Decompositions of dense graphs into small subgraphs

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joint work with Ben Barber, Daniela Kühn, Allan Lo

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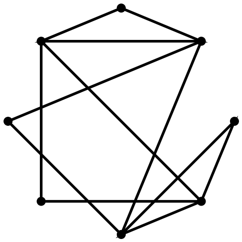
F-decompositions

Definition

A graph G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F.



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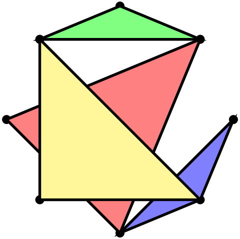

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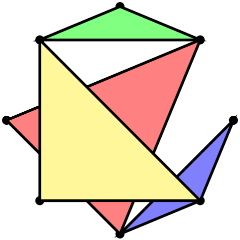

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A necessary condition

If G has an F-decomposition, then

(a) the number of edges in F divides the number of edges in G;

(b) $\gcd(F)$ divides $\gcd(G)$, where $\gcd(H)$ is the largest integer dividing the degree of every vertex of a graph H.

G is said to be *F-divisible* if G satisfies (a) and (b).

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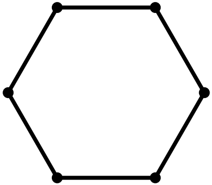

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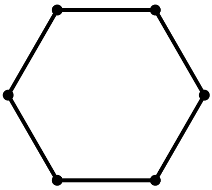

F-divisiblity is not sufficient for F-decomposition.

The problem of deciding whether a graph G has an F-decomposition is NP-complete if F contains a connected component with at least 3 edges.

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For n large, every F-divisible K_n has an F-decomposition.

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Generalization to hypergraph cliques:

Theorem (Keevash 2014$^+$)

For $r \leq q \ll n$, every complete r-uniform hypergraph on n vertices $K_n^{(r)}$ (subject to the necessary divisibility conditions) has a $K_q^{(r)}$-decomposition.

Triangle decompositions of graphs of large minimum degree

Conjecture (Nash-Williams 1970)

Every large triangle-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a triangle decomposition.

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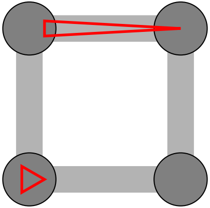
Extremal example: blow up each vertex of C_4 to a K_m (m odd and divisible by 3).



Each triangle has at least one edge in one of the four cliques but less than a third of the edges lie inside the cliques.

Decompositions of graphs of large minimum degree

Theorem (Gustavsson 1991, Keevash 2014$^+$)

For every graph F, there exist ε and n_0 such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1-\varepsilon)n$ has an F-decomposition.

For $F = K_r$, Gustavsson states $\varepsilon = 10^{-37} r^{-94}$.

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Theorem (Yuster 2002)

Let F be a bipartite graph with $\delta(F) = 1$. If G is F-divisible and $\delta(G) \geq \left(\frac{1}{2} + o(1)\right) n$, then G has an F-decomposition.

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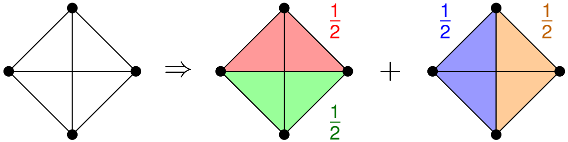
Theorem (Bryant and Cavenagh 2014$^+$)

If G is C_4-divisible and $\delta(G) \geq \left(\frac{31}{32} + o(1)\right) n$, then G has a C_4-decomposition.

Fractional decompositions

fractional F-decomposition of G: give every copy of F in G a weight $w(F) \in [0,1]$ such that $\sum_{F: e \in E(F)} w(F) = 1$ for each edge e of G



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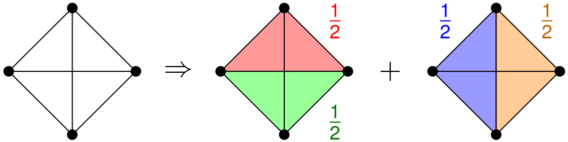

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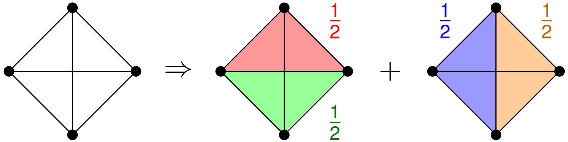

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fractional K_3-decomposition threshold:
Garaschuk (2014): $\delta_{frac}(K_3) \leq 0.956$
Dross (2015$^+$): $\delta_{frac}(K_3) \leq 0.9$

fractional K_r-decomposition threshold:
Yuster (2005): $\delta_{frac}(K_r) \leq 1 - \frac{1}{9r^{10}}$
Dukes (2012): $\delta_{frac}(K_r) \leq 1 - \frac{1}{16r^4}$
Barber, Kühn, Lo, Montgomery, Osthus (2015$^+$): $\delta_{frac}(K_r) \leq 1 - \frac{1}{10^4 r^{3/2}}$

From fractional to ‘real’ decompositions

Theorem (Barber, Kühn, Lo, Osthus 2014^{+})

Every large triangle-divisible graph G with $\delta(G) \geq (\delta_{frac}(K_3) + o(1))n$ has a triangle decomposition.

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Corollary

- Every large triangle-divisible graph G with $\delta(G) \geq (0.9 + o(1))n$ has a triangle decomposition.
- Every large K_r-divisible graph G with $\delta(G) \geq (1 - 1/10^4 r^{3/2})n$ has a K_r-decomposition.

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- Every large F-divisible graph G with $\delta(G) \geq (1 - c/|F|^2)n$ has an F-decomposition.

Proof idea

Theorem (Barber, Kühn, Lo, Osthus 2014$^+$)

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Will use:

Theorem (Haxell and Rödl 2001)

Every large graph G with $\delta(G) \geq \delta_{frac}(K_3)n$ can be decomposed into edge-disjoint copies of K_3 and a remainder R with εn^2 uncovered edges.

Problem: What to do with leftover?

Planning ahead

Plan

1. Take out highly structured subgraph A.
2. Take out many triangles to leave a sparse remainder R.
3. Use ‘structure’ of A to decompose $A \cup R$ into triangles.

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Approach: take A to be union of A_R over all possible sparse remainders R.

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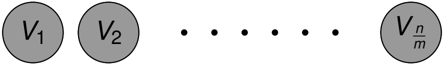
Far more than n^2 possibilities for R, so no hope of finding one absorber for each R.

Absorbers

Aim

Reduce the number of possible remainders R.



Let m be a large integer and equipartition the vertex set into $V_1, \ldots, V_{\frac{n}{m}}$ each of size m. Can we ensure that every edge of R is contained within some V_i?

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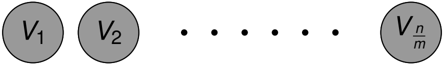

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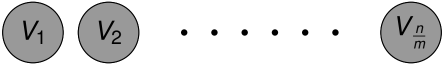

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Lemma

Yes.

Let R_i be the part of R contained within V_i.
For each i, there are at most $2^{\binom{m}{2}}$ possibilities for R_i.
So we only need to find $2^{\binom{m}{2}} n/m = O(n)$ absorbers.

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Useful fact

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Useful fact

- $\leftrightarrow$ is symmetric
- $\leftrightarrow$ is transitive—if $H_1 \leftrightarrow H_2 \leftrightarrow H_3$, then $H_1 \leftrightarrow H_3$

Moving

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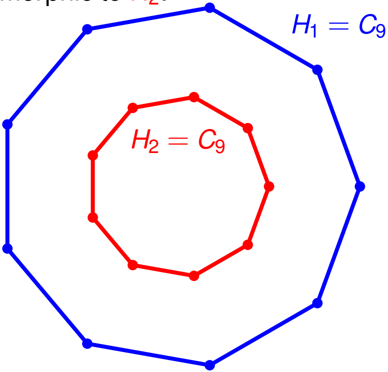
Suppose H_1 is isomorphic to H_2.



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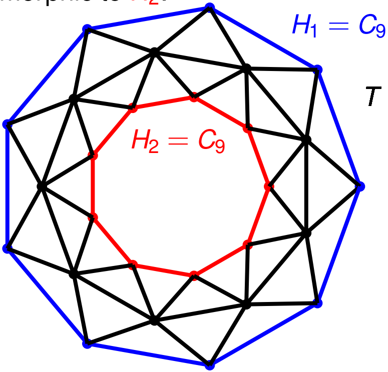

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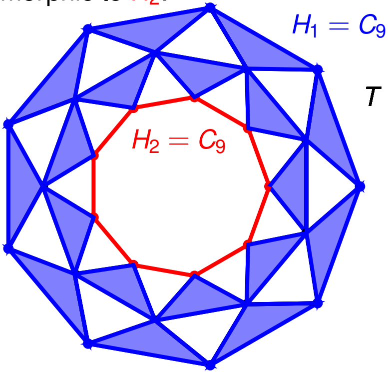

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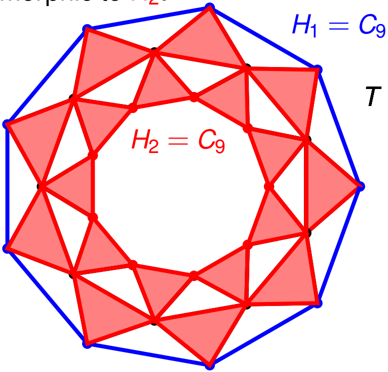

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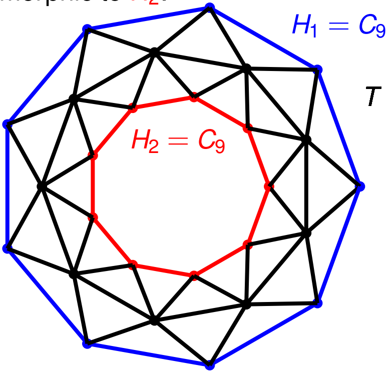
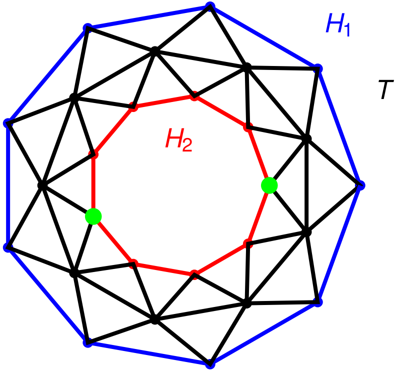


This allows us to 'move graphs around'.

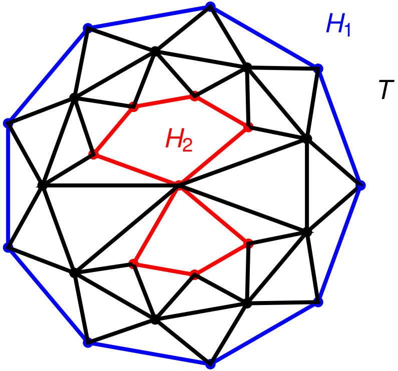
Identifying vertices

Identify the green vertices of H_2.



Identifying vertices

Identify the green vertices of H_2.



Note that T is still an (H_1, H_2)-transformer.

So we can ‘identify vertices’ (providing no multiple edges are created).

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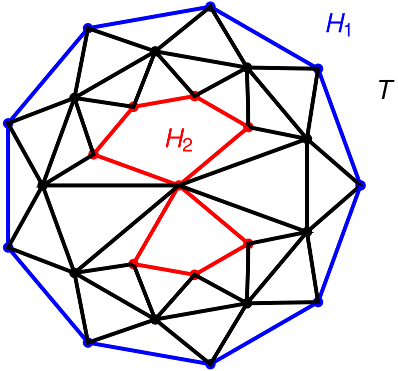
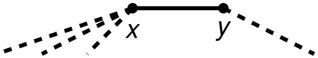


Note that T is still an (H_1, H_2)-transformer.
So we can 'identify vertices' (providing no multiple edges are created).
Since $\leftrightarrow$ is symmetric, we can 'split vertices' (providing the resulting graph is still triangle-divisible).

Subdividing an edge

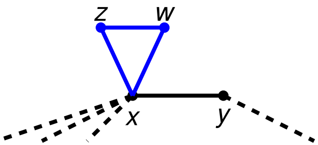
Let xy be an edge.



Subdividing an edge

Let xy be an edge.

1. Attach a triangle to x



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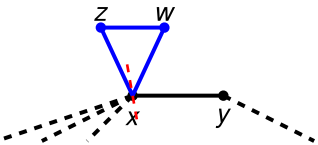

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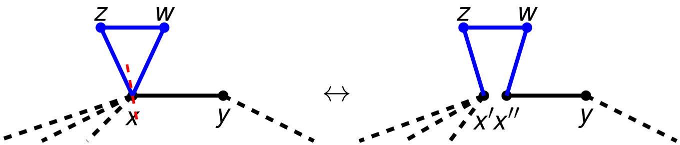

We can subdivide xy into a path of length 4.

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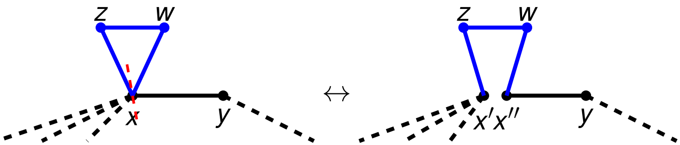


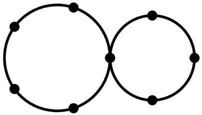
We can subdivide xy into a path of length 4.

So we can

- identify vertices;
- split a vertex;
- subdivide an edge.

Constructing an absorber for a given H

Suppose that $e(H) = 3k$.

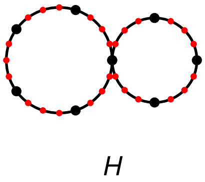


H

Constructing an absorber for a given H

Suppose that $e(H) = 3k$.

1. Subdivide all edges of H.



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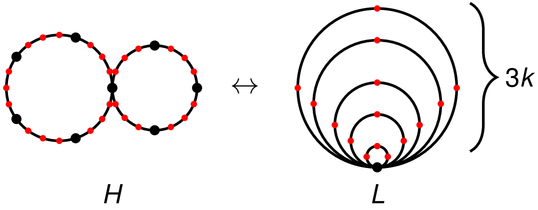
2. Identify all original vertices of H.



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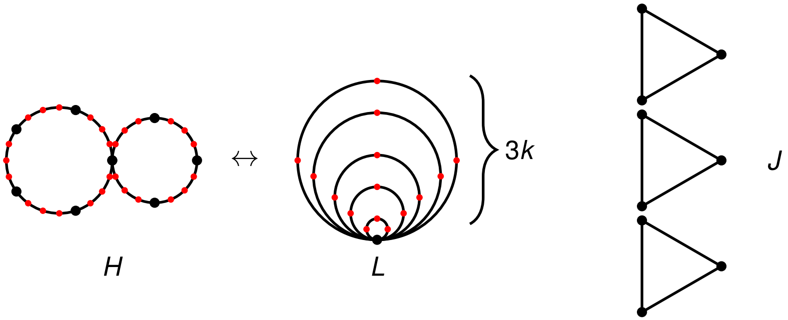
3. Let J be a union of k vertex-disjoint triangles.



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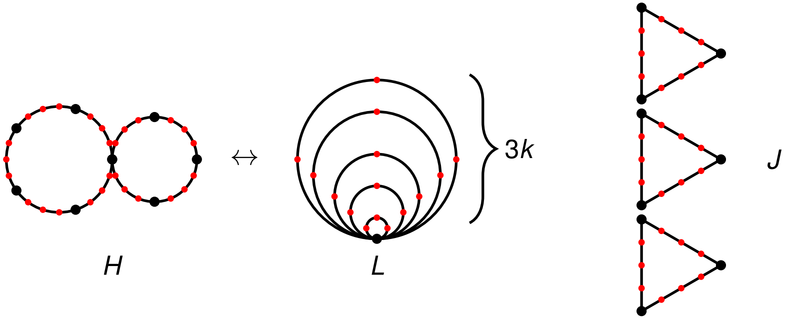
2. Identify all original vertices of H.
3. Let J be a union of k vertex-disjoint triangles.
4. Subdivide all edges of J.



Constructing an absorber for a given H

Suppose that $e(H) = 3k$.

1. Subdivide all edges of H.
2. Identify all original vertices of H.
3. Let J be a union of k vertex-disjoint triangles.
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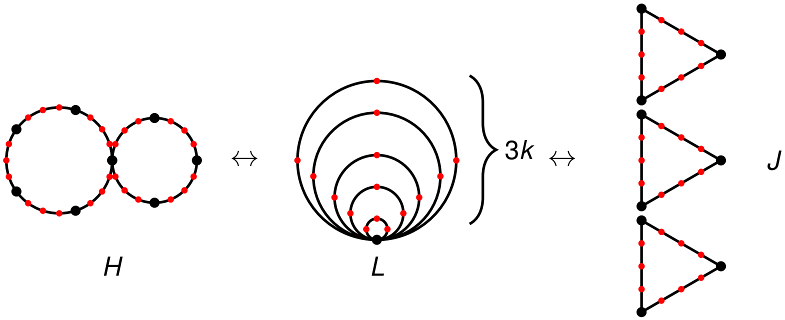

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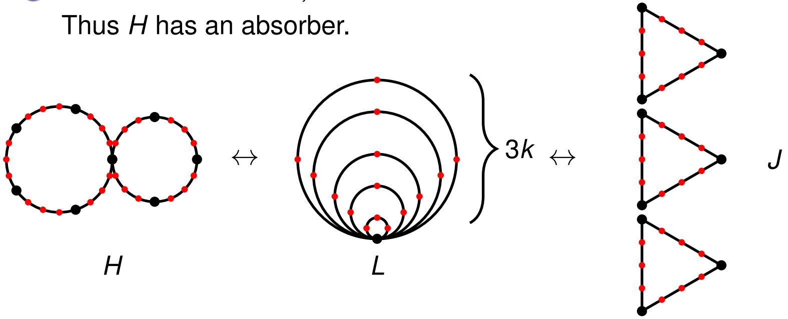
6. Since $\leftrightarrow$ is transitive, $H \leftrightarrow J$.
   Thus H has an absorber.



Open problem: better fractional thresholds

Theorem (Barber, Kühn, Lo, Osthus 2014$^+$)

Every large triangle-divisible graph G with $\delta(G) \geq (\delta_{frac}(K_3) + o(1))n$ has a triangle decomposition.

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Problem

Determine $\delta_{frac}(F)$, i.e. the minimum degree threshold for a graph G to have a fractional F-decomposition.

• For triangles, showing that $\delta_{frac}(K_3) = 3/4$ could be combined with our results to show the actual 'decomposition threshold' is $(3/4 + o(1))n$.

• Actually, showing that $3n/4$ guarantees '(fractional) almost decomposition' would suffice.

Decompositions into even cycles

Theorem (Barber, Kühn, Lo, Osthus 2014^{+})

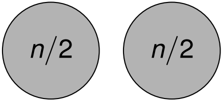
For n sufficiently large, every C_ℓ-divisible graph G on n vertices with

$$\delta(G) \geq \begin{cases} \left(\frac{2}{3} + o(1)\right) n & \text{if } \ell = 4, \\ \left(\frac{1}{2} + o(1)\right) n & \text{if } \ell \geq 6 \text{ is even,} \end{cases}$$

has a C_ℓ-decomposition.

asymptotically best possible

$\ell \geq 6$ even



$$\delta = n/2 - 1$$

neither component is C_ℓ-divisible, but entire graph is

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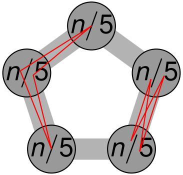
$\ell = 4$ (Kahn & Winkler)



$\delta = 3n/5 - 1$, odd number of edges in blown-up C_5

Decompositions into even cycles

Theorem (Barber, Kühn, Lo, Osthus 2014$^+$)

For n sufficiently large, every C_ℓ-divisible graph G on n vertices with

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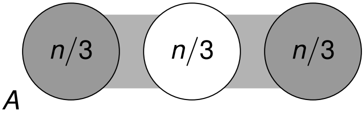
$\ell = 4$ (Taylor)



$\delta = 2n/3 - 2$

every C_4 has even number of edges inside A, but $e(A)$ is odd