d_{\mathcal{H}_{\text{aux}}}(w^i) = (1 \pm 2\beta)D$. Moreover, by (8.1), for every $e^* \in \mathcal{H}_{\text{unif}}$, we have $(1-\beta)D \leq d_{\mathcal{H}_{\text{aux}}}(e^*) \leq D$. In summary, we have the following.

(8.2) For every vertex $w \in V(\mathcal{H}_{\text{aux}})$, we have $d_{\mathcal{H}_{\text{aux}}}(w) = (1 \pm 2\beta)D$.

By the construction of $\mathcal{H}_{\text{aux}}$, we have

$$n \leq (1-\beta)D|V(\mathcal{H}')| + |\mathcal{H}_{\text{unif}}| \leq n' := |V(\mathcal{H}_{\text{aux}})| \leq D|V(\mathcal{H}_{\text{unif}})| + |\mathcal{H}_{\text{unif}}| \overset{(a),(b)}{\leq} n^7. \tag{8.3}$$

Let

$$\mathcal{F}^{\text{aux}} := \bigcup_{i=1}^{D} \{\varphi(i, A) \setminus \varphi(i, V(M_i)) \,:\, A \in \mathcal{F}_V\} \cup \{\psi^{-1}(F) \,:\, F \in \mathcal{F}_E\},$$

where $|\mathcal{F}^{\text{aux}}| \leq D|\mathcal{F}_V| + |\mathcal{F}_E| \leq n^{2\log n} \leq (n')^{2\log n'}$. Since (8.2) holds and $D \geq n^{1/2} \overset{(8.3)}{\geq} (n')^{1/14}$, we can apply Corollary 4.3, with n', $\mathcal{H}_{\text{aux}}$, D, 2β, γ, $\mathcal{F}^{\text{aux}}$ playing the roles of n, $\mathcal{H}$, D, κ, γ, $\mathcal{F}$, respectively, to obtain a matching N^* in $\mathcal{H}_{\text{aux}}$, such that $|A \setminus V(N^*)| = (\gamma \pm 8\beta)|A|$ for every $A \in \mathcal{F}^{\text{aux}}$ with $|A| \geq D^{1/20}$. This implies that for any $A \in \mathcal{F}^{\text{aux}}$,

$$|A \setminus V(N^*)| = \gamma|A| \pm (8\beta|A| + D^{1/20}). \tag{8.4}$$

For $i \in [D]$, let

$$N'_i := \{\psi(e^*) \,:\, e^* = \{v_1, \ldots, v_r\} \in E_{\text{meet}}\,,\, \{e^*, v_1^i, \ldots, v_r^i\} \in N^*\} \text{ and } N_i := N'_i \cup M_i.$$

Then $N'_1, \ldots, N'_D \subseteq \mathcal{H}'$ satisfy the following properties.

(i) For any $w \in V(\mathcal{H}')$, we have $w \in V(N'_i)$ if and only if $w^i = \varphi(i, w) \in V(N^*)$.

(ii) For any $e \in \mathcal{H}'$, we have $e \in \bigcup_{i=1}^{D} N'_i$ if and only if $\psi^{-1}(e) \in V(N^*)$. (Recall that $\psi^{-1}(e) \in \mathcal{H}_{\text{unif}} \subseteq V(\mathcal{H}_{\text{aux}})$.)

It is easy to see that $N'_1, \ldots, N'_D$ are pairwise edge-disjoint matchings in $\mathcal{H}'$, and that for every $i \in [D]$, we have $V(N'_i) \cap V(M_i) = \varnothing$. Moreover, recall that $M_1, M_2, \ldots, M_D \subseteq \mathcal{H} \setminus \mathcal{H}'$ are pairwise edge-disjoint. Altogether this implies that $N_1, \ldots, N_D$ are pairwise edge-disjoint matchings in $\mathcal{H}$. This proves (8.1.1).

For any $F \in \mathcal{F}_E$, since $\psi^{-1}(F) \subseteq \mathcal{H}_{\text{unif}} \subseteq V(\mathcal{H}^{\text{aux}})$ and $\psi^{-1}(F) \in \mathcal{F}^{\text{aux}}$, we have

$$|F \setminus \bigcup_{i=1}^{D} N'_i| \overset{(ii)}{=} |\psi^{-1}(F) \setminus V(N^*)| \overset{(8.4)}{=} \gamma|F| \pm (8\beta|F| + D^{1/20}) \leq \gamma|F| + \kappa \max(|F|, D).$$

Thus, (8.1.3) holds.

Finally, we prove (8.1.2). Let us consider any $A \in \mathcal{F}_V$ and $i \in [D]$. Since

$$|A \setminus V(N_i)| = |(A \setminus V(M_i)) \setminus V(N'_i)| \overset{(i)}{=} |\varphi(i, A \setminus V(M_i)) \setminus V(N^*)| = |(\varphi(i, A) \setminus \varphi(i, V(M_i))) \setminus V(N^*)|$$

and $\varphi(i, A) \setminus \varphi(i, V(M_i)) \in \mathcal{F}^{\text{aux}}$, by (8.4), we have $|A \setminus V(N'_i)| = \gamma|A| \pm \kappa n$. Thus, the matching N_i is (γ, κ)-pseudorandom with respect to $\mathcal{F}_V$, proving (8.1.2). □

In the next lemma, using the absorption lemmas from Section 7, we extend the matchings given by the previous lemma in such a way that each matching will cover all but at most one vertex of $V_+^{(1-\varepsilon)}(G)$, where the uncovered vertex of $V_+^{(1-\varepsilon)}(G)$ must lie in a prescribed defect set S. In principle, we could apply Lemma 8.1 directly with $D = (1-\rho)n$ to colour almost all of the non-reserved small edges, but in order to be able to apply Lemmas 7.6 and 7.7 to each matching, we actually need to partition the hypergraph into subhypergraphs of maximum degree at most κn, and we apply Lemma 8.1 to each part successively, alternating with applications of one of Lemma 7.6 or 7.7.

Lemma 8.2 (Main colouring lemma)**.** *Let $0 < 1/n_0 \ll 1/r, \xi, \beta \ll \gamma \ll \varepsilon, \rho \ll 1$, let $n \geq n_0$, and let $D \in [n^{2/3}, n]$. Let $\mathcal{H}$ be an n-vertex linear multi-hypergraph, let $G := \mathcal{H}^{(2)}$ and let $U := V_+^{(1-\varepsilon)}(G)$.*

C1 *Let $S \subseteq U$ satisfy $|S| \geq D + \gamma n$ if $|U| > (1-2\varepsilon)n$,*

C2 *let $R \subseteq E(G)$ be a $(\rho, \gamma, \xi, \varepsilon)$-absorber for $\mathcal{V}$ such that $U, V(\mathcal{H}) \in \mathcal{V}$,*

C3 *let $\mathcal{H}' \subseteq \mathcal{H} \setminus R$ be a linear multi-hypergraph such that $V(\mathcal{H}) = V(\mathcal{H}')$, every edge $e \in \mathcal{H}'$ satisfies $|e| \leq r$, and $d_{\mathcal{H}'}(w) = (1 \pm \beta)D$ for every vertex $w \in V(\mathcal{H}')$, and*

C4 *let $\mathcal{M} = \{M_1, \ldots, M_D\}$ be a set of edge-disjoint matchings in $\mathcal{H} \setminus (\mathcal{H}' \cup R)$ such that $|V(M_i)| \leq \beta D$ for every $i \in [D]$, and $|\{i \in [D] \,:\, e \cap V(M_i) \neq \varnothing\}| \leq \beta D$ for every edge $e \in \mathcal{H}'$.*

Then there exists a set $\mathcal{N} := \{N_1, \ldots, N_D\}$ of edge-disjoint matchings in $\mathcal{H}$ satisfying the following.

(8.2.1) For every $i \in [D]$, we have $N_i \supseteq M_i$ and $N_i \setminus M_i \subseteq \mathcal{H}' \cup R$.

(8.2.2) For every vertex $w \in V(\mathcal{H})$, $|E_R(w) \cap \bigcup_{k=1}^{D} N_k| \leq \gamma D$ and $|E_{\mathcal{H}'}(w) \setminus \bigcup_{k=1}^{D} N_k| \leq \gamma D$.

(8.2.3) If $|U| \le (1-2\varepsilon)n$, then $\mathcal{N}$ has perfect coverage of U. Otherwise, $\mathcal{N}$ has nearly-perfect coverage of U with defects in S.

Proof. Let $K := \lceil \kappa^{-1} \rceil$, where we choose κ so that $r^{-1}, \xi, \beta \ll \kappa \ll \gamma$. First, we find a partition of $\mathcal{H}'$ into pairwise edge-disjoint hypergraphs $\mathcal{H}'_1, \dots, \mathcal{H}'_K$ such that $\bigcup_{j=1}^K \mathcal{H}'_j = \mathcal{H}'$, and every vertex has degree $(1 \pm 2\beta)D/K$ in $\mathcal{H}'_i$ for $i \in [K]$. (To show the desired partition exists, consider a partition chosen uniformly at random.) Now, for each $i \in [K]$, we may choose n_i to be either $\lfloor D/K \rfloor$ or $\lceil D/K \rceil$ such that $\sum_{j=1}^K n_j = D$. Let us partition the set $[D]$ into K disjoint parts $I_1, \dots, I_K$ such that $|I_i| = n_i$. Then $n_i \le \kappa n$, and every vertex in $\mathcal{H}'_i$ has degree $(1 \pm 3\beta)n_i$.

Let us define the following statements for $0 \le j \le K$.

(i)$_j$ For any $1 \le k \le j$, there exists a set $\mathcal{N}_k := \{N_c : c \in I_k\}$ of n_k matchings in $\mathcal{H}$ such that $M_c \subseteq N_c$ and $N_c \setminus M_c \subseteq \mathcal{H}'_k \cup R$ for every $c \in I_k$. Moreover, the matchings in $\bigcup_{k=1}^j \mathcal{N}_k$ are pairwise edge-disjoint.

(ii)$_j$ For every $w \in V(\mathcal{H})$,

$$|E_R(w) \cap \bigcup_{k\in[j]} \bigcup_{N\in\mathcal{N}_k} N| \le \gamma \sum_{k\in[j]} n_k \quad \text{and} \quad |E_{\bigcup_{k=1}^j \mathcal{H}'_k}(w) \setminus \bigcup_{k\in[j]} \bigcup_{N\in\mathcal{N}_k} N| \le \gamma \sum_{k\in[j]} n_k.$$

(iii)$_j$ If $|U| \le (1-2\varepsilon)n$, then $\bigcup_{k=1}^j \mathcal{N}_k$ has perfect coverage of U. Otherwise, $\bigcup_{k=1}^j \mathcal{N}_k$ has nearly-perfect coverage of U with defects in S.

Using induction on j, we will show that (i)$_j$–(iii)$_j$ hold for $j = K$ which clearly proves the lemma. Note that (i)$_j$–(iii)$_j$ trivially hold for $j = 0$. Let $i \in [K]$, and suppose that (i)$_j$–(iii)$_j$ hold for $j = i-1$. Our goal is to find a collection $\mathcal{N}_i$ of n_i pairwise edge-disjoint matchings in $\mathcal{H}$ satisfying (i)$_j$–(iii)$_j$ for $j = i$.

Let $R_i := R \setminus \bigcup_{k=1}^{i-1} \bigcup_{N\in\mathcal{N}_k} N$, let $S_i := S \setminus \bigcup_{k=1}^{i-1} \bigcup_{N\in\mathcal{N}_k} (U \setminus V(N))$, and

$$\mathcal{W} := \mathcal{F}(R_i) \cup \{U, S_i\}, \text{ where } \mathcal{F}(R_i) := \{N_{R_i}(u) \cap U : u \in U\} \cup \{N_{R_i}(u) \setminus U : u \in U\}. \tag{8.5}$$

Now we apply Lemma 8.1 with $\mathcal{H}'_i, \mathcal{W}, \{E_{\mathcal{H}'_i}(w) : w \in V(\mathcal{H})\}, \{M_c : c \in I_i\}, \beta^{1/2}, \gamma/4, n_i$ playing the roles of $\mathcal{H}', \mathcal{F}_V, \mathcal{F}_E, \{M_1, \dots, M_D\}, \beta, \gamma, D$ to obtain a set $\mathcal{N}'_i := \{N'_c : c \in I_i\}$ of n_i pairwise edge-disjoint matchings in $\mathcal{H}$ such that the following hold.

(a)$_i$ For every $c \in I_i$, $N'_c \supseteq M_c$ and $N'_c \setminus M_c \subseteq \mathcal{H}'_i$. In particular, $N'_c \cap R = \varnothing$.

(b)$_i$ For every $c \in I_i$, N'_c is $(\gamma/4, \kappa)$-pseudorandom with respect to $\mathcal{W}$.

(c)$_i$ For every $w \in V(\mathcal{H})$, $|E_{\mathcal{H}'_i}(w) \setminus \bigcup_{c\in I_i} N'_c| \le \gamma n_i/2$.

(d)$_i$ For every $w \in V(\mathcal{H})$, the number of matchings in $\mathcal{N}'_i$ not covering w is at most

$$|\mathcal{N}'_i| - (d_{\mathcal{H}'_i}(w) - |E_{\mathcal{H}'_i}(w) \setminus \bigcup_{c\in I_i} N'_c|) \overset{\text{(c)}_i}{\le} n_i - (1-3\beta)n_i + \gamma n_i/2 \le \gamma n_i.$$

Now we show that for any given $c \in I_i$, $(\mathcal{H}, N'_c, R_i, S_i)$ is $(\rho, \varepsilon, \gamma/4, \kappa, \xi)$-absorbable by pseudorandomness of N'_c, as follows.

- Using the fact that R is a $(\rho, \gamma, \xi, \varepsilon)$-absorber for $\mathcal{V}$, (ii)$_j$ with $j = i-1$, and Observation 7.3, we deduce that R_i is a $(\rho, 2\gamma, \xi, \varepsilon)$-absorber for $\mathcal{V}$, showing (AB1).
- (a)$_i$ implies (AB2).
- By (b)$_i$, N'_c is $(\gamma/4, \kappa)$-pseudorandom with respect to $\mathcal{W}$, so (i) of (AB3) holds, as required.

Moreover, if $|U| > (1-2\varepsilon)n$, then $|S_i| \ge |S| - \sum_{k=1}^{i-1} n_k \ge D + \gamma n - D = \gamma n$, so we can apply either Lemma 7.6 or Lemma 7.7 depending on the size of U, with $\gamma/4, R_i, S_i, \mathcal{N}'_i$ playing the roles of $\gamma, R, S, \mathcal{N}$. This yields a set $\mathcal{N}_i := \{N_c : c \in I_i\}$ of n_i pairwise edge-disjoint matchings in $\mathcal{H}$ such that the following hold.

- For every $c \in I_i$, $N_c \supseteq N'_c$ and $N_c \setminus N'_c \subseteq R_i$. Since $N'_c \setminus M_c \subseteq \mathcal{H}'_i$ by (a)$_i$, this shows (i)$_j$ for $j = i$.
- By (c)$_i$, for any $w \in V(\mathcal{H})$, $|E_{\mathcal{H}'_i}(w) \setminus \bigcup_{c\in I_i} N_c| \le \gamma n_i/2$. Moreover, by (d)$_i$, all but at most γn_i of the matchings in $\mathcal{N}'_i$ cover w. This together with (a)$_i$ implies that $|E_{R_i}(w) \cap \bigcup_{c\in I_i} N_c| \le \gamma n_i$. This shows (ii)$_j$ for $j = i$.
- If $|U| \le (1-2\varepsilon)n$, then $\mathcal{N}_i$ has perfect coverage of U. Otherwise, $\mathcal{N}_i$ has nearly-perfect coverage of U with defects in $S_i \subseteq S$. This shows (iii)$_j$ for $j = i$. □

Lemma 8.2 colours most of the non-reserved small edges (as shown in (8.2.2)). We will use the following lemma to colour the remaining non-reserved small edges such that every colour class covers all but at most one vertex of $V_+^{(1-\varepsilon)}(G)$. Since the proportion of remaining non-reserved small edges is small, we can afford to be less efficient in the number of colours we use in this step in order to ensure that each colour class is small, which allows us to use Lemma 7.11 to extend them.

Lemma 8.3 (Leftover colouring lemma). *Let $0 < 1/n_0 \ll 1/r, \xi \ll \gamma \ll \rho, \varepsilon \ll 1$, let $n \geq n_0$, and let $D \in [n^{2/3}, n]$. Let $\mathcal{H}$ be an n-vertex linear hypergraph, let $G := \mathcal{H}^{(2)}$, and let $U := V_+^{(1-\varepsilon)}(G)$.*

L1 *Let C be a set of colours with $\gamma D/2 \leq |C| \leq \gamma D$,*

L2 *let $\mathcal{M} := \{M_c : c \in C\}$ be a set of pairwise edge-disjoint matchings in $\mathcal{H}$, where $|V(M_c)| \leq \gamma n/2$ for every $c \in C$,*

L3 *let $R \subseteq E(G) \setminus \bigcup_{c\in C} M_c$ be a $(\rho, 10\gamma, \xi, \varepsilon)$-absorber for $\mathcal{V} := \{V(\mathcal{H}), U\}$,*

L4 *let $\mathcal{H}_{\mathrm{rem}} \subseteq \mathcal{H} \setminus (R \cup \bigcup_{c\in C} M_c)$ such that $V(\mathcal{H}) = V(\mathcal{H}_{\mathrm{rem}})$, $\Delta(\mathcal{H}_{\mathrm{rem}}) \leq \gamma^2 D/20$, every edge $e \in \mathcal{H}_{\mathrm{rem}}$ satisfies $|e| \leq r$ and $|\{c \in C : e \cap V(M_c) \neq \varnothing\}| \leq \gamma^2 D/100$,*

L5 *let $S \subseteq U$ be a subset satisfying $|S| > (\gamma + \varepsilon)n$ if $|U| > (1 - 10\varepsilon)n$.*

Then there exists a set $\mathcal{N} := \{N_c : c \in C\}$ of pairwise edge-disjoint matchings such that the following hold.

(8.3.1) For any $c \in C$, we have $N_c \supseteq M_c$ and $\mathcal{H}_{\mathrm{rem}} \subseteq \bigcup_{c\in C}(N_c \setminus M_c) \subseteq \mathcal{H}_{\mathrm{rem}} \cup R$.

(8.3.2) If $|U| \leq (1 - 10\varepsilon)n$, then $\mathcal{N}$ has perfect coverage of U. Otherwise, $\mathcal{N}$ has nearly-perfect coverage of U with defects in S.

Proof. Let us choose κ such that $r^{-1}, \xi \ll \kappa \ll \gamma$. Let $t := \lceil 4\gamma^{-1} \rceil$ and $C_1, \ldots, C_t$ be a partition of C into t sets such that $|C_i| \geq \gamma^2 D/10$ for $1 \leq i \leq t$.

We will first show that for every $i \in [t]$, there exists a proper colouring of the edges of $\mathcal{H}_{\mathrm{rem}}$ using colours from C_i. To that end, let $C(e) := \{c \in C : e \cap V(M_c) \neq \varnothing\}$ for every edge $e \in \mathcal{H}_{\mathrm{rem}}$. Since $\Delta(\mathcal{H}_{\mathrm{rem}}) \leq \gamma^2 D/20$, $|C(e)| \leq \gamma^2 D/100$ and $|C_i| \geq \gamma^2 D/10$, we can apply Theorem 4.6 with $\gamma^2 D/20$ and $1/2$ playing the roles of D and α, respectively, to show that for every $i \in [t]$, there exists a proper edge-colouring $\psi_i : \mathcal{H}_{\mathrm{rem}} \to C_i$ such that $\psi_i(e) \notin C(e)$ for every $e \in \mathcal{H}_{\mathrm{rem}}$. By the definition of $C(e)$, this implies that $V(\psi_i^{-1}(c)) \cap V(M_c) = \varnothing$ for any $i \in [t]$ and $c \in C_i$.

Let us now define a proper edge-colouring $\psi : \mathcal{H}_{\mathrm{rem}} \to C$ by choosing $i(e) \in [t]$ uniformly and independently at random for each $e \in \mathcal{H}_{\mathrm{rem}}$ and setting $\psi(e) := \psi_{i(e)}(e)$. Fix an arbitrary colour $c \in C$. Then there is a unique $j \in [t]$ such that $c \in C_j$, and $|V(\psi^{-1}(c))| = \sum_{e\in\psi_j^{-1}(c)} |e|\mathbf{1}_{i(e)=j}$, so by the linearity of expectation, $\mathbb{E}[|V(\psi^{-1}(c))|] = \sum_{e\in\psi_j^{-1}(c)} |e| \cdot \mathbb{P}(i(e) = j) \leq n/t \leq \gamma n/4$.

Since $|V(\psi^{-1}(c))|$ is a weighted sum of independent indicator random variables with maximum weight at most r, by applications of Theorem 4.1 together with a union bound, it is easy to see that $|V(\psi^{-1}(c))| < \gamma n/2$ for all $c \in C$ with non-zero probability. Combining this with the fact that for every $c \in C$, $V(\psi^{-1}(c)) \cap V(M_c) = \varnothing$ and $|V(M_c)| \leq \gamma n/2$, it follows that there exists a proper edge-colouring $\psi : \mathcal{H}_{\mathrm{rem}} \to C$ such that $\{\psi^{-1}(c) \cup M_c : c \in C\}$ is a set of edge-disjoint matchings in $\mathcal{H}$ where for every $c \in C$, $|V(\psi^{-1}(c)) \cup V(M_c)| \leq \gamma n$. Thus, for every $c \in C$, $(\mathcal{H}, M_c \cup \psi^{-1}(c), R, S)$ is $(\rho, \varepsilon, \gamma, \kappa, \xi)$-absorbable by smallness of $M_c \cup \psi^{-1}(c)$. So we can apply Lemma 7.11 with $\{M_c \cup \psi^{-1}(c) : c \in C\}$ playing the role of $\mathcal{N}$ to obtain a set $\mathcal{N} = \{N_c : c \in C\}$ of pairwise edge-disjoint matchings in $\mathcal{H}$ such that the following hold.

- For every $c \in C$, $N_c \supseteq M_c \cup \psi^{-1}(c)$ and $N_c \setminus (M_c \cup \psi^{-1}(c)) \subseteq R$; thus (8.3.1) holds.
- If $|U| \leq (1 - 10\varepsilon)n$, then $\mathcal{N}$ has perfect coverage of U. Otherwise, $\mathcal{N}$ has nearly-perfect coverage of U with defects in S; thus (8.3.2) holds. □

9. Optimal edge-colourings

In this section we will prove colouring results (Lemma 9.2 and Corollary 9.6) which will be used to colour the leftover edges of the reservoir in the final step of the proof of Theorem 1.1.

9.1. Edge-colourings with forbidden lists.

The following observation follows easily from Hall's theorem.

Observation 9.1. *Let G be a bipartite graph with bipartition $\{A, B\}$, and let δ_A and δ_B be the minimum degrees of the vertices in A and B, respectively. If $|A| \leq |B|$ and $\delta_A + \delta_B \geq |A|$, then G has a matching covering A.*

Lemma 9.2. *Let $\delta \in (0, 1)$, let H be an n-vertex graph, let C be a set of colours satisfying $|C| \geq 7\delta n$, and for every $w \in V(H)$, let $C_w \subseteq C$ such that the following hold.*

(i) *For any $w \in V(H)$, $d_H(w) \leq |C| - |C_w|$.*

(ii) *There is a set $U \subseteq V(H)$ with $|U| \leq \delta n$ such that every edge of H is incident to a vertex of U.*

(iii) *For every vertex $w \in V(H)$, $|C_w| \leq \delta n$.*

(iv) *For every $c \in C$, $|\{w \in V(H) : c \in C_w\}| \leq \delta n$.*

Then there exists a proper edge-colouring $\phi : E(H) \to C$ such that every edge $uv \in E(H)$ satisfies $\phi(uv) \notin C_u \cup C_v$.

Proof. Let $U := \{u_1, \dots, u_t\}$, where $t := |U| \leq \delta n$. Let $\phi_0 : \varnothing \to C$ be an empty function, and for every $1 \leq j \leq t$, let us inductively define a proper edge-colouring $\phi_j : \bigcup_{k=1}^{j} E_H(u_k) \to C$ such that

(a)$_j$ $\phi_j(uv) \notin C_u \cup C_v$ for each $uv \in \bigcup_{k=1}^{j} E_H(u_k)$, and

(b)$_j$ ϕ_j is a proper edge-colouring extending ϕ_{j-1}.

Since every edge of H is incident to a vertex of U by (ii), $\phi := \phi_t$ satisfies the assertion of the lemma. Let $i \in [t]$, and suppose we have already defined ϕ_j satisfying both (a)$_j$ and (b)$_j$ for $j \in [i-1]$; now we aim to construct ϕ_i satisfying (a)$_j$ and (b)$_j$ for $j = i$.

For each $v \in V(H) \setminus \{u_1, \dots, u_{i-1}\}$, let $C_v^* := \phi_{i-1}(E_H(v) \cap \bigcup_{j=1}^{i-1} E_H(u_j))$ be the set of colours of edges incident to a vertex $v \in V(H)$ in ϕ_{i-1}. Since any vertex $v \in V(H) \setminus \{u_1, \dots, u_{i-1}\}$ is adjacent to at most $i-1$ vertices in $\{u_1, \dots, u_{i-1}\}$, we have

$$|C_v^*| \overset{(b)_j}{=} |E_H(v) \cap \bigcup_{j=1}^{i-1} E_H(u_j)| \leq i - 1 \leq \delta n. \tag{9.1}$$

Let $A := E_H(u_i) \setminus \bigcup_{j=1}^{i-1} E_H(u_j)$ and $B := C \setminus (C_{u_i} \cup C_{u_i}^*)$. Let G_i be an auxiliary bipartite graph with the bipartition $\{A, B\}$ such that $\{e, c\} \in E(G_i)$ for $e = u_i v \in A$ and $c \in B$ if and only if $c \notin C_v \cup C_v^*$. Thus, the following hold.

- $|A| \leq |B|$. Indeed, $|A| = d_H(u_i) - |E_H(u_i) \cap \bigcup_{j=1}^{i-1} E_H(u_j)|$ and $|B| \overset{(b)_j}{\geq} |C| - |C_{u_i}| - |E_H(u_i) \cap \bigcup_{j=1}^{i-1} E_H(u_j)| \overset{\text{(i)}}{\geq} |A|$.
- For each $e = u_i v \in A$, we have

$$d_{G_i}(e) = |B| - |C_v \cup C_v^*| \geq |C| - |C_{u_i} \cup C_{u_i}^*| - |C_v \cup C_v^*| \overset{(9.1),\ \text{(iii)}}{\geq} 3\delta n. \tag{9.2}$$

- For each $c \in B$, since there are at most $i-1$ edges in $\bigcup_{j=1}^{i-1} E_H(u_j)$ which could be assigned the colour c by ϕ_{i-1}, we have $|\{v \in V(H) : c \in C_v^*\}| \leq 2(i-1) \leq 2\delta n$. Thus

$$d_{G_i}(c) \geq |A| - |\{v \in V(H) : c \in C_v^*\}| - |\{w \in V(H) \,:\, c \in C_w\}| \overset{\text{(iv)}}{\geq} |A| - 3\delta n. \tag{9.3}$$

Let δ_A and δ_B be the minimum degrees of the vertices in A and B in G_i, respectively. Then by (9.2) and (9.3), we have $\delta_A + \delta_B \geq |A|$. Moreover, $|A| \leq |B|$, so there exists a matching M_i in G_i covering A by Observation 9.1.

For each $e \in A = E_H(u_i) \setminus \bigcup_{j=1}^{i-1} E_H(u_j)$, let $c_e \in B$ be the unique element such that $\{e, c_e\} \in E(M_i)$. Let us define $\phi_i(e) := \phi_{i-1}(e)$ for $e \in \bigcup_{j=1}^{i-1} E_H(u_j)$, and $\phi_i(e) := c_e$ for $e \in E_H(u_i) \setminus \bigcup_{j=1}^{i-1} E_H(u_j)$. Since $c_e \notin C_{u_i} \cup C_{u_i}^* \cup C_v \cup C_v^*$ for every $e = u_i v \in E_H(u_i) \setminus \bigcup_{j=1}^{i-1} E_H(u_j)$, ϕ_i is a proper edge-colouring satisfying (a)$_j$ and (b)$_j$ for $j = i$, as desired. □

9.2. **Edge-colouring pseudorandom graphs.** Here we derive an optimal colouring result for pseudorandom graphs (Corollary 9.6) from a result (Theorem 9.5) on the overfull subgraph conjecture, which in turn is a consequence of the main result in [35] on Hamilton decompositions of robustly expanding regular graphs.

Definition 9.3 (Lower regularity)**.** Let $\rho, \xi \in (0,1)$, and let G be an n-vertex graph. A set $R \subseteq E(G)$ is *lower (ρ, ξ, G)-regular* if for every pair of disjoint sets $S, T \subseteq V(G)$ with $|S|, |T| \geq \xi n$, we have $|E_G(S,T) \cap R| \geq \rho e_G(S,T) - \xi |S||T|$.

A graph H is lower (ρ, ξ)-regular if $E(H)$ is lower $(\rho, \xi, K_{v(H)})$-regular, i.e., for every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geq \xi v(H)$, we have $e_H(S,T) \geq (\rho - \xi)|S||T|$.

Observation 9.4 (Robustness of lower regularity)**.** *Let $\alpha, \xi, \rho \in (0,1)$, and let G be a graph. Then the following hold.*

(9.4.1) If $R \subseteq E(G)$ is lower (ρ, ξ, G)-regular and $R' \subseteq R$ satisfies $\Delta(R - R') \leq \alpha v(G)$, then R' is lower $(\rho, \xi + \alpha^{1/2}, G)$-regular.

(9.4.2) If G is lower (ρ, ξ)-regular and $G \subseteq H$ such that $v(H) \leq (1 + \alpha\xi) v(G)$, then H is lower $(\rho(1-\alpha)^2, \frac{\xi}{1-\alpha})$-regular.

Theorem 9.5 (Glock, Kühn, and Osthus [20])**.** *Let $0 < 1/n_0 \ll \nu, \varepsilon \ll p < 1$, and let $n \geq n_0$. Let G be an n-vertex graph that is lower (p, ε)-regular and satisfies $\Delta(G) - \delta(G) \leq \nu n$. Let $\mathrm{def}_G(v) := \Delta(G) - d_G(v)$*

for any $v \in V(G)$. If n is even and

$$\mathrm{def}_G(w) \le \sum_{v \in V(G)\setminus\{w\}} \mathrm{def}_G(v) \tag{9.4}$$

for some vertex $w \in V(G)$ with $d_G(w) = \delta(G)$, then $\chi'(G) = \Delta(G)$.

Even though the statement of [20, Theorem 1.6] requires G to have no overfull subgraph, it is shown in its proof that it suffices to assume that G satisfies (9.4) (see the remark below [20, Theorem 1.6]). We will use the following corollary of Theorem 9.5.

Corollary 9.6. *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$, and let $n \ge n_0$. Let G be an n-vertex graph that is lower (p, ε)-regular and satisfies $\Delta(G) - \delta(G) \le \eta n$. If there are at least $\Delta(G)$ vertices in G having degree less than $\Delta(G)$, then $\chi'(G) = \Delta(G)$.*

Proof. Note that (9.4) is equivalent to

$$2(\Delta(G) - \delta(G)) \le \sum_{v \in V(G)} (\Delta(G) - d_G(v)). \tag{9.5}$$

Since G is lower (p, ε)-regular, it is easy to see that $\Delta(G) \ge (p - 3\varepsilon)n$. Now we prove the corollary. First suppose n is even. Since G has at least $\Delta(G)$ vertices having degree less than $\Delta(G)$, we have $2(\Delta(G) - \delta(G)) \le 2\eta n \le (p - 3\varepsilon)n \le \Delta(G) \le \sum_{v \in V(G)}(\Delta(G) - d_G(v))$. Thus (9.5) holds, which implies that (9.4) holds. So we can apply Theorem 9.5 to show that $\chi'(G) = \Delta(G)$.

Now suppose n is odd. Let $t := \lfloor \Delta(G) - 2\eta n \rfloor$. Let G^* be a graph obtained from G by adding a new vertex v^* adjacent to exactly t vertices of G having degree less than $\Delta(G)$ in G. Then

(a) $\Delta(G^*) = \Delta(G)$,
(b) $\delta(G^*) = d_{G^*}(v^*) = t$,
(c) $|\{v \in V(G) : \Delta(G^*) > d_{G^*}(v)\}| \ge \Delta(G) - d_{G^*}(v^*)$, and
(d) G^* is lower $(p/4, 2\varepsilon)$-regular by (9.4.2).

This implies that

$$\sum_{v \in V(G^*)} (\Delta(G^*) - d_{G^*}(v)) \overset{(b)}{=} \sum_{v \in V(G)} (\Delta(G^*) - d_{G^*}(v)) + (\Delta(G^*) - \delta(G^*)) \overset{(a),(b),(c)}{\ge} 2(\Delta(G^*) - \delta(G^*)).$$

Thus, G^* satisfies (9.5), so it satisfies (9.4). Moreover (d) holds, and $\Delta(G^*) - \delta(G^*) \le 3\eta v(G^*)$ by (a) and (b), so applying Theorem 9.5 with $G^*, p/4, 2\varepsilon, 3\eta$ playing the roles of G, p, ε, ν, respectively, we deduce that $\Delta(G) \le \chi'(G) \le \chi'(G^*) = \Delta(G^*) \overset{(a)}{=} \Delta(G)$, as desired. □

10. Constructing reservoirs

In this section, we construct a set $R_{\mathrm{res}} \subseteq E(G)$ called a *reservoir* with several pseudorandom properties.

Definition 10.1 (Pseudorandom / Regularising reservoirs)**.** Let $\varepsilon, \rho, \xi \in (0,1)$, let $\mathcal{H}$ be an n-vertex linear hypergraph, let $G := \mathcal{H}^{(2)}$, let G' be the spanning subgraph consisting of the edges of G with at least one vertex in $V_+^{(1-\varepsilon)}(G)$, and let $\mathcal{V}$ be a set of subsets of $V(\mathcal{H})$.

- A subset $R_{\mathrm{res}} \subseteq E(G)$ is a $(\rho, \xi, \varepsilon, \mathcal{V})$-*pseudorandom reservoir* if
  (P1) for each $v \in V(\mathcal{H})$, $d_{R_{\mathrm{res}}}(v) = \rho d_G(v) \pm \xi n$, and
  (P2) $R_{\mathrm{res}} \cap E(G')$ is a $(\rho, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$.
- Suppose R_{abs} is a $(\rho/2, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$. A set $R_{\mathrm{reg}} \subseteq E(G) \setminus R_{\mathrm{abs}}$ is a $(\rho, \xi, \varepsilon, R_{\mathrm{abs}}, \mathcal{V})$-*regularising reservoir* if $R_{\mathrm{res}} := R_{\mathrm{abs}} \cup R_{\mathrm{reg}}$ satisfies the following.
  (R1) For each $v \in V_+^{(1-\varepsilon)}(G)$, $d_{R_{\mathrm{res}}}(v) = \rho d_G(v) \pm \xi n$, and
  (R2) for any $w \in V(\mathcal{H}) \setminus V_+^{(1-\varepsilon)}(G)$, $\max(\rho d_G(w), (\rho - 20\varepsilon)n) \le d_{R_{\mathrm{res}}}(w) \le \rho(1-\varepsilon)n + \xi n$.

Now we define various types of reservoirs. The type of reservoir we choose to use will depend on the structure of the hypergraph $\mathcal{H}$.

Definition 10.2 (Types of reservoirs)**.** Let $\varepsilon, \rho, \xi \in (0,1)$, let $\mathcal{H}$ be an n-vertex linear hypergraph, let $G := \mathcal{H}^{(2)}$, and let G' be the spanning subgraph consisting of the edges of G that are incident to a vertex in $V_+^{(1-\varepsilon)}(G)$.

For a collection $\mathcal{V}$ of subsets of $V(\mathcal{H})$, and $R_{\mathrm{res}} \subseteq E(G)$, we say R_{res} is

- a $(\rho, \xi, \varepsilon, \mathcal{V})$-*reservoir of Type* A_1 if R_{res} is a $(\rho, \xi, \varepsilon, \mathcal{V})$-pseudorandom reservoir,
- a $(\rho, \xi, \varepsilon, \mathcal{V})$-*reservoir of Type* A_2 if $R_{\mathrm{res}} = R_{\mathrm{abs}} \cup R_{\mathrm{reg}}$, where

- ◦ R_{abs} is a $(\rho/2, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$ that is also lower $(\rho/2, \xi, G')$-regular, and
- ◦ R_{reg} is a $(\rho, \xi, \varepsilon, R_{\mathrm{abs}}, \mathcal{V})$-regularising reservoir, and
- a $(\rho, \xi, \varepsilon, \mathcal{V})$-*reservoir of Type* B if R_{res} is a $(\rho, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$.

For brevity, we often omit the type if it is clear from the context.

We will use reservoirs of Type A_1 when $\mathcal{H}$ is neither (ρ, ε)-full nor FPP-extremal (which are defined in Definitions 2.2 and 2.3), reservoirs of Type A_2 when $\mathcal{H}$ is (ρ, ε)-full but not FPP-extremal, and reservoirs of Type B when $\mathcal{H}$ is FPP-extremal.

Now we show the existence of a suitable absorber, a pseudorandom reservoir, and a regularising reservoir.

Proposition 10.3 (The existence of a pseudorandom reservoir and an absorber)**.** *Let $0 < 1/n_0 \ll \xi, \varepsilon, \rho < 1$, and let $n \geq n_0$. Let $\mathcal{H}$ be an n-vertex linear hypergraph, let $G := \mathcal{H}^{(2)}$, and let G' be the spanning subgraph of G consisting of the edges of G incident to a vertex in $V_+^{(1-\varepsilon)}(G)$. If $\mathcal{V}$ is a collection of subsets of $V(\mathcal{H})$ such that $|\mathcal{V}| \leq n^{\log n}$, then there exists $R_{\mathrm{rnd}} \subseteq E(G)$ such that*

- *$R_{\mathrm{rnd}} \cap E(G')$ is a $(\rho, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$, and lower (ρ, ξ, G')-regular, and*
- *R_{rnd} is a $(\rho, \xi, \varepsilon, \mathcal{V})$-pseudorandom reservoir.*

In particular, R_{rnd} is a $(\rho, \xi, \varepsilon, \mathcal{V})$-reservoir of Type A_1 *and $R_{\mathrm{rnd}} \cap E(G')$ is a $(\rho, \xi, \varepsilon, \mathcal{V})$-reservoir of Type* B*.*

To prove Proposition 10.3, it suffices to consider a set $R_{\mathrm{rnd}} \subseteq E(G)$ of edges chosen independently and uniformly at random with probability ρ and apply the weighted Chernoff's inequality (Theorem 4.1) with all weights equal to 1.

Lemma 10.4 (The existence of a regularising reservoir)**.** *Let $0 < 1/n_0 \ll \xi \ll \varepsilon \ll \rho \ll 1$, and let $n \geq n_0$. Let $\mathcal{H}$ be an n-vertex linear hypergraph, and let $G := \mathcal{H}^{(2)}$. If $\mathcal{V}$ is a collection of subsets of $V(\mathcal{H})$ such that $|\mathcal{V}| \leq n^{\log n}$, $V(\mathcal{H}) \in \mathcal{V}$, and $\mathcal{H}$ is (ρ, ε)-full, then for any $(\rho/2, \xi, \xi, \varepsilon)$-absorber R_{abs} for $\mathcal{V}$, there exists a $(\rho, \xi, \varepsilon, R_{\mathrm{abs}}, \mathcal{V})$-regularising reservoir $R_{\mathrm{reg}} \subseteq E(G) \setminus R_{\mathrm{abs}}$.*

Proof. Let $U := V_+^{(1-\varepsilon)}(G)$ and let G' be the spanning subgraph of G consisting of the edges of G incident to a vertex of U. Let R_{abs} be a $(\rho/2, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$. Since $V(\mathcal{H}) \in \mathcal{V}$,

$$\text{for any } v \in V(G),\ d_{R_{\mathrm{abs}}}(v) = \rho d_{G'}(v)/2 \pm \xi n. \tag{10.1}$$

Let

$$U' := \{w \in V(\mathcal{H}) \setminus U \ : \ d_G(w) \geq (1 - 20\varepsilon\rho^{-1})n\}. \tag{10.2}$$

Since $\mathcal{H}$ is (ρ, ε)-full, we can choose a subset $S \subseteq V^{(n-1)}(G)$ with $|S| = \lceil (\rho - 20\varepsilon)n \rceil$. Note that every vertex of S is adjacent to all the other vertices of G.

For each vertex $w \in V(\mathcal{H}) \setminus (U \cup U')$, we choose $\lceil (\rho - 20\varepsilon)n - d_{R_{\mathrm{abs}}}(w) \rceil > 0$ edges of $E_{G'}(S, \{w\}) \setminus R_{\mathrm{abs}}$, and let $R' \subseteq E(G') \setminus R_{\mathrm{abs}}$ be the union of all such edges for all $w \in V(\mathcal{H}) \setminus (U \cup U')$. Then for any $w \in V(\mathcal{H}) \setminus (U \cup U')$, we have

$$\rho d_G(w) \overset{(10.2)}{\leq} (\rho - 20\varepsilon)n \leq d_{R_{\mathrm{abs}}}(w) + d_{R'}(w) \leq (\rho - 20\varepsilon)n + 1 \leq \rho(1 - \varepsilon)n. \tag{10.3}$$

For each vertex $w \in U \cup U'$, let us define

$$f(w) := \lfloor \rho d_G(w) + \xi n - d_{R_{\mathrm{abs}}}(w) - d_{R'}(w) \rfloor \text{ and } g(w) := f(w) - 1.$$

Claim 1. *There exists a (g, f)-factor R'' in $H := G[U \cup U'] - R_{\mathrm{abs}} = G[U \cup U'] - R_{\mathrm{abs}} - R'$.*

Proof of claim: Since $\varepsilon \ll \rho$, for any $w \in U \cup U'$, we have $d_G(w) \geq (1 - \rho)n$. Moreover, for any $w \in U \cup U'$, since $\mathcal{H}$ is (ρ, ε)-full, $d_{R'}(w) \leq |V(\mathcal{H}) \setminus U| \leq 10\varepsilon n \leq \rho n/10$, and by (10.1), $d_{R_{\mathrm{abs}}}(w) \leq \rho n/2 + \xi n$. Hence, for any $w \in U \cup U'$ we have

$$\frac{3\rho n}{10} \leq \rho(1 - \rho)n + \xi n - \left(\frac{\rho n}{2} + \xi n\right) - \frac{\rho n}{10} - 1 \leq f(w) \leq \rho n + \xi n < \frac{3\rho n}{2}.$$

Therefore, for any $w \in U \cup U'$,

$$f(w), g(w) \in [\rho n/4, 2\rho n]. \tag{10.4}$$

Moreover, for any $w \in U \cup U'$, we have

$$d_{G - R_{\mathrm{abs}}}(w) \geq (1 - \rho)n - \left(\frac{\rho n}{2} + \xi n\right) - \frac{\rho n}{10} \geq (1 - 2\rho)n. \tag{10.5}$$

By Lovász's (g, f)-factor Theorem [36], there exists a (g, f)-factor in H if

(i) $0 \leq g(w) < f(w) \leq d_H(w)$ for each $w \in U \cup U'$, and

(ii) for any pair of disjoint sets $S, T \subseteq V(H)$, $c(S,T) \geq 0$, where
$$c(S,T) := \sum_{t\in T}(d_H(t) - g(t)) + \sum_{s\in S} f(s) - e_H(S,T) = \sum_{t\in T}(d_{H-S}(t) - g(t)) + \sum_{s\in S} f(s).$$

Since $\mathcal{H}$ is (ρ,ε)-full and $\varepsilon \ll \rho$, note that

$$|V(\mathcal{H}) \setminus (U \cup U')| \leq |V(\mathcal{H}) \setminus U| \leq 10\varepsilon n \leq \rho n. \tag{10.6}$$

Hence, for any $w \in U \cup U'$, we have

$$d_H(w) \geq d_{G-R_{\mathrm{abs}}}(w) - |V(G) \setminus (U \cup U')| \overset{(10.5),(10.6)}{\geq} (1-2\rho)n - \rho n = (1-3\rho)n \overset{(10.4)}{\geq} f(w),$$

so (i) holds. Now, we verify (ii). If $|S| \leq (1-5\rho)n$, then for any $t \in T$, we have

$$d_{H-S}(t) \geq d_{G-R_{\mathrm{abs}}}(t) - |V(G) \setminus (U \cup U')| - |S| \overset{(10.5),(10.6)}{\geq} (1-2\rho)n - \rho n - |S| \geq 2\rho n \overset{(10.4)}{\geq} g(t),$$

so $c(S,T) \geq 0$. Hence we may assume that $|S| > (1-5\rho)n$, which implies that $|T| < 5\rho n$. Then

$$\sum_{t\in T} g(t) \overset{(10.4)}{\leq} |T| \cdot 2\rho n \leq 10\rho^2 n^2 \leq (1-5\rho)n \cdot \frac{\rho n}{4} < |S| \cdot \frac{\rho n}{4} \overset{(10.4)}{\leq} \sum_{s\in S} f(s),$$

so $c(S,T) \geq -\sum_{t\in T} g(t) + \sum_{s\in S} f(s) \geq 0$, proving (ii) and thus the claim. ♦

Finally, we show that $R_{\mathrm{reg}} := R' \cup R''$ is a $(\rho,\xi,\varepsilon,R_{\mathrm{abs}},\mathcal{V})$-regularising reservoir. Let $R_{\mathrm{res}} := R_{\mathrm{abs}} \cup R' \cup R''$. For each $w \in U \cup U'$, we have

$$d_{R_{\mathrm{res}}}(w) \geq (f(w) - 1) + d_{R_{\mathrm{abs}}}(w) + d_{R'}(w) \geq \rho d_G(w) + \xi n - 2,$$
$$d_{R_{\mathrm{res}}}(w) \leq f(w) + d_{R_{\mathrm{abs}}}(w) + d_{R'}(w) \leq \rho d_G(w) + \xi n.$$

Thus, (R1) holds, and for $w \in U'$, $(\rho - 20\varepsilon)n \overset{(10.2)}{\leq} \rho d_G(w) \leq d_{R_{\mathrm{res}}}(w) \leq \rho(1-\varepsilon)n + \xi n$, as required by (R2). Moreover, for any $w \in V(\mathcal{H}) \setminus (U \cup U')$, $d_{R_{\mathrm{res}}}(w) = d_{R_{\mathrm{abs}}}(w) + d_{R'}(w)$ since $R'' \subseteq E(H) \subseteq E(G[U \cup U'])$. Therefore, by (10.3), $\max(\rho d_G(w), (\rho - 20\varepsilon)n) \leq d_{R_{\mathrm{res}}}(w) \leq \rho(1-\varepsilon)n$, showing that (R2) holds for $w \in V(\mathcal{H}) \setminus (U \cup U')$. Thus, (R2) holds for all $w \in V(\mathcal{H}) \setminus U$. This completes the proof. □

Definition 10.5 (Regularised linear multi-hypergraph)**.** For an n-vertex linear hypergraph $\mathcal{H}$, let $\mathcal{H}_{\mathrm{reg}}$ be the linear multi-hypergraph obtained from $\mathcal{H}$ by adding $\max(0, n-3-d_{\mathcal{H}}(w))$ singleton edges incident to each $w \in V(\mathcal{H})$.

Recall that $\mathcal{H}_{\mathrm{small}} := \{e \in \mathcal{H} : |e| \leq r_1\}$. In order to be able to use Lemma 8.2, we need to embed $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}}$ into an almost-regular linear multi-hypergraph $\mathcal{H}'$ by adding singleton edges. In particular, for each vertex $w \in V(\mathcal{H})$, we add at most $\max(0, n-3-d_{\mathcal{H}}(w))$ singleton edges containing w, so that $\mathcal{H}' \subseteq \mathcal{H}_{\mathrm{reg}}$.

Lemma 10.6 (Regularising lemma)**.** *Let $0 < 1/n_0 \ll \xi, 1/r_1 \ll \beta, \varepsilon, \rho \ll 1$, and let $n \geq n_0$. Let $\mathcal{H}$ be an n-vertex linear hypergraph, and let $\mathcal{V}$ be a collection of subsets in $V(\mathcal{H})$ such that $V(\mathcal{H}) \in \mathcal{V}$. If either*
(i) R_{res} is a $(\rho,\xi,\varepsilon,\mathcal{V})$-reservoir of Type A_1 or A_2, or
(ii) R_{res} is a $(\rho,\xi,\varepsilon,\mathcal{V})$-reservoir of Type B and $3\rho \leq \varepsilon$,
then there exists a linear multi-hypergraph $\mathcal{H}' \subseteq \mathcal{H}_{\mathrm{reg}}$ such that
- *$\mathcal{H}'$ is obtained from $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}}$ by adding singleton edges, and*
- *for every $w \in V(\mathcal{H})$, we have $d_{\mathcal{H}'}(w) = (1-\rho)(n-1 \pm \beta n)$.*

Proof. Let $G := \mathcal{H}^{(2)}$, and let $U := V_+^{(1-\varepsilon)}(G)$. Since $\mathcal{H}$ is linear and every $w \in V(\mathcal{H})$ is contained in at most one singleton,

$$d_{\mathcal{H}\setminus E(G)}(w) \leq \frac{n-1-d_G(w)}{2} + 1 \leq (1-\rho)(n-1-d_G(w)) + 1. \tag{10.7}$$

We will show that for any $w \in V(\mathcal{H})$,

$$d_{\mathcal{H}\setminus R_{\mathrm{res}}}(w) \leq (1-\rho)(n-1) + \xi n + 1. \tag{10.8}$$

Let us first consider the case when R_{res} is a $(\rho,\xi,\varepsilon,\mathcal{V})$-reservoir of Type A_1 or A_2. In this case, for any $w \in V(\mathcal{H})$, we have

$$\rho d_G(w) - \xi n \leq d_{R_{\mathrm{res}}}(w) \leq (\rho + \xi)n. \tag{10.9}$$

Indeed, if R_{res} is of Type A_1, then (10.9) holds by (P1), and if R_{res} is of Type A_2, then (10.9) holds by (R1) and (R2). Now, by (10.7) and (10.9), every $w \in V(\mathcal{H})$ satisfies

$$d_{\mathcal{H}\setminus R_{\mathrm{res}}}(w) = d_{G-R_{\mathrm{res}}}(w) + d_{\mathcal{H}\setminus E(G)}(w) \overset{(10.9)}{\leq} (1-\rho)d_G(w) + \xi n + d_{\mathcal{H}\setminus E(G)}(w)$$

$$\overset{(10.7)}{\leq} (1-\rho)(n-1) + \xi n + 1, \tag{10.10}$$

proving (10.8) when R_{res} is a $(\rho, \xi, \varepsilon, \mathcal{V})$-reservoir of Type A_1 or A_2.

Now let us consider the case when R_{res} is a $(\rho, \xi, \varepsilon, \mathcal{V})$-reservoir of Type B. Let G' be the spanning subgraph of G consisting of the edges of G incident to a vertex of U. Since R_{res} is a $(\rho, \xi, \xi, \varepsilon)$-absorber for $\mathcal{V}$ and $V(\mathcal{H}) \in \mathcal{V}$, every $w \in V(\mathcal{H})$ satisfies

$$d_{R_{\mathrm{res}}}(w) = \rho d_{G'}(w) \pm \xi n. \tag{10.11}$$

If $w \in U$, then $d_{G'}(w) = d_G(w)$, so $d_{R_{\mathrm{res}}}(w) = \rho d_G(w) \pm \xi n$ and by the same reasoning as in (10.10), one can show that (10.8) holds if $w \in U$.

It remains to show that (10.8) holds for $w \in V(\mathcal{H}) \setminus U$. Indeed, every $w \in V(\mathcal{H}) \setminus U$ satisfies

$$\begin{aligned}
d_{\mathcal{H}\setminus R_{\mathrm{res}}}(w) &= d_{G'-R_{\mathrm{res}}}(w) + d_{G-E(G')-R_{\mathrm{res}}}(w) + d_{\mathcal{H}\setminus E(G)}(w) \\
&\overset{(10.11)}{\leq} (1-\rho)d_{G'}(w) + \xi n + d_{G-E(G')}(w) + d_{\mathcal{H}\setminus E(G)}(w) \\
&= (1-\rho)d_G(w) + \rho d_{G-E(G')}(w) + d_{\mathcal{H}\setminus E(G)}(w) + \xi n \\
&\overset{(10.7)}{\leq} (1-\rho)d_G(w) + \rho(1-\varepsilon)n + \frac{n-1-d_G(w)}{2} + 1 + \xi n \\
&= (1/2-\rho)d_G(w) + \rho(1-\varepsilon)n + \frac{n-1}{2} + \xi n + 1 \\
&\leq (1-\rho)(n-1) + \xi n + 1,
\end{aligned}$$

as desired. Note that the last inequality is equivalent to $(1/2-\rho)d_G(w) + \rho(1-\varepsilon)n \leq (1/2-\rho)(n-1)$, which holds since $d_G(w) \leq (1-\varepsilon)n$ and $3\rho \leq \varepsilon$.

Now let $k := \lfloor (1-\rho)(n-1) - \beta n/2 \rfloor$, and let $\mathcal{H}'$ be the linear multi-hypergraph obtained from $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}}$ as follows. For every vertex $w \in V(\mathcal{H})$ satisfying $d_{\mathcal{H}_{\mathrm{small}}\setminus R_{\mathrm{res}}}(w) < k$, we add $k - d_{\mathcal{H}_{\mathrm{small}}\setminus R_{\mathrm{res}}}(w)$ singleton edges containing w. Then, by (10.8), for every vertex $w \in V(\mathcal{H})$,

$$d_{\mathcal{H}'}(w) = (1-\rho)(n-1) \pm 2\beta n/3 = (1-\rho)(n-1 \pm \beta n), \tag{10.12}$$

as desired. Now we prove that $\mathcal{H}' \subseteq \mathcal{H}_{\mathrm{reg}}$ by showing that $\mathcal{H}'$ is obtained from $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}}$ by adding at most $\max(0, n-3-d_{\mathcal{H}}(w))$ singleton edges incident to each vertex $w \in V(\mathcal{H})$. Indeed, for any vertex $w \in V(\mathcal{H})$ with $d_{\mathcal{H}_{\mathrm{small}}\setminus R_{\mathrm{res}}}(w) < k$, we add at most

$$k - d_{\mathcal{H}_{\mathrm{small}}\setminus R_{\mathrm{res}}}(w) \overset{(10.9),(10.11)}{\leq} (1-\rho)n - \frac{\beta n}{2} - \left(d_{\mathcal{H}}(w) - \frac{2n}{r_1}\right) + (\rho+\xi)n \leq n - 3 - d_{\mathcal{H}}(w)$$

singleton edges incident to w, since $d_{\mathcal{H}}(w) - d_{\mathcal{H}_{\mathrm{small}}}(w) \leq 2n/r_1$ and $\xi, 1/r_1 \ll \beta$. This completes the proof of the lemma. □

11. Proof of Theorem 1.1

Now we are ready to prove our main theorem. As discussed in Section 2, the proof depends on the structure of $\mathcal{H}$. The relevant properties of $\mathcal{H}$ are captured by the following definition. (Recall that (ρ, ε)-full linear hypergraphs were introduced in Definition 2.2.)

Definition 11.1 (Types of hypergraphs and colourings)**.** Let $\mathcal{H}$ be a linear n-vertex hypergraph, and let $\phi : \mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}} \to [n]$ be a proper edge-colouring. We say ϕ is of Type A if it satisfies (6.1:a) of Theorem 6.1, and ϕ is of Type B if it satisfies (6.1:b) of Theorem 6.1. We say $(\mathcal{H}, \phi)$ is of

- Type A_1 if ϕ is of Type A, and $\mathcal{H}$ is not (ρ, ε)-full,
- Type A_2 if ϕ is of Type A, and $\mathcal{H}$ is (ρ, ε)-full,
- Type B if ϕ is of Type B.

Proof of Theorem 1.1. Recall the hierarchy of the parameters

$$0 < 1/n_0 \ll 1/r_0 \ll \xi \ll 1/r_1 \ll \beta \ll \kappa \ll \gamma_1 \ll \varepsilon_1 \ll \rho_1 \ll \sigma \ll \delta \ll \gamma_2 \ll \rho_2 \ll \varepsilon_2 \ll 1,$$

where r_0 and r_1 are integers. Let $n \geq n_0$, and let $\mathcal{H}$ be an n-vertex linear hypergraph. Without loss of generality, we may assume that $\mathcal{H}$ has no singleton edges. Our aim is to find n pairwise edge-disjoint matchings containing all of the edges of $\mathcal{H}$.

Let $G := \mathcal{H}^{(2)}$, let $U := V_+^{(1-\varepsilon)}(G)$, and let G' be the spanning subgraph of G consisting of the set of edges of G incident to a vertex of U.

Step 1. *Colour large and medium edges, and define the corresponding parameters.*

Let $\phi : \mathcal{H}_{\mathrm{med}} \cup \mathcal{H}_{\mathrm{large}} \to [n]$ be the proper edge-colouring given by Theorem 6.1.

Now we define some parameters depending on the type of $(\mathcal{H}, \phi)$ as follows.

- $\rho := \rho_1$, $\rho_{\mathrm{abs}} := \rho_1$, $\varepsilon := \varepsilon_1$, and $\gamma := \gamma_1$ if $(\mathcal{H}, \phi)$ is of Type A_1,
- $\rho := \rho_1$, $\rho_{\mathrm{abs}} := \rho_1/2$, $\varepsilon := \varepsilon_1$, and $\gamma := \gamma_1$ if $(\mathcal{H}, \phi)$ is of Type A_2, and
- $\rho := \rho_2$, $\rho_{\mathrm{abs}} := \rho_2$, $\varepsilon := \varepsilon_2$, and $\gamma := \gamma_2$ if $(\mathcal{H}, \phi)$ is of Type B.

Thus, we have

$$\kappa \ll \gamma \ll \rho, \rho_{\mathrm{abs}}, \varepsilon \ll 1. \tag{11.1}$$

Recall that difficult matchings were defined in Definition 7.10. The following claim is used in the later steps.

Claim 1. *The following hold.*

(1.1) For any edge $e \in \mathcal{H}$ such that $e \cap U \neq \varnothing$, we have $|e| \leq \varepsilon n$.

(1.2) For any edge $e \in \mathcal{H}_{\mathrm{small}}$, we have $|\{f \in \mathcal{H}_{\mathrm{large}} : e \cap f \neq \varnothing\}| \leq 2r_1 n/r_0$.

(1.3) If ϕ is of Type B, then $|U| \leq 2\delta n$, and there is no difficult colour class in ϕ.

(1.4) If ϕ is of Type A, *then there is at most one colour $c \in [n]$ such that $\phi^{-1}(c)$ contains a huge edge and $\phi^{-1}(c)$ is difficult. Moreover, if such a colour c exists, then $\phi^{-1}(c) = \{e\}$ for some huge edge e.*

Proof of claim: Let us first prove (1.1). For any edge $e \in \mathcal{H} \setminus E(G)$ containing a vertex $w \in U$, since $\mathcal{H}$ is linear, the vertex w is not adjacent (in G) to any vertex of e. Thus, $(1 - \varepsilon)n \leq d_G(w) \leq n - |e|$, implying that $|e| \leq \varepsilon n$, as desired.

Now we prove (1.2). Since $\mathcal{H}$ is linear, every vertex $w \in V(\mathcal{H})$ is incident to at most $n/(r_0 - 1)$ edges of $\mathcal{H}_{\mathrm{large}}$. Thus, $|\{f \in \mathcal{H}_{\mathrm{large}} : e \cap f \neq \varnothing\}| \leq |e|n/(r_0 - 1) \leq 2r_1 n/r_0$.

Now we show (1.3). Since there is a set of FPP-extremal edges in $\mathcal{H}$ with volume at least $1 - \delta$, we have $|U|(1 - \varepsilon)n/2 \leq |E(G)| \leq \delta\binom{n}{2}$. Thus, $|U| \leq 2\delta n$. If $\phi^{-1}(c)$ is difficult, then $|V(\phi^{-1}(c))| \geq 3|V(\mathcal{H}) \setminus U|/4 \geq n/2$, which is impossible since each colour class covers at most δn vertices by (6.1:b)(i), (6.1:b)(ii), and (6.1:b)(iii) of Theorem 6.1.

Finally, we prove (1.4). By (6.1:a)(i) of Theorem 6.1, every colour class of ϕ containing a huge edge consists of a unique edge. Suppose $\phi^{-1}(c_1)$ and $\phi^{-1}(c_2)$ are difficult colour classes of ϕ such that both of them contain a huge edge, and let $c_1 \neq c_2$. Then $|V(\mathcal{H}) \setminus U| \geq 2$ and there exist huge edges $e_1 \neq e_2$ in $\mathcal{H}$ such that $\phi(e_1) = c_1$ and $\phi(e_2) = c_2$. If $|V(\mathcal{H}) \setminus U| = 2$, then both e_1 and e_2 contain $V(\mathcal{H}) \setminus U$ since $\lceil 3|V(\mathcal{H}) \setminus U|/4 \rceil = 2$, contradicting the linearity of $\mathcal{H}$. Otherwise, if $|V(\mathcal{H}) \setminus U| \geq 3$, then for $i \in \{1, 2\}$, we have $|e_i \setminus U| \geq 3|V(\mathcal{H}) \setminus U|/4$, so $|(e_1 \cap e_2) \setminus U| \geq |V(\mathcal{H}) \setminus U|/2 > 1$, also contradicting the linearity of $\mathcal{H}$. ♦

Step 2. *Choose a reservoir R_{res} and a defect-set S.*

Let us define

$$\mathcal{V} := \{U, V(\mathcal{H})\} \cup \bigcup_{i=1}^{n} \{U \cup V(\phi^{-1}(i)), U \setminus V(\phi^{-1}(i))\}. \tag{11.2}$$

By Proposition 10.3 and Lemma 10.4, there exists $R_{\mathrm{res}} \subseteq E(G)$ such that

RES1 R_{res} is a $(\rho, \xi, \varepsilon, \mathcal{V})$-reservoir of Type i if $(\mathcal{H}, \phi)$ is of Type i, for $i \in \{\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}\}$, and

RES2 R_{res} contains a $(\rho_{\mathrm{abs}}, \xi, \xi, \varepsilon)$-absorber R_{abs} for $\mathcal{V}$. Moreover, if $(\mathcal{H}, \phi)$ is of Type A_2, then R_{abs} is lower $(\rho/2, \xi, G')$-regular, and if $(\mathcal{H}, \phi)$ is of Type B then $R_{\mathrm{res}} = R_{\mathrm{abs}}$.

Now let us define the 'defect' set S by

$$S := \begin{cases} U \setminus V^{(n-1)}(\mathcal{H}) & \text{if } (\mathcal{H}, \phi) \text{ is of Type } \mathrm{A}_1. \\ U & \text{if } (\mathcal{H}, \phi) \text{ is of Type } \mathrm{A}_2 \text{ or B.} \end{cases} \tag{11.3}$$

Since $|U| \geq (1 - 10\varepsilon)n$ holds only if ϕ is of Type A by (1.3) of Claim 1, and $\varepsilon \ll \rho$ holds if ϕ is of Type A, we can deduce the following.

$$\text{If } |U| > (1 - 10\varepsilon)n, \text{ then } |S| \geq (1 - \rho)n + 5\varepsilon n. \tag{11.4}$$

Step 3. *Define various subsets of colours.*

In this step, we will define various sets of colours, C_{med}, C_{diff}, C_{huge}, C_{main}, C_{buff}, C_{large} and C_{final}, such that $[n] = C_{\mathrm{med}} \cup C_{\mathrm{diff}} \cup C_{\mathrm{huge}} \cup C_{\mathrm{main}} \cup C_{\mathrm{buff}} \cup C_{\mathrm{final}}$.

In the following steps, roughly, our goal is to extend the colour classes $\phi^{-1}(c)$ for $c \in [n] \setminus C_{\rm final}$ in such a way that the maximum degree in the hypergraph of remaining uncoloured edges is at most $|C_{\rm final}|$. To that end, first, for each $c \in C_{\rm diff}$, we will extend the colour class $\phi^{-1}(c)$ to cover every vertex of $V^{(n-1)}(\mathcal{H})$ and all but at most five vertices of $V^{(n-2)}(\mathcal{H})$ using some edges of G. Then for each $c \in C_{\rm huge} \cup C_{\rm med}$ we will extend the colour class $\phi^{-1}(c)$ to cover all but at most one vertex of U using some edges of $R_{\rm abs}$, and finally, we will extend the colour classes $\phi^{-1}(c)$ for $c \in C_{\rm main} \cup C_{\rm buff}$ to contain all of the remaining edges in $\mathcal{H}_{\rm small} \setminus R_{\rm res}$, and the resulting colour classes are further extended using some edges of $R_{\rm abs}$.

Now we define the following parameters and sets of colours.

- Let $D := \lfloor (1-\rho)(n-1) \rfloor$ and $D' := \lfloor 10\gamma^{1/2} D \rfloor$.
- Let $C_{\rm med} \subseteq [n]$ be a set of at most γn colours such that $\phi(\mathcal{H}_{\rm med}) \subseteq C_{\rm med}$, which is guaranteed by (6.1:a)(ii) and (6.1:b)(ii) of Theorem 6.1.
- By (1.3) and (1.4) of Claim 1, there is at most one colour $c_{\rm diff} \in \phi(\mathcal{H}_{\rm huge}) \setminus C_{\rm med}$ such that $\phi^{-1}(c_{\rm diff})$ is difficult. If such a colour $c_{\rm diff}$ exists, then let $C_{\rm diff} := \{c_{\rm diff}\}$. Otherwise, let $C_{\rm diff} := \varnothing$.
- Let $C_{\rm huge} := \phi(\mathcal{H}_{\rm huge}) \setminus (C_{\rm med} \cup C_{\rm diff})$. Note that, since $\mathcal{H}$ is linear and for every $e \in C_{\rm huge}$, $|e| \geq \beta n/4$, we have

$$|C_{\rm huge} \cup C_{\rm diff}| \leq e(\mathcal{H}_{\rm huge}) \overset{(2.2)}{\leq} 8\beta^{-1}. \tag{11.5}$$

- Let $C_{\rm large} := \phi(\mathcal{H}_{\rm large}) \setminus (C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge})$.
- Let $C_{\rm main} \subseteq [n] \setminus (C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge})$ be a subset of size D that maximises $|C_{\rm large} \cap C_{\rm main}|$. Note that such a subset exists since $n - |C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge}| \geq n - \gamma n - 8\beta^{-1} \geq D$ by (11.1) and (11.5). In particular, if ϕ is of Type A, then $C_{\rm main} \supseteq C_{\rm large}$, since $|C_{\rm large}| \leq (1-\sigma)n < D$ by (6.1:a) of Theorem 6.1.
- Let $C_{\rm buff} \subseteq [n] \setminus (C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge} \cup C_{\rm main})$ be a subset of size D'. Note that such a subset exists since $n - |C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge} \cup C_{\rm main}| \geq n - \gamma n - 8\beta^{-1} - (1-\rho)n \geq \rho n/2 \geq D'$ by (11.5).
- Let $C_{\rm final} := [n] \setminus (C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge} \cup C_{\rm main} \cup C_{\rm buff})$.

We will use the following observations later.

T1 If ϕ is of Type A, then for any $c \in C_{\rm final}$, $\phi^{-1}(c) = \varnothing$, since $C_{\rm large} \subseteq C_{\rm main}$.

T2 If ϕ is of Type B, then for any $c \in C_{\rm final}$, $\phi^{-1}(c) \subseteq \mathcal{H}_{\rm large} \setminus \mathcal{H}_{\rm huge}$. Moreover, (6.1:b)(iii) of Theorem 6.1 implies that $|V(\phi^{-1}(c))| \leq \beta n$ for any $c \in C_{\rm final}$.

Also note that

$$(1-\rho)n \leq |C_{\rm med} \cup C_{\rm diff} \cup C_{\rm huge} \cup C_{\rm main} \cup C_{\rm buff}| \leq (1-\rho+15\gamma^{1/2})n, \tag{11.6}$$

since $|C_{\rm med}| \leq \gamma n$, $|C_{\rm main} \cup C_{\rm buff}| = D + D' \leq (1-\rho+10\gamma^{1/2})n$, and (11.5) holds.

Step 4. *Extend the colour classes in $\{\phi^{-1}(c) : c \in C_{\rm diff}\}$ using Lemma 7.12.*

In this step, for each $c \in C_{\rm diff}$, we extend the colour class $\phi^{-1}(c)$ to cover every vertex of $V^{(n-1)}(\mathcal{H})$ and all but at most five vertices of $V^{(n-2)}(\mathcal{H})$, by using only edges of G.

If $C_{\rm diff} \neq \varnothing$, then by (1.3) and (1.4) of Claim 1, $C_{\rm diff} = \{c_{\rm diff}\}$, ϕ is of Type A, and $\phi^{-1}(c_{\rm diff}) = \{e\}$ for some huge edge e. Applying Lemma 7.12 with $\phi^{-1}(c_{\rm diff})$ playing the role of M, either $\chi'(\mathcal{H}) \leq n$ or we have a set $\mathcal{M}_{\rm diff} := \{M_{c_{\rm diff}}\}$ such that the following holds.

D1 For each $c \in C_{\rm diff}$, $M_c \supseteq \phi^{-1}(c)$, $M_c \setminus \phi^{-1}(c) \subseteq E(G)$, M_c covers every vertex of $V^{(n-1)}(\mathcal{H})$, and $|V^{(n-2)}(\mathcal{H}) \setminus V(M_c)| \leq 5$.

Let us define

$$R_1 := R_{\rm abs} \setminus \bigcup_{c \in C_{\rm diff}} M_c \text{ and } S_1 := S \setminus \bigcup_{c \in C_{\rm diff}} (V^{(n-2)}(\mathcal{H}) \setminus V(M_c)).$$

Since $|C_{\rm diff}| \leq 1$ and **RES2** holds, by Observation 7.3, we have the following.

$$R_1 \text{ is a } (\rho_{\rm abs}, 10\xi, \xi, \varepsilon)\text{-absorber for } \mathcal{V}, \text{ so it is also a } (\rho_{\rm abs}, 3\gamma/2, \xi, \varepsilon)\text{-absorber for } \mathcal{V}. \tag{11.7}$$

$$\text{If } |U| > (1-10\varepsilon)n, \text{ then } |S_1| \overset{\textbf{D1}}{\geq} |S| - 5 \overset{(11.4)}{\geq} 2\varepsilon n > (\varepsilon + 3\gamma/2)n. \tag{11.8}$$

Step 5. *Extend the colour classes in $\{\phi^{-1}(c) : c \in C_{\rm huge} \cup C_{\rm med}\}$ using Lemma 7.11.*

In this step, for each $c \in C_{\rm huge} \cup C_{\rm med}$, we extend the colour class $\phi^{-1}(c)$ to cover all but at most one vertex of U, by using only edges in $R_1 \subseteq R_{\rm abs}$.

Combining (11.5) and the fact that $|C_{\rm med}| \leq \gamma n$, we have

$$|C_{\rm huge} \cup C_{\rm med}| \leq 3\gamma n/2. \tag{11.9}$$

First suppose $c \in C_{\text{huge}}$. Then we claim that $|V(\phi^{-1}(c)) \cap U| \le \varepsilon n$. Indeed, if ϕ is of Type A, then $\phi^{-1}(c)$ contains exactly one edge by (6.1:a)(i) of Theorem 6.1, so $|V(\phi^{-1}(c)) \cap U| \le \varepsilon n$ by (1.1) of Claim 1. Otherwise, if ϕ is of Type B, by (6.1:b)(i) of Theorem 6.1, we have $|V(\phi^{-1}(c))| \le \delta n < \varepsilon n$, so again $|V(\phi^{-1}(c)) \cap U| \le \varepsilon n$. Moreover, as noted in (11.7), R_1 is a $(\rho_{\text{abs}}, 3\gamma/2, \xi, \varepsilon)$-absorber for $\mathcal{V}$, and $U \cup V(\phi^{-1}(c)), U \setminus V(\phi^{-1}(c)), U, V(\mathcal{H}) \in \mathcal{V}$ by (11.2), so $(\mathcal{H}, \phi^{-1}(c), R_1, S_1)$ is $(\rho_{\text{abs}}, \varepsilon, 3\gamma/2, \kappa, \xi)$-absorbable by typicality of R_1 if $c \in C_{\text{huge}}$. Now suppose $c \in C_{\text{med}}$. Then $|V(\phi^{-1}(c))| \le \gamma n$ by (6.1:a)(ii) and (6.1:b)(ii) of Theorem 6.1. So, again, since R_1 is a $(\rho_{\text{abs}}, 3\gamma/2, \xi, \varepsilon)$-absorber for $\mathcal{V}$, and $U, V(\mathcal{H}) \in \mathcal{V}$ by (11.2), it follows that $(\mathcal{H}, \phi^{-1}(c), R_1, S_1)$ is $(\rho_{\text{abs}}, \varepsilon, 3\gamma/2, \kappa, \xi)$-absorbable by smallness of $\phi^{-1}(c)$ if $c \in C_{\text{med}}$. Moreover, since (11.8) and (11.9) hold, we can apply Lemma 7.11 with $\{\phi^{-1}(c) : c \in C_{\text{huge}} \cup C_{\text{med}}\}$, R_1, S_1, ρ_{abs}, and $3\gamma/2$ playing the roles of $\mathcal{N}$, R, S, ρ, and γ respectively, to obtain the set $\mathcal{M}_{\text{hm}} := \{M_c : c \in C_{\text{huge}} \cup C_{\text{med}}\}$ of pairwise edge-disjoint matchings in $\mathcal{H}$ such that the following hold.

HM1 For each $c \in C_{\text{huge}} \cup C_{\text{med}}$, M_c is edge-disjoint from the matchings in $\mathcal{M}_{\text{diff}}$, $M_c \supseteq \phi^{-1}(c)$ and $M_c \setminus \phi^{-1}(c) \subseteq R_{\text{abs}}$.

HM2 If $|U| \le (1 - 10\varepsilon)n$, then $\mathcal{M}_{\text{hm}}$ has perfect coverage of U. Otherwise, $\mathcal{M}_{\text{hm}}$ has nearly-perfect coverage of U with defects in S_1.

Now let us define

$$R_2 := R_1 \setminus \bigcup_{c \in C_{\text{huge}} \cup C_{\text{med}}} M_c \text{ and } S_2 := S_1 \setminus \bigcup_{c \in C_{\text{huge}} \cup C_{\text{med}}} (U \setminus V(M_c)).$$

By (11.7), (11.9), and Observation 7.3, the following hold.

(11.10) R_2 is a $(\rho_{\text{abs}}, 2\gamma, \xi, \varepsilon)$-absorber for $\mathcal{V}$.

(11.11) If $|U| > (1 - 10\varepsilon)n$, then $|S_2| \overset{(11.9)}{\ge} |S_1| - 3\gamma n/2 \overset{(11.8)}{\ge} |S| - 5 - 3\gamma n/2 \overset{(11.4)}{\ge} D + 4\varepsilon n$.

Step 6. *Colour most of the edges of $\mathcal{H}_{\text{small}} \setminus (R_{\text{res}} \cup \bigcup_{M \in \mathcal{M}_{\text{diff}}} M)$ by extending the colour classes in $\{\phi^{-1}(c) : c \in C_{\text{main}}\}$ using Lemma 8.2.*

In this step, we will first colour most of the edges in $\mathcal{H}_{\text{small}} \setminus (R_{\text{res}} \cup \bigcup_{M \in \mathcal{M}_{\text{diff}}} M)$ with colours from C_{main} by extending the colour classes in $\{\phi^{-1}(c) : c \in C_{\text{main}}\}$, and the resulting colour classes are further extended by using only edges of $R_2 \subseteq R_{\text{abs}}$. To do this, we will use Lemma 8.2. (Note that after this step there are only a few remaining uncoloured edges in $\mathcal{H}_{\text{small}} \setminus (R_{\text{res}} \cup \bigcup_{M \in \mathcal{M}_{\text{diff}}} M)$ incident to each vertex, which will be coloured in the next step.)

To be able to apply Lemma 8.2, we need to first embed $\mathcal{H}_{\text{small}} \setminus (R_{\text{res}} \cup \bigcup_{M \in \mathcal{M}_{\text{diff}}} M)$ into an almost-regular linear multi-hypergraph $\mathcal{H}^*$. In order to define $\mathcal{H}^*$, let $\mathcal{H}'$ be the linear multi-hypergraph obtained by applying Lemma 10.6 (Regularising lemma) with $\mathcal{H}$, R_{res}, and $\beta/2$ playing the roles of $\mathcal{H}$, R_{res}, and β, respectively. In particular, $\mathcal{H}'$ is a linear multi-hypergraph obtained from $\mathcal{H}_{\text{small}} \setminus R_{\text{res}}$ by adding singleton edges such that $\mathcal{H}' \subseteq \mathcal{H}_{\text{reg}}$ (which is defined in Definition 10.5). Now let $\mathcal{H}^* := \mathcal{H}' \setminus \bigcup_{M \in \mathcal{M}_{\text{diff}}} M$. Then it is clear that $\mathcal{H}^*$ can be obtained from $\mathcal{H}_{\text{small}} \setminus (R_{\text{res}} \cup \bigcup_{M \in \mathcal{M}_{\text{diff}}} M)$ by adding singleton edges, so we have the following.

(11.12) $\mathcal{H}^* \subseteq \mathcal{H}_{\text{reg}} \setminus R_{\text{res}} \subseteq \mathcal{H}_{\text{reg}} \setminus R_2$, and $d_{\mathcal{H}^*}(w) = (1 \pm 2\beta)D$ for any $w \in V(\mathcal{H})$.

Now we want to apply Lemma 8.2 with $\mathcal{H}_{\text{reg}}$, $\mathcal{H}^*$, $\{\phi^{-1}(c) : c \in C_{\text{main}}\}$, R_2, S_2, r_1, ρ_{abs}, 2β, and 2γ playing the roles of $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{M}$, R, S, r, ρ, β, and γ, respectively. To that end, we need to check that the assumptions **C1**–**C4** of Lemma 8.2 are satisfied.

First, (11.11) implies that if $|U| > (1-10\varepsilon)n$, then $|S_2| \ge D + 2\gamma n$, so **C1** holds. By (11.10) and (11.2), **C2** holds. Since every edge $e \in \mathcal{H}^*$ satisfies $|e| \le r_1$, **C3** follows from (11.12). Lastly, we show that **C4** holds. By the definition of C_{main}, for any $c \in C_{\text{main}}$, $\phi^{-1}(c)$ is either empty or is contained in $\mathcal{H}_{\text{large}}$, so $\phi^{-1}(c) \subseteq \mathcal{H}_{\text{reg}} \setminus (\mathcal{H}^* \cup R_2)$. Moreover, by (6.1:a)(iii) and (6.1:b)(iii) of Theorem 6.1, $|V(\phi^{-1}(c))| \le \beta n \le 2\beta D$. Furthermore, by (1.2) of Claim 1, for any $e \in \mathcal{H}^*$, $|\{c \in C_{\text{main}} : e \cap V(\phi^{-1}(c)) \neq \varnothing\}| \le 2r_1 n/r_0 \le 2\beta D$, as desired.

Thus, we can apply Lemma 8.2 to obtain a set $\mathcal{M}^*_{\text{main}} := \{M^*_c : c \in C_{\text{main}}\}$ of D edge-disjoint matchings in $\mathcal{H}_{\text{reg}}$ such that the following hold.

MA1* For any $c \in C_{\text{main}}$, we have $M^*_c \supseteq \phi^{-1}(c)$ and $M^*_c \setminus \phi^{-1}(c) \subseteq \mathcal{H}^* \cup R_2 \subseteq \mathcal{H}^* \cup R_{\text{abs}}$; in particular, M^*_c is edge-disjoint from all the matchings in $\mathcal{M}_{\text{diff}} \cup \mathcal{M}_{\text{hm}}$.

MA2* For any $w \in V(\mathcal{H})$,
- (i) $|E_{R_{\text{abs}}}(w) \cap \bigcup_{c \in C_{\text{main}}} M^*_c| \le 2\gamma D$, and
- (ii) $|E_{\mathcal{H}^*}(w) \setminus \bigcup_{c \in C_{\text{main}}} M^*_c| \le 2\gamma D$.

MA3* If $|U| \leq (1-2\varepsilon)n$, then $\mathcal{M}^*_{\mathrm{main}}$ has perfect coverage of U. Otherwise, $\mathcal{M}^*_{\mathrm{main}}$ has nearly-perfect coverage of U with defects in S_2.

For every $c \in C_{\mathrm{main}}$, let M_c be obtained from M^*_c by removing all singleton edges, and let $\mathcal{M}_{\mathrm{main}} := \{M_c : c \in C_{\mathrm{main}}\}$. Then, since $\mathcal{M}^*_{\mathrm{main}}$ is a set of matchings in $\mathcal{H}_{\mathrm{reg}} \supseteq \mathcal{H}$, $\mathcal{M}_{\mathrm{main}}$ is a set of matchings in $\mathcal{H}$. Since we obtain $\mathcal{H}_{\mathrm{small}} \setminus (R_{\mathrm{res}} \cup \bigcup_{M \in \mathcal{M}_{\mathrm{diff}}} M)$ after removing singleton edges from $\mathcal{H}^*$, **MA1*** and **MA2*** immediately imply **MA1** and **MA2** stated below. Let

$$\mathcal{H}_{\mathrm{rem}} := \mathcal{H}_{\mathrm{small}} \setminus (R_{\mathrm{res}} \cup \bigcup_{M \in \mathcal{M}_{\mathrm{diff}} \cup \mathcal{M}_{\mathrm{main}}} M) = \mathcal{H}_{\mathrm{small}} \setminus (R_{\mathrm{res}} \cup \bigcup_{c \in C_{\mathrm{diff}} \cup C_{\mathrm{main}}} M_c).$$

MA1 For any $c \in C_{\mathrm{main}}$, M_c is edge-disjoint from all the matchings in $\mathcal{M}_{\mathrm{diff}} \cup \mathcal{M}_{\mathrm{hm}}$, $M_c \supseteq \phi^{-1}(c)$, and $M_c \setminus \phi^{-1}(c) \subseteq \big(\mathcal{H}_{\mathrm{small}} \setminus (R_{\mathrm{res}} \cup \bigcup_{M \in \mathcal{M}_{\mathrm{diff}}} M)\big) \cup R_2$.

MA2 (i) For any $w \in V(\mathcal{H})$, $|E_{R_{\mathrm{abs}}}(w) \cap \bigcup_{c \in C_{\mathrm{main}}} M_c| \leq 2\gamma D$, and
(ii) $\Delta(\mathcal{H}_{\mathrm{rem}}) \leq 2\gamma D$.

Now let us define

$$R_3 := R_2 \setminus \bigcup_{c \in C_{\mathrm{main}}} M^*_c \text{ and } S_3 := S_2 \setminus \bigcup_{c \in C_{\mathrm{main}}} (U \setminus V(M^*_c)).$$

Since $|C_{\mathrm{main}}| = D$,

$$\text{if } |U| > (1-10\varepsilon)n, \text{ then } |S_3| \geq |S_2| - D \overset{(11.11)}{\geq} 4\varepsilon n. \tag{11.13}$$

Moreover, by (11.10), **MA2**(i) and Observation 7.3,

$$R_3 \text{ is a } (\rho_{\mathrm{abs}}, 4\gamma, \xi, \varepsilon)\text{-absorber for } \mathcal{V}. \tag{11.14}$$

Step 7. *Finish colouring the remaining uncoloured edges of $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}}$ by extending the colour classes in $\{\phi^{-1}(c) : c \in C_{\mathrm{buff}}\}$ using Lemma 8.3.*

Recall that $\mathcal{H}_{\mathrm{rem}} = \mathcal{H}_{\mathrm{small}} \setminus (R_{\mathrm{res}} \cup \bigcup_{c \in C_{\mathrm{diff}} \cup C_{\mathrm{main}}} M_c)$ is the hypergraph consisting of all the uncoloured edges in $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}}$. In this step, we will first colour all the edges of $\mathcal{H}_{\mathrm{rem}}$ with colours from the 'buffer' set C_{buff} by extending the colour classes in $\{\phi^{-1}(c) : c \in C_{\mathrm{buff}}\}$, and the resulting colour classes are further extended to cover all but at most one vertex of U, by using only edges of $R_3 \subseteq R_{\mathrm{abs}}$. To do this, we want to apply Lemma 8.3 with C_{buff}, $\{\phi^{-1}(c) : c \in C_{\mathrm{buff}}\}$, r_1, ρ_{abs}, R_3 and $10\gamma^{1/2}$ playing the roles of C, $\mathcal{M}$, r, ρ, R and γ, respectively. So now we check that the assumptions **L1**–**L5** of Lemma 8.3 are satisfied.

First, $|C_{\mathrm{buff}}| = D' = \lfloor 10\gamma^{1/2} D \rfloor$, so **L1** holds. For each $c \in C_{\mathrm{buff}}$, note that $\phi^{-1}(c)$ is either empty or is contained in $\mathcal{H}_{\mathrm{large}}$, so $|V(\phi^{-1}(c))| \leq \beta n \leq 5\gamma^{1/2} n$ by (6.1:a)(iii) or (6.1:b)(iii) of Theorem 6.1. Thus **L2** holds. Since (11.14) holds and $\{U, V(\mathcal{H})\} \subseteq \mathcal{V}$ by (11.2), it follows that **L3** holds. By (11.13), if $|U| > (1-10\varepsilon)n$, then $|S_3| \geq 4\varepsilon n \geq (10\gamma^{1/2} + \varepsilon)n$, so **L5** holds. Lastly, we show that **L4** holds. Note that **MA2**(ii) implies that $\Delta(\mathcal{H}_{\mathrm{rem}}) \leq 2\gamma D \leq (10\gamma^{1/2})^2 D/20$. Moreover, since for each $c \in C_{\mathrm{buff}}$, $\phi^{-1}(c)$ is either empty or is contained in $\mathcal{H}_{\mathrm{large}}$, we have $\mathcal{H}_{\mathrm{rem}} \subseteq \mathcal{H} \setminus (R_3 \cup \bigcup_{c \in C_{\mathrm{buff}}} \phi^{-1}(c))$, and for any $e \in \mathcal{H}_{\mathrm{rem}} \subseteq \mathcal{H}_{\mathrm{small}}$, we have $|\{c \in C_{\mathrm{buff}} : e \cap V(\phi^{-1}(c)) \neq \varnothing\}| \leq 2r_1 n/r_0 \leq (10\gamma^{1/2})^2 D/100$, by (1.2) of Claim 1, as desired.

Thus, by Lemma 8.3, we obtain a set of pairwise edge-disjoint matchings $\mathcal{M}_{\mathrm{buff}} := \{M_c : c \in C_{\mathrm{buff}}\}$ such that the following hold.

B1 For any $c \in C_{\mathrm{buff}}$, M_c is edge-disjoint from all the matchings in $\mathcal{M}_{\mathrm{diff}} \cup \mathcal{M}_{\mathrm{hm}} \cup \mathcal{M}_{\mathrm{main}}$, $M_c \supseteq \phi^{-1}(c)$ and $\mathcal{H}_{\mathrm{rem}} \subseteq \bigcup_{c \in C_{\mathrm{buff}}} (M_c \setminus \phi^{-1}(c)) \subseteq \mathcal{H}_{\mathrm{rem}} \cup R_3$. In particular, $\mathcal{H}_{\mathrm{small}} \setminus R_{\mathrm{res}} \subseteq \bigcup_{c \in C_{\mathrm{diff}} \cup C_{\mathrm{main}} \cup C_{\mathrm{buff}}} (M_c \setminus \phi^{-1}(c)) \subseteq \mathcal{H}_{\mathrm{small}}$.

B2 If $|U| \leq (1-10\varepsilon)n$, then $\mathcal{M}_{\mathrm{buff}}$ has perfect coverage of U. Otherwise, $\mathcal{M}_{\mathrm{buff}}$ has nearly-perfect coverage of U with defects in S_3.

Now we combine all the matchings constructed previously. Let us define

$$\mathcal{M}^*_{\mathrm{prev}} := \mathcal{M}_{\mathrm{diff}} \cup \mathcal{M}_{\mathrm{hm}} \cup \mathcal{M}^*_{\mathrm{main}} \cup \mathcal{M}_{\mathrm{buff}} \text{ and } \mathcal{M}_{\mathrm{prev}} := \mathcal{M}_{\mathrm{diff}} \cup \mathcal{M}_{\mathrm{hm}} \cup \mathcal{M}_{\mathrm{main}} \cup \mathcal{M}_{\mathrm{buff}},$$

where both $\mathcal{M}^*_{\mathrm{prev}}$ and $\mathcal{M}_{\mathrm{prev}}$ consist of pairwise edge-disjoint matchings by **HM1**, **MA1***, **MA1**, and **B1**. Let us also define

$$R_{\mathrm{final}} := R_{\mathrm{res}} \setminus \bigcup_{M \in \mathcal{M}_{\mathrm{prev}}} M, \quad G_{\mathrm{final}} := (V(\mathcal{H}), R_{\mathrm{final}}), \quad \text{and} \quad \mathcal{H}_{\mathrm{final}} := R_{\mathrm{final}} \cup \bigcup_{c \in C_{\mathrm{final}}} \phi^{-1}(c).$$

Step 8. *Analyse properties of G_{final} and $\mathcal{H}_{\mathrm{final}}$.*

In this step we will prove the following properties of G_{final} and $\mathcal{H}_{\mathrm{final}}$.

F1 $(1-\rho)n \leq |\mathcal{M}^*_{\mathrm{prev}}| = |\mathcal{M}_{\mathrm{prev}}| = n - |C_{\mathrm{final}}| \leq (1 - \rho + 15\gamma^{1/2})n$.

F2 If $(\mathcal{H},\phi)$ is of Type A_2, then $\Delta(G_{\mathrm{final}})-\delta(G_{\mathrm{final}})\leq 22\varepsilon n$, and G_{final} is lower $(\rho/4,\varepsilon^{1/3})$-regular.

F3 $\mathcal{H}_{\mathrm{final}}=\mathcal{H}\setminus\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M=\mathcal{H}\setminus\bigcup_{M\in\mathcal{M}^*_{\mathrm{prev}}}M$. Moreover, if ϕ is of Type A, $\mathcal{H}_{\mathrm{final}}=R_{\mathrm{final}}$.

First, since $\mathcal{M}_{\mathrm{prev}}=\{M_c : c\in C_{\mathrm{med}}\cup C_{\mathrm{diff}}\cup C_{\mathrm{huge}}\cup C_{\mathrm{main}}\cup C_{\mathrm{buff}}\}$ and $C_{\mathrm{final}}=[n]\setminus(C_{\mathrm{med}}\cup C_{\mathrm{diff}}\cup C_{\mathrm{huge}}\cup C_{\mathrm{main}}\cup C_{\mathrm{buff}})$, **F1** follows from (11.6).

Now we prove **F2**. First we show that for any $w\in V(\mathcal{H})$,

$$d_{R_{\mathrm{res}}\setminus R_{\mathrm{final}}}(w)\leq 15\gamma^{1/2}n. \tag{11.15}$$

Indeed, by **MA1** and **MA2**(i), $d_{R_{\mathrm{res}}\setminus R_{\mathrm{final}}}(w)\leq|\mathcal{M}_{\mathrm{diff}}|+|\mathcal{M}_{\mathrm{hm}}|+2\gamma n+|\mathcal{M}_{\mathrm{buff}}|$, and by (11.9), $|\mathcal{M}_{\mathrm{hm}}|\leq 3\gamma n/2$. The previous two inequalities together with the fact that $|\mathcal{M}_{\mathrm{diff}}|\leq 1$ and $|\mathcal{M}_{\mathrm{buff}}|=D'\leq 10\gamma^{1/2}n$, imply that (11.15) holds.

Now we show that $\Delta(G_{\mathrm{final}})-\delta(G_{\mathrm{final}})\leq 22\varepsilon n$. Indeed, since $R_{\mathrm{final}}\subseteq R_{\mathrm{res}}$ and R_{res} is a $(\rho,\xi,\varepsilon,\mathcal{V})$-reservoir of Type A_2, (R1) and (R2) of Definition 10.1 imply that $\Delta(G_{\mathrm{final}})\leq(\rho+\xi)n$. On the other hand, (R1) and (R2) of Definition 10.1 also imply that $d_{R_{\mathrm{res}}}(w)\geq(\rho-20\varepsilon)n$ for any $w\in V(\mathcal{H})$, so by (11.15), we have $d_{R_{\mathrm{final}}}(w)\geq d_{R_{\mathrm{res}}}(w)-15\gamma^{1/2}n\geq(\rho-20\varepsilon)n-15\gamma^{1/2}n\geq(\rho-21\varepsilon)n$ for any $w\in V(\mathcal{H})$, so $\delta(G_{\mathrm{final}})\geq(\rho-21\varepsilon)n$. Thus, $\Delta(G_{\mathrm{final}})-\delta(G_{\mathrm{final}})\leq(21\varepsilon+\xi)n\leq 22\varepsilon n$, as desired.

Now we show that G_{final} is lower $(\rho/4,\varepsilon^{1/3})$-regular. Since $(\mathcal{H},\phi)$ is of Type A_2, $\mathcal{H}$ is (ρ,ε)-full, so $|V(\mathcal{H})\setminus U|\leq 10\varepsilon n$. Thus, by (9.4.2), it suffices to show that $G_{\mathrm{final}}[U]$ is lower $(\rho/3,\varepsilon^{1/2})$-regular.

To that end, note that by **RES2**, R_{abs} is lower $(\rho/2,\xi,G')$-regular, so (11.15) and (9.4.1) imply that $R_{\mathrm{abs}}\cap R_{\mathrm{final}}$ is lower $(\rho/2,\gamma^{1/5},G')$-regular. Moreover, for any two disjoint sets $A,B\subseteq U$ with $|A|,|B|\geq\varepsilon^{1/2}|U|\geq\varepsilon^{1/2}n/2$, we have $e_{G'}(A,B)\geq(|A|-\varepsilon n)|B|\geq(1-2\varepsilon^{1/2})|A||B|$. Therefore, for any two disjoint sets $A,B\subseteq U$ with $|A|,|B|\geq\varepsilon^{1/2}|U|\geq\gamma^{1/5}n$, we have

$$e_{G_{\mathrm{final}}[U]}(A,B)\geq\frac{\rho}{2}e_{G'}(A,B)-\gamma^{1/5}|A||B|\geq\frac{\rho}{2}(1-2\varepsilon^{1/2})|A||B|-\gamma^{1/5}|A||B|\geq(\rho/3-\varepsilon^{1/2})|A||B|,$$

so $G_{\mathrm{final}}[U]$ is lower $(\rho/3,\varepsilon^{1/2})$-regular, completing the proof of **F2**.

Finally, we prove **F3**. Since $\mathcal{H}=\mathcal{H}_{\mathrm{small}}\cup(\mathcal{H}_{\mathrm{med}}\cup\mathcal{H}_{\mathrm{large}})$ and $\mathcal{H}_{\mathrm{final}}=R_{\mathrm{final}}\cup\bigcup_{c\in C_{\mathrm{final}}}\phi^{-1}(c)$, in order to show that $\mathcal{H}_{\mathrm{final}}=\mathcal{H}\setminus\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M$, it suffices to prove the following two statements.

$$\mathcal{H}_{\mathrm{small}}\setminus\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M=R_{\mathrm{final}}\quad\text{and}\quad(\mathcal{H}_{\mathrm{med}}\cup\mathcal{H}_{\mathrm{large}})\setminus\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M=\bigcup_{c\in C_{\mathrm{final}}}\phi^{-1}(c). \tag{11.16}$$

The first statement of (11.16) directly follows from the fact that $\mathcal{H}_{\mathrm{small}}\setminus R_{\mathrm{res}}\subseteq\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M$ which is guaranteed by **B1**. Now we show the second statement of (11.16). To that end, first note that **D1**, **HM1**, **MA1**, and **B1** together imply that for each $c\in[n]\setminus C_{\mathrm{final}}$, $\phi^{-1}(c)\subseteq M_c\in\mathcal{M}_{\mathrm{prev}}$, so $\bigcup_{c\in[n]\setminus C_{\mathrm{final}}}\phi^{-1}(c)\subseteq\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M$. Moreover, $\bigcup_{c\in[n]\setminus C_{\mathrm{final}}}(M_c\setminus\phi^{-1}(c))\subseteq\mathcal{H}_{\mathrm{small}}$, so $\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M\cap\bigcup_{c\in C_{\mathrm{final}}}\phi^{-1}(c)=\varnothing$. This proves that (11.16) holds, showing that $\mathcal{H}_{\mathrm{final}}=\mathcal{H}\setminus\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M$. Since $\mathcal{H}$ has no singleton edges, it immediately follows that $\mathcal{H}\setminus\bigcup_{M\in\mathcal{M}_{\mathrm{prev}}}M=\mathcal{H}\setminus\bigcup_{M\in\mathcal{M}^*_{\mathrm{prev}}}M$. Moreover, if ϕ is of Type A, then by **T1**, $\bigcup_{c\in C_{\mathrm{final}}}\phi^{-1}(c)=\varnothing$, so $\mathcal{H}_{\mathrm{final}}=R_{\mathrm{final}}$. This completes the proof of **F3**.

Step 9. *Bound the degrees of vertices in* $\mathcal{H}_{\mathrm{final}}$.

In this step we prove the following statements when bounding the number of colours used in the final step.

UC1 For any $w\in V(\mathcal{H})\setminus U$, $d_{\mathcal{H}_{\mathrm{final}}}(w)\leq|C_{\mathrm{final}}|-\rho\varepsilon n/4$.

UC2 For any $w\in U\setminus V^{(n-1)}(\mathcal{H})$, $d_{\mathcal{H}_{\mathrm{final}}}(w)\leq|C_{\mathrm{final}}|-1$.

UC3 For any $w\in V^{(n-1)}(\mathcal{H})$, $d_{\mathcal{H}_{\mathrm{final}}}(w)\leq|C_{\mathrm{final}}|$. Moreover, there are at least $|C_{\mathrm{final}}|$ vertices of degree less than $|C_{\mathrm{final}}|$ in $\mathcal{H}_{\mathrm{final}}$.

UC4 If $(\mathcal{H},\phi)$ is either Type A_1 or B, then for any $w\in V^{(n-1)}(\mathcal{H})$, $d_{\mathcal{H}_{\mathrm{final}}}(w)\leq|C_{\mathrm{final}}|-1$.

Now we prove **UC1**. First we show that for any $w\in V(\mathcal{H})\setminus U$, $d_{R_{\mathrm{res}}}(w)\leq\rho(1-\varepsilon)n+\xi n$. Indeed, if R_{res} is a $(\rho,\xi,\varepsilon,\mathcal{V})$-reservoir of Type A_1 or A_2, then it follows from (P1) or (R2). Otherwise, if R_{res} is a $(\rho,\xi,\varepsilon,\mathcal{V})$-reservoir of Type B, then it also follows since R_{res} is (ρ,ξ,G')-typical with respect to $\mathcal{V}\ni V(\mathcal{H})$. Thus, for any $w\in V(\mathcal{H})\setminus U$,

$$d_{G_{\mathrm{final}}}(w)\leq d_{R_{\mathrm{res}}}(w)\leq\rho(1-\varepsilon)n+\xi n\overset{\mathbf{F1}}{\leq}n-|\mathcal{M}_{\mathrm{prev}}|-\rho\varepsilon n/2\overset{\mathbf{F1}}{=}|C_{\mathrm{final}}|-\rho\varepsilon n/2. \tag{11.17}$$

Since w is incident to at most $2n/r_0\leq\rho\varepsilon n/4$ edges of $\mathcal{H}_{\mathrm{large}}$, and $\mathcal{H}_{\mathrm{final}}\setminus R_{\mathrm{final}}=\bigcup_{c\in C_{\mathrm{final}}}\phi^{-1}(c)\subseteq\mathcal{H}_{\mathrm{large}}$ by **T1** and **T2**, we deduce that $d_{\mathcal{H}_{\mathrm{final}}\setminus R_{\mathrm{final}}}(w)\leq\rho\varepsilon n/4$. This together with (11.17) implies that for any $w\in V(\mathcal{H})\setminus U$, $d_{\mathcal{H}_{\mathrm{final}}}(w)=d_{G_{\mathrm{final}}}(w)+d_{\mathcal{H}_{\mathrm{final}}\setminus R_{\mathrm{final}}}(w)\leq|C_{\mathrm{final}}|-\rho\varepsilon n/4$, proving **UC1**.

Before proving **UC2**, **UC3**, and **UC4**, we need to collect some facts. For any $w\in V(\mathcal{H})$, let $m(w)$ be the number of the matchings in $\mathcal{M}^*_{\mathrm{prev}}$ not covering w. Since $\mathcal{M}^*_{\mathrm{prev}}$ is a set of edge-disjoint matchings in

$\mathcal{H}_{\mathrm{reg}}$, and $\mathcal{H}_{\mathrm{reg}}$ is a linear multi-hypergraph obtained from $\mathcal{H}$ by adding $\max(0, n-3-d_{\mathcal{H}}(w))$ singleton edges incident to each vertex $w \in V(\mathcal{H})$ (where $\mathcal{H}$ has no singleton edges), we deduce that all but at most $m(w) + \max(0, n-3-d_{\mathcal{H}}(w))$ matchings in $\mathcal{M}_{\mathrm{prev}}$ cover w. Moreover, by **F3**, $\mathcal{H}_{\mathrm{final}} = \mathcal{H} \setminus \bigcup_{M \in \mathcal{M}_{\mathrm{prev}}} M$, so for any $w \in V(\mathcal{H})$ we have

$$\begin{aligned} d_{\mathcal{H}_{\mathrm{final}}}(w) &\le d_{\mathcal{H}}(w) - (|\mathcal{M}_{\mathrm{prev}}| - m(w) - \max(0, n-3-d_{\mathcal{H}}(w))) \\ &\overset{\mathbf{F1}}{=} |C_{\mathrm{final}}| + m(w) - \min(n - d_{\mathcal{H}}(w), 3). \end{aligned} \tag{11.18}$$

Recall from Step 4 that $S_1 = S \setminus \bigcup_{c \in C_{\mathrm{diff}}} (V^{(n-2)}(\mathcal{H}) \setminus V(M_c))$. Note that **D1** implies that every matching in $\mathcal{M}_{\mathrm{diff}}$ covers all of the vertices in $V^{(n-1)}(\mathcal{H})$, and **HM2**, **MA3***, and **B2** imply that $\mathcal{M}_{\mathrm{hm}} \cup \mathcal{M}^*_{\mathrm{main}} \cup \mathcal{M}_{\mathrm{buff}} = \mathcal{M}^*_{\mathrm{prev}} \setminus \mathcal{M}_{\mathrm{diff}}$ has nearly-perfect coverage of U with defects in S_1. In particular, every matching in $\mathcal{M}^*_{\mathrm{prev}} \setminus \mathcal{M}_{\mathrm{diff}}$ covers all the vertices in $U \setminus S_1$, every matching in $\mathcal{M}_{\mathrm{diff}}$ covers all the vertices in $V^{(n-1)}(\mathcal{H}) \setminus S_1$ and $|\mathcal{M}_{\mathrm{diff}}| \le 1$. Thus, we have

$$m(u) \le 1 \text{ for any } u \in U \setminus S_1, \text{ and } m(v) = 0 \text{ for any } v \in V^{(n-1)}(\mathcal{H}) \setminus S_1. \tag{11.19}$$

Now note that every vertex in S_1 is covered by all but at most one matching in $\mathcal{M}^*_{\mathrm{prev}} \setminus \mathcal{M}_{\mathrm{diff}}$, every matching in $\mathcal{M}_{\mathrm{diff}}$ covers all the vertices in $V_+^{(n-2)}(\mathcal{H}) \cap S_1$ and $|\mathcal{M}_{\mathrm{diff}}| \le 1$. So we have $m(u) \le 2$ for any $u \in S_1$ and $m(v) \le 1$ for any $v \in V_+^{(n-2)}(\mathcal{H}) \cap S_1$. Combining this with (11.19), we deduce that

$$m(u) \le 2 \text{ for any } u \in U \setminus V_+^{(n-2)}(\mathcal{H}), \text{ and } m(v) \le 1 \text{ for any } v \in V_+^{(n-2)}(\mathcal{H}). \tag{11.20}$$

Finally, if $|U| \le (1-10\varepsilon)n$ then $\mathcal{M}^*_{\mathrm{prev}} \setminus \mathcal{M}_{\mathrm{diff}}$ has perfect coverage of U by **HM2**, **MA3***, and **B2**. This combined with the fact that every matching in $\mathcal{M}_{\mathrm{diff}}$ covers all the vertices in $V^{(n-1)}(\mathcal{H})$ and $|\mathcal{M}_{\mathrm{diff}}| \le 1$, implies that

$$\text{if } |U| \le (1-10\varepsilon)n, \text{ then } m(u) \le 1 \text{ for } u \in U \setminus V^{(n-1)}(\mathcal{H}), \text{ and } m(v) = 0 \text{ for } v \in V^{(n-1)}(\mathcal{H}). \tag{11.21}$$

Now we are ready to prove **UC2**, **UC3**, and **UC4**. Note that (11.20) and (11.18) together imply that for any $v \in U \setminus V^{(n-1)}(\mathcal{H})$, $d_{\mathcal{H}_{\mathrm{final}}}(v) \le |C_{\mathrm{final}}| - 1$, thus **UC2** holds.

Now we prove **UC3**. Note that (11.20) and (11.18) together imply that for any $v \in V^{(n-1)}(\mathcal{H})$, $d_{\mathcal{H}_{\mathrm{final}}}(v) \le |C_{\mathrm{final}}|$. To prove the second statement of **UC3**, we will bound the number of vertices $v \in V^{(n-1)}(\mathcal{H})$ satisfying $m(v) = 0$. Since $\mathcal{M}^*_{\mathrm{prev}} \setminus \mathcal{M}_{\mathrm{diff}}$ has nearly-perfect coverage of U with defects in S_1, every matching in $\mathcal{M}^*_{\mathrm{prev}} \setminus \mathcal{M}_{\mathrm{diff}}$ covers all but at most one vertex in U. Moreover, every matching in $\mathcal{M}_{\mathrm{diff}}$ covers all of the vertices in $V^{(n-1)}(\mathcal{H})$. Thus, by **F1**, there are at most $|\mathcal{M}_{\mathrm{prev}}| = n - |C_{\mathrm{final}}|$ vertices $v \in V^{(n-1)}(\mathcal{H})$ satisfying $m(v) \ge 1$, so every other vertex of $V^{(n-1)}(\mathcal{H})$ has degree less than $|C_{\mathrm{final}}|$ by (11.18). This fact combined with **UC1** and **UC2** implies that there are at least $|C_{\mathrm{final}}|$ vertices with degree less than $|C_{\mathrm{final}}|$ in $\mathcal{H}_{\mathrm{final}}$, proving **UC3**.

Finally, if $(\mathcal{H}, \phi)$ is of Type A_1 then $S_1 \subseteq S = U \setminus V^{(n-1)}(\mathcal{H})$ by (11.3), so by (11.19) every vertex $v \in V^{(n-1)}(\mathcal{H})$ satisfies $m(v) = 0$. On the other hand, if $(\mathcal{H}, \phi)$ is of Type B, then by (1.3) of Claim 1, $|U| \le 2\delta n$, so again every vertex $v \in V^{(n-1)}(\mathcal{H})$ satisfies $m(v) = 0$ by (11.21). Thus, in either case we have $d_{\mathcal{H}_{\mathrm{final}}}(v) \le |C_{\mathrm{final}}| - 1$ by (11.18), proving **UC4**.

Step 10. *Colour $\mathcal{H}_{\mathrm{final}}$ with colours in C_{final}.*

We divide the proof into three cases depending on the type of $(\mathcal{H}, \phi)$. In each case, it suffices to show that there is a set $\mathcal{M}_{\mathrm{final}} = \{M_c : c \in C_{\mathrm{final}}\}$ of edge-disjoint matchings in $\mathcal{H}_{\mathrm{final}}$ such that for every $c \in C_{\mathrm{final}}$, $\phi^{-1}(c) \subseteq M_c$ and $\bigcup_{c \in C_{\mathrm{final}}} M_c = \mathcal{H}_{\mathrm{final}}$. Indeed, then $\mathcal{H} = \bigcup_{M \in \mathcal{M}_{\mathrm{prev}} \cup \mathcal{M}_{\mathrm{final}}} M$ by **F3**, so $\mathcal{M}_{\mathrm{prev}} \cup \mathcal{M}_{\mathrm{final}}$ would be the desired set of n pairwise edge-disjoint matchings in $\mathcal{H}$, proving Theorem 1.1.

Case 1: $(\mathcal{H}, \phi)$ is of Type A_1.

Note that in this case $\mathcal{H}_{\mathrm{final}} = R_{\mathrm{final}}$ by **F3**, and recall that $G_{\mathrm{final}} = (V(\mathcal{H}), R_{\mathrm{final}})$. Thus, by **UC1**, **UC2**, and **UC4**, $\Delta(G_{\mathrm{final}}) \le |C_{\mathrm{final}}| - 1$. Applying Vizing's theorem (Theorem 4.5) to G_{final}, we obtain a set $\mathcal{M}_{\mathrm{final}} = \{M_c : c \in C_{\mathrm{final}}\}$ of edge-disjoint matchings in G_{final} such that $\bigcup_{c \in C_{\mathrm{final}}} M_c = R_{\mathrm{final}} = \mathcal{H}_{\mathrm{final}}$. Moreover, by **T1**, for any $c \in C_{\mathrm{final}}$, $\phi^{-1}(c) = \varnothing$, so $\phi^{-1}(c) \subseteq M_c$ trivially holds, as desired. This completes the proof of Theorem 1.1 in the case when $(\mathcal{H}, \phi)$ is of Type A_1.

Case 2: $(\mathcal{H}, \phi)$ is of Type A_2.

Note that in this case $\mathcal{H}_{\mathrm{final}} = R_{\mathrm{final}}$ by **F3**. Thus, by **UC1**, **UC2**, and **UC3**, $\Delta(G_{\mathrm{final}}) \le |C_{\mathrm{final}}|$, and

$$\text{there are at least } |C_{\mathrm{final}}| \text{ vertices having degree less than } |C_{\mathrm{final}}| \text{ in } G_{\mathrm{final}}. \tag{11.22}$$

If $\Delta(G_{\text{final}}) \leq |C_{\text{final}}| - 1$, then we may apply Vizing's theorem (Theorem 4.5) to G_{final} to obtain the desired set $\mathcal{M}_{\text{final}} = \{M_c : c \in C_{\text{final}}\}$ of edge-disjoint matchings where for any $c \in C_{\text{final}}$, $\phi^{-1}(c) \subseteq M_c$, and $\bigcup_{c\in C_{\text{final}}} M_c = R_{\text{final}} = \mathcal{H}_{\text{final}}$ as in the previous case.

Otherwise, if $\Delta(G_{\text{final}}) = |C_{\text{final}}|$, then by (11.22) and **F2**, we can apply Corollary 9.6 with $\rho/4$, $\varepsilon^{1/3}$, 22ε, and G_{final} playing the roles of p, ε, η, and G, respectively, to obtain a set $\mathcal{M}_{\text{final}} = \{M_c : c \in C_{\text{final}}\}$ of edge-disjoint matchings such that $\bigcup_{c\in C_{\text{final}}} M_c = R_{\text{final}} = \mathcal{H}_{\text{final}}$. Moreover, by **T1**, for any $c \in C_{\text{final}}$, $\phi^{-1}(c) = \varnothing$, so $\phi^{-1}(c) \subseteq M_c$ trivially holds, as desired. This completes the proof of Theorem 1.1 in the case when $(\mathcal{H}, \phi)$ is of Type A_2.

Case 3: $(\mathcal{H}, \phi)$ is of Type B.

First, for each $w \in V(\mathcal{H})$, let us define $C_w := \{c \in C_{\text{final}} : w \in V(\phi^{-1}(c))\}$. Note that by definition, $\mathcal{H}_{\text{final}} \setminus R_{\text{final}} = \bigcup_{c\in C_{\text{final}}} \phi^{-1}(c)$, and by **T2**, for any $c \in C_{\text{final}}$, $\phi^{-1}(c) \subseteq \mathcal{H}_{\text{large}}$. Thus, for any vertex $w \in V(\mathcal{H})$, we have

$$|C_w| = d_{\mathcal{H}_{\text{final}}\setminus R_{\text{final}}}(w) \leq 2\delta n, \tag{11.23}$$

since ϕ is a proper edge-colouring, and w is incident to at most $2n/r_0 \leq 2\delta n$ edges of $\mathcal{H}_{\text{large}}$. Note that **UC1**, **UC2**, and **UC4** imply $\Delta(\mathcal{H}_{\text{final}}) \leq |C_{\text{final}}| - 1$, so for any vertex $w \in V(\mathcal{H})$, we have

$$d_{R_{\text{final}}}(w) = d_{\mathcal{H}_{\text{final}}}(w) - d_{\mathcal{H}_{\text{final}}\setminus R_{\text{final}}}(w) \overset{(11.23)}{\leq} |C_{\text{final}}| - 1 - |C_w|. \tag{11.24}$$

Now we apply Lemma 9.2 with G_{final}, C_{final}, and 2δ playing the roles of H, C, and δ, respectively. To that end, we need to check that the assumptions (i)–(iv) of Lemma 9.2 are satisfied. First, by **F1**, $|C_{\text{final}}| \geq (\rho - 15\gamma^{1/2})n \geq 14\delta n$. By (11.24), (i) of Lemma 9.2 holds. Now, by (1.3) of Claim 1, $|U| \leq 2\delta n$, and by Definition 7.2(i) and **RES2**, $R_{\text{final}} \subseteq R_{\text{res}} = R_{\text{abs}} \subseteq E(G')$. Thus, every edge of G_{final} is incident to a vertex of U by the definition of G' given before **Step 1**, so (ii) of Lemma 9.2 holds. By (11.23), (iii) of Lemma 9.2 holds. Finally, by **T2**, for any $c \in C_{\text{final}}$, $|\{w \in V(\mathcal{H}) : c \in C_w\}| = |V(\phi^{-1}(c))| \leq \beta n \leq 2\delta n$, so (iv) of Lemma 9.2 holds.

Hence, by applying Lemma 9.2, we obtain a proper edge-colouring $\psi : R_{\text{final}} \to C_{\text{final}}$ such that every $uv \in R_{\text{final}}$ satisfies $\psi(uv) \notin C_u \cup C_v$, implying that $V(\psi^{-1}(c)) \cap V(\phi^{-1}(c)) = \varnothing$ for every $c \in C_{\text{final}}$. Now let $M_c := \psi^{-1}(c) \cup \phi^{-1}(c)$ for each $c \in C_{\text{final}}$. Then the set $\mathcal{M}_{\text{final}} := \{M_c : c \in C_{\text{final}}\}$ consists of pairwise edge-disjoint matchings such that $\phi^{-1}(c) \subseteq M_c$ for each $c \in C_{\text{final}}$, and $\bigcup_{c\in C_{\text{final}}} M_c = R_{\text{final}} \cup \bigcup_{c\in C_{\text{final}}} \phi^{-1}(c) = \mathcal{H}_{\text{final}}$, as desired. This completes the proof of Theorem 1.1. □

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