# Minor-universal planar graphs without accumulation points

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**Abstract**

We show that the countably infinite union of infinite grids, $H$ say, is minor-universal in the class of all graphs that can be drawn in the plane without vertex accumulation points, i.e. that $H$ contains every such graph as a minor. Furthermore, we characterize the graphs that occur as minors of the infinite grid by a natural topological condition on their embeddings.

## 1 Introduction

In [3] we constructed a minor-universal planar graph—a planar graph that contains every other planar graph as a minor. Since every finite planar graph is a minor of some large enough finite grid, a first candidate for such a minor-universal planar graph might have been the infinite grid. It is easily seen however, that the grid does not work: the graph $H$ obtained from $K^4$ by joining to each of its four vertices infinitely many new vertices of degree one is planar but not a minor of the infinite grid.

Every drawing of $H$ has vertex accumulation points whereas every minor of the infinite grid has a drawing without them. Thus one might hope that the infinite grid is minor-universal at least for all VAP-free planar graphs, those that have a drawing without vertex accumulation points. Again this is not true: for example the graph that consists of two disjoint copies of the infinite grid is VAP-free but not a minor of the infinite grid.

One purpose of this paper is to show that infinitely many copies of the grid will do: the countably infinite disjoint union of infinite grids is a minor-universal VAP-free planar graph. This is best possible in the sense that every such minor-universal graph must have infinitely many components, each of which is a minor of the infinite grid.

Furthermore, we characterize the graphs that do occur as minors of the grid by more restrictive conditions on their embeddings: they are exactly those graphs that have a drawing without vertex accumulation points such that each edge accumulation point arises from edges with a common endvertex.

## 2 Terminology

Our basic notation follows Diestel [2], except that we allow the edges of plane graphs to be simple Jordan curves rather than just polygonal arcs. All graphs in this paper are countable.

A *vertex accumulation point* (*VAP*) of a plane graph $G$ is an accumulation point of the vertex set of $G$. A point $x \in \mathbb{R}^2$ is called an *edge accumulation point* of $G$ if every neighbourhood of $x$ meets infinitely many distinct edges of $G$. $G$ is called *VAP-free* if it does not have vertex accumulation points. Similarly, a planar graph is called *VAP-free* if it has a drawing without vertex accumulation points.

A graph $H$ is a *minor* of a graph $G$ if there is a family $V_x$, $x \in V(H)$ of disjoint nonempty connected subsets of $V(G)$ with the property that if $xy \in E(H)$ then there is a $V_x$-$V_y$ edge in $G$. The set $V_x$ is called the *branch set* of $x$. A graph obtained from $H$ by replacing the edges of $H$ by independent paths between their ends is a *subdivision* of $H$. If $G$ is a subdivision of $H$, we view $V(H)$ as a subset of $V(G)$ and call these vertices *branch vertices* of $G$.

We assume a fixed orientation of $\mathbb{R}^2$. If $C$ is a cycle embedded in $\mathbb{R}^2$, we denote by $\vec{C}$ the cycle $C$ oriented in the direction agreeing with this orientation of $\mathbb{R}^2$. We shall also call this direction *clockwise*. Hence, if $f$ is a face in a plane graph, and $\vec{e}$ is an oriented edge on the boundary of $f$, then $f$ lies on the *right* of $\vec{e}$ or on its *left* (or both) in a natural way. (Think of the direction agreeing with the orientation of $\mathbb{R}^2$ as a right turn.) We further assume that the above orientation is so that the inner face of any plane cycle $C$ lies on the right of every edge on $\vec{C}$. The inner face of a plane cycle $C$ will be denoted by $f(C)$.

Given an open set $A \subseteq \mathbb{R}^2$, we say that an edge $e$ runs *through* $A$ if $e$ is contained in $A$ except possibly its endpoints.

The infinite grid is the graph with vertex set $\mathbb{Z}^2$, two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ being adjacent if and only if $|x_1 - y_1| + |x_2 - y_2| = 1$.

## 3 A minor-universal VAP-free planar graph

The purpose of this section is to prove the following

**Theorem 1** *The countably infinite disjoint union of infinite grids is a minor-universal VAP-free planar graph. In particular, every connected VAP-free planar graph is a minor of the infinite grid.*

*Proof.* Since the infinite grid has a VAP-free embedding into an infinite strip of the plane of bounded width, the countably infinite disjoint union of infinite grids is VAP-free. Hence it suffices to prove that every connected VAP-free planar graph $G$ is a minor of the infinite grid.

We may assume that $G$ is an infinite connected VAP-free plane graph. Then there is a sequence $G_1 \subseteq G_2 \subseteq \dots$ of finite connected plane subgraphs of $G$ such that

(1) $G_{n+1}$ is obtained from $G_n$ either by adding a new vertex in the outer face of $G_n$ and joining it to some vertex of $G_n$, or by adding a new edge running through the outer face of $G_n$ that joins two vertices of $G_n$;

(2) $|G_1| = 1$;

(3) $\bigcup_{n=1}^{\infty} G_n = G$.

To prove the existence of such a sequence it suffices to show the following: if $G_k$ is a finite connected plane subgraph of $G$ such that all edges in $E(G) \setminus E(G_k)$ run through the outer face of $G_k$, and if $e$ is such an edge, then there exists a finite sequence $G_k \subseteq G_{k+1} \subseteq \cdots \subseteq G_{k+\ell}$ of plane subgraphs of $G$ satisfying (1) such that $e \in G_{k+\ell}$ and all edges in $E(G) \setminus E(G_{k+\ell})$ run through the outer face of $G_{k+\ell}$. This in turn follows at once from the connectedness of $G$ if at least one endvertex of $e$ does not belong to $G_k$. If both endvertices of $e$ belong to $G_k$, then since $G$ is VAP-free there are only finitely many edges in $E(G) \setminus E(G_k + e)$ that lie in an inner face of $G_k + e$. Thus in this case the assertion follows by induction on the number of those edges.

Let us inductively embed $G$ as a minor into a plane graph $H$ that is a minor of the infinite grid. Using compactness and the fact that $G$ is VAP-free we may thicken the point set $G_n$ together with its inner faces a little to obtain a topological disc that contains only the edges and vertices of $G$ already belonging to $G_n$. Note where the edges in $E(G) \setminus E(G_n)$ hit the boundary of this disc for the first time: the boundary of the disc may be partitioned into finitely many intervals such that all edges which have their first point of intersection with the boundary of the disc in a common interval emanate from a common endvertex in $G_n$. (Here an edge in $E(G) \setminus E(G_n)$ that joins two vertices of $G_n$ has two first points of intersection with the boundary of the disc—one for each of its endpoints.)

We may inductively assume that we have embedded $G_n$ as a minor into a finite connected plane graph $H_n$ of maximum degree at most four such that:

*For every interval $I$ of outgoing edges all emanating from some vertex $x \in G_n$, say, there is a* leaf $v(I)$ *of $H_n$ (a vertex of degree one on the boundary of the outer face of $H_n$) that is contained in the branch set of $x$. Furthermore, the oriented cyclic order of the intervals is reflected by the oriented cyclic order of their leaves when the point set $H_n$ together with its inner faces is thickened and viewed as a disc.*

Now consider how $G_{n+1}$ is obtained from $G_n$. If a vertex $y$ is added and joined to a vertex $x \in G_n$, then the edge $xy$ splits the interval $I$ it belongs to into two parts (possibly empty). Extend $H_n$ by joining three new leaves (placed in the outer face of $H_n$) to $v(I)$ so that the middle leaf starts a new branch set for $y$ and the (up to) two outer leaves are added to the branch set of $x$. Let $H_{n+1}$ be the plane graph thus obtained.

If $G_{n+1}$ was obtained from $G_n$ by adding a new edge $xy$ for $x, y \in G_n$, we may assume that the outer face of $G_{n+1}$ lies on the left of $\overrightarrow{xy}$. The edge $xy$ belongs to an interval $I$ of outgoing edges emanating from $x$ as well as to an interval $J$ of outgoing edges emanating from $y$. Since $G_n$ is connected, none of the outgoing edges belongs to the part of $I$ on the right of $\overrightarrow{xy}$ or the part of $J$ on the left of $\overrightarrow{yx}$. Join $v(I)$ to $v(J)$ so that the outer face of the plane graph thus obtained lies on the left of $\overrightarrow{v(I)v(J)}$. Join a new leaf to $v(I)$ and another to $v(J)$ and add them to the branch sets of $x$ and $y$ respectively. Let $H_{n+1}$ denote the plane graph thus obtained, and note that all leaves of $H_n$ belonging to intervals distinct from $I$ and $J$ lie on the boundary of the outer face of $H_{n+1}$.

Continuing in this way we obtain a minor embedding of $G$ into $\bigcup_{n=1}^{\infty} H_n =: H$. Note that the leaves of $H_n$ are the only vertices of $H_n$ which are incident with edges in $E(H) \setminus E(H_n)$. Furthermore, we may embed a suitable subdivision of $H_n$ into a finite grid such that only the branch vertices that correspond to leaves of $H_n$ lie on the boundary of the grid. It follows that the infinite grid contains a suitable subdivision of $H$ as a subgraph. □

**Remark.** The proof of Theorem 1 implies that every VAP-free planar graph $G$ has a VAP-free drawing whose edges are polygonal arcs. Indeed, suppose that $G$ is a connected VAP-free planar graph and $G'$ is a VAP-free drawing of G. Then one may inductively define a new VAP-free embedding of $G$ whose edges are polygonal arcs into an infinite strip of the plane of bounded length using a sequence $G'_1 \subseteq G'_2 \subseteq \dots$ of subgraphs of $G'$ as in the beginning of the proof of Theorem 1.

## 4 Minors of the infinite grid

We have seen that all minors of the infinite grid are VAP-free, but not every VAP-free planar graph is a minor of the infinite grid. In this section we characterize the graphs that do occur as minors of the grid by more restrictive conditions on their embeddings.

Let us consider the class $\mathcal{G}$ of all plane graphs $G$ such that every point $a \in \mathbb{R}^2$ has a neighbourhood that contains at most one vertex of $G$ and meets only edges incident with that vertex. Thus $G$ is VAP-free and every edge accumulation point arises from edges with a common endvertex. $\mathcal{G}$ is already considered in Bollobás [1, p. 24]. Note that the infinite grid lies in $\mathcal{G}$.

A family of pairwise disjoint cycles in a plane graph is *nested* if for every two cycles in the family, one of the cycles is contained in the inner face of the other cycle.

**Theorem 2** *Let $G$ be a planar graph. Then the following conditions are equivalent.*

*(i) $G$ is a minor of the infinite grid.*

*(ii) $G$ has a drawing in $\mathcal{G}$.*

*(iii) $G$ has a VAP-free drawing $G'$ with the property that if some component $D$ of $G'$ has an infinite family of nested cycles, then every component other than $D$ is finite.*

*Proof.* It is easily seen that (i) implies (ii). It is also straightforward to show that every drawing of $G$ in $\mathcal{G}$ satisfies the condition in (iii). Thus (ii) implies (iii). It remains to prove that (iii) implies (i). If $G$ has at most one infinite component, then (i) follows from Theorem 1 and the fact that every finite planar graph is a minor of some large enough finite grid. Thus we may assume that no component of $G'$ has an infinite family of nested cycles. We shall prove that every component of $G'$ is a minor of the graph $G^*$ that is obtained from the infinite grid by deleting the vertices $(0, i)$, for all $i \geq 1$. Since the union of countably infinite disjoint copies of $G^*$ is a minor of the infinite grid, this will imply (i).

Let $D$ be a component of $G'$. Since $D$ does not have an infinite family of nested cycles, it is either acyclic or it has a cycle $C$ such that any other cycle $C'$ in $D$ with $f(C) \subseteq f(C')$ meets $C$. If $D$ is acyclic, then clearly it is a minor of $G^*$. Thus we may assume that the latter alternative holds. Choose among the cycles $C'$ one that minimizes $|V(C) \cap V(C')|$, and call it $C_1$. Let $x_1$ be any vertex in $V(C) \cap V(C_1)$. Then every $C'$ contains $x_1$ (for if not, then $C_1 \cup C'$ would contain a cycle whose inner face contains $f(C)$ and which has less vertices in common with $C$ than $C_1$, contradicting the choice of $C_1$). Let $D_1$ be the subgraph of $D$ consisting of all vertices and edges that are contained in the closure of $f(C)$. We say that an edge $x_1 y \in E(D) \setminus E(D_1)$ *leads to the right (left) of* $x_1$ if there is a path $P$ of the form $x_1 y \dots z$ in $D$ such that $x_1 \neq z$ are the only vertices of $P$ in $D_1$ and the outer face of $D_1 \cup P$ lies on the left (right) of $\overrightarrow{x_1 y}$. Note that no edge lies both on the right and on the left of $x_1$ (otherwise there would be a cycle $C'$ in $D$ such that $f(C) \subseteq f(C')$ but $x_1 \notin C'$). Let $x_0$ be the predecessor of $x_1$ on $\vec{C}$. If we look at the edges incident with $x_1$ in clockwise order beginning with $x_0 x_1$, then the edges in $E(D) \setminus E(D_1)$ leading to the left of $x_1$ precede those leading to the right of $x_1$. We now partition the edges in $E(D) \setminus E(D_1)$ incident with $x_1$ into two sets: an edge $x_1 y$ is *on the left of* $x_1$ if it precedes all edges leading to the right of $x_1$. All other edges are said to be *on the right of* $x_1$. Note that there is no path $\vec{P}$ with vertices in $V(D) \setminus V(D_1)$ beginning at an endvertex of an edge on the left of $x_1$ and ending at an endvertex of an edge on the right of $x_1$ such that the outer face of $D[V(C) \cup V(P)]$ lies on the left of $\vec{P}$.

We now modify the proof of Theorem 1 in order to embed $D$ inductively as a minor into $G^*$. Again, since $D$ is VAP-free, there is a sequence $D_1 \subseteq D_2 \subseteq \dots$ of finite connected plane subgraphs of $D$ such that

(1) $D_{n+1}$ is obtained from $D_n$ either by adding a new vertex in the outer face of $D_n$ and joining it to some vertex of $D_n$, or by adding a new edge running through the outer face of $D_n$ that joins two vertices of $D_n$;

(2) $\bigcup_{n=1}^{\infty} D_n = D$.

We inductively embed $D$ as a minor into a plane graph $H$ that is a minor of $G^*$. As before we thicken the point set $D_n$ together with its inner faces a little to obtain a topological disc, and the boundary of the disc can be partitioned into finitely many intervals such that all edges which have their first point of intersection with the boundary of the disc in a common interval emanate from a common endvertex in $D_n$. Additionally this time we require that no interval contains the first points of intersection with the boundary of the disc of both an edge on the left of $x_1$ and an edge on the right of $x_1$.

Again, we may inductively assume that we have embedded $D_n$ as a minor into a finite connected plane graph $H_n$ such that:

*For every interval $I$ of outgoing edges all emanating from some vertex $x \in D_n$, say, there is a* leaf *$v(I)$ of $H_n$ that is contained in the branch set of $x$. Furthermore, the oriented cyclic order of the intervals is reflected by the oriented cyclic order of their leaves when the point set $H_n$ together with its inner faces is thickened and viewed as a disc.*

Here $H_1$ is obtained from $D_1$ by joining a new leaf (placed in the outer face of $D_1$) to every vertex on $C$, except that $x_1$ is joined to two leaves $x_1^\ell$ (for the edges on the left of $x_1$) and $x_1^r$ (for the edges on the right of $x_1$). Exactly as in the proof of Theorem 1 we now embed $D_{n+1}$ as a minor into a plane graph $H_{n+1}$ obtained from $H_n$. Thus $D$ is a minor of $\bigcup_{n=1}^{\infty} H_n =: H$. The crucial point now is that the outer face of $H_{n+1}$ lies on the left of both $\overrightarrow{x_1^\ell x_1}$ and $\overrightarrow{x_1 x_1^r}$. Using this, it is easy to embed $H$ inductively as a minor into $G^*$. (First embed $D_1$ as a minor into a finite subgrid of $G^*$ such that $(-1, 0)$, $(0, 0)$ and $(1, 0)$ lie on the boundary of the subgrid and also belong to the branch set of $x_1$. Gradually extend this to a minor-embedding of $H_n$ into $G^*$ such that leaves of $H_n$ belonging to intervals of edges on the left (right) of $x_1$ are embedded on the respective side of the ray in $G^*$ which passes through all $(0, i)$, $i \geq 1$.) □

**Remark.** Note that every minor of the infinite grid has a drawing in $\mathcal{G}$ whose edges are polygonal arcs. Together with Theorem 2 this implies that every graph with a drawing in $\mathcal{G}$ has also a drawing in $\mathcal{G}$ whose edges are polygonal arcs.

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## References

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