

k -ORDERED HAMILTON CYCLES IN DIGRAPHS

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ABSTRACT. Given a digraph D , let $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$ be the minimum semi-degree of D . D is k -ordered Hamiltonian if for every sequence s_1, \dots, s_k of distinct vertices of D there is a directed Hamilton cycle which encounters s_1, \dots, s_k in this order. Our main result is that every digraph D of sufficiently large order n with $\delta^0(D) \geq \lceil (n+k)/2 \rceil - 1$ is k -ordered Hamiltonian. The bound on the minimum semi-degree is best possible. An undirected version of this result was proved earlier by Kierstead, Sárközy and Selkow [10].

1. INTRODUCTION

The famous theorem of Dirac determines the smallest minimum degree of a graph which guarantees the existence of a Hamilton cycle. There are many subsequent results which investigate degree conditions that guarantee the existence of a Hamilton cycle with some additional properties. In particular, Chartrand (see [13]) introduced the notion of a Hamilton cycle which has to visit a given set of vertices in a prescribed order. More formally, we say that a graph G is k -ordered if for every sequence s_1, \dots, s_k of distinct vertices of G there is a cycle which encounters s_1, \dots, s_k in this order. G is k -ordered Hamiltonian if it contains a Hamilton cycle with this property. Kierstead, Sárközy and Selkow [10] showed that for all $k \geq 2$ every graph on $n \geq 11k - 3$ vertices of minimum degree at least $\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is k -ordered Hamiltonian. This bound on the minimum degree is best possible and proved a conjecture of Ng and Schultz [13]. Several related problems have subsequently been considered: for instance, the case when k is large compared to n was investigated in [6] (but has not been completely settled yet). Ore-type conditions were investigated in [13, 6, 5]. For more results in this direction, see the survey by Gould [8].

It seems that digraphs provide an equally natural setting for such problems. Our main result is a version of the result in [10] for digraphs. The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices. Given a digraph D , the *minimum semi-degree* $\delta^0(D)$ of D is the minimum of the minimum outdegree $\delta^+(D)$ of D and its minimum indegree $\delta^-(D)$.

Theorem 1. *For every $k \geq 3$ there is an integer $n_0 = n_0(k)$ such that every digraph D on $n \geq n_0$ vertices with $\delta^0(D) \geq \lceil (n+k)/2 \rceil - 1$ is k -ordered Hamiltonian.*

Our proof shows that one can take $n_0 := Ck^9$ where C is a sufficiently large constant. Note that if n is even and k is odd the bound on the minimum semi-degree is slightly larger than in the undirected case. However, it is best possible in all cases. In fact, if the minimum semi-degree is smaller, it turns out that D need not even be k -ordered. This is easy to see if k is even: let D be the digraph which consists of a complete digraph A of order $\lceil n/2 \rceil + k/2 - 1$ and a complete digraph B of order $\lfloor n/2 \rfloor + k/2$ which has precisely $k - 1$ vertices in common with A . Pick

vertices $s_1, s_3, \dots, s_{k-1} \in A - B$ and $s_2, s_4, \dots, s_k \in B - A$. Then D has no cycle which encounters s_1, \dots, s_k in this order. A similar construction also works if both k and n are odd. The construction in the remaining case is a little more involved, see [11] for details. Note that every Hamiltonian digraph is 2-ordered Hamiltonian, so the case when $k \leq 2$ in Theorem 1 is covered by the result of Ghouila-Houri [7] (Theorem 4 below) which implies that every digraph with minimum semi-degree at least $n/2$ contains a Hamilton cycle.

Theorem 1 can be used to deduce a version for edges which have to be traversed in a prescribed order by the Hamilton cycle: we say that a digraph D is *k-arc ordered Hamiltonian* if, for every sequence e_1, \dots, e_k of independent edges, D contains a Hamilton cycle which encounters e_1, \dots, e_k in this order. D is *k-arc Hamiltonian* if it contains a Hamilton cycle which encounters these edges in any order. D is called *Hamiltonian k-linked* if $|D| \geq 2k$ and if for every sequence $x_1, \dots, x_k, y_1, \dots, y_k$ of distinct vertices there are disjoint paths P_1, \dots, P_k in D such that P_i joins x_i to y_i and such that together all the P_i cover all the vertices of D . Thus every digraph D which is Hamiltonian k -linked is also k -arc ordered Hamiltonian. Indeed, if x_1y_1, \dots, x_ky_k are the (directed) edges our Hamilton cycle has to encounter then disjoint paths linking y_{i-1} to x_i for all $i = 1, \dots, k$ yield the required Hamilton cycle.

Corollary 2. *For all $k \geq 3$ there is an integer $n_0 = n_0(k)$ such that every digraph D on $n \geq n_0$ vertices with $\delta^0(D) \geq \lceil n/2 \rceil + k - 1$ is Hamiltonian k -linked and thus in particular k -arc ordered Hamiltonian.*

The examples in [11] show that in both parts of Corollary 2 the bound on the minimum semi-degree is best possible. In fact, if the minimum semi-degree is smaller then one cannot even guarantee the digraph to be k -arc ordered. A result of Bermond [3] (see also [2]) implies that if $\delta^0(D) \geq \lceil (n+k)/2 \rceil$ then D is k -arc Hamiltonian. It easily follows that if $\delta^0(D) \geq \lceil (n+1)/2 \rceil$, then D is Hamiltonian 1-linked, i.e. Hamiltonian connected (see [2]). This covers the case $k = 1$ of Corollary 2. As observed in [1, Thm 9.2.10], if $\delta^0(D) \geq \lceil n/2 \rceil + 1$, then D is Hamiltonian 2-linked, which covers the case $k = 2$ of Corollary 2.

Corollary 2 can easily be deduced from Theorem 1 as follows: let x_1, \dots, x_k and y_1, \dots, y_k be distinct vertices where we aim to link x_i to y_i for all i . Let D' be the digraph obtained from D by contracting x_i and y_{i-1} into a new vertex s_i whose outneighbourhood is that of x_i and whose inneighbourhood is that of y_{i-1} . More precisely, let $A := \{x_1, \dots, x_k, y_1, \dots, y_k\}$. Then D' is the digraph obtained from $D - A$ by adding new vertices s_1, \dots, s_k and defining the edges incident to these new vertices as follows. The outneighbours of s_i are the outneighbours of x_i in $V(D) \setminus A$ as well as all the s_j for all those $j \neq i-1$ for which y_j is an outneighbour of x_i in D (where $y_0 := y_k$). Similarly, inneighbours of s_i are the inneighbours of y_{i-1} in $V(D) \setminus A$ as well as all the s_j for all those $j \neq i$ for which x_j is an inneighbour of y_{i-1} in D . It is easy to check that $\delta^0(D') \geq \lceil (|D'| + k)/2 \rceil - 1$ and that a Hamilton cycle in D' which encounters s_1, \dots, s_k in this order corresponds to a spanning set of disjoint paths from x_i to y_i .

A result of Chen et al. [4, Theorem 10] implies that the smallest minimum degree which guarantees an undirected graph to be k -arc ordered Hamiltonian is $\lceil n/2 \rceil + k - 1$. (A graph is k -arc ordered Hamiltonian if for any sequence of k

independent oriented edges there exists a Hamilton cycle which encounters these edges in the given order and orientation.) The smallest minimum degree which forces a graph to be k -linked was determined by Kawarabayashi, Kostochka and Yu [9]. It is not clear whether the minimum degree for Hamiltonian k -linkedness is the same.

The main tool in our proof of Theorem 1 is a recent result by the first authors (Theorem 3 below), which shows that the degree condition in Theorem 1 at least guarantees a k -ordered cycle (but not necessarily a Hamiltonian one). The strategy of the proof of Theorem 1 is to consider such a cycle of maximal length and to show that it must be Hamiltonian. The same strategy was already applied in the proof of the undirected case in [10]. However, both parts of the strategy are more difficult in the digraph case: the existence of a k -ordered directed cycle (i.e. Theorem 3) already confirms a conjecture of Manoussakis [12] for large n . The Hamiltonicity of a k -ordered cycle of maximal length is easier to show in the undirected case as one can consider ‘local transformations’ of a given k -ordered cycle which reverse the orientation of certain segments of the cycle. This means that apart from some basic observations like Lemma 8 below our proof is quite different from that in [10].

Theorem 3. [11] *Let k and n be integers such that $k \geq 2$ and $n \geq 200k^3$. Then every digraph D on n vertices with $\delta^0(D) \geq \lceil (n+k)/2 \rceil - 1$ is k -ordered.*

2. NOTATION AND TOOLS

Given a digraph D , we write $V(D)$ for its vertex set, $E(D)$ for its edge set and $|D| := |V(D)|$ for its order. We write xy for the edge directed from x to y . More generally, if A and B are disjoint sets of vertices of D then an A - B edge is an edge of the form ab where $a \in A$ and $b \in B$. A digraph is *complete* if every pair of distinct vertices is joined by edges in both directions.

Given disjoint subdigraphs D_1 and D_2 of a digraph D such that $D_1 \cup D_2$ is spanning and a set $A \subseteq V(D_1)$, we write $N_{D_i}^+(A)$ for the set of all those vertices $x \in V(D_i) \setminus A$ which in the digraph D receive an edge from some vertex in A . $N_{D_i}^-(A)$ is defined similarly. If A consists of a single vertex x , we just write $N_{D_i}^+(x)$ etc. and put $d_{D_i}^+(x) := |N_{D_i}^+(x)|$ and $d_{D_i}^-(x) := |N_{D_i}^-(x)|$. So in particular, $N_D^+(x)$ is the outneighbourhood of x in D and $d_D^+(x)$ is its outdegree. Also, note that $N_{D_1}^+(x)$ is the outneighbourhood of x in the subdigraph $D[V(D_1)]$ of D induced by $V(D_1)$ and not its outneighbourhood in D_1 (where $x \in D_1$). We let $N_D(x) := N_D^+(x) \cup N_D^-(x)$.

If we refer to paths and cycles in digraphs then we always mean that they are directed without mentioning this explicitly. The *length* of a path is the number of its edges. Given two vertices $x, y \in D$, an x - y path is a path which is directed from x to y . Given two vertices x and y on a directed cycle C , we write xCy for the subpath of C from x to y . Similarly, given two vertices x and y on a directed path P such that x precedes y , we write xPy for the subpath of P from x to y .

A digraph D is *strongly connected* if for every ordered pair x, y of vertices of D there exists an x - y path. D is *Hamiltonian connected* if for every ordered pair x, y of vertices of D there exists a Hamilton path from x to y . (So Hamiltonian connectedness is the same as Hamiltonian 1-linkedness.)

We will often use the following result of Ghouila-Houri [7] which gives a sufficient condition for the existence of a Hamilton cycle in a digraph. In particular, it implies a version of Theorem 1 for $k \leq 2$ as any Hamiltonian digraph is 2-ordered Hamiltonian.

Theorem 4. *Suppose that D is a strongly connected digraph such that $d_D^+(x) + d_D^-(x) \geq |D|$ for every vertex $x \in D$. Then D is Hamiltonian.*

The next result of Overbeck-Larisch [14] provides a sufficient condition for a digraph to be Hamiltonian connected.

Theorem 5. *Suppose that D is a digraph such that $d_D^+(x) + d_D^-(y) \geq |D| + 1$ whenever xy is not an edge. Then D is Hamiltonian connected.*

3. PRELIMINARY RESULTS

Let D be a digraph satisfying the conditions of Theorem 1. Let $S = (s_1, \dots, s_k)$ by any sequence of $k \geq 3$ vertices of D . We will often view S as a set. An S -cycle in D is a cycle which encounters s_1, \dots, s_k in this order. So we have to show that D has a Hamiltonian S -cycle. Theorem 3 implies the existence of an S -cycle in D . Let C be a longest such cycle and suppose that C is not Hamiltonian. Let H be the subdigraph of D induced by all the vertices outside C . Our aim is to find a longer S -cycle by modifying C (yielding a contradiction). The purpose of this section is to collect the properties of C and H that we need in our proof of Theorem 1.

We let F be the set of all those vertices on C which receive an edge from some vertex in H and we let T be the set of all those vertices on C which send an edge to some vertex in H . Given $i \in \mathbb{N}$, we write F_i for the set of all those vertices on C which receive an edge from at least i vertices in H . Thus $F_1 = F$. T_i is defined similarly. Given a vertex x on C , we will denote its successor on C by x^+ and its predecessor by x^- .

Lemma 6. *H is Hamiltonian connected and $d_H^-(x) + d_H^+(y) \geq |H| + k - 2$ for all vertices $x, y \in H$. Moreover any digraph obtained from H by deleting at most 2 vertices is strongly connected and $k \leq |H| \leq \lfloor \frac{n-k}{2} \rfloor$.*

Proof. We first show that any two (not necessarily distinct) vertices $x, y \in H$ for which H contains an x - y path, P say, satisfy the degree condition in the lemma. To see this, note that no vertex in $N_C^-(x)$ is a predecessor of some vertex in $N_C^+(y)$. Indeed, if $v \in N_C^-(x)$ and $v^+ \in N_C^+(y)$ then by replacing the edge vv^+ with the path $vxPyv^+$ we obtain a longer S -cycle, a contradiction. But this means that $d_C^-(x) + d_C^+(y) \leq |C|$ and thus

$$(1) \quad d_H^-(x) + d_H^+(y) \geq 2 \left(\left\lceil \frac{n+k}{2} \right\rceil - 1 \right) - |C| \geq |H| + k - 2,$$

as required. However, as $k \geq 3$ this degree condition means that $N_H^-(x) \cap N_H^+(y) \neq \emptyset$ and so H contains an y - x path of length 2. Thus whenever H contains an x - y path it also contains a y - x path.

Now let x and z be any two vertices of H . What we have shown above applied with $y := x$ implies that $d_H^-(x) + d_H^+(x) \geq |H| + 1$ and thus $|N_H(x)| \geq (|H| + 1)/2$. Note that by the above x is joined to every vertex in $N_H(x)$ with paths in both directions. Similarly, $|N_H(z)| \geq (|H| + 1)/2$ and z is joined to every vertex in $N_H(z)$

with paths in both directions. As $|N_H(x) \cap N_H(z)| > 0$ this means that x is joined to z with paths in both directions, i.e. H is strongly connected. Together with (1) this in turn implies that $d_H^-(x) + d_H^+(z) \geq |H| + k - 2 \geq |H| + 1$ for all vertices $x, z \in H$. In particular, H is Hamiltonian connected by Theorem 5.

To show that any digraph H' obtained from H by deleting at most 2 vertices is strongly connected note that $d_{H'}^-(x) + d_{H'}^+(y) \geq |H'| - 1$ for every $x, y \in H'$. Thus if $x \neq y$ then either yx is an edge or H' contains an y - x path of length 2.

It now remains to prove the bounds on $|H|$. Consider any vertex $x \in H$. Then $2(|H| - 1) \geq d_H^-(x) + d_H^+(x) \geq |H| + k - 2$ and so $|H| \geq k$. For the upper bound, note that no vertex in T has a successor in F . Indeed, if v is such a vertex in T and v^+ is its successor then we could replace vv^+ with a path through H to obtain a longer S -cycle, a contradiction. But this means that some vertex of C must have all its inneighbours on C or all its outneighbours on C . Thus $|C| \geq \lceil (n + k)/2 \rceil$ and so $|H| \leq \lfloor (n - k)/2 \rfloor$. \square

Recall that the proof of Lemma 6 implies the following.

Corollary 7. *No vertex on C which lies in T has a successor in F .*

The next result deals with the case when the vertices $x_1 \in T$ and $x_2 \in F$ are further apart.

Lemma 8. *Suppose that $x_1, x_2 \in C$ are distinct and the interior of x_1Cx_2 does not contain a vertex from S . Then there are no distinct vertices $y_1, y_2 \in H$ such that $x_1y_1, y_2x_2 \in E(D)$.*

Proof. Suppose that such y_1, y_2 do exist. Furthermore, we may assume that x_1 and x_2 are chosen such that they satisfy all these properties and subject to this $|x_1Cx_2|$ is minimum. Let Q denote the set of all vertices in the interior of x_1Cx_2 . Then our choice of x_1 and x_2 implies that $N_C^-(y_1) \cap Q = \emptyset$ and $N_C^+(y_2) \cap Q = \emptyset$. Moreover, by Corollary 7 no vertex in $N_C^-(y_1)$ is a predecessor of some vertex in $N_C^+(y_2)$. Thus $d_C^-(y_1) + d_C^+(y_2) \leq |C| - |Q| + 1$ and so

$$n + k - 2 \leq d_D^-(y_1) + d_D^+(y_2) \leq |C| - |Q| + 1 + 2(|H| - 1) = n - |Q| + |H| - 1.$$

This implies that $|H| > |Q|$ and thus replacing the interior of x_1Cx_2 with a Hamilton path from y_1 to y_2 through H (which exists by Lemma 6) yields a longer S -cycle, a contradiction. \square

The next two results will be used in the proof of Lemma 11.

Lemma 9. *Let G be a digraph such that $d_G^+(x) + d_G^-(x) \geq |G| + 3$ for every vertex $x \in G$ and $d_G^+(x) + d_G^-(y) \geq |G| + 1$ for every pair of vertices $x, y \in G$. Let z_1 and z_2 be distinct vertices of G such that $z_1z_2 \notin E(G)$. Then there exists a vertex $a \in N_G^+(z_1) \cap N_G^-(z_2)$ such that $G - \{z_1, z_2, a\}$ is strongly connected.*

Proof. First note that $|N_G^+(z_1) \cap N_G^-(z_2)| \geq 3$ since $z_1z_2 \notin E(G)$. Pick $a_1, a_2, a_3 \in N_G^+(z_1) \cap N_G^-(z_2)$. We will show that one of these a_i can play the role of a . Let $G^* := G - \{z_1, z_2\}$. Note that $d_{G^*}^+(x) + d_{G^*}^-(x) \geq |G^*| + 1$ for every vertex $x \in G^*$ and $d_{G^*}^+(x) + d_{G^*}^-(y) \geq |G^*| - 1$ for every pair of vertices $x, y \in G^*$. In particular, the latter condition implies that G^* is strongly connected. Thus G^* has a Hamilton cycle C by Theorem 4. Let a_1^+ denote the successor of a_1 on C and let a_1^- be its

predecessor. Put $N^+ := N_{G^*}^+(a_1^-) \setminus \{a_1\}$ and $N^- := N_{G^*}^-(a_1^+) \setminus \{a_1\}$. Note that $|N^+|, |N^-| \geq 1$ since $d_{G^*}^+(a_1^-) + d_{G^*}^-(a_1^-) \geq |G^*| + 1$ and $d_{G^*}^+(a_1^+) + d_{G^*}^-(a_1^+) \geq |G^*| + 1$. Similarly $|N^+| + |N^-| \geq |G^*| - 3$. Clearly, if $a_1^- a_1^+$ is an edge or $N^+ \cap N^- \neq \emptyset$, then $G^* - a_1$ is strongly connected and so we can take a to be a_1 . So we may assume that neither of these is the case. But then $N^+ \cup N^- = V(G^*) \setminus \{a_1, a_1^+, a_1^-\}$. Let $v \in N^+$ be such that $|vCa_1^-|$ is maximal. Similarly, let $w \in N^-$ be such that $|a_1^+Cw|$ is maximal. Note that if $w \in vCa_1^-$ then $G^* - a_1$ is strongly connected. So we may assume that this is not the case. But then v must be the successor of w on C , N^+ must consist of precisely the vertices in $V(vCa_1^-) \setminus \{a_1^-\}$ and N^- must consist of precisely the vertices in $V(a_1^+Cw) \setminus \{a_1^+\}$.

Let $A^+ := N^+ \cup \{a_1^-\}$ and $A^- := N^- \cup \{a_1^+\}$. We may assume that G does not contain an A^+A^- edge as otherwise $G^* - a_1$ is strongly connected. We will now show $G^*[A^+]$ is complete and that a_1 receives an edge from every vertex in A^+ . So consider any vertex $x \in A^+$. Then $d_{G^*}^+(x) + d_{G^*}^-(a_1^+) \geq |G^*| - 1$. Together with the fact that there is no A^+A^- edge this shows that $N_{G^*}^+(x) = (A^+ \cup \{a_1\}) \setminus \{x\}$. Thus $G^*[A^+]$ is complete and a_1 receives an edge from every vertex in A^+ . Similarly one can show that $G^*[A^-]$ is complete and that a_1 sends an edge to every vertex in A^- .

Now consider a_2 and a_3 . If for example $a_2 \neq v, w$ then $G^* - a_2$ is strongly connected and so we can take a to be a_2 . As one can argue similarly for a_3 , we may assume that $v = a_2$ and $w = a_3$. If $a_1^+ a_1^-$ is an edge or $a_1 \in N_{G^*}^+(a_1^+) \cap N_{G^*}^-(a_1^-)$ then $G^* - a_2$ is strongly connected. (Here we used that $a_1^- \neq v = a_2$ since $|N^+| \geq 1$.) If this is not the case, then $d_{G^*}^+(a_1^+) + d_{G^*}^-(a_1^-) \geq |G^*| - 1$ implies the existence of some vertex $x \in N_{G^*}^+(a_1^+) \cap N_{G^*}^-(a_1^-)$ with $x \neq a_1$. If $x \in A^+$ then $a_1^+ x$ is an A^-A^+ edge avoiding $w = a_3$ and so $G^* - a_3$ is strongly connected. (Here we used that $a_1^+ \neq w = a_3$ since $|N^-| \geq 1$.) Similarly, if $x \in A^-$ then $G^* - a_2$ is strongly connected. Altogether, this shows that we can take a to be a_1, a_2 or a_3 . \square

Lemma 10. *Suppose that H contains a vertex v with $d_H^-(v) + d_H^+(v) \leq |H| + k - 1$. Suppose that $x_1, x_2 \in T$ and $y_1, y_2 \in F$ are distinct vertices on C . Then $x_1v, vy_1 \in E(D)$ or $x_2v, vy_2 \in E(D)$ (or both).*

Proof. Let F_v denote the set of all those vertices on C which receive an edge from v . Let T_v^+ denote the set of all those vertices on C whose predecessor sends an edge to v . Corollary 7 implies that $T_v^+ \cap F_v = \emptyset$. Since

$$d_C^-(v) + d_C^+(v) \geq 2 \left(\left\lceil \frac{n+k}{2} \right\rceil - 1 \right) - (|H| + k - 1) \geq |C| - 1$$

this shows that at most one vertex on C lies outside $T_v^+ \cup F_v$. Let z be the vertex in $V(C) \setminus (T_v^+ \cup F_v)$ (if it exists).

Suppose first that $z \notin F$ (this also covers the case when z does not exist). Then $z \neq y_1, y_2$. Also either $z \neq x_1^+$ or $z \neq x_2^+$. So let us assume that $z \neq x_1^+$ (the case when $z \neq x_2^+$ is similar). We will show that $x_1v, vy_1 \in E(D)$. So suppose first that $x_1v \notin E(D)$. Then $x_1^+ \notin T_v^+$ and thus $x_1^+ \in F_v$, a contradiction to Corollary 7. Similarly, if $vy_1 \notin E(D)$ then $y_1 \notin F_v$ and thus $y_1 \in T_v^+$, i.e. the predecessor of y_1 lies in T , contradicting Corollary 7.

So suppose next that $z \in F$ and thus, by Corollary 7, the predecessor of z does not lie in T . This in turn implies that $z \neq x_1^+, x_2^+$. Moreover either $z \neq y_1$ or $z \neq y_2$. So let us assume that $z \neq y_1$. Similarly as before one can show that $x_1v, vy_1 \in E(D)$. \square

In our proof of Theorem 1 we will frequently need two disjoint paths through H joining two given disjoint pairs of vertices on C in order to modify C into a longer S -cycle. The following lemma implies the existence of such paths provided that the pairs consist of vertices having sufficiently many neighbours in H (see also Corollary 12).

Lemma 11. *Suppose that $X_1, X_2 \subseteq T$ and $Y_1, Y_2 \subseteq F$ are disjoint subsets of $V(C)$ such that $|N_H^+(X_1)|, |N_H^+(X_2)| \geq 3$ and $|N_H^-(Y_1)|, |N_H^-(Y_2)| \geq 3$. Then there are disjoint X_i - Y_i paths P_i of length at least 2 and such that all inner vertices of P_1 and P_2 lie in H . Moreover, if $|H| \geq 15$ and if we even have that $|N_H^+(X_1)|, |N_H^+(X_2)| \geq 8$ and $|N_H^-(Y_1)|, |N_H^-(Y_2)| \geq 8$ then we can find such paths which additionally satisfy $|P_1 \cup P_2| \geq |H|/6$.*

Proof. By disregarding some neighbours if necessary we may assume that $|N_H^+(X_1)| = |N_H^+(X_2)| = |N_H^-(Y_1)| = |N_H^-(Y_2)|$. Our first aim is to show that for some $i \in \{1, 2\}$ there is an X_i - Y_i path P_i which satisfies the following properties:

- (i) The graph $H' := H - V(P_i)$ has a Hamilton cycle C' .
- (ii) All $x, y \in H'$ satisfy $d_{H'}^+(x) + d_{H'}^-(y) \geq |H'| - 2$.
- (iii) $3 \leq |P_i| \leq 5$, i.e. P_i contains at least 1 and at most 3 vertices from H .
- (iv) If $i = 1$ then $|N_H^+(X_2) \cap V(P_1)| \leq 2$ and $|N_H^-(Y_2) \cap V(P_1)| \leq 2$. If $i = 2$ then $|N_H^+(X_1) \cap V(P_2)| \leq 2$ and $|N_H^-(Y_1) \cap V(P_2)| \leq 2$.

If we have found such an i , say $i = 1$, then our aim is to use the Hamilton cycle C' in order to find P_2 . To prove the existence of such an i , recall that Lemma 6 implies $d_H^-(x) + d_H^+(y) \geq |H| + k - 2 \geq |H| + 1$ for every pair of vertices $x, y \in H$. Thus condition (ii) will hold automatically if (iii) holds.

Now suppose first that there exists a vertex $z_1 \in N_H^+(X_1) \cap N_H^-(Y_1)$. Take $i = 1$ and take P_1 to be any X_1 - Y_1 path whose interior consists precisely of z_1 . Then $d_{H'}^-(x) + d_{H'}^+(x) \geq |H'|$ for every $x \in H'$. As H' is strongly connected by Lemma 6 we can apply Theorem 4 to find a Hamilton cycle C' of H' . (If $|H'| = 2$ then C' will consist of just a double edge.) In the case when $N_H^+(X_2) \cap N_H^-(Y_2) \neq \emptyset$ we proceed similarly.

Now suppose that $N_H^+(X_1) \cap N_H^-(Y_1) = \emptyset$ and $N_H^+(X_2) \cap N_H^-(Y_2) = \emptyset$. Then Lemma 10 implies that $d_H^-(x) + d_H^+(x) \geq |H| + k \geq |H| + 3$ for every $x \in H$. If there is an $N_H^+(X_1)$ - $N_H^-(Y_1)$ edge z_1z_2 take $i := 1$ and take P_1 to be any X_1 - Y_1 path whose interior consists of this edge. Then $d_{H'}^-(x) + d_{H'}^+(x) \geq |H| - 1 = |H'| + 1$ for every $x \in H'$ and so again, as H' is strongly connected by Lemma 6, we can apply Theorem 4 to find a Hamilton cycle C' of H' . In the case when there is an $N_H^+(X_2)$ - $N_H^-(Y_2)$ edge we proceed similarly.

Thus we may assume that $N_H^+(X_i) \cap N_H^-(Y_i) = \emptyset$ and that there is no $N_H^+(X_i)$ - $N_H^-(Y_i)$ edge (for $i = 1, 2$). Pick any vertex $z_1 \in N_H^+(X_1)$ and let $z_2 \in N_H^-(Y_1)$ be a vertex such that $|N_H^+(X_2) \cap \{z_1, z_2\}| \leq 1$ and $|N_H^-(Y_2) \cap \{z_1, z_2\}| \leq 1$. (The fact that we can choose such a z_2 follows from $N_H^+(X_i) \cap N_H^-(Y_i) = \emptyset$ and our assumption that

the sizes of the $N_H^+(X_i)$ and the $N_H^-(Y_i)$ are equal.) Apply Lemma 9 with $G := H$ to find a vertex $z_3 \in N_H^+(z_1) \cap N_H^-(z_2)$ such that $H - \{z_1, z_2, z_3\}$ is strongly connected. Take $i := 1$ and P_1 to be any X_1 - Y_1 path whose interior consists of $z_1 z_3 z_2$. Then $d_{H'}^-(x) + d_{H'}^+(x) \geq |H| - 3 = |H'|$ for every $x \in H'$ and so again H' contains a Hamilton cycle C' by Theorem 4. Our choice of z_1 and z_2 implies that (iv) holds.

Altogether, this shows that in each case for some i there exists a path P_i satisfying (i)–(iv). We may assume that $i = 1$. As mentioned before, our aim now is to use the Hamilton cycle C' of H' in order to find an X_2 - Y_2 path P_2 through H' . In the case when $|N_H^+(X_2)|, |N_H^-(Y_2)| \geq 3$ this is trivial since by (iv) both $N_H^+(X_2)$ and $N_H^-(Y_2)$ meet H' in at least one vertex.

So suppose now that $|H| \geq 15$ and $|N_H^+(X_2)|, |N_H^-(Y_2)| \geq 8$ and thus we wish to find a long X_2 - Y_2 path. To do this, let $N^+ := N_H^+(X_2) \cap V(H')$ and $N^- := N_H^-(Y_2) \cap V(H')$. Thus $|N^+|, |N^-| \geq 6$ by (iv). Choose $a_1 \in N^+$ and $b_1 \in N^-$ to be distinct such that $|a_1 C' b_1|$ is maximum. If $|a_1 C' b_1| \geq |H'|/6$ then we can take P_2 to be any X_2 - Y_2 path whose interior consists of $a_1 C' b_1$. So we may assume that $|a_1 C' b_1| \leq |H'|/6$.

Note that the choice of a_1 and b_1 implies that $N^+, N^- \subseteq V(a_1 C' b_1)$. Moreover, all the vertices in N^+ must precede the vertices in N^- on $a_1 C' b_1$. (Indeed, if e.g. $a \in N^+$ and $b \in N^-$ are distinct vertices such that b precedes a , i.e. a lies on $b C' b_1$ then $|a C' b| \geq |H'| - |a_1 C' b_1| \geq |H'|/2$, contradicting the choice of a_1 and b_1 .) Thus $|N^+ \cap N^-| \leq 1$ and there are vertices $a_2, \dots, a_5 \in N^+$ and $b_2, \dots, b_5 \in N^-$ such that $a_1, \dots, a_5, b_5, \dots, b_1$ are distinct and appear on C' in this order. We now distinguish several cases.

Case 1. *There are $i, j \leq 4$ such that $a_i b_j$ is an edge.*

Note that $d_{H'}^+(a_5) \geq |H'|/2 - 1$ or $d_{H'}^-(b_5) \geq |H'|/2 - 1$ by (ii). Suppose that the former holds (the other case is similar). As $|a_1 C' b_1| \leq |H'|/6$ this means that a_5 has at least $|H'|/3 - 1$ outneighbours in the interior of $b_1 C' a_1$ and so we can find such an outneighbour v with $|v C' a_1| \geq |H'|/3$. But then we can take P_2 to be any X_2 - Y_2 path whose interior consists of $a_5 v C' a_i b_j$ (Figure 1).

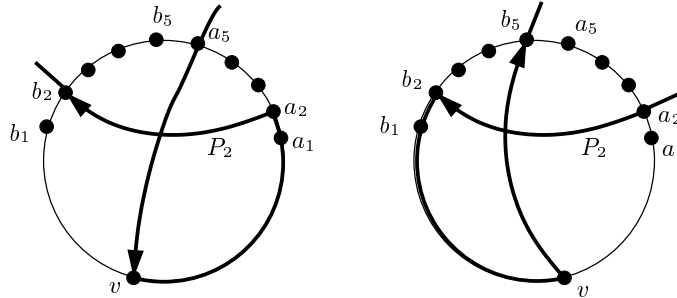


FIGURE 1. The path P_2 in Case 1. The left figure is for the subcase when $d_{H'}^+(a_5) \geq |H'|/2 - 1$ and the right figure is for the subcase when $d_{H'}^-(b_5) \geq |H'|/2 - 1$.

Case 2. *For all $i, j \leq 4$ $a_i b_j$ is not an edge.*

Case 2.1. *There exists some vertex $u \in N_{H'}^+(a_1) \cap N_{H'}^-(b_3)$.*

Note that $u \neq a_2, b_4$ since by our assumption neither a_2b_3 nor a_1b_4 is an edge. As before, either $d_{H'}^+(a_2) \geq |H'|/2 - 1$ or $d_{H'}^-(b_4) \geq |H'|/2 - 1$. Suppose that the former holds (the other case is similar).

If u lies in the interior of $a_1C'b_3$, let v be an outneighbour of a_2 in the interior of $b_3C'a_1$ with $|vC'a_1| \geq |H'|/3$. Then we can take P_2 to be any X_2 - Y_2 path whose interior consists of $a_2vC'a_1ub_3$.

So we may assume that u lies in the interior of $b_3C'a_1$. But then either the interior of $b_3C'u$ contains at least $|H'|/6 - 1$ outneighbours of a_2 or the interior of $uC'a_1$ contains at least $|H'|/6 - 1$ outneighbours of a_2 . If the former holds let v be any outneighbour of a_2 in the interior of $b_3C'u$ such that $|vC'u| \geq |H'|/6$ and take P_2 to be any X_2 - Y_2 path whose interior consists of $a_2vC'ub_3$ (see Figure 2). If the latter holds let v be any outneighbour of a_2 in the interior of $uC'a_1$ such that $|vC'a_1| \geq |H'|/6$ and take P_2 to be any X_2 - Y_2 path whose interior consists of $a_2vC'a_1ub_3$.

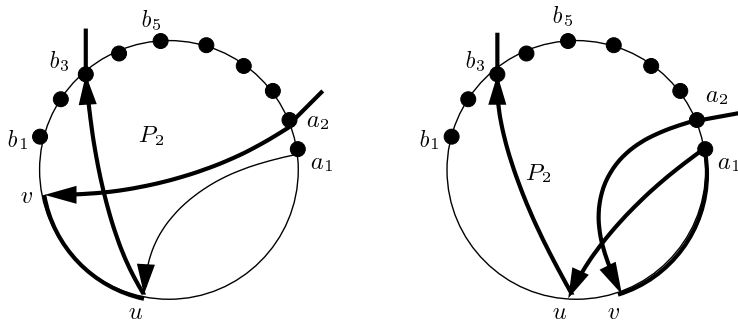


FIGURE 2. The path P_2 in Case 2.1 if u lies in the interior of $b_3C'a_1$. The left figure is for the subcase when the interior of $b_3C'u$ contains at least $|H'|/6 - 1$ outneighbours of a_2 . The right figure is for the subcase when the interior of $uC'a_1$ contains at least $|H'|/6 - 1$ outneighbours of a_2 .

Case 2.2. *There exists some vertex $u \in N_{H'}^+(a_3) \cap N_{H'}^-(b_1)$.*

This case is similar to Case 2.1 and we omit the details.

Case 2.3. *Both $N_{H'}^+(a_1) \cap N_{H'}^-(b_3)$ and $N_{H'}^+(a_3) \cap N_{H'}^-(b_1)$ are empty.*

Together with (ii) and our assumption that a_1b_3 is not an edge this implies that $N_{H'}^+(a_1) \cup N_{H'}^-(b_3) = V(H') \setminus \{a_1, b_3\}$. Since a_3b_3 is not an edge this means that a_1a_3 is an edge. Similarly it follows that b_3b_1 is an edge. But as before either $d_{H'}^+(a_2) \geq |H'|/2 - 1$ or $d_{H'}^-(b_2) \geq |H'|/2 - 1$. Suppose that the former holds (the other case is similar). Then we can find an outneighbour v of a_2 in the interior of $b_1C'a_1$ with $|vC'a_1| \geq |H'|/3$. But then we can take P_2 to be any X_2 - Y_2 path whose interior consists of $a_2vC'a_1a_3C'b_1$. \square

Lemma 11 immediately implies the following corollary, which is sometimes more convenient to apply.

Corollary 12. *Suppose that $x_1, x_2 \subseteq T_3$ and $y_1, y_2 \subseteq F_3$ are distinct vertices on C . Then D contains disjoint x_i - y_i paths P_i of length at least 2 such that all*

inner vertices of P_1 and P_2 lie in H . Moreover, if $|H| \geq 15$ and if we even have that $x_1, x_2 \subseteq T_8$ and $y_1, y_2 \subseteq F_8$ then we can find such paths which additionally satisfy $|P_1 \cup P_2| \geq |H|/6$.

The last of our preliminary results gives a lower bound on the sizes of T_3 and F_3 .

Lemma 13. *We have that $|T|, |F| \geq (n+k)/2 - |H|$. Moreover, $|T_3|, |F_3| \geq (n-k)/2 - |H|$ and $|T_3 \cup F_3| \geq |C| - |H| - 2k$.*

Proof. To see the bound on $|T|$, note that $d_C^-(x) \geq \delta^0(D) - (|H| - 1) \geq (n+k)/2 - |H|$ for every vertex $x \in H$ and so $|T| \geq (n+k)/2 - |H|$. The proof for $|F|$ is similar. To prove the bound on $|T_3|$, we double-count the number $e(T, H)$ of edges in D from T to $V(H)$. Since $d_C^-(x) \geq (n+k)/2 - |H|$ for any vertex $x \in H$ we have that $e(T, H) \geq |H|((n+k)/2 - |H|)$. On the other hand $e(T, H) \leq |T_3||H| + 2(|T| - |T_3|) = |T_3|(|H| - 2) + 2|T|$. Before we can use this to estimate $|T_3|$, we need an upper bound on $|T|$. For this, recall that $|F| \geq (n+k)/2 - |H|$. Together with Corollary 7 this shows that $|T| \leq |C| - |F| \leq (n-k)/2$. Altogether this gives

$$\begin{aligned} |T_3| &\geq \frac{|H|((n+k)/2 - |H|) - (n-k)}{|H| - 2} = \frac{(|H| - 2)(n-k)/2 - |H|(|H| - k)}{|H| - 2} \\ &\geq \frac{n-k}{2} - |H| = \frac{|C| - |H| - k}{2}. \end{aligned}$$

The proof for $|F_3|$ is similar. The bound on $|T_3 \cup F_3|$ follows since $|T_3 \cap F_3| \leq k$. Indeed, the latter holds since Lemma 8 implies that whenever $s, s' \in S$ are distinct and no vertex from S lies in the interior of sCs' then $T_3 \cap F_3$ meets sCs' in at most one vertex. \square

4. PROOF OF THEOREM 1

Throughout this section, we assume that the order n of our given digraph D is sufficiently large compared to k for our estimates to hold. We will also omit floors and ceilings whenever this does not affect the argument. Let S, C and H be as defined at the beginning of Section 3. Recall that we assume that C is not Hamiltonian and will show that we can extend C into a longer S -ordered cycle (which would yield a contradiction and thus would prove Theorem 1). Given consecutive vertices $s, s' \in S$, we call the path obtained from sCs' by deleting s' the *interval from s to s'* . Thus no vertex from S lies in the interior of sCs' and C consists of precisely $|S| = k$ disjoint intervals. In our proof of Theorem 1 we distinguish the following 4 cases according to the order of H . Recall that $|H| \geq k$ by Lemma 6.

Case 1. $k \leq |H| \leq 220k^3$.

Recall that $|T_3| \geq (n-k)/2 - |H| \geq n/3$ by Lemma 13 and so at least one of the k intervals of C must contain at least $n/(3k)$ vertices from T_3 . Suppose that this is the case for the interval I from s to s' . Recall that by Lemma 13 at most $|H| + 2k \leq 3|H|$ vertices of C do not lie in $T_3 \cup F_3$ and by Corollary 7 no vertex in F_3 is the successor of a vertex in T_3 . Since every maximal subpath of I consisting of vertices from T_3 is succeeded by at least one vertex outside $T_3 \cup F_3$, it follows that I contains a subpath A which consists entirely of vertices from T_3

and satisfies $|A| \geq n/(3k(3|H| + 1))$. Let A_1 be the subpath of A consisting of its initial $n/(20k|H|)$ inner vertices and let A_2 be the subpath of A consisting of its last $n/(20k|H|)$ inner vertices.

Let t be the first vertex of A . (So t^+ is the first vertex of A_1 .) Consider any vertex a on t^+Cs' . Lemma 8 implies that $a \notin F$. Thus $N_D^-(a) \subseteq V(C)$ and hence

$$(2) \quad d_C^-(a) \geq \delta^0(D) \geq (n+k)/2 - 1 \geq n+k-1 - |H| - |F| > |C| - |F|.$$

(To see the third inequality recall that $|F| \geq (n+k)/2 - |H|$ by Lemma 13.)

Case 1.1. *There are vertices $a_1 \in A_1$ and $a_2 \in A_2$ such that a_1a_2 is an edge.*

Inequality (2) applied with $a := a_1^+$ implies that there exists a vertex $w \in N_C^-(a_1^+)$ such that the successor w^+ of w lies in F . Recall that F avoids t^+Cs' and so w^+ must lie in $s'Cs' - s'$. Hence w must lie in $s'Cs' - t$ (and thus in the interior of a_2Ca_1). As $a_2^- \in V(A) \subseteq T_3$ and as H is Hamiltonian connected by Lemma 6, there is an $a_2^- - w^+$ path P whose interior consists of precisely all the vertices in H . But then the S -ordered cycle $a_1a_2Cwa_1^+Ca_2^-Pw^+Ca_1$ is Hamiltonian, contradicting the choice of C (see Figure 3).

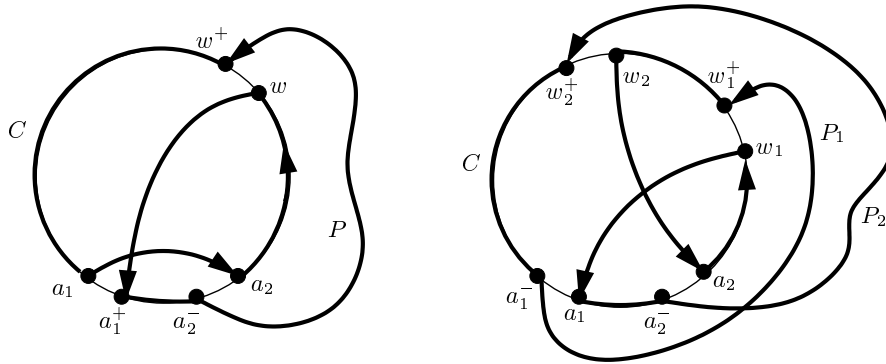


FIGURE 3. Extending C into a longer S -ordered cycle in Case 1.1 (left) and Case 1.2 (right).

Case 1.2. *There are no such vertices $a_1 \in A_1$ and $a_2 \in A_2$.*

Let F_3^- denote the set of all predecessors of vertices in F_3 . Recall that F avoids t^+Cs' . Thus F_3^- avoids $tCs' - s'$. Now consider any vertex a on A_2 . Then $N_D^-(a) \subseteq V(C)$ since $a \notin F$ and thus $N_D^-(a) \subseteq V(C) \setminus V(A_1)$ by our assumption. But then using that $|F_3| \geq (n-k)/2 - |H|$ by Lemma 13 and arguing similarly as in (2) one can show that $d_{C-A_1}^-(a) \geq n-1 - |H| - |F_3| = |C| - 1 - |F_3| = |C - A_1| - |F_3^-| + |A_1| - 1$. Together with the fact that $F_3^- \cap V(A_1) = \emptyset$ this gives

$$(3) \quad |N_{C-A_1}^-(a) \cap F_3^-| \geq |A_1| - 1 \geq n/(21k|H|).$$

Let I_1 be the subpath of the interval I preceding the first vertex in A_1 . So $I_1 = sCt$. Let I_2, \dots, I_k denote all the other intervals. For each $i = 1, \dots, k$ let G_i be the auxiliary bipartite graph whose vertex classes are $V(A_2)$ and $V(I_i) \cap F_3^-$ and in which $a \in V(A_2)$ is joined to $w \in V(I_i) \cap F_3^-$ if $wa \in E(D)$. Recall that F_3^- avoids $tCs' - s'$. Thus $F_3^- \subseteq V(I_1) \cup \dots \cup V(I_k)$ and so the edges of $G_1 \cup \dots \cup G_k$

correspond to the edges of D from F_3^- to A_2 . Together with (3) this implies that there is some i such that

$$e(G_i) \geq \frac{n|A_2|}{21k^2|H|} \geq \frac{n^2}{420k^3|H|^2} \geq 3n \geq 3|G_i|.$$

Thus G_i is not planar and so there are vertices $a_1, a_2 \in V(A_2)$ and $w_1, w_2 \in V(I_i) \cap F_3^-$ such that the edges w_1a_1, w_2a_2 ‘cross’ in G_i , i.e. such that w_1 lies in the interior of a_2Cw_2 and a_1 lies in the interior of w_2Ca_2 . Recall that $w_1^+, w_2^+ \in F_3$ by the definition of F_3^- and $a_1^-, a_2^- \in T_3$ as A_2 consisted of inner vertices of A . Thus we can apply Corollary 12 to obtain disjoint $a_j^-w_j^+$ paths P_j having all their inner vertices in H and such that each P_j contains at least one inner vertex (where $j = 1, 2$). Thus $a_1^-P_1w_1^+Cw_2a_2Cw_1a_1Ca_2^-P_2w_2^+Ca_1^-$ is an S -ordered cycle with at least $|C| + 2$ vertices (note that it contains all the vertices of C), contradicting the choice of C (see Figure 3).

Case 2. $220k^3 \leq |H| \leq n/2 - n/(50k)$.

The argument for this case is similar to that in Case 1. Recall that $|T_3| \geq (n - k)/2 - |H| \geq n/(60k)$ by Lemma 13 and so one of the k intervals of C must contain at least $n/(60k^2)$ vertices from T_3 . Suppose that this is the case for the interval I from s to s' . Let t be the first vertex on I that lies in T_3 . Let A be the set consisting of the last $n/(70k^2)$ vertices from T_3 lying in the interior of I . For each $a \in A$ let Q_a be the set of $220k^3$ vertices of C preceding a . Note that the definition of A implies that Q_a lies in the interior of I and that t precedes the first vertex of Q_a . Together with Lemma 8 this shows that F avoids t^+Cs' and thus all of $Q_a \cup \{a, a^+\}$. In particular, $a \notin F$. Thus $N_D^-(a) \subseteq V(C)$ and so a satisfies (2).

Case 2.1. *There is a vertex $a \in A$ for which a^+ receives an edge from some vertex $q \in Q_a$.*

Inequality (2) implies that there exists a vertex $w \in N_C^-(a)$ such that the successor w^+ of w lies in F . Note that w lies in the interior of aCq since F avoids $Q_a \cup \{a, a^+\}$. As $a \in A \subseteq T_3$ and as H is Hamiltonian connected by Lemma 6, there is an $a-w^+$ path P whose interior consists precisely of all the vertices in H . But then the cycle qa^+CwaPw^+Cq is S -ordered and contains $|H| - |Q_a| + 1 > 0$ more vertices than C , a contradiction.

Case 2.2. *There is no such vertex $a \in A$.*

This case is similar to Case 1.2. Let F_3^- denote the set of all predecessors of vertices in F_3 again. Let A^+ denote the set of all successors of vertices in A . Recall that F avoids t^+Cs' . Thus F_3^- avoids $tCs' - s'$ and thus in particular all the sets Q_a .

Consider any $a \in A$. Then $N_D^-(a^+) \subseteq V(C)$ since $a^+ \notin F$ by Corollary 7. Thus $N_D^-(a^+) \subseteq V(C) \setminus Q_a$ by our assumption. Hence similarly as in Case 1.2 one can show that $d_{C-Q_a}^-(a^+) \geq |C - Q_a| - |F_3^-| + |Q_a| - 1$. Together with the fact that $F_3^- \cap Q_a = \emptyset$ this gives

$$(4) \quad |N_{C-Q_a}^-(a^+) \cap F_3^-| \geq |Q_a| - 1 \geq 210k^3.$$

Let I_1 be the subpath of the interval I preceding the first vertex in A^+ . Let I_2, \dots, I_k denote all the other intervals. For each $i = 1, \dots, k$ let G_i be the auxiliary bipartite graph whose vertex classes are A^+ and $V(I_i) \cap F_3^-$ and in which $a^+ \in A^+$ is

joined to $w \in V(I_i) \cap F_3^-$ if wa^+ is an edge of D . Note that $F_3^- \subseteq V(I_1) \cup \dots \cup V(I_k)$ since F_3^- avoids $tCs' - s'$. Thus the edges of $G_1 \cup \dots \cup G_k$ correspond to the edges from F_3^- to A^+ . Together with (4) this implies that there is some i such that

$$e(G_i) \geq \frac{210k^3|A^+|}{k} = \frac{210k^3n}{70k^3} = 3n \geq 3|G_i|.$$

Thus G_i is not planar and so there are vertices $a_1^+, a_2^+ \in V(A^+)$ and $w_1, w_2 \in V(I_i) \cap F_3^-$ such that the edges $w_1a_1^+, w_2a_2^+$ cross. As in Case 1.2 we can apply Corollary 12 to obtain disjoint $a_j-w_j^+$ paths having all their inner vertices in H such that each P_j contains at least one inner vertex (where $j = 1, 2$ and a_j is the predecessor of a_j^+). Thus $a_1P_1w_1^+Cw_2a_2^+Cw_1a_1^+Ca_2P_2w_2^+Ca_1$ is an S -ordered cycle with at least $|C| + 2$ vertices (note that it contains all the vertices of C), contradicting the choice of C .

Case 3. $n/2 - n/(50k) \leq |H| \leq \lceil (n - k)/2 \rceil - 1$.

Our first aim is to find vertices x_1, x_2, y_1, y_2 on C with the following properties:

- (i) x_1, x_2, y_1, y_2 occur on C in this order and either all of these vertices are distinct or else $|\{x_1, x_2, y_1, y_2\}| = 3$ and $x_1 = y_2$.
- (ii) S avoids the interior of x_1Cx_2 , the interior of y_1Cy_2 as well as x_2 and y_1 .
- (iii) There are distinct vertices $h_1, h_2, h'_1, h'_2 \in H$ such that $x_1h_1, x_2h_2, h'_1y_1, h'_2y_2$ are edges.
- (iv) If $x_1 \neq y_2$ (and so x_1, x_2, y_1, y_2 are distinct) then there are disjoint x_i-y_i paths P_i of length at least 2 such that all inner vertices of P_1 and P_2 lie in H and $|P_1 \cup P_2| \geq |H|/6$.

To prove the existence of such vertices, suppose first that $|T_8| \geq k + 1$ and $|F_8| \geq k + 1$. Then we can find two vertices $x_1, x_2 \in T_8$ and two vertices $y_1, y_2 \in F_8$ satisfying (ii). Then these vertices automatically satisfy (iii). Lemma 8 implies that they also satisfy (i). Finally, if they are all distinct then Corollary 12 shows that they also satisfy (iv).

So suppose next that for example $|T_8| \leq k$ but $|F_8| \geq k + 1$. Pick $y_1, y_2 \in F_8$ as before. To find x_1 and x_2 , first note that each vertex $h \in H$ satisfies

$$d_C^-(h) \geq \delta^-(D) - (|H| - 1) \geq \lceil (n + k)/2 \rceil - 1 - \lceil (n - k)/2 \rceil + 2 = k + 1$$

and so h receives at least one edge from some vertex in $T \setminus T_8$. As each vertex in $T \setminus T_8$ sends an edge to at most 7 vertices in H , this means that there are at least $|H|/7$ independent edges from C to H . Thus the interior of some interval of C contains the endvertices of 16 of these independent edges which avoid y_1 and y_2 . Let X_1 be the set of the first 8 endvertices of these edges on this interval and let X_2 be the set of the next 8 endvertices. Then Lemma 11 implies that there are vertices $x_1 \in X_1$ and $x_2 \in X_2$ which together with y_1 and y_2 satisfy (iv). By construction, x_1, x_2, y_1, y_2 are all distinct and satisfy (ii) and (iii). Again, Lemma 8 implies that they also satisfy (i). The cases when $|T_8| \geq k + 1$ but $|F_8| \leq k$ and when $|T_8|, |F_8| \leq k$ are similar. So we have shown that there are vertices x_1, x_2, y_1, y_2 satisfying (i)-(iv).

In what follows, we will frequently use the fact that any vertex $x \in V(C) \setminus F_2$ receives an edge from all but at most

$$|C| - (\delta^-(D) - 1) \leq n/2 + n/(50k) - (n + k)/2 + 2 \leq n/(45k)$$

vertices of C . Similarly, any vertex $x \in V(C) \setminus T_2$ sends an edge to all but at most $n/(45k)$ vertices of C .

Case 3.1. $|x_1 C x_2| \geq n/(15k)$

Let A_2 be the set of $n/(40k)$ vertices which immediately precede x_2 and let A_1 be the set of $n/(40k)$ vertices which immediately precede A_2 . Corollary 7 implies that the successor x_2^+ of x_2 on C does not lie in F . Thus x_2^+ receives an edge from some vertex $a_1 \in A_1$ since it receives an edge from all but at most $n/(45k)$ vertices of C . Similarly, the predecessor y_2^- of y_2 does not lie in T and thus sends an edge to some vertex $a_2 \in A_2$. Lemma 6 now implies that H contains a Hamilton path P from h_2 to h_2' . But then the cycle $a_1 x_2^+ C y_2^- a_2 C x_2 h_2 P h_2' y_2 C a_1$ is S -ordered and contains all vertices of C except those in the interior of $a_1 C a_2$ (see Figure 4). But as $|H| > n/4 > |a_1 C a_2|$ this means that this new cycle is longer than C , a contradiction.

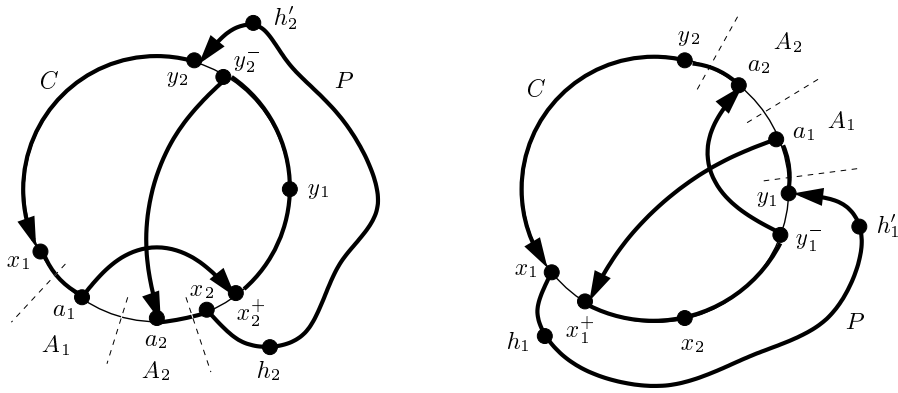


FIGURE 4. Extending C into a longer S -ordered cycle in Case 3.1 (left) and Case 3.2 (right).

Case 3.2. $|y_1 C y_2| \geq n/(15k)$

The proof of this case is similar to that of Case 3.1. Let A_1 be the set of $n/(40k)$ vertices which immediately succeed y_1 and let A_2 be the set of $n/(40k)$ vertices which immediately succeed A_1 . Then the predecessor y_1^- of y_1 sends an edge to some vertex $a_2 \in A_2$ and the successor x_1^+ of x_1 receives an edge from some vertex $a_1 \in A_1$. Then the S -ordered cycle $y_1^- a_2 C x_1 h_1 P h_1' y_1 C a_1 x_1^+ C y_1^-$ is longer than C , where P is a Hamilton path in H from h_1 to h_1' (see Figure 4).

Case 3.3. $|y_2 C x_1| \geq n/5$

Let Z be a segment of the interior of $y_2 C x_1$ such that $|Z| \geq n/(6k)$ and such that Z avoids S . Let Z_1 be the set consisting of the first $n/(40k)$ vertices on Z . Let Z_2 be the set consisting of the next $n/(40k)$ vertices and define Z_3, \dots, Z_6 similarly. As by Corollary 7 the predecessor y_1^- of y_1 does not lie in T it must send an edge to some vertex $z_4 \in Z_4$. Similarly the predecessor y_2^- of y_2 sends an edge to some vertex $z_2 \in Z_2$, the successor x_1^+ of x_1 receives an edge from some vertex $z_5 \in Z_5$ and the successor x_2^+ of x_2 receives an edge from some vertex $z_3 \in Z_3$. Now Lemma 8 implies that either $Z_1 \cap T_2 = \emptyset$ or $Z_6 \cap F_2 = \emptyset$ or both. If $Z_1 \cap T_2 = \emptyset$ then every

vertex in Z_1 sends an edge to Z_6 (since every vertex outside T_2 sends an edge to all but at most $n/(45k)$ vertices on C). Similarly, if $Z_6 \cap F_2 = \emptyset$ then every vertex in Z_6 receives an edge from some vertex in Z_1 . So in both cases we can find a Z_1 - Z_6 edge z_1z_6 . But then the cycle $x_1P_1y_1Cy_2^-z_2Cz_3x_2^+Cy_1^-z_4Cz_5x_1^+Cx_2P_2y_2Cz_1z_6Cx_1$ is S -ordered and contains at least $|P_1 \cup P_2| - 4 - (|Z| - 6) \geq |H|/6 - n/(6k) > 0$ vertices more than C , a contradiction (see Figure 5).

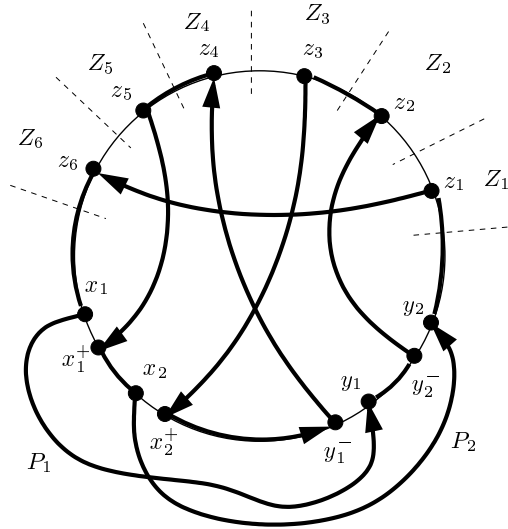


FIGURE 5. Extending C into a longer S -ordered cycle in Case 3.3.

Case 3.4. *None of Cases 3.1–3.3 holds.*

In this case we must have that $|x_2Cy_1| \geq n/5$ and can argue similarly as in Case 3.3 (see Figure 6). We omit the details.

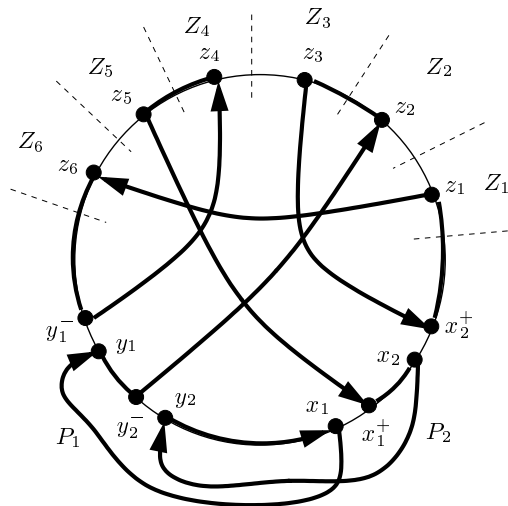


FIGURE 6. Extending C into a longer S -ordered cycle in Case 3.4.

Case 4. *None of Cases 1–3 holds.*

Together with Lemma 6 this implies that $n - k$ is even and $|H| = (n - k)/2$. So $|C| = (n + k)/2$. First note that any vertex $h \in H$ satisfies

$$(5) \quad d_C^+(h), d_C^-(h) \geq (n + k)/2 - 1 - (|H| - 1) = k.$$

Moreover, if $h, h' \in H$ are distinct and if $s \in S \cap N_C^-(h)$ then by Lemma 8 the special vertex s' succeeding s on C (i.e. the unique vertex $s' \in S$ for which S avoids the interior of sCs') cannot lie in $N_C^+(h')$. Thus $|S \cap N_C^-(h)| + |S \cap N_C^+(h')| \leq k$ and so

$$(6) \quad |N_C^-(h) \setminus S| + |N_C^+(h') \setminus S| \geq |N_C^-(h)| + |N_C^+(h')| - k \stackrel{(5)}{\geq} k.$$

Case 4.1. *There exists some vertex $x \in N_C^-(h) \setminus S$.*

First note that by Corollary 7 the successor x^+ of x on C does not lie in F . Thus $d_C^-(x^+) \geq \delta^0(D) = |C| - 1$ and so x^+ receives an edge from the predecessor x^- of x . Pick any vertex $y \in F \setminus \{x, x^-\}$. (Such a vertex exists since $|F| \geq 3$ by (5).) Note that $y \neq x^+$ since $x^+ \notin F$. By Corollary 7 the predecessor y^- of y does not send an edge to H and so y^-x must be an edge (since $d_C^+(y^-) = |C| - 1$). Now apply Lemma 6 to find an x - y path P of length at least 2 all whose inner vertices lie in H . Then $x^-x^+Cy^-xPyCx^-$ is an S -ordered cycle which is longer than C , a contradiction.

Case 4.2. *There is no vertex as in Case 4.1.*

Together with (6) this implies that we can find a vertex $x \in N_C^+(h') \setminus S$. We then argue similarly as in Case 4.1. This completes the proof of Theorem 1.

5. ACKNOWLEDGEMENT

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