On the decomposition threshold of a graph

Deryk Osthus

University of Birmingham

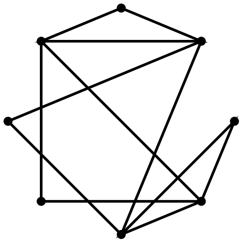
joint work with
Ben Barber, Stefan Glock, Daniela Kühn, ‘Allan Lo, Richard Montgomery, Amelia Taylor

May 2016

F-decompositions

Definition

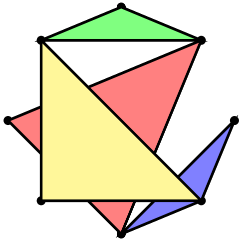
A graph G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F.



F-decompositions

Definition

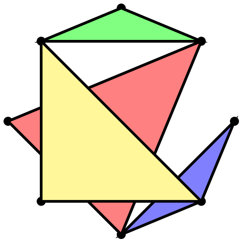
A graph G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F.



F-decompositions

Definition

A graph G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F.



Question

When does G have an F-decomposition?

Necessary conditions

Question

When does G have an F-decomposition?

If G has a triangle decomposition, then

(a) the number of edges of G is divisible by 3;

(b) every vertex has even degree.

Necessary conditions

Question

When does G have an F-decomposition?

If G has a triangle decomposition, then

(a) the number of edges of G is divisible by 3;

(b) every vertex has even degree.

A necessary condition

If G has an F-decomposition, then

(a) the number of edges in F divides the number of edges in G;

(b) $\gcd(F)$ divides $\gcd(G)$, where $\gcd(H)$ is the largest integer dividing the degree of every vertex of a graph H.

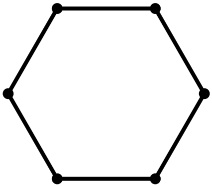
G is said to be ***F*-divisible** if G satisfies (a) and (b).

F-divisibility

F-divisibility

(a) the number of edges in F divides the number of edges in G

(b) $\gcd(F)$ divides $\gcd(G)$, where $\gcd(H)$ is the largest integer dividing the degree of every vertex of a graph H

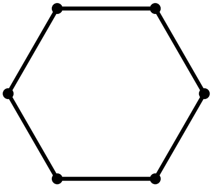


F-divisibility

F-divisibility

(a) the number of edges in F divides the number of edges in G

(b) $\gcd(F)$ divides $\gcd(G)$, where $\gcd(H)$ is the largest integer dividing the degree of every vertex of a graph H



F-divisiblity is not sufficient for F-decomposition.

The problem of deciding whether a graph G has an F-decomposition is NP-complete if F contains a connected component with at least 3 edges.

Decompositions of complete host graphs

Theorem (Kirkman 1847)

Every triangle-divisible K_n has a triangle decomposition. (i.e. $n \equiv 1, 3 \mod 6$)

Decompositions of complete host graphs

Theorem (Kirkman 1847)

Every triangle-divisible K_n has a triangle decomposition. (i.e. $n \equiv 1, 3 \mod 6$)

Theorem (Wilson 1975)

For n large, every F-divisible K_n has an F-decomposition.

Decompositions of complete host graphs

Theorem (Kirkman 1847)

Every triangle-divisible K_n has a triangle decomposition. (i.e. $n \equiv 1, 3 \mod 6$)

Theorem (Wilson 1975)

For n large, every F-divisible K_n has an F-decomposition.

Generalization to hypergraph cliques:

Theorem (Keevash 2014^{+})

For $r \leq q \ll n$, every complete r-uniform hypergraph on n vertices $K_n^{(r)}$ (subject to the necessary divisibility conditions) has a $K_q^{(r)}$-decomposition.

Triangle decompositions of graphs of large minimum degree

Conjecture (Nash-Williams 1970)

Every large triangle-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a triangle decomposition.

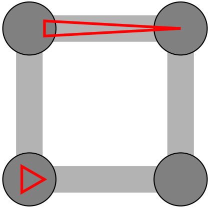
conjecture generalizes to K_r-decompositions

Triangle decompositions of graphs of large minimum degree

Conjecture (Nash-Williams 1970)

Every large triangle-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a triangle decomposition.

conjecture generalizes to K_r-decompositions
Extremal example: blow up each vertex of C_4 to a K_m (m odd and divisible by 3).



Each triangle has at least one edge in one of the four cliques but less than a third of the edges lie inside the cliques.

The decomposition threshold

Question

What if the host graph G is not complete?

The decomposition threshold

Question

What if the host graph G is not complete?

Decomposition threshold:

Definition

For a given graph F, let $\delta_{dec}(F)$ denote the smallest $\delta \in [0, 1]$ such that every F-divisible graph G with $\delta(G) \geq (\delta + o(1))|G|$ has an F-decomposition.

The decomposition threshold

Question

What if the host graph G is not complete?

Decomposition threshold:

Definition

For a given graph F, let $\delta_{dec}(F)$ denote the smallest $\delta \in [0, 1]$ such that every F-divisible graph G with $\delta(G) \geq (\delta + o(1))|G|$ has an F-decomposition.

Conjecture (Nash-Williams 1970)

$\delta_{dec}(K_3) = 3/4$.

Clique decompositions of graphs of large minimum degree

Definition

For a given graph F, let $\delta_{dec}(F)$ denote the smallest $\delta \in [0,1]$ such that every F-divisible graph G with $\delta(G) \geq (\delta + o(1))|G|$ has an F-decomposition.

Conjecture

$\delta_{dec}(K_r) = \frac{r}{r+1}$.

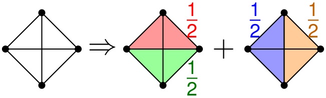
Theorem (Gustavsson 1991)

$\delta_{dec}(K_r) \leq 1 - 10^{-37} r^{-94}$

Yuster: $\delta_{dec}(tree) = 1/2$
Bryant & Cavenagh: $\delta_{dec}(C_4) \leq 31/32$

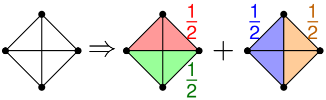
Fractional decompositions

fractional F-decomposition of G: give every copy of F in G a weight $w(F) \in [0,1]$ such that $\sum_{F: e \in E(F)} w(F) = 1$ for each edge e of G



Fractional decompositions

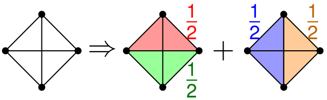
fractional F-decomposition of G: give every copy of F in G a weight $w(F) \in [0,1]$ such that $\sum_{F: e \in E(F)} w(F) = 1$ for each edge e of G



fractional decomposition threshold $\delta_{frac}(F)$: smallest $c \in [0,1]$ s.t. every large graph G with $\delta(G) \geq cn$ has fractional F-decomposition

Fractional decompositions

fractional F-decomposition of G: give every copy of F in G a weight $w(F) \in [0,1]$ such that $\sum_{F: e \in E(F)} w(F) = 1$ for each edge e of G



fractional decomposition threshold $\delta_{frac}(F)$: smallest $c \in [0,1]$ s.t. every large graph G with $\delta(G) \geq cn$ has fractional F-decomposition

fractional K_3-decomposition threshold:

Garaschuk (2014): $\delta_{frac}(K_3) \leq 0.956$

Dross (2015$^+$): $\delta_{frac}(K_3) \leq 0.9$

fractional K_r-decomposition threshold:

Yuster (2005): $\delta_{frac}(K_r) < 1 - \frac{1}{9r^{10}}$

Dukes (2012): $\delta_{frac}(K_r) < 1 - \frac{1}{16r^4}$

Barber, Kühn, Lo, Montgomery, Osthus (2015$^+$): $\delta_{frac}(K_r) < 1 - \frac{1}{10^4 r^{3/2}}$

From fractional to ‘real’ decompositions

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16$^+$)

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.

From fractional to ‘real’ decompositions

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16^{+})

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.
- $\delta_{dec}(F) \leq \max\{\delta_{frac}(F), 1 - \frac{1}{\chi(F)+1}\}$

From fractional to ‘real’ decompositions

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16^{+})

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.
- $\delta_{dec}(F) \leq \max\{\delta_{frac}(F), 1 - \frac{1}{\chi(F)+1}\}$

From fractional to ‘real’ decompositions

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16$^+$)

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.
- $\delta_{dec}(F) \leq \max\{\delta_{frac}(F), 1 - \frac{1}{\chi(F)+1}\}$

Corollary

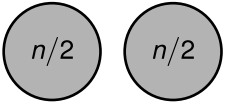
- $\delta_{dec}(K_r) \leq 0.9$
- $\delta_{dec}(K_r) \leq 1 - \frac{1}{10^4 r^{3/2}}$
- $\delta_{dec}(F) \leq 1 - \frac{1}{10^4 \chi(F)^{3/2}}$

Bipartite graphs

Theorem (Barber, Kühn, Lo, Osthus)

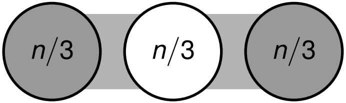
- $\delta_{dec}(C_4) = 2/3$
- $\delta_{dec}(C_{2\ell}) = 1/2$ for every $\ell \geq 3$.

$C_{2\ell}, \ell \geq 3$



$\delta = n/2 - 1$
neither component is
C_ℓ-divisible

C_4



$\delta = 2/3$

Theorem (Glock, Kühn, Lo, Montgomery, Osthus)

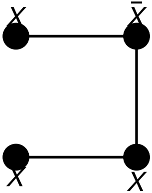
Let F be bipartite and connected. Then

$$\delta_{dec}(F) = \begin{cases} 1/2 & \text{if } \tau_{hcf}(F) = 1 \\ 2/3 & \text{otherwise} \end{cases}$$

In particular,

- $\delta_{dec}(K_{r,r}) = 2/3$ for $r \geq 2$
- $\delta_{dec}(K_{r,r+1}) = 1/2$ for $r \geq 2$
- $\delta_{dec}(\text{tree}) = 1/2$ (Yuster)
- $\delta_{dec}(Q_3) = 2/3$

A set $X \in V(F)$ is called C_4-supporting in F if there exist distinct $a, b \in X$ and $c, d \in V(F) \setminus X$ such that $ac, bd, cd \in E(F)$.

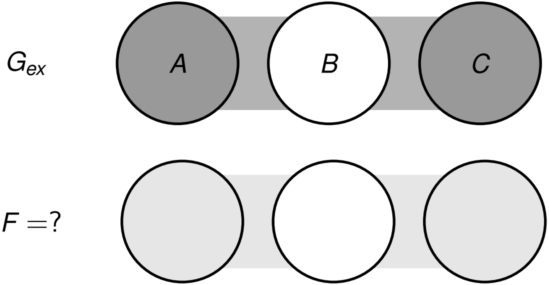


We define

$$\tau_{hcf}(F) := gcd\{e(F[X]) : X \in V(F) \text{ is not } C_4\text{-supporting in } F\}$$

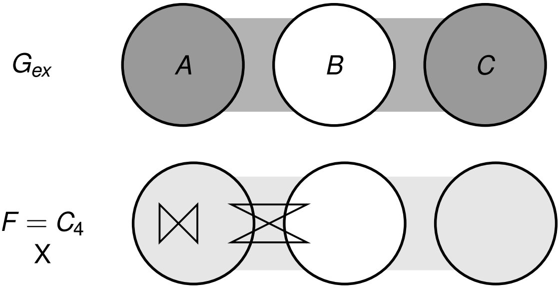
Motivation

$\tau_{hcf}(F) = 1$ if and only if there copies $F_1, \ldots, F_s$ of F in G_{ex} such that $\gcd\{e(F_1[A]), \ldots, e(F_s[A]) := \}1$.



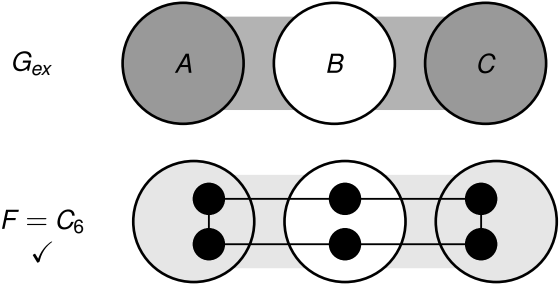
Motivation

$\tau_{hcf}(F) = 1$ if and only if there copies $F_1, \ldots, F_s$ of F in G_{ex} such that $\gcd\{e(F_1[A]), \ldots, e(F_s[A]) :=\}1$.



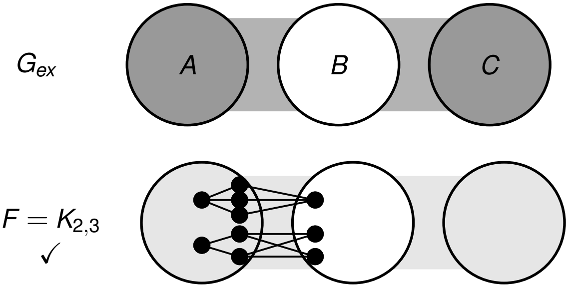
Motivation

$\tau_{hcf}(F) = 1$ if and only if there copies $F_1, \ldots, F_s$ of F in G_{ex} such that $\gcd\{e(F_1[A]), \ldots, e(F_s[A]) \; : =\}1$.



Motivation

$\tau_{hcf}(F) = 1$ if and only if there copies $F_1, \ldots, F_s$ of F in G_{ex} such that $\gcd\{e(F_1[A]), \ldots, e(F_s[A]) \; := \}1$.



Proof idea

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16$^+$)

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.

Proof idea

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16^{+})

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.
- $\delta_{dec}(F) \leq \max\{\delta_{frac}(F), 1 - \frac{1}{\chi(F)+1}\}$

Proof idea

Theorem (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2014/16$^+$)

- $\delta_{dec}(K_r) = \delta_{frac}(K_r)$.
- $\delta_{dec}(F) \leq \max\{\delta_{frac}(F), 1 - \frac{1}{\chi(F)+1}\}$

Use the existence of an approximate decomposition as a black box

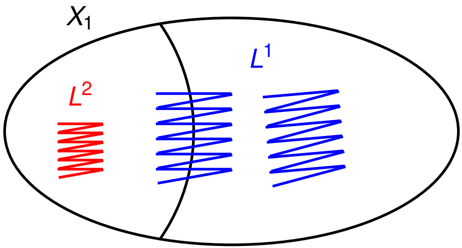
Theorem (Haxell and Rödl 2001)

Every large graph G with $\delta(G) \geq \delta_{frac}(F)n$ can be decomposed into edge-disjoint copies of F and a remainder R with εn^2 uncovered edges.

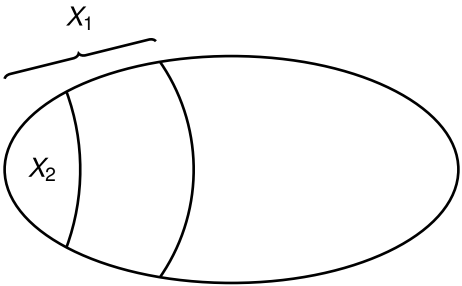
Absorption

1.) Remove sparse absorbing graph A from G,
2.) find approximate F-decomposition of $G - A$, call leftover L,
3.) hope that $L \cup A$ has an F-decomposition.
    Difficult! Use iterative absorption approach. Split up the absorbing process into many steps which gradually make leftover smaller and smaller.

Given an approximate decomposition with leftover L^1, construct 'lazy cleaner' graph L^1_{clean} so that $L^1 \cup L^1_{clean}$ contains copies $\mathscr{F}^1$ of F so that leftover edges $L^2 = L^1 \cup L^1_{clean} - \mathscr{F}^1$ lie entirely in X_1.



Repeat with X_2 inside X_1



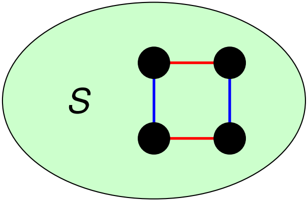
Repeat this ‘cover-down step’ until leftover L^t has bounded size and lies within X_t

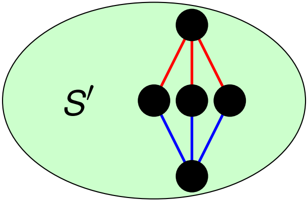
L^t having bounded size $\Rightarrow$ only boundedly many possibilities $R_1, \ldots, R_s$

$\Rightarrow$ suffices to find an ‘absorber’ A_i for each i, i.e. $A_i \cup R_i$ has an F-decomposition, but also A_i has an F-decomposition

Switchers

Building blocks for absorbers A_i: edge-switchers





C_4-switcher S

$S \cup$ red has F-dec

$S \cup$ blue has F-dec

double-star-switcher S'

$S' \cup$ red has F-dec

$S' \cup$ blue has F-dec

Think of red as ‘old’ leftover and of blue as ‘new’ leftover.

Discretization

Discretization

Theorem (Glock, Kühn, Lo, Montgomery, Osthus 2016$^+$)

Let F be a graph and $\chi := \chi(F)$.

(i) $\delta_F \leq \max\{\delta_F^*, 1 - 1/(\chi+1)\}$;

(ii) If $\chi \geq 5$, then $\delta_F \in \{\delta_F^*, 1 - 1/\chi, 1 - 1/(\chi+1)\}$;

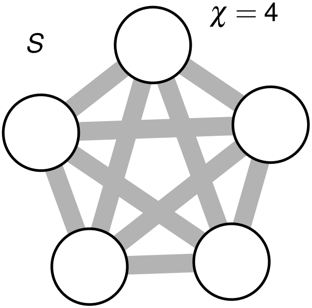
(iii) If $\chi = 2$, then $\delta_F \in \{0, 1/2, 2/3\}$.

Discretization

1.) Show $\delta_{dec}(F) \leq 1 - 1/(\chi + 1)$

2.) Suppose that $\delta_{dec}(F) < 1 - 1/(\chi + 1)$

$\Rightarrow$ Any almost complete (divisible) $(\chi + 1)$-partite graph has F-decomposition!



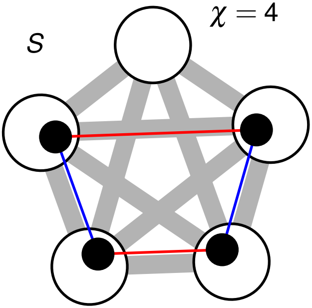
Discretization

1.) Show $\delta_{dec}(F) \leq 1 - 1/(\chi + 1)$

2.) Suppose that $\delta_{dec}(F) < 1 - 1/(\chi + 1)$

$\Rightarrow$ Any almost complete (divisible) $(\chi + 1)$-partite graph has F-decomposition!

$\Rightarrow$ $S \cup$ red and $S \cup$ blue both have F-decomposition



Discretization

1.) Show $\delta_{dec}(F) \leq 1 - 1/(\chi + 1)$

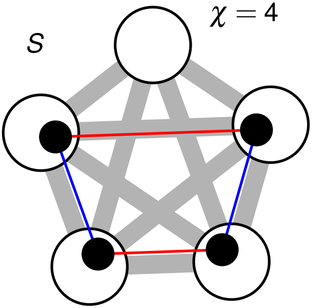
2.) Suppose that $\delta_{dec}(F) < 1 - 1/(\chi + 1)$

$\Rightarrow$ Any almost complete (divisible) $(\chi + 1)$-partite graph has F-decomposition!

$\Rightarrow$ $S \cup$ red and $S \cup$ blue both have F-decomposition

$\Rightarrow$ S is an C_4-switcher (similar for double-star-switcher)

But can find S (in G) at $\delta(G) \geq (1 - 1/\chi + o(1))|G|$



Discretization

1.) Show $\delta_{dec}(F) \leq 1 - 1/(\chi + 1)$

2.) Suppose that $\delta_{dec}(F) < 1 - 1/(\chi + 1)$

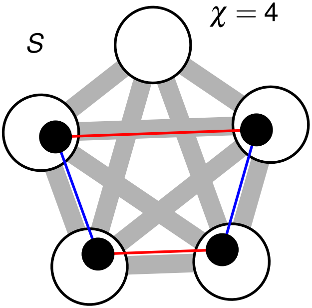
$\Rightarrow$ Any almost complete (divisible) $(\chi + 1)$-partite graph has F-decomposition!

$\Rightarrow$ $S \cup$ red and $S \cup$ blue both have F-decomposition

$\Rightarrow$ S is an C_4-switcher (similar for double-star-switcher)

But can find S (in G) at $\delta(G) \geq (1 - 1/\chi + o(1))|G|$

$\Rightarrow$ $\delta_{dec}(F) \leq 1 - 1/\chi$



Open problem: better fractional thresholds

Theorem (Barber, Kühn, Lo, Osthus 2014^{+})

Every triangle-divisible graph G with $\delta(G) \geq (\delta_{frac}(K_3) + o(1))n$ has a triangle decomposition.

Open problem: better fractional thresholds

Theorem (Barber, Kühn, Lo, Osthus 2014^{+})

Every triangle-divisible graph G with $\delta(G) \geq (\delta_{frac}(K_3) + o(1))n$ has a triangle decomposition.

Problem

Determine $\delta_{frac}(F)$, i.e. the minimum degree threshold for a graph G to have a fractional F-decomposition.

• For triangles, showing that $\delta_{frac}(K_3) = 3/4$ could be combined with our results to show the actual 'decomposition threshold' is $(3/4 + o(1))n$.

• Actually, showing that $3n/4$ guarantees '(fractional) almost decomposition' would suffice.

Decompositions of r-partite host graphs

r-partite G is locally balanced if every vertex has same degree into each class (apart from its own)

r-partite G has K_r-decomposition $\Longrightarrow$ G locally balanced

$\hat{\delta}(G)$:= 'partite minimum degree'

Decompositions of r-partite host graphs

r-partite G is locally balanced if every vertex has same degree into each class (apart from its own)

r-partite G has K_r-decomposition $\Longrightarrow$ G locally balanced

$\hat{\delta}(G)$:= 'partite minimum degree'

Theorem (Barber, Kühn, Lo, Osthus, Taylor 2015$^+$)

Let G be locally balanced r-partite graph with vertex classes of size n. If $\hat{\delta}(G) \geq (\delta_{frac}^{partite}(K_r) + o(1))n$ then G has a K_r-decomposition.

Chowla, Erdős and Straus 1960: case when G complete r-partite

Decompositions of r-partite host graphs

r-partite G is locally balanced if every vertex has same degree into each class (apart from its own)

r-partite G has K_r-decomposition $\Longrightarrow$ G locally balanced

$\hat{\delta}(G)$:= 'partite minimum degree'

Theorem (Barber, Kühn, Lo, Osthus, Taylor 2015$^+$)

Let G be locally balanced r-partite graph with vertex classes of size n. If $\hat{\delta}(G) \geq (\delta^{partite}_{frac}(K_r) + o(1))n$ then G has a K_r-decomposition.

Chowla, Erdős and Straus 1960: case when G complete r-partite

Dukes 2015$^+$: $\delta^{partite}_{frac}(K_3) \leq \frac{101}{104}$
Montgomery 2015$^+$: $\delta^{partite}_{frac}(K_r) \leq 1 - \frac{1}{10^6 r^3}$

Completion of Latin squares

Conjecture (Daykin & Häggkvist, 1984)

Every partially complete $n \times n$ Latin square in which every row, column, symbol is used at most $n/4$ times, can be completed to a Latin square.

follows from 3-partite version of Nash-Williams conjecture on K_3-decompositions

Completion of Latin squares

Conjecture (Daykin & Häggkvist, 1984)

Every partially complete $n \times n$ Latin square in which every row, column, symbol is used at most $n/4$ times, can be completed to a Latin square.

follows from 3-partite version of Nash-Williams conjecture on K_3-decompositions

Corollary

Conjecture true for large n if every row, column, symbol is used at most $3n/104$ times.

- improves previous bounds of Bartlett, Chetwynd & Häggkvist, Gustavsson
- get analogue or mutually orthogonal Latin squares