CLIQUE DECOMPOSITIONS OF MULTIPARTITE GRAPHS AND COMPLETION OF LATIN SQUARES

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ABSTRACT. Our main result essentially reduces the problem of finding an edge-decomposition of a balanced r-partite graph of large minimum degree into r-cliques to the problem of finding a fractional r-clique decomposition or an approximate one. Together with very recent results of Dukes as well as Montgomery on fractional decompositions into triangles and cliques respectively, this gives the best known bounds on the minimum degree which ensures an edge-decomposition of an r-partite graph into r-cliques (subject to trivially necessary divisibility conditions). The case of triangles translates into the setting of partially completed Latin squares and more generally the case of r-cliques translates into the setting of partially completed mutually orthogonal Latin squares.

1. INTRODUCTION

A *K_r-decomposition* of a graph G is a partition of its edge set $E(G)$ into cliques of order r. If G has a K_r-decomposition, then certainly $e(G)$ is divisible by $\binom{r}{2}$ and the degree of every vertex is divisible by $r-1$. A classical result of Kirkman [18] asserts that, when $r=3$, these two conditions ensure that K_n has a triangle decomposition (i.e. Steiner triple systems exist). This was generalized to arbitrary r (for large n) by Wilson [27] and to hypergraphs by Keevash [16]. Recently, there has been much progress in extending this from decompositions of complete host graphs to decompositions of graphs which are allowed to be far from complete. In this paper, we investigate this question in the r-partite setting. This is of particular interest as it implies results on the completion of partial Latin squares and more generally partial mutually orthogonal Latin squares.

1.1. Clique decompositions of r-partite graphs.

Our main result (Theorem 1.1) states that if G is (i) balanced r-partite, (ii) satisfies the necessary divisibility conditions and (iii) its minimum degree is at least a little larger than the minimum degree which guarantees an approximate decomposition into r-cliques, then G in fact has a decomposition into r-cliques. (Here an approximate decomposition is a set of edge-disjoint copies of K_r which cover almost all edges of G.) To state this result precisely, we need the following definitions.

We say that a graph or multigraph G on $(V_1,\dots,V_r)$ is *K_r-divisible* if G is r-partite with vertex classes $V_1,\dots,V_r$ and for all $1\le j_1,j_2\le r$ and every $v\in V(G)\setminus(V_{j_1}\cup V_{j_2})$,

$$d(v,V_{j_1})=d(v,V_{j_2}).$$

Note that in this case, for all $1\le j_1,j_2,j_3,j_4\le r$ with $j_1\neq j_2$, $j_3\neq j_4$, we automatically have $e(V_{j_1},V_{j_2})=e(V_{j_3},V_{j_4})$. In particular, $e(G)$ is divisible by $e(K_r)=\binom{r}{2}$.

Let G be an r-partite graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$. Let

$$\hat{\delta}(G):=\min\{d(v,V_j):1\le j\le r,\ v\in V(G)\setminus V_j\}.$$

An *η-approximate* K_r-decomposition of G is a set of edge-disjoint copies of K_r covering all but at most ηn^2 edges of G. We define $\hat{\delta}^{\eta}_{K_r}(n)$ to be the infimum over all δ such that every K_r-divisible graph G

Date: March 3, 2016.

The research leading to these results was partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreement no. 258345 (B. Barber and D. Kühn) and 306349 (D. Osthus). The research was also partially supported by the EPSRC, grant no. EP/M009408/1 (D. Kühn and D. Osthus), and by the Royal Society and the Wolfson Foundation (D. Kühn).

on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$ and $\hat{\delta}(G)\geq \delta n$ has an η-approximate K_r-decomposition. Let $\hat{\delta}^{\eta}_{K_r}:=\limsup_{n\to\infty}\hat{\delta}^{\eta}_{K_r}(n)$. So if $\varepsilon>0$ and G is sufficiently large, K_r-divisible and $\hat{\delta}(G)>(\hat{\delta}^{\eta}_{K_r}+\varepsilon)n$, then G has an η-approximate K_r-decomposition. Note that it is important here that G is K_r-divisible. Take, for example, the complete r-partite graph with vertex classes of size n and remove $\lceil \eta n\rceil$ edge-disjoint perfect matchings between one pair of vertex classes. The resulting graph G satisfies $\hat{\delta}(G)=n-\lceil \eta n\rceil$, yet has no η-approximate K_r-decomposition whenever $r\geq 3$.

Theorem 1.1. *For every $r\geq 3$ and every $\varepsilon>0$ there exists an $n_0\in\mathbb{N}$ and an $\eta>0$ such that the following holds for all $n\geq n_0$. Suppose G is a K_r-divisible graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$. If $\hat{\delta}(G)\geq(\hat{\delta}^{\eta}_{K_r}+\varepsilon)n$, then G has a K_r-decomposition.*

By a result of Haxell and Rödl [14], the existence of an approximate decomposition follows from that of a fractional decomposition. So together with very recent results of Dukes [10] as well as Montgomery [21] on fractional decompositions into triangles and cliques respectively, Theorem 1.1 implies the following explicit bounds. We discuss this derivation in Section 1.3.

Theorem 1.2. *For every $r\geq 3$ and every $\varepsilon>0$ there exists an $n_0\in\mathbb{N}$ such that the following holds for all $n\geq n_0$. Suppose G is a K_r-divisible graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$.*

(i) *If $r=3$ and $\hat{\delta}(G)\geq\left(\frac{101}{104}+\varepsilon\right)n$, then G has a K_3-decomposition.*
(ii) *If $r\geq 4$ and $\hat{\delta}(G)\geq\left(1-\frac{1}{10^6r^3}+\varepsilon\right)n$, then G has a K_r-decomposition.*

If G is the complete r-partite graph, this corresponds to a theorem of Chowla, Erdős and Straus [6]. A bound of $(1-1/(10^{16}r^{29}))n$ was claimed by Gustavsson [13]. The following conjecture seems natural (and is implicit in [13]).

Conjecture 1.3. *For every $r\geq 3$ there exists an $n_0\in\mathbb{N}$ such that the following holds for all $n\geq n_0$. Suppose G is a K_r-divisible graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$. If $\hat{\delta}(G)\geq(1-1/(r+1))n$, then G has a K_r-decomposition.*

A construction which matches the lower bound in Conjecture 1.3 is described in Section 3.1 (this construction also gives a similar lower bound on $\hat{\delta}^{\eta}_{K_r}$). In the non-partite setting, the triangle case is a long-standing conjecture by Nash-Williams [22] that every graph G on n vertices with minimum degree at least $3n/4$ has a triangle decomposition (subject to divisibility conditions). Barber, Kühn, Lo and Osthus [3] recently reduced its asymptotic version to proving an approximate or fractional version. Corresponding results on fractional triangle decompositions were proved by Yuster [28], Dukes [9], Garaschuk [11] and Dross [8].

More generally [3] also gives results for arbitrary graphs, and corresponding fractional decomposition results have been obtained by Yuster [28], Dukes [9] as well as Barber, Kühn, Lo, Montgomery and Osthus [2]. Further results on F-decompositions of non-partite graphs (leading on from [3]) have been obtained by Glock, Kühn, Lo, Montgomery and Osthus [12]. Amongst others, for any bipartite graph F, they asymptotically determine the minimum degree threshold which guarantees an F-decomposition.

1.2. **Mutually orthogonal Latin squares and K_r-decompositions of r-partite graphs.** A *Latin square* $\mathcal{T}$ of order n is an $n\times n$ grid of cells, each containing a symbol from $[n]$, such that no symbol appears twice in any row or column. It is easy to see that $\mathcal{T}$ corresponds to a K_3-decomposition of the complete tripartite graph $K_{n,n,n}$ with vertex classes consisting of the rows, columns and symbols.

Now suppose that we have a partial Latin square; that is, a partially filled in grid of cells satisfying the conditions defining a Latin square. When can it be completed to a Latin square? This natural question has received much attention. For example, a classical theorem of Smetaniuk [24] as well as Anderson and Hilton [1] states that this is always possible if at most $n-1$ entries have been made (this bound is best possible). The case $r=3$ of Conjecture 1.3 implies that, provided we have used each row, column and symbol at most $n/4$ times, it should also still be possible to complete a partial Latin square. This was conjectured by Daykin and Häggkvist [7]. (Note that this conjecture corresponds to the special case

of Conjecture 1.3 when $r = 3$ and the condition of G being K_r-divisible is replaced by that of G being obtained from $K_{n,n,n}$ by deleting edge-disjoint triangles.)

More generally, we say that two Latin squares R (red) and B (blue) drawn in the same $n \times n$ grid of cells are *orthogonal* if no blue symbol appears twice next to the same red symbol. In the same way that a Latin square corresponds to a K_3-decomposition of $K_{n,n,n}$, a pair of orthogonal Latin squares corresponds to a K_4-decomposition of $K_{n,n,n,n}$ where the vertex classes are rows, columns, red symbols and blue symbols. More generally, there is a natural bijection between sequences of $r-2$ *mutually orthogonal* Latin squares (where every pair from the sequence are orthogonal) and K_r-decompositions of complete r-partite graphs with vertex classes of equal size. Sequences of mutually orthogonal Latin squares are also known as *transversal designs*. Theorem 1.2 can be used to show the following (see Section 3.2 for details).

Theorem 1.4. *For every $r \geq 3$ and every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let*

$$c_r := \begin{cases} \frac{3}{104} & \text{if } r = 3, \\ \frac{9}{10^7 r^3} & \text{if } r \geq 4. \end{cases}$$

Let $\mathcal{T}_1, \dots, \mathcal{T}_{r-2}$ be a sequence of mutually orthogonal partial $n \times n$ Latin squares (drawn in the same $n \times n$ grid). Suppose that each row and column of the grid contains at most $(c_r - \varepsilon)n$ non-empty cells and each coloured symbol is used at most $(c_r - \varepsilon)n$ times. Then $\mathcal{T}_1, \dots, \mathcal{T}_{r-2}$ can be completed to a sequence of mutually orthogonal Latin squares.

The best previous bound for the triangle case $r = 3$ is due to Bartlett [4], who obtained a minimum degree bound of $(1 - 10^{-4})n$. This improved an earlier bound of Chetwynd and Häggkvist [5] as well as the one claimed by Gustavsson [13].

1.3. **Fractional and approximate decompositions.** A *fractional K_r-decomposition* of a graph G is a non-negative weighting of the copies of K_r in G such that the total weight of all the copies of K_r containing any fixed edge of G is exactly 1. Fractional decompositions are of particular interest to us because of the following result of Haxell and Rödl, of which we state only a very special case.

Theorem 1.5 (Haxell and Rödl [14])**.** *For every $r \geq 3$ and every $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices that has a fractional K_r-decomposition. Then G has an η-approximate K_r-decomposition.*

We define $\hat{\delta}^*_{K_r}(n)$ to be the infimum over all δ such that every K_r-divisible graph G on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$ and $\hat{\delta}(G) \geq \delta n$ has a fractional K_r-decomposition. Let $\hat{\delta}^*_{K_r} := \limsup_{n\to\infty} \hat{\delta}^*_{K_r}(n)$. Theorem 1.5 implies that, for every $\eta > 0$, $\hat{\delta}^\eta_{K_r} \leq \hat{\delta}^*_{K_r}$. Together with Theorem 1.1, this yields the following.

Corollary 1.6. *For every $r \geq 3$ and every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose G is a K_r-divisible graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. If $\hat{\delta}(G) \geq (\hat{\delta}^*_{K_r} + \varepsilon)n$, then G has a K_r-decomposition.*

In particular, to prove Conjecture 1.3 asymptotically, it suffices to show that $\hat{\delta}^*_{K_r} \leq 1 - 1/(r+1)$. For triangles, the best bound on the 'fractional decomposition threshold' is due to Dukes [10].

Theorem 1.7 (Dukes [10])**.** $\hat{\delta}^*_{K_3} \leq \frac{101}{104}$.

For arbitrary cliques, Montgomery obtained the following bound. Somewhat weaker bounds (obtained by different methods) are also proved in [10].

Theorem 1.8 (Montgomery [21])**.** *For every $r \geq 3$, $\hat{\delta}^*_{K_r} \leq 1 - \frac{1}{10^6 r^3}$.*

Note that together with Corollary 1.6, these results immediately imply Theorem 1.2.

This paper is organised as follows. In Section 2 we introduce some notation and tools which will be used throughout this paper. In Section 3 we give extremal constructions which support the bounds in Conjecture 1.3 and we provide a proof of Theorem 1.4. Section 4 outlines the proof of Theorem 1.1 and guides the reader through the remaining sections in this paper.

2. Notation and tools

Let G be a graph and let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a partition of $V(G)$. We write $G[U^1]$ for the subgraph of G induced by U^1 and $G[U^1, U^2]$ for the bipartite subgraph of G induced by the vertex classes U^1 and U^2. We will also sometimes write $G[U^1, U^1]$ for $G[U^1]$. We write $G[\mathcal{P}] := G[U^1, \dots, U^k]$ for the k-partite subgraph of G induced by the partition $\mathcal{P}$. We write $U^{<i}$ for $U^1 \cup \dots \cup U^{i-1}$. We say the partition $\mathcal{P}$ is *equitable* if its parts differ in size by at most one. Given a set $U \subseteq V(G)$, we write $\mathcal{P}[U]$ for the restriction of $\mathcal{P}$ to U.

Let G be a graph and let $U, V \subseteq V(G)$. We write $N_G(U, V) := \{v \in V : xv \in E(G) \text{ for all } x \in U\}$ and $d_G(U, V) := |N_G(U, V)|$. For $v \in V(G)$, we write $N_G(v, V)$ for $N_G(\{v\}, V)$ and $d_G(v, V)$ for $d_G(\{v\}, V)$. If U and V are disjoint, we let $e_G(U, V) := e(G[U, V])$.

Let G and H be graphs. We write $G - H$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$. We write $G \setminus H$ for the subgraph of G induced by the vertex set $V(G) \setminus V(H)$. We call a vertex-disjoint collection of copies of H in G an *H-matching*. If the H-matching covers all vertices in G, we say that it is *perfect*.

Throughout this paper, we consider a partition $V_1, \dots, V_r$ of a vertex set V such that $|V_j| = n$ for all $1 \leq j \leq r$. Given a set $U \subseteq V$, we write

$$U_j := U \cap V_j.$$

We write $K_r(k)$ for the complete r-partite graph with vertex classes of size k. We say that an r-partite graph G on $(V_1, \dots, V_r)$ is *balanced* if $|V_1| = \dots = |V_r|$.

We use hierarchies, for example $1/n \ll a \ll b < 1$, where constants are chosen from right to left. The notation $a \ll b$ means that there exists an increasing function f for which the result holds whenever $a \leq f(b)$. In order to simplify the presentation, we will not determine these functions explicitly.

Let $m, n, N \in \mathbb{N}$ with $m, n < N$. The *hypergeometric distribution* with parameters N, n and m is the distribution of the random variable X defined as follows. Let S be a random subset of $\{1, 2, \dots, N\}$ of size n and let $X := |S \cap \{1, 2, \dots, m\}|$. We will frequently use the following bounds, which are simple forms of Hoeffding's inequality.

Lemma 2.1 (see [15, Remark 2.5 and Theorem 2.10])**.** *Let $X \sim B(n, p)$ or let X have a hypergeometric distribution with parameters N, n, m. Then $\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq 2e^{-2t^2/n}$.*

Lemma 2.2 (see [15, Corollary 2.3 and Theorem 2.10])**.** *Suppose that X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}(X)| \geq a\mathbb{E}(X)) \leq 2e^{-a^2\mathbb{E}(X)/3}$.*

3. Extremal graphs and completion of Latin squares

3.1. **Extremal graphs.** The following proposition shows that the minimum degree bound conjectured in Conjecture 1.3 would be best possible. It also provides a lower bound on the approximate decomposition threshold $\hat{\delta}^{\eta}_{K_r}$ (and thus on the fractional decomposition threshold $\hat{\delta}^{*}_{K_r}$).

Proposition 3.1. *Let $r \in \mathbb{N}$ with $r \geq 3$ and let $\eta > 0$. For infinitely many n, there exists a K_r-divisible graph G on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$ and $\hat{\delta}(G) = \lceil (1 - 1/(r+1))n \rceil - 1$ which does not have a K_r-decomposition. Moreover, $\hat{\delta}^{\eta}_{K_r} \geq 1 - 1/(r+1) - \eta$.*

Proof. Let $m \in \mathbb{N}$ with $1/m \ll \eta$ and let $n := (r-1)m$. Let $\{U^1, \dots, U^{r-1}\}$ be a partition of $V_1 \cup \dots \cup V_r$ such that, for each $1 \leq i \leq r-1$ and each $1 \leq j \leq r$, $U^i_j = U^i \cap V_j$ has size m.

Let G_0 be the intersection of the complete r-partite graph on $(V_1,\dots,V_r)$ and the complete $(r-1)$-partite graph on $(U^1,\dots,U^{r-1})$. For each $1\le q\le m$ and each $1\le i\le r-1$, let H_q^i be a graph formed by starting with the empty graph on U^i and including a q-regular bipartite graph with vertex classes $(U^i_{j_1},U^i_{j_2})$ for each $1\le j_1<j_2\le r$. Let $H_q:=H_q^1\cup\dots\cup H_q^{r-1}$ and let $G_q:=G_0\cup H_q$. Observe that G_q is regular, K_r-divisible and

$$\hat\delta(G_q)=(r-2)m+q.$$

Now G_0 is $(r-1)$-partite, so every copy of K_r in G_q contains at least one edge of H_q. Therefore, any collection of edge-disjoint copies of K_r in G will leave at least

$$\begin{aligned}\ell(G_q):=e(G_q)-e(H_q)\binom{r}{2}&=\big((r-2)m+q-\binom{r}{2}q\big)\binom{r}{2}n\\&=(m-(r+1)q/2)(r-2)\binom{r}{2}n\end{aligned}$$

edges of G_q uncovered. Let $q_0:=\lceil 2m/(r+1)\rceil-1$. Then $\ell(G_{q_0})>0$, so G_{q_0} does not have a K_r-decomposition. Also,

$$\hat\delta(G_{q_0})=(r-2)m+\lceil 2m/(r+1)\rceil-1=\lceil(1-1/(r+1))n\rceil-1.$$

Now let $q_\eta:=\lceil 2m/(r+1)-\eta n\rceil$. We have $\hat\delta(G_{q_\eta})\ge(1-1/(r+1)-\eta)n$ and

$$\begin{aligned}\ell(G_{q_\eta})&\ge(m-(2m/(r+1)-\eta n+1)(r+1)/2)(r-2)\binom{r}{2}n\\&=(\eta n-1)(r+1)(r-2)r(r-1)n/4\ge 6(\eta n-1)n>\eta n^2.\end{aligned}$$

Thus, $\hat\delta^\eta_{K_r}\ge 1-1/(r+1)-\eta$. □

3.2. **Completion of mutually orthogonal Latin squares.** In this section, we give a proof of Theorem 1.4. Note that better bounds on the fractional decomposition threshold would immediately lead to better bounds on c_r. For any r-partite graph H on $(V_1,\dots,V_r)$, we let $\overline{H}$ denote the r-partite complement of H on $(V_1,\dots,V_r)$.

Proof of Theorem 1.4. By making ε smaller if necessary, we may assume that $\varepsilon\ll 1$. Let $n_0\in\mathbb{N}$ be such that $1/n_0\ll\varepsilon,1/r$. Use $\mathcal{T}_1,\dots,\mathcal{T}_{r-2}$ to construct a balanced r-partite graph G with vertex classes $V_j=[n]$ for $1\le j\le r$ as follows. For each $1\le i,j,k\le n$ and each $1\le m\le r-2$, if in $\mathcal{T}_m$ the cell (i,j) contains the symbol k, include a K_3 on the vertices $i\in V_{r-1}$, $j\in V_r$ and $k\in V_m$. (If the cell (i,j) is filled in different $\mathcal{T}_m$, this leads to multiple edges between $i\in V_{r-1}$ and $j\in V_r$, which we disregard.) For each $1\le i,j,k,k'\le n$ and each $1\le m<m'\le r-2$ such that the cell (i,j) contains symbol k in $\mathcal{T}_m$ and symbol k' in $\mathcal{T}_{m'}$, add an edge between the vertices $k\in V_m$ and $k'\in V_{m'}$.

If $r=3$, then G is an edge-disjoint union of copies of K_3, so G is K_3-divisible. Then $\overline{G}$ is also K_3-divisible and $\hat\delta(\overline{G})\ge(101/104+\varepsilon)n$. So we can apply Theorem 1.2 to find a K_3-decomposition of $\overline{G}$ which we can then use to complete $\mathcal{T}_1$ to a Latin square.

Suppose now that $r\ge 4$. Observe that G consists of an edge-disjoint union of cliques $H_1,\dots,H_q$ such that, for each $1\le i\le q$, H_i contains an edge of the form xy where $x\in V_{r-1}$ and $y\in V_r$. We have $q\le(c_r-\varepsilon)n^2$. We now show that we can extend G to a graph which has a K_r-decomposition. We will do this by greedily extending each H_i in turn to a copy H_i' of K_r. Suppose that $1\le p\le q$ and we have already found edge-disjoint $H_1',\dots,H_{p-1}'$. Given $v\in V(G)$, let $s(v,p-1)$ be the number of graphs in $\{H_1',\dots,H_{p-1}'\}\cup\{H_p,\dots,H_q\}$ which contain v. Suppose that $s(v,p-1)\le 10(c_r-\varepsilon^2)n/9$ for all $v\in V(G)$. For each $1\le j\le r$, let $B_j:=\{v\in V_j: s(v,p-1)\ge 10(c_r-\varepsilon)n/9\}$. We have

$$|B_j|\le\frac{q}{10(c_r-\varepsilon)n/9}\le\frac{9n}{10}.\tag{3.1}$$

Let $G_{p-1} := G \cup \bigcup_{i=1}^{p-1}(H_i' - H_i)$. Note that

$$\hat{\delta}(\overline{G}_{p-1}) \geq (1 - 10(c_r - \varepsilon^2)/9)n. \tag{3.2}$$

We will extend H_p to a copy of K_r as follows. Let $\{j_1, \dots, j_m\} = \{j : 1 \leq j \leq r \text{ and } V(H_p) \cap V_j = \emptyset\}$. For each j_i in turn, starting with j_1, choose one vertex x_{j_i} from the set $N_{\overline{G}_{p-1}}(V(H_p) \cup \{x_{j_1}, \dots, x_{j_{i-1}}\}, V_{j_i} \setminus B_{j_i})$. This is possible since (3.1) and (3.2) imply

$$d_{\overline{G}_{p-1}}(V(H_p) \cup \{x_{j_1}, \dots, x_{j_{i-1}}\}, V_{j_i} \setminus B_{j_i}) \geq (1/10 - (r-1)10(c_r - \varepsilon^2)/9)n > 0.$$

Let H_p' be the copy of K_r with vertex set $V(H_p) \cup \{x_j : 1 \leq j \leq r \text{ and } V(H_p) \cap V_j = \emptyset\}$. By construction, for every $v \in V(G)$, the number $s(v,p)$ of graphs in $\{H_1', \dots, H_p'\} \cup \{H_{p+1}, \dots, H_q\}$ which contain v satisfies $s(v,p) \leq 10(c_r - \varepsilon^2)n/9$.

Continue in this way to find edge-disjoint $H_1', \dots, H_q'$ such that $s(v,q) \leq 10(c_r - \varepsilon^2)n/9$. Let $G_q := \bigcup_{1 \leq i \leq q} H_i'$. We have $\hat{\delta}(\overline{G}_q) \geq (1 - 10(c_r - \varepsilon^2)/9)n = (1 - 1/10^6 r^3 + 10\varepsilon^2/9)n$ and, since G_q is an edge-disjoint union of copies of K_r, we know that $\overline{G}_q$ is K_r-divisible. So we can apply Theorem 1.2 to find a K_r-decomposition $\mathcal{F}$ of $\overline{G}_q$. Note that $\mathcal{F}' := \mathcal{F} \cup \bigcup_{1 \leq i \leq q} H_i'$ is a K_r-decomposition of the complete r-partite graph. Since $H_i \subseteq H_i'$ for each $1 \leq i \leq q$, we can use $\mathcal{F}'$ to complete $\mathcal{T}_1, \dots, \mathcal{T}_{r-2}$ to a sequence of mutually orthogonal Latin squares. □

4. Proof Sketch

Our proof of Theorem 1.1 builds on the proof of the main results of [3], but requires significant new ideas. In particular, the r-partite setting involves a stronger notion of divisibility (the non-partite setting simply requires that $r-1$ divides the degree of each vertex of G and that $\binom{r}{2}$ divides $e(G)$) and we have to work much harder to preserve it during our proof. This necessitates a delicate 'balancing' argument (see Section 10). In addition, we use a new construction for our absorbers, which allows us to obtain the best possible version of Theorem 1.1. (The construction of [3] would only achieve $1 - 1/3(r-1)$ in place of $1 - 1/(r+1)$.)

The idea behind the proof is as follows. We are assuming that we have access to a black box approximate decomposition result: given a K_r-divisible graph G on vertex classes of size n with $\hat{\delta}(G) \geq (\hat{\delta}^{\eta}_{K_r} + \varepsilon)n$ we can obtain an approximate K_r-decomposition that leaves only ηn^2 edges uncovered. We would like to obtain an exact decomposition by 'absorbing' this small remainder. By an *absorber* for a K_r-divisible graph H we mean a graph A_H such that both A_H and $A_H \cup H$ have a K_r-decomposition. For any fixed H we can construct an absorber A_H. But there are far too many possibilities for the remainder H to allow us to reserve individual absorbers for each in advance.

To bridge the gap between the output of the approximate result and the capabilities of our absorbers, we use an iterative absorption approach (see also [3] and [19]). Our guiding principle is that, since we have no control on the remainder if we apply the approximate decomposition result all in one go, we should apply it more carefully. More precisely, we begin by partitioning $V(G)$ at random into a large number of parts $U^1, \dots, U^k$. Since k is large, $G[U^1, \dots, U^k]$ still has high minimum degree, and, since the partition is random, each $G[U^i]$ also has high minimum degree. We first reserve a sparse and well structured subgraph J of $G[U^1, \dots, U^k]$, then we obtain an approximate decomposition of $G[U^1, \dots, U^k] - J$ leaving a sparse remainder H. We then use a small number of edges from the $G[U^i]$ to cover all edges of $H \cup J$ by copies of K_r. Let G' be the subgraph of G consisting of those edges not yet used in the approximate decomposition. Then all edges of G' lie in some $G'[U^i]$, and each $G'[U^i]$ has high minimum degree, so we can repeat this argument on each $G'[U^i]$. Suppose that we can iterate in this way until we obtain a partition $W_1 \cup \dots \cup W_m$ of $V(G)$ such that each W_i has size at most some constant M and all edges of G have been used in the approximate decomposition except for those contained entirely within some W_i. Then the remainder is a vertex-disjoint union of graphs $H_1, \dots, H_m$, with each H_i contained within W_i. At this point we have already achieved that the total leftover $H_1 \cup \dots \cup H_m$ has only $O(n)$ edges. More

importantly, the set of all possibilities for the graphs H_i has size at most $2^{M^2}m = O(n)$, which is a small enough number that we are able to reserve special purpose absorbers for each of them in advance (i.e. right at the start of the proof).

The above sketch passes over one genuine difficulty. Recall that $H \subseteq G[U^1, \dots, U^k]$ denotes the sparse remainder obtained from the approximate decomposition, which we aim to 'clean up' using a well structured graph J set aside at the beginning of the proof, i.e. we aim to cover all edges of $H \cup J$ with copies of K_r by using a few additional edges from the $G[U^i]$. So consider any vertex $v \in U_1^1$ (recall that $U_j^i = U^i \cap V_j$). In order to cover the edges in $H \cup J$ between v and U^2, we would like to find a perfect K_{r-1}-matching in $N(v) \cap U^2$. However, for this to work, the number of neighbours of v inside each of $U_2^2, \dots, U_r^2$ must be the same, and the analogue must hold with U^2 replaced by any of $U^3, \dots, U^k$. (This is in contrast to [3], where one only needs that the number of leftover edges between v and any of the parts U^i is divisible by r, which is much easier to achieve.) We ensure this balancedness condition by constructing a 'balancing graph' which can be used to transfer a surplus of edges or degrees from one part to another. This 'balancing graph' will be the main ingredient of J. Another difficulty is that whenever we apply the approximate decomposition result, we need to ensure that the graph is K_r-divisible. This means that we need to 'preprocess' the graph at each step of the iteration.

The rest of this paper is organised as follows. In Section 5, we present general purpose embedding lemmas that allow us to find a wide range of desirable structures within our graph. In Section 6, we detail the construction of our absorbers. In Section 7, we prove some basic properties of random subgraphs and partitions. In Section 8, we show how we can assume that our approximate decomposition result produces a remainder with low maximum degree rather than simply a small number of edges. In Section 9, we clean up the edges in the remainder using a few additional edges from inside each part of the current partition. However, we assume in this section that our remainder is balanced in the sense described above. In Section 10, we describe the balancing operation which ensures that we can make this assumption. Finally, in Section 11 we put everything together to prove Theorem 1.1.

5. Embedding lemmas

Let G be an r-partite graph on $(V_1, \dots, V_r)$ and let $\mathcal{P} = \{U^1, U^2, \dots, U^k\}$ be a partition of $V(G)$. Recall that $U_j^i := U^i \cap V_j$ for each $1 \le i \le k$ and each $1 \le j \le r$. We say that a graph (or multigraph) H is *$\mathcal{P}$-labelled* if:

(a) every vertex of H is labelled by one of: $\{v\}$ for some $v \in V(G)$; U_j^i for some $1 \le i \le k$, $1 \le j \le r$ or V_j for some $1 \le j \le r$;
(b) the vertices labelled by singletons (called *root vertices*) form an independent set in H, and each $v \in V(G)$ appears as a label $\{v\}$ at most once;
(c) for each $1 \le j \le r$, the set of vertices $v \in V(H)$ such that v is labelled L for some $L \subseteq V_j$ forms an independent set in H.

Any vertex which is not a root vertex is called a *free vertex*.

Let H be a $\mathcal{P}$-labelled graph and let H' be a copy of H in G. We say that H' is *compatible with its labelling* if each vertex of H gets mapped to a vertex in its label.

Given a graph H and $U \subseteq V(H)$ with $e(H[U]) = 0$, we define the *degeneracy of H rooted at U* to be the least d for which there is an ordering $v_1, \dots, v_b$ of the vertices of H such that

- there is an a such that $U = \{v_1, \dots, v_a\}$ (the ordering of U is unimportant);
- for $a < j \le b$, v_j is adjacent to at most d of the v_i with $1 \le i < j$.

The *degeneracy of a $\mathcal{P}$-labelled graph H* is the degeneracy of H rooted at U, where U is the set of root vertices of H.

In the proof of Lemma 10.9, we use the following special case of Lemma 5.1 from [3] to find copies of labelled graphs inside a graph G, provided their degeneracy is small. Moreover, this lemma allows us to assume that the subgraph of G used to embed these graphs has low maximum degree.

Lemma 5.1. *Let $1/n \ll \eta \ll \varepsilon, 1/d, 1/b \leq 1$ and let G be a graph on n vertices. Suppose that:*

(i) *for each $S \subseteq V(G)$ with $|S| \leq d$, $d_G(S, V(G)) \geq \varepsilon n$.*

Let $m \leq \eta n^2$ and let $H_1, \dots, H_m$ be labelled graphs such that, for every $1 \leq i \leq m$, every vertex of H_i is labelled $\{v\}$ for some $v \in V(G)$ or labelled by $V(G)$ and that property (b) *above holds for H_i. Moreover, suppose that:*

(ii) *for each $1 \leq i \leq m$, $|H_i| \leq b$;*
(iii) *for each $1 \leq i \leq m$, the degeneracy of H_i (rooted at the set of vertices labelled by singletons) is at most d;*
(iv) *for each $v \in V(G)$, there are at most ηn graphs H_i with some vertex labelled $\{v\}$.*

Then there exist edge-disjoint embeddings $\phi(H_1), \dots, \phi(H_m)$ of $H_1, \dots, H_m$ compatible with their labellings such that the subgraph $H := \bigcup_{i=1}^m \phi(H_i)$ of G satisfies $\Delta(H) \leq \varepsilon n$.

We will also use the following partite version of the lemma to find copies of $\mathcal{P}$-labelled graphs in an r-partite graph G. We omit the proof since it is very similar to the proof of Lemma 5.1 in [3] (for details, see [26]).

Lemma 5.2. *Let $1/n \ll \eta \ll \varepsilon, 1/d, 1/b, 1/k, 1/r \leq 1$ and let G be an r-partite graph on $(V_1, \dots, V_r)$ where $|V_1| = \dots = |V_r| = n$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of $V(G)$. Suppose that:*

(i) *for each $1 \leq i \leq k$ and each $1 \leq j \leq r$, if $S \subseteq V(G) \setminus V_j$ with $|S| \leq d$ then $d_G(S, U^i_j) \geq \varepsilon |U^i_j|$.*

Let $m \leq \eta n^2$ and let $H_1, \dots, H_m$ be $\mathcal{P}$-labelled graphs such that the following hold:

(ii) *for each $1 \leq i \leq m$, $|H_i| \leq b$;*
(iii) *for each $1 \leq i \leq m$, the degeneracy of H_i is at most d;*
(iv) *for each $v \in V(G)$, there are at most ηn graphs H_i with some vertex labelled $\{v\}$.*

Then there exist edge-disjoint embeddings $\phi(H_1), \dots, \phi(H_m)$ of $H_1, \dots, H_m$ in G which are compatible with their labellings such that $H := \bigcup_{1 \leq i \leq m} \phi(H_i)$ satisfies $\Delta(H) \leq \varepsilon n$. □

6. Absorbers

Let H be any r-partite graph on the vertex set $V = (V_1, \dots, V_r)$. An *absorber for H* is a graph A such that both A and $A \cup H$ have K_r-decompositions.

Our aim is to find an absorber for each small K_r-divisible graph H on V. The construction develops ideas in [3]. In particular, we will build the absorber in stages using transformers, introduced below, to move between K_r-divisible graphs.

Let H and H' be vertex-disjoint graphs. An *$(H, H')_r$-transformer* is a graph T which is edge-disjoint from H and H' and is such that both $T \cup H$ and $T \cup H'$ have K_r-decompositions. Note that if H' has a K_r-decomposition, then $T \cup H'$ is an absorber for H. So the idea is that we can use a transformer to transform a given H into a new graph H', then into H'' and so on, until finally we arrive at a graph which has a K_r-decomposition.

Let $V = (V_1, \dots, V_r)$. Throughout this section, given two r-partite graphs H and H' on V, we say that H' is a *partition-respecting copy* of H if there is an isomorphism $f : H \to H'$ such that $f(v) \in V_j$ for every vertex $v \in V(H) \cap V_j$.

Given r-partite graphs H and H' on V, we say that H' is *obtained from H by identifying vertices* if there exists a sequence of r-partite graphs $H_0, \dots, H_s$ on V such that $H_0 = H$, $H_s = H'$ and the following holds. For each $0 \leq i < s$, there exists $1 \leq j_i \leq r$ and vertices $x_i, y_i \in V(H_i) \cap V_{j_i}$ satisfying the following:

(i) $N_{H_i}(x_i) \cap N_{H_i}(y_i) = \emptyset$.
(ii) H_{i+1} is the graph which has vertex set $V(H_i) \setminus \{y_i\}$ and edge set $E(H_i \setminus \{y_i\}) \cup \{vx_i : vy_i \in E(H_i)\}$ (i.e., H_{i+1} is obtained from H_i by identifying the vertices x_i and y_i).

Condition (i) ensures that the identifications do not produce multiple edges. Note that if H and H' are r-partite graphs on V and H' is a partition-respecting copy of a graph obtained from H by identifying

vertices then there exists a graph homomorphism $\phi : H \to H'$ that is edge-bijective and maps vertices in V_j to vertices in V_j for each $1 \le j \le r$.

In the following lemma, we find a transformer between a pair of K_r-divisible graphs H and H' whenever H' can be obtained from H by identifying vertices.

Lemma 6.1. *Let $r \ge 3$ and $1/n \ll \eta \ll 1/s \ll \varepsilon, 1/b, 1/r \le 1$. Let G be an r-partite graph on $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Suppose that $\hat{\delta}(G) \ge (1 - 1/(r+1) + \varepsilon)n$. Let H and H' be vertex-disjoint K_r-divisible graphs on V with $|H| \le b$. Suppose further that H' is a partition-respecting copy of a graph obtained from H by identifying vertices. Let $B \subseteq V$ be a set of at most ηn vertices. Then G contains an $(H, H')_r$-transformer T such that $V(T) \cap B \subseteq V(H \cup H')$ and $|T| \le s^2$.*

In our proof of Lemma 6.1, we will use the following multipartite asymptotic version of the Hajnal–Szemerédi theorem.

Theorem 6.2 ([17] and [20])**.** *Let $r \ge 2$ and let $1/n \ll \varepsilon, 1/r$. Suppose that G is an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$ and $\hat{\delta}(G) \ge (1 - 1/r + \varepsilon)n$. Then G contains a perfect K_r-matching.*

Proof of Lemma 6.1. Let $\phi : H \to H'$ be a graph homomorphism from H to H' that is edge-bijective and maps vertices in V_j to V_j for each $1 \le j \le r$.

Let T be any graph defined as follows:

(a) For each $xy \in E(H)$, $Z^{xy} := \{z_j^{xy} : 1 \le j \le r \text{ and } x, y \notin V_j\}$ is a set of $r-2$ vertices. For each $x \in V(H)$, let $Z^x := \bigcup_{y \in N_H(x)} Z^{xy}$.
(b) For each $x \in V(H)$, S^x is a set of $(r-1)s$ vertices.
(c) For all distinct $e, e' \in E(H)$ and all distinct $x, x' \in V(H)$, the sets Z^e, $Z^{e'}$, S^x, $S^{x'}$ and $V(H \cup H')$ are disjoint.
(d) $V(T) := V(H) \cup V(H') \cup \bigcup_{e \in E(H)} Z^e \cup \bigcup_{x \in V(H)} S^x$.
(e) $E_H := \{xz : x \in V(H) \text{ and } z \in Z^x\}$.
(f) $E_{H'} := \{\phi(x)z : x \in V(H) \text{ and } z \in Z^x\}$.
(g) $E_Z := \{wz : e \in E(H) \text{ and } w, z \in Z^e\}$.
(h) $E_S := \{xv : x \in V(H) \text{ and } v \in S^x\}$.
(i) $E'_S := \{\phi(x)v : x \in V(H) \text{ and } v \in S^x\}$.
(j) For each $x \in V(H)$, F_1^x is a perfect K_{r-1}-matching on $S^x \cup Z^x$.
(k) For each $x \in V(H)$, F_2^x is a perfect K_{r-1}-matching on S^x.
(l) For each $x \in V(H)$, F_1^x and F_2^x are edge-disjoint.
(m) For each $x \in V(H)$, Z^x is independent in F_1^x.
(n) $E(T) := E_H \cup E_{H'} \cup E_Z \cup E_S \cup E'_S \cup \bigcup_{x \in V(H)} E(F_1^x \cup F_2^x)$.

Then

$$|T| = |H| + |H'| + \sum_{e \in E(H)} |Z^e| + \sum_{x \in V(H)} |S^x| = |H| + |H'| + (r-2)e(H) + (r-1)s|H| \le s^2.$$

Let T_1 be the subgraph of T with edge set $E_H \cup E_{H'} \cup E_Z$ and let $T_2 := T - T_1$. So $E(T_2) = E_S \cup E'_S \cup \bigcup_{x \in V(H)} E(F_1^x \cup F_2^x)$. In what follows, we will often identify certain subsets of the edge set of T with the subgraphs of T consisting of these edges. For example, we will write $E_S[\{x\}, S^x]$ for the subgraph of T consisting of all the edges in E_S between x and S^x. Note that there are several possibilities for T as we have several choices for the perfect K_{r-1}-matchings in (j) and (k).

Lemma 6.1 will follow from Claims 1 and 2 below.

Claim 1: *If T satisfies* (a)–(n)*, then T is an $(H, H')_r$-transformer.*

Proof of Claim 1. Note that $H \cup E_H \cup E_Z$ can be decomposed into $e(H)$ copies of K_r, where each copy of K_r has vertex set $\{x, y\} \cup Z^{xy}$ for some edge $xy \in E(H)$. Similarly, $H' \cup E_{H'} \cup E_Z$ can be decomposed into $e(H)$ copies of K_r.

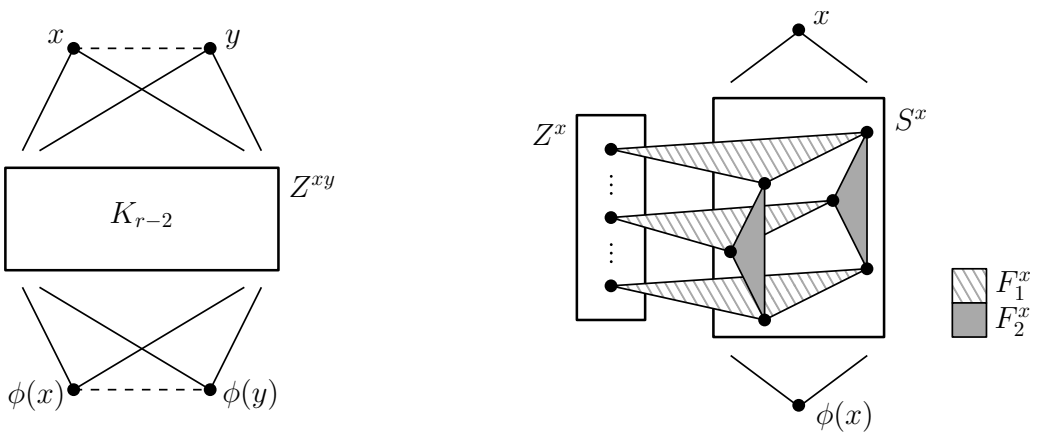

FIGURE 1. Left: Subgraph of T_1 associated with $xy \in E(H)$. Right: Subgraph of T_2 associated with $x \in V(H)$ in the case when $r = 4$.

For each $x \in V(H)$, note that $(E_{H'} \cup E'_S)[\{\phi(x)\}, S^x \cup Z^x] \cup F_1^x$ and $E_S[\{x\}, S^x] \cup F_2^x$ are edge-disjoint and have K_r-decompositions. Since

$$T_2 \cup E_{H'} = \bigcup_{x \in V(H)} \big((E_{H'} \cup E'_S)[\{\phi(x)\}, S^x \cup Z^x] \cup F_1^x\big) \cup \bigcup_{x \in V(H)} \big(E_S[\{x\}, S^x] \cup F_2^x\big),$$

it follows that $T_2 \cup E_{H'}$ has a K_r-decomposition. Similarly, for each $x \in V(H)$, $(E_H \cup E_S)[\{x\}, S^x \cup Z^x] \cup F_1^x$ and $E'_S[\{\phi(x)\}, S^x] \cup F_2^x$ are edge-disjoint and have K_r-decompositions, so $T_2 \cup E_H$ has a K_r-decomposition.

To summarise, $H \cup E_H \cup E_Z$, $H' \cup E_{H'} \cup E_Z$, $T_2 \cup E_H$ and $T_2 \cup E_{H'}$ all have K_r-decompositions. Therefore, $T \cup H = (H \cup E_H \cup E_Z) \cup (T_2 \cup E_{H'})$ has a K_r-decomposition, as does $T \cup H' = (H' \cup E_{H'} \cup E_Z) \cup (T_2 \cup E_H)$. Hence T is an $(H, H')_r$-transformer.

Claim 2: *G contains a graph T satisfying* (a)–(n) *such that $V(T) \cap B \subseteq V(H \cup H')$.*

Proof of Claim 2. We begin by finding a copy of T_1 in G. It will be useful to note that, for any graph T which satisfies (a)–(n), T_1 is r-partite with vertex classes $(V(H \cup H') \cap V_j) \cup \{z_j^{xy} : xy \in E(H) \text{ and } x, y \notin V_j\}$ where $1 \le j \le r$. Also, $T[V(H \cup H')]$ is empty and every vertex $z \in V(T_1) \setminus V(H \cup H')$ satisfies

$$d_{T_1}(z) = 2 + (r-3) + 2 = r + 1. \tag{6.1}$$

So T_1 has degeneracy $r+1$ rooted at $V(H \cup H')$. Since $\hat{\delta}(G) \ge (1 - 1/(r+1) + \varepsilon/2)n + |B|$, we can find a copy of T_1 in G such that $V(T_1) \cap B \subseteq V(H \cup H')$.

We now show that, after fixing T_1, we can extend T_1 to T by finding a copy of T_2. Consider any ordering $x_1, \dots, x_{|H|}$ on the vertices of H. Suppose we have already chosen $S^{x_1}, \dots, S^{x_{q-1}}$, $F_1^{x_1}, \dots, F_1^{x_{q-1}}$ and $F_2^{x_1}, \dots, F_2^{x_{q-1}}$ and we are currently embedding S^{x_q}. Let $B' := B \cup V(T_1) \cup \bigcup_{i=1}^{q-1} S^{x_i}$; that is, B' is the set of vertices that are unavailable for S^{x_q}, either because they have been used previously or they lie B. Note that $|B'| \le |T| + |B| \le 2\eta n$. We will choose suitable vertices for S^{x_q} in the common neighbourhood of x_q and $\phi(x_q)$.

To simplify notation, we write $x := x_q$ and assume that $x \in V_1$ (the argument is identical in the other cases). Choose a set $V' \subseteq (N_G(x) \cap N_G(\phi(x))) \setminus B'$ which is maximal subject to $|V'_2| = \cdots = |V'_r|$ (recall that $V'_j = V' \cap V_j$). Note that for each $2 \le j \le r$, we have

$$|V'_j| \ge (1 - 1/(r+1) + \varepsilon)n - (1/(r+1) - \varepsilon)n - |B'| \ge (1 - 2/(r+1))n.$$

Let $n' := |V'_2|$. For every $2 \le j \le r$ and every $v \in V(G) \setminus V_j$, we have

$$d_G(v, V'_j) \ge n' - (1/(r+1) - \varepsilon)n \ge (1 - 1/(r-1) + \varepsilon)n'. \tag{6.2}$$

Roughly speaking, we will choose S^x as a random subset of V'. For each $2 \le j \le r$, choose each vertex of V'_j independently with probability $p := (1+\varepsilon/8)s/n'$ and let S'_j be the set of chosen vertices. Note that, for each j, $\mathbb{E}(|S'_j|) = n'p = (1+\varepsilon/8)s$. We can apply Lemma 2.2 to see that

$$\begin{aligned}\mathbb{P}(||S'_j| - (1+\varepsilon/8)s| \ge \varepsilon s/8) &\le \mathbb{P}(||S'_j| - (1+\varepsilon/8)s| \ge \varepsilon \mathbb{E}(|S'_j|)/10)\\ &\le 2e^{-\varepsilon^2 s/300} \le 1/4(r-1). \end{aligned} \tag{6.3}$$

Given a vertex $v \in V(G)$ and $2 \le j \le r$ such that $v \notin V_j$, note that

$$\mathbb{E}(d_G(v, S'_j)) \overset{(6.2)}{\ge} (1 - 1/(r-1) + \varepsilon)n'p > (1 - 1/(r-1) + \varepsilon)s.$$

We will say that a vertex $v \in V(G)$ is *bad* if there exists $2 \le j \le r$ such that $v \notin V_j$ and $d_G(v, S'_j) < (1-1/(r-1)+3\varepsilon/4)s$, that is, the degree of v in S'_j is lower than expected. We can again apply Lemma 2.2 to see that

$$\begin{aligned}\mathbb{P}(d_G(v, S'_j) \le (1 - 1/(r-1) + 3\varepsilon/4)s) &\le \mathbb{P}(|d_G(v, S'_j) - \mathbb{E}(d_G(v, S'_j))| \ge \varepsilon s/4)\\ &\le \mathbb{P}(|d_G(v, S'_j) - \mathbb{E}(d_G(v, S'_j))| \ge \varepsilon \mathbb{E}(d_G(v, S'_j))/10)\\ &\le 2e^{-\varepsilon^2 s/600}.\end{aligned}$$

So $\mathbb{P}(v \text{ is bad}) \le 2(r-1)e^{-\varepsilon^2 s/600} \le e^{-s^{1/2}}$. Let $S' := \bigcup_{j=2}^r S'_j$. We say that the set S' is *bad* if $S' \cup Z^x$ contains a bad vertex. We have

$$\begin{aligned}\mathbb{P}(S' \text{ is bad}) &\le \sum_{v \in V'} \mathbb{P}(v \in S' \text{ and } v \text{ is bad}) + \sum_{v \in Z^x} \mathbb{P}(v \text{ is bad})\\ &= \sum_{v \in V'} \mathbb{P}(v \in S')\mathbb{P}(v \text{ is bad}) + \sum_{v \in Z^x} \mathbb{P}(v \text{ is bad})\\ &\le (n'p + (b-1)(r-2))e^{-s^{1/2}} \le 2se^{-s^{1/2}} \le 1/4.\end{aligned} \tag{6.4}$$

We apply (6.3) and (6.4) to see that with probability at least $1/2$, the set S' chosen in this way is not bad and, for each $2 \le j \le r$, we have $s \le |S'_j| \le (1+\varepsilon/4)s$. Choose one such set S'. Delete at most $\varepsilon s/4$ vertices from each S'_j to obtain sets S^x_j satisfying $|S^x_2| = \cdots = |S^x_r| = s$. Let $S^x := \bigcup_{j=2}^r S^x_j$. Since S' was not bad, for each $2 \le j \le r$ and each vertex $v \in (S^x \cup Z^x) \setminus V_j$,

$$d_G(v, S^x_j) \ge (1 - 1/(r-1) + 3\varepsilon/4)s - \varepsilon s/4 = (1 - 1/(r-1) + \varepsilon/2)s. \tag{6.5}$$

We now show that we can find F^x_1 and F^x_2 satisfying (j)–(m). Let $G^x := G[Z^x \cup S^x] - G[Z^x]$. Note that G^x is a balanced $(r-1)$-partite graph with vertex classes of size n_x where $s \le n_x \le s + (r-2)(b-1)/(r-1) < s + b$. Using (6.5), we see that

$$\hat{\delta}(G^x) \ge (1 - 1/(r-1) + \varepsilon/2)s \ge (1 - 1/(r-1) + \varepsilon/3)n_x.$$

So, using Theorem 6.2, we can find a perfect K_{r-1}-matching F^x_1 in G^x. Finally, let $G' := G - F^x_1$ and use (6.5) to see that

$$\hat{\delta}(G'[S^x]) \ge (1 - 1/(r-1) + \varepsilon/3)s.$$

So we can again apply Theorem 6.2, to find a perfect K_{r-1}-matching F^x_2 in $G'[S^x]$. In this way, we find a copy of T satisfying (a)–(n) such that $V(T) \cap B \subseteq V(H \cup H')$. □

We now construct our absorber by combining several suitable transformers.

Let H be an r-partite multigraph on $V = (V_1, \dots, V_r)$ and let $xy \in E(H)$. A *K_r-expansion of xy* is defined as follows. Consider a copy F_{xy} of K_r on vertex set $\{u_1, \dots, u_r\}$ such that $u_j \in V_j \setminus V(H)$ for all $1 \le j \le r$. Let j_1, j_2 be such that $x \in V_{j_1}$ and $y \in V_{j_2}$. Delete xy from H and $u_{j_1}u_{j_2}$ from F_{xy} and add edges joining x to u_{j_2} and joining y to u_{j_1}. Let $H_{\exp}$ be the graph obtained by K_r-expanding every edge of H, where the F_{xy} are chosen to be vertex-disjoint for different edges $xy \in E(H)$.

Fact 6.3. *Suppose that the graph H' is obtained from a graph H by K_r-expanding the edge $xy \in E(H)$ as above. Then the graph obtained from H' by identifying x and u_{j_1} is H with a copy of K_r attached to x.*

Let $h \in \mathbb{N}$. We define a graph M_h as follows. Take a copy of K_r on V (consisting of one vertex in each V_j) and replace each edge by h multiedges. Let M denote the resulting multigraph. Let $M_h := M_{\exp}$ be the graph obtained by K_r-expanding every edge of M. We have $|M_h| = r + hr\binom{r}{2}$. Note that M_h has degeneracy $r-1$. To see this, list all vertices in $V(M)$ (in any order) followed by the vertices in $V(M_h \setminus M)$ (in any order).

We will now apply Lemma 6.1 twice in order to find an $(H, M_h)_r$-transformer in G.

Lemma 6.4. *Let $r \ge 3$ and $1/n \ll \eta \ll 1/s \ll \varepsilon, 1/b, 1/r \le 1$. Let G be an r-partite graph on $V = (V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Suppose that $\hat{\delta}(G) \ge (1 - 1/(r+1) + \varepsilon)n$. Let H be a K_r-divisible graph on V with $|H| \le b$. Let $h := e(H)/\binom{r}{2}$. Let M_h' be a partition-respecting copy of M_h on V which is vertex-disjoint from H. Let $B \subseteq V$ be a set of at most ηn vertices. Then G contains an $(H, M_h')_r$-transformer T such that $V(T) \cap B \subseteq V(H \cup M_h')$ and $|T| \le 3s^2$.*

Proof. We construct a graph H_{att} as follows. Start with the graph H. For each edge of H, arbitrarily choose one of it endpoints x and attach a copy of K_r (found in $G \setminus ((V(H \cup M_h') \cup B) \setminus \{x\})$) to x. The copies of K_r should be chosen to be vertex-disjoint outside $V(H)$. Write H_{att} for the resulting graph. Let $H'_{\exp}$ be a partition-respecting copy of $H_{\exp}$ in $G \setminus (V(H_{\mathrm{att}} \cup M_h') \cup B)$. Note that we are able to find these graphs since both have degeneracy $r-1$ and $\hat{\delta}(G) \ge (1 - 1/(r+1) + \varepsilon)n$.

By Fact 6.3, H_{att} is a partition-respecting copy of a graph obtained from $H'_{\exp}$ by identifying vertices, and this is also the case for M_h'. To see the latter, for each $1 \le j \le r$, identify all vertices of $H'_{\exp}$ lying in V_j. (We are able to do this since these vertices are non-adjacent with disjoint neighbourhoods.)

Apply Lemma 6.1 to find an $(H'_{\exp}, H_{\mathrm{att}})_r$-transformer T' in $G - M_h'$ such that $V(T') \cap B \subseteq V(H)$ and $|T'| \le s^2$. Then apply Lemma 6.1 again to find an $(H'_{\exp}, M_h')_r$-transformer T'' in $G - (H_{\mathrm{att}} \cup T')$ such that $V(T'') \cap B \subseteq V(M_h')$ and $|T''| \le s^2$.

Let $T := T' \cup T'' \cup H'_{\exp} \cup (H_{\mathrm{att}} - H)$. Then T is edge-disjoint from $H \cup M_h'$. Note that

$$T \cup H = (T' \cup H_{\mathrm{att}}) \cup (T'' \cup H'_{\exp}) \quad \text{and}$$
$$T \cup M_h' = (T' \cup H'_{\exp}) \cup (T'' \cup M_h') \cup (H_{\mathrm{att}} - H),$$

both of which have K_r-decompositions. Therefore T is an $(H, M_h')_r$-transformer. Moreover, $|T| \le 3s^2$. Finally, observe that $V(T) \cap B = V(T' \cup T'' \cup H_{\mathrm{att}}) \cap B \subseteq V(H \cup M_h')$. □

We now have all of the necessary tools to find an absorber for H in G.

Lemma 6.5. *Let $r \ge 3$ and let $1/n \ll \eta \ll 1/s \ll \varepsilon, 1/b, 1/r \le 1$. Let G be an r-partite graph on $V = (V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Suppose that $\hat{\delta}(G) \ge (1 - 1/(r+1) + \varepsilon)n$. Let H be a K_r-divisible graph on V with $|H| \le b$. Let $B \subseteq V$ be a set of at most ηn vertices. Then G contains an absorber A for H such that $V(A) \cap B \subseteq V(H)$ and $|A| \le s^3$.*

Proof. Let $h := e(H)/\binom{r}{2}$. Let $G' := G \setminus (V(H) \cup B)$. Write hK_r for the graph consisting of h vertex-disjoint copies of K_r. Since $\hat{\delta}(G') \ge (1 - 1/(r+1) + \varepsilon/2)n$, we can choose vertex-disjoint (partition-respecting) copies of M_h and hK_r in G' (and call these M_h and hK_r again). Use Lemma 6.4

to find an $(H, M_h)_r$-transformer T' in $G - hK_r$ such that $V(T') \cap B \subseteq V(H)$ and $|T'| \leq 3s^2$. Apply Lemma 6.4 again to find an $(hK_r, M_h)_r$-transformer T'' in $G - (H \cup T')$ which avoids B and satisfies $|T''| \leq 3s^2$. It is easy to see that $T := T' \cup T'' \cup M_h$ is an $(H, hK_r)_r$-transformer.

Let $A := T \cup hK_r$. Note that both A and $A \cup H = (T \cup H) \cup hK_r$ have K_r-decompositions. So A is an absorber for H. Moreover, $V(A) \cap B \subseteq V(T') \cap B \subseteq V(H)$ and $|A| \leq s^3$. □

6.1. **Absorbing sets.** Let $\mathcal{H}$ be a collection of graphs on the vertex set $V = (V_1, \dots, V_r)$. We say that $\mathcal{A}$ is an *absorbing set for* $\mathcal{H}$ if $\mathcal{A}$ is a collection of edge-disjoint graphs and, for every $H \in \mathcal{H}$ and every K_r-divisible subgraph $H' \subseteq H$, there is a distinct $A_{H'} \in \mathcal{A}$ such that $A_{H'}$ is an absorber for H'.

Lemma 6.6. *Let $r \geq 3$ and $1/n \ll \eta \ll \varepsilon, 1/b, 1/r \leq 1$. Let G be an r-partite graph on $V = (V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Suppose that $\hat{\delta}(G) \geq (1 - 1/(r+1) + \varepsilon)n$. Let $m \leq \eta n^2$ and let $\mathcal{H}$ be a collection of m edge-disjoint graphs on $V = (V_1, \dots, V_r)$ such that each vertex $v \in V$ appears in at most ηn of the elements of $\mathcal{H}$ and $|H| \leq b$ for each $H \in \mathcal{H}$. Then G contains an absorbing set $\mathcal{A}$ for $\mathcal{H}$ such that $\Delta(\bigcup \mathcal{A}) \leq \varepsilon n$.*

We repeatedly use Lemma 6.5 and aim to avoid any vertices which have been used too often.

Proof. Enumerate the K_r-divisible subgraphs of all $H \in \mathcal{H}$ as $H_1, \dots, H_{m'}$. Note that each $H \in \mathcal{H}$ can have at most $2^{e(H)} \leq 2^{\binom{b}{2}}$ K_r-divisible subgraphs so $m' \leq 2^{\binom{b}{2}}\eta n^2$. For each $v \in V(G)$ and each $0 \leq j \leq m$, let $s(v, j)$ be the number of indices $1 \leq i \leq j$ such that $v \in V(H_i)$. Note that $s(v, j) \leq 2^{\binom{b}{2}}\eta n$.

Let $s \in \mathbb{N}$ be such that $\eta \ll 1/s \ll \varepsilon, 1/b, 1/r$. Suppose that we have already found absorbers $A_1, \dots, A_{j-1}$ for $H_1, \dots, H_{j-1}$ respectively such that $|A_i| \leq s^3$, for all $1 \leq i \leq j-1$, and, for every $v \in V(G)$,

$$d_{G_{j-1}}(v) \leq \eta^{1/2}n + (s(v, j-1) + 1)s^3, \tag{6.6}$$

where $G_{j-1} := \bigcup_{1 \leq i \leq j-1} A_i$. We show that we can find an absorber A_j for H_j in $G - G_{j-1}$ which satisfies (6.6) with j replacing $j - 1$.

Let $B := \{v \in V(G) : d_{G_{j-1}}(v) \geq \eta^{1/2}n\}$. We have

$$|B| \leq \frac{2e(G_{j-1})}{\eta^{1/2}n} \leq \frac{2m'\binom{s^3}{2}}{\eta^{1/2}n} \leq \frac{2^{\binom{b}{2}+1}\eta n^2 s^6}{\eta^{1/2}n} \leq \eta^{1/3}n.$$

We have

$$\begin{aligned}\hat{\delta}(G - G_{j-1}) &\overset{(6.6)}{\geq} (1 - 1/(r+1) + \varepsilon)n - \eta^{1/2}n - (s(v, j-1) + 1)s^3 \\ &\geq (1 - 1/(r+1) + \varepsilon)n - \eta^{1/2}n - (2^{\binom{b}{2}}\eta n + 1)s^3 > (1 - 1/(r+1) + \varepsilon/2)n.\end{aligned}$$

So we can apply Lemma 6.5 (with $\varepsilon/2$, $\eta^{1/3}$, $G - G_{j-1}$ and H_j playing the roles of ε, η, G and H) to find an absorber A_j for H_j in $G - G_{j-1}$ such that $V(A_j) \cap B \subseteq V(H_j)$ and $|A_j| \leq s^3$.

We now check that (6.6) holds with j replacing $j-1$. If $v \in V(G) \setminus B$, this is clear. Suppose then that $v \in B$. If $v \in V(A_j)$, then $v \in V(H_j)$ and $s(v, j) = s(v, j-1) + 1$. So in all cases,

$$d_{G_j}(v) \leq \eta^{1/2}n + (s(v, j) + 1)s^3.$$

Continue in this way until we have found an absorber A_i for each H_i. Then $\mathcal{A} := \{A_i : 1 \leq i \leq m'\}$ is an absorbing set. Using (6.6),

$$\Delta(\bigcup \mathcal{A}) = \Delta(G_{m'}) \leq \eta^{1/2}n + (2^{\binom{b}{2}}\eta n + 1)s^3 \leq \varepsilon n,$$

as required. □

7. Partitions and random subgraphs

In this section we consider a sequence $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ of successively finer partitions which will underlie our iterative absorption process. We will also construct corresponding sparse quasirandom subgraphs R_i which will be used to 'smooth out' the leftover from the approximate decomposition in each step of the process.

Let G be an r-partite graph on $(V_1, \dots, V_r)$. An (α, k, δ)-*partition* for G on $(V_1, \dots, V_r)$ is a partition $\mathcal{P} = \{U^1, \dots, U^k\}$ of $V(G)$ such that the following hold:

(Pa1) for each $1 \le j \le r$, $\{U_j^i : 1 \le i \le k\}$ is an equitable partition of V_j (recall that $U_j^i = U^i \cap V_j$);
(Pa2) for each $1 \le i \le k$, $|U_1^i| = \dots = |U_r^i|$;
(Pa3) for each $v \in V(G)$, each $1 \le i \le k$ and each $1 \le j \le r$,

$$|d_G(v, U_j^i) - d_G(v, V_j)/k| < \alpha |U_j^i|;$$

(Pa4) for each $1 \le i \le k$, each $1 \le j \le r$ and each $v \notin V_j$, $d_G(v, U_j^i) \ge \delta |U_j^i|$.

We say that $\mathcal{P} = \{U^1, \dots, U^k\}$ is a k-*partition* if it satisfies (Pa1) and (Pa2).

The following proposition guarantees a $(n^{-1/3}/2, k, \delta - n^{-1/3}/2)$-partition of any sufficiently large balanced r-partite graph G with $\hat{\delta}(G) \ge \delta n$. To prove this result, it suffices to consider an equitable partition $U_j^1, U_j^2, \dots, U_j^k$ of V_j chosen uniformly at random (with $|U_j^1| \le \dots \le |U_j^k|$).

Proposition 7.1. *Let $k, r \in \mathbb{N}$. There exists n_0 such that if $n \ge n_0$ and G is any r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$ and $\hat{\delta}(G) \ge \delta n$, then G has a $(\nu, k, \delta - \nu)$-partition, where $\nu := n^{-1/3}/2$.* □

We say that $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_\ell$ is an (α, k, δ, m)-*partition sequence* for G on $(V_1, \dots, V_r)$ if, writing $\mathcal{P}_0 := \{V(G)\}$,

(S1) for each $1 \le i \le \ell$, $\mathcal{P}_i$ refines $\mathcal{P}_{i-1}$;
(S2) for each $1 \le i \le \ell$ and each $W \in \mathcal{P}_{i-1}$, $\mathcal{P}_i[W]$ is an (α, k, δ)-partition for $G[W]$;
(S3) for each $1 \le i \le \ell$, all $1 \le j_1, j_2, j_3 \le r$ with $j_1 \ne j_2, j_3$, each $W \in \mathcal{P}_{i-1}$, each $U \in \mathcal{P}_i[W]$ and each $v \in W_{j_1}$,

$$|d_G(v, U_{j_2}) - d_G(v, U_{j_3})| < \alpha |U_{j_1}|;$$

(S4) for each $U \in \mathcal{P}_\ell$ and each $1 \le j \le r$, $|U_j| = m$ or $m - 1$.

Note that (S2) and (Pa2) together imply that $|U_{j_1}| = |U_{j_2}|$ for each $1 \le i \le \ell$, each $U \in \mathcal{P}_i$ and all $1 \le j_1, j_2 \le r$.

By successive applications of Proposition 7.1, we immediately obtain the following result which guarantees the existence of a suitable partition sequence (for details see [26]).

Lemma 7.2. *Let $k, r \in \mathbb{N}$ with $k \ge 2$ and let $0 < \alpha < 1$. There exists m_0 such that, for all $m' \ge m_0$, any K_r-divisible graph G on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n \ge km'$ and $\hat{\delta}(G) \ge \delta n$ has an $(\alpha, k, \delta - \alpha, m)$-partition sequence for some $m' \le m \le km'$.* □

Suppose that we are given a k-partition $\mathcal{P}$ of G. The following proposition finds a quasirandom spanning subgraph R of G so that each vertex in R has roughly the expected number of neighbours in each set $U \in \mathcal{P}$. The proof is an easy application of Lemma 2.1.

Proposition 7.3. *Let $1/n \ll \alpha, \rho, 1/k, 1/r \le 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Suppose that $\mathcal{P}$ is a k-partition for G. Let $\mathcal{S}$ be a collection of at most n^2 subsets of $V(G)$. Then there exists $R \subseteq G[\mathcal{P}]$ such that for all $1 \le j \le r$, all distinct $x, y \in V(G)$, all $U \in \mathcal{P}$ and all $S \in \mathcal{S}$:*

- $|d_R(x, U_j) - \rho d_{G[\mathcal{P}]}(x, U_j)| < \alpha |U_j|$;

- $|d_R(\{x,y\},U_j)-\rho^2 d_{G[\mathcal{P}]}(\{x,y\},U_j)|<\alpha|U_j|$;
- $|d_G(y,N_R(x,U_j))-\rho d_G(y,N_{G[\mathcal{P}]}(x,U_j))|<\alpha|U_j|$;
- $|d_R(y,S_j)-\rho d_{G[\mathcal{P}]}(y,S_j)|<\alpha n$. □

We need to reserve some quasirandom subgraphs R_i of G at the start of our proof, whilst the graph G is still almost balanced with respect to the partition sequence. We will add the edges of R_i back after finding an approximate decomposition of $G[\mathcal{P}_i]$ in order to assume the leftover from this approximate decomposition is quasirandom. The next lemma gives us suitable subgraphs for R_i.

Lemma 7.4. *Let $1/m \ll \alpha \ll \rho, 1/k, 1/r \leq 1$. Let G be an r-partite graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|$. Suppose that $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ is a $(1,k,0,m)$-partition sequence for G. Let $\mathcal{P}_0:=\{V(G)\}$ and, for each $0\leq q\leq \ell$, let $G_q:=G[\mathcal{P}_q]$. Then there exists a sequence of graphs $R_1,\dots,R_\ell$ such that $R_q\subseteq G_q-G_{q-1}$ for each q and the following holds. For all $1\leq q\leq \ell$, all $1\leq j\leq r$, all $W\in\mathcal{P}_{q-1}$, all distinct $x,y\in W$ and all $U\in\mathcal{P}_q[W]$:*

(i) $|d_{R_q}(x,U_j)-\rho d_{G_q}(x,U_j)|<\alpha|U_j|$;
(ii) $|d_{R_q}(\{x,y\},U_j)-\rho^2 d_{G_q}(\{x,y\},U_j)|<\alpha|U_j|$;
(iii) $d_{G'_{q+1}}(y,N_{R_q}(x,U_j))\geq \rho d_{G_{q+1}}(y,N_{G_q}(x,U_j))-3\rho^2|U_j|$, *where $G'_{q+1}:=G_{q+1}-R_{q+1}$ if $q\leq \ell-1$, $G'_{\ell+1}:=G$ and $G_{\ell+1}:=G$.*

Proof. For $1\leq q\leq \ell$, we say that the sequence of graphs $R_1,\dots,R_q$ is *good* if $R_i\subseteq G_i-G_{i-1}$ and for all $1\leq i\leq q$, all $1\leq j\leq r$, all $W\in\mathcal{P}_{i-1}$, all distinct $x,y\in W$ and all $U\in\mathcal{P}_i[W]$:

(a) (i) and (ii) hold (with q replaced by i);
(b) $|d_{G_{i+1}}(y,N_{R_i}(x,U_j))-\rho d_{G_{i+1}}(y,N_{G_i}(x,U_j))|<\alpha|U_j|$;
(c) if $i\leq q-1$, $d_{R_{i+1}}(y,N_{R_i}(x,U_j))<\rho d_{G_{i+1}}(y,N_{R_i}(x,U_j))+\alpha|U_j|$.

Suppose $1\leq q\leq \ell$ and we have found a good sequence of graphs $R_1,\dots,R_{q-1}$. We will find R_q such that $R_1,\dots,R_q$ is good. Let $W\in\mathcal{P}_{q-1}$, let $\mathcal{S}_1$ be the empty set and, if $q\geq 2$, let $W'\in\mathcal{P}_{q-2}$ be such that $W\subseteq W'$ and let $\mathcal{S}_q:=\{N_{R_{q-1}}(x,W):x\in W'\}$. Apply Proposition 7.3 (with $|W|/r$, $G_{q+1}[W]$, $\mathcal{P}_q[W]$ and $\mathcal{S}_q$ playing the roles of n, G, $\mathcal{P}$ and $\mathcal{S}$) to find $R_W\subseteq G_{q+1}[W][\mathcal{P}_q[W]]=G_q[W]$ such that:

$$\begin{aligned}
|d_{R_W}(x,U_j)-\rho d_{G_q}(x,U_j)|&<\alpha|U_j|,\\
|d_{R_W}(\{x,y\},U_j)-\rho^2 d_{G_q}(\{x,y\},U_j)|&<\alpha|U_j|,\\
|d_{G_{q+1}}(y,N_{R_W}(x,U_j))-\rho d_{G_{q+1}}(y,N_{G_q}(x,U_j))|&<\alpha|U_j|,\\
|d_{R_W}(y,S_j)-\rho d_{G_q}(y,S_j)|&<\alpha|W_j|,
\end{aligned}\tag{7.1}$$

for all $1\leq j\leq r$, all distinct $x,y\in W$, all $U\in\mathcal{P}_q[W]$ and all $S\in\mathcal{S}_q$. Set $R_q:=\bigcup_{W\in\mathcal{P}_{q-1}}R_W$. It is clear that $R_1,\dots,R_q$ satisfy (a) and (b). We now check that (c) holds when $1\leq i=q-1$. Let $1\leq j\leq r$, $W\in\mathcal{P}_{q-2}$, $x,y\in W$ be distinct and $U\in\mathcal{P}_{q-1}[W]$. If $y\notin U$, then $d_{R_q}(y,U_j)=0$ and so (c) holds. If $y\in U$, then $d_{R_q}(y,N_{R_{q-1}}(x,U))=d_{R_U}(y,N_{R_{q-1}}(x,U))$ and (c) follows by replacing W and S by U and $N_{R_{q-1}}(x,U)$ in property (7.1). So $R_1,\dots,R_q$ is good.

So G contains a good sequence of graphs $R_1,\dots,R_\ell$. We will now check that this sequence also satisfies (iii). If $q=\ell$, this follows immediately from (b). Let $1\leq q<\ell$, $1\leq j\leq r$, $W\in\mathcal{P}_{q-1}$, $x,y\in W$ be distinct and $U\in\mathcal{P}_q[W]$. We have

$$\begin{aligned}
d_{R_{q+1}}(y,N_{R_q}(x,U_j))&\overset{(c)}{<}\rho d_{G_{q+1}}(y,N_{R_q}(x,U_j))+\alpha|U_j|\\
&\overset{(b)}{<}\rho^2 d_{G_{q+1}}(y,N_{G_q}(x,U_j))+(\alpha\rho+\alpha)|U_j|\leq 2\rho^2|U_j|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d_{G'_{q+1}}(y,N_{R_q}(x,U_j))&=d_{G_{q+1}}(y,N_{R_q}(x,U_j))-d_{R_{q+1}}(y,N_{R_q}(x,U_j))\\
&\overset{(b)}{\geq}\rho d_{G_{q+1}}(y,N_{G_q}(x,U_j))-3\rho^2|U_j|.
\end{aligned}$$

So $R_1, \dots, R_\ell$ satisfy (i)–(iii). □

We apply Lemma 7.4 when $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is an $(\alpha, k, 1-1/r+\varepsilon, m)$-partition sequence for G to obtain the following result. For details of the proof, see [26].

Corollary 7.5. *Let $1/m \ll \alpha \ll \rho, 1/k \ll \varepsilon, 1/r \leq 1$. Let G be a K_r-divisible graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r|$. Suppose that $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is an $(\alpha, k, 1-1/r+\varepsilon, m)$-partition sequence for G. Let $\mathcal{P}_0 := \{V(G)\}$ and $G_q := G[\mathcal{P}_q]$ for $0 \leq q \leq \ell$. There exists a sequence of graphs $R_1, \dots, R_\ell$ such that $R_q \subseteq G_q - G_{q-1}$ for each $1 \leq q \leq \ell$ and the following holds. For all $1 \leq q \leq \ell$, all $1 \leq j, j' \leq r$, all $W \in \mathcal{P}_{q-1}$, all distinct $x, y \in W$ and all $U, U' \in \mathcal{P}_q[W]$:*

(i) $d_{R_q}(x, U_j) < \rho d_{G_q}(x, U_j) + \alpha |U_j|$;
(ii) $d_{R_q}(\{x, y\}, U_j) < (\rho^2 + \alpha)|U_j|$;
(iii) *if $x \notin U \cup U' \cup V_j \cup V_{j'}$,* $|d_{R_q}(x, U_j) - d_{R_q}(x, U'_{j'})| < 3\alpha |U_j|$;
(iv) *if $x \notin U$, $y \in U$ and $x, y \notin V_j$, then*

$$d_{G'_{q+1}}(y, N_{R_q}(x, U_j)) \geq \rho(1 - 1/(r-1)) d_{G_q}(x, U_j) + \rho^{5/4}|U_j|,$$

where $G'_{q+1} := G_{q+1} - R_{q+1}$ if $q \leq \ell - 1$ and $G'_{\ell+1} := G$. □

8. A remainder of low maximum degree

The aim of this section is to prove the following lemma which lets us assume that the remainder of G after finding an η-approximate decomposition has small maximum degree.

Lemma 8.1. *Let $1/n \ll \alpha \ll \eta \ll \gamma \ll \varepsilon < 1/r < 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$ and $\hat{\delta}(G) \geq (\hat{\delta}^{\eta}_{K_r} + \varepsilon)n$. Suppose also that, for all $1 \leq j_1, j_2 \leq r$ and every $v \notin V_{j_1} \cup V_{j_2}$,*

$$|d_G(v, V_{j_1}) - d_G(v, V_{j_2})| < \alpha n. \tag{8.1}$$

Then there exists $H \subseteq G$ such that $G - H$ has a K_r-decomposition and $\Delta(H) \leq \gamma n$.

Our strategy for the proof of Lemma 8.1 is as follows. We first remove a sparse random subgraph H of G and then choose an η-approximate K_r-decomposition of $G - H$. Now consider the remainder R obtained from G by deleting all edges in the copies of K_r in this decomposition. Suppose that v is a vertex whose degree in R is too high. Our aim will be to find a K_{r-1}-matching in a sparse random subgraph whose vertex set is the neighbourhood of v in G. Each vertex in this random subgraph sees, on average, at most $\rho d_G(v)/(r-1) \ll (1 - 1/(r-1) + \varepsilon) d_G(v)/(r-1)$ vertices in each other part, so Theorem 6.2 alone is of no use. But Theorem 6.2 can be combined with the Regularity lemma in order to find the desired K_r-matching.

8.1. Regularity.

In this section, we introduce a version of the Regularity lemma which we will use to prove Lemma 8.1.

Let G be a bipartite graph on (A, B). For non-empty sets $X \subseteq A$, $Y \subseteq B$, we define the *density of* $G[X, Y]$ to be $d_G(X, Y) := e_G(X, Y)/|X||Y|$. Let $\varepsilon > 0$. We say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon |A|$ and $|Y| \geq \varepsilon |B|$ we have

$$|d_G(A, B) - d_G(X, Y)| < \varepsilon.$$

The following simple result follows immediately from this definition.

Proposition 8.2. *Suppose that $0 < \varepsilon \leq \alpha \leq 1/2$. Let G be a bipartite graph on (A, B). Suppose that G is ε-regular with density d. If $A' \subseteq A, B' \subseteq B$ with $|A'| \geq \alpha |A|$ and $|B'| \geq \alpha |B|$ then $G[A', B']$ is ε/α-regular and has density greater than $d - \varepsilon$.* □

Proposition 8.2 shows that regularity is robust, that is, it is not destroyed by deleting a small number of vertices. The next observation allows us to delete a small number of edges at each vertex and still maintain regularity. The proof again follows from the definition.

Proposition 8.3. *Let $n \in \mathbb{N}$ and let $0 < \gamma \ll \varepsilon \leq 1$. Let G be a bipartite graph on (A, B) with $|A| = |B| = n$. Suppose that G is ε-regular with density d. Let $H \subseteq G$ with $\Delta(H) \leq \gamma n$ and let $G' := G - H$. Then G' is 2ε-regular and has density greater than $d - \varepsilon/2$.* □

The following proposition takes a graph G on $(V_1, \dots, V_r)$ where each pair of vertex classes induces an ε-regular pair and allows us to find a K_r-matching covering most of the vertices in G. Part (i) follows from Proposition 8.2 and the definition of regularity. For (ii), apply (i) repeatedly until only $\lceil \varepsilon^{1/r} n \rceil$ vertices remain uncovered in each V_j.

Proposition 8.4. *Let $1/n \ll \varepsilon \ll d, 1/r \leq 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Suppose that, for all $1 \leq j_1 < j_2 \leq r$, the graph $G[V_{j_1}, V_{j_2}]$ is ε-regular with density at least d.*

(i) *For each $1 \leq j \leq r$, let $W_j \subseteq V_j$ with $|W_j| = \lceil \varepsilon^{1/r} n \rceil$. Then $G[W_1, \dots, W_r]$ contains a copy of K_r.*
(ii) *The graph G contains a K_r-matching which covers all but at most $2r\varepsilon^{1/r} n$ vertices of G.* □

We will use a version of Szemerédi's Regularity lemma [25] stated for r-partite graphs. It is proved in the same way as the non-partite degree version.

Lemma 8.5 (Degree form of the r-partite Regularity lemma)**.** *Let $0 < \varepsilon < 1$ and $k_0, r \in \mathbb{N}$. Then there is an $N = N(\varepsilon, k_0, r)$ such that the following holds for every $0 \leq d < 1$ and for every r-partite graph G on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n \geq N$. There exists a partition $\mathcal{P} = \{U^0, \dots, U^k\}$ of $V(G)$, $m \in \mathbb{N}$ and a spanning subgraph G' of G satisfying the following:*

(i) *$k_0 \leq k \leq N$;*
(ii) *for each $1 \leq j \leq r$, $|U_j^0| \leq \varepsilon n$;*
(iii) *for each $1 \leq i \leq k$ and each $1 \leq j \leq r$, $|U_j^i| = m$;*
(iv) *for each $1 \leq j \leq r$ and each $v \in V(G)$, $d_{G'}(v, V_j) > d_G(v, V_j) - (d + \varepsilon)n$;*
(v) *for all but at most εk^2 pairs $U_{j_1}^{i_1}, U_{j_2}^{i_2}$ where $1 \leq i_1, i_2 \leq k$ and $1 \leq j_1 < j_2 \leq r$, the graph $G'[U_{j_1}^{i_1}, U_{j_2}^{i_2}]$ is ε-regular and has density either 0 or $> d$.*

We define the *reduced graph* R as follows. The vertex set of R is the set of clusters $\{U_j^i : 1 \leq i \leq k \text{ and } 1 \leq j \leq r\}$. For each $U, U' \in V(R)$, UU' is an edge of R if the subgraph $G'[U, U']$ is ε-regular and has density greater than d. Note that R is a balanced r-partite graph with vertex classes $W_j := \{U_j^i : 1 \leq i \leq k\}$ for $1 \leq j \leq r$. The following simple proposition relates the minimum degree of G and the minimum degree of R.

Proposition 8.6. *Suppose that $0 < 2\varepsilon \leq d \leq c/2$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$ and $\hat{\delta}(G) \geq cn$. Suppose that G has a partition $\mathcal{P} = \{U^0, \dots, U^k\}$ and a subgraph $G' \subseteq G$ as given by Lemma 8.5. Let R be the reduced graph of G. Then $\hat{\delta}(R) \geq (c - 2d)k$.* □

8.2. **Degree reduction.** At the beginning of our proof of Lemma 8.1, we will reserve a random subgraph H of G. Proposition 8.8 below ensures that we can partition the neighbourhood of each vertex so that H induces ε-regular graphs between these parts. In our proof of Proposition 8.8, we will use the following well-known result for which we omit the proof.

Proposition 8.7. *Let $1/n \ll \varepsilon \ll d, \rho \leq 1$. Let G be a bipartite graph on (A, B) with $|A| = |B| = n$. Suppose that G is ε-regular with density at least d. Let H be a graph formed by taking each edge of G independently with probability ρ. Then, with probability at least $1 - 1/n^2$, H is 4ε-regular with density at least $\rho d/2$.* □

Proposition 8.8. *Let $1/n \ll \alpha \ll 1/N \ll 1/k_0 \leq \varepsilon^* \ll d \ll \rho < \varepsilon, 1/r < 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$ and $\hat{\delta}(G) \geq (1 - 1/r + \varepsilon)n$. Suppose that for all $1 \leq j_1, j_2 \leq r$ and every $v \notin V_{j_1} \cup V_{j_2}$, $|d_G(v, V_{j_1}) - d_G(v, V_{j_2})| < \alpha n$. Then there exists $H \subseteq G$ satisfying the following properties:*

- (i) *For each $1 \leq j \leq r$ and each $v \in V(G)$, $|d_H(v, V_j) - \rho d_G(v, V_j)| < \alpha n$. In particular, for any $1 \leq j_1, j_2 \leq r$ such that $v \notin V_{j_1} \cup V_{j_2}$, $|d_H(v, V_{j_1}) - d_H(v, V_{j_2})| < 3\alpha n$.*
- (ii) *For each vertex $v \in V(G)$, there exists a partition $\mathcal{P}(v) = \{U^0(v), \dots, U^{k_v}(v)\}$ of $N_G(v)$ and $m_v \in \mathbb{N}$ such that:*
  - *$k_0 \leq k_v \leq N$;*
  - *for each $1 \leq j \leq r$, $|U_j^0(v)| \leq \varepsilon^* n$;*
  - *for each $1 \leq i \leq k_v$ and each $1 \leq j \leq r$ such that $v \notin V_j$, $|U_j^i(v)| = m_v$;*
  - *for each $1 \leq i \leq k_v$ and all $1 \leq j_1 < j_2 \leq r$ such that $v \notin V_{j_1} \cup V_{j_2}$, the graph $H[U_{j_1}^i(v), U_{j_2}^i(v)]$ is ε^*-regular with density greater than d.*

Roughly speaking, (ii) says that for each $v \in V(G)$ the reduced graph of $H[N_G(v)]$ has a perfect K_{r-1}-matching.

Proof. Let H be the graph formed by taking each edge of G independently with probability ρ. For each $1 \leq j \leq r$ and each $v \in V(G)$, Lemma 2.1 gives

$$\mathbb{P}(|d_H(v, V_j) - \rho d_H(v, V_j)| \geq \alpha n) \leq 2e^{-2\alpha^2 n} < 1/rn^2.$$

So the probability that there exist $1 \leq j \leq r$ and $v \in V(G)$ such that $|d_H(v, V_j) - \rho d_G(v, V_j)| \geq \alpha n$ is at most $rn/rn^2 = 1/n$. Let $1 \leq j_1, j_2 \leq r$. Note that if $v \notin V_{j_1} \cup V_{j_2}$ and $|d_H(v, V_j) - \rho d_G(v, V_j)| < \alpha n$ for $j = j_1, j_2$, then

$$|d_H(v, V_{j_1}) - d_H(v, V_{j_2})| < |\rho d_G(v, V_{j_1}) - \rho d_G(v, V_{j_2})| + 2\alpha n < 3\alpha n.$$

So H satisfies (i) with probability at least $1 - 1/n$.

We will now show that H satisfies (ii) with probability at least $1/2$. We find partitions of the neighbourhood of each vertex $v \in V(G)$ as follows. To simplify notation, we will assume that $v \in V_1$ (the argument is identical for the other cases). For all $2 \leq j_1, j_2 \leq r$, we have $|d_G(v, V_{j_1}) - d_G(v, V_{j_2})| < \alpha n$. So, there exists n_v and, for each $2 \leq j \leq r$, a subset $V_j(v) \subseteq N_G(v, V_j)$ such that $|V_j(v)| > d_G(v, V_j) - \alpha n$ and

$$|V_j(v)| = n_v \geq \hat{\delta}(G) \geq (1 - 1/r)n.$$

Let G_v denote the balanced $(r-1)$-partite graph $G[V_2(v), \dots, V_r(v)]$. Note that

$$\hat{\delta}(G_v) \geq n_v - \frac{n}{r} + \varepsilon n \geq \left(1 - \frac{1}{r-1} + \varepsilon\right) n_v. \tag{8.2}$$

Apply Lemma 8.5 (with $\varepsilon^*/4$, $2d/\rho$, k_0 and G_v playing the roles of ε, d, k_0 and G) to find a partition $\mathcal{Q}(v) = \{W^0(v), \dots, W^{k_v}(v)\}$ of $V(G_v)$ satisfying properties (i)–(v) of Lemma 8.5. Let $m_v := |W_2^1(v)|$. Let R_v denote the reduced graph corresponding to this partition. Proposition 8.6 together with (8.2) implies that

$$\hat{\delta}(R_v) \geq (1 - 1/(r-1) + \varepsilon/2)k_v.$$

So we can use Theorem 6.2 to find a perfect K_{r-1}-matching M_v in R_v. Let $U^0(v) := W^0(v) \cup (N_G(v) \setminus V(G_v))$. Note that for each $2 \leq j \leq r$, $|U_j^0| < |W_j^0| + \alpha n \leq \varepsilon^* n$. Let $\mathcal{P}(v) := \{U^0(v), \dots, U^{k_v}(v)\}$ be a partition of $N_G(v)$ which is chosen such that, for each $1 \leq i \leq k_v$, $\{U_j^i(v) : 2 \leq j \leq r\}$ induces a copy of K_{r-1} in M_v. By the definition of R_v, for each $1 \leq i \leq k_v$ and all $2 \leq j_1 < j_2 \leq r$, the graph $G[U_{j_1}^i(v), U_{j_2}^i(v)]$ is $\varepsilon^*/4$-regular with density greater than $2d/\rho$.

Fix $1 \leq i \leq k_v$ and $2 \leq j_1 < j_2 \leq r$. Proposition 8.7 (with m_v, $\varepsilon^*/4$, $2d/\rho$, $G[U_{j_1}^i(v), U_{j_2}^i(v)]$ and $H[U_{j_1}^i(v), U_{j_2}^i(v)]$ playing the roles of n, ε, d, G and H) gives that $H[U_{j_1}^i(v), U_{j_2}^i(v)]$ is ε^*-regular and has density greater than d with probability at least $1 - 1/m_v^2$.

We require the graph $H[U^i_{j_1}(v), U^i_{j_2}(v)]$ to be ε^*-regular with density greater than d for every edge $U^i_{j_1}(v)U^i_{j_2}(v) \in E(M_v)$. There are k_v choices for i and, for each i, there are $\binom{r-1}{2}$ choices for j_1 and j_2. So the probability that, for fixed $v \in V(G)$, there exists an edge $U^i_{j_1}(v)U^i_{j_2}(v) \in E(M_v)$ which fails to be ε^*-regular with density greater than d is at most

$$k_v r^2 \frac{1}{m_v^2} < \frac{1}{2rn}.$$

We multiply this probability by rn for each of the rn choices of v to see that H satisfies property (ii) with probability at least $1 - rn/2rn = 1/2$. Hence, the graph H satisfies both (i) and (ii) with probability at least $1/2 - 1/n > 0$. So we can choose such a graph H. □

In order to find an η-approximate K_r-decomposition in a graph G, we would like to use the definition of $\hat{\delta}^\eta_{K_r}$ which requires G to be K_r-divisible. The next proposition shows that, provided that $d_G(v, V_{j_1})$ is close to $d_G(v, V_{j_2})$ for all $1 \le j_1, j_2 \le r$ and $v \notin V_{j_1} \cup V_{j_2}$, G can be made K_r-divisible by removing only a small number of edges.

Proposition 8.9. *Let $1/n \ll \alpha \ll \gamma \ll 1/r < 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$ and $\hat{\delta}(G) \ge (1/2 + 2\gamma/r)n$. Suppose that, for all $1 \le j_1, j_2 \le r$ and every $v \in V(G) \setminus (V_{j_1} \cup V_{j_2})$, $|d_G(v, V_{j_1}) - d_G(v, V_{j_2})| < \alpha n$. Then there exists $H \subseteq G$ such that $G - H$ is K_r-divisible and $\Delta(H) \le \gamma n$.*

To prove Proposition 8.9, we require the following result whose proof is based on the Max-Flow-Min-Cut theorem.

Proposition 8.10. *Suppose that $1/n \ll \alpha \ll \xi \ll 1$. Let G be a bipartite graph on (A, B) with $|A| = |B| = n$. Suppose that $\delta(G) \ge (1/2 + 4\xi)n$. For every vertex $v \in V(G)$, let $n_v \in \mathbb{N}$ be such that $(\xi - \alpha)n \le n_v \le (\xi + \alpha)n$ and such that $\sum_{a \in A} n_a = \sum_{b \in B} n_b$. Then G contains a spanning graph G' such that $d_{G'}(v) = n_v$ for every $v \in V(G)$.*

Proof. We will use the Max-Flow-Min-Cut theorem. Orient every edge of G towards B and give each edge capacity one. Add a source vertex s^* which is attached to every vertex $a \in A$ by an edge of capacity n_a. Add a sink vertex t^* which is attached to every vertex in $b \in B$ by an edge of capacity n_b. Let $c_0 := \sum_{a \in A} n_a = \sum_{b \in B} n_b$. Note that an integer-valued c_0-flow corresponds to the desired spanning graph G' in G. So, by the Max-Flow-Min-Cut theorem, it suffices to show that every cut has capacity at least c_0.

Consider a minimal cut C. Let $S \subseteq A$ be the set of all vertices $a \in A$ for which $s^*a \notin C$ and let $T \subseteq B$ be the set of all $b \in B$ for which $bt^* \notin C$. Let $S' := A \setminus S$ and $T' := B \setminus T$. Then C has capacity

$$c := \sum_{s \in S'} n_s + e_G(S, T) + \sum_{t \in T'} n_t.$$

First suppose that $|S| \ge (1/2 - 2\xi)n$. In this case, since $\delta(G) \ge (1/2 + 4\xi)n$, each vertex in T receives at least $2\xi n$ edges from S. So

$$c \ge \sum_{t \in T'} n_t + 2|T|\xi n \ge \sum_{t \in T'} n_t + |T|(\xi + \alpha)n \ge c_0.$$

A similar argument works if $|T| \ge (1/2 - 2\xi)n$. Suppose then that $|S|, |T| < (1/2 - 2\xi)n$. Then $|S'|, |T'| > (1/2 + 2\xi)n$ and

$$c \ge \sum_{s \in S'} n_s + \sum_{t \in T'} n_t \ge (|S'| + |T'|)(\xi - \alpha)n > (n + 4\xi n)(\xi - \alpha)n \ge (\xi + \alpha)n^2 \ge c_0,$$

as required. □

We now use Proposition 8.10 to prove Proposition 8.9.

Proof of Proposition 8.9. For each $v \in V(G)$, let

$$m_v := \min\{d_G(v, V_j) : 1 \le j \le r \text{ with } v \notin V_j\}.$$

For each $1 \le j \le r$ and each $v \notin V_j$, let $a_{v,j} := d_G(v, V_j) - m_v$. Note that,

$$0 \le a_{v,j} < \alpha n. \tag{8.3}$$

For each $1 \le j \le r$, let $N_j := \sum_{v \in V_j} m_v$. We have, for any $1 \le j_1, j_2 \le r$,

$$\begin{aligned} |N_{j_1} - N_{j_2}| &= \left| \sum_{v \in V_{j_1}} (d_G(v, V_{j_2}) - a_{v,j_2}) - \sum_{v \in V_{j_2}} (d_G(v, V_{j_1}) - a_{v,j_1}) \right| \\ &= \left| \sum_{v \in V_{j_1}} a_{v,j_2} - \sum_{v \in V_{j_2}} a_{v,j_1} \right| \overset{(8.3)}{<} \alpha n^2. \end{aligned} \tag{8.4}$$

Let $N := \min\{N_j : 1 \le j \le r\}$ and, for each $1 \le j \le r$, let $M_j := N_j - N$. Note that (8.4) implies $0 \le M_j < \alpha n^2$. For each $1 \le j \le r$ and each $v \in V_j$, choose $p_v \in \mathbb{N}$ to be as equal as possible such that $\sum_{v \in V_j} p_v = M_j$. Then

$$0 \le p_v < \alpha n + 1. \tag{8.5}$$

Let $\xi := \gamma/2r$. For each $1 \le j \le r$ and each $v \notin V_j$, let

$$n_{v,j} := \lceil \xi n \rceil + a_{v,j} + p_v.$$

Using (8.3) and (8.5), we see that,

$$\xi n \le n_{v,j} \le (\xi + 3\alpha) n. \tag{8.6}$$

We will consider each pair $1 \le j_1 < j_2 \le r$ separately and choose $H_{j_1,j_2} = H[V_{j_1}, V_{j_2}]$. Fix $1 \le j_1 < j_2 \le r$ and observe that,

$$\begin{aligned} \sum_{v \in V_{j_1}} n_{v,j_2} &= \sum_{v \in V_{j_1}} (\lceil \xi n \rceil + a_{v,j_2} + p_v) = \lceil \xi n \rceil n + \sum_{v \in V_{j_1}} a_{v,j_2} + M_{j_1} \\ &= \lceil \xi n \rceil n + M_{j_1} + \sum_{v \in V_{j_1}} (d_G(v, V_{j_2}) - m_v) = \lceil \xi n \rceil n + M_{j_1} + e_G(V_{j_1}, V_{j_2}) - N_{j_1} \\ &= \lceil \xi n \rceil n - N + e_G(V_{j_1}, V_{j_2}) = \sum_{v \in V_{j_2}} n_{v,j_1}. \end{aligned}$$

Let $G_{j_1,j_2} := G[V_{j_1}, V_{j_2}]$ and note that $\delta(G_{j_1,j_2}) \ge (1/2 + 4\xi)n$. Apply Proposition 8.10 (with 3α, ξ, G_{j_1,j_2}, V_{j_1} and V_{j_2} playing the roles of α, ξ, G, A and B) to find $H_{j_1,j_2} \subseteq G_{j_1,j_2}$ such that $d_{H_{j_1,j_2}}(v) = n_{v,j_2}$ for every $v \in V_{j_1}$ and $d_{H_{j_1,j_2}}(v) = n_{v,j_1}$ for every $v \in V_{j_2}$.

Let $H := \bigcup_{1 \le j_1 < j_2 \le r} H_{j_1,j_2}$. By (8.6), we have $\Delta(H) \le 2r\xi n = \gamma n$. For any $1 \le j \le r$ and any $v \notin V_j$, we have

$$\begin{aligned} d_{G-H}(v, V_j) &= d_G(v, V_j) - d_H(v, V_j) = d_G(v, V_j) - n_{v,j} \\ &= d_G(v, V_j) - \lceil \xi n \rceil - d_G(v, V_j) + m_v - p_v = m_v - p_v - \lceil \xi n \rceil. \end{aligned}$$

So $G - H$ is K_r-divisible. □

We now have all the necessary tools to prove Lemma 8.1. This lemma finds an approximate K_r-decomposition which covers all but at most γn edges at any vertex.

Proof of Lemma 8.1. The lemma trivially holds if $r = 2$, so we may assume that $r \geq 3$. In particular, by Proposition 3.1, $\hat{\delta}(G) \geq (1 - 1/(r+1) + \varepsilon/2)n$. Choose constants N, k_0, ε^*, d and ρ satisfying

$$\eta \ll 1/N \ll 1/k_0 \leq \varepsilon^* \ll d \ll \rho \ll \gamma.$$

Apply Proposition 8.8 to find a subgraph $H_1 \subseteq G$ satisfying properties (i)–(ii).

Let $G_1 := G - H_1$. Using (8.1) and that H_1 satisfies Proposition 8.8(i), for all $1 \leq j_1, j_2 \leq r$ and each $v \notin V_{j_1} \cup V_{j_2}$,

$$\begin{aligned}|d_{G_1}(v, V_{j_1}) - d_{G_1}(v, V_{j_2})| &\leq |d_G(v, V_{j_1}) - d_G(v, V_{j_2})| + |d_{H_1}(v, V_{j_1}) - d_{H_1}(v, V_{j_2})| \\ &< \alpha n + 3\alpha n = 4\alpha n.\end{aligned}$$

Note also that $\hat{\delta}(G_1) \geq 3n/4$. So we can apply Proposition 8.9 (with G_1, 4α and $\gamma/2$ playing the roles of G, α and γ) to obtain $H_2 \subseteq G_1$ such that $G_1 - H_2$ is K_r-divisible and $\Delta(H_2) \leq \gamma n/2$. Then $\hat{\delta}(G_1 - H_2) \geq (\hat{\delta}^{\eta}_{K_r} + \varepsilon/2)n$, so we can find an η-approximate K_r-decomposition $\mathcal{F}$ of $G_1 - H_2$.

Let $G_2 := G_1 - H_2 - \bigcup \mathcal{F}$ be the graph consisting of all the remaining edges in $G_1 - H_2$. Let

$$B := \{v \in V(G) : d_{G_2}(v) > \eta^{1/2} n\}.$$

Note that

$$|B| \leq 2e(G_2)/\eta^{1/2} n \leq 2\eta^{1/2} n. \tag{8.7}$$

Let $\mathcal{F}_1 := \{F \in \mathcal{F} : F \cap B = \emptyset\}$ and let $G_3 := G - \bigcup \mathcal{F}_1$. If $v \in B$, then $N_{G_3}(v) = N_G(v)$. Suppose that $v \notin B$. For any $u \in B$, at most one copy of K_r in $\mathcal{F} \setminus \mathcal{F}_1$ can contain both u and v. So there can be at most $(r-1)|B|$ edges in $\bigcup(\mathcal{F} \setminus \mathcal{F}_1)$ that are incident to v and so

$$\begin{aligned}d_{G_3}(v) &\leq d_{H_1}(v) + d_{H_2}(v) + d_{G_2}(v) + (r-1)|B| \\ &\leq (r-1)(\rho + \alpha)n + \gamma n/2 + \eta^{1/2} n + 2(r-1)\eta^{1/2} n \leq \gamma n.\end{aligned} \tag{8.8}$$

Label the vertices of $B = \{v_1, v_2, \ldots, v_{|B|}\}$. We will use copies of K_r to cover most of the edges at each vertex v_i in turn. We do this by finding a K_{r-1}-matching M_i in $H_1[N_{G_3}(v_i)] = H_1[N_G(v_i)]$ in turn for each i. Suppose that we are currently considering $v := v_i$ and let $\mathcal{M} := \bigcup_{1 \leq j < i} M_j$. To simplify notation, we will assume that $v \in V_1$ (the proof in the other cases is identical).

Let $\mathcal{P}(v) = \{U^0(v), \ldots, U^{k_v}(v)\}$ be a partition of $N_G(v)$ satisfying Proposition 8.8(ii). We can choose a partition $\mathcal{Q}(v) = \{W^0(v), \ldots, W^{k_v}(v)\}$ of $N_G(v)$ and $m'_v \geq m_v - |B|$ such that, for each $1 \leq i \leq k_v$:

- $W^i(v) \subseteq U^i(v)$;
- $W^i(v) \cap B = \emptyset$;
- for each $2 \leq j \leq r$, $|W^i_j(v)| = m'_v$.

Note that, using (8.7), $|W^0(v)| \leq |U^0(v)| + |B| k_v r \leq r(\varepsilon^* n + 2\eta^{1/2} n k_v) \leq 2\varepsilon^* r n$.

By Proposition 8.8(ii), for each $1 \leq i \leq k_v$ and all $2 \leq j_1 < j_2 \leq r$, the graph $H_1[U^i_{j_1}(v), U^i_{j_2}(v)]$ is ε^*-regular with density greater than d. So Proposition 8.2 implies that $H_1[W^i_{j_1}(v), W^i_{j_2}(v)]$ is $2\varepsilon^*$-regular with density greater than $d/2$. Let $H'_1 := H_1 - \mathcal{M}$. Using (8.7), we have $\Delta(\mathcal{M}[W^i_{j_1}(v), W^i_{j_2}(v)]) \leq |B| \leq \eta^{1/3} m'_v$. So we can apply Proposition 8.3 (with m'_v, $\eta^{1/3}$ and $2\varepsilon^*$ playing the roles of n, γ and ε) to see that $H'_1[W^i_{j_1}(v), W^i_{j_2}(v)]$ is $4\varepsilon^*$-regular with density greater than $d/3$.

We use Proposition 8.4 (with m'_v, $4\varepsilon^*$, $d/3$ and $r-1$ playing the roles of n, ε, d and r) to find a K_{r-1}-matching covering all but at most $2(r-1)(4\varepsilon^*)^{1/(r-1)} m'_v$ vertices in $H'_1[W^i(v)]$ for each $1 \leq i \leq k_v$. Write M_i for the union of these K_{r-1}-matchings over $1 \leq i \leq k_v$. Note that M_i covers all but at most

$$|W^0(v)| + 2(r-1)(4\varepsilon^*)^{1/(r-1)} m'_v k_v \leq 2\varepsilon^* r n + 2(r-1)(4\varepsilon^*)^{1/(r-1)} n \leq \gamma n \tag{8.9}$$

vertices in $N_G(v)$.

Continue to find edge-disjoint $M_1, \dots, M_{|B|}$. For each $1 \le i \le |B|$, $M_i' := \{v_i \cup K : K \in M_i\}$ is an edge-disjoint collection of copies of K_r in G_3 covering all but at most γn edges at v_i in G. Write $\mathcal{M}' := \bigcup_{1 \le i \le |B|} M_i'$ and let $H := G_3 - \bigcup \mathcal{M}' = G - \bigcup(\mathcal{F}_1 \cup \mathcal{M}')$. Then $G - H = \bigcup(\mathcal{F}_1 \cup \mathcal{M}')$ has a K_r-decomposition and $\Delta(H) \le \gamma n$, by (8.8) and (8.9). □

9. Covering a pseudorandom remainder between vertex classes

After applying Lemma 8.1, we are left with a graph H such that $H[\mathcal{P}]$ has low maximum degree. We will add a suitable quasirandom graph R to H to be able to assume that the remainder $H' = R \cup H$ is actually quasirandom. The results in this section will allow us to cover any remaining edges in $H'[\mathcal{P}]$ using only a small number of edges from $H' - H'[\mathcal{P}]$. This is done by finding, for each $x \in V(G)$, suitable vertex-disjoint copies of K_{r-1} inside $H' - H'[\mathcal{P}]$ such that each copy of K_{r-1} forms a copy of K_r together with the edges incident to x in $H'[\mathcal{P}]$.

Lemma 9.1. *Let $r \ge 2$ and $1/n \ll 1/k, 1/r, \rho \le 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Let $q \le krn$ and let $W^1, \dots, W^q \subseteq V(G)$. Suppose that:*

- (i) *for each $1 \le i \le q$, there exists $1 \le j_i \le r$ and $n_i \in \mathbb{N}$ such that, for each $1 \le j \le r$, $|W_j^i| = 0$ if $j = j_i$ and $|W_j^i| = n_i$ otherwise;*
- (ii) *for each $1 \le i \le q$, $\hat{\delta}(G[W^i]) \ge (1 - 1/(r-1))n_i + 9kr^2\rho^{3/2}n$;*
- (iii) *for all $1 \le i_1 < i_2 \le q$, $|W^{i_1} \cap W^{i_2}| \le 2r\rho^2 n$;*
- (iv) *each $v \in V(G)$ is contained in at most $2k\rho n$ of the W^i.*

Then there exist edge-disjoint $T_1, \dots, T_q$ in G such that each T_i is a perfect K_{r-1}-matching in $G[W^i]$.

The W^i in Lemma 9.1 will play the role of vertex neighbourhoods later on. The proof of Lemma 9.1 is similar to that of Lemma 10.7 in [3], we include it here for completeness. We will use the following result.

Proposition 9.2 (Jain, see [23, Lemma 8])**.** *Let $X_1, \dots, X_n$ be Bernoulli random variables such that, for any $1 \le s \le n$ and any $x_1, \dots, x_{s-1} \in \{0, 1\}$,*

$$\mathbb{P}(X_s = 1 \mid X_1 = x_1, \dots, X_{s-1} = x_{s-1}) \le p.$$

Let $X = \sum_{s=1}^n X_i$ and let $B \sim B(n, p)$. Then $\mathbb{P}(X \ge a) \le \mathbb{P}(B \ge a)$ for any $a \ge 0$.

Proof of Lemma 9.1. Set $t := \lceil 8kr\rho^{3/2}n \rceil$. Let $G_i := G[W^i]$ for $1 \le i \le q$. Suppose we have already found $T_1, \dots T_{s-1}$ for some $1 \le s \le q$. We find T_s as follows.

Let $H_{s-1} := \bigcup_{i=1}^{s-1} T_i$ and $G_s' := G_s - H_{s-1}[W^s]$. If $\Delta(H_{s-1}[W^s]) > (r-2)\rho^{3/2}n$, let $T_1', \dots, T_t'$ be empty graphs on W^s. Otherwise, (ii) implies

$$\hat{\delta}(G_s') \ge \big(1 - \frac{1}{r-1}\big)n_s + 8kr^2\rho^{3/2}n \ge \big(1 - \frac{1}{r-1} + \rho^{3/2}\big)n_s + (r-2)(t-1)$$

and we can greedily find t edge-disjoint perfect K_{r-1}-matchings $T_1', \dots, T_t'$ in G_s' using Theorem 6.2. In either case, pick $1 \le i \le t$ uniformly at random and set $T_s := T_i'$. It suffices to show that, with positive probability,

$$\Delta(H_{s-1}[W^s]) \le (r-2)\rho^{3/2}n \qquad \text{for all } 1 \le s \le q.$$

Consider any $1 \le i \le q$ and any $w \in W^i$. For $1 \le s \le q$, let $Y_s^{i,w}$ be the indicator function of the event that T_s contains an edge incident to w in G_i. Let $X^{i,w} := \sum_{s=1}^q Y_s^{i,w}$. Note $d_{H_q}(w, W^i) \le (r-2)X^{i,w}$. So it suffices to show that, with positive probability, $X^{i,w} \le \rho^{3/2}n$ for all $1 \le i \le q$ and all $w \in W^i$.

Fix $1 \le i \le q$ and $w \in W^i$. Let $J^{i,w}$ be the set of indices $s \ne i$ such that $w \in W^s$; (iv) implies $|J^{i,w}| < 2k\rho n$. If $s \notin J^{i,w} \cup \{i\}$, then $w \notin W^s$ and $Y_s^{i,w} = 0$. So

$$X^{i,w} \le 1 + \sum_{s \in J^{i,w}} Y_s^{i,w}. \tag{9.1}$$

Let $s_1 < \cdots < s_{|J^{i,w}|}$ be an enumeration of $J^{i,w}$. For any $b \le |J^{i,w}|$, note that

$$d_{G_{s_b}}(w, W^i) \le |W^i \cap W^{s_b}| \overset{\text{(iii)}}{\le} 2r\rho^2 n.$$

So at most $2r\rho^2 n$ of the subgraphs T'_j that we picked in G'_{s_b} contain an edge incident to w in G_i. Thus

$$\mathbb{P}(Y^{i,w}_{s_b} = 1 \mid Y^{i,w}_{s_1} = y_1, \dots, Y^{i,w}_{s_{b-1}} = y_{b-1}) \le 2r\rho^2 n/t \le \rho^{1/2}/4k$$

for all $y_1, \dots, y_{b-1} \in \{0,1\}$ and $1 \le b \le |J^{i,w}|$. Let $B \sim B(|J^{i,w}|, \rho^{1/2}/4k)$. Using Proposition 9.2, Lemma 2.1 and that $|J^{i,w}| \le 2k\rho n$, we see that

$$\begin{aligned}\mathbb{P}(X^{i,w} > \rho^{3/2}n) &\overset{(9.1)}{\le} \mathbb{P}\Big(\sum_{s \in J^{i,w}} Y^{i,w}_s > 3\rho^{3/2}n/4\Big) \le \mathbb{P}(B > 3\rho^{3/2}n/4)\\ &\le \mathbb{P}(|B - \mathbb{E}(B)| > \rho^{3/2}n/4) \le 2e^{-\rho^2 n/16k}.\end{aligned}$$

There are at most $qrn \le kr^2n^2$ pairs (i,w), so there is a choice of $T_1, \dots, T_q$ such that $X^{i,w} \le \rho^{3/2}n$ for all $1 \le i \le q$ and all $w \in W^i$. □

The following is an immediate consequence of Lemma 9.1.

Corollary 9.3. *Let $r \ge 2$ and $1/n \ll 1/k, 1/r, \rho \le 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $U, W \subseteq V(G)$ be disjoint with $|W_1| = \cdots = |W_r| \ge \lfloor n/k \rfloor$. Suppose the following hold:*

(i) *for all $1 \le j_1, j_2 \le r$ and all $x \in U \setminus (V_{j_1} \cup V_{j_2})$, $d_G(x, W_{j_1}) = d_G(x, W_{j_2})$;*
(ii) *for all $1 \le j \le r$ and all $x \in U \setminus U_j$, $\hat{\delta}(G[N_G(x,W)]) \ge (1 - 1/(r-1))d_G(x, W_j) + 9kr\rho^{3/2}|W|$;*
(iii) *for all distinct $x, x' \in U$, $|N_G(x,W) \cap N_G(x',W)| \le 2\rho^2|W|$;*
(iv) *for all $y \in W$, $d_G(y, U) \le 2k\rho|W_1|$.*

Then there exists $G_W \subseteq G[W]$ such that $G[U,W] \cup G_W$ has a K_r-decomposition and $\Delta(G_W) \le 2kr\rho|W_1|$.

Proof. Let $q := |U|$ and let $u^1, \dots, u^q$ be an enumeration of U. For each $1 \le i \le q$, let $W^i := N_G(u^i, W)$. Note that $q \le kr|W_1|$. Apply Lemma 9.1 (with $G[W]$ and $|W_1|$ playing the roles of G and n) to obtain edge-disjoint perfect K_{r-1}-matchings T^i in each $G[W^i]$. Let $G_W := \bigcup_{i=1}^q T^i$. Then $G[U,W] \cup G_W$ has a K_r-decomposition. For each $y \in W$, we use (iv) to see that $d_{G_W}(y) \le (r-1)d_G(y,U) < 2kr\rho|W_1|$. □

If we are given a k-partition $\mathcal{P}$ of the r-partite graph G, we can apply Corollary 9.3 repeatedly with each $U \in \mathcal{P}$ playing the role of W to obtain the following result.

Corollary 9.4. *Let $r \ge 2$ and $1/n \ll \rho \ll 1/k, 1/r \le 1$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition for G. Suppose that the following hold for all $2 \le i \le k$:*

(i) *for all $1 \le j_1, j_2 \le r$ and all $x \in U^{<i} \setminus (V_{j_1} \cup V_{j_2})$, $d_G(x, U^i_{j_1}) = d_G(x, U^i_{j_2})$;*
(ii) *for all $1 \le j \le r$ and all $x \in U^{<i} \setminus V_j$, $\hat{\delta}(G[N_G(x,U^i)] \ge (1 - 1/(r-1))d_G(x, U^i_j) + 9kr\rho^{3/2}|U^i|$;*
(iii) *for all distinct $x, x' \in U^{<i}$, $|N_G(x,U^i) \cap N_G(x',U^i)| \le 2\rho^2|U^i|$;*
(iv) *for all $y \in U^i$, $d_G(y, U^{<i}) \le 2k\rho|U^i_1|$.*

Then there exists $G_0 \subseteq G - G[\mathcal{P}]$ such that $G[\mathcal{P}] \cup G_0$ has a K_r-decomposition and $\Delta(G_0) \le 3r\rho n$.

Proof. For each $2 \le i \le k$, let $G_i := G[U^{<i}, U^i] \cup G[U^i]$. Apply Corollary 9.3 to each G_i with $U^{<i}$, U^i playing the roles of U, W to obtain $G'_i \subseteq G[U^i]$ such that $G[U^{<i}, U^i] \cup G'_i$ has a K_r-decomposition and $\Delta(G'_i) \le 2kr\rho\lceil n/k \rceil \le 3r\rho n$. Let $G_0 := \bigcup_{i=2}^k G'_i$. Then $G[\mathcal{P}] \cup G_0$ has a K_r-decomposition and $\Delta(G_0) \le 3r\rho n$. □

10. Balancing graph

In our proof we will consider a sequence of successively finer partitions $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ in turn. When considering $\mathcal{P}_i$, we will assume the leftover is a subgraph of $G - G[\mathcal{P}_{i-1}]$ and aim to use Lemma 8.1 and then Corollary 9.4 to find copies of K_r such that the leftover is now contained in $G - G[\mathcal{P}_i]$ (i.e. inside the smaller partition classes). However, to apply Corollary 9.4 we need the leftover to be balanced with respect to the partition classes. In this section we show how this can be achieved.

Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. We say that a graph H on $(V_1, \dots, V_r)$ is *locally $\mathcal{P}$-balanced* if

$$d_H(v, U^i_{j_1}) = d_H(v, U^i_{j_2})$$

for all $1 \le i \le k$, all $1 \le j_1, j_2 \le r$ and all $v \in U^i \setminus (V_{j_1} \cup V_{j_2})$. Note that a graph which is locally $\mathcal{P}$-balanced is not necessarily K_r-divisible but that $H[U^i]$ is K_r-divisible for all $1 \le i \le k$.

Let $\gamma > 0$. A $(\gamma, \mathcal{P})$-*balancing graph* is a K_r-decomposable graph B on V such that the following holds. Let H be any K_r-divisible graph on V with:

(P1) $e(H \cap B) = 0$;
(P2) $|d_H(v, U^i_{j_1}) - d_H(v, U^i_{j_2})| < \gamma n$ for all $1 \le i \le k$, all $1 \le j_1, j_2 \le r$ and all $v \notin V_{j_1} \cup V_{j_2}$.

Then there exists $B' \subseteq B$ such that $B - B'$ has a K_r-decomposition and

$$d_{H \cup B'}(v, U^i_{j_1}) = d_{H \cup B'}(v, U^i_{j_2})$$

for all $2 \le i \le k$, all $1 \le j_1, j_2 \le r$ and all $v \in U^{<i} \setminus (V_{j_1} \cup V_{j_2})$.

Our aim in this section will be to prove Lemma 10.1 which finds a $(\gamma, \mathcal{P})$-balancing graph in a suitable graph G.

Lemma 10.1. *Let $1/n \ll \gamma \ll \gamma' \ll 1/k \ll \varepsilon \ll 1/r \le 1/3$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition for G. Suppose $d_G(v, U^i_j) \ge (1 - 1/(r+1) + \varepsilon)|U^i_j|$ for all $1 \le i \le k$, all $1 \le j \le r$ and all $v \notin V_j$. Then there exists $B \subseteq G$ which is a $(\gamma, \mathcal{P})$-balancing graph such that B is locally $\mathcal{P}$-balanced and $\Delta(B) < \gamma' n$.*

The balancing graph B will be made up of two graphs: B_{edge}, an edge balancing graph (which balances the total number of edges between appropriate classes), and B_{deg}, a degree balancing graph (which balances individual vertex degrees). These are described in Sections 10.1 and 10.2 respectively.

10.1. Edge balancing.

Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Let $\gamma > 0$. A $(\gamma, \mathcal{P})$-*edge balancing graph* is a K_r-decomposable graph B_{edge} on V such that the following holds. Let H be any K_r-divisible graph on V which is edge-disjoint from B_{edge} and satisfies (P2). Then there exists $B'_{\text{edge}} \subseteq B_{\text{edge}}$ such that $B_{\text{edge}} - B'_{\text{edge}}$ has a K_r-decomposition and

$$e_{H \cup B'_{\text{edge}}}(U^{i_1}_{j_1}, U^{i_2}_{j_2}) = e_{H \cup B'_{\text{edge}}}(U^{i_1}_{j_1}, U^{i_2}_{j_3})$$

for all $1 \le i_1 < i_2 \le k$ and all $1 \le j_1, j_2, j_3 \le r$ with $j_1 \ne j_2, j_3$.

In this section, we first construct and then find a $(\gamma, \mathcal{P})$-edge balancing graph in G.

For any multigraph G on W and any $e \in W^{(2)}$, let $m_G(e)$ be the multiplicity of the edge e in G. We say that a K_r-divisible multigraph G on $W = (W_1, \dots, W_r)$ is *irreducible* if G has no non-trivial K_r-divisible proper subgraphs; that is, for every $H \subsetneq G$ with $e(H) > 0$, H is not K_r-divisible. It is easy to see that there are only finitely many irreducible K_r-divisible multigraphs on W. In particular, this implies the following proposition.

Proposition 10.2. *Let $r \in \mathbb{N}$ and let $W = (W_1, \dots, W_r)$. Then there exists $N = N(W)$ such that every irreducible K_r-divisible multigraph on W has edge multiplicity at most N.* □

Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a partition of $V = (V_1, \dots, V_r)$. Take a copy K of $K_r(k)$ with vertex set $(W_1, \dots, W_r)$ where $W_j = \{w^1_j, \dots, w^k_j\}$ for each $1 \le j \le r$. For each $1 \le i \le k$, let $W^i := \{w^i_j : 1 \le j \le$

$r\}$. Given a graph H on V, we define an *excess multigraph* $\mathrm{EM}(H)$ on the vertex set $V(K)$ as follows. Between each pair of vertices $w_{j_1}^{i_1}$, $w_{j_2}^{i_2}$ such that $w_{j_1}^{i_1}w_{j_2}^{i_2} \in E(K)$ there are exactly

$$e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) - \min\{e_H(U_j^{i_1}, U_{j'}^{i_2}) : 1 \le j, j' \le r, j \ne j'\}$$

multiedges in $\mathrm{EM}(H)$.

Proposition 10.3. *Let $r \in \mathbb{N}$ with $r \ge 3$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let H be any K_r-divisible graph on V satisfying* (P2)*. Then the excess multigraph $\mathrm{EM}(H)$ has a decomposition into at most $3\gamma k^2 r^2 n^2$ irreducible K_r-divisible multigraphs.*

Proof. First, note that for any $1 \le i_1, i_2 \le k$, any $1 \le j_1, j_2, j_3 \le r$ with $j_1 \ne j_2, j_3$ and any $v \in U_{j_1}^{i_1}$, we have $|d_H(v, U_{j_2}^{i_2}) - d_H(v, U_{j_3}^{i_2})| < \gamma n$ by (P2). Therefore,

$$|e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) - e_H(U_{j_1}^{i_1}, U_{j_3}^{i_2})| < \gamma n |U_{j_1}^{i_1}| < \gamma n^2. \tag{10.1}$$

We claim that, for all $w_{j_1}^{i_1}w_{j_2}^{i_2} \in E(K)$,

$$m_{\mathrm{EM}(H)}(w_{j_1}^{i_1}w_{j_2}^{i_2}) < 3\gamma n^2. \tag{10.2}$$

Let $1 \le j_1', j_2' \le r$ with $j_1' \ne j_2'$. Let $1 \le j \le r$ with $j \ne j_1, j_1'$. Then

$$\begin{aligned} |e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) - e_H(U_{j_1'}^{i_1}, U_{j_2'}^{i_2})| &\le |e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) - e_H(U_{j_1}^{i_1}, U_j^{i_2})| + |e_H(U_{j_1}^{i_1}, U_j^{i_2}) - e_H(U_{j_1'}^{i_1}, U_j^{i_2})| \\ &\quad + |e_H(U_{j_1'}^{i_1}, U_j^{i_2}) - e_H(U_{j_1'}^{i_1}, U_{j_2'}^{i_2})| \\ &\overset{(10.1)}{<} 3\gamma n^2. \end{aligned}$$

So (10.2) holds.

We will now show that $\mathrm{EM}(H)$ is K_r-divisible. Consider any vertex $w_{j_1}^{i_1} \in V(\mathrm{EM}(H))$ and any $1 \le j_2, j_3 \le r$ such that $j_1 \ne j_2, j_3$. Note that, since H is K_r-divisible,

$$\begin{aligned} d_{\mathrm{EM}(H)}(w_{j_1}^{i_1}, W_{j_2}) &= \sum_{i=1}^{k} m_{\mathrm{EM}(H)}(w_{j_1}^{i_1}, w_{j_2}^{i}) \\ &= e_H(U_{j_1}^{i_1}, V_{j_2}) - \sum_{i=1}^{k} \min\{e_H(U_j^{i_1}, U_{j'}^{i}) : 1 \le j, j' \le r, j \ne j'\} \\ &= e_H(U_{j_1}^{i_1}, V_{j_3}) - \sum_{i=1}^{k} \min\{e_H(U_j^{i_1}, U_{j'}^{i}) : 1 \le j, j' \le r, j \ne j'\} \\ &= \sum_{i=1}^{k} m_{\mathrm{EM}(H)}(w_{j_1}^{i_1}, w_{j_3}^{i}) = d_{\mathrm{EM}(H)}(w_{j_1}^{i_1}, W_{j_3}). \end{aligned}$$

So $\mathrm{EM}(H)$ is K_r-divisible and therefore has a decomposition $\mathcal{F}$ into irreducible K_r-divisible multigraphs. By (10.2), there are at most $3\gamma n^2$ edges between any pair of vertices in $\mathrm{EM}(H)$, so $|\mathcal{F}| \le (3\gamma n^2)e(K) < 3\gamma k^2 r^2 n^2$. □

Let $N = N(V(K))$ be the maximum multiplicity of an edge in any irreducible K_r-divisible multigraph on $V(K) = (W_1, \dots, W_r)$ (N exists by Proposition 10.2). Label each vertex w_j^i of K by U_j^i. Let $K(N)$ be the labelled multigraph obtained from K by replacing each edge of K by N multiedges.

We now construct a $\mathcal{P}$-labelled graph which resembles the multigraph $K(N)$ (when we compare relative differences in the numbers of edges between vertices) and has lower degeneracy. Consider any edge $e = w_{j_1}^{i_1}w_{j_2}^{i_2} \in E(K(N))$. Let $\theta(e)$ be the graph obtained by the following procedure. Take a copy K_e of

$K[W^{i_1}, W^{i_2}] - w^{i_1}_{j_1} w^{i_2}_{j_2}$ (K_e inherits the labelling of $K[W^{i_1}, W^{i_2}]$). Note that $K[W^{i_1}, W^{i_2}]$ is a copy of K_r if $i_1 = i_2$ and a copy of the graph obtained from $K_{r,r}$ by deleting a perfect matching otherwise. Join $w^{i_1}_{j_1}$ to the copy of $w^{i_2}_{j_2}$ in K_e and join $w^{i_2}_{j_2}$ to the copy of $w^{i_1}_{j_1}$ in K_e. Write $\theta(e)$ for the resulting $\mathcal{P}$-labelled graph (so the vertex set of $\theta(e)$ consists of $w^{i_1}_{j_1}$, $w^{i_2}_{j_2}$ as well as all the vertices in K_e). Choose the graphs K_e to be vertex-disjoint for all $e \in E(K(N))$. For any $K' \subseteq K(N)$, let $\theta(K') := \bigcup\{\theta(e) : e \in E(K')\}$.

To see that the labelling of $\theta(K(N))$ is actually a $\mathcal{P}$-labelling, note that for any U^i_j, the set of vertices labelled U^i_j forms an independent set in $\theta(K(N))$. Moreover, note that $\theta(K(N))$ has degeneracy $r-1$. To see this, list its vertices in the following order. First list all the original vertices of $V(K)$. These form an independent set in $\theta(K(N))$. Then list the remaining vertices of $\theta(K(N))$ in any order. Each of these vertices has degree $r-1$ in $\theta(K(N))$, so the degeneracy of $\theta(K(N))$ can be at most $r-1$.

Proposition 10.4. *Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $J = \phi(\theta(K(N)))$ be a copy of $\theta(K(N))$ on V which is compatible with its $\mathcal{P}$-labelling. Then the following hold:*

(i) *J is K_r-divisible and locally $\mathcal{P}$-balanced;*
(ii) *for any multigraph $H \subseteq K(N)$, any $1 \le i_1, i_2 \le k$ and any $1 \le j_1 < j_2 \le r$,*

$$e_{\phi(\theta(H))}(U^{i_1}_{j_1}, U^{i_2}_{j_2}) = e_H(W^{i_1}, W^{i_2}) + m_H(w^{i_1}_{j_1} w^{i_2}_{j_2}).$$

Proof. We first prove that J is K_r-divisible. Consider any $x \in V(\theta(K(N)))$. If $x = w^i_j \in V(K)$, then $d_J(\phi(x), V_{j_1}) = Nk$ for all $1 \le j_1 \le r$ with $j_1 \neq j$ (since for each edge $w^i_j w^{i_1}_{j_1} \in E(K)$, x has exactly N neighbours labelled $U^{i_1}_{j_1}$ in $\theta(K(N))$). If $x \notin V(K)$, x must appear in a copy of K_e in $\theta(e)$ for some edge $e \in E(K(N))$. In this case, $d_J(\phi(x), V_j) = 1$ for all $1 \le j \le r$ such that $\phi(x) \notin V_j$. So J is K_r-divisible.

To see that J is locally $\mathcal{P}$-balanced, consider any $x \in V(\theta(K(N)))$. If $x = w^i_j \in V(K)$, then $\phi(x) \in U^i_j$ and $d_J(\phi(x), U^i_{j_1}) = N$ for all $1 \le j_1 \le r$ with $j_1 \neq j$. Otherwise, x must appear in a copy of K_e in $\theta(e)$ for some edge $e = w^{i_1}_{j_1} w^{i_2}_{j_2} \in E(K(N))$. Let i, j be such that $\phi(x) \in U^i_j$ (so $i \in \{i_1, i_2\}$). If $i_1 \neq i_2$, then $d_J(\phi(x), U^i_{j'}) = 0$ for all $1 \le j' \le r$. If $i_1 = i_2$, then $d_J(\phi(x), U^i_{j'}) = 1$ for all $1 \le j' \le r$ with $j' \neq j$. So J is locally $\mathcal{P}$-balanced. Thus (i) holds.

We now prove (ii). Let $1 \le i_1, i_2 \le k$ and $1 \le j_1 < j_2 \le r$. Consider any edge $w^i_j w^{i'}_{j'} \in E(K(N))$. The $\mathcal{P}$-labelling of $\theta(K(N))$ gives

$$e_{\phi(\theta(w^i_j w^{i'}_{j'}))}(U^{i_1}_{j_1}, U^{i_2}_{j_2}) = \begin{cases} 0 & \text{if } \{i, i'\} \neq \{i_1, i_2\}, \\ 2 & \text{if } \{(i,j),(i',j')\} = \{(i_1,j_1),(i_2,j_2)\}, \\ 1 & \text{otherwise.} \end{cases} \tag{10.3}$$

Let $H \subseteq K(N)$. Then (ii) follows from applying (10.3) to each edge in H. □

The following proposition allows us to use a copy of $\theta(K(N))$ to correct imbalances in the number of edges between parts $U^{i_1}_{j_1}$ and $U^{i_2}_{j_2}$ when $\mathrm{EM}(H)$ is an irreducible K_r-divisible multigraph.

Proposition 10.5. *Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let H be a graph on V such that $\mathrm{EM}(H) = I$ is an irreducible K_r-divisible multigraph. Let $J = \phi(\theta(K(N)))$ be a copy of $\theta(K(N))$ on V which is compatible with its $\mathcal{P}$-labelling and edge-disjoint from H. Then there exists $J' \subseteq J$ such that $J - J'$ is K_r-divisible and $H' := H \cup J'$ satisfies*

$$e_{H'}(U^{i_1}_{j_1}, U^{i_2}_{j_2}) = e_{H'}(U^{i_1}_{j_1}, U^{i_2}_{j_3}) \tag{10.4}$$

for all $1 \le i_1 < i_2 \le k$ and all $1 \le j_1, j_2, j_3 \le r$ with $j_1 \neq j_2, j_3$.

Proof. Recall that N denotes the maximum multiplicity of an edge in an irreducible K_r-divisible multigraph on $V(K)$. So we may view I as a subgraph of $K(N)$. Let $J' := J - \phi(\theta(I))$. For all

$1 \le i_1 < i_2 \le k$, let

$$p_{i_1,i_2} := \min\{e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) : 1 \le j_1, j_2, \le r, j_1 \ne j_2\}.$$

Proposition 10.4 gives, for all $1 \le i_1 < i_2 \le k$ and all $1 \le j_1, j_2 \le r$ with $j_1 \ne j_2$,

$$\begin{aligned} e_{J'}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) &= e_{\phi(\theta(K(N)))}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) - e_{\phi(\theta(I))}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) \\ &= e_{K(N)}(W^{i_1}, W^{i_2}) + N - (e_I(W^{i_1}, W^{i_2}) + m_I(w_{j_1}^{i_1} w_{j_2}^{i_2})) \\ &= e_{K(N)-I}(W^{i_1}, W^{i_2}) + N - m_I(w_{j_1}^{i_1} w_{j_2}^{i_2}). \end{aligned}$$

Recall that $I = \mathrm{EM}(H)$, so $e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) = m_I(w_{j_1}^{i_1} w_{j_2}^{i_2}) + p_{i_1,i_2}$ and

$$e_{H'}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) = e_H(U_{j_1}^{i_1}, U_{j_2}^{i_2}) + e_{J'}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) = e_{K(N)-I}(W^{i_1}, W^{i_2}) + N + p_{i_1,i_2}.$$

Note that the right hand side is independent of j_1, j_2. Thus (10.4) holds. □

The following proposition describes a $(\gamma, \mathcal{P})$-edge balancing graph based on the construction in Propositions 10.4 and 10.5

Proposition 10.6. *Let $k, r \in \mathbb{N}$ with $r \ge 3$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $J_1, \dots, J_\ell$ be a collection of $\ell \ge 3\gamma k^2 r^2 n^2$ copies of $\theta(K(N))$ on V which are compatible with their labellings. Let $\{A_1, \dots, A_m\}$ be an absorbing set for $J_1, \dots, J_\ell$ on V. Suppose that $J_1, \dots, J_\ell, A_1, \dots, A_m$ are edge-disjoint. Then $B_{\mathrm{edge}} := \bigcup_{i=1}^{\ell} J_i \cup \bigcup_{i=1}^{m} A_i$ is a $(\gamma, \mathcal{P})$-edge balancing graph.*

Proof. Let H be any K_r-divisible graph on V which is edge-disjoint from B_{edge} and satisfies (P2). Apply Proposition 10.3 to find a decomposition of $\mathrm{EM}(H)$ into a collection $\mathcal{I} = \{I_1, \dots, I_{\ell'}\}$ of irreducible K_r-divisible multigraphs, where $\ell' \le 3\gamma k^2 r^2 n^2 \le \ell$. If $\ell' = 0$, let $B'_{\mathrm{edge}} \subseteq B_{\mathrm{edge}}$ be the empty graph. If $\ell' > 0$, we proceed as follows to find B'_{edge}. Let $H_1, \dots, H_{\ell'}$ be graphs on V which partition the edge set of H and satisfy $\mathrm{EM}(H_s) = I_s$ for each $1 \le s \le \ell'$. (To find such a partition, for each $1 \le s < \ell'$ form H_s by taking one $U_{j_1}^{i_1} U_{j_2}^{i_2}$-edge from H for each edge $w_{j_1}^{i_1} w_{j_2}^{i_2}$ in I_s. Let $H_{\ell'}$ consist of all the remaining edges.)

Apply Proposition 10.5 for each $1 \le s \le \ell'$ with H_s and J_s playing the roles of H and J to find $J'_s \subseteq J_s$ such that $J_s - J'_s$ is K_r-divisible and $H'_s := H_s \cup J'_s$ satisfies

$$e_{H'_s}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) = e_{H'_s}(U_{j_1}^{i_1}, U_{j_3}^{i_2}) \tag{10.5}$$

for all $1 \le i_1 < i_2 \le k$ and all $1 \le j_1, j_2, j_3 \le r$ with $j_1 \ne j_2, j_3$. Let $B'_{\mathrm{edge}} := \bigcup_{s=1}^{\ell'} J'_s$. Then (10.5) implies that the graph $H' := H \cup B'_{\mathrm{edge}} = \bigcup_{s=1}^{\ell'} H'_s$ satisfies

$$e_{H'}(U_{j_1}^{i_1}, U_{j_2}^{i_2}) = e_{H'}(U_{j_1}^{i_1}, U_{j_3}^{i_2})$$

for all $1 \le i_1 < i_2 \le k$ and all $1 \le j_1, j_2, j_3 \le r$ with $j_1 \ne j_2, j_3$.

We now check that B_{edge} and $B_{\mathrm{edge}} - B'_{\mathrm{edge}}$ are K_r-decomposable. Recall that every absorber A_i is K_r-decomposable. Also recall that, for every $1 \le s \le \ell$, J_s is K_r-divisible, by Proposition 10.4(i). Since $\{A_1, \dots, A_m\}$ is an absorbing set, it contains a distinct absorber for each J_s. So for each $1 \le s \le \ell$, there exists a distinct $1 \le i_s \le m$ such that $A_{i_s} \cup J_s$ has a K_r-decomposition. Therefore B_{edge} is K_r-decomposable. To see that $B_{\mathrm{edge}} - B'_{\mathrm{edge}}$ is K_r-decomposable, recall that for each $1 \le s \le \ell'$, $J_s - J'_s$ is a K_r-divisible subgraph of J_s. So for each $1 \le s \le \ell$, there exists a distinct $1 \le j_s \le m$ such that, if $s \le \ell'$, $A_{j_s} \cup (J_s - J'_s)$ has a K_r-decomposition and, if $s > \ell'$, $A_{j_s} \cup J_s$ has a K_r-decomposition. So we can find a K_r-decomposition of

$$B_{\mathrm{edge}} - B'_{\mathrm{edge}} = \bigcup_{s=1}^{\ell'} (J_s - J'_s) \cup \bigcup_{s=\ell'+1}^{\ell} J_s \cup \bigcup_{s=1}^{m} A_m.$$

Therefore, B_{edge} is a $(\gamma, \mathcal{P})$-edge balancing graph. □

The next proposition finds a copy of this $(\gamma, \mathcal{P})$-edge balancing graph in G.

Proposition 10.7. *Let $1/n \ll \gamma \ll \gamma' \ll 1/k \ll \varepsilon \ll 1/r \leq 1/3$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition for G. Suppose that $d_G(v, U^i_j) \geq (1 - 1/(r+1) + \varepsilon)|U^i_j|$ for all $1 \leq i \leq k$, all $1 \leq j \leq r$ and all $v \notin V_j$. Then there exists a $(\gamma, \mathcal{P})$-edge balancing graph $B_{\mathrm{edge}} \subseteq G$ such that B_{edge} is locally $\mathcal{P}$-balanced and $\Delta(B_{\mathrm{edge}}) < \gamma' n$.*

Proof. Let γ_1 be such that $\gamma \ll \gamma_1 \ll \gamma'$. Recall that $\theta(K(N))$ is a $\mathcal{P}$-labelled graph with degeneracy $r-1$ and all vertices of $\theta(K(N))$ are free vertices. Also,

$$|\theta(K(N))| \leq |K| + 2re(K)N = kr + 2rk^2\binom{r}{2}N \leq k^2r^3N.$$

Let $\ell := \lceil 3\gamma k^2 r^2 n^2 \rceil \leq \gamma^{1/2} n^2$. We can apply Lemma 5.2 (with $\gamma^{1/2}$, γ_1, $r-1$, k^2r^3N and $\theta(K(N))$ playing the roles of η, ε, d, b and H_i) to find edge-disjoint copies $J_1, \dots, J_\ell$ of $\theta(K(N))$ in G which are compatible with their labellings and satisfy $\Delta(\bigcup_{i=1}^{\ell} J_i) \leq \gamma_1 n$.

Let $G' := G[\mathcal{P}] - \bigcup_{i=1}^{\ell} J_i$ and note that

$$\hat{\delta}(G') \geq (1 - 1/(r+1) + \varepsilon)n - \lceil n/k \rceil - \gamma_1 n \geq (1 - 1/(r+1) + \gamma')n.$$

Apply Lemma 6.6 (with γ_1, $\gamma'/2$, k^2r^3N and G' playing the roles of η, ε, b and G) to find an absorbing set $\mathcal{A}$ for $J_1, \dots, J_\ell$ in G' such that $\Delta(\bigcup \mathcal{A}) \leq \gamma' n/2$.

Let $B_{\mathrm{edge}} := \bigcup_{i=1}^{\ell} J_i \cup \bigcup \mathcal{A}$. Then B_{edge} is a $(\gamma, \mathcal{P})$-edge balancing graph by Proposition 10.6. Also, $\Delta(B_{\mathrm{edge}}) < \gamma' n$. Note that for each $1 \leq i \leq k$, $B_{\mathrm{edge}}[U^i] = \bigcup_{s=1}^{\ell} J_s[U^i]$ (this is the reason for finding $\mathcal{A}$ in $G[\mathcal{P}]$). Moreover, each J_s is locally $\mathcal{P}$-balanced by Proposition 10.4(i). Therefore B_{edge} is also locally $\mathcal{P}$-balanced. □

10.2. **Degree balancing.** Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition of the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $\gamma > 0$. A $(\gamma, \mathcal{P})$-*degree balancing graph* is a K_r-decomposable graph B_{deg} on V such that the following holds. Let H be any K_r-divisible graph on V satisfying:

(Q1) $e(H \cap B_{\mathrm{deg}}) = 0$;
(Q2) $e_H(U^{i_1}_{j_1}, U^{i_2}_{j_2}) = e_H(U^{i_1}_{j_1}, U^{i_2}_{j_3})$ for all $1 \leq i_1 < i_2 \leq k$ and all $1 \leq j_1, j_2, j_3 \leq r$ with $j_1 \neq j_2, j_3$;
(Q3) $|d_H(v, U^i_{j_2}) - d_H(v, U^i_{j_3})| < \gamma |U^i_{j_1}|$ for all $2 \leq i \leq k$, all $1 \leq j_1, j_2, j_3 \leq r$ with $j_1 \neq j_2, j_3$ and all $v \in U^{<i}_{j_1}$.

Then there exists $B'_{\mathrm{deg}} \subseteq B_{\mathrm{deg}}$ such that $B_{\mathrm{deg}} - B'_{\mathrm{deg}}$ has a K_r-decomposition and

$$d_{H \cup B'_{\mathrm{deg}}}(v, U^i_{j_1}) = d_{H \cup B'_{\mathrm{deg}}}(v, U^i_{j_2})$$

for all $2 \leq i \leq k$, all $1 \leq j_1, j_2 \leq r$ and all $v \in U^{<i} \setminus (V_{j_1} \cup V_{j_2})$.

We will build a degree balancing graph by combining smaller graphs which correct the degrees between two parts of the partition at a time. So, let us assume that the partition has only two parts, i.e., let $\mathcal{P} = \{U^1, U^2\}$ partition the vertex set $V = (V_1, \dots, V_r)$. We begin by defining those graphs which will form the basic gadgets of the degree balancing graph. Let D_0 be a copy of $K_r(3)$ with vertex classes $\{w^i_j : 1 \leq i \leq 3\}$ for $1 \leq j \leq r$. For each $1 \leq i \leq 3$, let $W^i := \{w^i_j : 1 \leq j \leq r\}$. We define a labelling $L : V(D_0) \to \{U^1_j, U^2_j : 1 \leq j \leq r\}$ as follows:

$$L(w^i_j) = \begin{cases} U^1_j & \text{if } i = 1, 2, \\ U^2_j & \text{if } i = 3. \end{cases}$$

Suppose that x, y are distinct vertices in $U^1_{j_1}$ where $1 \leq j_1 \leq r$. Obtain the $\mathcal{P}$-labelled graph $D_{x,y}$ by taking the labelled copy of D_0 and changing the label of $w^1_{j_1}$ to $\{x\}$ and $w^2_{j_1}$ to $\{y\}$. Let $1 \leq j_2 \leq r$ be

such that $j_2 \neq j_1$. Let $D^{j_2}_{x\to y}$ be the $\mathcal{P}$-labelled subgraph of $D_{x,y}$ which has as its vertex set

$$W^1 \cup \{w^2_{j_1}\} \cup (W^3 \setminus \{w^3_{j_1}\}),$$

contains all possible edges in $W^1 \setminus \{w^1_{j_1}\}$, all possible edges in $W^3 \setminus \{w^3_{j_1}\}$, all edges of the form $w^1_{j_1}w^3_j$ and $w^1_j w^2_{j_1}$ where $1 \leq j \leq r$ and $j \neq j_1, j_2$, as well as the edges $w^1_{j_1}w^1_{j_2}$ and $w^2_{j_1}w^3_{j_2}$. (Note that if we were to identify the vertices $w^1_{j_1}$ and $w^2_{j_1}$ we would obtain two copies of K_r which have only one vertex in common.)

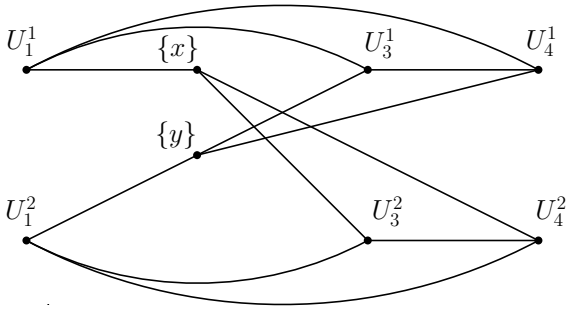

FIGURE 2. A copy of $D^1_{x\to y}$ when $r = 4$ and $x, y \in U^1_2$.

As in Section 10.1, we would like to reduce the degeneracy of $D_{x,y}$. The operation θ (which will be familiar from Section 10.1) replaces each edge of $D_{x,y}$ by a $\mathcal{P}$-labelled graph as follows. Consider any edge $e = w^{i_1}_{j_3}w^{i_2}_{j_4} \in E(D_{x,y})$. Take a labelled copy D_e of $D_0[W^{i_1}, W^{i_2}] - w^{i_1}_{j_3}w^{i_2}_{j_4}$ (D_e inherits the labelling of $D_0[W^{i_1}, W^{i_2}]$). Note that $D_0[W^{i_1}, W^{i_2}]$ is a copy of K_r if $i_1 = i_2$ and a copy of the graph obtained from $K_{r,r}$ by deleting a perfect matching otherwise. Join $w^{i_1}_{j_3}$ to the copy of $w^{i_2}_{j_4}$ in D_e and join $w^{i_2}_{j_4}$ to the copy of $w^{i_1}_{j_3}$ in D_e (so the vertex set of $\theta(e)$ consists of $w^{i_1}_{j_3}$, $w^{i_2}_{j_4}$ as well as all the vertices in D_e). Write $\theta(e)$ for the resulting $\mathcal{P}$-labelled graph. Choose the graphs D_e to be vertex-disjoint for all $e \in E(D_{x,y})$. For any $D' \subseteq D_{x,y}$, let $\theta(D') := \bigcup\{\theta(e) : e \in E(D')\}$. The graph $\theta(D_{x,y})$ has the following properties:

(θ1) $|\theta(D_{x,y})| \leq 3r + 2r3^2\binom{r}{2} \leq 10r^3$ (since we add at most $2re(K_r(3))$ new vertices to obtain $\theta(D_{x,y})$ from $D_{x,y}$);

(θ2) $\theta(D_{x,y})$ has degeneracy $r-1$ (to see this, take the original vertices of $D_{x,y}$ first, followed by the remaining vertices in any order).

Suppose that H is a graph on V and $x, y \in U^1_{j_1}$. Suppose that $d_H(x, U^2_{j_2})$ is currently too large and $d_H(y, U^2_{j_2})$ is too small. The next proposition allows us to use copies of $\theta(D^{j_2}_{x\to y})$ to 'transfer' some of this surplus from x to y.

Proposition 10.8. *Let $\mathcal{P} = \{U^1, U^2\}$ be a partition of the vertex set $V = (V_1, \dots, V_r)$. Let $1 \leq j_1, j_2 \leq r$ with $j_1 \neq j_2$ and suppose $x, y \in U^1_{j_1}$. Suppose that $D_1 = \phi(\theta(D_{x,y}))$ is a copy of $\theta(D_{x,y})$ on V which is compatible with its labelling. Let $D_2 := \phi(\theta(D^{j_2}_{x\to y})) \subseteq D_1$. Then the following hold:*

(i) *both D_1 and D_2 are K_r-divisible;*
(ii) *D_1 is locally $\mathcal{P}$-balanced;*
(iii) *for any $1 \leq j_3, j_4 \leq r$ with $j_4 \neq j_2$ and any $v \in U^1 \setminus (V_{j_3} \cup V_{j_4})$,*

$$d_{D_2}(v, U^2_{j_3}) - d_{D_2}(v, U^2_{j_4}) = \begin{cases} -1 & \text{if } v = x \text{ and } j_3 = j_2, \\ 1 & \text{if } v = y \text{ and } j_3 = j_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we show that (i) holds. Consider any $v \in V(\theta(D_{x,y}))$. If $v \in V(D_{x,y})$, then $d_{D_1}(\phi(v), V_j) = 3$ for all $1 \leq j \leq r$ such that $\phi(v) \notin V_j$. Otherwise, v appears in a copy of D_e for some edge $e \in E(D_{x,y})$

and $d_{D_1}(\phi(v), V_j) = 1$ for all $1 \le j \le r$ such that $\phi(v) \notin V_j$. So D_1 is K_r-divisible. For D_2, consider any $v \in V(\theta(D^{j_2}_{x\to y}))$. If $v \in V(D^{j_2}_{x\to y})$, then $d_{D_2}(\phi(v), V_j) = 1$ for all $1 \le j \le r$ with $\phi(v) \notin V_j$. Otherwise, v appears in a copy of D_e for some edge $e \in E(D^{j_2}_{x\to y})$ and $d_{D_2}(\phi(v), V_j) = 1$ for all $1 \le j \le r$ such that $\phi(v) \notin V_j$. So D_2 is K_r-divisible.

For (ii), consider any $v \in V(\theta(D_{x,y}))$. First suppose $v = w^i_j \in V(D_{x,y})$. If $i = 1, 2$, then $\phi(v) \in U^1_j$ and $d_{D_1}(\phi(v), U^1_{j'}) = 2$ for all $1 \le j' \le r$ with $j' \ne j$. If $i = 3$, then $\phi(v) \in U^2_j$ and $d_{D_1}(\phi(v), U^2_{j'}) = 1$ for all $1 \le j' \le r$ with $j' \ne j$. Otherwise, v must appear in a copy of D_e in $\theta(e)$ for some edge $e = w^{i_1}_{j_1} w^{i_2}_{j_2} \in E(D_{x,y})$. Let i, j be such that $\phi(v) \in U^i_j$. If $i_1, i_2 \in \{1, 2\}$ or if $i_1 = i_2 = 3$, then $d_{D_1}(\phi(v), U^i_{j'}) = 1$ for all $1 \le j' \le r$ with $j' \ne j$. Otherwise, $d_{D_1}(\phi(v), U^i_{j'}) = 0$ for all $1 \le j' \le r$. So D_1 is locally $\mathcal{P}$-balanced.

Property (iii) will follow from the $\mathcal{P}$-labelling of $\theta(D^{j_2}_{x\to y})$. Note that

$$d_{D_2}(x, U^2_{j'}) = \begin{cases} 0 & \text{if } j' \in \{j_1, j_2\}, \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad d_{D_2}(y, U^2_{j'}) = \begin{cases} 1 & \text{if } j' = j_2, \\ 0 & \text{otherwise.} \end{cases}$$

The only other edges ab in D_2 of the form U^1U^2 are those which appear in the image of D_e for some $e = w^i_j w^3_{j'} \in E(D^{j_2}_{x\to y})$ with $i = 1, 2$. Note that such e must be incident to x or y and that a and b are new vertices, i.e., $a, b \notin V(D^{j_2}_{x\to y})$. But for any $v \in \phi(D_e) \cap U^1$, we have $d_{D_2}(v, U^2_{j'}) = 1$ for every $1 \le j' \le r$ such that $\phi(v) \notin V_{j'}$. It follows that (iii) holds. □

In what follows, given a collection $\mathcal{D}$ of graphs and an embedding $\phi(D)$ for each $D \in \mathcal{D}$, we write $\phi(\mathcal{D}) := \{\phi(D) : D \in \mathcal{D}\}$.

Lemma 10.9. *Let $1/n \ll \gamma \ll \gamma' \le 1/r \le 1/3$. Let $V = (V_1, \dots, V_r)$ with $|V_1| = \dots = |V_r| = n$. Let $\mathcal{P} = \{U^1, U^2\}$ be a 2-partition of V. Let $1 \le j_1 \le r$. Then there exists $\mathcal{D} \subseteq \{\theta(D^j_{x\to y}) : x, y \in U^1_{j_1}, x \ne y, 1 \le j \le r, j \ne j_1\}$ such that the following hold.*

- (i) *$|\mathcal{D}| \le \gamma' n^2$.*
- (ii) *Each vertex $v \in V$ is a root vertex in at most $\gamma' n$ elements of $\mathcal{D}$.*
- (iii) *Suppose that, for each $D \in \mathcal{D}$, $\phi(D)$ is a copy of D on V which is compatible with its labelling. Suppose further that $\phi(D)$ and $\phi(D')$ are edge-disjoint for all distinct $D, D' \in \mathcal{D}$. Let H be any r-partite graph on V which is edge-disjoint from $\bigcup \phi(\mathcal{D})$ and satisfies* (Q2) *and* (Q3)*. Then there exists $\mathcal{D}' \subseteq \mathcal{D}$ such that $H' := H \cup \bigcup \phi(\mathcal{D}')$ satisfies the following. For all $v \in U^1_{j_1}$, and all $1 \le j_2, j_3 \le r$ such that $j_1 \ne j_2, j_3$,*

$$d_{H'}(v, U^2_{j_2}) = d_{H'}(v, U^2_{j_3})$$

and for all $1 \le j_2, j_3 \le r$ and all $v \in U^1 \setminus (V_{j_1} \cup V_{j_2} \cup V_{j_3})$,

$$d_{H'}(v, U^2_{j_2}) - d_{H'}(v, U^2_{j_3}) = d_H(v, U^2_{j_2}) - d_H(v, U^2_{j_3}).$$

In particular, H' satisfies (Q2) *and* (Q3)*.*

Proof. Let $p := \gamma'/4(r-1)$ and $m := |U^1_{j_1}|$. Define an auxiliary graph R on $U^1_{j_1}$ such that $\Delta(R) < 2pm$ and

$$|N_R(S)| \ge p^2 m/2 \tag{10.6}$$

for all $S \subseteq U^1_{j_1}$ with $|S| \le 2$. It is easy to find such a graph R; indeed, a random graph with edge probability p has these properties with high probability.

Let

$$\mathcal{D} := \{\theta(D^j_{x\to y}), \theta(D^j_{y\to x}) : xy \in E(R), 1 \le j \le r, j \ne j_1\}.$$

Each vertex of V appears as x or y in some $\theta(D^j_{x\to y})$ in $\mathcal{D}$ at most $2(r-1)\Delta(R) < 4(r-1)pm = \gamma' m$ times. In particular, this implies $|\mathcal{D}| \le \gamma' m^2$. So $\mathcal{D}$ satisfies (i) and (ii).

We now show that $\mathcal{D}$ satisfies (iii). Suppose that, for each $D \in \mathcal{D}$, $\phi(D)$ is a copy of D on V which is compatible with its labelling. Suppose further that $\phi(D)$ and $\phi(D')$ are edge-disjoint for all distinct $D, D' \in \mathcal{D}$. Let H be any r-partite graph on V which is edge-disjoint from $\bigcup \phi(\mathcal{D})$ and satisfies (Q2) and (Q3).

Let $j_{\min} := \min\{j : 1 \leq j \leq r, j \neq j_1\}$. For each $v \in U^1_{j_1}$ and each $j_{\min} < j \leq r$ such that $j \neq j_1$, let

$$f(v,j) := d_H(v, U^2_j) - d_H(v, U^2_{j_{\min}}). \tag{10.7}$$

By (Q3),

$$|f(v,j)| < \gamma(m+1) < 2\gamma m. \tag{10.8}$$

Let $U^+(j)$ be a multiset such that each $v \in U^1_{j_1}$ appears precisely $\max\{f(v,j), 0\}$ times. Let $U^-(j)$ be a multiset such that each $v \in U^1_{j_1}$ appears precisely $\max\{-f(v,j), 0\}$ times. Property (Q2) implies that $|U^+(j)| = |U^-(j)|$, so there is a bijection $g_j : U^+(j) \to U^-(j)$.

For each copy u' of u in $U^+(j)$, let $P_{u'}$ be a path of length two whose vertices are labelled, in order,

$$\{u\}, U^1_{j_1}, \{g_j(u')\}.$$

So $P_{u'}$ has degeneracy two. Let $\mathcal{S}_j := \{P_{u'} : u' \in U^+(j)\}$. It follows from (10.8) that each vertex is used as a root vertex at most $2\gamma m$ times in $\mathcal{S}_j$ and $|\mathcal{S}_j| \leq 2\gamma m^2$. Using (10.6), we can apply Lemma 5.1 (with m, 2, 3, 2γ, $p^2/2$ and R playing the roles of n, d, b, η, ε and G) to find a set of edge-disjoint copies $\mathcal{T}_j$ of the paths in $\mathcal{S}_j$ in R which are compatible with their labellings. (Note that we do not require the paths in $\mathcal{T}_j$ to be edge-disjoint from the paths in $\mathcal{T}_{j'}$ for $j \neq j'$.) We will view the paths in $\mathcal{T}_j$ as directed paths whose initial vertex lies in $U^+(j)$ and whose final vertex lies in $U^-(j)$.

For each $j_{\min} < j \leq r$ such that $j \neq j_1$, let $\mathcal{D}_j := \{\theta(D^j_{x \to y}) : \overrightarrow{xy} \in E(\bigcup \mathcal{T}_j)\}$. Let

$$\mathcal{D}' := \bigcup_{\substack{j_{\min} < j \leq r \\ j \neq j_1}} \mathcal{D}_j \subseteq \mathcal{D}.$$

It remains to show that $H' := H \cup \bigcup \phi(\mathcal{D}')$ satisfies (iii). For each $j_{\min} < j \leq r$ such that $j \neq j_1$, let $H_j := \bigcup \phi(\mathcal{D}_j)$. Consider any vertex $v \in U^1_{j_1}$ and let $j_{\min} < j_2 \leq r$ be such that $j_2 \neq j_1$. Now v will be the initial vertex in exactly $a := \max\{f(v,j_2), 0\}$ paths and the final vertex in exactly $b := \max\{-f(v,j_2), 0\} = a - f(v,j_2)$ paths in $\mathcal{T}_{j_2}$. Let c be the number of paths in $\mathcal{T}_{j_2}$ for which v is an internal vertex. By definition, H_{j_2} contains $a + c$ graphs $\phi(D)$ where D is of the form $\theta(D^{j_2}_{v \to y})$ for some $y \in U^1_{j_1}$. Also, H_{j_2} contains $b + c$ graphs $\phi(D)$ where D of the form $\theta(D^{j_2}_{x \to v})$ for some $x \in U^1_{j_1}$. Proposition 10.8(iii) then implies that

$$d_{H_{j_2}}(v, U^2_{j_2}) - d_{H_{j_2}}(v, U^2_{j_{\min}}) = (b+c) - (a+c) = -f(v,j_2). \tag{10.9}$$

For any $j_{\min} < j_3 \leq r$ such that $j_3 \neq j_1, j_2$, Proposition 10.8(iii) implies that

$$d_{H_{j_3}}(v, U^2_{j_2}) - d_{H_{j_3}}(v, U^2_{j_{\min}}) = 0. \tag{10.10}$$

Equations (10.9) and (10.10) imply that

$$d_{\bigcup \phi(\mathcal{D}')}(v, U^2_{j_2}) - d_{\bigcup \phi(\mathcal{D}')}(v, U^2_{j_{\min}}) = d_{H_{j_2}}(v, U^2_{j_2}) - d_{H_{j_2}}(v, U^2_{j_{\min}}) = -f(v,j_2),$$

which together with (10.7) gives

$$d_{H'}(v, U^2_{j_2}) - d_{H'}(v, U^2_{j_{\min}}) = d_H(v, U^2_{j_2}) - d_H(v, U^2_{j_{\min}}) - f(v,j_2) = 0. \tag{10.11}$$

Thus, for all $v \in U^1_{j_1}$ and all $1 \leq j_2, j_3 \leq r$ such that $j_1 \neq j_2, j_3$,

$$d_{H'}(v, U^2_{j_2}) = d_{H'}(v, U^2_{j_{\min}}) = d_{H'}(v, U^2_{j_3}).$$

Finally, consider any $1 \leq j_2, j_3 \leq r$ and any $v \in U^1 \setminus (V_{j_1} \cup V_{j_2} \cup V_{j_3})$. Proposition 10.8(iii) implies that

$$d_{\bigcup \phi(\mathcal{D}')}(v, U^2_{j_2}) - d_{\bigcup \phi(\mathcal{D}')}(v, U^2_{j_3}) = 0,$$

so

$$d_{H'}(v,U^2_{j_2}) - d_{H'}(v,U^2_{j_3}) = d_H(v,U^2_{j_2}) - d_H(v,U^2_{j_3}). \tag{10.12}$$

That H' satisfies (Q2) and (Q3) follows immediately from (10.11) and (10.12). □

Let $\mathcal{P} = \{U^1, U^2\}$ partition the vertex set $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. We say that a collection $\mathcal{D}$ of $\mathcal{P}$-labelled graphs is a (γ, γ')-*degree balancing set* for the pair (U^1, U^2) if the following properties hold. Suppose that, for each $D \in \mathcal{D}$, $\phi(D)$ is a copy of D on V which is compatible with its labelling. Suppose further that $\phi(D)$ and $\phi(D')$ are edge-disjoint for all distinct $D, D' \in \mathcal{D}$.

(a) Each $D \in \mathcal{D}$ has degeneracy at most $r-1$ and $|D| \le 10r^3$.
(b) $|\mathcal{D}| \le \gamma' n^2$.
(c) Each vertex $v \in V$ is a root vertex in at most $\gamma' n$ elements of $\mathcal{D}$.
(d) For each $D \in \mathcal{D}$, $\phi(D)$ is K_r-divisible and locally $\mathcal{P}$-balanced.
(e) Let H be any r-partite graph on V which is edge-disjoint from $\bigcup \phi(\mathcal{D})$ and satisfies (Q2) and (Q3). Then, for each $D \in \mathcal{D}$, there exists $D' \subseteq D$ such that $\phi(D')$ is K_r-divisible and, if $\mathcal{D}' := \{D' : D \in \mathcal{D}\}$ and $H' := H \cup \bigcup \phi(\mathcal{D}')$, then
$$d_{H'}(v,U^2_{j_1}) = d_{H'}(v,U^2_{j_2})$$
for all $1 \le j_1, j_2 \le r$ and all $v \in U^1 \setminus (V_{j_1} \cup V_{j_2})$.

The following result describes a (γ, γ')-degree balancing set based on the gadgets constructed so far.

Proposition 10.10. *Let $1/n \ll \gamma \ll \gamma' \le 1/r \le 1/3$. Let $V = (V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $\mathcal{P} = \{U^1, U^2\}$ be a 2-partition for V. Then (U^1, U^2) has a (γ, γ')-degree balancing set.*

Proof. Apply Lemma 10.9 for each $1 \le j_1 \le r$ with γ'/r playing the role of γ' to find sets $\mathcal{D}_{j_1} \subseteq \{\theta(D^j_{x\to y}) : x, y \in U^1_{j_1}, x \ne y, 1 \le j \le r, j \ne j_1\}$ satisfying the properties (i)–(iii). Let $\mathcal{D}$ consist of one copy of $\theta(D_{x,y})$ for each $\theta(D^j_{x\to y})$ in $\bigcup_{j=1}^r \mathcal{D}_j$. We claim that $\mathcal{D}$ is a (γ, γ')-degree balancing set. Note that each $\theta(D_{x,y})$ satisfies $|\theta(D_{x,y})| \le 10r^3$ and has degeneracy at most $r-1$ by (θ1) and (θ2), so (a) holds. For each $1 \le j \le r$, $|\mathcal{D}_j| \le \gamma' n^2/r$, so (b) holds. Also, each vertex $v \in V$ is used as a root vertex in at most $\gamma' n/r$ elements of each $\mathcal{D}_j$. Since $\theta(D_{x,y})$ and $\theta(D^j_{x\to y})$ have the same set of root vertices, (c) holds. Property (d) follows from Proposition 10.8(i) and (ii).

It remains to show that (e) is satisfied. Suppose that, for each $D \in \mathcal{D}$, $\phi(D)$ is a copy of D on V which is compatible with its labelling. Suppose further that $\phi(D)$ and $\phi(D')$ are edge-disjoint for all distinct $D, D' \in \mathcal{D}$. Let H be any r-partite graph on V which is edge-disjoint from $\bigcup \phi(\mathcal{D})$ and satisfies (Q2) and (Q3). Using property (iii) of $\mathcal{D}_1$ in Lemma 10.9, we can find $\mathcal{D}'_1 \subseteq \mathcal{D}_1$ such that $H_1 := H \cup \bigcup \phi(\mathcal{D}'_1)$ satisfies (Q2), (Q3) and
$$d_{H_1}(v,U^2_{j_1}) = d_{H_1}(v,U^2_{j_2})$$
for all $v \in U^1_1$ and all $2 \le j_1, j_2 \le r$. We can then find $\mathcal{D}'_2 \subseteq \mathcal{D}_2$ such that $H_2 := H_1 \cup \bigcup \phi(\mathcal{D}'_2)$ satisfies (Q2), (Q3) and
$$d_{H_2}(v,U^2_{j_1}) = d_{H_2}(v,U^2_{j_2})$$
for all $v \in U^1_j$ where $j = 1, 2$ and all $1 \le j_1, j_2 \le r$ with $j \ne j_1, j_2$. Continuing in this way, we eventually find $\mathcal{D}'_r \subseteq \mathcal{D}_r$ such that $H_r := H_{r-1} \cup \bigcup \phi(\mathcal{D}'_{r-1})$ satisfies
$$d_{H_r}(v,U^2_{j_1}) = d_{H_r}(v,U^2_{j_2}) \tag{10.13}$$
for all $1 \le j_1, j_2 \le r$ and all $v \in U^1 \setminus (V_{j_1} \cup V_{j_2})$.

For each $D \in \mathcal{D}_j$, if $D \in \mathcal{D}'_j$, then let $D' := D$; otherwise let D' be the empty graph. Let $\mathcal{D}' := \{D' : D \in \bigcup_{j=1}^r \mathcal{D}_j\}$. For each $D' \in \mathcal{D}'$, D' is either empty or of the form $\theta(D^j_{x\to y})$, so $\phi(D')$ is K_r-divisible by Proposition 10.8(i). By (10.13), $\mathcal{D}'$ satisfies (e). So $\mathcal{D}$ satisfies (a)–(e) and is a (γ, γ')-degree balancing set for (U^1, U^2). □

The following result finds copies of the degree balancing sets described in the previous proposition.

Proposition 10.11. *Let $1/n \ll \gamma \ll \gamma' \ll 1/k \ll \varepsilon \ll 1/r \leq 1/3$. Let G be an r-partite graph on $(V_1, \dots, V_r)$ with $|V_1| = \cdots = |V_r| = n$. Let $\mathcal{P} = \{U^1, \dots, U^k\}$ be a k-partition for G. Suppose that $d_G(v, U^i_j) \geq (1 - 1/(r+1) + \varepsilon)|U^i_j|$ for all $1 \leq i \leq k$, all $1 \leq j \leq r$ and all $v \notin V_j$. Then there exists a $(\gamma, \mathcal{P})$-degree balancing graph $B_{\deg} \subseteq G$ such that $B_{\deg}$ is locally $\mathcal{P}$-balanced and $\Delta(B_{\deg}) < \gamma' n$.*

Proof. Choose γ_1, γ_2 such that $\gamma \ll \gamma_1 \ll \gamma_2 \ll \gamma'$. Proposition 10.10 describes a (γ, γ_1^2)-degree balancing set $\mathcal{D}_{i_1,i_2}$ for each pair (U^{i_1}, U^{i_2}) with $1 \leq i_1 < i_2 \leq k$. Let $\mathcal{D} := \bigcup_{1 \leq i_1 < i_2 \leq k} \mathcal{D}_{i_1,i_2}$. We have $|\mathcal{D}| \leq k^2 \gamma_1^2 n^2 \leq \gamma_1 n^2$ and each vertex is used as a root vertex in at most $k^2 \gamma_1^2 n \leq \gamma_1 n$ elements of $\mathcal{D}$. By (a), we can apply Lemma 5.2 (with γ_1, γ_2, $r-1$ and $10r^3$ playing the roles of η, ε, d and b) to find edge-disjoint copies $\phi(D)$ of each $D \in \mathcal{D}$ in G which are compatible with their labellings and satisfy $\Delta(\bigcup \phi(\mathcal{D})) \leq \gamma_2 n$.

Let $G' := G[\mathcal{P}] - \bigcup \phi(\mathcal{D})$ and note that

$$\hat{\delta}(G') \geq (1 - 1/(r+1) + \varepsilon)n - \lceil n/k \rceil - \gamma_2 n \geq (1 - 1/(r+1) + \gamma')n.$$

Apply Lemma 6.6 (with γ_2, $\gamma'/2$, $10r^3$ and G' playing the roles of η, ε, b and G) to find an absorbing set $\mathcal{A}$ for $\phi(\mathcal{D})$ in G' such that $\Delta(\bigcup \mathcal{A}) \leq \gamma' n/2$.

Let $B_{\deg} := \bigcup \phi(\mathcal{D}) \cup \bigcup \mathcal{A}$. Then, $\Delta(B_{\deg}) < \gamma' n$. For all $1 \leq i_1 < i_2 \leq k$, $\mathcal{D}_{i_1,i_2}$ is a degree balancing set so $\bigcup \phi(\mathcal{D}_{i_1,i_2})$ is locally $\mathcal{P}$-balanced by (d). Since $B_{\deg}[U^i] = \bigcup \phi(\mathcal{D})[U^i]$ for each $1 \leq i \leq k$, the graph $B_{\deg}$ must also be locally $\mathcal{P}$-balanced.

We now check that $B_{\deg}$ is a $(\gamma, \mathcal{P})$-degree balancing graph. Let H be any K_r-divisible graph on V satisfying (Q1)–(Q3). Consider any $1 \leq i_1 < i_2 \leq k$. Note that $H[U^{i_1} \cup U^{i_2}]$ satisfies (Q1)–(Q3). Since $\mathcal{D}_{i_1,i_2}$ is a (γ, γ')-degree balancing set for (U^{i_1}, U^{i_2}), there exist $D' \subseteq D$ for each $D \in \mathcal{D}_{i_1,i_2}$ such that $\phi(D')$ is K_r-divisible and, if $\mathcal{D}'_{i_1,i_2} := \{D' : D \in \mathcal{D}_{i_1,i_2}\}$ and $H'_{i_1,i_2} := H \cup \bigcup \phi(\mathcal{D}'_{i_1,i_2})$, then

$$d_{H'_{i_1,i_2}}(v, U^{i_2}_{j_1}) = d_{H'_{i_1,i_2}}(v, U^{i_2}_{j_2})$$

for all $1 \leq j_1, j_2 \leq r$ and all $v \in U^{i_1} \setminus (V_{j_1} \cup V_{j_2})$. Let $B'_{\deg} := \bigcup_{1 \leq i_1 < i_2 \leq k} \phi(\mathcal{D}'_{i_1,i_2})$ and let $H' := H \cup B'_{\deg}$. Note that $V(\bigcup \phi(\mathcal{D}'_{i_1,i_2})) \subseteq U^{i_1} \cup U^{i_2}$ for all $1 \leq i_1 < i_2 \leq k$. So we have $d_{H'}(v, U^i_{j_1}) = d_{H'}(v, U^i_{j_2})$ for all $2 \leq i \leq k$, all $1 \leq j_1, j_2 \leq r$ and all $v \in U^{<i} \setminus (V_{j_1} \cup V_{j_2})$.

It remains to show that $B_{\deg}$ and $B_{\deg} - B'_{\deg}$ both have K_r-decompositions. Recall that $\mathcal{A}$ is an absorbing set for $\phi(\mathcal{D})$. So, for any K_r-divisible subgraph D^* of any graph in $\phi(\mathcal{D})$, $\mathcal{A}$ contains an absorber for D^*. Also, A is K_r-decomposable for each $A \in \mathcal{A}$. Since $\phi(D)$ is K_r-divisible for each $D \in \mathcal{D}$ by (d), we see that $B_{\deg}$ has a K_r-decomposition. Note that, for each $D \in \mathcal{D}_{i_1,i_2}$, $\phi(D')$ is K_r-divisible by (e) and hence $\phi(D) - \phi(D')$ is also K_r-divisible. So

$$B_{\deg} - B'_{\deg} = \bigcup \mathcal{A} \cup \bigcup_{D \in \mathcal{D}} (\phi(D) - \phi(D'))$$

has a K_r-decomposition. Therefore, $B_{\deg}$ is a $(\gamma, \mathcal{P})$-degree balancing graph. □

10.3. **Finding the balancing graph.** Finally, we combine the edge balancing graph and degree balancing graph from Propositions 10.7 and 10.11 respectively to find a $(\gamma, \mathcal{P})$-balancing graph in G.

Proof of Lemma 10.1. Choose constants γ_1 and γ_2 such that $\gamma \ll \gamma_1 \ll \gamma_2 \ll \gamma'$. First apply Proposition 10.7 to find a $(\gamma, \mathcal{P})$-edge balancing graph $B_{\text{edge}} \subseteq G$ such that B_{edge} is locally $\mathcal{P}$-balanced and $\Delta(B_{\text{edge}}) < \gamma_1 n$. Now $G' := G - B_{\text{edge}}$ satisfies $d_{G'}(v, U^i_j) \geq (1 - 1/(r+1) + \varepsilon/2)|U^i_j|$ for all $v \notin V_j$, so we can apply Proposition 10.11 to find a $(\gamma_2, \mathcal{P})$-degree balancing graph $B_{\deg} \subseteq G'$ such that $B_{\deg}$ is locally $\mathcal{P}$-balanced and $\Delta(B_{\deg}) < \gamma' n/2$. Let $B := B_{\text{edge}} \cup B_{\deg}$. Then $\Delta(B) < \gamma' n$ and B is locally $\mathcal{P}$-balanced. Also, since both B_{edge} and $B_{\deg}$ are K_r-decomposable, B is K_r-decomposable.

We now show that B is a $(\gamma,\mathcal{P})$-balancing graph. Let H be any K_r-divisible graph on V satisfying (P1) and (P2). Since B_{edge} is a $(\gamma,\mathcal{P})$-edge balancing graph, there exists $B'_{\mathrm{edge}}\subseteq B_{\mathrm{edge}}$ such that $B_{\mathrm{edge}}-B'_{\mathrm{edge}}$ has a K_r-decomposition and $H_1:=H\cup B'_{\mathrm{edge}}$ satisfies

$$e_{H_1}(U^{i_1}_{j_1},U^{i_2}_{j_2})=e_{H_1}(U^{i_1}_{j_1},U^{i_2}_{j_3})$$

for all $1\le i_1<i_2\le k$ and all $1\le j_1,j_2,j_3\le r$ with $j_1\neq j_2,j_3$.

Note that H_1 is K_r-divisible. Also

$$|d_{H_1}(v,U^i_{j_2})-d_{H_1}(v,U^i_{j_3})|\le |d_H(v,U^i_{j_2})-d_H(v,U^i_{j_3})|+\Delta(B_{\mathrm{edge}})<\gamma n+\gamma_1 n\le \gamma_2|U^i_{j_1}|$$

for all $2\le i\le k$, all $1\le j_1,j_2,j_3\le r$ with $j_1\neq j_2,j_3$ and all $v\in U^{<i}_{j_1}$. So H_1 satisfies (Q1)–(Q3) with H_1 and γ_2 replacing H and γ. Now, $B_{\deg}$ is a $(\gamma_2,\mathcal{P})$-degree balancing graph so there exists $B'_{\deg}\subseteq B_{\deg}$ such that $B_{\deg}-B'_{\deg}$ has a K_r-decomposition and $H_2:=H_1\cup B'_{\deg}$ satisfies

$$d_{H_2}(v,U^i_{j_1})=d_{H_2}(v,U^i_{j_2})$$

for all $2\le i\le k$, all $1\le j_1,j_2\le r$ and all $v\in U^{<i}\setminus(V_{j_1}\cup V_{j_2})$.

Let $B':=B'_{\mathrm{edge}}\cup B'_{\deg}$. Then $B-B'=(B_{\mathrm{edge}}-B'_{\mathrm{edge}})\cup(B_{\deg}-B'_{\deg})$ has a K_r-decomposition. Note that $H\cup B'=H_2$. So B is a $(\gamma,\mathcal{P})$-balancing graph. □

11. Proof of Theorem 1.1

In this section, we prove our main result, Theorem 1.1. The idea is to take a suitable partition $\mathcal{P}$ of $V(G)$, cover all edges in $G[\mathcal{P}]$ by edge-disjoint copies of K_r and then absorb all remaining edges using an absorber which we set aside at the start of the process. However, for the final step to work, we need that the classes of $\mathcal{P}$ have bounded size. A key step towards this is the following lemma which, for a partition $\mathcal{P}$ into a bounded number of parts, finds an approximate K_r-decomposition which covers all edges of $G[\mathcal{P}]$. We then iterate this lemma inductively to get a similar lemma where the parts have bounded size (see Lemma 11.2).

Lemma 11.1. *Let $1/n\ll\alpha\ll\eta\ll\rho\ll 1/k\ll\varepsilon\ll 1/r\le 1/3$. Let G be a K_r-divisible graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$. Let $\mathcal{P}$ be a k-partition for G. For each $x\in V(G)$, each $U\in\mathcal{P}$ and each $1\le j\le r$, let $0\le d_{x,U_j}\le|U_j|$. Let $G_0\subseteq G-G[\mathcal{P}]$, $G_1:=G-G_0$ and $R\subseteq G[\mathcal{P}]$. Suppose the following hold for all $U,U'\in\mathcal{P}$ and all $1\le j,j_1,j_2\le r$ such that $j\neq j_1,j_2$:*

(a) *for all $x\in U_j$, $|d_G(x,U_{j_1})-d_G(x,U_{j_2})|<\alpha|U_j|$;*
(b) *for all $x\notin V_j$, $d_{G_1}(x,U_j)\ge(\hat\delta^{\eta}_{K_r}+\varepsilon)|U_j|$;*
(c) *for all $x\in V(G)$, $d_R(x,U_j)<\rho d_{x,U_j}+\alpha|U_j|$;*
(d) *for all distinct $x,y\in V(G)$, $d_R(\{x,y\},U_j)<(\rho^2+\alpha)|U_j|$;*
(e) *for all $x\notin U\cup U'\cup V_{j_1}\cup V_{j_2}$, $|d_R(x,U_{j_1})-d_R(x,U'_{j_2})|<3\alpha|U_{j_1}|$;*
(f) *for all $x\notin U$ and all $y\in U$ such that $x,y\notin V_j$,*

$$d_{G_1}(y,N_R(x,U_j))\ge\rho(1-1/(r-1))d_{x,U_j}+\rho^{5/4}|U_j|.$$

Then there is a subgraph $H\subseteq G_1-G[\mathcal{P}]$ such that $G[\mathcal{P}]\cup H$ has a K_r-decomposition and $\Delta(H)\le 4r\rho n$.

To prove Lemma 11.1, we apply Lemma 8.1 to cover almost all the edges of $G[\mathcal{P}]$. We then balance the leftover using Lemma 10.1. The remaining edges in $G[\mathcal{P}]$ can then be covered using Corollary 9.4. The graph R in Lemma 11.1 forms the main part of the graph G in Corollary 9.4. Conditions (c)–(f) ensure that R is 'quasirandom'.

Proof. Write $\mathcal{P}=\{U^1,\dots,U^k\}$. Let $G_2:=G_1-R=G-G_0-R$. Note that Proposition 3.1 together with (b) and (c) implies that for any $1\le i\le k$, any $1\le j\le r$ and any $x\notin V_j$,

$$d_{G_2}(x,U^i_j)\ge(\hat\delta^{\eta}_{K_r}+\varepsilon-2\rho)|U^i_j|\ge(1-1/(r+1)+\varepsilon/2)|U^i_j|.$$

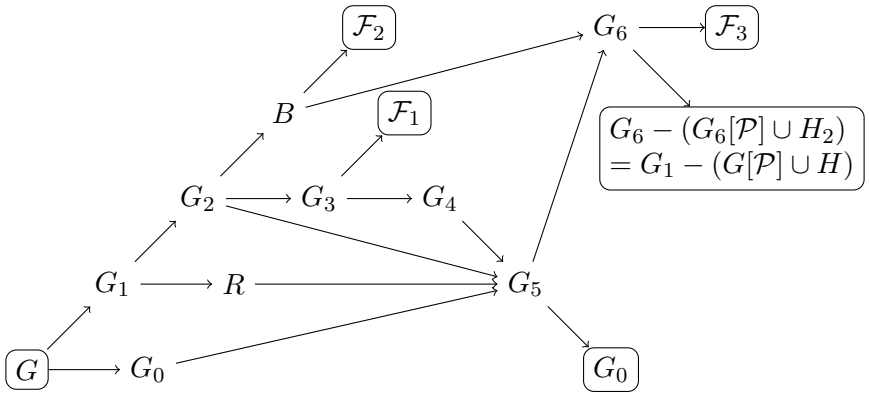

FIGURE 3. Outline for Proof of Lemma 11.1.

Choose constants γ_1, γ_2 such that $\eta \ll \gamma_1 \ll \gamma_2 \ll \rho$. Apply Lemma 10.1 (with γ_1, γ_2, $\varepsilon/2$, k, G_2, $\mathcal{P}$ playing the roles of γ, γ', ε, k, G, $\mathcal{P}$) to find a $(\gamma_1, \mathcal{P})$-balancing graph $B \subseteq G_2$ such that

$$\Delta(B) < \gamma_2 n \tag{11.1}$$

and B is locally $\mathcal{P}$-balanced. As B is also K_r-decomposable, for all $1 \leq j_1, j_2 \leq r$ and all $x \notin V_{j_1} \cup V_{j_2}$,

$$d_{B[\mathcal{P}]}(x, V_{j_1}) = d_{B[\mathcal{P}]}(x, V_{j_2}). \tag{11.2}$$

Let $G_3 := G_2[\mathcal{P}] - B = G[\mathcal{P}] - R - B$. Then (b), (c) and (11.1) give

$$\hat{\delta}(G_3) \geq (\hat{\delta}^{\eta}_{K_r} + \varepsilon)n - \lceil n/k \rceil - 2\rho n - \gamma_2 n \geq (\hat{\delta}^{\eta}_{K_r} + \varepsilon/2)n.$$

Consider any $1 \leq j_1, j_2 \leq r$ and any $x \notin V_{j_1} \cup V_{j_2}$. Using (a), (e) and (11.2), we have

$$\begin{aligned} |d_{G_3}(x, V_{j_1}) - d_{G_3}(x, V_{j_2})| &\leq |d_{G[\mathcal{P}]}(x, V_{j_1}) - d_{G[\mathcal{P}]}(x, V_{j_2})| + |d_R(x, V_{j_1}) - d_R(x, V_{j_2})| \\ &< \alpha n + 3\alpha n = 4\alpha n. \end{aligned}$$

So we can apply Lemma 8.1 (with 4α, η, $\gamma_1/2$, $\varepsilon/2$, G_3 playing the roles of α, η, γ, ε, G) to find $G_4 \subseteq G_3$ such that $G_3 - G_4$ has a K_r-decomposition $\mathcal{F}_1$ and

$$\Delta(G_4) \leq \gamma_1 n/2. \tag{11.3}$$

The graphs G, $G_3 - G_4$ and B are all K_r-divisible (and $G_3 - G_4$ and B are edge-disjoint), so

$$G_5 := G - (G_3 - G_4) - B = (G - G[\mathcal{P}] - B) \cup G_4 \cup R$$

must also be K_r-divisible. Note that $e(G_5 \cap B) = 0$ and $G_5[\mathcal{P}] = G_4 \cup R$. Consider any $1 \leq i \leq k$, any $1 \leq j_1, j_2 \leq r$ and any $x \notin V_{j_1} \cup V_{j_2}$. If $x \notin U^i$, (11.3) and (e) give

$$\begin{aligned} |d_{G_5}(x, U^i_{j_1}) - d_{G_5}(x, U^i_{j_2})| &= |d_{G_4 \cup R}(x, U^i_{j_1}) - d_{G_4 \cup R}(x, U^i_{j_2})| \\ &\leq \Delta(G_4) + |d_R(x, U^i_{j_1}) - d_R(x, U^i_{j_2})| < (\gamma_1/2 + 3\alpha)n < \gamma_1 n. \end{aligned}$$

If $x \in U^i$, then we use (a), that B is locally $\mathcal{P}$-balanced and that $G_4, R \subseteq G[\mathcal{P}]$ to see that

$$\begin{aligned} |d_{G_5}(x, U^i_{j_1}) - d_{G_5}(x, U^i_{j_2})| &\leq |d_G(x, U^i_{j_1}) - d_G(x, U^i_{j_2})| + |d_B(x, U^i_{j_1}) - d_B(x, U^i_{j_2})| \\ &< \alpha n \leq \gamma_1 n. \end{aligned}$$

So (P1) and (P2) in Section 10 hold with G_5 and γ_1 replacing H and γ. Since B is a $(\gamma_1, \mathcal{P})$-balancing graph, there exists $B' \subseteq B$ such that $B - B'$ has a K_r-decomposition $\mathcal{F}_2$ and, for all $2 \leq i \leq k$, all $1 \leq j_1, j_2 \leq r$ and all $x \in U^{<i} \setminus (V_{j_1} \cup V_{j_2})$,

$$d_{G_5 \cup B'}(x, U^i_{j_1}) = d_{G_5 \cup B'}(x, U^i_{j_2}). \tag{11.4}$$

Write $H_1 := \bigcup_{i=1}^{k}(B - B')[U^i]$ and let

$$G_6 := G_5 \cup B' - G_0 = (G - G[\mathcal{P}] - G_0 - B) \cup R \cup G_4 \cup B'.$$

Note that

$$G_6[\mathcal{P}] = R \cup G_4 \cup B'[\mathcal{P}] = G_5[\mathcal{P}] \cup B'[\mathcal{P}]. \tag{11.5}$$

We now check conditions (i)–(iv) of Corollary 9.4 (with G_6 playing the role of G). Since $G_0 \subseteq G - G[\mathcal{P}]$, (i) follows immediately from (11.4). For (ii), suppose that $2 \le i \le k$ and $x \in U^{<i}$. For any $1 \le j \le r$, using (c), (11.3) and (11.1), we have

$$\begin{aligned} d_{G_6}(x, U^i_j) &\overset{(11.5)}{\le} d_R(x, U^i_j) + \Delta(G_4) + \Delta(B) < \rho d_{x,U^i_j} + \alpha|U^i_j| + \gamma_1 n/2 + \gamma_2 n \\ &\le \rho d_{x,U^i_j} + 2\gamma_2 n. \end{aligned} \tag{11.6}$$

Consider any $y \in N_{G_6}(x, U^i)$. Note that $G_6[U^i] = G_1[U^i] - (B - B')[U^i]$. So, for any $1 \le j \le r$ such that $x, y \notin V_j$, we have

$$\begin{aligned} d_{G_6}(y, N_{G_6}(x, U^i_j)) &\ge d_{G_6}(y, N_R(x, U^i_j)) \ge d_{G_1}(y, N_R(x, U^i_j)) - \Delta(B) \\ &\overset{\text{(f)},(11.1)}{\ge} (1 - 1/(r-1))\rho d_{x,U^i_j} + \rho^{5/4}|U^i_j| - \gamma_2 n \\ &\overset{(11.6)}{\ge} (1 - 1/(r-1)) d_{G_6}(x, U^i_j) + \rho^{5/4}|U^i_j| - 3\gamma_2 n \\ &> (1 - 1/(r-1)) d_{G_6}(x, U^i_j) + 9kr\rho^{3/2}|U^i|. \end{aligned}$$

So (ii) holds.

To see that G_6 satisfies property (iii) of Corollary 9.4, note that for all $2 \le i \le k$ and all distinct $x, x' \in U^{<i}$, (d), (11.1), (11.3) and (11.5) imply that

$$\begin{aligned} |N_{G_6}(x, U^i) \cap N_{G_6}(x', U^i)| &\le d_R(\{x, x'\}, U^i) + \Delta(G_4) + \Delta(B) \\ &< (\rho^2 + \alpha)|U^i| + \gamma_1 n/2 + \gamma_2 n \le 2\rho^2|U^i|. \end{aligned}$$

Finally, by (c), (11.1), (11.3) and (11.5), for any $y \in U^i$, we have that

$$d_{G_6}(y, U^{<i}) \le \Delta(R) + \Delta(G_4) + \Delta(B) \le 3\rho n/2 \le 2k\rho|U^i_1|,$$

and (iv) holds. Hence we can apply Corollary 9.4 to G_6 to find a subgraph $H_2 \subseteq G_6 - G_6[\mathcal{P}]$ such that $G_6[\mathcal{P}] \cup H_2$ has a K_r-decomposition $\mathcal{F}_3$ and $\Delta(H_2) \le 3r\rho n$. Set $H := H_1 \cup H_2 \subseteq G_1 - G[\mathcal{P}]$. We have $\Delta(H) \le \Delta(H_1) + \Delta(H_2) \le \Delta(B) + \Delta(H_2) \le 4r\rho n$. Now,

$$\begin{aligned} G[\mathcal{P}] \cup H &= G_2[\mathcal{P}] \cup R \cup H = G_3 \cup R \cup H \cup B[\mathcal{P}] \\ &= \bigcup \mathcal{F}_1 \cup G_4 \cup R \cup H \cup B[\mathcal{P}] = \bigcup \mathcal{F}_1 \cup G_5[\mathcal{P}] \cup H_1 \cup H_2 \cup B[\mathcal{P}] \\ &= \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) \cup G_5[\mathcal{P}] \cup H_2 \cup B'[\mathcal{P}] \overset{(11.5)}{=} \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) \cup G_6[\mathcal{P}] \cup H_2 \\ &= \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3). \end{aligned}$$

So $G[\mathcal{P}] \cup H$ has a K_r-decomposition $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. □

We now iterate Lemma 11.1, applying it to each partition $\mathcal{P}_i$ in a partition sequence $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ for G. This allows us to cover all of the edges in $G[\mathcal{P}_\ell]$ by edge-disjoint copies of K_r, leaving only a small remainder in $\bigcup_{U\in\mathcal{P}_\ell} G[U]$.

Lemma 11.2. *Let $1/m \ll \alpha \ll \eta \ll \rho \ll 1/k \ll \varepsilon \ll 1/r \le 1/3$. Let G be a K_r-divisible graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n$. Let $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ be a $(1,k,\hat{\delta}^\eta_{K_r}+\varepsilon/2,m)$-partition sequence for G. For each $1\le q\le \ell$, each $1\le j\le r$, each $U\in\mathcal{P}_q$ and each $x\in V(G)$, let $0\le d_{x,U_j}\le |U_j|$ be given. Let $\mathcal{P}_0:=\{V(G)\}$ and, for each $0\le q\le \ell$, let $G_q:=G[\mathcal{P}_q]$. Let $R_1,\dots,R_\ell$ be a sequence of graphs such that $R_q\subseteq G_q-G_{q-1}$ for each q. Suppose the following hold for all $1\le q\le\ell$, all $1\le j,j_1,j_2\le r$ such that $j\ne j_1,j_2$, all $W\in\mathcal{P}_{q-1}$, all distinct $x,y\in W$ and all $U,U'\in\mathcal{P}_q[W]$:*

(i) *if $q\ge 2$, $\mathcal{P}_q[W]$ is a $(1,k,\hat{\delta}^\eta_{K_r}+\varepsilon)$-partition for $G[W]$;*
(ii) *if $x\in U_j$, $|d_G(x,U_{j_1})-d_G(x,U_{j_2})|<\alpha|U_j|$;*
(iii) *$d_{R_q}(x,U_j)<\rho d_{x,U_j}+\alpha|U_j|$;*
(iv) *$d_{R_q}(\{x,y\},U_j)<(\rho^2+\alpha)|U_j|$;*
(v) *if $x\notin U\cup U'\cup V_{j_1}\cup V_{j_2}$, $|d_{R_q}(x,U_{j_1})-d_{R_q}(x,U'_{j_2})|<3\alpha|U_{j_1}|$;*
(vi) *if $x\notin U$, $y\in U$ and $x,y\notin V_j$, then*

$$d_{G'_{q+1}}(y,N_{R_q}(x,U_j))\ge \rho(1-1/(r-1))d_{x,U_j}+\rho^{5/4}|U_j|$$

where $G'_{q+1}:=G_{q+1}-R_{q+1}$ if $q\le\ell-1$ and $G'_{\ell+1}:=G$.

Then there is a subgraph $H\subseteq\bigcup_{U\in\mathcal{P}_\ell}G[U]$ such that $G-H$ has a K_r-decomposition.

Proof. We will use induction on ℓ. If $\ell=1$, apply Lemma 11.1 (with $\varepsilon/2$, $\mathcal{P}_1$, R_1 and the empty graph playing the roles of ε, $\mathcal{P}$, R and G_0) to find $H'\subseteq G-G[\mathcal{P}_1]$ such that $G[\mathcal{P}_1]\cup H'$ has a K_r-decomposition. Letting $H:=G-G[\mathcal{P}_1]-H'\subseteq\bigcup_{U\in\mathcal{P}_\ell}G[U]$, shows the result holds for $\ell=1$.

Suppose then that $\ell\ge 2$ and the result holds for all smaller ℓ. Note that for each $1\le j\le r$, each $x\notin V_j$ and each $U\in\mathcal{P}_1$, $d_{G[\mathcal{P}_2]-R_2}(x,U_j)\ge(\hat{\delta}^\eta_{K_r}+\varepsilon/3)|U_j|$, since R_2 satisfies (iii) and $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ is a $(1,k,\hat{\delta}^\eta_{K_r}+\varepsilon/2,m)$-partition sequence for G. So we may apply Lemma 11.1 (with $\varepsilon/3$, $\mathcal{P}_1$, R_1, G and $(G-G[\mathcal{P}_2])\cup R_2$ playing the roles of ε, $\mathcal{P}$, R, G and G_0) to find $H'\subseteq G[\mathcal{P}_2]-(G[\mathcal{P}_1]\cup R_2)$ such that $G[\mathcal{P}_1]\cup H'$ has a K_r-decomposition $\mathcal{F}_1$ and $\Delta(H')\le 4r\rho n$. Let $G^*:=G-G[\mathcal{P}_1]-H'=G-\bigcup\mathcal{F}_1$, so G^* is K_r-divisible. Observe that $G^*=\bigcup_{U\in\mathcal{P}_1}G^*[U]$, so $G^*[U]$ is K_r-divisible for each $U\in\mathcal{P}_1$.

Consider any $U\in\mathcal{P}_1$. We check that

$$G^*[U],\mathcal{P}_2[U],\dots,\mathcal{P}_\ell[U],R_2[U],\dots,R_\ell[U]$$

satisfy the conditions of Lemma 11.2. Since $\Delta(H')\le 4r\rho n\le\varepsilon n/4k^2$, $\mathcal{P}_2[U]$ is a $(1,k,\hat{\delta}^\eta_{K_r}+\varepsilon/2)$-partition for $G^*[U]$. For any $3\le q\le\ell$ and any $W\in\mathcal{P}_{q-1}$, $G^*[W]=G[W]$ since $H'\subseteq G[\mathcal{P}_2]$. So (i) holds and $\mathcal{P}_2[U],\dots,\mathcal{P}_\ell[U]$ is a $(1,k,\hat{\delta}^\eta_{K_r}+\varepsilon/2,m)$-partition sequence for $G^*[U]$. For (ii), note that for any $2\le q\le\ell$, any $1\le j\le r$, any $U'\in\mathcal{P}_q[U]$ and any $x\in U'$, $d_{G^*}(x,U'_j)=d_G(x,U'_j)$. Conditions (iii)–(v) are automatically satisfied. To see that (vi) holds, note that for any $2\le q\le\ell$ and any $U'\in\mathcal{P}_q[U]$, $G^*_{q+1}[U']=G_{q+1}[U']$ since $H'\subseteq G[\mathcal{P}_2]$. So we can apply the induction hypothesis to $G^*[U],\mathcal{P}_2[U],\dots,\mathcal{P}_\ell[U],R_2[U],\dots,R_\ell[U]$ to obtain a subgraph $H_U\subseteq\bigcup_{U'\in\mathcal{P}_\ell[U]}G^*[U']$ such that $G^*[U]-H_U$ has a K_r-decomposition $\mathcal{F}_U$. Set $H:=\bigcup_{U\in\mathcal{P}_1}H_U$. Then, $H\subseteq\bigcup_{U\in\mathcal{P}_\ell}G[U]$ and $G-H$ has a K_r-decomposition $\mathcal{F}_1\cup\bigcup_{U\in\mathcal{P}_1}\mathcal{F}_U$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $n_0\in\mathbb{N}$ and $\eta>0$ be such that $1/n_0\ll\eta\ll\varepsilon$ and choose additional constants η_1, m', α, ρ and k such that

$$1/n_0\ll\eta_1\ll 1/m'\ll\alpha\ll\eta\ll\rho\ll 1/k\ll\varepsilon.$$

Let G be any K_r-divisible graph on $(V_1,\dots,V_r)$ with $|V_1|=\dots=|V_r|=n\geq n_0$ and $\hat{\delta}(G)\geq(\hat{\delta}^{\eta}_{K_r}+\varepsilon)n$. Apply Lemma 7.2 to find an $(\alpha,k,\hat{\delta}^{\eta}_{K_r}+\varepsilon-\alpha,m)$-partition sequence $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ for G where $m'\leq m\leq km'$. So in particular, by (S3), for each $1\leq q\leq\ell$, all $1\leq j_1,j_2,j_3\leq r$ with $j_1\neq j_2,j_3$, each $U\in\mathcal{P}_q$ and each $x\in U_{j_1}$,

$$|d_G(x,U_{j_2})-d_G(x,U_{j_3})|<\alpha|U_{j_1}|. \tag{11.7}$$

Let $\mathcal{P}_0:=\{V(G)\}$ and $G_q:=G[\mathcal{P}_q]$ for $0\leq q\leq\ell$. Note that $\hat{\delta}^{\eta}_{K_r}+\varepsilon-\alpha\geq 1-1/r+\varepsilon$ (with room to spare) by Proposition 3.1. So we can apply Corollary 7.5 to find a sequence of graphs $R_1,\dots,R_\ell$ such that $R_q\subseteq G_q-G_{q-1}$ for each $1\leq q\leq\ell$ and the following holds. For all $1\leq q\leq\ell$, all $1\leq j,j'\leq r$, all $W\in\mathcal{P}_{q-1}$, all distinct $x,y\in W$ and all $U,U'\in\mathcal{P}_q[W]$,

$$\begin{aligned}
d_{R_q}(x,U_j)&<\rho d_{G_q}(x,U_j)+\alpha|U_j|;\\
d_{R_q}(\{x,y\},U_j)&<(\rho^2+\alpha)|U_j|;\\
|d_{R_q}(x,U_j)-d_{R_q}(x,U'_{j'})|&<3\alpha|U_j| \quad \text{if } x\notin U\cup U'\cup V_j\cup V_{j'};\\
d_{G'_{q+1}}(y,N_{R_q}(x,U_j))&\geq\rho(1-1/(r-1))d_{G_q}(x,U_j)+\rho^{5/4}|U_j| \quad \text{if } x\notin U, y\in U \text{ and } x,y\notin V_j,
\end{aligned} \tag{11.8}$$

where $G'_{q+1}:=G_{q+1}-R_{q+1}$ if $q\leq\ell-1$ and $G'_{\ell+1}:=G$.

Let $\mathcal{H}:=\{G[U]:U\in\mathcal{P}_\ell\}$. Each $H\in\mathcal{H}$ satisfies $|H|\leq rm$. Note that

$$\hat{\delta}(G[\mathcal{P}_1]-R_1)\geq(\hat{\delta}^{\eta}_{K_r}+\varepsilon)n-\lceil n/k\rceil-2\rho n>(1-1/(r+1)+\varepsilon/2)n.$$

So we can apply Lemma 6.6 (with η_1, α, rm and $G[\mathcal{P}_1]-R_1$ playing the roles of η, ε, b and G) to find an absorbing set $\mathcal{A}$ for $\mathcal{H}$ inside $G[\mathcal{P}_1]-R_1$ such that $A^*:=\bigcup\mathcal{A}$ satisfies $\Delta(A^*)\leq\alpha n$.

Let $G^*:=G-A^*$. Note that both G and A^* are K_r-divisible, so G^* is K_r-divisible. Since $\Delta(A^*)\leq\alpha n$ and $A^*\subseteq G[\mathcal{P}_1]$, $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ is an $(1,k,\hat{\delta}^{\eta}_{K_r}+\varepsilon/2,m)$-partition sequence for G^*. For each $1\leq q\leq\ell$, each $1\leq j\leq r$, each $U\in\mathcal{P}_q$ and each $x\in V(G)$, set $d_{x,U_j}:=d_{G_q}(x,U_j)$. Using (11.7), (11.8) and that $A^*\subseteq G[\mathcal{P}_1]$, we see that G^*, the partition sequence $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ and the sequence of graphs $R_1,\dots,R_\ell$ satisfy properties (i)–(vi) of Lemma 11.2 (with $\varepsilon-\alpha$ playing the role of ε). So we may apply Lemma 11.2 to find $H\subseteq\bigcup_{U\in\mathcal{P}_\ell}G^*[U]$ such that G^*-H has a K_r-decomposition $\mathcal{F}_1$.

Note that H is a K_r-divisible subgraph of $\bigcup_{U\in\mathcal{P}_\ell}G[U]$, so for each $U\in\mathcal{P}_\ell$, $H[U]\subseteq G[U]$ is K_r-divisible. Since $\mathcal{A}$ is an absorbing set for $\mathcal{H}$, it contains a distinct absorber for each $H[U]$. So $H\cup A^*$ has a K_r-decomposition $\mathcal{F}_2$. Thus $G=(G^*-H)\cup(H\cup A^*)$ has a K_r-decomposition $\mathcal{F}_1\cup\mathcal{F}_2$. □

Acknowledgements

We are grateful to Peter Dukes and Richard Montgomery for helpful discussions.

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