# FRACTIONAL CLIQUE DECOMPOSITIONS OF DENSE GRAPHS AND HYPERGRAPHS

BEN BARBER, DANIELA KÜHN, ALLAN LO, RICHARD MONTGOMERY AND DERYK OSTHUS

Abstract. Our main result is that every graph $G$ on $n \geq 10^4 r^3$ vertices with minimum degree $\delta(G) \geq (1-1/10^4 r^{3/2})n$ has a fractional $K_r$-decomposition. Combining this result with recent work of Barber, Kühn, Lo and Osthus leads to the best known minimum degree thresholds for exact (non-fractional) $F$-decompositions for a wide class of graphs $F$ (including large cliques). For general $k$-uniform hypergraphs, we give a short argument which shows that there exists a constant $c_k > 0$ such that every $k$-uniform hypergraph $G$ on $n$ vertices with minimum codegree at least $(1 - c_k/r^{2k-1})n$ has a fractional $K_r^{(k)}$-decomposition, where $K_r^{(k)}$ is the complete $k$-uniform hypergraph on $r$ vertices. (Related fractional decomposition results for triangles have been obtained by Dross and for hypergraph cliques by Dukes as well as Yuster.) All the above new results involve purely combinatorial arguments. In particular, this yields a combinatorial proof of Wilson's theorem that every large $F$-divisible complete graph has an $F$-decomposition.

## 1. Introduction and results

### 1.1. (Fractional) decompositions of graphs.

We say that a $k$-uniform hypergraph $G$ has an *$F$-decomposition* if its edge set $E(G)$ can be partitioned into copies of $F$. A natural relaxation is that of a fractional decomposition. To define this, let $\mathcal{F}(G)$ be the set of copies of $F$ in $G$. A *fractional $F$-decomposition* is a function $\omega : \mathcal{F}(G) \to [0,1]$ such that, for each $e \in E(G)$,

$$\sum_{F \in \mathcal{F}(G)\colon e \in E(F)} \omega(F) = 1. \tag{1.1}$$

Note that every $F$-decomposition is a fractional $F$-decomposition where $\omega(F) \in \{0,1\}$. As a partial converse, Haxell and Rödl [8] used Szemerédi's regularity lemma to show that the existence of a fractional $F$-decomposition of a graph $G$ implies the existence of an approximate $F$-decomposition of $G$, i.e. a set of edge-disjoint copies of $F$ in $G$ which cover almost all edges of $G$ (their main result is more general than this). Rödl, Schacht, Siggers and Tokushige [12] later generalised this result to $k$-uniform hypergraphs.

The study of $F$-decompositions of cliques is central to design theory and has a long and rich history. In 1847, Kirkman [10] showed that $K_n$ has a $K_3$-decomposition if and only if $n \equiv 1, 3 \mod 6$. More generally, we say that a graph $G$ is *$F$-divisible* if $e(F)$ divides $e(G)$ and the greatest common divisor of the degrees of $F$ divides the degree of every vertex of $G$. If $G$ has an $F$-decomposition then it is certainly $F$-divisible. Wilson [13, 14, 15, 16] proved that if $G$ is a large complete graph, then this necessary condition is also sufficient.

---

*Date*: July 17, 2015.

The research leading to these results was partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreement n. 258345 (B. Barber, D. Kühn and R. Montgomery) and 306349 (D. Osthus). The research was also partially supported by the EPSRC, grant no. EP/M009408/1 (D. Kühn and D. Osthus).

For a given graph $F$, it is probably not possible to find a satisfactory characterization of all graphs $G$ which have an $F$-decomposition. This is supported by the fact that Dor and Tarsi [2] proved that determining whether a graph $G$ has an $F$-decomposition is NP-complete if $F$ has a connected component with at least 3 edges. However, it is natural to ask whether one can extend Kirkman's result and Wilson's theorem to all dense graphs. In particular, Nash-Williams made the following conjecture on triangle decompositions.

**Conjecture 1.1** (Nash-Williams [11]). *There exists $N \in \mathbb{N}$ so that for all $n \geq N$, if $G$ is a $K_3$-divisible graph on $n$ vertices and $\delta(G) \geq 3n/4$, then $G$ has a $K_3$-decomposition.*

There has been considerable recent progress towards this conjecture. The first result towards the conjecture was obtained by Gustavsson [7] who showed that, for every fixed graph $F$, there exists $\varepsilon = \varepsilon(F) > 0$ and $n_0 = n_0(F)$ such that every $F$-divisible graph $G$ on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (1-\varepsilon)n$ has an $F$-decomposition. The bound on $\varepsilon(F)$ claimed by Gustavsson is around $10^{-37}|F|^{-94}$.

Recently, Barber, Kühn, Lo and Osthus [1] significantly improved the bound on $\varepsilon(F)$ by establishing a connection to fractional decompositions. For a graph $F$ and $n \in \mathbb{N}$, let $\delta^*_F(n)$ be the infimum over all $\delta$ such that every graph $G$ on $n$ vertices with $\delta(G) \geq \delta n$ has a fractional $F$-decomposition. We call $\delta^*_F := \limsup_{n\to\infty} \delta^*_F(n)$ the *fractional $F$-decomposition threshold*. The main results in [1] imply the following.

**Theorem 1.2** (Barber, Kühn, Lo and Osthus [1]). *Let $F$ be a graph, let $\varepsilon > 0$ and let $n$ be sufficiently large. Let $G$ be an $F$-divisible graph on $n$ vertices and suppose that at least one of the following holds.*

(i) *$\delta(G) \geq (\delta+\varepsilon)n$, where $\delta := \max\{\delta^*_{K_{\chi(F)}}, 1 - 1/6e(F)\}$.*

(ii) *$F$ is $d$-regular and $\delta(G) \geq (\delta+\varepsilon)n$, where $\delta := \max\{\delta^*_{K_{\chi(F)}}, 1 - 1/3d\}$.*

(iii) *$F = C_\ell$, where $\ell \geq 3$ is odd, and $\delta(G) \geq (\delta^*_{C_\ell} + \varepsilon)n$.*

*Then $G$ has an $F$-decomposition.*

Furthermore, asymptotically optimal results for even cycles have been obtained in [1] and for all bipartite graphs with a leaf by Yuster [17]. Note that by Theorem 1.2(iii) it suffices to show that $\delta^*_{K_3} \leq 3/4$ in order to prove Conjecture 1.1 asymptotically. Determining $\delta^*_{K_r}$ is therefore an important problem, as well as being interesting in its own right. The best current result towards the triangle case is due to Dross [3], who gave a very short and elegant argument showing that $\delta^*_{K_3} \leq 0.913$. This improves previous bounds of Yuster [18] Dukes [4, 5] and Garaschuk [6]. For $r \geq 4$, Yuster [18] proved that $\delta^*_{K_r} \leq 1 - 1/9r^{10}$; this was subsequently improved by Dukes [4, 5] who showed that $\delta^*_{K_r} \leq 1 - 2/9r^2(r-1)^2$. On the other hand, a construction showing $\delta^*_{K_r} \geq 1 - 1/(r+1)$ is described in [18]. Our main result gets substantially closer to this lower bound for large $r$.

**Theorem 1.3.** *The following holds for any integers $r \geq 3$ and $n \geq 10^4 r^3$. If $G$ is a graph on $n$ vertices and $\delta(G) \geq (1 - 1/10^4 r^{3/2})n$, then $G$ has a fractional $K_r$-decomposition.*

In order to clarify the presentation, we have made no attempt to optimise the constant $10^4$ appearing in Theorem 1.3. Along the way, we also obtain a comparatively short and simple proof that $\delta(G) \geq (1 - 1/10^5 r^2)n$ guarantees a fractional $K_r$-decomposition (see Theorem 6.1).

Together with Theorem 1.2, we immediately obtain the following corollary. Note that (iii) is a special case of (ii).

**Corollary 1.4.** *Let $F$ be a graph, let $\varepsilon > 0$ and let $n$ be sufficiently large. Let $G$ be an $F$-divisible graph on $n$ vertices such that at least one of the following holds.*

(i) $\delta(G) \geq (1 - 1/10^4|F|^2)n$.
(ii) *$F$ is $d$-regular and* $\delta(G) \geq (1 - 1/10^4(d+1)^{3/2} + \varepsilon)n$.
(iii) $F = K_r$ *and* $\delta(G) \geq (1 - 1/10^4 r^{3/2} + \varepsilon)n$.

*Then $G$ has an $F$-decomposition.*

An obvious open problem is to improve the bounds in Theorem 1.3 (and thus in Corollary 1.4). Furthermore, in view e.g. of Theorem 1.2(iii) it would also be very interesting to obtain better bounds on the fractional decomposition threshold for odd cycles.

1.2. **(Fractional) decompositions of hypergraphs.** Our methods also extend to $k$-uniform hypergraphs with $k \geq 3$. For a $k$-uniform hypergraph $G$, the *minimum codegree* $\delta_{k-1}(G)$ of $G$ is the minimum over all $(k-1)$-subsets $S$ of $V(G)$ of the number of edges containing all the vertices in $S$. For a $k$-uniform hypergraph $F$ and $n \in \mathbb{N}$, let $\delta^*_F(n)$ be the infimum over all $\delta$ such that every $k$-uniform hypergraph $G$ on $n$ vertices with $\delta_{k-1}(G) \geq \delta n$ has a fractional $F$-decomposition. We again call $\delta^*_F := \limsup_{n\to\infty} \delta^*_F(n)$ the *fractional $F$-decomposition threshold*. For $r \geq k \geq 2$, let $K_r^{(k)}$ denote the complete $k$-uniform hypergraph on $r$ vertices. For $r \geq k \geq 2$, Yuster [19] proved that $\delta^*_{K_r^{(k)}} \leq 1 - 1/6^{kr}$. Dukes [4, 5] improved this to $\delta^*_{K_r^{(k)}} \leq 1 - 1/(2 \cdot 3^k \binom{r}{k}^2)$. We give a short combinatorial proof for a similar bound (which is slightly better when $r$ is large).

**Theorem 1.5.** *Given $r, k \in \mathbb{N}$ with $r > k \geq 2$, let $\delta := \frac{k!}{2^{k+3}k^2 r^{2k-1}}$ and let $n > 1/\delta$. Then any $k$-uniform hypergraph $G$ on $n$ vertices with $\delta_{k-1}(G) \geq (1-\delta)n$ has a fractional $K_r^{(k)}$-decomposition.*

Note that for graphs, Theorem 1.5 gives weaker bounds than those discussed in the previous subsection.

In a recent breakthrough, Keevash [9] proved that every sufficiently large $K_n^{(k)}$ satisfying the necessary divisibility conditions has a $K_r^{(k)}$-decomposition. This settled a question regarding the existence of designs going back to the 19th century. Moreover, his results also extend to hypergraphs with minimum codegree at least $(1-\varepsilon)n$, for an unspecified $\varepsilon > 0$. Theorem 1.5 may help to obtain explicit bounds on $\varepsilon$.

1.3. **Proof idea and organization of the paper.** The proof by Dukes [4, 5] that $\delta^*_{K_r} \leq 1 - \Omega(1/r^4)$ is based on tools from linear algebra. To prove Theorem 1.3 we build on the combinatorial approach of Dross [3]. The latter argument begins with a uniform weighting of the triangles in a graph $G$ with high minimum degree (this idea is actually already implicit in [4]). This uniform weighting can be shown to be 'close' to a fractional triangle decomposition of $G$. Then the idea is to use the max-flow min-cut theorem to make the necessary adjustments to this weighting to obtain a fractional triangle decomposition. Our methods begin with a similar initial weighting, but avoid using the max-flow min-cut theorem. Theorem 1.5 is obtained by generalising (simplified versions of) these methods to hypergraphs; we give a more detailed sketch in Section 2. We then prove Theorem 1.5 in Section 3, before proving Theorem 1.3 in Sections 4, 5 and 7–9. In Section 6 we combine the results of Sections 4 and 5 to give a short proof of Theorem 6.1, a weaker form of Theorem 1.3 with $r^2$ in place of $r^{3/2}$.

Our argument here and that in [1] is purely combinatorial. So the proofs of Theorem 1.3 and Theorem 1.2 together yield a combinatorial proof of Wilson's theorem [13, 14, 15, 16] that every large $F$-divisible clique has an $F$-decomposition. (The original proof as well as that by Keevash [9] made use of algebraic tools.)

1.4. **Notation.** Given $k \geq 2$, a *$k$-uniform hypergraph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite set (the vertex set) and $E(G)$ is a set of $k$-element subsets of $V(G)$ (the edge set). Given a $k$-uniform hypergraph $G$ and $S \subseteq V(G)$ with $|S| \leq k-1$, we let $N(S) := \{T \subseteq V(G) \setminus S : T \cup S \in E(G)\}$ and write $d(S) := |N(S)|$. We let $N^c(S) := \{T \subseteq V(G) : |T| = k - |S|, T \cup S \notin E(G)\}$. For $1 \leq j \leq k-1$, we write $\delta_j(G) := \min\{d(S) : S \subseteq V(G), |S| = j\}$ for the *minimum $j$-degree* ($\delta_{k-1}(G)$ is also known as the *minimum codegree*).

Given $r \geq k \geq 2$, we write $\mathcal{K}_r^{(k)}(G)$ for the set of copies of $K_r^{(k)}$ in $G$. If $G$ is clear from the context, we just write $\mathcal{K}_r^{(k)}$; if $k = 2$, then we just write $\mathcal{K}_r$. We write $k_r = k_r(G) := |\mathcal{K}_r^{(k)}(G)|$ for the number of $r$-cliques in $G$. (For $r < 0$, we let $k_r := 0$.) For each $S \subseteq V(G)$ and $r \in \mathbb{N}$, let $\kappa_S^{(r)} := |\{K \in \mathcal{K}_r : S \subseteq V(K)\}|$. For an edge $e$, we often write $\kappa_e^{(r)}$ for $\kappa_{V(e)}^{(r)}$. For $r, k \in \mathbb{N}$, we write $(r)_k := r(r-1)\cdots(r-k+1)$ for the $k$th falling factorial of $r$.

For a graph $G$ and $x \in V(G)$, we write $N(x) := \{y \in V(G) : xy \in E(G)\}$ for the neighbourhood of $x$ and $d(x) := |N(x)|$ for the degree of $x$. We let $N^c(x) = \{y \in V(G) : xy \notin E(G)\}$ (note that this includes $x$ itself). For $S \subseteq V(G)$, we write $G[S]$ for the subgraph of $G$ induced by $S$, and abbreviate $\mathcal{K}_r(G[S])$ by $\mathcal{K}_r[S]$. Given any event $A$, we let

$$\mathbf{1}_A := \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

By a *weighting* of the $r$-cliques in $G$ we mean a function $\omega : \mathcal{K}_r^{(k)} \to \mathbb{R}$. The *weight* of a clique $K$ is $\omega(K)$. For $e \in E(G)$, the *weight over $e$* is $\sum_{K \in \mathcal{K}_r(G) : e \in E(K)} \omega(K)$.

## 2. Sketch of proof

Here we present a sketch proof of Theorem 1.5, which will also form the backbone of the proof of Theorem 1.3. For simplicity, we describe the argument for graphs (which generalises straightforwardly to the hypergraph case).

As $\delta(G)$ is large, for each $e \in E(G)$, $G$ has many $r$-cliques containing $e$. In fact, all edges $e$ are contained in approximately the same number of $r$-cliques. More precisely, there is some small $\alpha > 0$ such that $(1-\alpha)k_{r-2} \leq \kappa_e^{(r)} \leq k_{r-2}$ for any $e \in E(G)$ (see Proposition 3.2). An appropriately scaled uniform weighting of the $r$-cliques of $G$ is therefore already close to a fractional decomposition of $G$, in the sense that the total weight over each edge is close to 1. We seek to perturb the weight of each $r$-clique so that the total weight over each edge becomes exactly 1.

For each $e \in E(G)$, we consider an 'edge-gadget' $\psi_e$ that permits us to alter the weight over $e$ without altering the weight over any other edge. This edge-gadget adds weight to some $r$-cliques and removes weight from other $r$-cliques so that the change in weight cancels out over every edge except for $e$. Formally speaking, an *edge-gadget* for the edge $e$ is a weighting $\psi_e : \mathcal{K}_r \to \mathbb{R}$ such that, for each $f \in E(G)$,

$$\sum_{K \in \mathcal{K}_r :\, f \in E(K)} \psi_e(K) = \mathbf{1}_{\{e=f\}}.$$

For $c \in \mathbb{R}$, the function $c \cdot \psi_e$ corresponds to adding weight $c$ over $e$. Our aim is to use these edge-gadgets $\psi_e$ (for $e \in E(G)$) to correct the weights over the edges without reducing the weight of any one clique so far that it becomes negative.

We construct a basic edge-gadget $\psi_e$ as follows. Let $J$ be an $(r+2)$-clique of $G$ that contains $e$ (which exists since the minimum degree is large). There are three types of edges in $J$, determined by how many vertices they share with $e$. Accordingly, write $E_j := \{f \in E(J) : |V(f) \cap V(e)| = j\}$. (Note that $E_2 = \{e\}$.) Similarly, there are three types of cliques in $\mathcal{K}_r(J)$, determined by how many vertices they share with $e$. Accordingly, write $S_j := \{K \in \mathcal{K}_r(J) : |V(K) \cap V(e)| = j\}$. We first increase the weight of every $r$-clique in $S_2$ by $1/|S_2| = 1/\binom{r}{2}$. This has the effect of increasing the weight over every edge of $J$, and this increase only depends on whether the edge is in $E_2$, $E_1$ or $E_0$. The weight over $e$ is now 1 as desired, but the weight over the edges in $E_1$ and $E_0$ is also positive. We now correct the weight over every edge in $E_1$ by reducing the weight of every clique in $S_1$ by the same amount. The weight over each edge in $E_2 \cup E_1$ is now as desired, so it remains only to correct the weights of each edge in $E_0$. But the edges in $E_0$ form a clique $K^*$, and the weight over every edge in $E_0$ is identical, so it can be made equal to zero by adjusting the weight of $K^*$. This completes the construction of $\psi_e$.

If we use the basic edge-gadget $\psi_e$ as described above to adjust the weight over each edge $e$, then we might have to make large adjustments to the weights of some cliques—large enough that these weights would become negative and prevent us obtaining a fractional decomposition. To avoid making too large an adjustment to the weight of any $r$-clique, we will therefore, for each edge $e$, use many different edge-gadgets $\psi_e$ to correct the weight over $e$, making a small adjustment using each $\psi_e$ and spreading the adjustments over as many $r$-cliques as possible. To be precise, note that in the previous paragraph we have actually defined an edge-gadget, $\psi_e^J$ say, for each $(r+2)$-clique $J$ containing $e$, and there are $\kappa_e^{(r+2)}$ such cliques $J$. So we set $\psi_e$ to be the average over of the edge-gadgets $\psi_e^J$, that is, $\psi_e := \sum_{J \in \mathcal{K}_{r+2} : e \in E(J)} \psi_e^J / \kappa_e^{(r+2)}$.

This simple argument can already be used to find fractional $K_r$-decompositions of graphs on $n$ vertices with minimum degree at least $(1 - c/r^3)n$ for some absolute constant $c$. The argument generalises straightforwardly to hypergraphs, and we use it to prove Theorem 1.5 in Section 3.

In order to prove Theorem 1.3, we introduce two additional ideas. Firstly we introduce an additional preprocessing step which allows us to limit the adjustments we need to make to the weight over most of the edges. This leaves us most concerned with the problem of correcting the weight over a small fraction of 'bad' edges. The naive averaging argument would then ask for a large adjustment to the weight of cliques that contain many bad edges. But the proportion of such gadgets which use many bad edges is small. Hence we can avoid using these gadgets and thus reduce the maximum adjustment that might be required for each $r$-clique. This allows us to obtain fractional $K_r$-decompositions provided that $\delta(G) \geq (1 - c/r^2)n$ for some absolute constant $c$. We prove this in Sections 4–6.

Secondly, we introduce a 'vertex-gadget' that allows us to increase the weight over every edge at a vertex by the same amount simultaneously (see Section 7). In return for this reduction in flexibility we are able to make these adjustments more efficiently, with smaller changes to the weights of cliques. By further analysing the pattern of changes required to the weights over the edges in Section 8, we use this vertex-gadget to make an initial adjustment before using edge-gadgets to make the final adjustment. We put together these ideas and results from Sections 4, 5, 7 and 8 to prove Theorem 1.3 in Section 9.

## 3. Fractional decompositions of hypergraphs

3.1. **Basic tools.** We first observe that if a $k$-uniform hypergraph has large minimum codegree, then for all $\ell < k$ its minimum $\ell$-degree is also large.

**Proposition 3.1.** *Let $k \in \mathbb{N}$ with $k \geq 2$, let $0 < \delta < 1$ and let $G$ be a $k$-uniform hypergraph on $n$ vertices. Suppose that $\delta_{k-1}(G) \geq (1-\delta)n$. Then for every $\ell \leq k-1$, $\delta_\ell(G) \geq (1-\delta)\binom{n-\ell}{k-\ell}$.*

*Proof.* Choose $S \subseteq V(G)$ with $|S| = \ell$ such that $d(S) = \delta_\ell(G)$. Let $\mathcal{T} := \{(T, e) : S \subseteq T \subseteq V(e), e \in E(G), |T| = k-1\}$. Then

$$\binom{n-\ell}{k-\ell-1} \cdot \delta_{k-1}(G) \leq |\mathcal{T}| = d(S) \cdot (k-\ell),$$

hence

$$\delta_\ell(G) = d(S) \geq \frac{(1-\delta)n}{k-\ell}\binom{n-\ell}{k-\ell-1} \geq (1-\delta)\binom{n-\ell}{k-\ell}. \qquad \square$$

We shall use the following bounds on the number of $r$-cliques and the number of $r$-cliques containing a fixed edge.

**Proposition 3.2.** *Let $n > r > k \geq 2$, let $1/n < \delta < 1$ and let $G$ be a $k$-uniform hypergraph on $n$ vertices with $\delta_{k-1}(G) \geq (1-\delta)n$. Then*

$$\left(1 - \binom{r}{k}\delta\right)\binom{n}{r} \leq k_r \leq \binom{n}{r} \leq \frac{n^r}{r!} \tag{3.1}$$

*and, for any $e \in E(G)$,*

$$k_{r-k} - \frac{2k\delta n^{r-k}}{(r-k)!}\binom{r}{k-1} \leq \kappa_e^{(r)} \leq k_{r-k}. \tag{3.2}$$

*Proof.* We first prove (3.1). The upper bound is clear. To see the lower bound, consider constructing a clique one vertex at a time. Since each new vertex must form an edge with all $(k-1)$-subsets of the previously chosen vertices, the number of $r$-cliques is at least

$$\begin{aligned}(n)_{k-1} \cdot (1-\delta)n \cdot (1-k\delta)n \cdot (1 - \tbinom{k+1}{k-1}\delta)n \cdots (1 - \tbinom{r-1}{k-1}\delta)n/r!\\ \geq (1 - \sum_{s=k}^{r} \tbinom{s-1}{k-1}\delta)(n)_r/r! = (1 - \tbinom{r}{k}\delta)\tbinom{n}{r}.\end{aligned}$$

We now verify (3.2). We have that $\kappa_e^{(r)} = k_{r-k} - g(e)$, where $g(e)$ is the number of $K \in \mathcal{K}_{r-k}^{(k)}$ such that $V(e) \cup V(K)$ does not induce an $r$-clique in $G$. This happens when either $V(e) \cap V(K) \neq \emptyset$, or when there is a non-edge $f$ of $G$ contained in $V(e) \cup V(K)$. The number of $K \in \mathcal{K}_{r-k}^{(k)}$ with $V(e) \cap V(K) \neq \emptyset$ is at most $k \cdot k_{r-k-1}$. And for a fixed non-edge $f$ of $G$, the number of $K \in \mathcal{K}_{r-k}^{(k)}$ such that $V(e) \cap V(K) = \emptyset$ and $V(f) \subseteq V(e) \cup V(K)$ is at most $k_{r-k-|V(f)\setminus V(e)|}$ (which is 0 if $r < k + |V(f) \setminus V(e)|$). Thus

$$\begin{aligned} g(e) &\le k \cdot k_{r-k-1} + \sum_{j=1}^{k-1} \sum_{S \subseteq V(e): |S|=k-j} |N^c(S)| k_{r-k-j} \\ &\le \frac{kn^{r-k-1}}{(r-k-1)!} + \sum_{j=1}^{\min\{k-1,r-k\}} \binom{k}{k-j} \cdot \frac{\delta n^j}{j!} \cdot \frac{n^{r-k-j}}{(r-k-j)!} \\ &\le 2\delta n^{r-k} \sum_{j=1}^{\min\{k-1,r-k\}} \frac{\binom{k}{k-j}}{j!(r-k-j)!} = \frac{2\delta n^{r-k}}{(r-k)!} \sum_{j=1}^{k-1} \binom{k}{k-j}\binom{r-k}{j} \\ &\le \frac{2k\delta n^{r-k}}{(r-k)!} \sum_{j=0}^{k-1} \binom{k}{k-1-j}\binom{r-k}{j} = \frac{2k\delta n^{r-k}}{(r-k)!}\binom{r}{k-1}, \end{aligned}$$

where the second inequality uses (3.1) and Proposition 3.1. □

Let $G$ be a $k$-uniform hypergraph. An *edge-weighting* of $G$ is a function $\omega : E(G) \to \mathbb{R}$. For the rest of this section, it will be convenient to view the set of edge-weightings of $G$ as an $e(G)$-dimensional vector space $\Omega(G)$. This space has a natural basis $\{1_e : e \in E(G)\}$, where

$$1_e(f) := \begin{cases} 1 & \text{if } e = f, \\ 0 & \text{otherwise.} \end{cases}$$

We shall identify $1_e$ with $e$ itself, and sums of edges with the corresponding subgraphs of $G$; thus we write $H = \sum_{e\in E(H)} e$ for every subgraph $H$ of $G$. Let $\Omega_r(G) := \{\sum_{K\in\mathcal{K}_r^{(k)}} \omega(K)K : \omega(K) \in \mathbb{R}\}$ be the subspace of $\Omega(G)$ spanned by the $r$-cliques of $G$ and let $\Omega_r^+(G) := \{\sum_{K\in\mathcal{K}_r^{(k)}} \omega(K)K : \omega(K) \ge 0\}$. We claim that if $G \in \Omega_r^+(G)$, then $G$ has a *fractional* $K_r^{(k)}$*-decomposition*. Indeed, observe that, if $G \in \Omega_r^+(G)$, then there is an $\omega : \mathcal{K}_r^{(k)} \to \mathbb{R}_{\ge 0}$ such that

$$\sum_{e\in E(G)} e = G = \sum_{K\in\mathcal{K}_r^{(k)}} \omega(K)K = \sum_{K\in\mathcal{K}_r^{(k)}} \omega(K) \sum_{e\in E(K)} e = \sum_{e\in E(G)} \Big( \sum_{K\in\mathcal{K}_r^{(k)}: e\in E(K)} \omega(K) \Big) e;$$

that is, the weight over each edge is exactly 1. Moreover, since no weight is negative it must be the case that $\omega$ is a function from $\mathcal{K}_r^{(k)}(G)$ to $[0, 1]$.

3.2. **Adding weight over an edge.** We now describe the basic edge-gadget that allows us to increase or decrease weight over a single edge by adjusting the weights of a suitable set of $r$-cliques.

**Proposition 3.3.** *Let $r > k$. There are $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ so that the following holds. Let $J$ be a copy of $K_{k+r}^{(k)}$ and let $e \in E(J)$. Then the weighting $\omega : \mathcal{K}_r^{(k)}(J) \to \mathbb{R}$ defined by $\omega(K) := \alpha_{|V(e)\cap V(K)|}$ satisfies*

(i) $e = \sum_{K\in\mathcal{K}_r^{(k)}(J)} \omega(K)K$,

(ii) *if $|V(e) \cap V(K)| = i$, then $|\omega(K)| \le \frac{2^{k-i}(k-i)!}{\binom{r-k+i}{i}}$.*

As discussed in Section 2, the idea of the proof is that to increase the weight over the edge $e$, we first increase the weight of every $r$-clique containing $e$. This puts too much weight over the edges that share $k-1$ vertices with $e$, so we remove this weight by decreasing the weight on each $r$-clique that shares $k-1$ vertices with $e$. Continuing in this fashion we eventually obtain a (signed) weighting of the $r$-cliques of $J$ such that the net weight over each edge $f$ is non-zero if and only if $e = f$.

*Proof of Proposition 3.3.* Let

$$\Omega^e(J) := \{ \sum_{f \in E(J)} \omega_f f : \omega_{f_1} = \omega_{f_2} \text{ if } |V(e) \cap V(f_1)| = |V(e) \cap V(f_2)| \}$$

be the $(k+1)$-dimensional subspace of $\Omega(J)$ in which the weight of each edge depends only on the size of its intersection with $e$. For $0 \le i \le k$, let $E_i := \sum_{f \in E(J):|V(e)\cap V(f)|=i} f$. The $E_i$ are a natural basis for $\Omega^e(J)$. For $0 \le j \le k$, let $F_j := \sum_{K \in \mathcal{K}_r^{(k)}(J):|V(e)\cap V(K)|=j} K$. We claim that the $F_j$ also form a basis for $\Omega^e(J)$. To see this, we first calculate $F_0, \dots, F_k$ in terms of $E_0, \dots, E_k$. Write $F_j = \sum_{i=0}^k a_{ij} E_i$. Then $a_{ij}$ is the number of ways to extend an edge meeting $e$ in $i$ vertices to an $r$-clique meeting $e$ in $j$ vertices. Explicitly,

$$a_{ij} = \binom{k-i}{j-i}\binom{r-(k-i)}{r-j-(k-i)} = \binom{k-i}{j-i}\binom{r-k+i}{j}. \tag{3.3}$$

In particular, $a_{ij} = 0$ for $i > j$ and $a_{ij} \neq 0$ when $i = j$, so the matrix $(a_{ij})$ is upper triangular with non-zero diagonal entries and hence is invertible. Thus there exist $\alpha_0, \dots, \alpha_k$ such that

$$e = E_k = \sum_{j=0}^k \alpha_j F_j \tag{3.4}$$

$$= \sum_{j=0}^k \alpha_j \sum_{i=0}^k a_{ij} E_i = \sum_{i=0}^k \big(\sum_{j=0}^k a_{ij}\alpha_j\big) E_i = \sum_{i=0}^k \big(\sum_{j=i}^k a_{ij}\alpha_j\big) E_i. \tag{3.5}$$

Set $\omega(K) = \alpha_{|V(e)\cap V(K)|}$. Then (3.4) proves (i). To see (ii) first note that, by (3.3) and (3.5), $\alpha_k = 1/a_{kk} = 1/\binom{r}{k}$, and, for $0 \le i < k$,

$$\sum_{j=i}^k \binom{k-i}{j-i}\binom{r-k+i}{j}\alpha_j = \sum_{j=i}^k a_{ij}\alpha_j = 0. \tag{3.6}$$

We shall prove by induction on $k - i$ that $|\alpha_i| \le 2^{k-i}(k-i)!/\binom{r-k+i}{i}$. This holds with equality for $\alpha_k$, so assume that $0 \le i < k$. Then by induction,

$$|\alpha_i| \overset{(3.6)}{\le} \sum_{j=i+1}^k \frac{\binom{k-i}{j-i}\binom{r-k+i}{j}}{\binom{r-k+i}{i}}|\alpha_j| \le \sum_{j=i+1}^k \frac{\binom{k-i}{j-i}\binom{r-k+i}{j}}{\binom{r-k+i}{i}} \frac{2^{k-j}(k-j)!}{\binom{r-k+j}{j}}$$

$$\le \frac{2^{k-i}(k-i)!}{\binom{r-k+i}{i}} \sum_{j=i+1}^k \frac{1}{2^{j-i}(j-i)!} \le \frac{2^{k-i}(k-i)!}{\binom{r-k+i}{i}},$$

as required. □

3.3. **Proof of Theorem 1.5.** We are now ready to put everything together to prove Theorem 1.5.

*Proof of Theorem 1.5.* Let $\kappa := \sum_{e \in E(G)} \kappa_e^{(r)}/e(G)$ be the average value of $\kappa_e^{(r)}$, and let $w := 1/\kappa$. By (3.2) of Proposition 3.2,

$$|\kappa - \kappa_e^{(r)}| \le \frac{2k\delta n^{r-k}}{(r-k)!}\binom{r}{k-1}. \tag{3.7}$$

Observe also that (in $\Omega(G)$)

$$\sum_{K\in\mathcal{K}_r^{(k)}(G)} K = \sum_{e\in E(G)} \kappa_e^{(r)} e. \tag{3.8}$$

By Proposition 3.3, for every $K \in \mathcal{K}_r^{(k)}(G)$ and every $e \in E(G)$, there exists $\omega_K^e$ with

$$|\omega_K^e| \leq \frac{2^{k-j}(k-j)!}{\binom{r-k+j}{j}}, \quad \text{where } j = |V(e)\cap V(K)|, \tag{3.9}$$

and such that for every $J \in \mathcal{K}_{r+k}^{(k)}(G)$ with $e \in E(J)$,

$$e = \sum_{K\in\mathcal{K}_r^{(k)}(J)} \omega_K^e K. \tag{3.10}$$

Thus

$$\begin{aligned}
G &= \sum_{e\in E(G)} \kappa w e = \sum_{e\in E(G)} (\kappa_e^{(r)} w e + (\kappa - \kappa_e^{(r)}) w e)\\
&\overset{(3.8),(3.10)}{=} \sum_{K\in\mathcal{K}_r^{(k)}(G)} wK + \sum_{e\in E(G)} \frac{(\kappa-\kappa_e^{(r)})w}{\kappa_e^{(r+k)}} \sum_{(J\in\mathcal{K}_{r+k}^{(k)}(G): e\in E(J))} \sum_{K\in\mathcal{K}_r^{(k)}(J)} \omega_K^e K\\
&= \sum_{K\in\mathcal{K}_r^{(k)}(G)} \Bigg(w + \sum_{(J\in\mathcal{K}_{r+k}^{(k)}(G): K\subseteq J)} \sum_{e\in E(J)} \frac{\omega_K^e(\kappa-\kappa_e^{(r)})w}{\kappa_e^{(r+k)}}\Bigg) K,
\end{aligned}$$

and it suffices to show that $w + \sum_{J\in\mathcal{K}_{r+k}^{(k)}(G):K\subseteq J} \sum_{e\in E(J)} \frac{\omega_K^e(\kappa-\kappa_e^{(r)})w}{\kappa_e^{(r+k)}} \geq 0$ for every $K \in \mathcal{K}_r^{(k)}(G)$. (Indeed, then $G \in \Omega_r^+(G)$ and so $G$ has a fractional $K_r^{(k)}$-decomposition by our remarks at the end of Section 3.1.) So fix $K \in \mathcal{K}_r^{(k)}(G)$, let $J \in \mathcal{K}_{r+k}^{(k)}(G)$ with $K \subseteq J$ and let $e \in E(J)$. By Proposition 3.2,

$$\kappa_e^{(r+k)} \geq (1 - \tbinom{r}{k}\delta)(n)_r/r! - \tfrac{2k\delta n^r}{r!}\tbinom{r+k}{k-1} \geq \tfrac{n^r}{2r!}. \tag{3.11}$$

Now by (3.7), (3.9) and (3.11), if $j = |V(e)\cap V(K)|$ then

$$\left|\frac{\omega_K^e(\kappa-\kappa_e^{(r)})}{\kappa_e^{(r+k)}}\right| \leq \frac{\frac{2^{k-j}(k-j)!}{\binom{r-k+j}{j}}\cdot\frac{2k\delta n^{r-k}}{(r-k)!}\binom{r}{k-1}}{n^r/2r!} \leq \frac{2^{k-j+2}k^2r^{2k-j-1}\delta}{\binom{k}{j}n^k},$$

hence

$$\begin{aligned}
\left|\sum_{(J\in\mathcal{K}_{r+k}^{(k)}(G):K\subseteq J)} \sum_{e\in E(J)} \frac{\omega_K^e(\kappa-\kappa_e^{(r)})w}{\kappa_e^{(r+k)}}\right| &\leq \frac{n^k}{k!}\sum_{j=0}^{k}\binom{k}{k-j}\binom{r}{j}\frac{2^{k-j+2}k^2r^{2k-j-1}\delta}{\binom{k}{j}n^k}\cdot w\\
&\leq \frac{2^{k+2}k^2r^{2k-1}\delta w}{k!}\sum_{j=0}^{k}\frac{1}{2^j j!} \leq w,
\end{aligned}$$

as required. □

In some cases it is possible to sharpen the computations in the proof of Theorem 1.5 to lower the minimum codegree that guarantees the existence of a fractional $K_r^{(k)}$-decomposition. Of particular interest is the case where $r = k+1$. In this case, equation (3.6) can be solved exactly to obtain $\alpha_j = (-1)^{k-j}/(k+1)\binom{k}{j}$. Redoing the computation above with these correct values for $\omega_K^e$ shows that a minimum codegree of $(1 - 1/k^2(k+1)2^{2k+1})n$ already guarantees a fractional $K_{k+1}^{(k)}$-decomposition, a substantial improvement over substituting $r = k+1$ into our general result.

## 4. Bounds on the number of cliques

We now turn to the special case of graphs. As for the more general case of hypergraphs, we shall be interested in the number $\kappa_e^{(r)}$ of $r$-cliques containing an edge $e$. In this section we first prove the following proposition relating the number of cliques of different sizes in a graph with high minimum degree, which is used repeatedly throughout the paper.

**Proposition 4.1.** *Let $r, n \in \mathbb{N}$, $\delta := 1/2r$, and let $G$ be a graph on $n$ vertices with $\delta(G) \geq (1-\delta)n$. Then, for each $i \in [r]$,*

$$k_{r-i} \leq (2r/n)^i k_r.$$

*Proof.* Let $i \in [r]$. For each clique $K \in \mathcal{K}_{r-i}$, using the minimum degree of $G$, the number of cliques in $\mathcal{K}_r$ containing $K$ is at least

$$\frac{1}{i!}\prod_{j=1}^{i}(n - (r-i+j-1)\delta n) \geq \frac{1}{i!}\left(\frac{n}{2}\right)^i.$$

Each clique $K \in \mathcal{K}_r$ contains $\binom{r}{i}$ cliques in $\mathcal{K}_{r-i}$. Therefore,

$$\frac{1}{i!}\left(\frac{n}{2}\right)^i k_{r-i} \leq \binom{r}{i} k_r \leq \frac{r^i}{i!} k_r,$$

and thus $k_{r-i} \leq (2r/n)^i k_r$. □

Our next lemma gives a range of bounds on the number of cliques containing a fixed smaller clique (and, in particular then, an edge).

**Lemma 4.2.** *Let $r, n \in \mathbb{N}$ and $\delta \leq 1/2r$, and let $G$ be a graph with $n$ vertices and $\delta(G) \geq (1-\delta)n$. Then, for each integer $t < r$ and each subset $Z \subseteq V(G)$, with $|Z| = t$ and $G[Z] \in \mathcal{K}_t$, we have*

(i) $|\kappa_Z^{(r)} - k_{r-t}| \leq 2t\delta r k_{r-t}$, *and*
(ii) $\left|\kappa_Z^{(r)} - k_{r-t} + |\bigcup_{z\in Z} N^c(z)| k_{r-t-1}\right| \leq 6(t\delta r)^2 k_{r-t}$.
(iii) *For each $xy \in E(G)$, we have*

$$\left|\kappa_{xy}^{(r)} - k_{r-2} - \sum_{i=1}^{3}(-1)^i \sum_{Y \subseteq N^c(x)\cup N^c(y): |Y|=i} \kappa_Y^{(r-2)}\right| \leq 11(\delta r)^4 k_{r-2}.$$

*Proof.* Given $Z \subseteq V(G)$ with $|Z| = t$ and $G[Z] \in \mathcal{K}_t$, we can obtain an $r$-clique $K$ containing $Z$ by extending $Z$ by the vertex set of an $(r-t)$-clique which lies in $\bigcap_{z\in Z} N(z)$. By the inclusion-exclusion principle,

$$\begin{aligned}\kappa_Z^{(r)} &= k_{r-t} - \left|\left\{K \in \mathcal{K}_{r-t} : V(K) \cap \bigcup_{z\in Z} N^c(z) \neq \emptyset\right\}\right| \\ &= k_{r-t} + \sum_{i=1}^{r-t}(-1)^i \sum_{Y\subseteq \bigcup_{z\in Z} N^c(z):\, |Y|=i} \kappa_Y^{(r-t)}.\end{aligned}$$

So by the Bonferroni inequalities, for each $\ell \leq r-t+1$,

$$\Big|\kappa_Z^{(r)} - k_{r-t} - \sum_{i=1}^{\ell-1}(-1)^i \sum_{Y\subseteq \bigcup_{z\in Z} N^c(z)\colon |Y|=i} \kappa_Y^{(r-t)}\Big|$$

$$\leq \sum_{Y\subseteq \bigcup_{z\in Z} N^c(z)\colon |Y|=\ell} \kappa_Y^{(r-t)} \leq \binom{t\delta n}{\ell} k_{r-t-\ell} \leq \frac{(t\delta n)^\ell}{\ell!} k_{r-t-\ell} \leq \frac{(2t\delta r)^\ell}{\ell!} k_{r-t}, \quad (4.1)$$

where we have used Proposition 4.1 in the final inequality. As $\ell$ increases, we obtain an increasingly accurate estimate for $\kappa_Z^{(r)}$ (provided $t$ is not too large). In particular, setting $\ell = 1$ we gain (i), and setting $\ell = 4$ in the case where $|Z| = 2$ we gain (iii).

Finally, for (ii), using (i) with clique size $r-t$ and set $\{x\}$ for each $x \in \bigcup_{z\in Z} N^c(z)$, we have

$$\begin{aligned}\Big|\kappa_Z^{(r)} - k_{r-t} + \Big|\bigcup_{z\in Z} N^c(z)\Big| k_{r-t-1}\Big| &\leq \Big|\kappa_Z^{(r)} - k_{r-t} + \sum_{x\in\bigcup_{z\in Z} N^c(z)} \kappa_{\{x\}}^{(r-t)}\Big| \\ &\quad + \sum_{x\in\bigcup_{z\in Z} N^c(z)} |\kappa_{\{x\}}^{(r-t)} - k_{r-t-1}| \\ &\overset{(4.1),(\mathrm{i})}{\leq} 2(t\delta r)^2 k_{r-t} + \Big|\bigcup_{z\in Z} N^c(z)\Big| 2\delta r k_{r-t-1} \\ &\leq 2t^2\delta^2 r^2 k_{r-t} + \delta t n \cdot 4\delta r^2 k_{r-t}/n \leq 6t^2\delta^2 r^2 k_{r-t},\end{aligned}$$

where we have used Proposition 4.1 in the penultimate inequality. □

As noted in Section 2, we shall want to construct edge-gadgets using only some of the $r$-cliques in the graph $G$. We will in fact have a small subset $X \subseteq V(G)$, and wish to avoid using $r$-cliques which have a large intersection with $X$. Our final result of this section demonstrates that there are not many such cliques.

**Proposition 4.3.** *Let $r \geq 3$, $n \in \mathbb{N}$ and $\delta := 1/600r^{3/2}$. Let $G$ be a graph on $n$ vertices with $\delta(G) \geq (1-\delta)n$, and let $X \subseteq V(G)$ with $|X| \leq \delta rn$. Let*

$$\mathcal{A} := \{K \in \mathcal{K}_r : |V(K)\cap X| \geq r^{1/2}\}.$$

*Then $|\mathcal{A}| \leq k_r/r^2$.*

*Proof.* Let $t := \lceil r^{1/2}\rceil$. Using Proposition 4.1, we have that

$$\begin{aligned}|\mathcal{A}| &\leq \sum_{i=t}^{r}\binom{\delta rn}{i} k_{r-i} \leq \sum_{i=t}^{r}\frac{(\delta rn)^i}{i!}k_{r-i} \leq \sum_{i=t}^{r}\frac{(2\delta r^2)^i}{i!}k_r = \sum_{i=t}^{r}\frac{(r^{1/2}/300)^i}{i!}k_r \\ &\leq r\frac{(r^{1/2}/300)^t}{t!}k_r \leq r(t/300)^t(e/t)^t k_r \leq rk_r/100^t \leq k_r/r^2.\end{aligned}$$

□

## 5. Adding weight over an edge

Recall from Section 2 that, in order to turn our initial uniform weighting into a fractional clique decomposition, our aim is to construct edge-gadgets which adjust the weight over an edge $e$ by adjusting the weights of some $r$-cliques. In our proof of Theorem 1.5 (for $k=2$), we implicitly used an edge-gadget $\psi_e$ that was the average of a basic edge-gadget $\psi_e^J$ over all cliques $J \in \mathcal{K}_{r+2}$ containing $e$ (defined more explicitly in Section 2). This averaging ensured that the weight of any given clique was not altered so much that it became negative. Using some simple preprocessing (namely removing $r$-cliques one-by-one until any further removal violates the minimum degree condition) we can reduce the total adjustment we need to make to the initial weighting. In fact, for most of the edges we will only need to make small adjustments, leaving us most concerned with certain 'bad' edges. Moreover, for each edge $e \in E(G)$, the edge-gadget $\psi_e$ requires larger adjustments to be made to those cliques whose intersection with $V(e)$ is larger. Thus we are limited by the adjustment we ask from cliques which contain many vertices in bad edges. By avoiding basic edge-gadgets which require adjustment to the weight of such cliques, we can reduce the minimum degree condition needed for these techniques to work.

In this section, we give sufficient conditions on a subset $\mathcal{A} \subseteq \mathcal{K}_r$ to ensure we can construct good edge-gadgets that only change the weights of cliques in $\mathcal{A}$.

**Definition 5.1.** *Given a graph $G$ we say that $\mathcal{A} \subseteq \mathcal{K}_r$ is* well-distributed *if, for each $e \in E(G)$, there are at least $k_r/2$ sets $A \subseteq V(G) \setminus V(e)$ for which $|A| = r$ and, for each subset $B \subseteq V(e) \cup A$ with $|B| = r$, $G[B] \in \mathcal{A}$.*

Informally, $\mathcal{A}$ is well-distributed if the $r$-cliques it contains can be used to build many different basic edge-gadgets $\psi_e$ for each edge $e$.

**Lemma 5.2.** *Let $r \geq 3$ and let $G$ be a graph on $n$ vertices. Suppose that $k_r > 0$ and that $\mathcal{A} \subseteq \mathcal{K}_r$ is well-distributed. Then for each edge $e \in E(G)$ there exists a function $\psi_e : \mathcal{A} \to \mathbb{R}$ so that the following holds.*

(i) *For all $e, f \in E(G)$,*
$$\sum_{K \in \mathcal{A}\colon f \in E(K)} \psi_e(K) = \mathbf{1}_{\{e=f\}}.$$

(ii) *For all $K \in \mathcal{A}$ and $e \in E(G)$, if $i = |V(K) \cap V(e)|$, then $|\psi_e(K)| \leq 6n^i/r^i k_r$.*

*Proof.* The proof idea is similar to that of Proosition 3.3. For each edge $e \in E(G)$, let $\mathcal{H}_e$ be the set of sets $A \subseteq V(G) \setminus V(e)$ for which $|A| = r$ and, for each subset $B \subseteq V(e) \cup A$ with $|B| = r$, $G[B] \in \mathcal{A}$. As $\mathcal{A}$ is well-distributed, $|\mathcal{H}_e| \geq k_r/2$. For each clique $K \in \mathcal{A}$, let $\alpha_{e,K}$ be the number of sets $A \in \mathcal{H}_e$ for which $K \in \mathcal{K}_r[A \cup V(e)]$. For each edge $e \in E(G)$ and clique $K \in \mathcal{A}$, let
$$\phi_e(K) := \begin{cases} \frac{2}{r(r-1)} & \text{if } |V(K) \cap V(e)| = 2, \\ -\frac{r-2}{r(r-1)} & \text{if } |V(K) \cap V(e)| = 1, \\ \frac{r-2}{r} & \text{if } |V(K) \cap V(e)| = 0, \end{cases}$$
and let $\psi_e(K) := \alpha_{e,K}\phi_e(K)/|\mathcal{H}_e|$. We will now show that $\psi_e$ satisfies the requirements of the lemma.

Firstly, let $e, f \in E(G)$, and $A \in \mathcal{H}_e$ with $V(f) \subseteq A \cup V(e)$. If $f = e$, then, as $|A| = r$, there are $\binom{r}{2}$ $r$-cliques $K \in \mathcal{K}_r[A \cup V(e)]$ with $f = e \in E(K)$. Thus
$$\sum_{K \in \mathcal{K}_r[A \cup V(e)]\colon f \in E(K)} \phi_e(K) = 1.$$

If $f$ and $e$ share precisely one vertex, then for each $i \in \{1, 2\}$ there are $\binom{r-1}{i}$ $r$-cliques $K \in \mathcal{K}_r[A \cup V(e)]$ with $f \in E(K)$ and $|V(K) \cap V(e)| = i$. Thus
$$\sum_{K \in \mathcal{K}_r[A \cup V(e)]\colon f \in E(K)} \phi_e(K) = \binom{r-1}{2}\frac{2}{r(r-1)} - (r-1)\frac{r-2}{r(r-1)} = 0.$$

If $f$ and $e$ share no vertices, then for each $i \in \{0, 1, 2\}$ there are $\binom{2}{2-i}\binom{r-2}{i}$ cliques $K \in \mathcal{K}_r[A \cup V(e)]$ with $f \in E(K)$ and $|V(K) \cap V(e)| = i$. Thus

$$\sum_{K \in \mathcal{K}_r[A \cup V(e)]:\, f \in E(K)} \phi_e(K) = \binom{r-2}{2} \frac{2}{r(r-1)} - 2(r-2)\frac{r-2}{r(r-1)} + \frac{r-2}{r} = 0.$$

Therefore,

$$\begin{aligned} \sum_{K \in \mathcal{A}:\, f \in E(K)} \psi_e(K) &= \sum_{K \in \mathcal{A}: f \in E(K)} \frac{1}{|\mathcal{H}_e|} \sum_{A \in \mathcal{H}_e : K \in \mathcal{K}_r[A \cup V(e)]} \phi_e(K) \\ &= \frac{1}{|\mathcal{H}_e|} \sum_{A \in \mathcal{H}_e} \sum_{K \in \mathcal{K}_r[A \cup V(e)] : f \in E(K)} \phi_e(K) = \frac{1}{|\mathcal{H}_e|} \sum_{A \in \mathcal{H}_e} \mathbf{1}_{\{e=f\}} = \mathbf{1}_{\{e=f\}}, \end{aligned}$$

as required.

Secondly, fix an edge $e \in E(G)$ and a clique $K \in \mathcal{A}$, and let $i := |V(K) \cap V(e)|$. There are at most $\binom{n}{i}$ sets $A \in \mathcal{H}_e$ for which $K \in \mathcal{K}_r[A \cup V(e)]$, and thus $\alpha_{e,K} \leq n^i$. As mentioned previously, we have $|\mathcal{H}_e| \geq k_r/2$, and we can observe that $|\phi_e(K)| \leq 3/r^i$. Therefore,

$$|\psi_e(K)| \leq (2n^i/k_r)|\phi_e(K)| \leq 6n^i/r^i k_r. \qquad \square$$

We will initially weight each clique with $1/\kappa$, where $\kappa := k_{r-2} - 2\delta n k_{r-3}$. As we will see later (in Lemma 8.1), this gives an almost fractional $K_r$-decomposition of $G$. Let $\pi : E(G) \to \mathbb{R}$ record the amount of weight we wish to add over each edge to achieve a fractional $K_r$-decomposition. We wish to know whether we can make these adjustments using edge-gadgets while keeping the weights on the $r$-cliques positive. The next lemma, Lemma 5.4, says that we can make these adjustments while changing the weight of each clique by no more than $1/2\kappa$, provided that the adjustments given by $\pi$ are on average quite small and $\pi$ is sufficiently 'smooth'. That is, $|\pi|$ is not significantly above average for any edge, and the average of $|\pi|$ around each vertex is even more restricted. Before we state Lemma 5.4, we formalise these properties by the following definition.

**Definition 5.3.** *Given a graph $G$ and $r \in \mathbb{N}$, a function $\pi : E(G) \to \mathbb{R}$ is* $r$-smooth *if*

(A1) *for each edge $xy \in E(G)$, $|\pi(xy)| \leq 1/10^4$,*
(A2) *for each vertex $x \in V(G)$, $\sum_{y \in N(x)} |\pi(xy)| \leq n/10^4 r$, and*
(A3) $\sum_{xy \in E(G)} |\pi(xy)| \leq n^2/10^4 r^2$.

Note that (A1) does not imply (A2), and (A2) does not imply (A3).

The intuition behind the definition of smoothness is as follows. To construct a basic edge-gadget $\phi_e$, we increased only the weight over $e$ by first increasing the weight of some $r$-cliques containing $e$, then making further adjustments to cancel out the change in weight over every other edge of these cliques. These cancellations introduce an inherent inefficiency and mean that we can only hope to correct errors of average size $O(1/r^2)$ (cf. (A3)), although we can handle slightly larger localised errors (cf. (A1) and (A2)).

**Lemma 5.4.** *Let $r \geq 4$, $n \in \mathbb{N}$ and $0 \leq \delta \leq 1/24r$. Let $G$ be a graph with $n$ vertices and $\delta(G) \geq (1-\delta)n$. Let $\kappa := k_{r-2} - 2\delta n k_{r-3}$, and let $\pi : E(G) \to \mathbb{R}$ be $r$-smooth. Then there exists a function $\omega : \mathcal{K}_r \to \mathbb{R}$ so that $|\omega(K)| \leq 1/2\kappa$ for all $K \in \mathcal{K}_r$ and, for each $e \in E(G)$,*

$$\sum_{K \in \mathcal{K}_r:\, e \in E(K)} \omega(K) = \pi(e).$$

*Proof.* Let $\gamma := 1/10^4 r^2$, and let

$$\mathcal{A} := \Big\{ K \in \mathcal{K}_r : \sum_{e' \in E(K)} |\pi(e')| \leq 72r^2\gamma \ \text{ and } \sum_{e' \in E(G): |V(K)\cap V(e')| \geq 1} |\pi(e')| \leq 48rn\gamma \Big\}. \tag{5.1}$$

We will show that $\mathcal{A}$ is well-distributed, and then define $\omega$ using the edge-gadgets $\psi_e$ obtained by applying Lemma 5.2 with $\mathcal{A}$.

Note that, using Proposition 4.1,

$$k_{r-2} \geq \kappa \geq k_{r-2} - 4\delta r k_{r-2} \geq 5k_{r-2}/6 > 0. \tag{5.2}$$

For each $e \in E(G)$, let

$$\mathcal{H}_e := \{A \subseteq V(G) \setminus V(e) : |A| = r \ \text{ and } \ G[A \cup V(e)] \in \mathcal{K}_{r+2}\}, \tag{5.3}$$

and note that, using Lemma 4.2(i), we have

$$|\mathcal{H}_e| = \kappa_{V(e)}^{(r+2)} \geq k_r - 4\delta(r+2)k_r \geq 3k_r/4. \tag{5.4}$$

Let

$$\mathcal{H}_{e,1} := \Big\{ A \in \mathcal{H}_e : \sum_{e' \in E(G[A \cup V(e)])} |\pi(e')| \leq 72r^2\gamma \Big\}.$$

**Claim 5.5.** *For each $e \in E(G)$, $|\mathcal{H}_e \setminus \mathcal{H}_{e,1}| \leq k_r/8$.*

*Proof of Claim 5.5.* For each $i \in \{0,1,2\}$, each edge $e' \in E(G)$ with $|V(e') \cap V(e)| = i$ is in at most $k_{r+i-2}$ of the graphs $G[A \cup V(e)]$, with $A \in \mathcal{H}_e$. We therefore have that, using Proposition 4.1 and (A1)–(A3),

$$\begin{aligned} |\mathcal{H}_e \setminus \mathcal{H}_{e,1}|(72r^2\gamma) &\leq \sum_{A \in \mathcal{H}_e} \sum_{e' \in E(G[A \cup V(e)])} |\pi(e')| \leq \sum_{i=0}^{2} \sum_{e' : |V(e) \cap V(e')| = i} k_{r+i-2}|\pi(e')| \\ &\leq n^2\gamma k_{r-2} + 2rn\gamma k_{r-1} + r^2\gamma k_r \leq 9r^2\gamma k_r, \end{aligned}$$

hence $|\mathcal{H}_e \setminus \mathcal{H}_{e,1}| \leq k_r/8$. □

Now let

$$\mathcal{H}_{e,2} := \Big\{ A \in \mathcal{H}_e : \sum_{e' \in E(G): |V(e') \cap (A \cup V(e))| \geq 1} |\pi(e')| \leq 48rn\gamma \Big\}.$$

**Claim 5.6.** *For each $e \in E(G)$, $|\mathcal{H}_e \setminus \mathcal{H}_{e,2}| \leq k_r/8$.*

*Proof of Claim 5.6.* Let $e' \in E(G)$. If $V(e) \cap V(e') = \emptyset$, then there are at most $2k_{r-1}$ sets $A \in \mathcal{H}_e$ for which $|V(e') \cap (A \cup V(e))| \geq 1$. If $|V(e) \cap V(e')| \geq 1$, then $|V(e') \cap (A \cup V(e))| \geq 1$ for every $A \in \mathcal{H}_e$, and $|\mathcal{H}_e| \leq k_r$. We therefore have that, using Proposition 4.1, (A2) and (A3),

$$\begin{aligned} |\mathcal{H}_e \setminus \mathcal{H}_{e,2}|(48rn\gamma) &\leq \sum_{A \in \mathcal{H}_e} \sum_{e' : |V(e') \cap (A \cup V(e))| \geq 1} |\pi(e')| \\ &\leq \sum_{e' \in E(G): |V(e) \cap V(e')| = 0} 2k_{r-1}|\pi(e')| + \sum_{e' \in E(G): |V(e) \cap V(e')| \geq 1} k_r|\pi(e')| \\ &\leq 2n^2\gamma k_{r-1} + 2rn\gamma k_r \leq 6rn\gamma k_r, \end{aligned}$$

hence $|\mathcal{H}_e \setminus \mathcal{H}_{e,2}| \leq k_r/8$. □

For each $e \in E(G)$, let $\bar{\mathcal{H}}_e := \mathcal{H}_{e,1} \cap \mathcal{H}_{e,2}$, so that by (5.4) and Claims 5.5 and 5.6, we have $|\bar{\mathcal{H}}_e| \geq 3k_r/4 - k_r/4 \geq k_r/2$. We can now check that the set $\mathcal{A}$ defined by (5.1) is well-distributed.

For each $A \in \bar{\mathcal{H}}_e$ and every $r$-clique $K \in \mathcal{K}_r[A \cup V(e)]$ we have from the definition of $\mathcal{H}_{e,1}$ and $\mathcal{H}_{e,2}$ that $K \in \mathcal{A}$. Since $|\bar{\mathcal{H}}_e| \geq k_r/2$ for each edge $e \in V(G)$, this implies that $\mathcal{A}$ is well-distributed. Thus by Lemma 5.2, for each $e \in E(G)$, there exists a function $\psi_e : \mathcal{A} \to \mathbb{R}$ so that the following holds.

(a) If $e' \in E(G)$, then $\sum_{K\in\mathcal{A}:\, e'\in E(K)} \psi_e(K) = \mathbf{1}_{\{e'=e\}}$.

(b) For each $K \in \mathcal{A}$, if $i = |V(K) \cap V(e)|$, then $|\psi_e(K)| \leq 6n^i/r^ik_r$.

Now, for each $K \in \mathcal{A}$, let

$$\omega(K) := \sum_{e\in E(G)} \psi_e(K)\pi(e), \tag{5.5}$$

and for each $K \in \mathcal{K}_r \setminus \mathcal{A}$, let $\omega(K) := 0$. Then, for each $e \in E(G)$,

$$\sum_{K\in\mathcal{K}_r:e\in E(K)} \omega(K) = \sum_{e'\in E(G)} \sum_{K\in\mathcal{A}:e\in E(K)} \psi_{e'}(K)\pi(e') \overset{\text{(a)}}{=} \sum_{e'\in E(G)} \mathbf{1}_{\{e'=e\}}\pi(e') = \pi(e),$$

as required. Moreover, (b), (5.5), (5.1), (A3), (5.2) and Proposition 4.1 together imply that, for each $K \in \mathcal{A}$,

$$\begin{aligned}|\omega(K)| &\leq \sum_{e\in E(K)} 6|\pi(e)|n^2/r^2k_r + \sum_{e\in E(G):|V(K)\cap V(e)|=1} 6|\pi(e)|n/rk_r + \sum_{e\in E(G)} 6|\pi(e)|/k_r \\ &\leq 6(72r^2\gamma)n^2/r^2k_r + 6(48rn\gamma)n/rk_r + 6(n^2\gamma)/k_r \\ &\leq 1000n^2\gamma/k_r = n^2/10r^2k_r \leq 2/5k_{r-2} \leq 1/2\kappa. \qquad \square\end{aligned}$$

## 6. Fractional $K_r$-decompositions when $\delta(G) \geq (1 - 1/10^5r^2)n$.

The aim of this section is to prove Theorem 1.3 under the stronger assumption that $\delta(G) \geq (1 - \delta)n$ with $\delta := 1/10^5r^2$ (see Theorem 6.1 below). We include a proof of this intermediate bound as it follows easily from Lemma 5.4, and shows how we will make use of that lemma.

As noted in Section 5, after initially weighting the $r$-cliques uniformly with value $1/\kappa$ (where $\kappa := k_{r-2} - 2\delta nk_{r-3}$), Lemma 5.4 permits us to move a $\Omega(1/r^2)$ proportion of the weight over the edges around (subject to certain constraints) without making any of the $r$-clique weights negative. We will see, using Lemma 4.2, that we only need to adjust a $O(\delta r)$ proportion of the weight over each edge to turn our initial uniform weighting into a fractional $K_r$-decomposition. Thus in the case when $\delta = O(1/r^3)$ is suitably small we can apply Lemma 5.4 to make this adjustment. This corresponds to the argument presented in Section 3.

However, if we carry out some initial preprocessing (removing $r$-cliques until the minimum degree condition would be violated by any further removal) we can reduce the overall proportion of weight over the edges that we might need to move to $O(\delta^2r^2)$. This allows us to use Lemma 5.4 even in the case when $\delta = O(1/r^2)$ is sufficiently small.

**Theorem 6.1.** *Let $r \geq 4$ and let $G$ be a graph with $n \geq 10^6r^4$ vertices and $\delta(G) \geq (1 - 1/10^5r^2)n$. Then $G$ has a fractional $K_r$-decomposition.*

*Proof.* Let $\delta := 1/10^5r^2$. We may assume that we cannot remove any $r$-cliques from $G$ while maintaining minimum degree at least $(1-\delta)n$. Indeed, by removing a sequence of $r$-cliques from $G$ we can find a subgraph $H$ for which $\delta(H) \geq (1 - \delta)n$ but for which removing any

$r$-clique violates this minimum degree condition; if $H$ has a fractional $K_r$-decomposition, then clearly $G$ does also. Therefore, writing $X := \{x \in V(G) : d(x) \geq (1-\delta)n + r - 1\}$, we may assume that $G[X]$ is $K_r$-free. As $G[X]$ has minimum degree at least $(1 - \delta n/|X|)|X|$, by Turán's theorem, $|X| \leq \delta(r-1)n$.

For each edge $e \in E(G)$, we have, by Lemma 4.2(ii), that

$$\left|\kappa_e^{(r)} - k_{r-2} + |N^c(x) \cup N^c(y)| k_{r-3}\right| \leq 24(\delta r)^2 k_{r-2}. \tag{6.1}$$

Let $\kappa := k_{r-2} - 2\delta n k_{r-3}$, so that, by Proposition 4.1, $\kappa \geq (1-4\delta r)k_{r-2} \geq 9k_{r-2}/10$. For each $e \in E(G)$, let $\pi(e) := \kappa_e^{(r)} - \kappa$, so that, by (6.1), we have

$$|\pi(e)| \leq (2\delta n - |N^c(x) \cup N^c(y)|)k_{r-3} + 24(\delta r)^2 k_{r-2}. \tag{6.2}$$

In particular, together with Proposition 4.1 this implies that

$$|\pi(e)| \leq 2\delta n k_{r-3} + 24(\delta r)^2 k_{r-2} \leq (4\delta r + 24(\delta r)^2)k_{r-2} \leq 9k_{r-2}/10^5 r \leq \kappa/10^4 r.$$

For each $x \in V(G)$, then, $\sum_{y \in N(x)} |\pi(xy)/\kappa| \leq n/10^4 r$. Furthermore, using (6.2), Proposition 4.1, and our assumption that $n \geq 10^6 r^4$,

$$\begin{aligned}
2\sum_{e \in E(G)} |\pi(e)| &= \sum_{x \in V(G)} \sum_{y \in N(x)} |\pi(xy)| \\
&\leq \sum_{x \in V(G)} \sum_{y \in N(x)} (2\delta n - |N^c(x) \cup N^c(y)|)k_{r-3} + 24n^2(\delta r)^2 k_{r-2} \\
&\leq 2\sum_{x \in X} \sum_{y \in N(x)} 2\delta n k_{r-3} + \sum_{x \notin X} \sum_{y \in N(x) \setminus X} (2\delta n - |N^c(x) \cup N^c(y)|)k_{r-3} + 24\delta^2 r^2 n^2 k_{r-2} \\
&\leq 4\delta^2 r n^3 k_{r-3} + \sum_{x \notin X} \sum_{y \in N(x) \setminus X} (2r + |N^c(x) \cap N^c(y)|)k_{r-3} + 24\delta^2 r^2 n^2 k_{r-2} \\
&\leq 8\delta^2 r^2 n^2 k_{r-2} + 2rn^2 k_{r-3} + \sum_{x \notin X} \sum_{z \in N^c(x)} |N^c(z)| k_{r-3} + 24\delta^2 r^2 n^2 k_{r-2} \\
&\leq 32\delta^2 r^2 n^2 k_{r-2} + 4r^2 n k_{r-2} + \delta^2 n^3 k_{r-3} \\
&\leq n^2 k_{r-2}/10^5 r^2 + n^2 k_{r-2}/10^5 r^2 + 2\delta^2 r n^2 k_{r-2} \leq 9n^2 k_{r-2}/10^5 r^2 \leq n^2 \kappa/10^4 r^2.
\end{aligned}$$

Therefore, the function $\pi/\kappa : E(G) \to \mathbb{R}$ is $r$-smooth. Thus Lemma 5.4 implies that there is a function $\omega' : \mathcal{K}_r \to \mathbb{R}$ so that, for each $e \in E(G)$, $\sum_{K \in \mathcal{K}_r : e \in E(K)} \omega'(K) = \pi(e)/\kappa$, and, for each $K \in \mathcal{K}_r$, $|\omega'(K)| \leq 1/2\kappa$.

Define $\omega : \mathcal{K}_r \to \mathbb{R}$ by setting $\omega(K) := 1/\kappa - \omega'(K)$ for each $K \in \mathcal{K}_r$. Then, for each $e \in E(G)$,

$$\sum_{K \in \mathcal{K}_r : e \in E(K)} \omega(K) = \frac{\kappa_e^{(r)} - \pi(e)}{\kappa} = 1,$$

and, for each $K \in \mathcal{K}_r$, $\omega(K) \geq 1/\kappa - 1/2\kappa \geq 0$. Therefore, $\omega$ is a fractional $K_r$-decomposition of $G$. □

## 7. Adding weight around a vertex

After our initial preprocessing of the graph $G$ and the initial weighting of the $r$-cliques with $1/\kappa$, where $\kappa := k_{r-2} - 2\delta n k_{r-3}$, we may need to add/subtract on average a $\Omega(\delta^2 r^2)$ proportion of the weight over each edge. Our edge-gadgets can only add/subtract weight over each edge if it is on average $O(1/r^2)$. Thus the techniques in Section 6 require $\delta = O(1/r^2)$.

In order to increase the size of $\delta$, in this section we introduce 'vertex-gadgets', defined explicitly below, which in this set-up are capable of adding/subtracting $\Omega(1/r)$ of the weight over each edge. However, while this is more efficient than using edge-gadgets, the vertex-gadgets can only change the weight of every edge around some vertex simultaneously by the same amount.

For a vertex $x \in V(G)$, a *vertex-gadget* is a function $\xi_x : \mathcal{K}_r \to \mathbb{R}$ such that for each edge $e \in E(G)$,

$$\sum_{K\in\mathcal{K}_r\colon e\in E(K)} \xi_x(K) = \begin{cases} 1 & \text{if } x \in V(e),\\ 0 & \text{if } x \notin V(e).\end{cases}$$

In the next lemma, we show that, for each vertex $x \in V(G)$, there exists a function $\phi_x : \mathcal{K}_r \to \mathbb{R}$ such that

(i) for each $e \in E(K)$ with $x \notin V(e)$, $\sum_{K\in\mathcal{K}_r\colon e\in E(K)} \phi_x(K) = 0$, and
(ii) for each $y \in N(x)$, $\sum_{K\in\mathcal{K}_r\colon xy\in E(K)} \phi_x(K)$ is close to 1.

Thus $\phi_x$ is almost a vertex-gadget. We will then use edge-gadgets to make the requisite corrections to $\phi_x$ to obtain an actual vertex-gadget—see Lemma 7.3. (Thus we actually define $\phi_x$ on a certain subset $\mathcal{A} \subseteq \mathcal{K}_r$ instead of $\mathcal{K}_r$ so that we can make these adjustments efficiently.)

**Lemma 7.1.** *Let $r \geq 4$, $0 < \delta \leq 1/600r^{3/2}$ and $n \geq 32r^3$. Let $G$ be a graph on $n$ vertices with $\delta(G) \geq (1-\delta)n$. Let $X := \{x \in V(G) : d_G(x) \geq (1-\delta)n + r - 1\}$ and suppose that $|X| \leq \delta(r-1)n$. Let $\mathcal{A} := \{K \in \mathcal{K}_r : |V(K) \cap X| \leq r^{1/2} + 2\}$. Then, for each vertex $x \in V(G)$, there exists a function $\phi_x : \mathcal{A} \to \mathbb{R}$ for which the following holds, where, for each $y \in N(x)$, we let $\tau_{x,y} := 1 - \sum_{K\in\mathcal{A}\colon xy\in E(K)} \phi_x(K)$.*

(B1) *If $x \in V(G)$ and $e \in E(G)$ with $x \notin V(e)$, then $\sum_{K\in\mathcal{A}\colon e\in E(K)} \phi_x(K) = 0$.*
(B2) *For all $x \in V(G)$ and $y \in N(x)$, $|\tau_{x,y}| \leq 1/r^{1/2}$.*
(B3) *For each $x \in V(G)$, $\sum_{y\in N(x)} |\tau_{x,y}| \leq n/r$.*
(B4) *For all $K \in \mathcal{A}$ and $x \in V(G)$, if $i = |V(K) \cap \{x\}|$, then $|\phi_x(K)| \leq 2n^{i+1}/r^{i+1}k_r$.*

*Proof.* For each vertex $x \in V(G)$, let $\mathcal{H}_x$ be the set of sets $A \subseteq V(G) \setminus \{x\}$ for which $|A| = r$, $G[A \cup \{x\}] \in \mathcal{K}_{r+1}$ and $|A \cap X| \leq r^{1/2} + 1$. For each $x \in V(G)$ and $K \in \mathcal{K}_r$, let $\alpha_{x,K}$ be the number of sets $A \in \mathcal{H}_x$ for which $K \in \mathcal{K}_r[A \cup \{x\}]$, and let

$$\psi_x(K) := \begin{cases} \frac{1}{r-1} & \text{if } x \in V(K),\\ -\frac{r-2}{r-1} & \text{if } x \notin V(K).\end{cases}$$

For each $x \in V(G)$, let $w_x := k_{r-1} - (n - d(x) + \delta n)k_{r-2}$. Note that, by Proposition 4.1,

$$w_x \geq k_{r-1} - 2\delta n k_{r-2} \geq k_{r-1} - 4\delta r k_{r-1} \geq 7k_{r-1}/8. \tag{7.1}$$

For each $K \in \mathcal{A}$, let $\phi_x(K) := \alpha_{x,K}\psi_x(K)/w_x$. We will now show that $\phi_x$ satisfies the requirements of the lemma.

First, let $x \in V(G)$ and let $e \in E(G)$ with $x \notin V(e)$. If $A \in \mathcal{H}_x$ and $V(e) \subseteq A \cup \{x\}$, then, for $i \in \{0,1\}$, there are $\binom{r-2}{i}$ cliques $K \in \mathcal{K}_r$ with $K \in \mathcal{K}_r[A \cup \{x\}]$, $e \in E(K)$ and $|V(K) \cap \{x\}| = i$. Thus,

$$\sum_{K\in\mathcal{K}_r[A\cup\{x\}]\colon e\in E(K)} \psi_x(K) = (r-2)\frac{1}{r-1} - \frac{r-2}{r-1} = 0.$$

Therefore if $x \in V(G)$ and $e \in E(G)$ with $x \notin V(e)$, we have

$$\begin{aligned}\sum_{K\in\mathcal{A}:\, e\in E(K)} \phi_x(K) &= \sum_{K\in\mathcal{A}:\, e\in E(K)} \frac{1}{w_x} \sum_{A\in\mathcal{H}_x:\, K\in\mathcal{K}_r[A\cup\{x\}]} \psi_x(K)\\ &= \frac{1}{w_x}\sum_{A\in\mathcal{H}_x}\sum_{K\in\mathcal{K}_r[A\cup\{x\}]:\, e\in E(K)} \psi_x(K) = 0.\end{aligned}$$

(In the second equality we use that each $K \in \mathcal{K}_r[A\cup\{x\}]$ lies in $\mathcal{A}$ by the definition of $\mathcal{H}_x$.) Therefore (B1) holds.

Now let $x \in V(G)$ and $y \in N(x)$. If $A \in \mathcal{H}_x$ and $y \in A$, then there are $r-1$ cliques $K \in \mathcal{K}_r$ with $K \in \mathcal{K}_r[A\cup\{x\}]$ and $xy \in E(K)$. Thus,

$$\sum_{K\in\mathcal{K}_r[A\cup\{x\}]:\, xy\in E(K)} \psi_x(K) = 1.$$

Let $w_{x,y}$ be the number of sets $A \subseteq V(G)$ for which $A \in \mathcal{H}_x$ and $y \in A$. Then

$$\sum_{K\in\mathcal{A}:\, xy\in E(K)} \phi_x(K) = \frac{1}{w_x}\sum_{A\in\mathcal{H}_x}\sum_{K\in\mathcal{K}_r[A\cup\{x\}]:\, xy\in E(K)} \psi_x(K) = \frac{w_{x,y}}{w_x}. \tag{7.2}$$

In the last equality we use that each $K \in \mathcal{K}_r[A\cup\{x\}]$ (with $xy \in E(K)$) lies in $\mathcal{A}$ by the definition of $\mathcal{H}_x$.

**Claim 7.2.** *For each $x \in V(G)$ and $y \in N(x)$,*

$$|w_{x,y} - w_x + (n - d(y) - \delta n)k_{r-2}| \le |N^c(x)\cap N^c(y)|k_{r-2} + (24\delta^2(r+1)^2 + 2/r^2)k_{r-1}.$$

*Proof of Claim 7.2.* By Proposition 4.3, there are at most $k_{r-1}/(r-1)^2 \le 2k_{r-1}/r^2$ cliques $K \in \mathcal{K}_{r-1}$ for which $|X \cap V(K)| \ge r^{1/2}$. Note that if $xy \in E(G)$, $K'$ is an $(r+1)$-clique containing $xy$, and $|(V(K')\setminus\{x,y\})\cap X| \le r^{1/2}$ then $V(K')\setminus\{x\} \in \mathcal{H}_x$. Thus

$$\left|w_{x,y} - \kappa^{(r+1)}_{xy}\right| \le 2k_{r-1}/r^2. \tag{7.3}$$

Then, by Lemma 4.2(ii) and (7.3), we have that

$$\left|w_{x,y} - k_{r-1} + |N^c(x)\cup N^c(y)|k_{r-2}\right| \le 24(\delta(r+1))^2 k_{r-1} + 2k_{r-1}/r^2.$$

Thus,

$$\begin{aligned}|w_{x,y} - w_x + (n-d(y)-\delta n)k_{r-2}| &= |w_{xy} - k_{r-1} + (2n - d(x) - d(y))k_{r-2}|\\ &\le \left|w_{x,y} - k_{r-1} + |N^c(x)\cup N^c(y)|k_{r-2}\right| + \big||N^c(x)\cup N^c(y)| - (2n-d(x)-d(y))\big|k_{r-2}\\ &\le (24\delta^2(r+1)^2 + 2/r^2)k_{r-1} + |N^c(x)\cap N^c(y)|k_{r-2}.\end{aligned}$$ □

By Claim 7.2, for each $x \in V(G)$ and $y \in N(x)$, using Proposition 4.1, we have

$$\begin{aligned}|w_{x,y} - w_x| &\le \delta n k_{r-2} + \delta n k_{r-2} + (24\delta^2(r+1)^2 + 2/r^2)k_{r-1}\\ &\le (4\delta r + 24\delta^2(r+1)^2 + 2/r^2)k_{r-1} \le k_{r-1}/2r^{1/2}.\end{aligned} \tag{7.4}$$

For each $x \in V(G)$ and $y \in N(x)$, recall that

$$\tau_{x,y} = 1 - \sum_{K\in\mathcal{A}:\, xy\in E(K)} \phi_x(K) \overset{(7.2)}{=} (w_x - w_{x,y})/w_x. \tag{7.5}$$

Therefore (7.1), (7.4) and (7.5) together imply that for each $x \in V(G)$ and $y \in N(x)$, $|\tau_{x,y}| \le 1/r^{1/2}$, and thus (B2) holds.

By Claim 7.2, we have, for each $x \in V(G)$, that

$$\begin{aligned}\sum_{y\in N(x)} |w_x - w_{x,y}| &\le \sum_{y\in V(G)} \big(|n-d(y)-\delta n| + |N^c(x)\cap N^c(y)|\big)k_{r-2} + (24\delta^2(r+1)^2 + 2/r^2)nk_{r-1} \\ &\le \Big(|X|\delta n + rn + \sum_{z\in N^c(x)} |N^c(z)|\Big)k_{r-2} + (24\delta^2(r+1)^2+2/r^2)nk_{r-1} \\ &\le (\delta^2(r-1)n^2 + rn + \delta^2 n^2)k_{r-2} + (24\delta^2(r+1)^2+2/r^2)nk_{r-1} \\ &\overset{\text{P4.1}}{\le} \big(2\delta^2 r^2 + 2r^2/n + 2r\delta^2 + 24\delta^2(r+1)^2 + 2/r^2\big)nk_{r-1} \le 7nk_{r-1}/8r,\end{aligned}$$

where the final inequality is due to the fact that $\delta \le 1/600r^{3/2}$ and $n \ge 32r^3$. Together with (7.1) and (7.5) this implies that, for each $x \in V(G)$, $\sum_{y\in N(x)} |\tau_{x,y}| = \sum_{y\in N(x)} |w_{xy} - w_x|/w_x \le n/r$, which proves (B3).

Finally, for each vertex $x \in V(G)$ and clique $K \in \mathcal{K}_r$, setting $i := |V(K)\cap\{x\}|$, there are at most $\binom{n}{i}$ sets $A \in \mathcal{H}_x$ for which $K \in \mathcal{K}_r[A\cup\{x\}]$, and thus $\alpha_{x,K} \le n^i$. Moreover, $k_r \le nk_{r-1}/r$ and $|\psi_x(K)| \le 4/3r^i$. Together with (7.1), this implies that

$$|\phi_x(K)| \le 8n^i|\psi_x(K)|/7k_{r-1} \le 2n^{i+1}/r^{i+1}k_r. \qquad \square$$

Consider the function $\phi_x$ given by Lemma 7.1. Note that for each $y \in N(x)$

$$\sum_{K\in\mathcal{K}_r:\, xy\in E(K)} \phi_x(K) = 1 - \tau_{x,y}.$$

To modify $\phi_x$ into a vertex-gadget, we will add weight $\tau_{x,y}$ to each edge $xy$ using our edge-gadgets. This is achieved by the next lemma.

**Lemma 7.3.** *Let $r \ge 4$, $0 < \delta \le 1/600r^{3/2}$ and $n \ge 32r^3$. Let $G$ be a graph on $n$ vertices with $\delta(G) \ge (1-\delta)n$. Let $X := \{x \in V(G) : d_G(x) \ge (1-\delta)n + r - 1\}$ and suppose $|X| \le \delta(r-1)n$. Let $\mathcal{A} := \{K \in \mathcal{K}_r : |V(K)\cap X| \le r^{1/2}+2\}$. Then for each vertex $x \in V(G)$, there exists a function $\xi_x : \mathcal{A} \to \mathbb{R}$ so that the following holds.*

(i) *If $x \in V(G)$ and $e \in E(G)$, then*

$$\sum_{K\in\mathcal{A}:\, e\in E(K)} \xi_x(K) = \begin{cases} 1 & \text{if } x \in V(e), \\ 0 & \text{if } x \notin V(e). \end{cases}$$

(ii) *For all $K \in \mathcal{A}$ and $x \in V(G)$, if $i = |V(K)\cap\{x\}|$, then $|\xi_x(K)| \le 80n^{i+1}/r^{i+1}k_r$.*

The efficiency of a vertex-gadget $\xi_x$ from Lemma 7.3 can be compared to the efficiency of an edge-gadget $\psi_{xy}$ from Lemma 5.2 as follows. If a clique $K$ is disjoint from $\{x,y\}$, then (ii) in Lemma 5.2 says that $|\psi_{xy}(K)| \le 6/k_r$, while (ii) in Lemma 7.3 says that $|\xi_x(K)| \le 80n/rk_r$; so $\xi_x$ may change the weight of the clique by an extra factor of $n/r$. However, $\psi_{xy}$ changes the weight of only one edge by 1, while $\xi_x$ changes the weight of $|N(x)| \ge (1-\delta)n$ edges by 1. As the edge-gadgets can move a $\Omega(1/r^2)$ proportion of the weight, this indicates that the vertex-gadgets can move a $\Omega(1/r)$ proportion of the weight.

*Proof.* For each vertex $x \in V(G)$, let $\phi_x$ be the function from Lemma 7.1 for which (B1)–(B4) hold with the set $\mathcal{A}$. The function $\phi_x$ is an approximation to the function $\xi_x$ we require. For each $x \in V(G)$ and $y \in N(x)$, we let

$$\tau_{x,y} := 1 - \sum_{K\in\mathcal{A}:\, xy\in E(K)} \phi_x(K). \qquad (7.6)$$

As discussed above, this records the adjustments we will need to make to $\phi_x$ in order to obtain $\xi_x$. We will make these adjustments using Lemma 5.2.

For each $e \in E(G)$, let

$$\mathcal{H}_e := \{A \subseteq V(G) \setminus V(e) : G[A \cup V(e)] \in K_{r+2} \text{ and } |A \cap X| \leq r^{1/2}\}.$$

By Proposition 4.3, there are at most $k_r/r^2$ sets $A \subseteq V(G)$ with $G[A] \in \mathcal{K}_r$ and $|A \cap X| \geq r^{1/2}$. For each edge $xy \in E(G)$, using Lemma 4.2(i), we have

$$|\mathcal{H}_{xy}| \geq \kappa_{xy}^{(r+2)} - k_r/r^2 \geq k_r - 4\delta(r+2)k_r - k_r/r^2 \geq 3k_r/4. \tag{7.7}$$

For each $e \in E(G)$ and $x \in V(G)$, let

$$\mathcal{H}_{e,x} := \Big\{A \in \mathcal{H}_e : \sum_{y \in (A \cup V(e)) \cap N(x)} |\tau_{x,y}| \leq 12\Big\}. \tag{7.8}$$

**Claim 7.4.** *For all $e \in E(G)$ and $x \in V(G)$, $|\mathcal{H}_{e,x}| \geq k_r/2$.*

*Proof of Claim 7.4.* For each $i \in \{0,1\}$ and each $e \in E(G)$, each $y \in V(G)$ with $|\{y\} \cap V(e)| = i$ is in at most $k_{r+i-1}$ of the sets $A \cup V(e)$ with $A \in \mathcal{H}_e$. We therefore have for all $e \in E(G)$ and $x \in V(G)$, that

$$\begin{aligned} 12|\mathcal{H}_e \setminus \mathcal{H}_{e,x}| &\overset{(7.8)}{\leq} \sum_{A \in \mathcal{H}_e \setminus \mathcal{H}_{e,x}} \sum_{y \in (A \cup V(e)) \cap N(x)} |\tau_{x,y}| \leq \sum_{y \in N(x) \cap V(e)} |\tau_{x,y}| k_r + \sum_{y \in N(x) \setminus V(e)} |\tau_{x,y}| k_{r-1} \\ &\overset{(B2),(B3)}{\leq} 2k_r/r^{1/2} + nk_{r-1}/r \overset{P4.1}{\leq} 3k_r. \end{aligned}$$

Therefore, $|\mathcal{H}_e \setminus \mathcal{H}_{e,x}| \leq k_r/4$. Thus, by (7.7), $|\mathcal{H}_{e,x}| \geq 3k_r/4 - k_r/4 \geq k_r/2$. □

For each $x \in V(G)$, let

$$\mathcal{A}_x := \Big\{K \in \mathcal{A} : \sum_{y \in V(K) \cap N(x)} |\tau_{x,y}| \leq 12\Big\}. \tag{7.9}$$

For all $e \in E(G)$, $x \in V(G)$, $A \in \mathcal{H}_{e,x}$ and cliques $K \in \mathcal{K}_r[A \cup V(e)]$, we have by (7.8) and the definition of $\mathcal{A}$, $\mathcal{A}_x$, $\mathcal{H}_e$ and $\mathcal{H}_{e,x}$, that $K \in \mathcal{A}_x$. Together with Claim 7.4, this implies that $\mathcal{A}_x$ is well-distributed. Thus, for each $x \in V(G)$ and each $e \in E(G)$, by Lemma 5.2, there exists a function $\psi_e^x : \mathcal{A}_x \to \mathbb{R}$ so that the following hold.

(a) If $e, e' \in E(G)$, then $\sum_{K \in \mathcal{A}_x :\, e' \in E(K)} \psi_e^x(K) = \mathbf{1}_{\{e'=e\}}$.

(b) For all $K \in \mathcal{A}_x$ and $e \in E(G)$, if $i = |V(K) \cap V(e)|$, then $|\psi_e^x(K)| \leq 6n^i/r^i k_r$.

For all $x \in V(G)$ and $e \in E(G)$, extend $\psi_e^x$ by setting $\psi_e^x(K) := 0$ for each $K \in \mathcal{A} \setminus \mathcal{A}_x$. For all $K \in \mathcal{A}$ and $x \in V(G)$, let

$$\xi_x(K) := \phi_x(K) + \sum_{z \in N(x)} \psi_{xz}^x(K)\tau_{x,z}. \tag{7.10}$$

We will now show that the functions $\xi_x$ have the required properties. Consider any $x \in V(G)$. Firstly, for each $y \in N(x)$, by (7.10), (a) and (7.6),

$$\begin{aligned} \sum_{K \in \mathcal{A} :\, xy \in E(K)} \xi_x(K) &= \sum_{K \in \mathcal{A} :\, xy \in E(K)} \phi_x(K) + \sum_{z \in N(x)} \mathbf{1}_{\{xz=xy\}} \tau_{x,z} \\ &= \sum_{K \in \mathcal{A} :\, xy \in E(K)} \phi_x(K) + \tau_{x,y} = 1, \end{aligned}$$

and for each edge $e \in E(G)$ with $x \notin V(e)$, by (B1) and (a),

$$\sum_{K\in\mathcal{A}\colon e\in E(K)} \xi_x(K) = \sum_{K\in\mathcal{A}\colon e\in E(K)} \phi_x(K) + \sum_{z\in N(x)} \mathbf{1}_{\{xz=e\}}\tau_{x,z} = 0+0 = 0.$$

Therefore, (i) is satisfied.

It remains to prove (ii) for all $x \in V(G)$ and $K \in \mathcal{A}$, which we do separately for $K \in \mathcal{A}\setminus\mathcal{A}_x$ and $K \in \mathcal{A}_x$.

If $K \in \mathcal{A}\setminus\mathcal{A}_x$, then $\psi_e^x(K) = 0$ for each $e \in E(G)$. Therefore, by (7.10), $\xi_x(K) = \phi_x(K)$. Together with (B4), this in turn implies that, if $i = |V(K)\cap\{x\}|$, then $|\xi_x(K)| \le 2n^{i+1}/r^{i+1}k_r$.

If $K \in \mathcal{A}_x$, then let $i := |V(K)\cap\{x\}|$. Note that if $z \in N(x)$ then $|\{x,z\}\cap V(K)| = i + |\{z\}\cap V(K)|$. Together with (7.10), (B4) and (b), this implies that

$$\begin{aligned}|\xi_x(K)| &\le 2n^{i+1}/r^{i+1}k_r + \sum_{z\in N(x)\cap V(K)} |\tau_{x,z}|(6n^{i+1}/r^{i+1}k_r) + \sum_{z\in N(x)\setminus V(K)} |\tau_{x,z}|(6n^{i}/r^{i}k_r)\\ &\overset{(7.9),\ (B3)}{\le} 2n^{i+1}/r^{i+1}k_r + 12(6n^{i+1}/r^{i+1}k_r) + 6n^{i+1}/r^{i+1}k_r = 80n^{i+1}/r^{i+1}k_r.\end{aligned}$$

□

## 8. Number of cliques containing a specified edge

Recall that, after some initial preprocessing of the graph $G$, we give each $r$-clique weight $1/\kappa$, where $\kappa := k_{r-2} - 2\delta n k_{r-3}$. This is not far from a fractional $K_r$-decomposition, and we aim to transform it into a fractional $K_r$-decomposition by correcting the weight over each edge using edge- and vertex-gadgets. For Theorem 1.3 we will have $\delta = \Theta(1/r^{3/2})$, and as before we may need to move a $\Omega(\delta^2 r^2) = \Omega(1/r)$ proportion of the weight around to correct the weight over each edge. Our best technique is to use vertex-gadgets which are indeed capable of moving a $\Omega(1/r)$ proportion of the weights over the edges, but only certain adjustments can be made using such gadgets.

The adjustment to be made to the weight over each edge $xy$ is $(\kappa_{xy}^{(r)}/\kappa) - 1$. In this section, we will break this adjustment down into $\sigma^*(x) + \sigma^*(y) + \pi^*(xy)$ so that on average $\sigma^*(x) = O(1/r)$ and $\pi^*(xy) = O(1/r^2)$. Hence we will be able to adjust the weight over each edge $xy$ by $\sigma^*(x) + \sigma^*(y)$ using vertex-gadgets and by $\pi^*(xy)$ using edge-gadgets.

We find such functions in the following lemma (where $(\sigma+\gamma)/\kappa$ and $\pi/\kappa$ correspond to $\sigma^*$ and $\pi^*$), before showing that the error term depending on the edges is $r$-smooth in Lemma 8.3, so that it can be corrected using the edge-gadgets (via Lemma 5.4).

In this section, we additionally require the notation that, for sets $A, B \subseteq V(G)$,

$$\bar{e}(A,B) := |\{(x,y) : x \in A, y \in B, xy \notin E(G)\}|.$$

**Lemma 8.1.** *Let $r \ge 5$ and $\delta := 1/10^4 r^{3/2}$. Suppose that $G$ is a graph on $n$ vertices with $\delta(G) \ge (1-\delta)n$. Let $X := \{x \in V(G) : d_G(x) \ge (1-\delta)n + r - 1\}$ and suppose that $|X| \le \delta(r-1)n$. For each $x \in V(G)$, let $\gamma(x) := (\delta n - |N^c(x)|)k_{r-3}$. Let $\kappa := k_{r-2} - 2\delta n k_{r-3}$. Let $\pi_1, \pi_2 : E(G) \to \mathbb{R}$ be functions defined by*

$$\begin{aligned}\pi_1(xy) &:= \delta n \sum_{z_1\in N^c(x)} |N^c(z_1)|k_{r-5} + \delta n \sum_{z_2\in N^c(y)} |N^c(z_2)|k_{r-5}\\ &\quad - \sum_{z_1\in N^c(x)}\sum_{z_2\in N^c(y)} |N^c(z_1)\cup N^c(z_2)|k_{r-5} + (\delta n - |N^c(x)|)(\delta n - |N^c(y)|)k_{r-4} \qquad (8.1)\end{aligned}$$

*and*

$$\pi_2(xy) := \Big(e(N^c(x))(|N^c(y)| - \delta n) + e(N^c(y))(|N^c(x)| - \delta n)\Big)k_{r-5}. \qquad (8.2)$$

*Then there exist functions $\sigma : V(G) \to \mathbb{R}$ and $\pi : E(G) \to \mathbb{R}$ so that the following hold.*

(i) *For each $xy \in E(G)$,*

$$\kappa^{(r)}_{xy} = \kappa + \gamma(x) + \gamma(y) + \sigma(x) + \sigma(y) + \pi(xy).$$

(ii) *For each $x \in V(G)$, $|\sigma(x)| \le k_{r-2}/10^4 r$.*

(iii) *For each $xy \in E(G)$,*

$$|\pi(xy)| \le |\pi_1(xy)| + |\pi_2(xy)| + 2|N^c(x) \cap N^c(y)|k_{r-3} + 203(\delta r)^4 k_{r-2} + 3\bar{e}(N^c(x), N^c(y))k_{r-4}.$$

*Proof.* By Lemma 4.2(iii), we have for each $xy \in E(G)$ that

$$\left|\kappa^{(r)}_{xy} - k_{r-2} - \sum_{i=1}^{3}(-1)^i \sum_{Z\subseteq N^c(x)\cup N^c(y)\,:\,|Z|=i} \kappa^{(r-2)}_Z\right| \le 11(\delta r)^4 k_{r-2}.$$

Together with Proposition 4.1 this implies that, for each $xy \in E(G)$,

$$\begin{aligned}\left|\kappa^{(r)}_{xy} - k_{r-2} - \sum_{i=1}^{3}(-1)^i \sum_{j=0}^{i} \sum_{Z_1\subseteq N^c(x)\,:\,|Z_1|=j}\ \sum_{Z_2\subseteq N^c(y)\setminus Z_1\,:\,|Z_2|=i-j} \kappa^{(r-2)}_{Z_1\cup Z_2}\right| & \qquad (8.3)\\ \le 11(\delta r)^4 k_{r-2} + \sum_{j=0}^{2}(2^{j+1}-1) \sum_{z\in N^c(x)\cap N^c(y)}\ \sum_{Z'\subseteq (N^c(x)\cup N^c(y))\setminus\{z\}\,:\,|Z'|=j} \kappa^{(r-2)}_{\{z\}\cup Z'} &\\ \le 11(\delta r)^4 k_{r-2} + |N^c(x)\cap N^c(y)|\left(k_{r-3} + 3\cdot 2\delta n k_{r-4} + 7\binom{2\delta n}{2}k_{r-5}\right) &\\ \le 11(\delta r)^4 k_{r-2} + |N^c(x)\cap N^c(y)|(1 + 12\delta r + 56(\delta r)^2)k_{r-3} &\\ \le 11(\delta r)^4 k_{r-2} + 2|N^c(x)\cap N^c(y)|k_{r-3}, &\end{aligned}$$

where in the first inequality we are bounding the extra contribution to the sum from those $Z_1 \cup Z_2$ that meet $N^c(x) \cap N^c(y)$. Thus for each $xy \in E(G)$, we have

$$|\kappa^{(r)}_{xy} - k_{r-2} - S_1(x) - S_1(y) - S_2 - S_3| \le 11(\delta r)^4 k_{r-2} + 2|N^c(x)\cap N^c(y)|k_{r-3}, \qquad (8.4)$$

where

$$S_2 = S_2(xy) := \sum_{z_1\in N^c(x)}\ \sum_{z_2\in N^c(y)\setminus\{z_1\}} \kappa^{(r-2)}_{\{z_1,z_2\}} \qquad (8.5)$$

$$S_3 = S_3(xy) := -\sum_{j=1}^{2}\ \sum_{Z_1\subseteq N^c(x):|Z_1|=j}\ \sum_{Z_2\subseteq N^c(y)\setminus Z_1:|Z_2|=3-j} \kappa^{(r-2)}_{Z_1\cup Z_2} \qquad (8.6)$$

and, for each $z \in V(G)$,

$$S_1(z) := \sum_{i=1}^{3}(-1)^i \sum_{Z\subseteq N^c(z):|Z|=i} \kappa^{(r-2)}_Z.$$

Here $S_1(x)$ and $S_1(y)$ count the contributions to the sum in (8.3) from those $Z_1 \cup Z_2$ with one of $Z_1$ or $Z_2$ empty, and $S_2, S_3$ count the contributions from those $Z_1 \cup Z_2$ with $Z_1, Z_2$ both non-empty and $|Z_1 \cup Z_2| = 2$ or 3 respectively. In order to estimate $\kappa^{(r)}_{xy}$ we will now estimate $S_1(x)$, $S_1(y)$, $S_2$, and $S_3$.

We will first estimate $S_1(x)$, for each $x \in V(G)$, for which we let

$$\sigma_1(x) := S_1(x) - \gamma(x) + \delta n k_{r-3} \tag{8.7}$$

$$= \Big( - \sum_{z \in N^c(x)} \kappa^{(r-2)}_{\{z\}} + |N^c(x)| k_{r-3} \Big) + \sum_{i=2}^{3} (-1)^i \sum_{Z \subseteq N^c(x) : |Z| = i} \kappa^{(r-2)}_Z. \tag{8.8}$$

**Claim 8.2.** *For each $x \in V(G)$, $|\sigma_1(x)| \leq 8(\delta r)^2 k_{r-2}$.*

*Proof of Claim 8.2.* By Lemma 4.2(i), for each $z \in V(G)$, $|\kappa^{(r-2)}_{\{z\}} - k_{r-3}| \leq 2\delta r k_{r-3}$. Together with Proposition 4.1, this implies that

$$\Big| \sum_{z \in N^c(x)} \kappa^{(r-2)}_{\{z\}} - |N^c(x)| k_{r-3} \Big| \leq \delta n \cdot 2\delta r k_{r-3} \leq 4(\delta r)^2 k_{r-2}. \tag{8.9}$$

Moreover, using Proposition 4.1,

$$\Big| \sum_{i=2}^{3} (-1)^i \sum_{Z \subseteq N^c(x) : |Z| = i} \kappa^{(r-2)}_Z \Big| \leq \binom{\delta n}{2} k_{r-4} + \binom{\delta n}{3} k_{r-5} \leq 4(\delta r)^2 k_{r-2}. \tag{8.10}$$

The claim follows from (8.8), (8.9) and (8.10). □

For each $x \in V(G)$, let

$$\sigma_2(x) := \delta n(|N^c(x)| - \delta n/2) k_{r-4} - \delta n \sum_{z_1 \in N^c(x)} |N^c(z_1)| k_{r-5}. \tag{8.11}$$

Note that, by Proposition 4.1, we have that

$$|\sigma_2(x)| \leq (\delta n)^2 k_{r-4}/2 + (\delta n)^3 k_{r-5} \leq 4\delta^2 r^2 k_{r-2}. \tag{8.12}$$

We will now estimate $|S_2 - \sigma_2(x) - \sigma_2(y)|$. If $z_1 z_2 \in E(G)$, then, by Lemma 4.2(ii),

$$\big| \kappa^{(r-2)}_{\{z_1,z_2\}} - k_{r-4} + |N^c(z_1) \cup N^c(z_2)| k_{r-5} \big| \leq 24(\delta r)^2 k_{r-4}. \tag{8.13}$$

If $z_1 z_2 \notin E(G)$, then $\kappa^{(r-2)}_{\{z_1,z_2\}} = 0$. Therefore, by (8.13) and Proposition 4.1, for each $xy \in E(G)$ we have

$$\Big| \sum_{z_1 \in N^c(x)} \sum_{z_2 \in N^c(y) \setminus \{z_1\}} \kappa^{(r-2)}_{\{z_1,z_2\}} - \sum_{z_1 \in N^c(x)} \sum_{z_2 \in N^c(y) \,:\, z_1 z_2 \in E(G)} (k_{r-4} - |N^c(z_1) \cup N^c(z_2)| k_{r-5}) \Big|$$
$$\leq 24\delta^4 r^2 n^2 k_{r-4} \leq 96(\delta r)^4 k_{r-2},$$

so that, using (8.5),

$$\Big| S_2 - |N^c(x)||N^c(y)| k_{r-4} + \sum_{z_1 \in N^c(x)} \sum_{z_2 \in N^c(y)} |N^c(z_1) \cup N^c(z_2)| k_{r-5} \Big|$$
$$\leq 96(\delta r)^4 k_{r-2} + \bar{e}(N^c(x), N^c(y)) k_{r-4}, \tag{8.14}$$

where we have used the fact that $k_{r-4} \geq |N^c(z_1) \cup N^c(z_2)| k_{r-5}$ by Proposition 4.1. Note that, by (8.1) and (8.11), for each $xy \in E(G)$,

$$\pi_1(xy) + \sigma_2(x) + \sigma_2(y) = |N^c(x)||N^c(y)| k_{r-4} - \sum_{z_1 \in N^c(x)} \sum_{z_2 \in N^c(y)} |N^c(z_1) \cup N^c(z_2)| k_{r-5}.$$

Together with (8.14), this implies that for each $xy \in E(G)$ we have

$$|S_2 - \sigma_2(x) - \sigma_2(y)| \leq |\pi_1(xy)| + 96(\delta r)^4 k_{r-2} + \bar{e}(N^c(x), N^c(y)) k_{r-4}. \tag{8.15}$$

Now, for each $x \in V(G)$, let

$$\sigma_3(x) := -e(N^c(x))\delta n k_{r-5}. \tag{8.16}$$

Note that for each $x \in V(G)$, by Proposition 4.1,

$$|\sigma_3(x)| \leq (\delta n)^3 k_{r-5}/2 \leq 4\delta^3 r^3 k_{r-2}. \tag{8.17}$$

We will now estimate $|S_3 - \sigma_3(x) - \sigma_3(y)|$. If $G[\{z_1, z_2, z_3\}] \in \mathcal{K}_3$, then, by Lemma 4.2(i), we have $|\kappa^{(r-2)}_{\{z_1,z_2,z_3\}} - k_{r-5}| \leq 6\delta r k_{r-5}$. Therefore,

$$\begin{aligned}
&\Big|\sum_{z_1\in N^c(x)}\ \sum_{\{z_2,z_3\}\subseteq N^c(y)\setminus\{z_1\}} \kappa^{(r-2)}_{\{z_1,z_2,z_3\}} - e(N^c(y))|N^c(x)|k_{r-5}\Big| \\
&\qquad = \Big|\sum_{z_1\in N^c(x)}\Big(\sum_{\{z_2,z_3\}\subseteq N^c(y)\setminus\{z_1\}} \kappa^{(r-2)}_{\{z_1,z_2,z_3\}} - \sum_{\{z_2,z_3\}\subseteq N^c(y):z_2z_3\in E(G)} k_{r-5}\Big)\Big| \\
&\qquad \leq \bar{e}(N^c(x), N^c(y))(\delta n)\cdot k_{r-5} + (\delta n)^3 \cdot 6\delta r k_{r-5} \\
&\qquad \leq \bar{e}(N^c(x), N^c(y))k_{r-4} + 48(\delta r)^4 k_{r-2},
\end{aligned} \tag{8.18}$$

where the last inequality is due to Proposition 4.1 and the fact that $\delta r \leq 1/2$. Similarly,

$$\begin{aligned}
&\Big|\sum_{z_1\in N^c(y)}\ \sum_{\{z_2,z_3\}\subseteq N^c(x)\setminus\{z_1\}} \kappa^{(r-2)}_{\{z_1,z_2,z_3\}} - e(N^c(x))|N^c(y)|k_{r-5}\Big| \\
&\qquad\qquad \leq \bar{e}(N^c(x), N^c(y))k_{r-4} + 48(\delta r)^4 k_{r-2}.
\end{aligned} \tag{8.19}$$

Note that, by (8.2) and (8.16), for each $xy \in E(G)$,

$$\pi_2(xy) - \sigma_3(x) - \sigma_3(y) = e(N^c(x))|N^c(y)|k_{r-5} + e(N^c(y))|N^c(x)|k_{r-5}.$$

Together with (8.6), (8.18) and (8.19), this implies that

$$|S_3 - \sigma_3(x) - \sigma_3(y)| \leq |\pi_2(xy)| + 2\bar{e}(N^c(x), N^c(y))k_{r-4} + 96(\delta r)^4 k_{r-2}. \tag{8.20}$$

For each $x \in V(G)$, let

$$\sigma(x) := \sigma_1(x) + \sigma_2(x) + \sigma_3(x), \tag{8.21}$$

and for each edge $xy \in E(G)$, let

$$\pi(xy) := \kappa^{(r)}_{xy} - \kappa - \gamma(x) - \gamma(y) - \sigma(x) - \sigma(y). \tag{8.22}$$

Then (i) holds. Note that, for each $x \in V(G)$, by Claim 8.2, (8.12), (8.17) and (8.21)

$$|\sigma(x)| \leq 8\delta^2 r^2 k_{r-2} + 4\delta^2 r^2 k_{r-2} + 4\delta^3 r^3 k_{r-2} \leq 13\delta^2 r^2 k_{r-2} \leq k_{r-2}/10^4 r,$$

and thus (ii) holds.

Note that $\pi(xy) = \kappa^{(r)}_{xy} - k_{r-2} - S_1(x) - S_1(y) - \sum_{i=2}^3(\sigma_i(x) + \sigma_i(y))$ by (8.7), (8.21) and (8.22). Together with (8.4), (8.15) and (8.20) this shows that for each $xy \in E(G)$ we have

$$\begin{aligned}
|\pi(xy)| &\leq \Big|\kappa^{(r)}_{xy} - k_{r-2} - S_1(x) - S_1(y) - S_2 - S_3\Big| + \Big|S_2 - \sigma_2(x) - \sigma_2(y)\Big| + \Big|S_3 - \sigma_3(x) - \sigma_3(y)\Big| \\
&\leq |\pi_1(xy)| + |\pi_2(xy)| + 2|N^c(x) \cap N^c(y)|k_{r-3} + 203(\delta r)^4 k_{r-2} + 3\bar{e}(N^c(x), N^c(y))k_{r-4}
\end{aligned}$$

and thus (iii) holds. □

Given a function $\pi : E(G) \to \mathbb{R}$ with the properties in Lemma 8.1, we wish to use Lemma 5.4 to add the weight $\pi(e)/\kappa$ to each edge $e$. We must therefore check that $\pi/\kappa$ is $r$-smooth.

**Lemma 8.3.** *Let $r \geq 25$, $\delta := 1/10^4 r^{3/2}$ and $n \geq 10^4 r^3$. Suppose that $G$ is a graph on $n$ vertices with $\delta(G) \geq (1-\delta)n$. Let $X := \{x \in V(G) : d_G(x) \geq (1-\delta)n + r - 1\}$ and suppose that $|X| \leq \delta(r-1)n$. Let $\kappa := k_{r-2} - 2\delta n k_{r-3}$ and let $\pi_1, \pi_2 : E(G) \to \mathbb{R}$ be the functions defined in the statement of Lemma 8.1. Suppose that $\pi : E(G) \to \mathbb{R}$ satisfies*

$$\begin{aligned} |\pi(xy)| \leq |\pi_1(xy)| + |\pi_2(xy)| + 2|N^c(x) \cap N^c(y)| k_{r-3} \\ + 203(\delta r)^4 k_{r-2} + 3\bar{e}(N^c(x), N^c(y)) k_{r-4}. \end{aligned} \tag{8.23}$$

*Then the function $\pi/\kappa$ is $r$-smooth.*

*Proof.* We will show that $\pi/\kappa$ is $r$-smooth using a sequence of claims. Note first, using Proposition 4.1, that $\kappa \geq k_{r-2} - 4\delta r k_{r-2} \geq 9k_{r-2}/10$.

**Claim 8.4.** *For each $xy \in E(G)$, $|\pi(xy)| \leq \kappa/10^4$. That is, $\pi/\kappa$ satisfies* (A1) *in the definition of $\pi/\kappa$ being $r$-smooth.*

*Proof of Claim 8.4.* Note that $|N^c(z_1) \cup N^c(z_2)| = |N^c(z_1)| + |N^c(z_1)| - |N^c(z_1) \cap N^c(z_2)|$ for each $z_1, z_2 \in V(G)$. Therefore, for each $xy \in E(G)$ we have by (8.1) that

$$\begin{aligned} \pi_1(xy) = (\delta n - |N^c(y)|) \sum_{z_1 \in N^c(x)} |N^c(z_1)| k_{r-5} + (\delta n - |N^c(x)|) \sum_{z_2 \in N^c(y)} |N^c(z_2)| k_{r-5} \\ + \sum_{z_1 \in N^c(x)} \sum_{z_2 \in N^c(y)} |N^c(z_1) \cap N^c(z_2)| k_{r-5} + (\delta n - |N^c(x)|)(\delta n - |N^c(y)|) k_{r-4}. \end{aligned} \tag{8.24}$$

So $|\pi_1(xy)| \leq 3(\delta n)^3 k_{r-5} + (\delta n)^2 k_{r-4}$. By (8.2), we have $|\pi_2(xy)| \leq (\delta n)^3 k_{r-5}$. Therefore, by (8.23), Proposition 4.1 and the fact that $r^{1/2} \geq 5$, we have

$$\begin{aligned} |\pi(xy)| &\leq 4(\delta n)^3 k_{r-5} + 203(\delta r)^4 k_{r-2} + 4(\delta n)^2 k_{r-4} + 2|N^c(x) \cap N^c(y)| k_{r-3} \\ &\leq (32(\delta r)^3 + 203(\delta r)^4 + 16(\delta r)^2) k_{r-2} + 2|N^c(x) \cap N^c(y)| k_{r-3} \\ &\leq 20(\delta r)^2 k_{r-2} + 2|N^c(x) \cap N^c(y)| k_{r-3} \qquad (8.25) \\ &\leq k_{r-2}/10^5 + 4\delta r k_{r-2} = (1/10^5 + 4/10^4 r^{1/2}) k_{r-2} \\ &\leq 9k_{r-2}/10^5 \leq \kappa/10^4. \end{aligned}$$

□

**Claim 8.5.** *For each vertex $x \in V(G)$, $\sum_{y \in N(x)} |\pi(xy)| \leq \kappa n/10^4 r$. That is, $\pi/\kappa$ satisfies* (A2) *in the definition of $\pi/\kappa$ being $r$-smooth.*

*Proof of Claim 8.5.* By (8.25) and Proposition 4.1, we have, for each $x \in V(G)$, that

$$\begin{aligned} \sum_{y \in N(x)} |\pi(xy)| &\leq 20(\delta r)^2 n k_{r-2} + 2 \sum_{y \in N(x)} |N^c(x) \cap N^c(y)| k_{r-3} \\ &\leq n k_{r-2}/10^5 r + 2 \sum_{z \in N^c(x)} |N^c(z)| k_{r-3} \leq n k_{r-2}/10^5 r + 2\delta^2 n^2 k_{r-3} \\ &\leq n k_{r-2}/10^5 r + 4\delta^2 r n k_{r-2} \leq 9n k_{r-2}/10^5 r \leq \kappa n/10^4 r. \end{aligned}$$

□

**Claim 8.6.** *We have $\sum_{x \in V(G)} \sum_{y \in N(x)} |\pi_1(xy)| \leq n^2 k_{r-2}/10^5 r^2$.*

*Proof of Claim 8.6.* Note that, as $10^4 \delta^2 n \geq 1$,

$$\sum_{x \in V(G)} (\delta n - |N^c(x)|) \leq \sum_{x \in X} \delta n + \sum_{x \notin X} r \leq \delta^2 r n^2 + rn \leq 10^5 \delta^2 r n^2. \tag{8.26}$$

Note also that

$$\sum_{y\in V(G)}\sum_{z_1\in N^c(y)}|N^c(z_1)|\le \delta n\sum_{z_1\in V(G)}|N^c(z_1)|\le \delta^2 n^3. \tag{8.27}$$

Therefore, by (8.26) and (8.27),

$$\sum_{x\in V(G)}\sum_{y\in N(x)}\Big((\delta n-|N^c(x)|)\sum_{z_1\in N^c(y)}|N^c(z_1)|\Big)\le 10^5\delta^4 rn^5. \tag{8.28}$$

Note also that

$$\begin{aligned}\sum_{x\in V(G)}\sum_{y\in N(x)}\sum_{z_1\in N^c(x)}\sum_{z_2\in N^c(y)}|N^c(z_1)\cap N^c(z_2)| &\le (\delta n)^2\sum_{z_1\in V(G)}\sum_{z_2\in V(G)}|N^c(z_1)\cap N^c(z_2)|\\ &\le (\delta n)^2\sum_{z\in V(G)}|N^c(z)|^2\le \delta^4 n^5 \qquad (8.29)\end{aligned}$$

Furthermore, by (8.26),

$$\sum_{x\in V(G)}\sum_{y\in N(x)}(\delta n-|N^c(x)|)(\delta n-|N^c(y)|)\le\Big(\sum_{x\in V(G)}(\delta n-|N^c(x)|)\Big)^2\le(10^5\delta^2 rn^2)^2. \tag{8.30}$$

Therefore, by (8.24), (8.28), (8.29), (8.30) and Proposition 4.1

$$\begin{aligned}\sum_{x\in V(G)}\sum_{y\in N(x)}|\pi_1(xy)| &\le 2\cdot 10^5\delta^4 rn^5 k_{r-5}+\delta^4 n^5 k_{r-5}+(10^5\delta^2 rn^2)^2 k_{r-4}\\ &\le \big(16\cdot 10^5(\delta r)^4+8\delta^4 r^3+4\cdot 10^{10}(\delta r)^4\big)n^2 k_{r-2}\\ &\le 10^{11}(\delta r)^4 n^2 k_{r-2}=n^2k_{r-2}/10^5r^2.\end{aligned}$$

□

**Claim 8.7.** *We have* $\sum_{x\in V(G)}\sum_{y\in N(x)}|\pi_2(xy)|\le n^2k_{r-2}/10^5r^2$.

*Proof of Claim 8.7.* Note that, from (8.2), for each $x\in V(G)$ and $y\in N(x)$,

$$|\pi_2(xy)|\le(\delta n-|N^c(y)|)\delta^2n^2k_{r-5}+(\delta n-|N^c(x)|)\delta^2n^2k_{r-5}.$$

Together with (8.26) and Proposition 4.1, this implies that

$$\begin{aligned}\sum_{x\in V(G)}\sum_{y\in N(x)}|\pi_2(xy)| &\le 2n\sum_{x\in V(G)}(\delta n-|N^c(x)|)\delta^2n^2k_{r-5}\\ &\le 2n\cdot 10^5\delta^2rn^2\cdot\delta^2n^2k_{r-5}\le 10^7(\delta r)^4n^2k_{r-2}\le n^2k_{r-2}/10^5r^2.\end{aligned}$$

□

**Claim 8.8.** *We have* $\sum_{x\in V(G)}\sum_{y\in N(x)}|N^c(x)\cap N^c(y)|k_{r-3}\le n^2k_{r-2}/10^5r^2$.

*Proof of Claim 8.8.* We have that, using Proposition 4.1,

$$\begin{aligned}\sum_{x\in V(G)}\sum_{y\in N(x)}|N^c(x)\cap N^c(y)|k_{r-3} &\le\sum_{x\in V(G)}\sum_{z\in N^c(x)}|N^c(z)|k_{r-3}\le\delta^2n^3k_{r-3}\\ &\le 2\delta^2rn^2k_{r-2}\le n^2k_{r-2}/10^5r^2.\end{aligned}$$

□

**Claim 8.9.** *We have* $\sum_{x\in V(G)}\sum_{y\in N(x)}\bar e(N^c(x),N^c(y))k_{r-4}\le n^2k_{r-2}/10^5r^2$.

*Proof of Claim 8.9.* Note that

$$\sum_{x\in V(G)}\sum_{y\in N(x)}\bar e(N^c(x),N^c(y))\le|\{(x,z_1,z_2,y)\in V(G)^4: xz_1,z_1z_2,z_2y\notin E(G)\}|\le n(\delta n)^3,$$

so by Proposition 4.1,

$$\sum_{x\in V(G)}\sum_{y\in N(x)} \bar{e}(N^c(x), N^c(y))k_{r-4} \le \delta^3 n^4 k_{r-4} \le 4\delta^3 r^2 n^2 k_{r-2} \le n^2 k_{r-2}/10^5 r^2. \qquad \square$$

Now (8.23) and Claims 8.6–8.9 together imply that

$$2\sum_{e\in E(G)} |\pi(e)| = \sum_{x\in V(G)}\sum_{y\in N(x)} |\pi(xy)| \le 203(\delta r)^4 n^2 k_{r-2} + 7n^2 k_{r-2}/10^5 r^2 \le 2n^2\kappa/10^4 r^2.$$

Thus $\pi/\kappa$ satisfies (A3) in the definition of $\pi/\kappa$ being $r$-smooth. This completes the proof that $\pi/\kappa$ is $r$-smooth. $\square$

## 9. Proof of Theorem 1.3

We now combine our results and techniques to prove Theorem 1.3. After some initial preprocessing, we give each clique a uniform weighting before using Lemma 8.1 to break down the adjustments that need to be made to the weight over each edge. We carry out the (potentially) larger adjustments using our vertex-gadgets from Lemma 7.3, while the finer adjustments are shown to be $r$-smooth by Lemma 8.3 and can thus be made using Lemma 5.4; making these corrections gives a fractional $K_r$-decomposition of the graph.

*Proof of Theorem 1.3.* First note that, for $r \le 24$, $1/10^4 r^{3/2} \le 1/64r^3$ (with room to spare), so the result follows from Theorem 1.5 with $k = 2$. So we may assume that $r \ge 25$.

Let $\delta := 1/10^4 r^{3/2}$ and $X := \{x \in V(G) : d(x) \ge (1-\delta)n + r - 1\}$. As in the proof of Theorem 6.1, we may assume that $G[X]$ is $K_r$-free and that, similarly, $|X| \le \delta(r-1)n$.

Let $\kappa := k_{r-2} - 2\delta n k_{r-3}$, and, for each vertex $x \in V(G)$, let

$$\gamma(x) := (\delta n - |N^c(x)|)k_{r-3}.$$

By Lemmas 8.1 and 8.3, there are functions $\sigma : V(G) \to \mathbb{R}$ and $\pi : E(G) \to \mathbb{R}$, so that the following hold.

(i) For each edge $xy \in E(G)$, $\kappa^{(r)}_{xy} = \kappa + \gamma(x) + \gamma(y) + \sigma(x) + \sigma(y) + \pi(xy)$.
(ii) For each vertex $x \in V(G)$, $|\sigma(x)| \le k_{r-2}/10^4 r$.
(iii) The function $\pi/\kappa$ is $r$-smooth.

By Lemma 5.4, there exists a weighting $\omega' : \mathcal{K}_r \to \mathbb{R}$ so that the following hold.

(iv) For each $e \in E(G)$, $\sum_{K\in\mathcal{K}_r : e\in E(K)} \omega'(K) = \pi(e)/\kappa$.
(v) For each $K \in \mathcal{K}_r$, $|\omega'(K)| \le 1/2\kappa$.

Let $\mathcal{A} := \{K \in \mathcal{K}_r : |V(K)\cap X| \le r^{1/2} + 2\}$. By Lemma 7.3, for each $x \in V(G)$, there is a function $\xi_x : \mathcal{A} \to \mathbb{R}$, so that

(vi) If $x \in V(G)$ and $e \in E(G)$, then $\sum_{K\in\mathcal{A}:\, e\in E(K)} \xi_x(K) = \mathbf{1}_{\{x\in V(e)\}}$.
(vii) For each $K \in \mathcal{A}$, and $x \in V(G)$, if $i = |V(K)\cap\{x\}|$, then $|\xi_x(K)| \le 80n^{i+1}/r^{i+1}k_r$.

Extend each $\xi_x$ by letting $\xi_x(K) := 0$ for each $K \in \mathcal{K}_r \setminus \mathcal{A}$. Define a function $\omega : \mathcal{K}_r \to \mathbb{R}$ by

$$\omega(K) := \frac{1}{\kappa}\Big(1 - \kappa\cdot\omega'(K) - \sum_{x\in V(G)} (\gamma(x)+\sigma(x))\xi_x(K)\Big). \tag{9.1}$$

We now check that $\omega$ gives a fractional $K_r$-decomposition of $G$.

Firstly, for each edge $xy \in E(G)$, by (9.1), the definition of $\kappa^{(r)}_{xy}$, (iv) and (vi), and then by (i), we have

$$\sum_{K \in \mathcal{K}_r : xy \in E(K)} \omega(K) = \frac{1}{\kappa}\Big(\kappa^{(r)}_{xy} - \pi(xy) - \sum_{v \in V(G)} (\gamma(v) + \sigma(v))\mathbf{1}_{\{v \in \{x,y\}\}}\Big) = 1.$$

Secondly note that, for each $x \in V(G)$, $|\gamma(x)| \leq \delta n k_{r-3}$, and thus, by (ii) and Proposition 4.1,

$$|\gamma(x) + \sigma(x)| \leq \delta n k_{r-3} + k_{r-2}/10^4 r \leq (8\delta r^3 + 4r/10^4)k_r/n^2 \leq 9r^{3/2}k_r/10^4 n^2. \tag{9.2}$$

Furthermore, if $x \in V(G) \setminus X$, then $|\gamma(x)| \leq r k_{r-3}$, and thus by (ii), Proposition 4.1 and the fact that $n \geq 10^4 r^3$,

$$|\gamma(x) + \sigma(x)| \leq r k_{r-3} + k_{r-2}/10^4 r \leq (8r^4/n + 4r/10^4)k_r/n^2 \leq 12rk_r/10^4 n^2. \tag{9.3}$$

Therefore, if $K \in \mathcal{A}$, then, by the definition of $\mathcal{A}$, (9.2), (9.3) and the fact that $r \geq 25$,

$$\begin{aligned}\sum_{x \in V(K)} |\gamma(x) + \sigma(x)| &\leq \sum_{x \in V(K)\cap X} |\gamma(x) + \sigma(x)| + \sum_{x \in V(K)\setminus X} |\gamma(x) + \sigma(x)| \\ &\leq (r^{1/2} + 2) \cdot 9k_r r^{3/2}/10^4 n^2 + r \cdot 12rk_r/10^4 n^2 \leq 3r^2 k_r/10^3 n^2. \end{aligned} \tag{9.4}$$

Furthermore, (9.2), (9.3), and the fact that $|X| \leq \delta(r-1)n$ together imply that

$$\sum_{x \in V(G)} |\gamma(x) + \sigma(x)| \leq \delta(r-1)n \cdot 9r^{3/2}k_r/10^4 n^2 + n \cdot 12rk_r/10^4 n^2 \leq 2rk_r/10^3 n. \tag{9.5}$$

So for each clique $K \in \mathcal{A}$, we have

$$\begin{aligned}\Big|\sum_{x \in V(G)} (\gamma(x) + \sigma(x))\xi_x(K)\Big| &\overset{\text{(vii)}}{\leq} \sum_{x \in V(K)} |\gamma(x) + \sigma(x)|(80n^2/r^2 k_r) + \sum_{x \in V(G)\setminus V(K)} |\gamma(x) + \sigma(x)|(80n/rk_r) \\ &\overset{(9.4),(9.5)}{\leq} (3r^2 k_r/10^3 n^2)(80n^2/r^2 k_r) + (2rk_r/10^3 n)(80n/rk_r) \leq 1/2. \end{aligned} \tag{9.6}$$

If $K \in \mathcal{K}_r \setminus \mathcal{A}$, then as $\xi_x(K) = 0$ for each $x \in V(G)$, we have $|\sum_{x \in V(G)}(\gamma(x) + \sigma(x))\xi_x(K))| = 0$. Therefore, by (9.1), (v), and (9.6), for each $K \in \mathcal{K}_r$, $\omega(K) \geq (1 - 1/2 - 1/2)/\kappa \geq 0$, as required. □

## References


[1] B. Barber, D. Kühn, A. Lo, and D. Osthus. Edge-decompositions of graphs with high minimum degree. *arXiv:1410.5750*, 2014.

[2] D. Dor and M. Tarsi. Graph decomposition is NP-complete: a complete proof of Holyer's conjecture. *SIAM J. Comput.*, 26(4):1166–1187, 1997.

[3] F. Dross. Fractional triangle decompositions in graphs with large minimum degree. *arXiv:1503.08191*, 2015.

[4] P. Dukes. Rational decomposition of dense hypergraphs and some related eigenvalue estimates. *Linear Algebra Appl.*, 436(9):3736–3746, 2012.

[5] P. Dukes. Corrigendum to "Rational decomposition of dense hypergraphs and some related eigenvalue estimates" [Linear Algebra Appl. 436 (9) (2012) 3736–3746]. *Linear Algebra Appl.*, 467:267–269, 2015.

[6] K. Garaschuk. *Linear methods for rational triangle decompositions*. PhD thesis, University of Victoria, 2014.

[7] T. Gustavsson. *Decompositions of large graphs and digraphs with high minimum degree*. PhD thesis, Univ. of Stockholm, 1991.

[8] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, 2001.

[9] P. Keevash. The existence of designs. *arXiv:1401.3665*, 2014.
[10] T. P. Kirkman. On a problem in combinations. *Cambridge and Dublin Math. J*, 2:191–204, 1847.
[11] C. St. J. A. Nash-Williams. An unsolved problem concerning decomposition of graphs into triangles. In *Combinatorial Theory and its Applications III*, 1179–183. North Holland, 1970.
[12] V. Rödl, M. Schacht, M. H. Siggers, and N. Tokushige. Integer and fractional packings of hypergraphs. *J. Combin. Theory Ser. B*, 97(2):245–268, 2007.
[13] R. M. Wilson. An existence theory for pairwise balanced designs. I. Composition theorems and morphisms. *J. Combin. Theory Ser. A*, 13:220–245, 1972.
[14] R. M. Wilson. An existence theory for pairwise balanced designs. II. The structure of PBD-closed sets and the existence conjectures. *J. Combin. Theory Ser. A*, 13:246–273, 1972.
[15] R. M. Wilson. An existence theory for pairwise balanced designs. III. Proof of the existence conjectures. *J. Combin. Theory Ser. A*, 18:71–79, 1975.
[16] R.M. Wilson. Decompositions of complete graphs into subgraphs isomorphic to a given graph. In *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*, 647–659. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.
[17] R. Yuster. The decomposition threshold for bipartite graphs with minimum degree one. *Random Structures Algorithms*, 21(2):121–134, 2002.
[18] R. Yuster. Asymptotically optimal $K_k$-packings of dense graphs via fractional $K_k$-decompositions. *J. Combin. Theory Ser. B*, 95(1):1–11, 2005.
[19] R. Yuster. Fractional decompositions of dense hypergraphs. *Bull. Lond. Math. Soc.*, 39(1):156–166, 2007.

School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK
*E-mail address*: {b.a.barber, d.kuhn, s.a.lo, r.h.montgomery, d.osthus}@bham.ac.uk