Embedding cycles of given length in oriented graphs

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Abstract

Kelly, Kühn and Osthus conjectured that for any $\ell \geq 4$ and the smallest number $k \geq 3$ that does not divide ℓ, any large enough oriented graph G with $\delta^+(G), \delta^-(G) \geq \lfloor |V(G)|/k \rfloor + 1$ contains a directed cycle of length ℓ. We prove this conjecture asymptotically for the case when ℓ is large enough compared to k and $k \geq 7$. The case when $k \leq 6$ was already settled asymptotically by Kelly, Kühn and Osthus.

1 Introduction

Oriented graphs are obtained from undirected graphs by giving each edge a direction. The *minimum semidegree* $\delta^0(G)$ of an oriented graph G is the minimum of its *minimum outdegree* $\delta^+(G)$ and of its *minimum indegree* $\delta^-(G)$. A *directed* cycle is a cycle in which all the edges are oriented consistently. Directed paths and walks are defined analogously. An *ℓ-cycle* is a directed cycle of length exactly ℓ. The girth $g(G)$ of an oriented graph G is the smallest number ℓ so that G contains an ℓ-cycle.

A central problem in this area is the following conjecture by Caccetta and Häggkvist [3].

Conjecture 1.1 (Caccetta-Häggkvist)**.** *An oriented graph on n vertices with minimum outdegree d has girth at most $\lceil \frac{n}{d} \rceil$.*

Despite much work over the years by a large number of researchers even the case $\lceil \frac{n}{d} \rceil = 3$ remains open. We refer to [10] for a survey on the topic and to [5] for the currently best bound for the case $\lceil \frac{n}{d} \rceil = 3$. The natural and related question of what minimum semidegree in an oriented graph forces cycles of length exactly $\ell \geq 4$ was raised in [6].

Conjecture 1.2 (Kelly, Kühn, Osthus)**.** *Let $\ell \geq 4$ be an integer and let $k \geq 3$ be minimal such that k does not divide ℓ. Then there exists an integer $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lfloor n/k \rfloor + 1$ contains an ℓ-cycle.*

Note that k in Conjecture 1.2 must be of the form p^s for some prime p and some $s \in \mathbb{N}$. If true Conjecture 1.2 is tight as a blow-up of a k-cycle has minimum semidegree $\lfloor \frac{n}{k} \rfloor$ and there is no ℓ-cycle (since k does not divide ℓ). Proposition 2.11 shows that the condition on n being sufficiently large is necessary.

In [6] Kelly, Kühn and Osthus proved Conjecture 1.2 exactly for $k = 3$ and proved it asymptotically for $k = 4$ and $\ell \geq 42$ as well as $k = 5$ and $\ell \geq 2550$. They also showed that a bound of $\lfloor n/3 \rfloor + 1$ suffices for any $\ell \geq 4$. In this paper we prove Conjecture 1.2 asymptotically for the case when $k \geq 7$ and ℓ is sufficiently large compared to k.

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Theorem 1.3. *Let $k \geq 7$ and $\ell \geq 10^7 k^6$. Suppose that k is the minimal integer greater than 2 that does not divide ℓ. Then for all $\eta > 0$ there exists an integer $n_1 = n_1(\eta, \ell)$ such that every oriented graph G on $n \geq n_1$ vertices with minimum semidegree $\delta^0(G) \geq (1+\eta)n/k$ contains an ℓ-cycle.*

Instead of an ℓ-cycle, one may want to find a cycle of length ℓ with some given orientation of its edges. In this case the semidegree condition seems to depend on the so-called cycle-type. Given an arbitrary orientation of a cycle C, the *cycle-type* $t(C)$ of C is the absolute value of the number of edges with clockwise orientation minus the number of edges with anticlockwise orientation. So an ℓ-cycle has cycle-type ℓ and oriented cycles with cycle-type 0 are precisely those for which there is digraph homomorphism into a directed path. Moreover if $t(C) \geq 3$ then $t(C)$ is the maximum length of a directed cycle into which there is a digraph homomorphism of C. The authors of [6] made the following conjecture.

Conjecture 1.4 (Kelly, Kühn, Osthus)**.** *Let C be an arbitrarily oriented cycle of length $\ell \geq 4$ and cycle-type $t(C) \geq 4$. Let $k \geq 3$ be minimal such that k does not divide $t(C)$. Then there exists an integer $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lfloor n/k \rfloor + 1$ contains C.*

Again the blow-up of a k-cycle shows that Conjecture 1.4 is tight. As observed in [6], Conjecture 1.2 would imply an approximate version of Conjecture 1.4. The same argument shows that Theorem 1.3 (together with the cases $k = 3, 4, 5$ of Conjecture 1.2 proved in [6]) implies an approximate version of Conjecture 1.4 for the case when the cycle-type $t(C)$ is sufficiently large with respect to k. More precisely Theorem 1.3 implies the following statement.

Theorem 1.5. *Let $k \geq 3$ and $t \geq 10^7 k^6$. Suppose that k is the minimal integer greater than 2 that does not divide t. Then for all $\eta > 0$ and all ℓ there exists an integer $n_1 = n_1(\eta, \ell)$ such that every oriented graph G on $n \geq n_1$ vertices with minimum semidegree $\delta^0(G) \geq (1+\eta)n/k$ contains any oriented cycle C of length ℓ and cycle-type $t(C) = t$.*

Note that if there exists no cycle of length ℓ with cycle-type t (for example in the case when $\ell < t$), then Theorem 1.5 is vacuously true.

For completeness we give an outline of the proof of Theorem 1.5. Let G be as in Theorem 1.5. Apply a version of Szemerédi's regularity lemma for directed graphs to G (such a directed version was proved by Alon and Shapira [1]) to obtain a directed cluster graph H' with similar minimum semidegree as in G, i.e. $\delta^0(H') \geq (1+\eta/2)|V(H')|/k$. However H' needs not to be oriented, but for every double edge of H' one can select one of the two edges randomly (with suitable probability) in order to obtain an oriented spanning subgraph H of H' which still satisfies $\delta_0(H) \geq (1+\eta/4)|V(H)|/k$ (see Lemma 3.2 in [7] for a proof). The oriented graph H satisfies the conditions of Theorem 1.3 (with $\eta/4$ playing the role of η) and thus contains a t-cycle. Now one can apply an oriented version of the embedding lemma to find a copy of C within the subgraph of G which corresponds to the t-cycle in H. (For the embedding lemma we refer the reader for example to Lemma 7.5.2. in [4].)

We conclude the introduction by raising several open problems. The first and perhaps most interesting one is whether we can replace the semidegree condition by an outdegree condition in Theorems 1.3 and 1.5. The second question is to find the exact bound for the semidegree condition in more cases. The smallest open case is for $\ell = 6$, as the case when $\ell \geq 4$ with $\ell \not\equiv 0 \mod 3$ is solved in [6]. Finally, can we prove these results without the use of the regularity lemma? The proofs of Theorems 1.3 and 1.5 rely on a version of the regularity lemma (in the proof of Theorem 1.3 the regularity lemma comes in because it was used in [6] to prove

Lemma 2.1). Therefore the bound on n_1 is huge. Proving these theorems without the help of the regularity lemma would significantly reduce the bound on n_1. Related problems can be found in the survey [8].

2 Proof of Theorem 1.3

Throughout this section the numbers k and ℓ are fixed and satisfy the assumptions of Theorem 1.3, i.e.

$$k \geq 7,\ \ell \geq 10^7 k^6 \text{ and } k \text{ is the minimal integer greater than 2 that does not divide } \ell. \tag{$*$}$$

By the following lemma from [6] it suffices to find a closed directed walk of length ℓ instead of an ℓ-cycle.

Lemma 2.1 (Kelly, Kühn, Osthus). *Let $\ell \geq 3$ be an integer and $c > 0$. Suppose that there exists an integer n_0 such that every oriented graph H on $n > n_0$ vertices with $\delta^0(H) \geq cn$ contains a closed directed walk of length ℓ. Then for each $\varepsilon > 0$ there exists $n_1 = n_1(\varepsilon, \ell, n_0)$ such that if G is an oriented graph on $n \geq n_1$ vertices with $\delta^0(G) \geq (c+\varepsilon)n$ then G contains an ℓ-cycle.*

The proof of Lemma 2.1 is similar to the argument showing that Theorem 1.3 implies Theorem 1.5. As there, one first applies the regularity lemma for directed graphs to G to obtain a directed cluster graph H'. The next step is then to find an oriented cluster graph H. As before $\delta^0(H) \geq c|V(H)|$ and so H contains a closed directed walk of length ℓ, which can then easily be converted to an ℓ-cycle in G.

Proposition 2.2. *Let k and ℓ be as in $(*)$. If H is an oriented graph of girth less than k, then it contains a closed directed walk of length ℓ.*

Proof. Let $s < k$ be such that H contains an s-cycle C. Then $(*)$ implies that s divides ℓ. Thus winding ℓ/s times around the cycle C gives a closed directed walk of length ℓ. □

By Lemma 2.1 and Proposition 2.2 we may assume that our oriented graph H has girth at least k. Our next aim is to show that in this case H contains a directed path of length at most $64k$ between any two vertices in H. For this we use the following result by Shen [9].

Theorem 2.3 (Shen). *An oriented graph H on n vertices with minimum outdegree $\delta^+(H) \geq d$ has girth*

$$g(H) \leq 3\left\lceil \left(\ln \frac{2+\sqrt{7}}{3}\right) \frac{n}{d} \right\rceil.$$

Corollary 2.4. *Let $k \in \mathbb{N}$ with $k \geq 7$. Then an oriented graph H on n vertices with minimum outdegree $\delta^+(H) \geq 63n/32k$ has girth $g(H) < k$.*

Proof. For $k \geq 10 > \frac{10^3}{101}$, Theorem 2.3 implies that the girth of H is at most

$$3\left\lceil \left(\ln \frac{2+\sqrt{7}}{3}\right) \frac{32k}{63} \right\rceil \leq 3\left(\left(\ln \frac{2+\sqrt{7}}{3}\right) \frac{32k}{63} + 1\right) \leq 0.67k + \frac{3 \cdot 101k}{10^3} = 0.973k.$$

For the values $k = 7, 8, 9$, from Theorem 2.3 we obtain that the girth of H is at most 6. □

Lemma 2.5. *Let $k \in \mathbb{N}$ with $k \geq 7$ and let H be an oriented graph on $n \geq 64k$ vertices, with $\delta^0(H) > n/k$ and with girth $g(H) \geq k$. Then there is a directed path of length at most $64k$ from any vertex x of H to any other vertex y of H.*

Proof. We claim that any subset A of order $a \leq \frac{n}{2}$ contains a vertex with at least $n/64k \geq 1$ outneighbours outside A. Suppose not. Then the minimum outdegree of $H[A]$ is at least $\frac{63n}{64k} \geq \frac{63a}{32k}$. Now Corollary 2.4 implies that the girth of $H[A]$ (and thus of H) is less than k, a contradiction.

Set $A_1 := N^+(x) \cup \{x\}$. We have $|A_1| > n/k$. If $|A_1| \leq n/2$ then by the above claim there is a vertex $v_1 \in A_1$ with $|N^+(v_1) \setminus A_1| \geq n/64k$. In this case set $A_2 := A_1 \cup N^+(v_1)$ and continue in this way. After at most $32k$ steps, we have $|A_i| > n/2$, for some $i \leq 32k$.

Set $B_1 := N^-(y) \cup \{y\}$ and proceed analogously to obtain that $|B_j| > n/2$ for some $j \leq 32k$. Then $A_i \cap B_j \neq \emptyset$ and there exists a directed path of length at most $64k$ from x to y whose vertices lie in $A_i \cup B_j$. □

Lemma 2.6. *Let k and ℓ be as in $(*)$. Let H be an oriented graph which contains a directed path of length at most $64k$ from any vertex to any other vertex. If H contains a closed walk W with a edges going forward and b edges going backwards, for some $a \neq b$ with $a + b = |V(W)| < k$, then there is a closed directed walk of length ℓ in H.*

Proof. Let W, a and b be as in the lemma. We may assume without loss of generality that

$$a > b. \tag{1}$$

We now consider any maximal subwalk in W with all edges oriented backwards. Let x and y denote the endvertices of this walk such that all the edges are oriented from x towards y. Let P_{yx} be a directed path in H from y to x of length at most $64k$. We find such a path for each maximal subwalk of W that is oriented backwards. Let P be the union of all these paths P_{yx} and $l(P)$ be the sum of their lengths. So $l(P) < \frac{k}{2} \cdot 64k = 32k^2$.

There exists a closed directed walk W_1 of length $a + l(P)$ in H consisting of the edges of W going forwards and P. Also there exists a closed directed walk W_2 of length $a + b + 2l(P)$ using each edge of the closed walk W exactly once and each path P_{yx} exactly twice (see Figure 1). In the case when $b = 0$ we have $W = W_1 = W_2$.

By (1) we have $2l(W_1) - l(W_2) = a - b > 0$, where $l(W_1)$ and $l(W_2)$ denote the lengths of the walks W_1 and W_2, respectively. Let $h := gcd(2l(W_1), l(W_2)))$ be the greatest common divisor of $2l(W_1)$ and $l(W_2)$. Thus

$$0 < h = gcd(a - b, l(W_2)) \leq a - b < k. \tag{2}$$

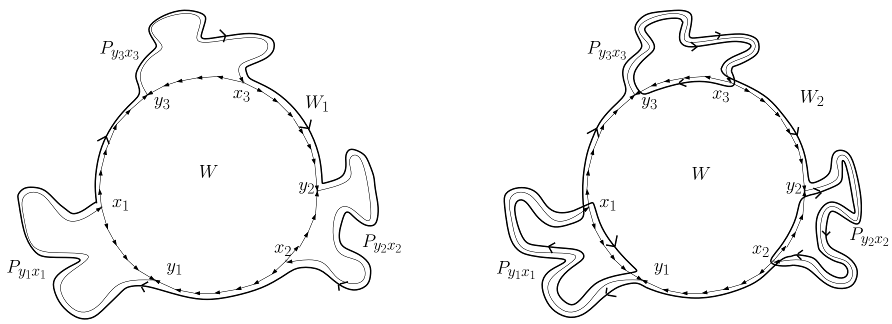
By Bézout's identity there are $p, q \in \mathbb{Z}$ such that

$$q \cdot 2l(W_1) + p \cdot l(W_2) = h. \tag{3}$$

Moreover we may assume that $0 \leq q < l(W_2) < k + 64k^2$ and $-p < 2l(W_1) < 2k + 64k^2$. Indeed, by replacing q with $q + s \cdot l(W_2)$ and p with $p - s \cdot 2l(W_1)$ for a suitable $s \in \mathbb{Z}$ we may assume that $0 \leq q < l(W_2)$. But then

$$-p \overset{(3)}{=} (q \cdot 2l(W_1) - h)/l(W_2) < 2l(W_1). \tag{4}$$

From $(*)$ and (2) we obtain that h divides ℓ. Let $\ell' = \frac{\ell}{h}$, $r = \lfloor \frac{\ell'}{2l(W_1)+l(W_2)} \rfloor$, $u = 2rh + (\ell' - (2l(W_1) + l(W_2))r)2q$ and $v = rh + (\ell' - (2l(W_1) + l(W_2))r)p$. Let W^* be the closed walk obtained by winding around W_1 u times and winding



(a) The walk W_1 uses only edges from the walk W going forwards and the paths P_{yx} otherwise.

(b) The walk W_2 uses every edge of the walk W once and each path P_{yx} twice.

Figure 1: The two different walks W_1 and W_2. For simplicity the figure illustrates the case when the closed walk W is a cycle.

around W_2 v times. Note that W^* is well defined. For this, first note that $q \geq 0$ implies that $u \geq 0$. Secondly, by (4) we also have

$$\begin{aligned} v &\geq rh - (\ell' - (2l(W_1) + l(W_2))r)(2k + 64k^2) \\ &> h\left(\frac{\ell'}{2l(W_1) + l(W_2)} - 1\right) - (2l(W_1) + l(W_2))(2k + 64k^2) \\ &\geq h\frac{\ell'}{2l(W_1) + l(W_2)} - h - (3k + 128k^2)(2k + 64k^2) \\ &\overset{(*)\&(2)}{>} \frac{\ell}{129k^2} - k - 129k^2 \cdot 65k^2 \overset{(*)}{\geq} 0. \end{aligned}$$

The total length of the closed directed walk W^* is

$$\begin{aligned} &u \cdot l(W_1) + v \cdot l(W_2) \\ &= rh(2l(W_1) + l(W_2)) + (\ell' - (2l(W_1) + l(W_2))r)(q \cdot 2l(W_1) + p \cdot l(W_2)) \\ &\overset{(3)}{=} rh(2l(W_1) + l(W_2)) + (\ell' - (2l(W_1) + l(W_2))r)h \\ &= \ell' h = \ell, \end{aligned}$$

as required. □

Corollary 2.7. *Let k and ℓ be as in $(*)$. Suppose that H is an oriented graph which contains a directed path of length at most $64k$ from any vertex to any other vertex. If for some odd $s < k$ the graph H contains some orientation of a cycle of length s, then H contains a closed directed walk of length ℓ.*

Recall that we may assume that our oriented graph H has girth at least k. Thus Lemma 2.5, Corollary 2.7 and the following result of Andrásfai, Erdős and Sós (see Remark 1.6 in [2]) together imply that we may assume that the underlying graph of H is bipartite.

Theorem 2.8 (Andrásfai, Erdős, Sós)**.** *Let $k \geq 4$. Any undirected graph on n vertices which contains no odd cycle of length less than k and has minimum degree $\delta > \frac{2n}{k}$ is bipartite.*

Lemma 2.9. *Let $k \in \mathbb{N}$ be odd or divisible by 4 and let H be an oriented bipartite graph on n vertices with $\delta^0(H) > n/k$. Then there are natural numbers $a \neq b$ with $a+b < k$ such that H contains a closed walk W of length $a+b$ with a edges oriented forwards and b edges oriented backwards.*

Proof. Assume for a contradiction that H contains no such walk W. Let V_1 and V_2 be the colour classes of H and without loss of generality assume that $|V_1| \geq |V_2|$.

Pick an arbitrary vertex $x \in V_1$. Let $X_1 := N^+(x)$, $Y_1 := N^-(x)$, and for each $i \geq 1$, set $X_{i+1} := N^+(X_i)$ and $Y_{i+1} := N^-(Y_i)$. Now observe that $X_i \cap Y_j = \emptyset$ whenever $i, j < \frac{k}{2}$ or both $i \neq j$ and $\max\{i, j\} = \lceil \frac{k}{2} \rceil$. Indeed, this holds since our assumption on H implies that H contains no directed cycle of length less than k and since there is no cycle of length k when k is odd, as H is bipartite.

Claim. *For all $i \neq j$ with $i, j \leq \lceil \frac{k}{2} \rceil$, we have $X_i \cap X_j = \emptyset$ and $Y_i \cap Y_j = \emptyset$.*

We prove only the first instance of the claim as the second is done analogously. Suppose for a contradiction that $v \in X_i \cap X_j$, for some $j < i \leq \lceil \frac{k}{2} \rceil$. By definition of X_i there is a directed walk P of length i from x to v and by definition of X_j there is a directed walk P' of length j from x to v. Then $P \cup P'$ forms a closed walk of length $i + j \leq 2\lceil \frac{k}{2} \rceil - 1$, with i edges going in one direction and j edges going in the other direction. If $i + j = k$, then k is odd and this contradicts the assumption that H is bipartite. If $i + j < k$ this contradicts our assumption that for all $a \neq b$ with $a + b < k$ there is no closed walk W with a edges going forwards and b edges going backwards. This proves the claim.

Observe that for any $i \leq \lceil \frac{k}{2} \rceil$, we have $|X_i|, |Y_i| > \frac{n}{k}$. Consider first the case when k is odd. We obtain

$$|V(H)| \geq \left| \left(\dot{\bigcup}_{i<\lceil \frac{k}{2} \rceil} (X_i \dot{\cup} Y_i) \right) \dot{\cup} (X_{\lceil \frac{k}{2} \rceil} \cup Y_{\lceil \frac{k}{2} \rceil}) \right| > \frac{k-1}{2} \cdot \frac{2n}{k} + \frac{n}{k} = n,$$

a contradiction. Now consider the case when 4 divides k. Then we get

$$|V_2| \geq \left| \dot{\bigcup}_{i \leq \frac{k}{4}} (X_{2i-1} \dot{\cup} Y_{2i-1}) \right| > \frac{k}{4} \cdot \frac{2n}{k} = \frac{n}{2},$$

a contradiction to the assumption that $|V_2| \leq |V_1|$ and thus $|V_2| \leq \frac{n}{2}$. This finishes the proof of Lemma 2.9. □

Let us summarise the above observations in the following lemma.

Lemma 2.10. *Let k and ℓ be as in $(*)$. Then any oriented graph H on $n \geq 64k$ vertices with $\delta^0(H) > n/k$ contains a closed directed walk of length ℓ.*

Proof. Let k, ℓ and H be as in the lemma. By Proposition 2.2 we may assume that H has girth at least k. This together with the assumption on the minimum semidegree $\delta^0(H) > n/k$ and Lemma 2.5 imply that H contains a directed path of length at most $64k$ between any ordered pair of vertices. Using Corollary 2.7 we may assume that the underlying graph of H contains no odd cycle of length less than k. Also observe that the underlying graph of H has minimum degree greater than $2n/k$. Now we can apply Theorem 2.8 to the underlying graph of H and deduce that H is bipartite. Recall that $(*)$ implies that $k = p^s \geq 4$ for some prime p and some $s \in \mathbb{N}$. In particular, k is either odd or divisible by 4. Thus Lemma 2.9 implies that H contains a closed walk of length $a + b$ with a edges going forwards and b edges going backwards, for some $a \neq b$ with $a + b < k$. Finally this together with Lemma 2.6 and our above observation that there must be a directed path of length at most $64k$ between any ordered pair of vertices of H imply that H contains a closed directed walk of length ℓ, as required. □

Proof of Theorem 1.3. Apply Lemma 2.1 with $c = \frac{1+\eta/2}{k}$, $n_0 = 64k$ and $\varepsilon = \eta/2k$ to obtain $n_1 \in \mathbb{N}$.

Let G be as in Theorem 1.3. Recall that by Lemma 2.1, in order to find an ℓ-cycle in G, it suffices to show that any oriented graph H on $n > 64k$ vertices and minimum semidegree $\delta^0(H) \geq (1+\eta/2)n/k$ contains a closed directed walk of length ℓ. This is done in Lemma 2.10. □

Finally, the following proposition shows that the condition that n_0 is large in Conjecture 1.2 is necessary for $\ell > 4$.

Proposition 2.11. *Let $\ell > 4$ be an even integer and $k > 2$ be minimal such that k does not divide ℓ. Then there exists a graph G on $\lfloor \frac{k-1}{2} \rfloor (\ell - 2) + 1$ vertices with $\delta^0(G) \geq \lfloor \frac{n}{k} \rfloor + 1$ that does not contain any cycle of length greater than $\ell - 1$.*

Proof. Let G be the union of $\lfloor \frac{k-1}{2} \rfloor$ regular tournaments on $\ell - 1$ vertices sharing a single vertex. Then G does not contain any cycle of length greater than $\ell - 1$, $\delta^0(G) = \frac{\ell-2}{2}$ and $n = \lfloor \frac{k-1}{2} \rfloor (\ell-2)+1$. Also

$$\left\lfloor \frac{n}{k} \right\rfloor \leq \left\lfloor \frac{\frac{k-1}{2}(\ell-2)+1}{k} \right\rfloor = \left\lfloor \frac{\ell-2}{2} - \frac{\ell-4}{2k} \right\rfloor \leq \frac{\ell-2}{2} - 1 = \delta^0(G) - 1 \,.$$

□

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