Decompositions of dense graphs into small subgraphs

Deryk Osthus

joint work with Ben Barber, Daniela Kühn, Allan Lo

University of Birmingham

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When does $G$ have an $F$-decomposition?
Necessary conditions

Question

When does \( G \) have an \( F \)-decomposition?

If \( G \) has a triangle decomposition, then

(a) the number of edges of \( G \) is divisible by 3;
(b) every vertex has even degree.
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(a) the number of edges of $G$ is divisible by 3;
(b) every vertex has even degree.

A necessary condition

If $G$ has an $F$-decomposition, then

(a) the number of edges in $F$ divides the number of edges in $G$;
(b) $\gcd(F)$ divides $\gcd(G)$, where $\gcd(H)$ is the largest integer dividing the degree of every vertex of a graph $H$.

$G$ is said to be $F$-divisible if $G$ satisfies (a) and (b).
### $F$-divisibility

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(b) $\gcd(F)$ divides $\gcd(G)$, where $\gcd(H)$ is the largest integer dividing the degree of every vertex of a graph $H$
F-divisibility

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$F$-divisibility is not sufficient for $F$-decomposition.

The problem of deciding whether a graph $G$ has an $F$-decomposition is NP-complete if $F$ contains a connected component with at least 3 edges.
Decompositions of complete host graphs

Theorem (Kirkman 1847)
Every triangle-divisible $K_n$ has a triangle decomposition. (i.e. $n \equiv 1, 3 \mod 6$)

Theorem (Wilson 1975)
For $n$ large, every $F$-divisible $K_n$ has an $F$-decomposition.

Generalization to hypergraph cliques:
Theorem (Keevash 2014)
For $r \leq q \ll n$, every complete $r$-uniform hypergraph on $n$ vertices $K^{(r)}_n$ (subject to the necessary divisibility conditions) has a $K^{(r)}_q$-decomposition.
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Conjecture (Nash-Williams 1970)

Every large triangle-divisible graph $G$ on $n$ vertices with $\delta(G) \geq 3n/4$ has a triangle decomposition.

Conjecture generalizes to $K_r$-decompositions.
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conjecture generalizes to $K_r$-decompositions

Extremal example: blow up each vertex of $C_4$ to a $K_m$ ($m$ odd and divisible by 3).

Each triangle has at least one edge in one of the four cliques but less than a third of the edges lie inside the cliques.
Theorem (Gustavsson 1991, Keevash 2014)$^+$

For every graph $F$, there exist $\varepsilon$ and $n_0$ such that every $F$-divisible graph $G$ on $n \geq n_0$ vertices with $\delta(G) \geq (1 - \varepsilon)n$ has an $F$-decomposition.

For $F = K_r$, Gustavsson states $\varepsilon = 10^{-37} r^{-94}$. 
Decompositions of graphs of large minimum degree

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Theorem (Yuster 2002)
Let $F$ be a bipartite graph with $\delta(F) = 1$. If $G$ is $F$-divisible and $\delta(G) \geq \left( \frac{1}{2} + o(1) \right) n$, then $G$ has an $F$-decomposition.

Theorem (Bryant and Cavenagh 2014⁺)
If $G$ is $C_4$-divisible and $\delta(G) \geq \left( \frac{31}{32} + o(1) \right) n$, then $G$ has a $C_4$-decomposition.
Fractional $F$-decomposition of $G$: give every copy of $F$ in $G$ a weight $w(F) \in [0, 1]$ such that $\sum_{F:e \in E(F)} w(F) = 1$ for each edge $e$ of $G$.
**Fractional decompositions**

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![Diagram of fractional decomposition]

**fractional decomposition threshold $\delta_{frac}(F)$:** smallest $c \in [0, 1]$ s.t. every large graph $G$ with $\delta(G) \geq cn$ has fractional $F$-decomposition.
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**Fractional $K_3$-decomposition threshold:**
- Garaschuk (2014): $\delta_{frac}(K_3) \leq 0.956$
- Dross (2015$^+$): $\delta_{frac}(K_3) \leq 0.9$

**Fractional $K_r$-decomposition threshold:**
- Yuster (2005): $\delta_{frac}(K_r) \leq 1 - \frac{1}{9r^{10}}$
- Dukes (2012): $\delta_{frac}(K_r) \leq 1 - \frac{1}{16r^4}$
- Barber, Kühn, Lo, Montgomery, Osthus (2015$^+$): $\delta_{frac}(K_r) \leq 1 - \frac{1}{10^4r^{3/2}}$
Theorem (Barber, Kühn, Lo, Osthus 2014\textsuperscript{+})

Every large triangle-divisible graph $G$ with $\delta(G) \geq (\delta_{\text{frac}}(K_3) + o(1))n$ has a triangle decomposition.
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If $F$ is $r$-regular, then every large $F$-divisible graph $G$ with $\delta(G) \geq (\max\{\delta_{frac}(F), 1 - 1/3r\} + o(1))n$ has a $F$-decomposition.
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Corollary

- Every large triangle-divisible graph $G$ with $\delta(G) \geq (0.9 + o(1))n$ has a triangle decomposition.
- Every large $K_r$-divisible graph $G$ with $\delta(G) \geq (1 - 1/10^4 r^{3/2})n$ has a $K_r$-decomposition.
From fractional to ‘real’ decompositions

Theorem (Barber, Kühn, Lo, Osthus 2014+)

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- Every large $F$-divisible graph $G$ with $\delta(G) \geq (1 - c/|F|^2)n$ has an $F$-decomposition.
Proof idea

**Theorem (Barber, Kühn, Lo, Osthus 2014+)**

Every large triangle-divisible graph $G$ with $\delta(G) \geq (\delta_{\frac{\Delta}{3}}(K_3) + o(1))n$ has a triangle decomposition.

Will use:

**Theorem (Haxell and Rödl 2001)**

Every large graph $G$ with $\delta(G) \geq \delta_{\frac{\Delta}{3}}(K_3)n$ can be decomposed into edge-disjoint copies of $K_3$ and a remainder $R$ with $\epsilon n^2$ uncovered edges.

Problem: What to do with leftover?
Proof idea

Theorem (Barber, Kühn, Lo, Osthus 2014+)

Every large triangle-divisible graph $G$ with $\delta(G) \geq \left( \delta_{frac}(K_3) + o(1) \right)n$ has a triangle decomposition.

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Problem: What to do with leftover?
Planning ahead

Plan

1. Take out highly structured subgraph $A$.
2. Take out many triangles to leave a sparse remainder $R$.
3. Use ‘structure’ of $A$ to decompose $A \cup R$ into triangles.

Assume now that all graphs are triangle-divisible.

Definition

An absorber is a graph $A$ such that $A \cup R$ has a triangle decomposition for any sparse graph $R$.

An absorber for a graph $R$ is a graph $A_R$ such that $A_R$ and $A_R \cup R$ both have a triangle decomposition.

Approach: take $A$ to be union of $A_R$ over all possible sparse remainders $R$.

Far more than $n^2$ possibilities for $R$, so no hope of finding one absorber for each $R$. 
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Far more than $n^2$ possibilities for $R$, so no hope of finding one absorber for each $R$. 
Absorbers

**Aim**

Reduce the number of possible remainders $R$.

Let $m$ be a large integer and equipartition the vertex set into $V_1, \ldots, V_{\frac{n}{m}}$ each of size $m$. Can we ensure that every edge of $R$ is contained within some $V_i$?
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Lemma

Yes.
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Lemma

Yes.

Let \( R_i \) be the part of \( R \) contained within \( V_i \).

For each \( i \), there are at most \( 2^{\binom{m}{2}} \) possibilities for \( R_i \).

So we only need to find \( 2^{\binom{m}{2}} \frac{n}{m} = O(n) \) absorbers.
Definition

A graph $T$ is an $(H_1, H_2)$-transformer if both $H_1 \cup T$ and $T \cup H_2$ have triangle decompositions.
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Proposition
If $H_2$ has a triangle decomposition, then $T \cup H_2$ is an absorber for $H_1$.  

$H_1 \leftrightarrow H_2$ if there is an $(H_1, H_2)$-transformer.  

Useful fact $\leftrightarrow$ is symmetric $\leftrightarrow$ is transitive—if $H_1 \leftrightarrow H_2 \leftrightarrow H_3$, then $H_1 \leftrightarrow H_3$.  

Transformers

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Suppose $H_1$ is isomorphic to $H_2$. 

$H_1 = C_9$

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This allows us to ‘move graphs around’.
Identify the green vertices of $H_2$. 

Note that $T$ is still an $(H_1, H_2)$-transformer. So we can 'identify vertices' (providing no multiple edges are created). Since $\leftrightarrow$ is symmetric, we can 'split vertices' (providing the resulting graph is still triangle-divisible).
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Identifying vertices

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Subdividing an edge

Let $xy$ be an edge.
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- Attach a triangle to \( x \)
Let $xy$ be an edge.

1. Attach a triangle to $x$
2. Split the vertex $x$. 

We can subdivide $xy$ into a path of length 4. So we can identify vertices; split a vertex; subdivide an edge.
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- identify vertices;
- split a vertex;
- subdivide an edge.
Suppose that $e(H) = 3k$. 

$H$
Constructing an absorber for a given $H$

Suppose that $e(H) = 3k$.

1. Subdivide all edges of $H$.

2. Identify all original vertices of $H$.

3. Let $J$ be a union of $k$ vertex-disjoint triangles.

4. Subdivide all edges of $J$.

5. Identify all original vertices of $J$.

6. Since $\leftrightarrow$ is transitive, $H \leftrightarrow J$.

Thus $H$ has an absorber.
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![Diagram showing the construction process](image)
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Open problem: better fractional thresholds

Theorem (Barber, Kühn, Lo, Osthus 2014+)

Every large triangle-divisible graph $G$ with $\delta(G) \geq (\delta_{frac}(K_3) + o(1))n$ has a triangle decomposition.

Problem

Determine $\delta_{frac}(F)$, i.e. the minimum degree threshold for a graph $G$ to have a fractional $F$-decomposition.

• For triangles, showing that $\delta_{frac}(K_3) = 3/4$ could be combined with our results to show the actual 'decomposition threshold' is $(3/4 + o(1))n$.

• Actually, showing that $3n/4$ guarantees '(fractional) almost decomposition' would suffice.
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- Actually, showing that $3n/4$ guarantees ‘(fractional) almost decomposition’ would suffice.
Theorem (Barber, Kühn, Lo, Osthus 2014+)

For $n$ sufficiently large, every $C_\ell$-divisible graph $G$ on $n$ vertices with

$$\delta(G) \geq \begin{cases} 
\left(\frac{2}{3} + o(1)\right)n & \text{if } \ell = 4, \\
\left(\frac{1}{2} + o(1)\right)n & \text{if } \ell \geq 6 \text{ is even},
\end{cases}$$

has a $C_\ell$-decomposition.

asymptotically best possible

$$\ell \geq 6 \text{ even}$$

$$n/2 \quad n/2$$

$$\delta = n/2 - 1$$

neither component is $C_\ell$-divisible, but entire graph is
Decompositions into even cycles

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asymptotically best possible

$$\ell = 4 \text{ (Kahn & Winkler)}$$

$\delta = 3n/5 - 1$, odd number of edges in blown-up $C_5$
Decompositions into even cycles

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asymptotically best possible

\[
\ell = 4 \text{ (Taylor)}
\]

\[
\delta = 2n/3 - 2
\]

every \( C_4 \) has even number of edges inside \( A \), but \( e(A) \) is odd