HAMILTON DECOMPOSITIONS OF REGULAR TOURNAMENTS

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Abstract. We show that every sufficiently large regular tournament can almost completely be decomposed into edge-disjoint Hamilton cycles. More precisely, for each $\eta > 0$ every regular tournament $G$ of sufficiently large order $n$ contains at least $(1/2 - \eta)n$ edge-disjoint Hamilton cycles. This gives an approximate solution to a conjecture of Kelly from 1968. Our result also extends to almost regular tournaments.

1. Introduction

A Hamilton decomposition of a graph or digraph $G$ is a set of edge-disjoint Hamilton cycles which together cover all the edges of $G$. The topic has a long history but some of the main questions remain open. In 1892, Walecki showed that the edge set of the complete graph $K_n$ on $n$ vertices has a Hamilton decomposition if $n$ is odd (see e.g. [2, 23] for the construction). If $n$ is even, then $n$ is not a factor of $(n\choose 2)$, so clearly $K_n$ does not have such a decomposition. Walecki’s result implies that a complete digraph $G$ on $n$ vertices has a Hamilton decomposition if $n$ is odd. More generally, Tillson [29] proved that a complete digraph $G$ on $n$ vertices has a Hamilton decomposition if and only if $n \neq 4, 6$.

A tournament is an orientation of a complete graph. We say that a tournament is regular if every vertex has equal in- and outdegree. Thus regular tournaments contain an odd number $n$ of vertices and each vertex has in- and outdegree $(n-1)/2$. The following beautiful conjecture of Kelly (see e.g. [4, 7, 24]), which has attracted much attention, states that every regular tournament has a Hamilton decomposition:

Conjecture 1 (Kelly). Every regular tournament on $n$ vertices can be decomposed into $(n - 1)/2$ edge-disjoint Hamilton cycles.

In this paper we prove an approximate version of Kelly’s conjecture.

Theorem 2. For every $\eta > 0$ there exists an integer $n_0$ so that every regular tournament on $n \geq n_0$ vertices contains at least $(1/2 - \eta)n$ edge-disjoint Hamilton cycles.

In fact, we prove the following stronger result, where we consider orientations of almost complete graphs which are almost regular. An oriented graph is obtained from an undirected graph by orienting its edges. So it has at most one edge between every pair of vertices, whereas a digraph may have an edge in each direction.
Theorem 3. For every \( \eta_1 > 0 \) there exist \( n_0 = n_0(\eta_1) \) and \( \eta_2 = \eta_2(\eta_1) > 0 \) such that the following holds. Suppose that \( G \) is an oriented graph on \( n \geq n_0 \) vertices such that every vertex in \( G \) has in- and outdegree at least \( (1/2 - \eta_2)n \). Then \( G \) contains at least \( (1/2 - \eta_1)n \) edge-disjoint Hamilton cycles.

The minimum semidegree \( \delta^0(G) \) of an oriented graph \( G \) is the minimum of its minimum outdegree and its minimum indegree. So the minimum semidegree of a regular tournament on \( n \) vertices is \( (n - 1)/2 \). Most of the previous partial results towards Kelly’s conjecture have been obtained by giving bounds on the minimum semidegree of an oriented graph which guarantees a Hamilton cycle. This approach was first used by Jackson [16], who showed that every regular tournament on at least 5 vertices contains a Hamilton cycle and a Hamilton path which are edge-disjoint. Zhang [30] then showed that every such tournament contains two edge-disjoint Hamilton cycles. Improved bounds on the value of \( \delta^0(G) \) which forces a Hamilton cycle were then found by Thomassen [27], Häggkvist [13], Häggkvist and Thomason [14] as well as Kelly, Kühn and Osthus [19]. Finally, Keevash, Kühn and Osthus [18] showed that every sufficiently large oriented graph \( G \) on \( n \) vertices with \( \delta^0(G) \geq (3n - 4)/8 \) contains a Hamilton cycle. This bound on \( \delta^0(G) \) is best possible and confirmed a conjecture of Häggkvist [13]. Note that this result implies that every sufficiently large regular tournament on \( n \) vertices contains at least \( n/8 \) edge-disjoint Hamilton cycles. This was the best bound so far towards Kelly’s conjecture.

Kelly’s conjecture has also been verified for \( n \leq 9 \) by Alspach (see the survey [6]). A result of Frieze and Krivelevich [12] states that Theorem 3 holds for ‘quasi-random’ tournaments. As indicated below, we will build on some of their ideas in the proof of Theorem 3.

Jackson [16] also introduced the following bipartite version of Kelly’s conjecture (both versions are also discussed e.g. in the Handbook article by Bondy [7]). A bipartite tournament is an orientation of a complete bipartite graph.

Conjecture 4 (Jackson). Every regular bipartite tournament has a Hamilton decomposition.

An undirected version of Conjecture 4 was proved independently by Auerbach and Laskar [3], as well as Heteyei [15]. However, a bipartite version of Theorem 3 does not hold, because there are almost regular bipartite tournaments which do not even contain a single Hamilton cycle. (Consider for instance the following ‘blow-up’ of a 4-cycle: the vertices are split into 4 parts \( A_0, \ldots, A_3 \) whose sizes are almost but not exactly equal, and we have all edges from \( A_i \) to \( A_{i+1} \), with indices modulo 4.)

Kelly’s conjecture has been generalized in several directions. For instance, given an oriented graph \( G \), define its excess by

\[
\text{ex}(G) := \sum_{v \in V(G)} \max\{d^+(v) - d^-(v), 0\},
\]

where \( d^+(v) \) denotes the number of outneighbours of the vertex \( v \), and \( d^-(v) \) the number of its innighbours. Pullman (see e.g. Conjecture 8.25 in [7]) conjectured that if \( G \) is an oriented graph such that \( d^+(v) + d^-(v) = d \) for all vertices \( v \) of \( G \), where \( d \) is odd, then \( G \) has a decomposition into \( \text{ex}(G) \) directed paths. To see
that this would imply Kelly’s conjecture, let \( G \) be the oriented graph obtained from a regular tournament by deleting a vertex. Another generalization was made by Bang-Jensen and Yeo [5], who conjectured that every \( k \)-edge-connected tournament has a decomposition into \( k \) spanning strong digraphs.

In [27], Thomassen also formulated the following weakening of Kelly’s conjecture.

**Conjecture 5 (Thomassen).** If \( G \) is a regular tournament on \( 2k + 1 \) vertices and \( A \) is any set of at most \( k - 1 \) edges of \( G \), then \( G - A \) has a Hamilton cycle.

In [22], we proved a result on the existence of Hamilton cycles in ‘robust expander digraphs’ which implies Conjecture 5 for large tournaments (see [22] for details). [27] also contains the related conjecture that for any \( \ell \geq 2 \), there is an \( f(\ell) \) so that every strongly \( f(\ell) \)-connected tournament contains \( \ell \) edge-disjoint Hamilton cycles.

It seems likely that the techniques developed in this paper will also be useful in solving further problems. In fact, Christofides, Kühn and Osthus [9] used similar ideas to prove approximate versions of the following two long-standing conjectures of Nash-Williams [25, 26]:

**Conjecture 6 (Nash-Williams [25]).** Let \( G \) be a \( 2d \)-regular graph on at most \( 4d + 1 \) vertices, where \( d \geq 1 \). Then \( G \) has a Hamilton decomposition.

**Conjecture 7 (Nash-Williams [26]).** Let \( G \) be a graph on \( n \) vertices with minimum degree at least \( n/2 \). Then \( G \) contains \( n/8 + o(n) \) edge-disjoint Hamilton cycles.

(Actually, Nash-Williams initially formulated Conjecture 7 with the term \( n/8 \) replaced by \( n/4 \), but Babai found a counterexample to this.)

Another related problem was raised by Erdős (see [27]), who asked whether almost all tournaments \( G \) have at least \( \delta^{\text{th}}(G) \) edge-disjoint Hamilton cycles. Note that an affirmative answer would not directly imply that Kelly’s conjecture holds for almost all regular tournaments, which would of course be an interesting result in itself. There are also a number of corresponding questions for random undirected graphs (see e.g. [12]).

After giving an outline of the argument in the next section, we will state a directed version of the Regularity lemma and some related results in Section 3. Section 4 contains statements and proofs of several auxiliary results, mostly on (almost) 1-factors in (almost) regular oriented graphs. The proof of Theorem 3 is given in Section 5.

2. Sketch of the proof of Theorem 3

Suppose we are given a regular tournament \( G \) on \( n \) vertices and our aim is to ‘almost’ decompose it into Hamilton cycles. One possible approach might be the following: first remove a spanning regular oriented subgraph \( H \) whose degree \( \gamma n \) satisfies \( \gamma \ll 1 \). Let \( G' \) be the remaining oriented subgraph of \( G \). Now consider a decomposition of \( G' \) into 1-factors \( F_1, \ldots, F_r \) (which clearly exists). Next, try to transform each \( F_i \) into a Hamilton cycle by removing some of its edges and adding some suitable edges of \( H \). This is of course impossible if many of the \( F_i \) consist of many cycles. However, an auxiliary result of Frieze and Krivelevich in [12] implies
that we can ‘almost’ decompose $G'$ so that each 1-factor $F_i$ consists of only a few cycles.

If $H$ were a ‘quasi-random’ oriented graph, then (as in [12]) one could use it to successively ‘merge’ the cycles of each $F_i$ into Hamilton cycles using a ‘rotation-extension’ argument: delete an edge of a cycle $C$ of $F_i$ to obtain a path $P$ from $a$ to $b$, say. If there is an edge of $H$ from $b$ to another cycle $C'$ of $F_i$, then extend $P$ to include the vertices of $C'$ (and similarly for $a$). Continue until there is no such edge. Then (in $H$) the current endvertices of the path $P$ have many neighbours on $P$. One can use this together with the quasi-randomness of $H$ to transform $P$ into a cycle with the same vertices as $P$. Now repeat this, until we have merged all the cycles into a single (Hamilton) cycle. Of course, one has to be careful to maintain the quasi-randomness of $H$ in carrying out this ‘rotation-extension’ process for the successive $F_i$ (the fact that $F_i$ contains only few cycles is important for this).

The main problem is that $G$ need not contain such a spanning ‘quasi-random’ subgraph $H$. So instead, in Section 5.1 we use Szemerédi’s regularity lemma to decompose $G$ into quasi-random subgraphs. We then choose both our 1-factors $F_i$ and the graph $H$ according to the structure of this decomposition. More precisely, we apply a directed version of Szemerédi’s regularity lemma to obtain a partition of the vertices of $G$ into a bounded number of clusters $V_i$ so that almost all of the bipartite subgraphs spanned by ordered pairs of clusters are quasi-random (see Section 3.3 for the precise statement). This then yields a reduced digraph $R$, whose vertices correspond to the clusters, with an edge from one cluster $U$ to another cluster $W$ if the edges from $U$ to $W$ in $G$ form a quasi-random graph. (Note that $R$ need not be oriented.) We view $R$ as a weighted digraph whose edge weights are the densities of the corresponding ordered pair of clusters. We then obtain an unweighted multidigraph $R_m$ from $R$ as follows: given an edge $e$ of $R$ joining a cluster $U$ to $W$, replace it with $K = K(e)$ copies of $e$, where $K$ is approximately proportional to the density of the ordered pair $(U, W)$. It is not hard to show that $R_m$ is approximately regular (see Lemma 11). If $R_m$ were regular, then it would have a decomposition into 1-factors, but this assumption may not be true. However, we can show that $R_m$ can ‘almost’ be decomposed into ‘almost’ 1-factors. In other words, there exist edge-disjoint collections $\mathcal{F}_1, \ldots, \mathcal{F}_r$ of vertex-disjoint cycles in $R_m$ such that each $\mathcal{F}_i$ covers almost all of the clusters in $R_m$ (see Lemma 15).

Now we choose edge-disjoint oriented spanning subgraphs $C_1, \ldots, C_r$ of $G$ so that each $C_i$ corresponds to $\mathcal{F}_i$. For this, consider an edge $e$ of $R$ from $U$ to $W$ and suppose for example that $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are the only $\mathcal{F}_i$ containing copies of $e$ in $R_m$. Then for each edge of $G$ from $U$ to $W$ in turn, we assign it to one of $C_1, C_2$ and $C_3$ with equal probability. Then with high probability, each $C_i$ consists of bipartite quasi-random oriented graphs which together form a disjoint union of ‘blown-up’ cycles. Moreover, we can arrange that all the vertices have degree close to $\beta m$ (here $m$ is the cluster size and $\beta$ a small parameter which does not depend on $i$). We now remove a small proportion of the edges from $G$ (and thus from each $C_i$) to form oriented subgraphs $H_1^+, H_1^-, H_2, H_3, i, H_4, H_5, i$ of $G$, where $1 \leq i \leq r$. Ideally, we would like to show that each $C_i$ can almost be decomposed into Hamilton cycles. Since the $C_i$ are edge-disjoint, this would yield the required result.
One obvious obstacle is that the $C_i$ need not be spanning subgraphs of $G$ (because of the exceptional set $V_0$ returned by the regularity lemma and because the $F_i$ are not spanning.) So in Section 5.2 we add suitable edges between $C_i$ and the leftover vertices to form edge-disjoint oriented spanning subgraphs $G_i$ of $G$ where every vertex has degree close to $\beta m$. (The edges of $H^-_1$ and $H^+_1$ are used in this step.) But the distribution of the edges added in this step may be somewhat ‘unbalanced’, with some vertices of $C_i$ sending out or receiving too many of them. In fact, as discussed at the beginning of Section 5.4, we cannot even guarantee that $G_i$ has a single 1-factor. We overcome this new difficulty by adding carefully chosen further edges (from $H_2$ this time) to each $G_i$ which compensate the above imbalances.

Once these edges have been added, in Section 5.5 we can use the max-flow min-cut theorem to almost decompose each $G_i$ into 1-factors $F_{i,j}$. (This is one of the points where we use the fact that the $C_i$ consist of quasi-random graphs which form a union of blown-up cycles.) Moreover, (i) the number of cycles in each of these 1-factors is not too large and (ii) most of the cycles inherit the structure of $F_i$. More precisely, (ii) means that most vertices $u$ of $C_i$ have the following property: let $U$ be the cluster containing $u$, and let $U^+$ be the successor of $U$ in $F_i$. Then the successor $u^+\ of\ u$ in $F_{i,j}$ lies in $U^+$.

In Section 5.6 we can use (i) and (ii) to merge the cycles of each $F_{i,j}$ into a 1-factor $F'_{i,j}$ consisting only of a bounded number of cycles – for each cycle $C$ of $F_i$, all the vertices of $G_i$ which lie in clusters of $C$ will lie in the same cycle of $F'_{i,j}$. We will apply a rotation-extension argument for this, where the additional edges (i.e. those not in $F_{i,j}$) come from $H_{3,i}$. Finally, in Section 5.7 we will use the fact that $R_m$ contains many short paths to merge each $F'_{i,j}$ into a single Hamilton cycle. The additional edges will come from $H_4$ and $H_{5,i}$ this time.

3. Notation and the Diregularity Lemma

3.1. Notation. Throughout this paper we omit floors and ceilings whenever this does not affect the argument. Given a graph $G$, we denote the degree of a vertex $x \in V(G)$ by $d_G(x)$ and the maximum degree of $G$ by $\Delta(G)$. Given two vertices $x$ and $y$ of a digraph $G$, we write $xy$ for the edge directed from $x$ to $y$. We denote by $N_G^+(x)$ the set of all outneighbours of $x$. So $N_G^+(x)$ consists of all those $y \in V(G)$ for which $xy \in E(G)$. We have an analogous definition for $N_G^-(x)$. Given a multidigraph $G$, we denote by $N_G^+(x)$ the multiset of vertices where a vertex $y \in V(G)$ appears $k$ times in $N_G^+(x)$ if $G$ contains precisely $k$ edges from $x$ to $y$. Again, we have an analogous definition for $N_G^-(x)$. We will write $N^+(x)$ for example, if this is unambiguous.

Given a vertex $x$ of a digraph or multidigraph $G$, we write $d^+_G(x) := |N^+(x)|$ for the outdegree of $x$, $d^-_G(x) := |N^-(x)|$ for its indegree and $d(x) := d^+(x) + d^-(x)$ for its degree. The maximum of the maximum outdegree $\Delta^+(G)$ and the maximum indegree $\Delta^-(G)$ is denoted by $\Delta^0(G)$. The minimum semidegree $\delta^0(G)$ of $G$ is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. Throughout the paper we will use $d^+_G(x)$, $\delta^+(G)$ and $N^+_G(x)$ as ‘shorthand’ notation. For example, $\delta^+(G) \geq \delta^+(H)/2$ is read as $\delta^+(G) \geq \delta^+(H)/2$ and $\delta^-(G) \geq \delta^-(H)/2$. 


A 1-factor of a multidigraph $G$ is a collection of vertex-disjoint cycles in $G$ which together cover all the vertices of $G$. Given $A,B \subseteq V(G)$, we write $e_G(A,B)$ to denote the number of edges in $G$ with startpoint in $A$ and endpoint in $B$. Similarly, if $G$ is an undirected graph, we write $e_G(A,B)$ for the number of all edges between $A$ and $B$. Given a multiset $X$ and a set $Y$ we define $X \cap Y$ to be the multiset where $x$ appears as an element precisely $k$ times in $X \cap Y$ if $x \in X$, $x \in Y$ and $x$ appears precisely $k$ times in $X$. We write $a = b \pm \varepsilon$ for $a \in [b - \varepsilon, b + \varepsilon]$.

3.2. A Chernoff bound. We will often use the following Chernoff bound for binomial and hypergeometric distributions (see e.g. [17, Corollary 2.3 and Theorem 2.10]). Recall that the binomial random variable with parameters $(n, p)$ is the sum of $n$ independent Bernoulli variables, each taking value 1 with probability $p$ or 0 with probability $1 - p$. The hypergeometric random variable $X$ with parameters $(n, m, k)$ is defined as follows. We let $N$ be a set of size $n$, fix $S \subseteq N$ of size $|S| = m$, pick a uniformly random $T \subseteq N$ of size $|T| = k$, then define $X = |T \cap S|$. Note that $\mathbb{E}X = km/n$.

**Proposition 8.** Suppose $X$ has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2\mathbb{E}X}{12}}$.

3.3. The Diregularity lemma. In the proof of Theorem 3 we will use the directed version of Szemerédi’s Regularity lemma. Before we can state it we need some more notation and definitions. The density of an undirected bipartite graph $G$ with vertex classes $A$ and $B$ is defined to be

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$ 

We will write $d(A, B)$ if this is unambiguous. Given any $\varepsilon, \varepsilon' > 0$, we say that $G$ is $[\varepsilon, \varepsilon']$-regular if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| < \varepsilon'$. In the case when $\varepsilon = \varepsilon'$ we say that $G$ is $\varepsilon$-regular.

Given $d \in [0, 1]$ we say that $G$ is $(\varepsilon, d)$-super-regular if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) = d \pm \varepsilon$ and, furthermore, if $d_G(a) = (d \pm \varepsilon)|A|$ for all $a \in A$ and $d_G(b) = (d \pm \varepsilon)|B|$ for all $b \in B$. Note that this is a slight variation of the standard definition.

Given disjoint vertex sets $A$ and $B$ in a digraph $G$, we write $(A, B)_G$ for the oriented bipartite subgraph of $G$ whose vertex classes are $A$ and $B$ and whose edges are all the edges from $A$ to $B$ in $G$. We say $(A, B)_G$ is $[\varepsilon, \varepsilon']$-regular and has density $d'$ if this holds for the underlying undirected bipartite graph of $(A, B)_G$. (Note that the ordering of the pair $(A, B)_G$ is important here.) In the case when $\varepsilon = \varepsilon'$ we say that $(A, B)_G$ is $\varepsilon$-regular and has density $d'$. Similarly, given $d \in [0, 1]$ we say $(A, B)_G$ is $(\varepsilon, d)$-super-regular if this holds for the underlying undirected bipartite graph.

The Diregularity lemma is a variant of the Regularity lemma for digraphs due to Alon and Shapira [1]. Its proof is similar to the undirected version. We will use the degree form of the Diregularity lemma which can be derived from the standard version in the same manner as the undirected degree form (see [21] for a sketch of the latter).
Lemma 9 (Degree form of the Diregularity lemma). For every \( \varepsilon \in (0, 1) \) and every integer \( M' \) there are integers \( M \) and \( m_0 \) such that if \( G \) is a digraph on \( n \geq m_0 \) vertices and \( d \in [0, 1] \) is any real number, then there is a partition of the vertex set of \( G \) into \( V_0, V_1, \ldots, V_L \) and a spanning subdigraph \( G' \) of \( G \) such that the following holds:

- \( M' \leq L \leq M \),
- \( |V_0| \leq \varepsilon n \),
- \( |V_1| = \cdots = |V_L| =: m \),
- \( d_{G'}^{+}(x) > d_{G}^{+}(x) - (d + \varepsilon)n \) for all vertices \( x \in V(G) \),
- for all \( i = 1, \ldots, L \) the digraph \( G'[V_i] \) is empty,
- for all \( 1 \leq i, j \leq L \) with \( i \neq j \) the pair \( (V_i, V_j)_{G'} \) is \( \varepsilon \)-regular and has density either 0 or at least \( d \).

We call \( V_1, \ldots, V_L \) clusters, \( V_0 \) the exceptional set and the vertices in \( V_0 \) exceptional vertices. We refer to \( G' \) as the pure digraph. The last condition of the lemma says that all pairs of clusters are \( \varepsilon \)-regular in both directions (but possibly with different densities). The reduced digraph \( R \) of \( G \) with parameters \( \varepsilon, d \) and \( M' \) is the digraph whose vertices are \( V_1, \ldots, V_L \) and in which \( V_iV_j \) is an edge precisely when \( (V_i, V_j)_{G'} \) is \( \varepsilon \)-regular and has density at least \( d \).

The next result shows that we can partition the set of edges of an \( \varepsilon \)-(super)-regular pair into edge-disjoint subgraphs such that each of them is still (super)-regular.

Lemma 10. Let \( 0 < \varepsilon \ll d_0 \ll 1 \) and suppose \( K \geq 1 \). Then there exists an integer \( m_0 = m_0(\varepsilon, d_0, K) \) such that for all \( d \geq d_0 \) the following holds.

(i) Suppose that \( G = (A, B) \) is an \( \varepsilon \)-regular pair of density \( d \) where \( |A| = |B| = m \geq m_0 \). Then there are \( [K] \) edge-disjoint spanning subgraphs \( S_1, \ldots, S_K \) of \( G \) such that each \( S_i \) is \( [\varepsilon, 4\varepsilon/K] \)-regular of density \( (d + 2\varepsilon)/K \).

(ii) If \( K = 2 \) and \( G = (A, B) \) is \( (\varepsilon, d) \)-super-regular with \( |A| = |B| = m \geq m_0 \), then there are two edge-disjoint spanning subgraphs \( S_1 \) and \( S_2 \) of \( G \) such that each \( S_i \) is \( (2\varepsilon, d/2) \)-super-regular.

Proof. We first prove (i). Suppose we have chosen \( m_0 \) sufficiently large. Initially set \( E(S_i) = \emptyset \) for each \( i = 1, \ldots, [K] \). We consider each edge of \( G \) in turn and add it to each \( E(S_i) \) with probability \( 1/K \), independently of all other edges of \( G \). So the probability that \( xy \) is added to none of the \( S_i \) is \( 1 - [K]/K \). Moreover, \( \mathbb{E}(e(S_i)) = e(G)/K = dm^2/K \).

Given \( X \subseteq A \) and \( Y \subseteq B \) with \( |X|, |Y| \geq \varepsilon m \) we have that \( |d_G(X, Y) - d| < \varepsilon \). Thus

\[
\frac{1}{K} (d - \varepsilon)|X||Y| < \mathbb{E}(e_{S_i}(X, Y)) < \frac{1}{K} (d + \varepsilon)|X||Y|
\]

for each \( i \). Proposition 8 for the binomial distribution implies that with high probability \( (d - 2\varepsilon)|X||Y|/K < e_{S_i}(X, Y) < (d + 2\varepsilon)|X||Y|/K \) for each \( i \leq [K] \) and every \( X \subseteq A \) and \( Y \subseteq B \) with \( |X|, |Y| \geq \varepsilon m \). Such \( S_i \) are as required in (i).

The proof of (ii) is similar. Indeed, as in (i) one can show that with high probability any \( X \subseteq A \) and \( Y \subseteq B \) with \( |X|, |Y| \geq \varepsilon m \) satisfy \( d_{S_i}(X, Y) = d/2 \pm 2\varepsilon \) (for \( i = 1, 2 \)). Moreover, each vertex \( a \in A \) satisfies \( \mathbb{E}(d_{S_i}(a)) = d_G(a)/2 = (d \pm \varepsilon)m/2 \).
(for \(i = 1, 2\)) and similarly for the vertices in \(B\). So again Proposition 8 for the binomial distribution implies that with high probability \(d_S(a) = (d/2 \pm 2\varepsilon)m\) for all \(a \in A\) and \(d_S(b) = (d/2 \pm 2\varepsilon)m\) for all \(b \in B\). Altogether this shows that with high probability both \(S_1\) and \(S_2\) are \((2\varepsilon, d/2)\)-super-regular.

\[\square\]

Suppose \(0 < 1/M' \ll \varepsilon \ll \beta \ll d \ll 1\) and let \(G\) be a digraph. Let \(R\) and \(G'\) denote the reduced digraph and pure digraph respectively, obtained by applying Lemma 9 to \(G\) with parameters \(\varepsilon, d\) and \(M'\). For each edge \(V_iV_j\) of \(R\) we write \(d_{i,j}\) for the density of \((V_i, V_j)_{G'}\). (So \(d_{i,j} \geq d\).) The reduced multidigraph \(R_m\) of \(G\) with parameters \(\varepsilon, \beta, d\) and \(M'\) is obtained from \(R\) by setting \(V(R_m) := V(R)\) and adding \([d_{i,j}/\beta]\) directed edges from \(V_i\) to \(V_j\) whenever \(V_iV_j \in E(R)\).

We will always consider the reduced multidigraph \(R_m\) of a digraph \(G\) whose order is sufficiently large in order to apply Lemma 10 to any pair \((V_i, V_j)_{G'}\) of clusters with \(V_iV_j \in E(R)\). Let \(K := d_{i,j}/\beta\) and \(S_{i,j}1, \ldots, S_{i,j}n\) be the spanning subgraphs of \((V_i, V_j)_{G'}\) obtained from Lemma 10. (So each \(S_{i,j,k}\) is \(\varepsilon\)-regular of density \(\beta \pm \varepsilon\).) Let \((V_iV_j)1, \ldots, (V_iV_j)_{n}\) denote the directed edges from \(V_i\) to \(V_j\) in \(R_m\). We associate each \((V_iV_j)_k\) with the edges in \(S_{i,j,k}\).

**Lemma 11.** Let \(0 < 1/M' \ll \varepsilon \ll \beta \ll d \ll c_1 \leq c_2 < 1\) and let \(G\) be a digraph of sufficiently large order \(n\) with \(\delta^0(G) \geq c_1n\) and \(\Delta^0(G) \leq c_2n\). Apply Lemma 9 with parameters \(\varepsilon, d\) and \(M'\) to obtain a pure digraph \(G'\) and a reduced digraph \(R\) of \(G\). Let \(R_m\) denote the reduced multidigraph of \(G\) with parameters \(\varepsilon, \beta, d\) and \(M'\). Then

\[
\delta^0(R_m) > (c_1 - 3d)\frac{|R_m|}{\beta} \quad \text{and} \quad \Delta^0(R_m) < (c_2 + 2\varepsilon)\frac{|R_m|}{\beta}.
\]

Note the corresponding upper bound would not hold if we considered \(R\) instead of \(R_m\) here.

**Proof.** Given any \(V_i, V_j \in V(R)\), let \(d_{i,j}\) denote the density of \((V_i, V_j)_{G'}\). Then

\[
(c_1 - 2d)|R| \leq \frac{(c_1 - 2d)nm}{m^2} \leq \sum_{v \in V_i} \frac{d^+_G(v) - |V_0|}{m^2} \leq \sum_{V_j \in V(R)} d_{i,j}
\]

by Lemma 9. Thus

\[
d^+_m(V_i) = \sum_{V_j \in V(R_m)} \left\lfloor \frac{d_{i,j}}{\beta} \right\rfloor \geq \frac{1}{\beta} \sum_{V_j \in V(R)} d_{i,j} - |R_m| \geq (c_1 - 2d - \beta)\frac{|R_m|}{\beta} \geq (c_1 - 3d)\frac{|R_m|}{\beta}.
\]

So indeed \(\delta^+(R_m) > (c_1 - 3d)|R_m|/\beta\). Similar arguments can be used to show that \(\delta^-(R_m) > (c_1 - 3d)|R_m|/\beta\) and \(\Delta^0(R_m) < (c_2 + 2\varepsilon)|R_m|/\beta\). \(\square\)

We will also need the well-known fact that for any cycle \(C\) of the reduced multigraph \(R_m\) we can delete a small number of vertices from the clusters in \(C\) in order to ensure that each edge of \(C\) corresponds to a super-regular pair. We include a proof for completeness.
Lemma 12. Let \( C = V_{j_1} \ldots V_{j_s} \) be a cycle in the reduced multigraph \( R_m \) as in Lemma 11. For each \( t = 1, \ldots, s \) let \( (V_{j_t}, V_{j_{t+1}})_{k_t} \) denote the edge of \( C \) which joins \( V_{j_t} \) to \( V_{j_{t+1}} \) (where \( V_{j_{t+1}} := V_{j_1} \)). Then we can choose subclusters \( V'_{j_t} \subseteq V_{j_t} \) of size \( m' := (1 - 4\varepsilon)m \) such that \( (V'_{j_t}, V'_{j_{t+1}})_{s_{j_t,j_{t+1},k_t}} \) is \((10\varepsilon, \beta)\)-super-regular (for each \( t = 1, \ldots, s \)).

Proof. Recall that for each \( t = 1, \ldots, s \) the digraph \( S_{j_t,j_{t+1},k_t} \) corresponding to the edge \( (V_{j_t}, V_{j_{t+1}})_{k_t} \) of \( C \) is \( \varepsilon \)-regular and has density \( \beta \pm \varepsilon \). So \( V_{j_t} \) contains at most \( 2\varepsilon m \) vertices whose outdegree in \( S_{j_t,j_{t+1},k_t} \) is either at most \( (\beta - 2\varepsilon)m \) or at least \( (\beta + 2\varepsilon)m \). Similarly, there are at most \( 2\varepsilon m \) vertices in \( V_{j_t} \) whose indegree in \( S_{j_t-1,j_t,k_{t-1}} \) is either at most \( (\beta - 2\varepsilon)m \) or at least \( (\beta + 2\varepsilon)m \). Let \( V'_{j_t} \) be a set of size \( m' \) obtained from \( V_{j_t} \) by deleting all these vertices (and some additional vertices if necessary). It is easy to check that \( V'_{j_t} \) are subclusters as required.

Finally, we will use the following crude version of the fact that every \( [\varepsilon, \varepsilon'] \)-regular pair contains a subgraph of given maximum degree \( \Delta \) whose average degree is close to \( \Delta \).

Lemma 13. Suppose that \( 0 < 1/n \ll \varepsilon', \varepsilon \ll d_0 \leq d_1 \ll 1 \) and that \((A, B)\) is an \([\varepsilon, \varepsilon']\)-regular pair of density \( d_1 \) with \( n \) vertices in each class. Then \((A, B)\) contains a subgraph \( H \) whose maximum degree is at most \( d_0 n \) and whose average degree is at least \( d_0 n / 8 \).

Proof. Let \( A'' \subseteq A \) be the set of vertices of degree at least \( 2d_1 n \) and define \( B'' \) similarly. Then \( |A''|, |B''| \leq \varepsilon n \). Let \( A' := A \setminus A'' \) and \( B' := B \setminus B'' \). Then \((A', B')\) is still \([2\varepsilon, 2\varepsilon']\)-regular of density at least \( d_1 / 2 \). Now consider a spanning subgraph \( H \) of \((A', B')\) which is obtained from \((A', B')\) by including each edge with probability \( d_0 / 3d_1 \). So the expected degree of every vertex is at most \( 2d_0 n / 3 \) and the expected number of edges of \( H \) is at least \( d_0 (n - \varepsilon n)^2 / 6 \). Now apply the Chernoff bound on the binomial distribution in Proposition 8 to each of the vertex degrees and to the total number of edges in \( H \) to see that with high probability \( H \) has the desired properties.

4. Useful results

4.1. 1-factors in multidigraphs. Our main aim in this subsection is to show that the reduced multidigraph \( R_m \) contains a collection of \('almost\' 1-factors which together cover almost all the edges of \( R_m \) (see Lemma 15). To prove this we will need the following result which implies \( R_m \) contains many edges between any two sufficiently large sets. The second part of the lemma will be used in Section 4.5.

Lemma 14. Let \( 0 < 1/n \ll 1/M' \ll \varepsilon \ll \beta \ll \eta \ll d \ll c, d' \ll 1 \). Suppose that \( G \) is an oriented graph of order \( n \) with \( \delta^0(G) \geq (1/2 - \eta)n \). Let \( R \) and \( R_m \) denote the reduced digraph and the reduced multidigraph of \( G \) obtained by applying Lemma 9 (with parameters \( \varepsilon, d, M' \) and \( \varepsilon, \beta, d, M' \) respectively). Let \( L := |R| = |R_m| \). Then the following properties hold.
(i) Let $X \subseteq V(R_m)$ be such that $\delta^0(R_m[X]) \geq (1/2 - c)|X|/\beta$. Then for all (not necessarily disjoint) subsets $A$ and $B$ of $X$ of size at least $(1/2 - c)|X|$ there are at least $|X|^2/(60\beta)$ directed edges from $A$ to $B$ in $R_m$.

(ii) Let $R'$ denote the spanning subdigraph of $R$ obtained by deleting all edges which correspond to pairs of density at most $d'$ (in the pure digraph $G'$). Then $\delta^0(R') \geq (1/2 - 2d')L$ and for all (not necessarily disjoint) subsets $A$ and $B$ of $V(R')$ of size at least $(1/2 - c)L$ there are at least $L^2/60$ directed edges from $A$ to $B$ in $R'$.

Proof. We first prove (i). Recall that for every edge $V_iV_j$ of $R$ there are precisely $|d_{i,j}/\beta|$ edges from $V_i$ to $V_j$ in $R_m$, where $d_{i,j}$ denotes the density of $(V_i,V_j)_{G'}$. But $d_{i,j} + d_{j,i} \leq 1$ since $G$ is oriented and so $R_m$ contains at most $1/\beta$ edges between $V_i$ and $V_j$ (here we count the edges in both directions).

By deleting vertices from $A$ and $B$ if necessary we may assume that $|A| = |B| = (1/2 - c)|X|$. We will distinguish two cases. Suppose first that $|A \cap B| > |X|/5$ and let $Y := A \setminus B$. Define $Y := X \setminus Y$ and $A \cup B := X \setminus (A \cup B)$. Then

$$2e(A, B) \geq 2e(Y) = \sum_{V \in Y} d_{R_m}[X](V) - e(Y, \overline{Y}) = e(\overline{Y}, Y) \geq |Y|(1 - 2c)|X|/\beta - |Y|(|X| - |Y|)/\beta = |Y|(|X| - 2c|X|)/\beta \geq |X|^2/(30\beta).$$

So suppose next that $|A \cap B| \leq |X|/5$. Then $|A \cup B| \leq |X| - |A| - |B| + |A \cap B| \leq (1/5 + 2c)|X|$. Therefore,

$$e(A, B) \geq \sum_{V \in A} d_{R_m}[X](V) - e(A, \overline{A \cup B}) - e(A) \geq |A|(1/2 - c)|X|/\beta - |A||A \cup B|/\beta - |A|^2/(2\beta) \geq |A|((1/2 - c) - (1/5 + 2c) - (1/2 - c)/2)|X|/\beta \geq |X|^2/(60\beta),$$

as required.

To prove (ii) we consider the weighted digraph $R'_w$ obtained from $R'$ by giving each edge $V_iV_j$ of $R'$ weight $d_{i,j}$. Given a cluster $V_i$, we write $w^+(V_i)$ for the sum of the weights of all edges sent out by $V_i$ in $R'_w$. We define $w^-(V_i)$ similarly and write $w^0(R'_w)$ for the minimum of $\min\{w^+(V_i), w^-(V_i)\}$ over all clusters $V_i$. Note that $\delta^0(R') \geq w^0(R'_w)$. Moreover, Lemma 9 implies that $d^+_{G' \setminus V_0}(x) > (1/2 - 2d)n$ for all $x \in V(G' \setminus V_0)$. Thus each $V_i \in V(R')$ satisfies

$$(1/2 - 2d)n |x| \leq e_{G'}(V_i, V(G' \setminus V_0)) \leq m^2 w^+(V_i) + (d'm^2)L$$

and so $w^+(V_i) \geq (1/2 - 2d - d')(1/2) > (1/2 - 2d)$. Arguing in the same way for inweights gives us $\delta^0(R') \geq w^0(R'_w) \geq (1/2 - 2d')L$. Let $A, B \subseteq V(R')$ be as in (ii). Similarly as in (i) (setting $\beta := 1$ and $X := V(R')$ in the calculations) one can show that the sum of all weights of the edges from $A$ to $B$ in $R'_w$ is at least $L^2/60$. But this implies that $R'$ contains at least $L^2/60$ edges from $A$ to $B$. 

\[\square\]

**Lemma 15.** Let $0 < 1/n \ll 1/M' \ll \varepsilon \ll \beta \ll \eta \ll d \ll c \ll 1$. Suppose that $G$ is an oriented graph of order $n$ with $\delta^0(G) \geq (1/2 - \eta)n$. Let $R_m$ denote the reduced multidigraph of $G$ with parameters $\varepsilon, \beta, d$ and $M'$ obtained by applying Lemma 9.
Let $r := (1/2 - c)|R_m|/\beta$. Then there exist edge-disjoint collections $\mathcal{F}_1, \ldots, \mathcal{F}_r$ of vertex-disjoint cycles in $R_m$ such that each $\mathcal{F}_i$ covers all but at most $c|R_m|$ of the clusters in $R_m$.

**Proof.** Let $L := |R_m|$. Since $\Delta^0(G) \leq n - \delta^0(G) \leq (1/2 + \eta)n$, Lemma 11 implies that

$$
\delta^0(R_m) \geq (1/2 - 4d)\frac{L}{\beta} \quad \text{and} \quad \Delta^0(R_m) \leq (1/2 + 2\eta)\frac{L}{\beta}.
$$

First we find a set of clusters $X \subseteq V(R)$ with the following properties:

- $|X| = cL$,
- $|N_{R_m}^\pm(V_i) \cap X| = (1/2 \pm 5d)\frac{L}{\beta}$ for all $V_i \in V(R_m)$.

We obtain $X$ by choosing a set of $cL$ clusters uniformly at random. Then each cluster $V_i$ satisfies

$$
\mathbb{E}(|N_{R_m}^\pm(V_i) \cap X|) = c|N_{R_m}^\pm(V_i)| \overset{(2)}{=} c(1/2 \pm 4d)\frac{L}{\beta}.
$$

Proposition 8 for the hypergeometric distribution now implies that with nonzero probability $X$ satisfies our desired conditions. (Recall that $N_{R_m}^\pm(V_i)$ is a multiset. Formally Proposition 8 does not apply to multisets. However, for each $j = 1, \ldots, 1/\beta$ we can apply Proposition 8 to the set of all those clusters which appear at least $j$ times in $N_{R_m}^\pm(V_i)$, and similarly for $N_{R_m}^\pm(V_i)$.)

Note that

$$
d_{R_m \setminus X}^\pm(V_i) = \left(\frac{1}{2} - \frac{c}{2} \pm 5d\right)\frac{L}{\beta}
$$

for each $V_i \in V(R_m \setminus X)$. We now add a small number of temporary edges to $R_m \setminus X$ in order to turn it into an $r'$-regular multigraph where $r' := (\frac{1}{2} - \frac{c}{2} \pm 5d)\frac{L}{\beta}$. We do this as follows. As long as $R_m \setminus X$ is not $r'$-regular there exist $V_i, V_j \in V(R_m \setminus X)$ such that $V_i$ has outdegree less than $r'$ and $V_j$ has indegree less than $r'$. In this case we add an edge from $V_i$ to $V_j$. (Note we may have $i = j$, in which case we add a loop.)

We decompose the edge set of $R_m \setminus X$ into $r'$ 1-factors $\mathcal{F}_1', \ldots, \mathcal{F}_{r'}'$. (To see that we can do this, consider the bipartite multigraph $H$ where both vertex classes $A, B$ consist of a copy of $V(R_m \setminus X)$ and we have $s$ edges between $a \in A$ and $b \in B$ if there are precisely $s$ edges from $a$ to $b$ in $R_m \setminus X$, including the temporary edges. Then $H$ is regular and so has a perfect matching. This corresponds to a 1-factor $\mathcal{F}_1'$. Now remove the edges of $\mathcal{F}_1'$ from $H$ and continue to find $\mathcal{F}_2', \ldots, \mathcal{F}_{r'}'$ in the same way.) Since at each cluster we added at most $20d\frac{L}{\beta}$ temporary edges, all but at most $20\sqrt{dL}$ of the $\mathcal{F}_i'$ contain at most $\sqrt{dL}$ temporary edges. By relabeling if necessary we may assume that $\mathcal{F}_1', \ldots, \mathcal{F}_{r'}'$ are such 1-factors. We now remove the temporary edges from each of these 1-factors, though we still refer to the digraphs obtained in this way as $\mathcal{F}_1', \ldots, \mathcal{F}_{r'}'$. So each $\mathcal{F}_i'$ spans $R_m \setminus X$ and consists of cycles and at most $\sqrt{dL}$ paths.
Our aim is to use the clusters in $X$ to piece up these paths into cycles in order to obtain edge-disjoint directed subgraphs $\mathcal{F}_1, \ldots, \mathcal{F}_r$ of $R_m$ where each $\mathcal{F}_i$ is a collection of vertex-disjoint cycles and $\mathcal{F}_i \subseteq \mathcal{F}_i$.

Let $P'_1, \ldots, P'_\ell$ denote all the paths lying in one of $\mathcal{F}'_1, \ldots, \mathcal{F}'_j$ (so $\ell \leq \sqrt{dL^2}/\beta$). Our next task is to find edge-disjoint paths and cycles $P_1, \ldots, P_\ell$ of length $5$ in $R_m$ with the following properties.

(i) If $P'_j$ consists of a single cluster $V_j \in V(R)$ then $P_j$ is a cycle consisting of
4 clusters in $X$ as well as $V_j$.

(ii) If $P'_j$ is a path of length $\geq 1$ then $P_j$ is a path whose startpoint is the
endpoint of $P'_j$. Similarly the endpoint of $P_j$ is the startpoint of $P'_j$.

(iii) If $P'_j$ is a path of length $\geq 1$ then the internal clusters in the path $P_j$ lie
in $\bar{X}$.

(iv) If $P'_{j_1}$ and $P'_{j_2}$ lie in the same $\mathcal{F}'_i$ then $P_{j_1}$ and $P_{j_2}$ are vertex-disjoint.

So conditions (i)–(iii) imply that $P'_j \cup P_j$ is a directed cycle for each $1 \leq j \leq \ell$. Assuming we have found such paths and cycles $P_1, \ldots, P_\ell$, we define $\mathcal{F}_1, \ldots, \mathcal{F}_r$ as follows. Suppose $P'_{j_1}, \ldots, P'_{j_t}$ are the paths in $\mathcal{F}'_i$. Then we obtain $\mathcal{F}_i$ from $\mathcal{F}_i'$ by adding the paths and cycles $P_{j_1}, \ldots, P_{j_t}$ to $\mathcal{F}_i$. Condition (iv) ensures that the $\mathcal{F}_i$ are indeed collections of vertex-disjoint cycles.

It remains to show the existence of $P_1, \ldots, P_\ell$. Suppose that for some $j \leq \ell$ we have already found $P_1, \ldots, P_{j-1}$ and now need to define $P_j$. Consider $P'_j$ and suppose it lies in $\mathcal{F}'_k$. Let $V_a$ denote the startpoint of $P'_j$ and $V_b$ its endpoint.

We call an edge $(V_i, V_j)_k$ in $R_m$ free if it has not been used in one of $P_1, \ldots, P_{j-1}$. Let $B$ be the set of all those clusters $V \in X$ for which at least $c|X|/\beta$ of the edges at $V$ in $R_m[X]$ are not free. Our next aim is to show that $B$ is small. More precisely,

$$|B| \leq d^{1/4}L.$$

To see this, note that $3(j - 1) \leq 3\ell \leq 3\sqrt{dL^2}/\beta$ edges of $R_m[X]$ lie in one of $P_1, \ldots, P_{j-1}$. Thus, $2 \cdot 3\sqrt{dL^2}/\beta \geq \frac{c|X|}{\beta}|B| = \frac{c^2L|B|}{\beta}$. (The extra factor of 2 comes from the fact that we may have counted edges at the vertices in $B$ twice.) Since $c \gg d$ this implies that $|B| \leq d^{1/4}L$, as desired. We will only use clusters in $X' := X \setminus B$ when constructing $P_j$. Note that $V_a$ receives at most $|B|/\beta \leq d^{1/4}L/\beta$ edges from $B$ in $R_m$.

Since we added at most $20dL/\beta$ temporary edges to $R_m \setminus X$ per cluster, $V_a$ can be the startpoint or endpoint of at most $20dL/\beta$ of the paths $P'_1, \ldots, P'_{j-1}$. Thus $V_a$ lies in at most $20dL/\beta$ of the paths and cycles $P_1, \ldots, P_{j-1}$. In particular, at most $40dL/\beta$ edges at $V_a$ in $R_m$ are not free. We will avoid such edges when constructing $P_j$.

For each of $P_1, \ldots, P_{j-1}$ we have used 4 clusters in $X$. Let $P'_{j_1}, \ldots, P'_{j_t}$ denote the paths which lie in $\mathcal{F}'_i$ (so $t \leq \sqrt{dL}$). Thus at most $4\sqrt{dL}$ clusters in $X$ already lie in the paths and cycles $P_{j_1}, \ldots, P_{j_t}$. So for $P_j$ to satisfy (iv), the inneighbour of $V_a$ on $P_j$ must not be one of these clusters. Note that $V_a$ receives at most $4\sqrt{dL}/\beta$ edges in $R_m$ from these clusters.
Thus in total we cannot use \( d^{1/4}\sqrt{dL}/\beta + 40dL/\beta + 4\sqrt{dL}/\beta \leq 2d^{1/4}L/\beta \) of the edges which \( V_a \) receives from \( X \) in \( R_m \). But \( |N_{\overline{R}_m}(V_a) \cap X| \geq (\frac{1}{2} - 5d)cL/\beta \gg 2d^{1/4}L/\beta \) and so we can still choose a suitable cluster \( V_{\overline{a}} \) in \( N_{\overline{R}_m}(V_a) \cap X \) which will play the role of the inneighbour of \( V_a \) on \( P_j \). Let \((V_a - V_a)k_5\) denote the corresponding free edge in \( R_m \) which we will use in \( P_j \).

A similar argument shows that we can find a cluster \( V_{b+} \neq V_{a-} \) to play the role of the outneighbour of \( V_b \) on \( P_j \). So \( V_{b+} \in X' \), \( V_{b+} \) does not lie on any of \( P_{j1}, \ldots, P_{jt} \) and there is a free edge \((V_b V_{b+})k_5\) in \( R_m \).

We need to choose the outneighbour \( V_{b++} \) of \( V_{b+} \) on \( P_j \) such that \( V_{b++} \in X' \setminus \{V_{a-}\}, V_{b++} \) has not been used in \( P_{j1}, \ldots, P_{jt} \) and there is a free edge from \( V_{b+} \) to \( V_{b++} \) in \( R_m \). Let \( A_1 \) denote the set of all clusters in \( X' \) which satisfy these conditions. Since \( V_{b+} \in X' \) at most \( c|X'|/\beta \) edges at \( V_{b+} \) in \( R_m[X] \) are not free. So \( V_{b+} \) sends out at least \( (\frac{1}{2} - 5d)|X|/\beta - c\frac{|X|}{\beta} - \frac{|Bu(V_{a-})|}{\beta} \geq (\frac{1}{2} - 2c)|X|/\beta \) free edges to \( X' \setminus \{V_{a-}\} \) in \( R_m \). On the other hand, as before one can show that \( V_{b+} \) sends at most \( 4\sqrt{dL}/\beta \) edges to clusters in \( X' \) which already lie in \( P_{j1}, \ldots, P_{jt} \). Hence, \( |A_1| \geq \beta[(\frac{1}{2} - 2c)|X|/\beta - 4\sqrt{dL}/\beta] \geq (1/2 - 3c)|X| \).

Similarly we need to choose the inneighbour \( V_{a--} \) of \( V_{a-} \) on \( P_j \) such that \( V_{a--} \in X' \setminus \{V_{b+}\}, V_{a--} \) has not been used in \( P_{j1}, \ldots, P_{jt} \) and so that \( R_m \) contains a free edge from \( V_{a--} \) to \( V_{a-} \). Let \( A_2 \) denote the set of all clusters in \( X' \) which satisfy these conditions. As before one can show that \( |A_2| \geq (1/2 - 3c)|X| \).

Recall that \( 6\beta(R_m[X]) \geq (1/2 - 5d)|X|/\beta \) by our choice of \( X \). Thus Lemma 14(iii) implies that \( R_m[X] \) contains at least \( |X|^2/(60\beta) = c^2L^2/(60\beta) \) edges from \( A_1 \) to \( A_2 \). Since all but at most \( 5 \leq 5\sqrt{dL}/\beta \) edges of \( R_m \) are free, there is a free edge \((V_{b++} V_{a--})k_3\) from \( A_1 \) to \( A_2 \). Let \((V_{b+} V_{b++})k_2\) be a free edge from \( V_{b+} \) to \( V_{b++} \) in \( R_m \) and let \((V_{a--} V_{a-})k_4\) be a free edge from \( V_{a--} \) to \( V_{a-} \) (such edges exist by definition of \( A_1 \) and \( A_2 \)). We take \( P_j \) to be the directed path or cycle which consists of the edges \((V_b V_{b+})k_1, (V_b+ V_{b++})k_2, (V_{b++} V_{a--})k_3, (V_{a--} V_{a-})k_4 \) and \((V_{a-} V_{a})k_5\).

4.2. Spanning subgraphs of super-regular pairs. Frieze and Krivelevich [12] showed that every \((\varepsilon, \beta)\)-super-regular pair \( \Gamma \) contains a regular subgraph \( \Gamma' \) whose density is almost the same as that of \( \Gamma \). The following lemma is an extension of this, where we can require \( \Gamma' \) to have a given degree sequence, as long as this degree sequence is almost regular.

**Lemma 16.** Let \( 0 < 1/m < \varepsilon \ll \beta \ll \alpha' \ll \alpha \ll 1 \). Suppose that \( \Gamma = (U, V) \) is an \((\varepsilon, \beta + \varepsilon)\)-super-regular pair where \(|U| = |V| = m \). Define \( \tau := (1 - \alpha)\beta m \). Suppose we have a non-negative integer \( x_i \leq \alpha' \beta m \) associated with each \( u_i \in U \) and a non-negative integer \( y_i \leq \alpha' \beta m \) associated with each \( v_i \in V \) such that \( \sum_{u_i \in U} x_i = \sum_{v_i \in V} y_i \). Then \( \Gamma \) contains a spanning subgraph \( \Gamma' \) in which \( c_i := \tau - x_i \) is the degree of \( u_i \in U \) and \( d_i := \tau - y_i \) is the degree of \( v_i \in V \).

**Proof.** We first obtain a directed network \( N \) from \( \Gamma \) by adding a source \( s \) and a sink \( t \). We add an edge \( su_i \) of capacity \( c_i \) for each \( u_i \in U \) and an edge \( v_it \) of capacity \( d_i \) for each \( v_i \in V \). We give all the edges in \( \Gamma \) capacity 1 and direct them from \( U \) to \( V \).
Our aim is to show that the capacity of any cut is at least \( \sum_{u_i \in U} c_i = \sum_{v_i \in V} d_i \).
By the max-flow min-cut theorem this would imply that \( N \) admits a flow of value \( \sum_{u_i \in U} c_i \), which by construction of \( N \) implies the existence of our desired subgraph \( \Gamma' \).

So consider any \((s, t)\)-cut \((S, \bar{S})\) where \( S = \{s\} \cup S_1 \cup S_2 \) with \( S_1 \subseteq U \) and \( S_2 \subseteq V \). Let \( \bar{S}_1 := U \setminus S_1 \) and \( \bar{S}_2 := V \setminus S_2 \). The capacity of this cut is

\[
\sum_{u_i \in S_1} c_i + \sum_{v_i \in S_2} d_i + e(S_1, \bar{S}_2)
\]
and so our aim is to show that

\[
(3) \quad e(S_1, \bar{S}_2) \geq \sum_{u_i \in S_1} c_i - \sum_{v_i \in S_2} d_i.
\]

Now

\[
(4) \quad \sum_{u_i \in S_1} c_i - \sum_{v_i \in S_2} d_i \leq |S_1|(1 - \alpha)\beta m - |S_2|(1 - \alpha - \alpha')\beta m
\]
and similarly

\[
(5) \quad \sum_{u_i \in S_1} c_i - \sum_{v_i \in S_2} d_i = \sum_{v_i \in S_2} d_i - \sum_{u_i \in S_1} c_i \leq |\bar{S}_2|(1 - \alpha)\beta m - |\bar{S}_1|(1 - \alpha - \alpha')\beta m.
\]

By (4) we may assume that \(|S_1| \geq (1 - 2\alpha')|S_2|\). (Since otherwise \(\sum_{u_i \in S_1} c_i - \sum_{v_i \in S_2} d_i < 0\) and thus (3) is satisfied.) Similarly by (5) we may assume that \(|\bar{S}_2| \geq (1 - 2\alpha)|\bar{S}_1|\). Let \(\alpha^* := \alpha'/\alpha\). We now consider several cases.

**Case 1.** \(|S_1|, |\bar{S}_2| \geq \varepsilon m \) and \(|S_1| \geq (1 + \alpha^*)|S_2|\).

Since \(\Gamma\) is \((\varepsilon, \beta + \varepsilon)\)-super-regular we have that

\[
e(S_1, \bar{S}_2) \geq \beta|S_1|(m - |S_2|) \geq \beta m(|S_1| - |S_2|)
\]
\[
= (|S_1|(1 - \alpha)\beta m - |S_2|(1 - \alpha - \alpha')\beta m) + \alpha\beta m|S_1| - (\alpha + \alpha')\beta m|S_2|
\]
\[
\geq |S_1|(1 - \alpha)\beta m - |S_2|(1 - \alpha - \alpha')\beta m.
\]
(The last inequality follows since \(\alpha|S_1| \geq (\alpha + \alpha')|S_2|\).) Together with (4) this implies (3).

**Case 2.** \(|S_1|, |\bar{S}_2| \geq \varepsilon m, |S_1| < (1 + \alpha^*)|S_2|\) and \(|S_2| \leq (1 - \alpha^*)m\).

Again since \(\Gamma\) is \((\varepsilon, \beta + \varepsilon)\)-super-regular we have that

\[
e(S_1, \bar{S}_2) \geq \beta|S_1|(m - |S_2|) = \beta|S_1||\bar{S}_2|.
\]
As before, to prove (3) we will show that

\[
e(S_1, \bar{S}_2) \geq |S_1|(1 - \alpha)\beta m - |S_2|(1 - \alpha - \alpha')\beta m.
\]
Thus by (6) it suffices to show that \(\alpha m|S_1| - |S_1||S_2| + (1 - \alpha - \alpha')m|S_2| \geq 0\).
We know that \(|S_2|(1 - \alpha - \alpha') \geq |S_1|(1 - \alpha - \alpha^*)\) since \((1 + \alpha^*)|S_2| > |S_1|\). Hence, \(\alpha|S_1| - |S_1|(1 - \alpha^*) + |S_2|(1 - \alpha - \alpha') \geq 0\). So \(\alpha m|S_1| - |S_1||S_2| + (1 - \alpha - \alpha')m|S_2| \geq 0\) as \(|S_2| \leq (1 - \alpha^*)m\). So indeed (3) is satisfied.

**Case 3.** \(|S_1|, |\bar{S}_2| \geq \varepsilon m, |S_1| < (1 + \alpha^*)|S_2|\) and \(|S_2| > (1 - \alpha^*)m\).
By (5) in order to prove (3) it suffices to show that
\[ e(S_1, S_2) \geq |S_2| (1 - \alpha) \beta m - |S_1| (1 - \alpha - \alpha') \beta m. \]
Since (6) also holds in this case, this means that it suffices to show that \( \alpha |S_2| m - |S_1||S_2| + (1 - \alpha - \alpha') |S_1| m \geq 0. \) Since \( |S_1| \geq (1 - 2\alpha') |S_2| \) and \( |S_2| > (1 - \alpha^m) m \) we have that \( |S_1| > (1 - \alpha) m. \) Thus \( \alpha |S_2| m \geq |S_1||S_2| \) and so indeed (3) holds.

**Case 4.** \( |S_1| < \varepsilon m \) and \( |S_2| \geq \varepsilon m. \)
Since \( |S_1| \geq (1 - 2\alpha') |S_2| \) we have that \( |S_2| \leq 2\varepsilon m. \) Hence,
\[ e(S_1, S_2) \geq \beta m |S_1| - |S_1||S_2| \geq (\beta - 2\varepsilon) m |S_1| \geq (1 - \alpha) \beta m |S_1| \]
and so by (4) we see that (3) is satisfied, as desired.

**Case 5.** \( |S_1| \geq \varepsilon m \) and \( |S_2| < \varepsilon m. \)
Similarly as in Case 4 it follows that \( e(S_1, S_2) \geq (1 - \alpha) \beta m |S_2| \) and so by (5) we see that (3) is satisfied, as desired.

Note that we have considered all possible cases since we cannot have that \( |S_1|, |S_2| < \varepsilon m. \) Indeed, if \( |S_1|, |S_2| < \varepsilon m \) then \( |S_2| \geq (1 - \varepsilon) m \) and as \( |S_1| \geq (1 - 2\alpha') |S_2| \) this implies \( |S_1| \geq (1 - 2\alpha')(1 - \varepsilon) m, \) a contradiction. \( \square \)

4.3. Special 1-factors in graphs and digraphs. It is easy to see that every regular oriented graph \( G \) contains a 1-factor. The following result states that if \( G \) is also dense, then (i) we can guarantee a 1-factor with few cycles. Such 1-factors have the advantage that we can transform them into a Hamilton cycle by adding/deleting a comparatively small number of edges. (ii) implies that even if \( G \) contains a sparse ‘bad’ subgraph \( H \), then there will be a 1-factor which does not contain ‘too many’ edges of \( H \).

**Lemma 17.** Let \( 0 < \theta_1, \theta_2, \theta_3 < 1/2 \) and \( \theta_1/\theta_3 < \theta_2. \) Let \( G \) be a \( \rho \)-regular oriented graph whose order \( n \) is sufficiently large and where \( \rho := \theta_3 n. \) Suppose \( A_1, \ldots, A_5 n \) are sets of vertices in \( G \) with \( \alpha_i := |A_i| \geq n^{1/2}. \) Let \( H \) be an oriented subgraph of \( G \) such that \( d_H^+(x) \leq \theta_1 n \) for all \( x \in A_i \) (for each \( i \)). Then \( G \) has a 1-factor \( F \) such that

(i) \( F \) contains at most \( n/(\log n)^{1/5} \) cycles;
(ii) For each \( i, \) at most \( \theta_2 \alpha_i \) edges of \( H \cap F \) are incident to \( A_i. \)

To prove this result we will use ideas similar to those used by Friese and Krivelevich [12]. In particular, we will use the following bounds on the number of perfect matchings in a bipartite graph.

**Theorem 18.** Suppose that \( B \) is a bipartite graph whose vertex classes have size \( n \) and \( d_1, \ldots, d_n \) are the degrees of the vertices in one of these vertex classes. Let \( \mu(B) \) denote the number of perfect matchings in \( B. \) Then
\[ \mu(B) \leq \prod_{k=1}^{n} (d_k!)^{1/d_k}. \]
Furthermore, if $B$ is $\rho$-regular then
$$
\mu(B) \geq \left( \frac{\rho}{n} \right)^n n!.
$$

The upper bound in Theorem 18 was proved by Brégman [8]. The lower bound is a consequence of the Van der Waerden conjecture which was proved independently by Egorychev [10] and Falikman [11].

We will deduce (i) from the following result in [20], which in turn is similar to Lemma 2 in [12].

**Lemma 19.** For all $\theta \leq 1$ there exists $n_0 = n_0(\theta)$ such that the following holds. Let $B$ be a $\theta n$-regular bipartite graph whose vertex classes $U$ and $W$ satisfy $|U| = |W| =: n \geq n_0$. Let $M_1$ be any perfect matching from $U$ to $W$ which is disjoint from $B$. Let $M_2$ be a perfect matching chosen uniformly at random from the set of all perfect matchings in $B$. Let $F = M_1 \cup M_2$ be the resulting 2-factor. Then the probability that $F$ contains more than $n/(\log n)^{1/5}$ cycles is at most $e^{-n}$.

**Proof of Lemma 17.** Consider the $\rho$-regular bipartite graph $B$ whose vertex classes $V_1, V_2$ are copies of $V(G)$ and where $x \in V_1$ is joined to $y \in V_2$ if $xy$ is a directed edge in $G$. Note that every perfect matching in $B$ corresponds to a 1-factor of $G$ and vice versa. Let $\mu(B)$ denote the number of perfect matchings of $B$. Then
$$
\mu(B) \geq \left( \frac{\rho}{n} \right)^n n! \geq \left( \frac{\rho}{n} \right)^n \left( \frac{n}{e} \right)^n = \left( \frac{\rho}{e} \right)^n
$$
by Theorem 18. Here we have also used Stirling’s formula which implies that for sufficiently large $m$,
$$
\left( \frac{m}{e} \right)^m \leq m! \leq \left( \frac{m}{e} \right)^{m+1}.
$$
We now count the number $\mu_i(G)$ of 1-factors of $G$ which contain more than $\theta_2 a_i$ edges of $H$ which are incident to $A_i$. Note that
$$
\mu_i(G) \leq \left( \frac{2a_i}{\theta_2 a_i} \right)(\theta_1 n)^{\theta_2 a_i}(\rho!)^{(n-\theta_2 a_i)/\rho}.
$$
Indeed, the term $\left( \frac{2a_i}{\theta_2 a_i} \right)(\theta_1 n)^{\theta_2 a_i}$ in (9) gives an upper bound for the number of ways we can choose $\theta_2 a_i$ edges from $H$ which are incident to $A_i$ such that no two of these edges have the same startpoint and no two of these edges have the same endpoint. The term $(\rho!)^{(n-\theta_2 a_i)/\rho}$ in (9) uses the upper bound in Theorem 18 to give a bound on the number of 1-factors in $G$ containing $\theta_2 a_i$ fixed edges. Now
$$
(\rho!)^{(n-\theta_2 a_i)/\rho} \leq \left( \frac{\rho}{e} \right)^{(1+1/\rho)(n-\theta_2 a_i)} \leq \left( \frac{\rho}{e} \right)^{n-\theta_2 a_i+1/\theta_3}
$$
since $\rho = \theta_3 n$ and
$$
\left( \frac{e}{\rho} \right)^{\theta_2 a_i-1/\theta_3} \leq \left( \frac{2e}{\theta_3 n} \right)^{\theta_2 a_i}
$$
since \( a_i \geq n^{1/2} \). Furthermore,
\[
(12) \quad \left( \frac{2a_i}{\theta_2 a_i} \right) \leq \frac{(2a_i)^{\theta_2 a_i}}{(\theta_2)^{\theta_2 a_i}} \leq \left( \frac{2e}{\theta_2} \right)^{\theta_2 a_i}.
\]
So by (9) we have that
\[
(10), (12) \quad \mu_i(G) \leq \left( \frac{2e}{\theta_2} \right)^{\theta_2 a_i} (\theta_1 n)^{\theta_2 a_i} \left( \frac{\theta}{e} \right)^{n-\theta_2 a_i+1/\theta_3}
\]
\[
(11) \quad \leq \left( \frac{2e}{\theta_2} \frac{\theta_1 n}{\theta_3 n} \right)^{\theta_2 a_i} \left( \frac{\theta}{e} \right)^{n} \leq \left( \frac{4e^2 \theta_1}{\theta_2 \theta_3} \right)^{\theta_2 a_i} \mu(B) \ll \frac{\mu(B)}{5n}
\]
since \( \theta_1 / \theta_3 < \theta_2 \), \( a_i \geq n^{1/2} \) and \( n \) is sufficiently large.

Now we apply Lemma 19 to \( B \) where \( M_1 \) is the identity matching (i.e. every vertex in \( V_1 \) is matched to its copy in \( V_2 \)). Then a cycle of length \( 2 \ell \) in \( M_1 \cup M_2 \) corresponds to a cycle of length \( \ell \) in \( G \). So, since \( n \) is sufficiently large, the number of \( 1 \)-factors of \( G \) containing more than \( n/(\log n)^{1/5} \) cycles is at most \( e^{-n} \mu(B) \). So there exists a \( 1 \)-factor \( F \) of \( G \) which satisfies (i) and (ii). \( \square \)

### 4.4. Rotation-Extension lemma

The following lemma will be a useful tool when transforming \( 1 \)-factors into Hamilton cycles. Given such a \( 1 \)-factor \( F \), we will obtain a path \( P \) by cutting up and connecting several cycles in \( F \) (as described in the proof sketch in Section 2). We will then apply the lemma to obtain a cycle \( C \) containing precisely the vertices of \( P \).

**Lemma 20.** Let \( 0 < 1/m \ll \varepsilon \ll \gamma < 1 \). Let \( G \) be an oriented graph on \( n \geq 2m \) vertices. Suppose that \( U \) and \( V \) are disjoint subsets of \( V(G) \) of size \( m \) with the following property:
\[
(13) \quad \text{If } S \subseteq U, T \subseteq V \text{ are such that } |S|, |T| \geq \varepsilon m \text{ then } e_G(S,T) \geq \gamma |S||T|/2.
\]
Suppose that \( P = u_1 \ldots u_k \) is a directed path in \( G \) where \( u_1 \in V \) and \( u_k \in U \). Let \( X \) denote the set of inneighbours \( u_i \) of \( u_k \) which lie on \( P \) so that \( u_i \in U \) and \( u_{i+1} \in V \). Similarly let \( Y \) denote the set of outneighbours \( u_i \) of \( u_k \) which lie on \( P \) so that \( u_i \in V \) and \( u_{i-1} \in U \). Suppose that \( |X|, |Y| \geq \gamma m \). Then there exists a cycle \( C \) in \( G \) containing precisely the vertices of \( P \) such that \( |E(C)| \leq 5 \). Furthermore, \( E(P) \setminus E(C) \) consists of edges from \( X \) to \( X^+ \) and edges from \( Y^- \) to \( Y \). (Here \( X^+ \) is the set of successors of vertices in \( X \) on \( P \) and \( Y^- \) is the set of predecessors of vertices in \( Y \) on \( P \).)

**Proof.** Clearly we may assume that \( u_k u_1 \notin E(G) \). Let \( X_1 \) denote the set of the first \( \gamma m/2 \) vertices in \( X \) along \( P \) and \( X_2 \) the set of the last \( \gamma m/2 \) vertices in \( X \) along \( P \). We define \( Y_1 \) and \( Y_2 \) analogously. So \( X_1, X_2 \subseteq U \) and \( Y_1, Y_2 \subseteq V \). We have two cases to consider.

**Case 1.** All the vertices in \( X_1 \) precede those in \( Y_2 \) along \( P \).
Partition \( X_1 = X_{11} \cup X_{12} \) where \( X_{11} \) denotes the set of the first \( \gamma m/4 \) vertices in \( X_1 \) along \( P \). We partition \( Y_2 \) into \( Y_{21} \) and \( Y_{22} \) analogously. Let \( X_{12} \) denote the set of
successors on $P$ of the vertices in $X_{12}$ and $Y_{21}^−$ the set of predecessors of the vertices in $Y_{21}$. So $X_{12}^+ \subseteq V$ and $Y_{21}^− \subseteq U$. Further define

- $X_{11}^+ := \{u_i \mid u_{i−1} \in X_{11} \text{ and } \exists \text{ edge from } u_{i−1} \text{ to } X_{12}^+\}$ and
- $Y_{22}^- := \{u_i \mid u_{i+1} \in Y_{22} \text{ and } \exists \text{ edge from } Y_{21}^− \text{ to } u_{i+1}\}$

So $X_{11}^+ \subseteq V$ and $Y_{22}^- \subseteq U$.

From (13) it follows that $|X_{11}^+| \geq \frac{(\gamma/2)(\gamma m/4)|X_{12}^+|}{|X_{12}^+|} \geq \varepsilon m$ and similarly $|Y_{22}^-| \geq \varepsilon m$. Since $u_i \in X_{11}^+$, by definition of $X_{11}^+$ it follows that $G$ contains an edge $u_i u_j$ from $Y_{22}^−$ to $X_{11}^+$. Since $u_i \in X_{11}^+$ and the latter two edges go from $X_{12}^+$ to the successors of the vertices $u_{j−1}$ and $u_{i+1}$. The former two edges go from $X$ to $X^+$ and the latter two from $Y^−$ to $Y$.

**Case 2.** All the vertices in $Y_1^−$ precede those in $X_2$ along $P$.

Let $Y_1^−$ be the predecessors of the vertices in $Y_1$ and $X_2^+$ the successors of the vertices in $X_2$ on $P$. So $|Y_1^−| = |X_2^+| = \gamma m/2$ and $Y_1^− \subseteq U$ and $X_2^+ \subseteq V$. Thus by (13) there exists an edge $u_i u_j \in E(G)$ from $Y_1^−$ to $X_2^+$. Again, it is easy to check that the cycle

$$C = u_1 \ldots u_{i−1} u_j u_{j+1} \ldots u_k u_{i+1} u_{i+2} \ldots u_{j−1} u_1$$

has the desired properties (see Figure 1). For example, $E(P) \setminus E(C)$ consists of the edges $u_{i−1} u_j$, $u_{j−1} u_j$, $u_j u_{j−1}$, and $u_j u_{j−1}$. The former two edges go from $X$ to $X^+$ and the latter two from $Y^−$ to $Y$.

![Figure 1. The cycle $C$ from Case 1](image)

**4.5. Shifted walks.** Suppose $R$ is a digraph and $F$ is a collection of vertex-disjoint cycles with $V(F) \subseteq V(R)$. A closed shifted walk $W$ in $R$ with respect to $F$ is a walk in $R \cup F$ of the form

$$W = c_1^+ C_1 c_2^+ C_2 c_2 \ldots c_s^+ C_s c_{s−1} c_{s−1} C_{s−1} \ldots c_1^+ c_s c_{s−1} C_{s−1} \ldots c_1^+,$$

where

- $\{C_1, \ldots, C_s\}$ is the set of all cycles in $F$;
- $c_i$ lies on $C_i$ and $c_i^+$ is the successor of $c_i$ on $C_i$ for each $1 \leq i \leq s$;
- $c_i c_{i+1}$ is an edge of $R$ (here $c_{s+1}^+ := c_1^+$).
Note that the cycles \( C_1, \ldots, C_s \) are not necessarily distinct. If a cycle \( C_i \) in \( F \) appears exactly \( t \) times in \( W \) we say that \( C_i \) is traversed \( t \) times. Note that a closed shifted walk \( W \) has the property that for every cycle \( C \) of \( F \), every vertex of \( C \) is visited the same number of times by \( W \). The next lemma will be used in Section 5.7 to combine cycles of \( G \) which correspond to different cycles of \( F \) into a single (Hamilton) cycle. Shifted walks were introduced in [19], where they were used for a similar purpose.

Lemma 21. Let \( 0 < 1/n \ll 1/M' \ll \varepsilon \ll \eta \ll d \ll c \ll d' \ll 1 \). Suppose that \( G \) is an oriented graph of order \( n \) with \( \delta^0(G) \geq (1/2 - \eta)n \). Let \( R \) denote the reduced digraph of \( G \) with parameters \( \varepsilon, d \) and \( M' \) obtained by applying Lemma 9. Let \( L := |R| \). Let \( R' \) denote the spanning subgraph of \( R \) obtained by deleting all edges which correspond to pairs of density at most \( d' \) in the pure digraph \( G' \). Let \( F \) be a collection of vertex-disjoint cycles with \( V(F) \subseteq V(R') \) and \( |V(F)| \geq (1 - c)L \). Then \( R' \) contains a closed shifted walk with respect to \( F \) so that each cycle \( C \) in \( F \) is traversed at most \( 3L \) times.

**Proof.** Let \( C_1, \ldots, C_t \) denote the cycles of \( F \). We construct our closed shifted walk \( W \) as follows: for each cycle \( C_i \), choose an arbitrary vertex \( a_i \) lying on \( C_i \) and let \( a_i^+ \) denote its successor on \( C_i \). Let \( U_i := N^+_R(a_i) \cap V(F) \) and let \( U_i^- \) be the set of predecessors of \( U_i \) on \( F \). Similarly, let \( V_i := N^-_R(a_i) \cap V(F) \) and let \( V_i^+ \) be the set of successors of \( V_i \) on \( F \). Since \( \delta^0(R') \geq (1/2 - 2d')L \) by Lemma 14(ii), we have \( |U_i^-| = |U_i| \geq (1/2 - 3d')L \) and \( |V_i^+| = |V_i| \geq (1/2 - 3d')L \). So by Lemma 14(ii) there is an edge \( u_i^- v_i^+_{i+1} \) from \( U_i^- \) to \( V_i^+_{i+1} \) in \( R' \). Then we obtain a walk \( W_i \) from \( a_i^+ \) to \( a_{i+1} \) by first traversing \( C_i \) to reach \( a_i \), then use the edge from \( a_i \) to the successor \( u_i \) of \( a_i^- \), then traverse the cycle in \( F \) containing \( u_i \) as far as \( u_i^- \), then use the edge \( u_i^- v_i^+_{i+1} \), then traverse the cycle in \( F \) containing \( v_i^+_{i+1} \) as far as \( v_{i+1}^+ \), and finally use the edge \( v_{i+1}^- a_{i+1} \). (Here \( a_{t+1} := a_1 \)) \( W \) is obtained by concatenating the \( W_i \).

5. Proof of Theorem 3

5.1. Applying the Diregularity lemma. Without loss of generality we may assume that \( 0 < \eta_1 \ll 1 \). Define further constants satisfying

\begin{equation}
0 < 1/M' \ll \varepsilon \ll \beta \ll \eta_2 \ll d \ll c \ll c' \ll \gamma_1 \ll \gamma_2 \ll \gamma_3 \ll \gamma_4 \ll \gamma_5 \ll d' \ll \gamma \ll \eta_1.
\end{equation}

Let \( G \) be an oriented graph of order \( n \gg M' \) such that \( \delta^0(G) \geq (1/2 - \eta_2)n \). Apply the Diregularity lemma (Lemma 9) to \( G \) with parameters \( \varepsilon, d \) and \( M' \) to obtain clusters \( V_1, \ldots, V_L \) of size \( m \), an exceptional set \( V_0 \), a pure digraph \( G' \) and a reduced digraph \( R \) (so \( L = |R| \)). Let \( R' \) be the spanning subdigraph of \( R \) whose edges correspond to pairs of density at least \( d' \). So \( V_i V_j \) is an edge of \( R' \) if \( (V_i, V_j) \in G' \) has density at least \( d' \).

Let \( R_m \) denote the reduced multidigraph of \( G \) with parameters \( \varepsilon, \beta, d \) and \( M' \). For each edge \( V_i V_j \) of \( R \) let \( d_{i,j} \) denote the density of the \( \varepsilon \)-regular pair \( (V_i, V_j) \). Recall that each edge \( (V_i V_j)_k \in E(R_m) \) is associated with the \( k \)th spanning subgraph \( S_{i,j,k} \).
of \((V, V_j)_{|'}\) obtained by applying Lemma 10 with parameters \(\varepsilon, d_{i,j}\) and \(K := d_{i,j}/\beta\). Each \(S_{i,j,k}\) is \(\varepsilon\)-regular with density \(\beta \pm \varepsilon\). Lemma 11 implies that
\[
\delta^0(R_m) \geq (1/2 - 4d)\frac{L}{\beta} \quad \text{and} \quad \Delta^0(R_m) \leq (1/2 + 2\eta_2)\frac{L}{\beta}.
\]
(The second inequality holds since \(\Delta^0(G) \leq n - \delta^0(G) \leq (1/2 + \eta_2)n\).) Apply Lemma 15 to \(R_m\) in order to obtain
\[
r := (1 - \gamma)L/2\beta
\]
equidistant collections \(F_1, \ldots, F_r\) of vertex-disjoint cycles in \(R_m\) such that each \(F_i\) contains all but at most \(cL\) of the clusters in \(R_m\). Let \(V_{0,i}\) denote the set of all those vertices in \(G\) which do not lie in clusters covered by \(F_i\). So \(V_0 \subseteq V_{0,i}\) for all \(1 \leq i \leq r\) and \(|V_{0,i}| \leq |V_0| + cLm \leq (\varepsilon + c)n\). We now apply Lemma 12 to each cycle in \(F_i\) to obtain subclusters of size \(m' := (1 - 4\varepsilon)m\) such that the edges of \(F_i\) now correspond to \((10\varepsilon, \beta)\)-super-regular pairs. By removing one extra vertex from each cluster if necessary we may assume that \(m'\) is even. All vertices not belonging to the chosen subclusters of \(F_i\) are added to \(V_{0,i}\). So now
\[
|V_{0,i}| \leq 2cn.
\]
We refer to the chosen subclusters as the clusters of \(F_i\) and still denote these clusters by \(V_1, \ldots, V_L\). (This is a slight abuse of notation since the clusters of \(F_i\) might be different from those of \(F_i'\).) Thus an edge \((V_j, V_j)_k\) in \(F_i\) corresponds to the \((10\varepsilon, \beta)\)-super-regular pair \(S_{j1,j2,k} := (V_{j1}, V_{j2})_{S_{j1,j2,k}}\).

Let \(C_i\) denote the oriented subgraph of \(G\) whose vertices are all those vertices belonging to clusters in \(F_i\) such that for each \((V_j, V_j)_k \in E(F_i)\) the edges between \(V_{j1}\) and \(V_{j2}\) are precisely all the edges in \(S_{j1,j2,k}\). Clearly \(C_1, \ldots, C_r\) are edge-disjoint.

We now define ‘random’ edge-disjoint oriented subgraphs \(H^+_1, H^-_1, H_2, H_{3,i}, H_4\) and \(H_{5,i}\) of \(G\) (for each \(i = 1, \ldots, r\)). \(H^+_1\) and \(H^-_1\) will be used in Section 5.2 to incorporate the exceptional vertices in \(V_{0,i}\) into \(C_i\). \(H_2\) will be used to choose the skeleton walks in Section 5.4. The \(H_{3,i}\) will be used in Section 5.6 to merge certain cycles. \(H_4\) and the \(H_{5,i}\) will be used in Section 5.7 to find our almost decomposition into Hamilton cycles. We will choose these subgraphs to satisfy the following properties:

**Properties of \(H^+_1\) and \(H^-_1\).**

- \(H^+_1\) is a spanning oriented subgraph of \(G\).

- For all \(x \in V(H^+_1), \gamma_1 n \leq d^+_1(x) \leq 2\gamma_1 n\).

- For all \(x \in V(H^+_1)\) and each \(1 \leq i \leq r\), \(|N^+_1(x) \cap V_{0,i}| \leq 4\gamma_1|V_{0,i}|\).

- \(H^-_1\) satisfies analogous properties.

**Properties of \(H_2\).**

- The vertex set of \(H_2\) consists of precisely all those vertices of \(G\) which lie in a cluster of \(R\) (i.e. \(V(H_2) = V(G) \setminus V_0\)).

- For each edge \((V_j, V_j)_k\) of \(R_m\), \(H_2\) contains a spanning oriented subgraph of \(S_{j1,j2,k}\) which forms an \(\varepsilon\)-regular pair of density at least \(\gamma_2\).
• All edges of $H_2$ belong to one of these $\varepsilon$-regular pairs.
• For all $x \in V(H_2)$, $d_{H_2}^+(x) \leq 2\gamma_2 n$.

Properties of each $H_{3,i}$.
• The vertex set of $H_{3,i}$ consists of precisely all those vertices of $G$ which lie in a cluster of $\mathcal{F}_i$ (i.e. $V(H_{3,i}) = V(G) \setminus V_{0,i}$).
• For each edge $(V_{j_1},V_{j_2})_k$ of $\mathcal{F}_i$, $H_{3,i}$ contains a spanning oriented subgraph of $S'_{j_1,j_2,k}$ which forms a $(\sqrt{\varepsilon}/2, 2\gamma_3 \beta)$-super-regular pair.
• All edges in $H_{3,i}$ belong to one of these pairs.
• Let $H_3$ denote the union of all the oriented graphs $H_{3,i}$. The last two properties together with (16) imply that $d_{H_3}^+(x) \leq 3\gamma_3 n$ for all $x \in V(H_3)$.

Properties of $H_4$.
• The vertex set of $H_4$ consists of precisely all those vertices of $G$ which lie in a cluster of $R'$ (i.e. $V(H_4) = V(G) \setminus V_0$).
• For each edge $V_{j_1}V_{j_2}$ of $R'$, $(V_{j_1},V_{j_2})_4$ is $\varepsilon$-regular of density at least $\gamma_4 d'$.
• All edges in $H_4$ belong to one of these $\varepsilon$-regular pairs.
• For all $x \in V(H_4)$, $d_{H_4}^+(x) \leq 2\gamma_4 n$.

Properties of each $H_{5,i}$.
• The vertex set of $H_{5,i}$ consists of precisely all those vertices of $G$ which lie in a cluster of $\mathcal{F}_i$.
• For each edge $(V_{j_1},V_{j_2})_k$ of $\mathcal{F}_i$, $H_{5,i}$ contains a spanning oriented subgraph of $S'_{j_1,j_2,k}$ which forms a $(\sqrt{\varepsilon}/2, 2\gamma_5 \beta)$-super-regular pair.
• All edges in $H_{5,i}$ belong to one of these pairs.
• Let $H_5$ denote the union of all the oriented graphs $H_{5,i}$. The last two properties together with (16) imply that $d_{H_5}^+(x) \leq 3\gamma_5 n$ for all $x \in V(H_5)$.

Properties of each $S'_{i,j,k}$.
• For each edge $(V_{j_1},V_{j_2})_k$ of $\mathcal{F}_i$, the oriented subgraph obtained from $S'_{j_1,j_2,k}$ by removing all the edges in $H^+_1, H^-_1, H_2, \ldots, H_5$ is $(\varepsilon^{1/3}, \beta_1)$-super-regular for some $\beta_1$ with

$$
(1 - \gamma) \beta \leq \beta_1 \leq \beta.
$$

The existence of $H^+_1, H^-_1, H_2, H_{3,i}, H_4$ and $H_{5,i}$ can be shown by considering suitable random subgraphs of $G$ and applying the Chernoff bound in Proposition 8. For example, to show that $H^+_1$ exists, consider a random subgraph of $G$ which is obtained by including each edge of $G$ with probability $3\gamma_1$. Similarly, to define $H_2$ choose every edge in $S_{j_1,j_2,k}$ with probability $3\gamma_2/2$ (for all $S_{j_1,j_2,k}$) and argue as in the proof of Lemma 10. Note that since $H_4$ only consists of edges between pairs of clusters $V_{j_1}, V_{j_2}$ which form an edge in $R'$, the densities of oriented subgraphs obtained from the $S'_{j_1,j_2,k}$ by deleting all the edges in $H^+_1, H^-_1, H_2, \ldots, H_5$ will not be close enough to each other. Indeed, if $V_{j_1}V_{j_2} \notin E(R')$, then the corresponding density will be larger. However, for such pairs we can delete approximately a further
\(\gamma_4\)-proportion of the edges to ensure this property holds. Again, the deletion is done by considering a random subgraph obtained by deleting edges with probability \(\gamma_4\).

We now remove the edges in \(H_1^+, H_1^-, H_2, \ldots, H_5\) from each \(C_i\). We still refer to the subgraphs of \(C_i\) and \(S'_{j_1,j_2,k}\) thus obtained as \(C_i\) and \(S'_{j_1,j_2,k}\).

5.2. Incorporating \(V_{0,i}\) into \(C_i\). Our ultimate aim is to use each of the \(C_i\) as a ‘framework’ to piece together roughly \(\beta_1 m'\) Hamilton cycles in \(G\). In this section we will incorporate the vertices in \(V_{0,i}\), together with some edges incident to these vertices, into \(C_i\). For each \(i = 1, \ldots, r\), let \(G_i\) denote the oriented spanning subgraph of \(G\) obtained from \(C_i\) by adding the vertices of \(V_{0,i}\). So initially \(G_i\) contains no edges with a start- or endpoint in \(V_{0,i}\). We now wish to add edges to \(G_i\) so that

(i) \(d_{G_i}^+(x) \geq (1 - \sqrt{c})\beta_1 m'\) where \(x\) has neighbours only in \(C_i\), for all \(x \in V_{0,i}\);

(ii) \(|N_{G_i}^+(y) \cap V_{0,i}| \leq \sqrt{c}\beta_1 m'\) for all \(y \in V(C_i)\);

(iii) \(G_1, \ldots, G_r\) are edge-disjoint.

For each \(x \in V(G)\) we define \(L_x := \{i \mid x \in V_{0,i}\}\) and let \(L := |L_x|\). Let

\[
B' := \left\{ x \in V(G) \mid L_x \geq \frac{\gamma_1 n}{2\beta_1 m'} \right\}.
\]

We now consider the vertices in \(B'\) and \(V(G)\setminus B'\) separately.

First consider any \(x \in V(G)\setminus B'\). Let \(p := 2\beta_1 m' / \gamma_1 n\) and consider each edge \(e\) sent out by \(x\) in \(H_1^+\). With probability \(L_x p \leq 1\) we will assign \(e\) to exactly one of the \(G_i\) with \(i \in L_x\). More precisely, for each \(i \in L_x\) we assign \(e\) to \(G_i\) with probability \(p\). So the probability \(e\) is not assigned to any of the \(G_i\) is \(1 - L_x p \geq 0\). We randomly distribute the edges of \(H_1^-\) received by \(x\) in an analogous way amongst all the \(G_i\) with \(i \in L_x\).

We proceed similarly for all the vertices in \(V(G)\setminus B'\), with the random choices being independent for different such vertices. Since \(H_1^+\) and \(H_1^-\) are edge-disjoint from each other and from all the \(C_i\), the oriented graphs obtained from \(G_1, \ldots, G_r\) in this way will still be edge-disjoint. Moreover, \(\mathbb{E}(d_{G_i}^+(x)) \geq \gamma_1 np\) and \(\mathbb{E}(d_{G_i[V_{0,i}]}^+(x)) \leq |V_{0,i}| p \leq 2cnp\) for every \(x \in V(G)\setminus B'\) and each \(i \in L_x\). Thus

\[
\text{(19)} \quad \mathbb{E}(|N_{G_i}^+(x) \cap V(C_i)|) \geq (\gamma_1 - 2c)p \geq \beta_1 m'.
\]

Let \(B_i := V_{0,i} \setminus B'\) and \(\bar{B}_i := V_{0,i} \setminus B'\). Since \(|N_{H_1^+ \cup H_1^-}^+(y) \cap V_{0,i}| \leq 8\gamma_1|V_{0,i}|\) for every \(y \in V(C_i)\) (by definition of \(H_1^+\) and \(H_1^-\)) we have that

\[
\text{(20)} \quad \mathbb{E}(|N_{G_i}^+(y) \cap \bar{B}_i|) \leq 8\gamma_1|V_{0,i}| p \leq 32c\beta_1 m'.
\]

Applying the Chernoff bound in Proposition 8 (for the binomial distribution) for each \(i\) and summing up the error probabilities for all \(i\) we see that with nonzero probability the following properties hold:

- (19) implies that \(|N_{G_i}^+(x) \cap V(C_i)| \geq (1 - \sqrt{c})\beta_1 m'\) for every \(x \in \bar{B}_i\).
- (20) implies that \(|N_{G_i}^+(y) \cap \bar{B}_i| \leq \sqrt{c}\beta_1 m'/2\) for every \(y \in V(C_i)\).

For each \(i\) we delete all the edges with both endpoints in \(V_{0,i}\) from \(G_i\).
Having dealt with the vertices in \( V(G) \setminus B' \), let us now consider any \( x \in B' \). We call each edge of \( G \) with startpoint \( x \) free if it does not lie in any of \( C_i, H_1^+, H_1^-, H_2, \ldots, H_5 \) (for all \( i = 1, \ldots, r \)) and if the endpoint is not in \( B' \). Note that
\[
|B'| \frac{\gamma_1 n}{2\beta_1 m'} \leq \sum_{i=1}^{r} |V_{0,i}| \leq 2cn \leq cn \frac{L}{\beta}.
\]
and so \( |B'| \leq \frac{2cn}{\gamma_1} \). So the number of free edges sent out by \( x \) is at least
\[
(1/2 - \eta_2)n - (\beta_1 + \varepsilon^{1/3})m'(r - L_x) - 4\gamma_1 n - 2\gamma_2 n - 3\gamma_3 n - 2\gamma_4 n - 3\gamma_5 n - |B'|
\]
\[
\geq (1/2 - \eta_2)n - (\beta + \varepsilon^{1/3})m'(1 - \gamma) + L_x \beta_1 m' - 4\gamma_5 n - \frac{2cn}{\gamma_1}
\]
\[
\geq (1/2 - \eta_2)n - \left( \frac{\varepsilon^{1/3} n}{2\beta} + \frac{n}{2} \right) + \frac{\gamma n}{4} + L_x \beta_1 m' - 5\gamma_5 n \geq L_x \beta_1 m'.
\]
We consider \( L_x \beta_1 m' \) of these free edges sent out by \( x \) and distribute them randomly amongst all the \( G_i \) with \( i \in L_x \). More precisely, each such edge is assigned to \( G_i \) with probability \( 1/L_x \) (for each \( i \in L_x \)). So for each \( i \in L_x \),
\[
\mathbb{E}(d^+_{G_i}(x)) = \beta_1 m'
\]
and
\[
\mathbb{E}(d^+_{G_i[V_{0,i}]}(x)) \leq |V_{0,i}| \frac{1}{L_x} \leq 2cn \left( \frac{2\beta_1 m'}{\gamma_1 n} \right) = \frac{4c\beta_1 m'}{\gamma_1} \ll \sqrt{c} \beta_1 m'/4.
\]
We can introduce an analogous definition of a free edge at \( x \) but for edges whose endpoint is \( x \). As above we randomly distribute \( L_x \beta_1 m' \) such edges amongst all the \( G_i \) with \( i \in L_x \). Thus for each \( i \in L_x \),
\[
\mathbb{E}(d^-_{G_i}(x)) = \beta_1 m' \quad \text{and} \quad \mathbb{E}(d^-_{G_i[V_{0,i}]}(x)) \ll \sqrt{c} \beta_1 m'/4.
\]
We proceed similarly for all vertices in \( B' \), with the random choices being independent for different vertices \( x \in B' \). (Note that every edge of \( G \) is free with respect to at most one vertex in \( B' \).) Then using the lower bound on \( L_x \) for all \( x \in B' \) we have
\[
\mathbb{E}(|N^\pm_{G_i}(y) \cap B_i|) \leq |V_{0,i}| \frac{2\beta_1 m'}{\gamma_1 n} \leq \sqrt{c} \beta_1 m'/4
\]
for each \( i = 1, \ldots, r \) and all \( y \in V(C_i) \). As before, applying the Chernoff type bound in Proposition 8 for each \( i \) and summing up the failure probabilities over all \( i \) shows that with nonzero probability the following properties hold:

- (21)–(23) imply that \( |N^\pm_{G_i}(x) \cap V(C_i)| \geq (1 - \sqrt{c}) \beta_1 m' \) for each \( x \in B_i \).
- (24) implies that \( |N^\pm_{G_i}(y) \cap B_i| \leq \sqrt{c} \beta_1 m'/2 \) for each \( y \in V(C_i) \).

Together with the properties of \( G_i \) established after choosing the edges at the vertices in \( V(G) \setminus B' \) it follows that \( |N^\pm_{G_i}(x) \cap V(C_i)| \geq (1 - \sqrt{c}) \beta_1 m' \) for every \( x \in V_{0,i} \) and \( |N^\pm_{G_i}(y) \cap V_{0,i}| \leq \sqrt{c} \beta_1 m' \) for every \( y \in V(C_i) \). Furthermore, \( G_1, \ldots, G_r \) are still edge-disjoint since when dealing with the vertices in \( B' \) we only added free edges. By
discarding any edges assigned to $G_i$ which lie entirely in $V_{0,i}$ we can ensure that (i) holds. So altogether (i)–(iii) are satisfied, as desired.

### 5.3. Randomly splitting the $G_i$.

As mentioned in the previous section we will use each of the $G_i$ to piece together roughly $\beta_1 m'$ Hamilton cycles of $G$. We will achieve this by firstly adding some more special edges to each $G_i$ (see Section 5.4) and then almost decomposing each $G_i$ into 1-factors. However, in order to use these 1-factors to create Hamilton cycles we will need to ensure that no 1-factor contains a 2-path with start- and endpoint in $V_{0,i}$, and midpoint in $C_i$. Unfortunately $G_i$ might contain such paths. To avoid them, we will ‘randomly split’ each $G_i$.

We start by considering a random partition of each $V \in E(\mathcal{F}_i)$. Using the Chernoff bound in Proposition 8 for the hypergeometric distribution one can show that there exists a partition of $V$ into subclusters $V'$ and $V''$ so that the following conditions hold:

- $|V'|, |V''| = m'/2$ for each $V \in E(\mathcal{F}_i)$.
- $|N_G^\pm(x) \cap V'| \geq (1/2 - \sqrt{c})\beta_1 m'$ and $|N_G^\pm(x) \cap V''| \geq (1/2 + \sqrt{c})\beta_1 m'$ for each $x \in V_{0,i}$. (Here $V' := \bigcup_{V \in E(\mathcal{F}_i)} V'$ and $V'' := \bigcup_{V \in E(\mathcal{F}_i)} V''$.)

Recall that each edge $(V_1 V_2) k \in E(\mathcal{F}_i)$ corresponds to the $(\varepsilon^{1/3}, \beta_1)$-super-regular pair $S'_{j_1 j_2 k}$. Let $\beta_2 := \beta_1/2$. So

$$(1/2 - \gamma)\beta \leq \beta_2 \leq \beta/2. \tag{18}$$

Apply Lemma 10(ii) to obtain a partition $E'_{j_1 j_2 k}, E''_{j_1 j_2 k}$ of the edge set of $S'_{j_1 j_2 k}$ so that the following condition holds:

- The edges of $E'_{j_1 j_2 k}$ and $E''_{j_1 j_2 k}$ both induce an $(\varepsilon^{1/4}, \beta_2)$-super-regular pair which spans $S'_{j_1 j_2 k}$.

We now partition $G_i$ into two oriented spanning subgraphs $G'_i$ and $G''_i$ as follows.

- The edge set of $G'_i$ is the union of all $E'_{j_1 j_2 k}$ (over all edges $(V_1 V_2) k$ of $\mathcal{F}_i$) together with all the edges in $G_i$ from $V_{0,i}$ to $V'$, and all edges in $G_i$ from $V''$ to $V_{0,i}$.
- The edge set of $G''_i$ is the union of all $E''_{j_1 j_2 k}$ (over all edges $(V_1 V_2) k$ of $\mathcal{F}_i$) together with all the edges in $G_i$ from $V_{0,i}$ to $V''$, and all edges in $G_i$ from $V'$ to $V_{0,i}$.

Note that neither $G'_i$ nor $G''_i$ contains the type of 2-paths we wish to avoid. For each $i = 1, \ldots, r$ we use Lemma 10(ii) to partition the edge set of each $H_{3,i}$ to obtain edge-disjoint oriented spanning subgraphs $H'_{3,i}$ and $H''_{3,i}$ so that the following condition holds:

- For each edge $(V_1 V_2) k$ in $\mathcal{F}_i$, both $H'_{3,i}$ and $H''_{3,i}$ contain a spanning oriented subgraph of $S'_{j_1 j_2 k}$ which is $(\sqrt{\varepsilon}, \gamma_3, \beta')$-super-regular. Moreover, all edges in $H'_{3,i}$ and $H''_{3,i}$ belong to one of these pairs.

Similarly we partition the edge set of each $H_{5,i}$ to obtain edge-disjoint oriented spanning subgraphs $H'_{5,i}$ and $H''_{5,i}$ so that the following condition holds:
• For each edge $(V_j,V_{j+1})$ in $\mathcal{F}_i$, both $H'_i$ and $H''_i$ contain a spanning oriented subgraph of $S'_{j,j+1,k}$ which is $(\sqrt{\epsilon}, \gamma_5 \beta)$-super-regular. Moreover, all edges in $H'_i$ and $H''_i$ belong to one of these pairs.

We pair $H'_{3,i}$ and $H''_{5,i}$ with $G'_i$ and pair $H''_{4,i}$ and $H'_{5,i}$ with $G''_i$. We now have 2r edge-disjoint oriented subgraphs of $G$, namely $G'_1, G'_2, \ldots, G'_r, G''_1, G''_2, \ldots, G''_r$. To simplify notation, we relabel these oriented graphs as $G_1, \ldots, G_r$ where

$$r' := 2r \ (16) = (1 - \gamma)L/\beta.$$ \hfill (26)

We similarly relabel the oriented graphs $H'_{3,1}, H'_{3,2}, \ldots, H'_{3,r}, H''_{3,r}$ as $H'_{3,1}, \ldots, H'_{3,r}$, and relabel $H''_{5,1}, H''_{5,2}, \ldots, H''_{5,r}, H''_{5,r}$ as $H''_{5,1}, \ldots, H''_{5,r}$ in such a way that $H'_{3,i}$ and $H''_{5,i}$ are the oriented graphs which we paired with $G_i$. For each $i$ we still use the notation $\mathcal{F}_i, C_i$ and $V_{0,i}$ in the usual way. Now (i) from Section 5.2 becomes

$$(i') \ d^i_{G_i}(x) \geq (1/2 - \sqrt{\epsilon})\beta m'$$

where $x$ has neighbours only in $C_i$, for all $x \in V_{0,i}$, while (ii) and (iii) remain valid.

### 5.4. Adding skeleton walks to the $G_i$.

Note that all vertices (including the vertices of $V_{0,i}$) in each $G_i$ now have in- and outdegree close to $\beta m'$. In Section 5.5 our aim is to find a $\tau$-regular oriented subgraph of $G_i$, where

$$\tau := (1 - \gamma)\beta m'.$$ \hfill (27)

However, this may not be possible: suppose for instance that $V_{0,i}$ consists of a single vertex $x$, $\mathcal{F}_i$ consists of 2 cycles $C$ and $C'$ and that all outneighbours of $x$ lie on $C$ and all inneighbours lie on $C'$. Then $G_i$ does not even contain a 1-factor. A similar problem arises if for example $V_{0,i}$ consists of a single vertex $x$, $\mathcal{F}_i$ consists of a single cycle $C = V_1 \ldots V_8$, all outneighbours of $x$ lie in the cluster $V_2$ and all inneighbours in the cluster $V_8$. Note that in both situations, the edges between $V_{0,i}$ and $C_i$ are not ‘well-distributed’ or ‘balanced’. To overcome this problem, we add further edges to $C_i$ which will ‘balance out’ the edges between $C_i$ and $V_{0,i}$ which we added previously. These edges will be part of the skeleton walks which we define below. To motivate the definition of the skeleton walks it may be helpful to consider the second example above: Suppose that we add an edge $e$ from $V_1$ to $V_9$. Then $G_i$ now has a 1-factor. In general, we cannot find such an edge, but it will turn out that we can find a collection of 5 edges fulfilling the same purpose.

A skeleton walk $S$ in $G$ with respect to $G_i$ is a collection of distinct edges $x_1 x_2$, $x_2 x_3$, $x_3 x_4$, $x_4 x_5$ and $x_5 x_1$ of $G$ with the following properties:

- $x_1 \in V_{0,i}$ and all vertices in $V(S) \setminus \{x_1\}$ lie in $C_i$.
- Given some $2 \leq j \leq 5$, let $V \in V(\mathcal{F}_i)$ denote the cluster in $\mathcal{F}_i$ containing $x_j$ and let $C$ denote the cycle in $\mathcal{F}_i$ containing $V$. Then $x_j \in V^-$, where $V^-$ is the predecessor of $V$ on $C$.

Note that whenever $S$ is a union of edge-disjoint skeleton walks and $V$ is a cluster in $\mathcal{F}_i$, then number of edges in $S$ whose endpoint is in $V$ is the same as the number of edges in $S$ whose startpoint is in $V^-$. As indicated above, this ‘balanced’ property will be crucial when finding a $\tau$-regular oriented subgraph of $G_i$ in Section 5.5.
The 2nd, 3rd and 4th edge of each skeleton walk $S$ with respect to $G_i$ will lie in the ‘random’ graph $H_2$ chosen in Section 5.1. More precisely, each of these three edges will lie in a ‘slice’ $H_{2,i}$ of $H_2$ assigned to $G_i$. We will now partition $H_2$ into these ‘slices’ $H_{2,1}, \ldots, H_{2,r'}$. To do this, recall that any edge $(V_{j_1}, V_{j_2})_k$ in $R_m$ corresponds to an $\varepsilon$-regular pair of density at least $\gamma_2 \beta$ in $H_2$. Here $V_{j_1}$ and $V_{j_2}$ are viewed as clusters in $R_m$, so $|V_{j_1}| = |V_{j_2}| = m$. Apply Lemma 10(i) to each such pair of clusters to find edge-disjoint oriented subgraphs $H_{2,1}, \ldots, H_{2,r'}$ of $H_2$ so that for each $H_{2,i}$ all the edges $(V_{j_1}, V_{j_2})_k$ in $R_m$ correspond to $[\varepsilon, 5\beta \varepsilon/L]$-regular pairs with density at least $(\gamma_2 \beta - 2\varepsilon)\beta/L \geq \gamma_2 \beta^2/2L$ in $H_{2,i}$.

Recall that by (i’) in Section 5.3 each vertex $x \in V_{0,i}$ has at least $(1/2 - \sqrt{c})\beta_1 m' \geq \tau$ outneighbours in $C_i$ and at least $(1/2 - \sqrt{c})\beta_1 m'$ inneighbours in $C_i$. We pair $\tau$ of these outneighbours $x^+$ with distinct inneighbours $x^-$. For each of these $\tau$ pairs $x^+, x^-$ we wish to find a skeleton walk with respect to $G_i$ whose 1st edge is $xx^+$ and whose 5th edge is $x^-x$. We denote the union of these $\tau$ pairs $xx^+, x^-x$ of edges over all $x \in V_{0,i}$ by $T_i$.

In Section 5.3 we partitioned each cluster $V \in V(F_i)$ into subclusters $V'$ and $V''$. We next show how to choose the skeleton walks for all those $G_i$ for which each edge in $G_i$ with startpoint in $V_{0,i}$ has its endpoint in $V'$ (and so each edge in $G_i$ with endpoint in $V_{0,i}$ has startpoint in $V''$). The other case is similar, one only has to interchange $V'$ and $V''$.

**Claim 22.** We can find a set $S_i$ of $\tau |V_{0,i}|$ skeleton walks with respect to $G_i$, one for each pair of edges in $T_i$, such that $S_i$ has the following properties:

(i) For each skeleton walk in $S_i$, its 2nd, 3rd and 4th edge all lie in $H_{2,i}$ and all these edges have their startpoint in $V''$ and endpoint in $V'$.

(ii) Any two of the skeleton walks in $S_i$ are edge-disjoint.

(iii) Every $y \in V(C_i)$ is incident to at most $e^{1/5 \beta_2 m'}$ edges belonging to the skeleton walks in $S_i$.

Note that $|S_i| = |T_i| = \tau |V_{0,i}| \leq 2e\beta_2 m'n$ by (17) and (27). To find $S_i$, we will first find so-called shadow skeleton walks (here the internal edges are edges of $R_m$ instead of $G$). More precisely, a shadow skeleton walk $S'$ with respect to $G_i$ is a collection of two edges $x_1x_2, x_5x_1$ of $G$ and three edges $(X_2X_3)_{k_2}, (X_3X_4)_{k_3}, (X_4^{-1}X_5)_{k_4}$ of $R_m$ with the following properties:

- $x_1x_2, x_5x_1$ is a pair in $T_i$.
- $x_2 \in X_2, x_5^{-1} \in X_5^{-1}$ and each $X_j$ is a vertex of a cycle in $F_i$ and $X_j^{-1}$ is the predecessor of $X_j$ on that cycle.

Note that in the second condition we slightly abused the notation: as $X_j$ is a cluster in $R_m$, it only corresponds to a cluster in $F_i$ (which has size $m'$ and is a subcluster of the one in $R_m$). However, in order to simplify our exposition, we will use the same notation for a cluster in $R_m$ as for the cluster in $F_i$ corresponding to it.

We refer to the edge $(X_j^{-1}X_{j+1})_{k_j}$ as the $j$th edge of the shadow skeleton walk $S'$. Given a collection $S'$ of shadow skeleton walks (with respect to $G_i$) we say an edge of $R_m$ is **bad** if it is used at least $B := e^{1/4 \beta_2 (m')^2/L}$ times in $S'$, and **very bad** if it is used at least $10B$ times in $S'$. We say an edge from $V$ to $U$ in $R_m$ is $(V, +)$-**bad**
Proof. Suppose that we have already found \( V \) and \( S \) skeleton walks for \( U \), there exists an edge from \( \sqrt{cB} \) at most \( \sqrt{cB} \) most in \( F \).

Similarly we define \( W \), let \( x^+ \in V^+ \) and \( x^- \in W^- \), where \( V^+, W^- \in V(\mathcal{F}_i) \). Let \( V \) denote the predecessor of \( V^+ \) in \( \mathcal{F}_i \) and \( W \) the successor of \( W^- \) in \( \mathcal{F}_i \). We define \( V^+ \) to consist of all those clusters \( U \in V(\mathcal{F}_i) \) for which there exists an edge from \( V \) to \( U \) in \( R_m \) which is not \( (V,+)\)-bad. By definition of \( G_i \) (condition (ii) in Section 5.2), each \( y \in V(G_i) \) has at most \( \sqrt{c\beta m} \) inneighbours in \( V_{0,i} \) in \( G_i \). So the number of \( (V,+)-\)bad edges is at most

\[
\frac{\sqrt{c\beta m}}{B} = \frac{\sqrt{c\beta m}}{c^{1/4}\beta L} = \frac{c^{1/4}\beta L}{\beta^2} \leq \frac{c^{1/4}L}{\beta}.
\]

Together with (15) this implies that

\[
|V^+| \geq (1/2 - 4d - c^{1/4})L \geq (1/2 - 2c^{1/4})L.
\]

Similarly we define \( W^- \) to consist of all those clusters \( U \in V(\mathcal{F}_i) \) for which there exists an edge from \( U \) to \( V \) in \( R_m \) which is not \( (W,-)\)-bad. Again, \( |W^-| \geq (1/2 - 2c^{1/4})L \). Let \( V \) denote the set of those clusters which are the predecessors in \( \mathcal{F}_i \) of a cluster in \( V^+ \). Similarly let \( V \) denote the set of those clusters which are the successors in \( \mathcal{F}_i \) of a cluster in \( V^- \). So \( |V| = |V^+| \) and \( |W| = |W^+| \). By Lemma 14(i) applied with \( X = V(R_m) \) there exist at least \( L^2/60\beta \) edges in \( R_m \) from \( V \) to \( W \). On the other hand, the number of bad edges is at most

\[
\frac{3\tau|V_{0,i}|^{(17),(27)}}{B} \leq \frac{6\beta2m'cn}{c^{1/4}\beta^2(m')^2/L} \leq \frac{7c^{3/4}\beta L^2}{\beta^2} \leq \frac{7c^{3/4}L^2}{\beta}.
\]

So we can choose an edge \((XY)_k\) from \( V \) to \( W \) in \( R_m \) which is not bad. Let \( X^+ \) denote the successor of \( X \) in \( \mathcal{F}_i \) and \( Y^- \) the predecessor of \( Y \) in \( \mathcal{F}_i \). Thus \( X^+ \in V^+ \) and \( Y^- \in W^- \) and so there is an edge \((VX^+)_k\) in \( R_m \) which is not \( (V,+)\)-bad and an edge \((Y^-W)_{k''}\) which is not \( (W,-)\)-bad. Let \( S' \) be the shadow skeleton walk consisting of the edges \( xx^+ \), \((VX^+)_k\), \((XY)_k\), \((Y^-W)_{k''}\), and \( x^-x \). Then we can add \( S' \) to our collection of \( \ell \) skeleton walks that we have found already. \( \square \)

We now use Claim 23 to prove Claim 22.

Proof of Claim 22. We apply Claim 23 to obtain a collection \( S'_i \) of shadow skeleton walks. We will replace each edge of \( R_m \) in these shadow skeleton walks with a distinct edge of \( H_{2,i} \) to obtain our desired collection \( S_i \) of skeleton walks.
Recall that each edge $(VW)_k$ in $R_m$ corresponds to an $[\varepsilon, 5\varepsilon \beta/L]$-regular pair of density at least $\gamma_2 \beta^2/2L$ in $H_{2,i}$. Thus in $H_{2,i}$ the edges from $V''$ to $W'$ induce a $[3\varepsilon, 10\varepsilon \beta/L]$-regular pair of density $d_1 \geq \gamma_2 \beta^2/3L$. (Here $V', V''$ and $W', W''$ are the partitions of $V$ and $W$ chosen in Section 5.3.) Let $d_0 := 80B/(m'2)^2$ and note that $d_0 \leq d_1$. So we can now apply Lemma 13 to $(V'', W')_{H_{2,i}}$ to obtain a subgraph $H_{2,i}'[V'', W']$ with maximum degree at most $d_0 m'/2$ and at least $d_0 (m'/2)^2 / 8 = 10B$ edges. We do this for all those edges in $R_m$ which are used in a shadow skeleton walk in $S'_i$.

Since no edge in $R_m$ is very bad, for each $S' \in S'_i$ we can replace an edge $(VW)_k$ in $S'$ with a distinct edge $e$ from $V''$ to $W'$ lying in $H_{2,i}'[V'', W']$. Thus we obtain a collection $S_i$ of skeleton walks that satisfies properties (i) and (ii) of Claim 22. Note that by the construction of $S_i$ every vertex $y \in V(C_i)$ is incident to at most $d_0 m'L/(2\beta) \ll c^{1/5} \beta m'/2$ edges which play the role of a 2nd, 3rd or 4th edge in a skeleton walk in $S_i$. Condition (ii) in Section 5.2 implies that $y$ is incident to at most $2\sqrt{c} \beta m'$ edges which play the role of a 1st or 5th edge in a skeleton walk in $S_i$. So in total $y$ is incident to at most $c^{1/5} \beta m'/2 + 2 \sqrt{c} \beta m' \leq c^{1/5} \beta m'$ edges of the skeleton walks in $S_i$. Hence (iii) and thus the entire claim is satisfied.

We now add the edges of the skeleton walks in $S_i$ to $G_i$. Moreover, for each $x \in V_{0,i}$ we delete all those edges at $x$ which do not lie in a skeleton walk in $S_i$.

5.5. Almost decomposing the $G_i$ into 1-factors. Our aim in this section is to find a suitable collection of 1-factors in each $G_i$ which together cover almost all the edges of $G_i$. In order to do this, we first choose a $\tau$-regular spanning oriented subgraph $G_i^*$ of $G_i$ and then apply Lemma 17 to $G_i^*$.

We will refer to all those edges in $G_i$ which lie in a skeleton walk in $S_i$ as red, and all other edges in $G_i$ as white. Given $V \in V(\mathcal{F}_i)$ and $x \in V$, we denote by $N^+_w(x)$ the set of all those vertices which receive a white edge from $x$ in $G_i$. Similarly we denote by $N^-_w(x)$ the set of all those vertices which send out a white edge to $x$ in $G_i$. So $N^+_w(x) \subseteq V^+$ and $N^-_w(x) \subseteq V^-$, where $V^+$ and $V^-$ are the successor and the predecessor of $V$ in $\mathcal{F}_i$. Note that $G_i$ has the following properties:

(a1) $d_{G_i}^+(x) = \tau$ for each $x \in V_{0,i}$. Moreover, $x$ does not have any in- or outneighbours in $V_{0,i}$.

(a2) Every path in $G_i$ consisting of two red edges has its midpoint in $V_{0,i}$.

(a3) For each $((V_jV_j^+)_k \in E(\mathcal{F}_i)$ the white edges in $G_i$ from $V_j$ to $V_j^+$ induce a $(\varepsilon^{1/4}, \beta_2)$-super-regular pair $(V_j, V_j^+)$ of $G_i$.

(a4) Every vertex $u \in V(C_i)$ receives at most $c^{1/5} \beta m'$ red edges and sends out at most $c^{1/5} \beta m'$ red edges in $G_i$.

(a5) In total, the vertices in $G_i$ lying in a cluster $V_j \in V(\mathcal{F}_i)$ send out the same number of red edges as the vertices in $V_j^+$ receive.

In order to find our $\tau$-regular spanning oriented subgraph of $G_i$, consider any edge $((V_jV_j^+)_k \in E(\mathcal{F}_i)$. Given any $u_\ell \in V_j$, let $x_\ell$ denote the number of red edges sent out by $u_\ell$ in $G_i$. Similarly given any $v_\ell \in V_j^+$, let $y_\ell$ denote the number of red edges received by $v_\ell$ in $G_i$. By (a4) we have that $x_\ell, y_\ell \leq c^{1/5} \beta m'$ and by (a5) we have...
Thus we can apply Lemma 16 to obtain an oriented spanning subgraph of \((V_j, V_j^+)\) in which each \(u \in V_j\) has outdegree \(\tau - x\) and each \(v \in V_j^+\) has indegree \(\tau - y\). We apply Lemma 16 to each \((V_j, V_j^+)\) in \(E(F_i)\). The union of all these oriented subgraphs together with the red edges in \(G_i\) clearly yield a \(\tau\)-regular oriented subgraph \(G_i^*\) of \(G_i\), as desired.

We will use the following claim to almost decompose \(G_i^*\) into 1-factors with certain useful properties.

**Claim 24.** Let \(G^*\) be a spanning \(\rho\)-regular oriented subgraph of \(G_i\) where \(\rho \geq \gamma/2m'\). Then \(G^*\) contains a 1-factor \(F^*\) with the following properties:

1. \(F^*\) contains at most \(n/(\log n)^{1/5}\) cycles.
2. For each \(V_j \in V(F_i)\), \(F^*\) contains at most \(m'\) red edges incident to vertices in \(V_j\).
3. Let \(F^*_{red}\) denote the set of vertices which are incident to a red edge in \(F^*\). Then \(|F^*_{red} \cap N_{H_{3,i}}^\pm(x)| \leq 2c'\gamma/2m'\) for each \(x \in V(C_i)\).
4. \(|F^*_{red} \cap N_{w}^\pm(x)| \leq 2c'\beta/2m'\) for each \(x \in V(C_i)\).

**Proof.** A direct application of Lemma 17 to \(G^*\) proves the claim. Indeed, we apply the lemma with \(\theta_1 = (c^{1/5}/2m')/n\), \(\theta_2 = c'\), \(\theta_3 = \rho/n \geq (\gamma/2m')/n\) and with the oriented spanning subgraph of \(G^*\) whose edge set consists precisely of the red edges in \(G^*\) playing the role of \(H\). Furthermore, the clusters in \(V(F_i)\) together with the sets \(N_{w}^\pm(x)\) and \(N_{H_{3,i}}^\pm(x)\) for each \(x \in V(C_i)\) play the role of the \(A_j\).

Repeatedly applying Claim 24 we obtain edge-disjoint 1-factors \(F_{i,1}, \ldots, F_{i,\psi}\) of \(G_i\) satisfying conditions (i)–(iv) of the claim, where

\[
\psi := (1 - 2\gamma)\beta/2m'.
\]

Our aim is now to transform each of the \(F_{i,j}\) into a Hamilton cycle using the edges of \(H_{5,i}, H_4\) and \(H_{5,i}\).

5.6. **Merging the cycles in \(F_{i,j}\) into a bounded number of cycles.** Let \(D_1, \ldots, D_\xi\) denote the cycles in \(F_i\) and define \(V_{G_i}(D_k)\) to be the set of vertices in \(G_i\) which lie in clusters in the cycle \(D_k\). In this subsection, for each \(i\) and \(j\) we will merge the cycles in \(F_{i,j}\) to obtain a 1-factor \(F'_{i,j}\) consisting of at most \(\xi\) cycles.

Recall from Section 5.5 that we call the edges of \(G_i\) which lie on a skeleton walk in \(S_i\) red and the non-red edges of \(G_i\) white. We call the edges of the ‘random’ oriented graph \(H_{3,i}\) defined in Section 5.1 green. (Recall that \(H_{3,i}\) was modified in Section 5.3.) We will use the edges from \(H_{3,i}\) to obtain 1-factors \(F'_{i,1}, \ldots, F'_{i,\psi}\) for each \(i\) with the following properties:

1. If \(i \neq i'\) or \(j \neq j'\) then \(F'_{i,j}\) and \(F'_{i',j'}\) are edge-disjoint.
2. For each \(V \in V(F_i)\) all \(x \in V\) which send out a white edge in \(F_{i,j}\) lie on the same cycle \(C\) in \(F'_{i,j}\).
\((\beta_3)\) \(|E(F'_{i,j})| \leq 6n/(\log n)^{1/5}\) for all \(i\) and \(j\). Moreover, \(|E(F'_{i,j})\setminus E(F_{i,j})|\) consists of green and white edges only.

\((\beta_4)\) For every edge in \(F_{i,j}\) both endvertices lie on the same cycle in \(F'_{i,j}\).

\((\beta_5)\) All the red edges in \(F_{i,j}\) still lie in \(F'_{i,j}\).

Before showing the existence of 1-factors satisfying \((\beta_1)-(\beta_5)\), we will derive two further properties \((\beta_6)\) and \((\beta_7)\) from them which we will use in the next subsection. So suppose that \(F'_{i,j}\) is a 1-factor satisfying the above conditions. Consider any cluster \(V \in V(\mathcal{F}_i)\). Claim 24(ii) implies that \(F_{i,j}\) contains at most \(c'm'\) red edges with startpoint in \(V\). So the cycle \(C\) in \(F_{i,j}\) which contains all vertitces \(x \in V\) sending out a white edge in \(F_{i,j}\) must contain at least \((1-c')m'\) such vertitces \(x\). In particular there are at least \((1-c')m' > c'm'\) vertices \(y \in V^+\) which lie on \(C\). So some of these vertices \(y\) send out a white edge in \(F_{i,j}\). But by \((\beta_2)\) this means that \(C\) contains all those vertices \(y \in V^+\) which send out a white edge in \(F_{i,j}\). Repeating this argument shows that \(C\) contains all vertices in \(V(D_k)\) which send out a white edge in \(F_{i,j}\) (here \(D_k\) is the cycle on \(F_i\) that contains \(V\)). Furthermore, by property \((\beta_3)\), \(C\) contains all vertices in \(V(D_k)\) which receive a white edge in \(F_{i,j}\). By property \((\alpha_2)\) in Section 5.5 no vertex of \(C_i\) is both the a startpoint of a red edge in \(G_i\) and an endpoint of a red edge in \(G_i\). So this implies that all vertices in \(V(G_i(D_k))\) lie on \(C\). Thus if we obtain 1-factors \(F'_{i,1}, \ldots, F'_{i,\psi}\) satisfying \((\beta_1)-(\beta_5)\) then the following conditions also hold:

\((\beta_6)\) For each \(j = 1, \ldots, \psi\) and each \(k = 1, \ldots, \xi\) all the vertices in \(V(G(D_k))\) lie on the same cycle in \(F'_{i,j}\).

\((\beta_7)\) For each \(V \in V(\mathcal{F}_i)\) and each \(j = 1, \ldots, \psi\) at most \(c'm'\) vertices in \(V\) lie on a red edge in \(F'_{i,j}\).

(Condition \((\beta_7)\) follows from Claim 24(ii) and the ‘moreover’ part of \((\beta_3)\).)

For every \(i\), we will define the 1-factors \(F'_{i,1}, \ldots, F'_{i,\psi}\) sequentially. Initially, we let \(F'_{i,j} = F_{i,j}\). So the \(F'_{i,j}\) satisfy all conditions except \((\beta_2)\). Next, we describe how to modify \(F'_{i,1}\) so that it also satisfies \((\beta_2)\).

Recall from Section 5.3 that for each edge \((VV^+)\) of \(\mathcal{F}_i\) the pair \((V,V^+)\) is \((\sqrt{\varepsilon}, \gamma_3\beta)\)-super-regular and thus \(\delta^\pm \geq (\gamma_3\beta - \sqrt{\varepsilon})m' \geq \gamma_3\beta m'/2\). Furthermore, whenever \(V \in V(\mathcal{F}_i)\) and \(x \in V\), the outneighbourhood of \(x\) in \(G_{3,i}\) lies in \(V^+\) and the inneighbourhood of \(x\) in \(G_{3,i}\) lies in \(V^-\). Let \(H'_{3,i}\) denote the oriented spanning subgraph of \(H_{3,i}\) whose edge set consists of those edges \(xy\) of \(H_{3,i}\) for which \(x\) is not a startpoint of a red edge in our current 1-factor \(F'_{i,1}\) and \(y\) is not an endpoint of a red edge in \(F'_{i,1}\). Consider a white edge \(xy\) in \(F'_{i,1}\). Claim 24(iii) implies that \(x\) sends out most \(2c'\gamma_3\beta m'\) green edges \(xz\) in \(H_{3,i}\) which do not lie in \(H'_{3,i}\). So \(d^+_{H_{3,i}}(x) \geq (1/2 - 2c')\gamma_3\beta m'.\) Similarly, \(d^-_{H'_{3,i}}(y) \geq (1/2 - 2c')\gamma_3\beta m'.\) (However, if \(uv\) is a red edge in \(F'_{i,1}\) then \(d^+_{H'_{3,i}}(u) = d^-_{H'_{3,i}}(v) = 0\).) Thus we have the following properties of \(H_{3,i}\) and \(H'_{3,i}\):

\((\gamma_1)\) For each \(V \in V(\mathcal{F}_i)\) all the edges in \(H_{3,i}\) sent out by vertices in \(V\) go to \(V^+\).

\((\gamma_2)\) If \(xy\) is a white edge in \(F'_{i,1}\) then \(d^+_{H'_{3,i}}(x), d^-_{H'_{3,i}}(y) \geq \gamma_3\beta m'/3\).
(γ3) Consider any $V \in V(F_i)$. Let $S \subseteq V$ and $T \subseteq V^+$ be such that $|S|, |T| \geq \sqrt{m'}$. Then $e_{H_{3,i}}(S, T) \geq \gamma_3 |S| |T| / 2$.

If $F_{i,1}'$ does not satisfy (β2), then it contains cycles $C \neq C^*$ such that there is a cluster $V \in V(F_i)$ and white edges $xy$ on $C$ and $x^*y^*$ on $C^*$ with $x, x^* \in V$ and $y, y^* \in V^+$.

We have 3 cases to consider. Firstly, we may have a green edge $xz \in E(H_{3,i}')$ such that $z$ lies on a cycle $C' \neq C$ in $F_{i,1}'$. Then $z \in V^+$ and $z$ is the endpoint of a white edge in $F_{i,1}'$ (by (γ1) and the definition of $H_{3,i}'$). Secondly, there may be a green edge $wy^* \in E(H_{3,i}')$ such that $w$ lies on a cycle $C' \neq C^*$ in $F_{i,1}'$. So here $w \in V$ is the startpoint of a white edge in $F_{i,1}'$. If neither of these cases hold, then $N_{H_{3,i}'}(x)$ lies on $C$ and $N_{H_{3,i}'}^{-}(y^*)$ lies on $C^*$. Since $d_{H_{3,i}'}^{+}(x), d_{H_{3,i}'}^{-}(y^*) \geq \gamma_3 \beta m'/3$ by (γ2), we can use (γ3) to find a green edge $x'y'$ from $N_{H_{3,i}'}^{-}(y^*)$ to $N_{H_{3,i}'}^{+}(x)$. Then $x' \in V$, $y' \in V^+$, $x'$ is the startpoint of a white edge in $F_{i,1}'$ and $y'$ is the endpoint of a white edge in $F_{i,1}'$.

We will only consider the first of these 3 cases. The other cases can be dealt with analogously: In the second case $w$ plays the role of $x$ and $y^*$ plays the role of $z$. In the third case $x'$ plays the role of $x$ and $y'$ plays the role of $z$.

So let us assume that the first case holds, i.e. there is a green edge $xz \in E(H_{3,i}')$ such that $z$ lies on a cycle $C' \neq C$ in $F_{i,1}'$ and $z$ lies on a white edge $wz$ on $C'$. Let $P$ denote the directed path $(C \cup C' \cup \{xz\}) \setminus \{xy, wz\}$ from $y \in V^+$ to $w \in V$. Suppose that the endpoint $w$ of $P$ lies on a green edge $wv \in E(H_{3,i}')$ such that $v$ lies outside $P$. Then $v \in V^+$ is the endpoint of a white edge $uv$ lying on the cycle $C''$ in $F_{i,1}'$ which contains $v$. We extend $P$ by replacing $P$ and $C''$ with $(P \cup C'' \cup \{vu\}) \setminus \{uv\}$. We make similar extensions if the startpoint $y$ of $P$ has an inneighbour in $H_{3,i}'$ outside $P$. We repeat this ‘extension’ procedure as long as we can. Let $P$ denote the path obtained in this way, say $P$ joins $a \in V^+$ to $b \in V$. Note that $a$ must be the endpoint of a white edge in $F_{i,1}'$ and $b$ the startpoint of a white edge in $F_{i,1}'$.

We will now apply a ‘rotation’ procedure to close $P$ into a cycle. By (γ2) $a$ has at least $\gamma_3 \beta m'/3$ inneighbours in $H_{3,i}'$ and $b$ has at least $\gamma_3 \beta m'/3$ outneighbours in $H_{3,i}'$ and all these in- and outneighbours lie on $P$ since we could not extend $P$ any further. Let $X := N_{H_{3,i}'}^{-}(a)$ and $Y := N_{H_{3,i}'}^{+}(b)$. So $|X|, |Y| \geq \gamma_3 \beta m'/3$ and $X \subseteq V$ and $Y \subseteq V^+$ by (γ1). Moreover, whenever $c \in X$ and $c^+$ is the successor of $c$ on $P$, then either $cc^+$ was a white edge in $F_{i,1}'$ or $cc^+ \in E(H_{3,i}')$. Thus in both cases $c^+ \in V^+$. So the set $X^+$ of successors in $P$ of all the vertices in $X$ lies in $V^+$ and no vertex in $X$ sends out a red edge in $P$. Similarly one can show that the set $Y^-$ of predecessors in $P$ of all the vertices in $Y$ lies in $V$ and no vertex in $Y$ receives a red edge in $P$. Together with (γ3) this shows that we can apply Lemma 20 with $P \cup H_{3,i}'$ playing the role of $G$ and $V^+$ playing the role of $V$ and $V$ playing the role of $U$ to obtain a cycle $\hat{C}$ containing precisely the vertices of $P$ such that $|E(\hat{C}) \setminus E(P)| \leq 5$, $E(\hat{C}) \setminus E(\hat{C}) \subseteq E(H_{3,i}')$ and such that $E(P) \setminus E(\hat{C})$ consists of edges from $X$ to $X^+$ and edges from $Y^-$ to $Y$. Thus $E(P) \setminus E(\hat{C})$ contains no red
edges. Replacing $P$ with $\hat{C}$ gives us a 1-factor (which we still call $F'_{i,j}$) with fewer cycles. Also note that if the number of cycles is reduced by $\ell$, then we use at most $\l+5 \leq 6\ell$ edges in $H_{3,i}$ to achieve this. So $F'_{i,j}$ still satisfies all requirements with the possible exception of $(\beta_2)$. If it still does not satisfy $(\beta_2)$, we will repeatedly apply this ‘rotation-extension’ procedure until the current 1-factor $F'_{i,1}$ also satisfies $(\beta_2)$. However, we need to be careful since we do not want to use edges of $H_{3,i}$ several times in this process. Simply deleting the edges we use may not work as $(\gamma_2)$ might fail later on (when we will repeat the above process for $F'_{i,j}$ with $j > 1$).

So each time we modify $F'_{i,1}$, we also modify $H_{3,i}$ as follows. All the edges from $H_{3,i}$ which are used in $F'_{i,1}$ are removed from $H_{3,i}$. All the edges which are removed from $F'_{i,1}$ in the rotation-extension procedure are added to $H_{3,i}$. (Note that by $(\beta_5)$ we never add red edges to $H_{3,i}$.) When we refer to $H_{3,i}$, we always mean the ‘current’ version of $H_{3,i}$, not the original one. Furthermore, at every step we still refer to an edge of $H_{3,i}$ as green, even if initially the edge did not lie in $H_{3,i}$. Similarly at every step we refer to the non-red edges of our current 1-factor as white, even if initially they belonged to $H_{3,i}$.

Note that if we added a green edge $xz$ into $F'_{i,1}$, then $x$ lost an outneighbour in $H_{3,i}$, namely $z$. However, $(\beta_5)$ implies that we also moved some (white) edge $xy$ of $F'_{i,1}$ to $H_{3,i}$, where $y$ lies in the same cluster $V^+ \in V(F_i)$ as $z$ (here $x \in V$). So we still have that $\delta^+(H_{3,i}) \geq \gamma_3 \beta m'/3$. Similarly, at any stage $\delta^-(H_{3,i}) \geq \gamma_3 \beta m'/3$.

When $H_{3,i}$ is modified, then $H'_{3,i}$ is modified accordingly. This will occur if we add some white edges to $H_{3,i}$ whose start or endpoint lies on a red edge in $F'_{i,1}$. However, Claim 24(iv) implies that at any stage we still have

$$d^+_{H'_{3,i}}(x), d^+_{H'_{3,i}}(y) \geq (1/2 - 2c')\gamma_3 \beta m' - 2c\beta_2 m' \geq \gamma_3 \beta m'/3.$$ 

Also note that by $(\beta_3)$, the modified version of $H_{3,i}$ still satisfies

$$e_{H_{3,i}}(S,T) \geq (\gamma_3 \beta - \sqrt{\varepsilon})|S||T| - 6n/(\log n)^{1/5} \geq \gamma_3 |S||T|/2.$$ 

So $H_{3,i}$ and $H'_{3,i}$ will satisfy $(\gamma_1)$–$(\gamma_3)$ throughout and thus the above argument still works. So after at most $n/(\log n)^{1/5}$ steps $F'_{i,1}$ will also satisfy $(\beta_2)$.

Suppose that for some $1 < j \leq \psi$ we have found 1-factors $F'_{i,1}, \ldots, F'_{i,j-1}$ satisfying $(\beta_1)$–$(\beta_5)$. We can now carry out the rotation-extension procedure for $F'_{i,j}$ in the same way as for $F'_{i,1}$ until $F'_{i,j}$ also satisfies $(\beta_2)$. In the construction of $F'_{i,j}$, we do not use the original $H_{3,i}$, but the modified version obtained in the construction of $F'_{i,j-1}$. We then introduce the oriented spanning subgraph $H'_{3,i}$ of $H_{3,i}$ similarly as before (but with respect to the current 1-factor $F'_{i,j}$). Then all the above bounds on these graphs still hold, except that in the middle expression of (29) we multiply the term $6n/(\log n)^{1/5}$ by $j$ to account for the total number of edges removed from $H_{3,i}$ so far. But this does not affect the next inequality. So eventually, all the $F'_{i,j}$ will satisfy $(\beta_1)$–$(\beta_5)$.

5.7. Merging the cycles in $F'_{i,j}$ to obtain Hamilton cycles. Our final aim is to piece together the cycles in $F'_{i,j}$, for each $i$ and $j$, to obtain edge-disjoint Hamilton
cycles of $G$. Since we have $\psi$ 1-factors $F'_{i,1},\ldots,F'_{i,\psi}$ for each $G_i$, in total we will find

$$\psi' \geq (1 - 2\gamma)\beta m'(1 - \gamma)L/\beta \geq (1 - 2\gamma)(1/2 - \gamma)m'L$$

edge-disjoint Hamilton cycles of $G_i$ as desired.

Recall that $R'$ was defined in Section 5.1. Given any $i$, apply Lemma 21 to obtain a closed shifted walk

$$W_i = U_1^+D'_1U_1U_2^+D'_2U_2\ldots U_s^+D'_sU_{s-1}U_{s-1}^+D'_sU_sU_1^+$$

in $R'$ with respect to $F_i$ such that each cycle in $F_i$ is traversed at most 3L times. So $\{D'_1,\ldots,D'_s\}$ is the set of all cycles in $F_i$, $U_k^+$ is the successor of $U_k$ on $D'_k$ and $U_kU_{k+1}^+ \in E(R')$ for each $k = 1,\ldots,s$ (where $U_{s+1} := U_1$). Moreover,

$$s \leq 3L^2.$$  

For each 1-factor $F'_{i,j}$ we will now use the edges of $H_4$ and $H_{5,i}$ to obtain a Hamilton cycle $C_{i,j}$ with the following properties:

(i) If $i \neq i'$ or $j \neq j'$ then $C_{i,j}$ and $C'_{i',j'}$ are edge-disjoint.

(ii) $E(C_{i,j})$ consists of edges from $F'_{i,j}$, $H_4$ and $H_{5,i}$ only.

(iii) There are at most 3$L^2$ edges from $H_4$ lying in $C_{i,j}$.

(iv) There are at most 3$L^2 + 5$ edges from $H_{5,i}$ lying in $C_{i,j}$.

For each $j$, we will use $W_i$ to ‘guide’ us how to merge the cycles in $F'_{i,j}$ into the Hamilton cycle $C_{i,j}$. Suppose that we have already defined $\ell < \psi' \psi$ of the Hamilton cycles $C'_{i',j'}$ satisfying (i)–(iv), but have yet to define $C_{i,j}$. We remove all those edges which have been used in these $\ell$ Hamilton cycles from both $H_4$ and $H_{5,i}$.

For each $V \in V(F_i)$, we denote by $V_w$ the subcluster of $V$ containing all those vertices which do not lie on a red edge in $F'_{i,j}$. We refer to $V_w$ as the white subcluster of $V$. Thus $|V_w| \geq (1 - c')m'$ by property ( stain $\beta$ in Section 5.6. Note that the outneighbours of the vertices in $V_w$ on $F'_{i,j}$ all lie in $V^+$ while their inneighbours lie in $V^-$. For each $k = 1,\ldots,s$ we will denote the white subcluster of a cluster $U_k$ by $U_{k,w}$. We use similar notation for $U_k^+$ and $U_k^-$. Consider any $UV \in E(R')$. Recall that $U$ and $V$ are viewed as clusters of size $m$ in $R'$, but when considering $F_i$ we are in fact considering subclusters of $U$ and $V$ of size $m'$. When viewed as clusters in $R'$, $UV$ initially corresponded to an $\varepsilon$-regular pair of density at least $\gamma_4d'$ in $H_4$. Thus when viewed as clusters in $F_i$, $UV$ initially corresponded to a $2\varepsilon$-regular pair of density at least $\gamma_4d'/2$ in $H_4$. Moreover, initially the edges from $U_w$ to $V_w$ in $H_4$ induce a $3\varepsilon$-regular pair of density at least $\gamma_4d'/3$. However, we have removed all the edges lying in the $\ell$ Hamilton cycles $C'_{i',j'}$ which we have defined already. Property (iii) implies that we have removed at most $3L^2\ell \leq 3L^2n$ edges from $H_4$. Thus we have the following property:

(\delta') Given any $UV \in E(R')$, let $S \subseteq U_w$, $T \subseteq V_w$ be such that $|S|,|T| \geq 3\varepsilon m'$.

Then $e_{H_4}(S,T) \geq \gamma_4d'|S||T|/4$. 
When constructing $C_{i,j}$ we will remove at most $3L^2$ more edges from $H_4$. But since $(\delta_1)$ is far from being tight, it will hold throughout the argument below. Similarly, the initial definition of $H_{5,i}$ (c.f. Section 5.3) and (iv) together imply the following property:

$(\delta_2)$ Consider any edge $VV^+ \in E(F_i)$. Let $S \subseteq V$ and $T \subseteq V^+$ be such that $|S|, |T| \geq \sqrt{\varepsilon m'}$. Then $e_{H_{5,i}}(S,T) \geq \gamma_5\beta|S||T|/2$.

We now construct $C_{i,j}$ from $F'_{i,j}$. Condition $(\beta_6)$ in Section 5.6 implies that, for each $k = 1, \ldots, s$, every vertex in $V_G(D'_k)$ lies on the same cycle, $C'_k$ say, in $F'_{i,j}$. Let $x_1 \in U_{1,w}$ be such that $x_1$ has at least $\gamma_4d'|U_{2,w}|/4 \geq \gamma_4d'm'/5$ outneighbours in $H_4$ which lie in $U_{2,w}$. By $(\delta_1)$ all but at most $3\varepsilon m'$ vertices in $U_{1,w}$ have this property.

Note that the outneighbour in $F'_{i,j}$ of any such vertex lies in $U_{1,w}$. However, by $(\delta_2)$ all but at most $\sqrt{\varepsilon m'}$ vertices in $U_{1,w}$ have at least $\gamma_5\beta|U_{1,w}|/2 \geq \gamma_5\beta m'/3$ inneighbours in $H_{5,i}$ which lie in $U_{1,w}$. Thus we can choose $x_1$ with the additional property that its outneighbour $y_1 \in U_{1,w}$ in $F'_{i,j}$ has at least $\gamma_5\beta m'/3$ inneighbours in $H_{5,i}$ which lie in $U_{1,w}$.

Let $P$ denote the directed path $C'_1 - x_1y_1$ from $y_1$ to $x_1$. We now have two cases to consider.

Case 1. $C'_1 \neq C'_2$.

Note that $x_1$ has at least $\gamma_4d'm'/5 - \varepsilon m' \geq \gamma_4d'm'/6$ outneighbours $y'_2 \in U_{2,w}^+$ in $H_4$ such that the inneighbour of $y'_2$ in $F'_{i,j}$ lies in $U_{2,w}$. However, by $(\delta_1)$ all but at most $3\varepsilon m'$ vertices in $U_{2,w}$ have at least $\gamma_4d'm'/5$ outneighbours in $H_4$ which lie in $U_{2,w}$. Thus we can choose an outneighbour $y'_2 \in U_{2,w}^+$ of $x_1$ in $H_4$ such that its inneighbour $x'_2$ of $y'_2$ in $F'_{i,j}$ lies in $U_{2,w}$ and $x'_2$ has at least $\gamma_4d'm'/5$ outneighbours in $H_4$ which lie in $U_{2,w}$. We extend $P$ by replacing it with $(P \cup C'_2 \cup \{x_1y'_2\}) \setminus \{x'_2y'_2\}$.

Case 2. $C'_1 = C'_2$.

In this case the vertices in $V_G(D'_k)$ already lie on $P$. We will use the following claim to modify $P$.

Claim 25. There is a vertex $y_2 \in U_{2,w}^+$ such that:

- $x_1y_2 \in E(H_4)$.
- The predecessor $x_2$ of $y_2$ on $P$ lies in $U_{2,w}$.
- There is an edge $x_2y'_2$ in $H_{5,i}$ such that $y'_2 \in U_{2,w}^+$ and $y_2$ precedes $y'_2$ on $P$ (but need not be its immediate predecessor).
- The predecessor $x'_2$ of $y'_2$ on $P$ lies in $U_{2,w}$.
- $x'_2$ has at least $\gamma_4d'm'/5$ outneighbours in $H_4$ which lie in $U_{3,w}^+$.

Proof. Since $x_1$ has at least $\gamma_4d'm'/5$ outneighbours in $H_4$ which lie in $U_{2,w}^+$, at least $\gamma_4d'm'/5 - \varepsilon m' \geq \gamma_4d'm'/6$ of these outneighbours $y$ are such that the predecessor $x$ of $y$ on $P$ lies in $U_{2,w}$ and at least $\gamma_4d'm'/5$ outneighbours of $x$ in $H_4$ lie in $U_{3,w}^+$. This follows since all such vertices $y$ have their predecessor on $P$ lying in $U_2$ (since $y \in U_{2,w}^+$), since $|U_{2,w}| \geq (1 - \varepsilon)m'$ and since by $(\delta_1)$ all but at most $3\varepsilon m'$ vertices in $U_{2,w}$ have at least $\gamma_4d'm'/5$ outneighbours in $U_{3,w}^+$. Let $Y_2$ denote
the set of all such vertices $y$, and let $X_2$ denote the set of all such vertices $x$. So $|X_2| = |Y_2| \geq \gamma_4d'm'/6$, $X_2 \subseteq U_{2,w}$, $Y_2 \subseteq U_{2,w}^+ \cap N_{H_d}^+(x_1)$. Let $X_2^*$ denote the set of the first $\gamma_4d'm'/12$ vertices in $X_2$ on $P$ and $Y_2^*$ the set of the last $\gamma_4d'm'/12$ vertices in $Y_2$ on $P$. Then $(\delta_2)$ implies the existence of an edge $x_2y_2'$ from $X_2^*$ to $Y_2^*$ in $H_{5,i}$. Then the successor $y_2'$ of $x_2$ on $P$ satisfies the claim. 

Let $x_2, y_2, x_2'$ and $y_2'$ be as in Claim 25. We modify $P$ by replacing $P$ with 

$$(P \cup \{x_1y_2, x_2y_2'\}) \backslash \{x_2y_2, x_2'y_2'\}$$

(see Figure 2).

In either of the above cases we obtain a path $P$ from $y_1$ to some vertex $x_2' \in U_{2,w}$ which has at least $\gamma_4d'm'/5$ outneighbours in $H_4$ lying in $U_{3,w}^+$. We can repeat the above process: If $C_3' \neq C_1'$, $C_2'$ then we extend $P$ as in Case 1. If $C_3' = C_1'$ or $C_3' = C_2'$ then we modify $P$ as in Case 2. In both cases we obtain a new path $P$ which starts in $y_1$ and ends in some $x_3' \in U_{3,w}$ that has at least $\gamma_4d'm'/5$ outneighbours in $H_4$ lying in $U_{4,w}^+$. We can continue this process, for each $C_k'$ in turn, until we obtain a path $P$ which contains all the vertices in $C_1', \ldots, C_k'$ (and thus all the vertices in $G$), starts in $y_1$ and ends in some $x_s' \in U_{s,w}$ having at least $\gamma_4d'm'/5$ outneighbours in $H_4$ which lie in $U_{1,w}^+$.

**Claim 26.** There is a vertex $y_1' \in U_1^+ \backslash \{y_1\}$ such that:

- $x_s'y_1' \in E(H_4)$.
- The predecessor $x_1'$ of $y_1'$ on $P$ lies in $U_{1,w}$.
- There is an edge $x_1'y_1''$ in $H_{5,i}$ such that $y_1'' \in U_{1,w}^+$ and $y_1'$ precedes $y_1''$ on $P$. 
- The predecessor $x_1''$ of $y_1''$ on $P$ lies in $U_{1,w}$.
- $x_1''$ has at least $\gamma_53m'/3$ outneighbours in $H_{5,i}$ which lie in $U_{1,w}^+$.

**Proof.** The proof is almost identical to that of Claim 25 except that we apply $(\delta_2)$ to ensure that $x_1''$ has at least $\gamma_53m'/3$ outneighbours in $H_{5,i}$ which lie in $U_{1,w}^+$.

Let $x_1', y_1', x_1''$ and $y_1''$ be as in Claim 26. We modify $P$ by replacing it with the path 

$$(P \cup \{x_s'y_1', x_1'y_1''\}) \backslash \{x_1'y_1', x_1''y_1''\}$$

from $y_1$ to $x_1''$. So $P$ is a Hamilton path in $G$ which is edge-disjoint from the $\ell$ Hamilton cycles $C_{i', j'}$ already defined. In each of the $s$ steps in our construction of $P$ we have added at most one edge from each of $H_4$ and $H_{5,i}$. So by (30) $P$ contains at most $3L^2$ edges from $H_4$ and at most $3L^2$ edges from $H_{5,i}$. All other edges of

**Figure 2.** The modified path $P$ in Case 2
$P$ lie in $F'_{i,j}$. Recall that $y_1$ has at least $\gamma_5 \beta m'/3$ inneighbours in $H_{5,i}$ which lie in $U_{1,w}$ and $x_1''$ has at least $\gamma_5 \beta m'/3$ outneighbours in $H_{5,i}$ which lie in $U_{1,w}^+$. Thus we can apply Lemma 20 to $P \cup H_{5,i}$ with $U_{1,w}^+$ playing the role of $V$ and $U_1$ playing the role of $U$ to obtain a Hamilton cycle $C_{i,j}$ in $G$ where $|E(C_{i,j}) \setminus E(P)| \leq 5$. By construction, $C_{i,j}$ satisfies (i)–(iv). Thus we can indeed find $(1/2 - \eta_1)n$ Hamilton cycles in $G$, as desired.

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References

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