A BLOW-UP LEMMA FOR APPROXIMATE DECOMPOSITIONS

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Abstract. We develop a new method for constructing approximate decompositions of dense graphs into sparse graphs and apply it to longstanding decomposition problems. For instance, our results imply the following. Let G be a quasi-random n-vertex graph and suppose $H_1, \dots, H_s$ are bounded degree n-vertex graphs with $\sum_{i=1}^{s} e(H_i) \le (1 - o(1))e(G)$. Then $H_1, \dots, H_s$ can be packed edge-disjointly into G. The case when G is the complete graph K_n implies an approximate version of the tree packing conjecture of Gyárfás and Lehel for bounded degree trees, and of the Oberwolfach problem.

We provide a more general version of the above approximate decomposition result which can be applied to super-regular graphs and thus can be combined with Szemerédi's regularity lemma. In particular our result can be viewed as an extension of the classical blow-up lemma of Komlós, Sárkőzy and Szemerédi to the setting of approximate decompositions.

1. Introduction

1.1. **Packings and decompositions.** Questions on packings and decompositions have a long history, going back to the 19th century. For instance, the existence of Steiner triple systems (proved by Kirkman in 1847) corresponds to a decomposition of the edge set of the complete graph K_n on n vertices into triangles (if $n \equiv 1$ or $3 \mod 6$). A fundamental theorem of Wilson [39] generalizes this to decompositions of K_n into arbitrary graphs H of fixed size: for any graph H, if n is sufficiently large and satisfies trivially necessary divisibility conditions, then K_n has a decomposition into edge-disjoint copies of H. Similarly, Walecki's theorem goes back to 1892 and gives a decomposition of K_n into Hamilton cycles (if n is odd). Recently, there has been some exciting progress in this area, often involving the use of probabilistic techniques.

Here, a *decomposition* of a graph G into graphs $H_1, \dots, H_s$ is a set of pairwise edge-disjoint copies of $H_1, \dots, H_s$ in G which together cover all edges of G. Conversely, we say that graphs $H_1, \dots, H_s$ *pack into* G if there are copies of $H_1, \dots, H_s$ in G so that these copies are pairwise edge-disjoint. We informally refer to an 'approximate decomposition' or a 'near-optimal packing' if there is a packing which leaves only a small proportion of the edges of G uncovered.

There are several beautiful conjectures which have driven a large amount of research in the area. A prime example is the tree packing conjecture of Gyárfás and Lehel, which would guarantee a decomposition of a complete graph into a suitable given collection of trees.

Conjecture 1.1 (Gyárfás and Lehel [20])**.** *Given trees $T_1, \dots, T_n$, where for each $i \in [n]$, the tree T_i has i vertices, the complete graph K_n has a decomposition into copies of $T_1, \dots, T_n$.*

A related conjecture, made by Ringel, concerns decompositions of complete graphs into copies of a single tree: for every tree T on n vertices, K_{2n+1} has a decomposition into copies of T. There are a large number of partial results on Conjecture 1.1, some focusing on special classes of trees and some on embedding a (small) proportion of the trees (see e.g. [4, 6, 15, 20, 22, 36, 40]).

Possibly the most striking results towards Conjecture 1.1 have been obtained for the case of bounded degree trees. In particular, a recent result by Böttcher, Hladkỳ, Piguet and Taraz [8] allows for approximate decompositions of K_n into bounded degree trees that are permitted to be almost spanning. More precisely, their main result states that for all $\varepsilon > 0$ and $\Delta \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that whenever $n \ge n_0$ and $T_1, \dots, T_s$ is any family of trees with $|T_i| \le n$,

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$\Delta(T_i) \le \Delta$ and $\sum_{i=1}^{s} e(T_i) \le \binom{n}{2}$, then $T_1, \dots, T_s$ pack into $K_{(1+\varepsilon)n}$. Note that this implies an approximate version of Conjecture 1.1 for bounded degree trees (it is approximate both in the sense that none of the trees is spanning and that they do not form a decomposition). The result in [8] was strengthened by Messuti, Rödl and Schacht [33] to approximate decompositions of complete graphs into almost spanning graphs of bounded degree which are 'separable' (roughly speaking, a graph is separable if it can be split into bounded size components by removing a small proportion of its vertices). Independently to us, Ferber, Lee and Mousset [18] generalized this further by proving that one can obtain an approximate decomposition of K_n into separable graphs of bounded degree which are allowed to be spanning. In particular, this means that one can always obtain an approximate decomposition of K_n into spanning trees of bounded degree.

Our first result is in fact more general and guarantees an approximate decomposition of a dense quasi-random graph G into arbitrary bounded degree graphs. More precisely, we say that a graph G on n vertices is (ε, p)-*quasi-random* if all vertices v of G have degree $d_G(v) = (1 \pm \varepsilon)pn$ and all pairs of distinct vertices u, v have $(1 \pm \varepsilon)p^2 n$ common neighbours. (In fact, the latter condition is only required for almost all pairs, see Section 8.)

Theorem 1.2. *For all $\Delta \in \mathbb{N}$ and $0 < p_0, \alpha \le 1$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \ge n_0$ and $p \ge p_0$. Suppose $H_1, \dots, H_s$ are n-vertex graphs with $\Delta(H_\ell) \le \Delta$ for all $\ell \in [s]$ and $\sum_{\ell=1}^{s} e(H_\ell) \le (1-\alpha)\binom{n}{2}p$. Suppose that G is an (ε, p)-quasi-random graph on n vertices. Then $H_1, \dots, H_s$ pack into G.*

Moreover, if in addition $\sum_{\ell=1}^{s} e(H_\ell) \ge (1-2\alpha)\binom{n}{2}p$, then writing $\phi(H_\ell)$ for the copy of H_ℓ in this packing of $H_1, \dots, H_s$ in G and writing $J := G - (\phi(H_1) \cup \dots \cup \phi(H_s))$, we have $\Delta(J) \le 4\alpha pn$.

This immediately implies the corresponding result (asymptotically almost surely) for the binomial random graph $G_{n,p}$ where p is constant and for the complete graph K_n. Note that the case of $G = K_n$ in Theorem 1.2 extends the previous results in [8, 18, 33] mentioned above.

The case $G = K_n$ also yields an approximate version of the longstanding 'Oberwolfach problem' (proposed by Ringel in 1967). Given an arbitrary union F of vertex-disjoint cycles altogether spanning n vertices where n is odd, the Oberwolfach problem asks for a decomposition of K_n into copies of F. Bryant and Scharaschkin [11] have recently provided an affirmative answer for infinitely many n.

The famous Bollobás-Eldridge-Catlin conjecture states that if G_1 and G_2 are graphs on n vertices and $(\Delta(G_1)+1)(\Delta(G_2)+1) < n+1$, then G_1 and G_2 can be packed into K_n. Bollobás, Kostochka and Nakprasit [7] investigated versions of this conjecture for packing many graphs of bounded degeneracy. The case $G = K_n$ of Theorem 1.2 can be viewed as an asymptotically optimal version of the conjecture for packing many graphs of bounded degree.

1.2. **The blow-up lemma.** Combined with Szemerédi's regularity lemma [38], the blow-up lemma of Komlós, Sárközy and Szemerédi [25] has had a major impact on extremal graph theory. Roughly speaking, Szemerédi's regularity lemma guarantees a partition of any dense graph into a bounded number of random-like bipartite subgraphs, while the blow-up lemma allows to embed bounded degree graphs H into such random-like host graphs G. (Here H is allowed to have the same number of vertices as G, i.e. the blow-up lemma can be used to find spanning subgraphs.) These two tools can be combined to find spanning structures in dense graphs, and within the last 20 years, they have lead to a series of very strong results. Early striking results concern spanning trees [27] and powers of Hamilton cycles [26], more recent results include H-factors [28, 29] and embeddings of bounded degree graphs of small bandwidth [9]. For further results, see e.g. the survey [30].

Here we develop a 'blow-up lemma for approximate decompositions'. This version of the blow-up lemma will not only guarantee a single copy of H inside the host graph G, but will guarantee a collection of pairwise edge-disjoint copies of H in G such that together all these copies of H cover almost all edges of G. In fact, our result is even more general – we show that, essentially, any not too large collection of graphs $H_1, \dots, H_s$ of uniformly bounded maximum degree pack into G.

As with the classical blow-up lemma, for this it is natural to consider a partite setting and to assume that both the graphs H_ℓ and the host graph G 'share' a common equipartition. We need the following standard definitions. A bipartite graph G with vertex classes A and B is called (ε, d)-*regular* if for all $A' \subseteq A, B' \subseteq B$ with $|A'| > \varepsilon|A|$ and $|B'| > \varepsilon|B|$ we have

$$\frac{e(G[A', B'])}{|A'||B'|} = d \pm \varepsilon.$$

We say that G is (ε, d)-*super-regular* if it is (ε, d)-regular and $d_G(v) = (d \pm \varepsilon)|A|$ for all $v \in B$ and $d_G(v) = (d \pm \varepsilon)|B|$ for all $v \in A$. It is well known and easy to see that super-regularity is a weaker requirement than (a bipartite version of) quasi-randomness. We can now state our second result.

Theorem 1.3. *For all $0 < d_0, \alpha \leq 1$ and $\Delta, r \in \mathbb{N}$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq d_0$. Suppose $H_1, \dots, H_s$ are r-partite graphs such that each H_ℓ has vertex classes $X_1, \dots, X_r$ of size n and $\Delta(H_\ell) \leq \Delta$. Suppose that G is an r-partite graph with vertex classes $V_1, \dots, V_r$ of size n, where $G[V_i, V_j]$ is (ε, d)-super-regular for all $1 \leq i \neq j \leq r$. If $\sum_{\ell=1}^s e(H_\ell) \leq (1-\alpha)e(G)$, then $H_1, \dots, H_s$ pack into G.*

An application of Szemerédi's regularity lemma to an arbitrary dense graph G is naturally associated with a 'reduced graph' R, whose vertices correspond to the clusters of the regularity partition and whose edges correspond to those pairs of clusters which in G induce an ε-regular graph of significant density. This reduced graph may not be complete and the number of clusters may be relatively large. The following result is designed with such a situation in mind. (A corresponding extension of the classical blow-up lemma to this setting was proved by Csaba [12] in the context of the Bollobás-Eldridge-Catlin conjecture, see Theorem 3.6.)

Theorem 1.4. *For all $0 < \alpha, \eta, d_0 \leq 1$ and $\Delta, \Delta_R \in \mathbb{N}$ there exists $\varepsilon > 0$ so that for all $r \in \mathbb{N}$ there exists $n_0 = n_0(\varepsilon, r) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq d_0$. Let $s \in \mathbb{N}$ be such that $s \leq \eta^{-1}n$. Suppose that R is a graph on $[r]$ with $\Delta(R) \leq \Delta_R$. Suppose that $H_1, \dots, H_s$ are r-partite graphs such that each H_ℓ has vertex classes $X_1, \dots, X_r$ of size n and satisfies $\Delta(H_\ell) \leq \Delta$. Further, suppose that $\sum_{\ell=1}^s e(H_\ell[X_i, X_j]) \leq (1-\alpha)dn^2$ for all $ij \in E(R)$ and $H_\ell[X_i, X_j]$ is empty if $ij \notin E(R)$. Suppose finally that G is an r-partite graph with vertex classes $V_1, \dots, V_r$ of size n, where $G[V_i, V_j]$ is (ε, d)-super-regular for all $ij \in E(R)$. Then $H_1, \dots, H_s$ pack into G.*

In Section 8, we will formulate even more general versions: we for instance allow arbitrary densities for the pairs $G[V_i, V_j]$ and do not require that the vertex classes have exactly equal size. Such a setting allows us to derive an approximate version of the bipartite analogue of the tree packing conjecture, formulated by Hobbs, Bourgeois and Kasiraj [22] in 1986 (see Conjecture 8.4 and Corollary 8.6).

Furthermore, in the main result of Section 6 (Theorem 6.1), we can also require that the embeddings of the H_ℓ satisfy additional restrictions: for instance, for the vertices of H_ℓ we can specify certain 'target sets' in G into which these vertices will be embedded. In a forthcoming paper, Joos, Kim, Kühn and Osthus use this to prove several 'exact' packing results, including the tree packing conjecture (Conjecture 1.1) for all bounded degree trees.

1.3. **Related results and further applications.** Kirkman's result on Steiner triple systems was recently generalized by Keevash [24], who showed that every sufficiently large quasi-random graph G of sufficient density has a decomposition into K_r for fixed r, provided the obvious necessary divisibility conditions hold (here the quasi-randomness assumption is stronger than the one in Theorem 1.2). Similarly, it is natural to consider decompositions of graphs of large minimum degree into fixed subgraphs (see e.g. [5]).

Kühn and Osthus [31, 32] extended Walecki's theorem on Hamilton decompositions of complete graphs to the setting of a 'robustly expanding' regular host graph G. (The robust expansion condition is considerably weaker than that of quasi-randomness or ε-regularity and also applies to all graphs of degree cn for $c > 1/2$.) In [13], this result is used as a tool to prove several decomposition and packing conjectures involving Hamilton cycles and perfect matchings. Also,

in [19], it is used to derive optimal decomposition results for dense quasi-random graphs into other structures, including linear forests. A different generalization of Walecki's theorem is given by the Alspach conjecture, which states that for odd n, the complete graph K_n should have a decomposition into cycles $C^1, \dots, C^s$, provided that $\sum_{\ell=1}^{s} |C^\ell| = \binom{n}{2}$. This was recently confirmed by Bryant, Horsley and Pettersson [10].

Of particular interest are decompositions into H-factors (also referred to as 'resolvable designs'). Here an *H-factor* in a graph G is a set of disjoint copies of H which together cover all vertices of G. A classical result of Ray-Chaudhuri and Wilson [35] states that if $H = K_k$, if n is a sufficiently large multiple of k and if $n-1$ is a multiple of $k-1$, then K_n admits a decomposition into H-factors. Dukes and Ling [17] resolved the more general problem of decomposing a complete graph K_n into H-factors for arbitrary but fixed graphs H (subject to divisibility conditions – note that, unlike in Wilson's theorem [39], for some graphs H, such as $H = K_{1,2t+1}$, those are never fulfilled). Related results (which avoid this divisibility issue) were obtained by Alon and Yuster [3].

The Hamilton-Waterloo problem (which in general is wide open and generalizes the Oberwolfach problem) asks for a decomposition of K_n into C_ℓ-factors and $C_{\ell'}$-factors, where ℓ and ℓ' as well as the number of cycle factors of a given type are given (and n is odd). Several special cases have received considerable attention, such as triangle factors versus Hamilton cycles (see e.g. [14]). Theorem 1.2 immediately implies an approximate solution to this problem. Moreover, we can combine Theorem 1.2 with the results in [31, 32] to obtain a general decomposition result into factors, which solves the Hamilton-Waterloo problem if a significant proportion of the factors are Hamilton cycles.

Corollary 1.5. *For all $\Delta \in \mathbb{N}$ and $0 < p_0, \beta \leq 1$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $p \geq p_0$. Suppose $r_\ell \leq \Delta$ for all $\ell \in [s]$ and $\sum_{\ell=1}^{s} r_\ell = pn$. Suppose $H_1, \dots, H_s$ are n-vertex graphs so that H_ℓ is r_ℓ-regular for all $\ell \in [s]$ and that $H_1, \dots, H_{\beta n}$ are Hamilton cycles. Suppose further that G is an (ε, p)-quasi-random graph on n vertices which is pn-regular. Then G has a decomposition into $H_1, \dots, H_s$.*

1.4. **Organization of the paper.** In the next section, we give a sketch of the main ideas of the proof. In Section 3 we then collect the tools that we need later. In Section 4 we establish some properties of a typical matching in a super-regular bipartite graph. These will be used in Section 5 in order to prove a 'uniform blow-up lemma' (Lemma 5.1). This lemma forms the core of the paper and embeds a 'near-regular' graph H into G in a sufficiently 'random-like' fashion. In Section 6 we use Lemma 5.1 in order to construct the desired packing when each H_ℓ is near-regular (Theorem 6.1). In Section 7 we will reduce the problem of finding a packing of bounded degree graphs $H_1, \dots, H_s$ to the case when each H_ℓ is near-regular. In Section 8 we combine Theorem 6.1 with the results from Section 7. In particular, we will deduce Theorems 1.2–1.4 as well as Corollary 1.5.

2. Sketch of the method

In this section, we highlight some of the main ideas of the argument towards Theorem 1.3. Suppose we are given a balanced r-partite graph G with vertex classes $V_1, \dots, V_r$ of size n such that the bipartite graph $G[V_i, V_j]$ is (ε, d)-super-regular for all $1 \leq i \neq j \leq r$. For simplicity, suppose H is a balanced r-partite graph with vertex classes $X_1, \dots, X_r$ of size n, such that the bipartite graph $H[X_i, X_j]$ is k-regular for all $1 \leq i \neq j \leq r$ (where k is a large constant). Our aim is to find an approximate H-decomposition of G, i.e. a set of $(1 - o(1))dn/k$ edge-disjoint copies of H in G so that X_i is mapped to V_i for all $i \in [r]$.

The classical blow-up lemma guarantees one such copy of H in G. In fact, one can repeatedly apply the blow-up lemma to obtain $\varepsilon^2 n$ (say) edge-disjoint copies of H in G, but after this, we can no longer guarantee that the remaining subgraph of G is ε'-regular for small enough ε'.

We overcome this as follows: we will prove a 'uniform blow-up lemma' which returns a random copy $\phi(H)$ of H in G so that $\phi(H)$ behaves like a uniformly distributed random subgraph of G (see Lemma 5.1). Then the hope would be that with high probability, the graph $G - \phi(H)$ is still ε-regular. One desirable property towards this goal would be that for almost all edges e of

G, the probability that e lies in $\phi(H)$ is close to $p := e(H)/e(G) \sim k/(dn)$. (Clearly, one cannot achieve this for *every* edge of G, as for example G may have an edge which does not lie in a triangle.)

Towards this, we will use the characterization of (ε, d)-super-regular graphs in terms of co-degrees: for all $j \neq i$ and all vertices $u \in V_i$, we have that $|N_G(u) \cap V_j| = (d \pm \varepsilon)n$. Also, for almost all pairs $u, v \in V_i$, the common neighbourhood $N_G(u, v) \cap V_j$ has size $(d^2 \pm 3\varepsilon)n$. Assume for simplicity that we have equality everywhere, i.e. for all pairs $u, v \in V_i$, we have $|N_G(u, v) \cap V_j| = d^2 n$ for all $j \neq i$. Assume also that for every edge of G the probability that e lies in $\phi(H)$ is exactly $p = k/(dn)$, independently of all other edges. Note that for any copy $\phi(H)$ in G, the bipartite pairs of $G - E(\phi(H))$ have density $d_1 := d - k/n$. Our uniform blow-up lemma would then find a copy $\phi(H)$ of H in G where we expect that

$$|N_{G-E(\phi(H))}(u, v)| = d^2 n(1 - p)^2 = d_1^2 n. \tag{2.1}$$

This is of course exactly what one would expect if H were a completely random subgraph of G (of the same density). It turns out that property (B1) of our uniform blow-up lemma can be used to prove a surprisingly accurate approximation to the idealized formula in (2.1) (see Claim 6.9).

Removing $E(\phi(H))$ from G now leaves a graph whose regularity parameters are much better than if we had removed this copy greedily. So ideally one would keep applying this uniform blow-up lemma, maintaining (after the removal of i copies) an (ε_i, d_i)-super-regular graph G_i with $d_i = d - ik/n$ and $\varepsilon_i \sim \varepsilon \ll d_i$. However, due to the fact that we need to allow i to be linear in n, this seems to be extremely challenging, if not infeasible.

So instead, we pursue a 'nibble-based' approach: we remove our copies of H in a large (but constant) number T of 'rounds'. For each $t \in [T]$, at the beginning of Round t, let G^t be the graph of currently available edges (i.e. those which do not lie in a copy of H selected in a previous round) and suppose that G^t is (ε_t, d_t)-super-regular for $\varepsilon_t \ll d_t$. Let $\gamma \sim d/(kT)$ (then γ will be a small constant). Within each round, we remove γn copies $\phi_1(H), \dots, \phi_{\gamma n}(H)$ of H from G^t, each chosen randomly in G^t (and independently of the other copies) according to the uniform blow-up lemma. So the edge sets of $\phi_i(H)$ will usually have a small but significant overlap for different i. On the other hand, if after Round t we let G^{t+1} be the graph obtained by removing the edges of all the $\phi_i(H)$, then one can show that G^{t+1} is still $(\varepsilon_{t+1}, d_{t+1})$-super-regular for $\varepsilon_{t+1} \ll d_{t+1}$. This means we can continue with a new embedding round for G^{t+1}. It turns out that we can in fact carry on with this approach until we have the required number of copies $\phi_i(H)$. Note that these copies of H will be edge-disjoint if they are constructed in different rounds.

However, we still need to resolve overlaps between the edge sets of the $\phi_i(H)$ which are constructed within the same round. In other words, we need to modify $\phi_i(H)$ into an embedding $\phi_i'(H)$ such that all the $\phi_i'(H)$ are pairwise edge-disjoint. For this, call any edge of $\phi_i(H)$ which also belongs to some other $\phi_j(H)$ a conflict edge. Let $W_i \subseteq V(G)$ be the vertices which have distance at most one to an endvertex of a conflict edge in $\phi_i(H)$. For technical reasons, we also enlarge W_i by adding a small random set of vertices. Then we still have $|W_i|/n \ll 1$. Remove any edges from $\phi_i(H)$ which are incident to W_i. We now patch up the resulting partial embeddings by using edges from a sparse 'patching graph' $P \subseteq G$ which we set aside at the beginning of the proof. For this patching process to work, we need each $\phi_i(H)$ to be 'compatible' with P. For instance, this means that if $w \in W_i$, if $v_1, v_2, v_3 \notin W_i$ and if $v_\ell w \in E(\phi_i(H))$ for each $\ell \in [3]$, then in the graph P, v_1, v_2, v_3 need to have many common neighbours in W_i (these are then potential candidates for the new image of w in $\phi_i'(H)$). This compatibility will already be ensured during the construction of $\phi_i(H)$ – in the proof of the uniform blow-up lemma, we will disregard any embeddings $\phi_i(H)$ which are not compatible with P. Lemma 5.13 formalizes the above description (we will deduce it from the uniform blow-up lemma).

The core of the current paper is the uniform blow-up lemma (Lemma 5.1) described above. To prove this lemma, we develop an approach by Rödl and Ruciński [37] (who designed it to give an alternative proof of the classical blow-up lemma). We will find a copy of H as the union of a bounded number of matchings. For this, we first apply the Hajnal-Szemerédi theorem to

the square H^2 of H in order to obtain a refined partition of $V(H)$ into classes Y_i, where the bipartite subgraph $H[Y_i, Y_j]$ is a (possibly empty) matching for each pair Y_i, Y_j of classes. We also find a corresponding partition of each V_i into subclasses U_j. We will embed each class Y_j into U_j in a single round. For this, at each round, we have a 'candidacy graph' A_i^j with vertex classes Y_j and U_j, where a vertex $x \in Y_j$ is adjacent to $v \in U_j$ if after Round i, v is still a good candidate for $\phi(x)$. Initially, we may take A_0^j to be the complete bipartite graph, and as the embedding progresses, the candidacy graphs grow sparser, as additional constraints come from the partial embedding. For instance, if we embed a neighbour y of x in Round i, then the number of candidates for $\phi(x)$ is expected to shrink by a factor of d. In particular, we always have $A_{i+1}^j \subseteq A_i^j$. As indicated in the previous paragraph, we also make sure that only embeddings which are compatible with the patching graph P are permitted by the candidacy graph A_{i+1}^j. Crucially, we will be able to show that each A_i^j is super-regular – in particular, this means that A_i^j contains a perfect matching. When embedding Y_i in Round i, we will choose such a perfect matching σ in A_{i-1}^i uniformly at random and embed $x \in Y_i$ to $v = \sigma(x)$. The key difficulty in proving Lemma 5.1 is in proving property (B1), which allows us to derive an approximation to (2.1).

The above nibble process together with the patching is carried out in Section 6, to prove our main decomposition theorem (Theorem 6.1). Theorem 6.1 however assumes that H has the property that each graph $H[X_i, X_j]$ is (extremely close to being) regular. Of course, in general we do not want to assume that H has this approximate regularity property. So in Section 7 we show that even if H does not satisfy this approximate regularity property, then we can pack together a large but bounded number of copies of H (in a suitably random fashion) into a new graph H' to which we can apply Theorem 6.1. In Section 8 we use this to deduce (from Theorem 6.1) several further results about packing arbitrary graphs H of bounded degree.

3. Notation and tools

3.1. **Notation.** For $s \in \mathbb{N}$ let $[s] := \{1, \dots, s\}$. For a graph G and an edge set E, let $G - E$ denote the graph G' with $V(G') = V(G), E(G') = E(G) \setminus E$. Given another graph H we let $G - H := G - E(H)$. As usual, $|G|$, $e(G)$ and $\Delta(G)$ will denote the number of vertices, edges and the maximum degree in G, respectively. Given a set $W \subseteq V = V(G)$, we let $N_G(W) = \bigcap_{v \in W} N_G(v)$, where $N_G(v)$ is the neighbourhood of v in G. We write $G[W]$ for the induced subgraph of G on W; when G is r-partite with partition classes $V_1, \dots, V_r$ and $W_i \subseteq V_i$ for all $i \in [r]$ we also write $G[W_1, \dots, W_r] := G[W_1 \cup \dots \cup W_r]$. We say an rn-vertex graph G is *r-equipartite* if G admits an r-partition into independent sets $V_1, \dots, V_r$ with $|V_i| = n$.

Let R be a graph on $[r]$. We say that a graph G admits a vertex partition $(R, V_1, \dots, V_r)$ if $V_1, \dots, V_r$ form a partition of $V(G)$ into independent sets and $G[V_i, V_j]$ is empty if $ij \notin E(R)$. If $\mathcal{V} := \{V_1, \dots, V_r\}$ then we also say that G admits the vertex partition $(R, \mathcal{V})$. Given a symmetric $r \times r$ matrix $\vec{k}$ with entries $k_{i,j} \in \mathbb{N}$ and $C \in \mathbb{N}$, we say that G is *$(R, \vec{k}, C)$-near-equiregular* with respect to $(V_1, \dots, V_r)$ if

- G admits the vertex partition $(R, V_1, \dots, V_r)$,
- $||V_i| - |V_j|| \le C$ for all $i \neq j \in [r]$,
- for each $ij \in E(R)$ all vertices in $G[V_i, V_j]$ have degree $k_{i,j}$ except for at most $Ck_{i,j}$ vertices having degree $k_{i,j} + 1$.

If R is the complete graph on $[r]$, $k_{i,j} = k$ for all $i \neq j \in [r]$ and G is $(R, \vec{k}, C)$-near-equiregular then we also say that G is *(r, k, C)-near-equiregular*.

We write R_K for the K-fold blow-up of R. So if R is a graph on $[r]$, then R_K is the graph on $[Kr]$ obtained by replacing each vertex i of R by the set of vertices $\{(i-1)K + 1, \dots, iK\}$ and replacing each edge of R by a copy of $K_{K,K}$. We record the following observation for future reference, where $(R_K)^2$ denotes the square of R_K.

Observation 3.1. *Suppose $r, K \in \mathbb{N}$ and R is a graph on $[r]$ with $\delta(R) \ge 1$. Let J_i be the set of vertices of R_K which corresponds to the vertex i of R. Let I be an independent set of $(R_K)^2$. Then $|J_i \cap I| \le 1$.*

For $a, b, c \in \mathbb{R}$ we write $a = b \pm c$ if $b - c \leq a \leq b + c$. In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is typically the order of a graph), then this means that there are non-decreasing functions $f^* : (0,1] \to (0,1]$, $g^* : (0,1] \to (0,1]$ and $h^* : (0,1] \to (0,1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f^*(c)$, $a \leq g^*(b)$ and $1/n \leq h^*(a)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way.

The following functions h', h, g', g, q_*, f, q will be used in the course of the proof. The values of q_*, f, q depend on a further constant $w \in \mathbb{N}$, and we shall only use these after the value of w has been fixed. Let

$$\begin{aligned} h'(a) &:= a^{1/10}, & h(a) &:= a^{1/20}, & g'(a) &:= a^{1/120}, & g(a) &:= a^{1/300}, \\ q_*(a) &:= a^{(1/300)^{w+1}}, & f(a) &:= a^{(1/300)^{w+2}}, & q(a) &:= a^{(1/300)^{w+3}}. \end{aligned} \tag{3.1}$$

Here and elsewhere we denote by $g^b(a)$ the b-fold iteration of $g(a)$ (and similarly for the other functions). Note that for small enough a we have $h'(a) < h(a) < g'(a) < g(a) < q_*(a) < f(a) < q(a)$.

3.2. **Probabilistic estimates.** We shall need the concentration inequalities of Azuma and Chernoff-Hoeffding. $X_0, \dots, X_N$ is a *martingale* if for all $n \in [N]$, $\mathbb{E}[X_n \mid X_0, \dots, X_{n-1}] = X_{n-1}$. We say it is *$c$-Lipschitz* if $|X_n - X_{n-1}| \leq c$ holds for all $n \in [N]$.

Our applications of Azuma's inequality will mostly involve *exposure martingales* (also known as Doob martingales). These are martingales of the form $X_i := \mathbb{E}[X \mid Y_1, \dots, Y_i]$, where X and $Y_1, \dots, Y_i$ are some previously defined random variables.

Theorem 3.2 (Azuma's inequality)**.** *Suppose that $\lambda, c > 0$ and that $X_0, \dots, X_N$ is a c-Lipschitz martingale. Then*

$$\mathbb{P}[|X_N - X_0| \geq \lambda] \leq 2e^{\frac{-\lambda^2}{2Nc^2}}. \tag{3.2}$$

For $n \in \mathbb{N}$ and $0 \leq p \leq 1$ we write $Bin(n,p)$ to denote the binomial distribution with parameters n and p. For $m, n, N \in \mathbb{N}$ with $m, n < N$ the *hypergeometric distribution* with parameters N, n and m is the distribution of the random variable X defined as follows. Let S be a random subset of $\{1, 2, \dots, N\}$ of size n and let $X := |S \cap \{1, 2, \dots, m\}|$. We will use the following bound, which is a simple form of Chernoff-Hoeffding's inequality.

Lemma 3.3 (see [23, Remark 2.5 and Theorem 2.10])**.** *Let $X \sim Bin(n,p)$ or let X have a hypergeometric distribution with parameters N, n, m. Then $\mathbb{P}[|X - \mathbb{E}(X)| \geq t] \leq 2e^{-2t^2/n}$.*

We shall need the following two inequalities for bounding tails of random variables in terms of the binomial distribution.

Proposition 3.4 (Jain, see [34, Lemma 8])**.** *Let $B \sim Bin(n,p)$, and let $X_1, \dots, X_n$ be Bernoulli random variables such that, for any $s \in [n]$ and any $x_1, \dots, x_{s-1} \in \{0,1\}$ we have*

$$\mathbb{P}\left[X_s = 1 \mid X_1 = x_1, \dots, X_{s-1} = x_{s-1}\right] \leq p.$$

Then $\mathbb{P}[\sum_{i=1}^n X_i \geq a] \leq \mathbb{P}[B \geq a]$ for any $a \geq 0$.
Likewise, if for any $s \in [n]$ and any $x_1, \dots, x_{s-1} \in \{0,1\}$ we have

$$\mathbb{P}\left[X_s = 1 \mid X_1 = x_1, \dots, X_{s-1} = x_{s-1}\right] \geq p,$$

then $\mathbb{P}[\sum_{i=1}^n X_i \leq a] \leq \mathbb{P}[B \leq a]$ for any $a \geq 0$.

3.3. **Graph Theory tools.** In the preparation stages of our proof we shall apply the Hajnal-Szemerédi theorem. Given a set X of size n, an *equitable r-partition* of X is a partition of X into sets of size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$ (note that the number of sets of each size is uniquely determined). An *equitable k-colouring* of a graph G is an equitable partition of $V(G)$ into k independent sets.

Theorem 3.5 (Hajnal-Szemerédi [21])**.** *Every graph G with $\Delta(G) \leq k$ admits an equitable $(k+1)$-colouring.*

Let R be a graph on $[r]$ and suppose that G is a graph with vertex partition $(R, V_1, \dots, V_r)$. Let $\vec{d}$ be a symmetric $r \times r$ matrix with entries $d_{i,j}$. We say that G is $(\varepsilon, \vec{d})$*-super-regular with respect to* $(R, V_1, \dots, V_r)$ if $G[V_i, V_j]$ is $(\varepsilon, d_{i,j})$-super-regular whenever $ij \in E(R)$. We say that an r-partite graph G with partition classes $V_1, \dots, V_r$ is (ε, d)*-super-regular with respect to* $V_1, \dots, V_r$ if each $G[V_i, V_j]$ is (ε, d)-super-regular for all distinct $i, j \in [r]$.

Let us now recall the statement of Csaba's extension of the blow-up lemma [12]. It will be convenient (though not essential) to use it in Section 7 (alternatively, we could have used the more general Lemma 5.1).

Theorem 3.6. [12] *Suppose $0 < \varepsilon \ll 1/k, d, 1/\Delta_R$ and that R is a graph on $[r]$ with $\Delta(R) \leq \Delta_R$. Suppose $\vec{d}$ is a symmetric $r \times r$ matrix with entries in $[0,1]$ such that $\min_{ij \in E(R)} d_{i,j} \geq d$. Suppose H is an r-partite graph with partition $(R, X_1, \dots, X_r)$ and G is an r-partite graph with partition $(R, V_1, \dots V_r)$ such that for all $i \in [r]$ we have $|X_i| = |V_i|$. Suppose finally that $\Delta(H) \leq k$ and G is $(\varepsilon, \vec{d})$-super-regular with respect to $(R, V_1, \dots, V_r)$. Then G contains a copy of H, in which for each $i \in [r]$ the vertices of V_i correspond to the vertices of X_i.*

The next statements are standard facts about graph regularity.

Proposition 3.7. *Suppose $G[A, B]$ is (ε, d)-regular and $B' \subseteq B$ with $|B'| \geq \varepsilon |B|$. Then all but at most $2\varepsilon|A|$ vertices in A have degree $(d \pm 2\varepsilon)|B'|$ in B'.*

Proposition 3.8. *Suppose $G[A, B]$ is (ε, d)-regular, and $A' \subseteq A, B' \subseteq B$ with $|A'|/|A|, |B'|/|B| \geq \delta$. Then $G[A', B']$ is $(\varepsilon/\delta, d)$-regular.*

Proposition 3.9. *Let $k \geq 4$ and let $0 < 1/n \ll \varepsilon \ll 1/k, d, 1/(C+1)$. Suppose that $G[A, B]$ is (ε, d)-super-regular with $|A| = |B| \pm C$ and $|A|, |B| \geq n$. If F is a spanning subgraph of G such that for each $v \in V(G)$, $d_G(v) - d_F(v) < k\varepsilon n$, then F is $(3\sqrt{k\varepsilon}/2, d)$-super-regular. In particular, F is $(k\sqrt{\varepsilon}, d)$-super-regular.*

The following lemma states that a super-regular graph can be, at the cost of increasing ε, edge-decomposed into two sparser graphs, each of which is also super-regular.

Lemma 3.10. *Suppose $0 < \beta \leq d \leq 1$ and $0 < 1/n \ll \varepsilon \ll \beta, d, d - \beta$, and that $G[A, B]$ is an (ε, d)-super-regular graph with $|A|, |B| \geq n$. Then there exists a spanning subgraph P of G such that P is $(2\varepsilon, \beta)$-super-regular and $G - P$ is $(2\varepsilon, d - \beta)$-super-regular.*

Proof. For each edge $e \in E(G)$, we select e with probability β/d, all choices being independent. Let P be the spanning subgraph of G formed by the selected edges. It is a straightforward exercise to check that the above conditions are indeed satisfied. □

The following two statements establish a link between codegree and graph regularity. The first one is due to Duke, Lefmann and Rödl [16] (a similar result is proved in [1]), the converse provided by Proposition 3.12 follows immediately from the definitions.

Theorem 3.11. [16] *Suppose $0 < \varepsilon < 2^{-200}$. Suppose $G = G[A, B]$ is a bipartite graph with $|A| > 2/\varepsilon$, and let $d := \frac{e(G)}{|A||B|}$. Let D be the collection of all pairs $\{x, x'\}$ of vertices of A for which*

(i) *$d(x), d(x') > (d - \varepsilon)|B|$, and*
(ii) *$|N_G(x) \cap N_G(x')| < (d + \varepsilon)^2 |B|$.*

Then if $|D| > \frac{1}{2}(1 - 5\varepsilon)|A|^2$, the graph G is $(\varepsilon^{1/6}, d)$-regular.

Proposition 3.12. *Suppose $0 < 1/n \ll \varepsilon \ll d$ and that $G[A,B]$ is (ε,d)-super-regular with $|A|,|B| \geq n$. Then all but at most $\varepsilon|A|^2$ vertex pairs $\{x,x'\} \subseteq A$ satisfy $|N_G(x) \cap N_G(x')| = (d^2 \pm 3\varepsilon)|B|$.*

4. Random Perfect Matchings

It is well-known that every (ε,d)-super-regular balanced bipartite graph G contains a perfect matching. In this section we show that the number of the perfect matchings containing a given edge e is roughly the same for all $e \in E(G)$, i.e. each edge has the same likelihood of appearing in a random perfect matching.

Given a bipartite graph $G[U,W]$, we write $M(G)$ for the set of all perfect matchings of G, $M_e(G)$ for the set of all perfect matchings containing a given edge $e \in E(G)$, and set $M'_e(G) := M(G) \setminus M_e(G)$. For $\sigma \in M(G)$, we often abuse the notion and think of σ as a bijection from U to W (where $\sigma(u) = w$ if and only if uw is an edge in the perfect matching σ).

We will use the following result of Rödl and Ruciński [37], which implies that the edges of a random perfect matching are uniformly distributed with respect to large sets of vertices.

Theorem 4.1. [37] *Suppose $0 < 1/n \ll c \ll \varepsilon \ll d \leq 1$ and $h'(a) = a^{1/10}$ is as defined in (3.1). Let $G[U,W]$ be an (ε,d)-super-regular bipartite graph with $|U| = |W| = n$ and let $S \subseteq U, T \subseteq W$ with $|S| = sn, |T| = tn$ and $s,t \geq h'(\varepsilon)$. Then at least $(1-(1-c)^n)|M(G)|$ perfect matchings σ of G satisfy $|\sigma(S) \cap T| = (1 \pm h'(\varepsilon))stn$.*

We also use the following result of Alon, Rödl and Ruciński [2] on the number of perfect matchings in (super-)regular graphs (which was also a tool in the proof of Theorem 4.1).

Theorem 4.2. [2] *Let $0 < 1/n \ll \varepsilon \ll d \leq 1$, and let $G[U,W]$ be an (ε,d)-super-regular bipartite graph with $|U| = |W| = n$. Then*

$$(d-2\varepsilon)^n n! \leq |M(G)| \leq (d+2\varepsilon)^n n!.$$

If $G[U,W]$ is (ε,d)-regular bipartite graph, then

$$|M(G)| \leq (d+3\varepsilon)^n n!.$$

We now use Theorem 4.1 to prove a 'localised' version of it.

Theorem 4.3. *Suppose $0 < 1/n \ll \varepsilon \ll d \leq 1$ and $h(a) = a^{1/20}$ is as defined in (3.1). Let $G[U,W]$ be an (ε,d)-super-regular bipartite graph with $|U| = |W| = n$. Then*

$$\frac{|M_e(G)|}{|M(G)|} = (1 \pm h(\varepsilon))\frac{1}{dn}.$$

In other words, if we choose a perfect matching σ of G uniformly at random, then for any edge $uv \in E(G)$ with $u \in U$ and $v \in W$,

$$\mathbb{P}[\sigma(u) = v] = (1 \pm h(\varepsilon))\frac{1}{dn}.$$

Proof. For $\sigma \in M(G)$ and distinct vertices $u_1,u_2,u_3 \in U$ such that $\sigma(u_1)u_2, \sigma(u_2)u_3, \sigma(u_3)u_1 \in E(G)$, we define the (u_1,u_2,u_3)-*switch* $S_{u_1,u_2,u_3}(\sigma)$ of the matching σ to be a new matching σ' where

$$\sigma'(u) := \begin{cases} \sigma(u) & \text{if } u \notin \{u_1,u_2,u_3\}, \\ \sigma(u_3) & \text{if } u = u_1, \\ \sigma(u_1) & \text{if } u = u_2, \\ \sigma(u_2) & \text{if } u = u_3. \end{cases}$$

If $\sigma' \in M(G)$ is the (u_1,u_2,u_3)-switch of $\sigma \in M(G)$ for some $u_1,u_2,u_3 \in U$, we also say that σ' is a *switch of* σ.

Our aim is to estimate $|M_e(G)|/|M'_e(G)|$ for a given edge $e = uv$. To do so, we consider the auxiliary bipartite graph H with bipartition $A_1 := M_e(G)$, $A_2 := M'_e(G)$ such that $\sigma\sigma' \in E(H)$ if and only if $\sigma \in A_1, \sigma' \in A_2$ and σ' is a switch of σ. So $V(H) = M(G)$. Let

$$A' := \{\sigma \in M(G) : \sigma(N_G(v')) \cap N_G(u') = (d^2 \pm 2h'(\varepsilon))n \text{ for all } u' \in U, v' \in W\}.$$

Choose a new constant c such that $1/n \ll c \ll \varepsilon$. Since G is (ε, d)-super-regular, $|N_G(u')| = (d \pm \varepsilon)n$ for all $u' \in V(G)$. This together with Theorem 4.1 implies that

$$|A'| \geq (1 - n^2(1-c)^n)|V(H)|. \tag{4.1}$$

Note that for all $\sigma_1 \in A_1$

$$d_H(\sigma_1) = |\{S_{u,u_1,u_2}(\sigma_1) : vu_1, \sigma_1(u_1)u_2, \sigma_1(u_2)u \in E(G), u_1, u_2 \in U \setminus \{u\} \text{ with } u_1 \neq u_2\}|, \tag{4.2}$$

and for all $\sigma_2 \in A_2$

$$d_H(\sigma_2) = |\{S_{u,u_1,\sigma_2^{-1}(v)}(\sigma_2) : u_1\sigma_2(u), \sigma_2(u_1)\sigma_2^{-1}(v) \in E(G), u_1 \in U \setminus \{u, \sigma_2^{-1}(v)\}\}|. \tag{4.3}$$

Thus for any $\sigma_1 \in A_1, \sigma_2 \in A_2$, we have

$$d_H(\sigma_1) \leq n^2 \quad \text{and} \quad d_H(\sigma_2) \leq n. \tag{4.4}$$

Let us now estimate the degree of vertices of A' in the graph H. For $\sigma_1 \in A_1$,

$$d_H(\sigma_1) \stackrel{(4.2)}{=} |\{(u_1, u_2) \in (U \setminus \{u\}) \times (U \setminus \{u\}) : vu_1, \sigma_1(u_1)u_2, \sigma_1(u_2)u \in E(G), u_1 \neq u_2\}|.$$

Since $u_1 \in N_G(v)$ we have $|N_G(v)| = (d \pm \varepsilon)n$ choices for u_1. Once u_1 is fixed, we have to choose u_2 such that $u_2 \in N_G(\sigma_1(u_1))$ and $\sigma_1(u_2) \in N_G(u)$. So if $\sigma_1 \in A'$, there are $(d^2 \pm 2h'(\varepsilon))n$ choices for u_2 once u_1 is fixed. Hence we obtain that

$$\text{if } \sigma_1 \in A_1 \cap A', \text{ then } d_H(\sigma_1) = (d \pm \varepsilon)n \cdot (d^2 \pm 2h'(\varepsilon))n = (d^3 \pm 3h'(\varepsilon))n^2. \tag{4.5}$$

For $\sigma_2 \in A_2$,

$$d_H(\sigma_2) \stackrel{(4.3)}{=} |\{u_1 \in U \setminus \{u, \sigma_2^{-1}(v)\} : u_1\sigma_2(u), \sigma_2(u_1)\sigma_2^{-1}(v) \in E(G)\}|.$$

Thus we count the number of choices of u_1 such that $u_1 \in N_G(\sigma_2(u))$ and $\sigma_2(u_1) \in N_G(\sigma_2^{-1}(v))$. Similarly as before, if $\sigma_2 \in A'$, there are $(d^2 \pm 2h'(\varepsilon))n$ choices for u_1. Hence we obtain that

$$\text{if } \sigma_2 \in A_2 \cap A', \text{ then } d_H(\sigma_2) = (d^2 \pm 2h'(\varepsilon))n. \tag{4.6}$$

Thus

$$\begin{aligned} |E(H)| &= \sum_{\sigma_1 \in A_1 \cap A'} d_H(\sigma_1) + \sum_{\sigma_1 \in A_1 \setminus A'} d_H(\sigma_1) \stackrel{(4.4),(4.5)}{=} (d^3 \pm 3h'(\varepsilon))n^2|A_1 \cap A'| \pm n^2|A_1 \setminus A'| \\ &= (d^3 \pm 3h'(\varepsilon))n^2|A_1| \pm 2n^2|A_1 \setminus A'| \stackrel{(4.1)}{=} (d^3 \pm 3h'(\varepsilon))n^2|A_1| \pm 2n^4(1-c)^n|V(H)|, \end{aligned}$$

and

$$\begin{aligned} |E(H)| &= \sum_{\sigma_2 \in A_2 \cap A'} d_H(\sigma_2) + \sum_{\sigma_2 \in A_2 \setminus A'} d_H(\sigma_2) \stackrel{(4.4),(4.6)}{=} (d^2 \pm 2h'(\varepsilon))n|A_2 \cap A'| \pm n|A_2 \setminus A'| \\ &\stackrel{(4.1)}{=} (d^2 \pm 2h'(\varepsilon))n|A_2| \pm 2n^3(1-c)^n|V(H)|. \end{aligned}$$

This in turn implies that

$$(d^3 \pm 3h'(\varepsilon))n^2|A_1| = (d^2 \pm 2h'(\varepsilon))n|A_2| \pm 3n^4(1-c)^n(|A_1| + |A_2|).$$

Since $1/n \ll c$, we conclude that $|A_2| = (d \pm 7h'(\varepsilon)/d^2)n|A_1|$. Since $h(\varepsilon) = \sqrt{h'(\varepsilon)}$ by (3.1), this in turn implies that

$$\frac{|M_e(G)|}{|M(G)|} = \frac{|A_1|}{|A_1| + |A_2|} = \frac{|A_1|}{|A_1| + (d \pm 7h'(\varepsilon)/d^2)n|A_1|} = (1 \pm h(\varepsilon))\frac{1}{dn}.$$

□

From Theorems 4.1–4.3 we now deduce further properties of a random perfect matching which we will make frequent use of in Section 5.

Lemma 4.4. *Suppose $0 < 1/n \ll \varepsilon \ll d' < d \leq 1$ and $0 < c \ll d' \leq d/9$. Let $h(a) = a^{1/20}$ be as defined in (3.1) and let $G[U, W]$ be an (ε, d)-super-regular bipartite graph with $|U| = |W| = n$. Then a perfect matching σ of G chosen uniformly at random satisfies the following.*

(M1) *For every $u \in U$ and every $S \subseteq N_G(u) \subseteq W$ with $|S| = sn$,*

$$\mathbb{P}[\sigma(u) \in S] = (1 \pm h(\varepsilon))\frac{s}{d}.$$

(M2) *For every $w \in W$ and every $S \subseteq N_G(w) \subseteq U$ with $|S| = sn$,*

$$\mathbb{P}[w \in \sigma(S)] = (1 \pm h(\varepsilon))\frac{s}{d}.$$

(M3) *Let $G' \subseteq G$ be a subgraph of G with $V(G') = V(G)$ such that $\Delta(G') \leq d'n$. Then*

$$\mathbb{P}\left[|\{u \in U : u\sigma(u) \in E(G')\}| > \frac{8d'}{d}n\right] < (1-c)^n.$$

Proof. To show (M1) and (M2), we use Theorem 4.3 to conclude that

$$\mathbb{P}[\sigma(u) \in S] = \sum_{w \in S} \mathbb{P}[\sigma(u) = w] = (1 \pm h(\varepsilon))\frac{|S|}{dn},$$

and

$$\mathbb{P}[w \in \sigma(S)] = \sum_{u \in S} \mathbb{P}[\sigma(u) = w] = (1 \pm h(\varepsilon))\frac{|S|}{dn}.$$

To prove (M3), let $U_\sigma := \{u \in U : u\sigma(u) \in E(G')\}$ and let $m := 8d'n/d$. For given $U' \subseteq U$ with $|U'| = m$, we shall now find an upper bound for the number of perfect matchings σ in G such that $U' \subseteq U_\sigma$. For each vertex $u \in U'$, there are at most $d'n$ ways to choose $\sigma(u)$ such that $u\sigma(u) \in E(G')$. After having chosen $\sigma(u)$ for all $u \in U'$, by Proposition 3.8, we are left with a $(\sqrt{\varepsilon}, d)$-regular graph $G[U \setminus U', W \setminus \sigma(U')]$, which contains at most $(d+3\sqrt{\varepsilon})^{n-m}(n-m)!$ perfect matchings by Theorem 4.2. Thus for any set $U' \subseteq U$ with $|U'| = m$, the number of perfect matchings σ in G which satisfy $U' \subseteq U_\sigma$ is at most $(d'n)^m(d+3\sqrt{\varepsilon})^{n-m}(n-m)!$. Since there are at most $\binom{n}{m}$ choices for U', the number of perfect matchings σ with $|U_\sigma| \geq m$ is at most

$$\binom{n}{m}(d'n)^m(d+3\sqrt{\varepsilon})^{n-m}(n-m)!.$$

By Theorem 4.2, we know $|M(G)| \geq (d-2\varepsilon)^n n!$. Thus,

$$\begin{aligned}
\mathbb{P}\left[|\{u \in U : u\sigma(u) \in E(G')\}| > \frac{8d'}{d}n\right] &\leq \frac{\binom{n}{m}(d'n)^m(d+3\sqrt{\varepsilon})^{n-m}(n-m)!}{(d-2\varepsilon)^n n!} \\
&= \left(\frac{d+3\sqrt{\varepsilon}}{d-2\varepsilon}\right)^{n-m} \frac{(d'n)^m}{(d-2\varepsilon)^m m!} \\
&\leq (1+\varepsilon^{1/3})^n \frac{d^m(m/8)^m}{(d/2)^m(m/3)^m} \\
&\leq (1+\varepsilon^{1/3})^n \left(\frac{3}{4}\right)^{8d'n/d} \leq (1-c)^n.
\end{aligned}$$

To obtain the third line we use that $8d'n = dm$ and $m! > (m/3)^m$ for large m. To obtain the final inequality we use that $\varepsilon, c \ll d', d$. □

5. A uniform blow-up lemma

Our goal in this section is to establish a probabilistic version of the classical blow-up lemma, which finds a 'uniformly distributed' copy of H in G. While the classical version asserts that, informally, a super-regular graph G contains a copy of any bounded degree graph H, here we show that such a copy of H can be selected to have some additional 'random-like' properties. Ideally this would take a similar form as Theorem 4.3: for a randomly chosen copy of H in G, each edge of G has the same probability of being in H. Unfortunately this is false. (Suppose H is a triangle factor and that $e \in E(G)$ does not lie in a triangle.) However we will be able to show that a suitable randomised algorithm gives a copy of H in G which is randomly distributed in a very strong sense (see Lemma 5.1).

First we need the following definitions. Let ϕ be an embedding of a graph H into a graph G and let $G' \subseteq G$. When applying Lemma 5.1, G' will be a 'bad' graph which we would like to avoid as much as possible when embedding H. We define

$$\begin{aligned}
&\phi_E(H,G,G') := \phi(E(H)) \cap E(G'),\\
&\phi(H,G,G') := \{v \in V(G) : \text{ there exists } e \in \phi_E(H,G,G') \text{ such that } v \in e\},\\
&\phi_2(H,G,G') := \{v \in V(G) : \text{ there exists } u \in V(G) \text{ such that } uv \in \phi(E(H)), u \in \phi(H,G,G')\}.
\end{aligned} \tag{5.1}$$

Note that $\phi(H,G,G') \subseteq \phi_2(H,G,G')$.

We would also like our embedding of H to be well behaved with respect to additional given graphs A_0 and P (where A_0 encodes the initial candidates for the images of each vertex of H, and P is the patching graph which will be used to adjust the embeddings later on). To formalise this, let R be a graph on $[r]$. Let H be a graph and let $\mathcal{X} = \{X_1, \dots, X_r\}$ be a partition of $V(H)$. We say that a (not necessarily uniform) hypergraph N with vertex set $V(H) = X_1 \cup \dots \cup X_r$ and edge set $\{N_x : x \in V(H)\}$ is an $(H, R, \mathcal{X})$-*candidacy hypergraph* if all $x, y \in V(H)$ satisfy the following properties:

- if $x \in X_i$, then $|N_x \cap X_j| \le 1$ for any $j \in [r]$, and $|N_x \cap X_j| = 0$ if $j \notin N_R(i)$,
- $N_H(x) \subseteq N_x$,
- $y \in N_x$ if and only if $x \in N_y$.

Notice that the above definition is only applicable to graphs H which admit the vertex partition $(R, X_1, \dots, X_r)$ with $\Delta(H[X_i, X_j]) \le 1$ for all $i, j \in [r]$ (and thus $\Delta(H) \le \Delta(R)$). Let P be a graph admitting vertex partition $(R, \mathcal{V})$ where $\mathcal{V} = (V_1, \dots, V_r)$. Suppose that $|X_i| = |V_i|$ for all $i \in [r]$ and that $\phi : V(H) \to V(P)$ is a bijection, mapping each X_i to V_i. Let A_0 be a graph on $V(P) \cup V(H)$. We call a bipartite graph F on $V(F) = (V(H), V(P))$ an $(H, P, R, A_0, \phi, \mathcal{X}, \mathcal{V}, N)$-*candidacy bigraph* if it satisfies the following conditions.

(CB1) N is an $(H, R, \mathcal{X})$-candidacy hypergraph.
(CB2) $N_F(x) \subseteq N_{A_0}(x) \cap \bigcap_{y \in N_x} N_P(\phi(y)) \cap V_i$ for all $x \in X_i$.

Note that in particular $E(F[X_i, V_j]) = \emptyset$ for all $i \neq j$. A candidacy bigraph F will always encode permissible images for embeddings, i.e. we may only embed x to v if $xv \in E(F)$.

For a given graph R on $[r]$ and a symmetric $r \times r$ matrix $\vec{\beta}$ with entries $\beta_{i,j}$, we denote

$$p(R, \vec{\beta}, i) := \prod_{\ell \in N_R(i)} \beta_{i,\ell}. \tag{5.2}$$

This will be convenient for measuring the densities of the bipartite subgraphs of the candidacy bigraph F. Recall that R_K denotes the K-fold blow-up of R. Now we can state our main result in this section.

Lemma 5.1 (Uniform blow-up lemma)**.** *Suppose*

$$0 < 1/n \ll c \ll \varepsilon \ll \gamma \ll \beta, d, d_0, 1/k, 1/\Delta_R, 1/(C+1) \text{ and } 1/n \ll 1/r.$$

Let $K := (k+1)^2\Delta_R$, let $w := K^2\Delta_R^2(\Delta_R+1)$ and let f be the function defined in (3.1)*. Suppose that R is a graph on $[r]$ with $\Delta(R) = \Delta_R$. Let $\vec{d}, \vec{\beta}, \vec{k}$ be symmetric $r \times r$ matrices such that $d = \min_{ij \in E(R)} d_{i,j}$, $\beta = \min_{ij \in E(R)} \beta_{i,j}$, $k = \max_{ij \in E(R)} k_{i,j}$ and $k_{i,j} \in \mathbb{N}$ for all $ij \in E(R)$. Suppose that the following hold.*

- *G is an $(\varepsilon, \vec{d})$-super-regular graph with respect to $(R, \mathcal{V})$, where $\mathcal{V} = (V_1, \dots, V_r)$, $\max_{i \in [r]} |V_i| = n$ and $n - C \le |V_i| \le n$ for all $i \in [r]$.*
- *P is an $(\varepsilon, \vec{\beta})$-super-regular graph with respect to $(R, \mathcal{V})$.*
- *H is an $(R, \vec{k}, C)$-near-equiregular graph admitting the vertex partition $(R, \mathcal{X})$ with $\mathcal{X} = (X_1, \dots, X_r)$ where $|X_i| = |V_i|$.*
- *A_0 is a bipartite graph with bipartition $(V(H), V(G))$ such that $N_{A_0}(X_i) = V_i$ and $A_0[X_i, V_i]$ is (ε, d_0)-super-regular for each $i \in [r]$.*

Then there exists a randomised algorithm (the uniform embedding algorithm) which succeeds with probability at least $1-(1-c)^n$ *in finding an embedding* ϕ *of* H *into* G *such that* $\phi(X_i)=V_i$ *for each* $i\in[r]$ *and* $\phi(x)\in N_{A_0}(x)$ *for each* $x\in V(H)$*. Conditional on being successful, this algorithm returns* $(\phi,\mathcal{Y},\mathcal{U},F,N)$ *with the following properties.*

(B1) *For all* $ij\in E(R)$*,* $v\in V_i$ *and* $S\subseteq V_j\cap N_G(v)$ *with* $|S|>f(\varepsilon)n$*,*

$$\mathbb{E}\left[|N_{\phi(H)}(v)\cap S|\right]=(1\pm f(\varepsilon))\frac{k_{i,j}|S|}{d_{i,j}n}.$$

(B2) $\mathcal{U}=\{U_1,\dots,U_{Kr}\}$ *is a partition refining* $V_1,\dots,V_r$ *and* $\mathcal{Y}=\{Y_1,\dots,Y_{Kr}\}$ *is a partition refining* $X_1,\dots,X_r$ *with* $|Y_i|=|U_i|$*, with* $\max_{i\in[Kr]}|Y_i|=\lceil n/K\rceil$ *and* $|Y_i|-|Y_j|\le C$ *for all* $i\neq j$*.* F *is a disjoint union of bipartite graphs* $F_1,\dots,F_{Kr}$ *such that each* F_j *has bipartition* (Y_j,U_j)*.* N *is a hypergraph with vertex set* $V(H)$ *and a hyperedge* N_x *for each* $x\in V(H)$*. Moreover, the following conditions hold.*

(B2.1) N *is an* $(H,R_K,\mathcal{Y})$*-candidacy hypergraph with* $\max\{|N_x|:N_x\in N\}\le K\Delta_R$*.*

(B2.2) *For all* $j\in[Kr]$ *and* $x\in Y_j$*,* $N_{F_j}(x)\subseteq U_j\cap N_{A_0}(x)\cap\bigcap_{y\in N_x}N_P(\phi(y))$*, thus* F *is a* $(H,P,R_K,A_0,\phi,\mathcal{Y},\mathcal{U},N)$*-candidacy bigraph.*

(B2.3) P *is* $(\varepsilon^{1/3},\vec{\beta}')$*-super-regular with respect to* $(R_K,\mathcal{U})$*, where* $\vec{\beta}'$ *is the symmetric* $Kr\times Kr$ *matrix with entries* $\beta'_{\ell,\ell'}:=\beta_{i,j}$ *whenever* $i=\lceil\ell/K\rceil$ *and* $j=\lceil\ell'/K\rceil$*.*

(B2.4) *For each* $j\in[Kr]$ *the graph* F_j *is* $(f(\varepsilon),d_0p(R_K,\vec{\beta}',j))$*-super-regular.*

(B3) *For all* $u\neq v\in V(G)$*,* $\mathbb{P}[N_H(\phi^{-1}(u))\cap N_H(\phi^{-1}(v))\neq\emptyset]\le 1/\sqrt{n}$*.*

(B4) *Let* G'' *be a subgraph of* G *with* $V(G'')=V(G)$ *such that* $\Delta(G'')\le\gamma n$*.*

(B4.1) *For any vertex* $v\in V(G)$*,* $\mathbb{P}[v\in\phi_2(H,G,G'')]\le\gamma^{1/2}$*.*

(B4.2) $\mathbb{P}[|\{u:u\in U_j,u\in\phi_2(H,G,G'')\}|\le\gamma^{3/5}n$ *for all* $j\in[Kr]]\ge 1-(1-2c)^n$*.*

(B5) *Suppose* $v_1,\dots,v_s\in V_i$ *where* $i\in[r]$ *with* $s\le K$*. For all* $j\in N_R(i)$ *and* $v\in V_j$*, let* B_v *be the random variable such that*

$$B_v:=\begin{cases}1 & \text{if there exists } \ell\in[s] \text{ with } v_\ell v\in\phi(E(H)),\\ 0 & \text{otherwise.}\end{cases}$$

Then for all $j\in N_R(i)$ *and all but at most* $2f(\varepsilon)n$ *vertices* $v\in N_G(v_1,\dots,v_s)\cap V_j$ *we have that*

$$\mathbb{P}[B_v=1]=(1\pm 2f(\varepsilon))\,\frac{k_{i,j}s}{d_{i,j}n}.$$

(B6) *For all* $i\in[r]$ *and all sets* $Q\subseteq X_i$ *and* $W\subseteq V_i$ *with* $|Q|,|W|>f(\varepsilon)n$*,*

$$\mathbb{P}\left[|\phi(Q)\cap W|=\frac{(1\pm f(\varepsilon))|Q||W|}{n}\right]\ge 1-(1-2c)^n.$$

Note that it is property (B1) that most intuitively encapsulates the 'randomness' of the embedding ϕ. Informally speaking, it says that a sufficiently large set S in the neighbourhood of v will contain approximately as many neighbours of v in the embedded copy of H as expected. As mentioned earlier, the graphs F_i in (B2) can be viewed as 'candidacy graphs'. If they are super-regular, this means that the embedding ϕ is well behaved with respect to the patching graph P. We will need this when replacing some edges of $\phi(E(H))$ by edges of P. When applying Lemma 5.1 in Section 6, G'' in (B4) will play the role of the union of certain previously embedded graphs. (B4) shows that the overlap of these previous embeddings with the new embedding $\phi(H)$ is small. Note in (B5), $N_{\phi(H)}(v)\cap\{v_1,\dots,v_s\}\neq\emptyset$ if and only if $B_v=1$. So (B5) can be viewed as a 'localized' version of (B1) which applies to most vertices v and small sets $S=\{v_1,\dots,v_s\}$. The case $s=1$ of (B5) can also be viewed as a generalization of Theorem 4.3 from matchings to arbitrary bounded degree graphs H which holds for most edges e of G. We will only use the case $s\le 2$ of (B5). (B6) will be used in order to prove property (T3) of the packing in Theorem 6.1.

To prove Lemma 5.1 we shall introduce the *Slender graph algorithm* and the *Uniform embedding algorithm*. Before that we shall state the Four graphs lemma of Rödl and Ruciński [37], which is required for the analysis of the Slender graph algorithm and was a key tool in their proof of the classical blow-up lemma. Roughly speaking, it asserts the following: given three

super-regular graphs forming a 'blown-up triangle', choosing a random perfect matching M in one of the pairs and then contracting the edges of M yields a super-regular pair.

More precisely, suppose that W_1, W_2, W_3 are disjoint sets of size n, and that for each pair i, j with $1 \le i < j \le 3$ we have a bipartite graph F_{ij} with partition (W_i, W_j). We say that the triple of graphs (F_{12}, F_{13}, F_{23}) is $(\varepsilon, d_{12}, d_{13}, d_{23})$-*super-regular* if

(R1) each F_{ij} is $(2\varepsilon, d_{ij})$-super-regular,
(R2) every edge $uv \in E(F_{12})$ (where $u \in W_1$, $v \in W_2$) satisfies

$$|N_{F_{13}}(u) \cap N_{F_{23}}(v)| = (d_{13}d_{23} \pm 2\varepsilon)n.$$

Moreover, given a bijection (which might be inducing a perfect matching) $\sigma : W_1 \to W_2$ in F_{12} we define a fourth graph A_σ with bipartition (W_1, W_3), where $uv \in E(A_\sigma)$ with $u \in W_1$ if and only if both $uv \in E(F_{13})$ and $\sigma(u)v \in E(F_{23})$.

Lemma 5.2 (Four graphs lemma [37])**.** *Suppose* $0 < 1/n \ll c \ll \varepsilon \ll d_{12}, d_{13}, d_{23}, 1/(C+1) \le 1$. *Let* g' *be the function defined in* (3.1). *Let* (F_{12}, F_{13}, F_{23}) *be a* $(\varepsilon, d_{12}, d_{13}, d_{23})$*-super-regular triple of graphs with vertex sets* W_1, W_2, W_3*, each of size* n*. Then if we choose a perfect matching* $\sigma : W_1 \to W_2$ *in* F_{12} *uniformly at random, then* A_σ *is* $(g'(\varepsilon), d_{13}d_{23})$*-super-regular with probability at least* $1 - (1-c)^n$.
Moreover, suppose some $x_1, \dots, x_C \in W_1, y_1, \dots, y_C \in W_2$ *are distinct and for each* $i \in [C]$ *they satisfy*

$$|N_{F_{13}}(x_i) \cap N_{F_{23}}(y_i)| = (d_{13}d_{23} \pm 2\varepsilon)n. \tag{5.3}$$

Then assigning $\sigma(x_i) := y_i$ *and choosing a perfect matching in* $F_{12} \setminus \{x_1, \dots, x_C, y_1, \dots, y_C\}$ *uniformly at random, we still obtain, with probability at least* $1-(1-c)^n$*, a bijection* $\sigma : W_1 \to W_2$ *for which* A_σ *is* $(g'(\varepsilon), d_{13}d_{23})$*-super-regular.*

Note that we do not require $y_i \in N_{F_{12}}(x_i)$.

Proof. The original proof (based on Theorem 4.1) from [37] carries over. It is straightforward to check that artificially assigning $\sigma(x_i) := y_i$ does not affect the proof other than slightly increasing the value of g'. □

Valid input. We say $\mathcal{S} = (G, P, H, R_*, A_0, \mathcal{U}, \mathcal{Y}, \mathcal{I}, c, \varepsilon, d_0, \vec{d}, \vec{\beta}, K, \Delta_R, C)$ is a valid input for the Slender graph algorithm if the following holds, where m denotes the size of the largest partition class in $\mathcal{U}$ and $\mathcal{Y}$, and $r := |R_*|/K$.

(V1) Both $\vec{d}$ and $\vec{\beta}$ are symmetric $Kr \times Kr$ matrices where $d := \min_{ij \in E(R_*)} d_{i,j}$ and $\beta := \min_{ij \in E(R_*)} \beta_{i,j}$. Moreover, $1/m \ll c \ll \varepsilon \ll \beta, d, d_0, 1/K, 1/\Delta_R, 1/(C+1) \le 1$, and $1/m \ll 1/r$.
(V2) $\mathcal{I} = (I_1, \dots, I_w)$ is a partition of $[Kr]$ with $w := K^2\Delta_R^2(\Delta_R + 1)$.
(V3) R_* is a graph on $[Kr]$ such that each $I_{i'} \in \mathcal{I}$ is an independent set of R_*. Moreover, $\Delta(R_*) \le K\Delta_R$ and $\Delta(R_*[I_{i'}, I_{j'}]) \le 1$ for all $i', j' \in [w]$.
(V4) G is a $(\varepsilon, \vec{d})$-super-regular graph with respect to $(R_*, \mathcal{U})$ where $\mathcal{U} = (U_1, \dots, U_{Kr})$ and $m - C \le |U_i| \le m$ for all $i \in [Kr]$.
(V5) P is a $(\varepsilon, \vec{\beta})$-super-regular graph with respect to $(R_*, \mathcal{U})$.
(V6) A_0 is a disjoint union of bipartite graphs $A_0(1), \dots, A_0(Kr)$ such that each $A_0(i)$ has bipartition (Y_i, U_i) and $A_0(i) = A_0[Y_i, U_i]$ is (ε, d_0)-super-regular for each $i \in [Kr]$.
(V7) H is a graph admitting vertex partition $(R_*, \mathcal{Y})$ with $\mathcal{Y} = (Y_1, \dots, Y_{Kr})$ such that $|Y_i| = |U_i|$ for $i \in [Kr]$ and $H[Y_i, Y_j]$ is a matching of size $\min\{|Y_i|, |Y_j|\}$.

We now introduce the Slender graph algorithm for graphs G, P, H, R_*, A_0, partitions $\mathcal{U}, \mathcal{Y}, \mathcal{I}$ and constants $c, \varepsilon, d_0, K, \Delta_R, C$ and $Kr \times Kr$ matrices $\vec{\beta}, \vec{d}$ forming a valid input (R_* will play the role of R_K in Lemma 5.1). As indicated by (V7) it views H as a union of perfect matchings. Their vertex classes are embedded into G in w rounds, where w is defined as in (V2). The vertex classes of H to be embedded in a given round correspond to an independent set of R_*, so their embeddings do not 'interfere' with each other. We will choose the images of the vertices in

$x \in Y_i$ by choosing a random perfect matching in an auxiliary graph $A(i)$, where xv is an edge in $A(i)$ if v is a suitable candidate for the image $\phi(x)$ of x (see (5.8)).

The Slender graph algorithm on $(G, P, H, R_*, A_0, \mathcal{U}, \mathcal{Y}, \mathcal{I}, c, \varepsilon, d_0, \vec{d}, \vec{\beta}, K, \Delta_R, C)$**.**

Preparation Round. The goal of the preparation round is to reduce the problem to the setting where the vertex classes all have equal size by adding additional artificial vertices to the smaller classes.

For all $i \in [Kr]$ and $j \in [m - |Y_i|]$ we introduce additional 'artificial' vertices $y_{i,j}, u_{i,j}$. Let Y_i' be the union of Y_i together with the $y_{i,j}$ and define U_i' similarly. We now define graphs G', P', H' and A_0' with $G \subseteq G', P \subseteq P', H \subseteq H', A_0 \subseteq A_0'$ and $V(G') := V(P') := \bigcup_{i=1}^{Kr} U_i'$, $V(H') := \bigcup_{i=1}^{Kr} Y_i'$ and $V(A_0') := V(H') \cup V(G')$ as follows. For all vertices $u_{i,j} \in U_i'$ and $u \in U_\ell$ with $i\ell \in E(R_*)$ we include the edge $uu_{i,j}$ in $E(G')$ independently with probability $d_{i,\ell}$ and include the edge $uu_{i,j}$ in $E(P')$ independently with probability $\beta_{i,\ell}$. Also, for all vertices $x \in Y_i$ and $u_{i,j} \in U_i'$, we include the edge $xu_{i,j}$ in $E(A_0')$ independently with probability d_0. Similarly for all $y_{i,j} \in Y_i', u \in U_i$, we include the edge $uy_{i,j}$ in $E(A_0')$ independently with probability d_0. For all $1 \le i \ne j \le Kr$ with $ij \in E(R_*)$ add at most C missing edges to extend the matching $H[Y_i', Y_j']$ to a perfect matching $H'[Y_i', Y_j']$. We define the following events.

- (P1) $G'[V(G)] = G$, $P'[V(P)] = P$ and $A_0'[V(G) \cup V(H)] = A_0$.
- (P2) G' is $(2\varepsilon, \vec{d})$-super-regular with respect to $(R_*, U_1', \dots, U_{Kr}')$ and P' is $(2\varepsilon, \vec{\beta})$-super-regular with respect to $(R_*, U_1', \dots, U_{Kr}')$. Also, $A_0'[Y_i', U_i']$ is $(2\varepsilon, d_0)$-super-regular for all $i \in [Kr]$.
- (P3) For all $i \in [Kr]$ the following holds for any set $Q_1 = \{q_1, q_2, \dots, q_s\}$ of $s \le K\Delta_R$ vertices such that $q_\ell \in U_{j_\ell}' \setminus U_{j_\ell}$ and $j_\ell \in N_{R_*}(i)$ for every $\ell \in [s]$, any set $Q_2 \subseteq V(G)$ with $|Q_2| \le K\Delta_R$ and any vertex $y \in Y_i'$:
  - (i) $|N_{G'}(Q_1) \cap N_{G'}(Q_2) \cap N_{A_0'}(y)| = (\prod_{\ell=1}^{s} d_{i,j_\ell}) |N_{G'}(Q_2) \cap N_{A_0'}(y)| \pm 2\varepsilon m.$
  - (ii) $|N_{P'}(Q_1) \cap N_{P'}(Q_2) \cap N_{A_0'}(y)| = (\prod_{\ell=1}^{s} \beta_{i,j_\ell}) |N_{P'}(Q_2) \cap N_{A_0'}(y)| \pm 2\varepsilon m.$
- (P4) $H'[V(H)] = H$ and $H'[Y_i', Y_j']$ is a perfect matching for all $ij \in E(R_*)$.

Note that (P1) and (P4) always hold. If (P2) or (P3) does not hold, then we end the algorithm with **failure of type 1**. So if failure of type 1 does not occur, then (P1)–(P4) hold.

Round 0**.** Here we initialise the settings for the algorithm. Let f_0 be an empty partial embedding which sends no vertex in H' to no vertex in G'. Recall from (V2) that $\mathcal{I} = (I_1, \dots, I_w)$ and for all $i' \in [w]$ let $I_1^{i'} := \bigcup_{j'=1}^{i'} I_{j'}$. For all $x \in V(H')$ and $i' \in [w]$ let

$$N_{i'}(x) := \bigcup_{j \in I_1^{i'}} Y_j' \cap N_{H'}(x).$$

So $N_{i'}(x) \subseteq \bigcup_{j \in I_1^{i'} \cap N_{R_*}(\ell)} Y_j'$ for all $x \in U_\ell'$. Also since $\Delta(R_*[I_{i'}, I_{j'}]) \le 1$ by (V3) and $H'[Y_i', Y_j']$ is empty if $ij \notin E(R_*)$ and a perfect matching if $ij \in E(R_*)$, we have $|\bigcup_{j \in I_{j'}} Y_j \cap N_{i'}(x)| \le 1$ for $i', j' \in [w]$. Thus $|N_{i'}(x)| = |N_{i'}(y)| \le i'$ for all $x, y \in Y_j'$, $i' \in [w]$ and $j \in [Kr]$.

Let g be the function defined in (3.1). So $g(a) = a^{1/300} \ge (4K\Delta_R)^{1/120} a^{1/240} = g'(4K\Delta_R\sqrt{a})$ for $a \ll 1/K, 1/\Delta_R$. For all $t \in [w]$ let

$$\xi_t := g^t(2\varepsilon) \text{ and let } \xi_0 := 2\varepsilon. \tag{5.4}$$

So

$$\xi_{t+1} \ge g'(4K\Delta_R\sqrt{\xi_t}) \quad \text{for any } t \in [w]. \tag{5.5}$$

For all $j \in [Kr]$, let $A_0^j := A_0'[Y_j', U_j']$ and $B_0^j := A_0^j$ and define

$$p(\vec{d}, j, 0) := d_0 \text{ and } p(\vec{\beta}, j, 0) := d_0. \tag{5.6}$$

So (P2) implies that A_0^j is $(\xi_0, p(\vec{d}, j, 0))$-super-regular and B_0^j is $(\xi_0, p(\vec{\beta}, j, 0))$-super-regular for each $j \in [Kr]$. A_0^j and B_0^j encode the initial constraints (or candidates) for the embedding. Move to Round 1.

Round i'. Given an embedding $f_{i'-1}$ of $H'[\bigcup_{j\in I_1^{i'-1}} Y_j']$ into $G'[\bigcup_{j\in I_1^{i'-1}} U_j']$ constructed in Round $i'-1$, we aim to extend it to $f_{i'} : H'[\bigcup_{j\in I_1^{i'}} Y_j'] \to G'[\bigcup_{j\in I_1^{i'}} U_j']$.

Assume that for each $j \in [Kr]$ the bipartite graphs $A_{i'-1}^j$ and $B_{i'-1}^j$ on (Y_j', U_j') constructed in Round $i'-1$ are such that

- $A_{i'-1}^j$ is $(\xi_{i'-1}, p(\vec{d}, j, i'-1))$-super-regular,
- $B_{i'-1}^j$ is $(\xi_{i'-1}, p(\vec{\beta}, j, i'-1))$-super-regular.

Here the candidacy graph $A_{i'-1}^j$ encodes the constraints accumulated until Round $i'-1$ that we have for embedding Y_j' into U_j' in order to extend $f_{i'-1}$ to $f_{i'}$. The candidacy graph $B_{i'-1}^j$ tracks analogous constraints for the patching graph P which we will use in Section 6 to resolve overlaps between embeddings of distinct graphs H. Recall that $|N_{R_*}(j) \cap I_{i'}| \le 1$ for $i' \in [w]$, and $j \in [Kr]$ by (V3). So for any symmetric $Kr \times Kr$ matrix $\vec{z}$ we can let

$$p(\vec{z}, j, i') := \begin{cases} z_{j,\ell} p(\vec{z}, j, i'-1) & \text{if } N_{R_*}(j) \cap I_{i'} = \{\ell\}, \\ p(\vec{z}, j, i'-1) & \text{if } N_{R_*}(j) \cap I_{i'} = \emptyset. \end{cases} \tag{5.7}$$

We will apply this with $\vec{d}$ and $\vec{\beta}$ playing the role of $\vec{z}$. (5.7) will be used to update the densities of the candidacy graphs after each round. For $i \in I_{i'}$, we define a spanning subgraph $A(i)$ of $A_{i'-1}^i$ which contains all those edges $xv \in E(A_{i'-1}^i)$ with $x \in Y_i'$ and $v \in U_i'$ satisfying

$$\begin{aligned} &|N_{A_{i'-1}^j}(x_j) \cap N_{G'}(v)| = (p(\vec{d}, j, i') \pm 2\xi_{i'-1})m \quad \text{and} \\ &|N_{B_{i'-1}^j}(x_j) \cap N_{P'}(v)| = (p(\vec{\beta}, j, i') \pm 2\xi_{i'-1})m \end{aligned} \tag{5.8}$$

for every $j \in N_{R_*}(i)$, where $\{x_j\} := N_{H'}(x) \cap Y_j'$. Excluding the edges not satisfying (5.8) will enable us to apply the four graphs lemma with $A(i)$ playing the role of F_{12}.

We now choose an embedding $\sigma_i(x)$ for all $x \in Y_i'$ with $i \in I_{i'}$. Let $\sigma_i(y_{i,j}) := u_{i,j}$ for all $i \in I_{i'}$ and $j \in [m - |Y_i|]$ and extend σ_i to a bijection from Y_i' to U_i' by choosing a perfect matching of $A(i)[Y_i, U_i]$ uniformly at random (such a perfect matching will be shown to exist in Lemma 5.4(i)). We now extend the current embedding. Let

$$f_{i'}(x) := \begin{cases} f_{i'-1}(x) & \text{if } x \in Y_i' \text{ for } i \in I_1^{i'-1}, \\ \sigma_i(x) & \text{if } x \in Y_i' \text{ for } i \in I_{i'}. \end{cases} \tag{5.9}$$

We now update the candidacy graphs to ensure that they incorporate the new constraints arising from the embeddings in the current round. For each $j \in [Kr]$, we let $A_{i'}^j$ be the bipartite graph on (Y_j', U_j') with the edge set

$$E(A_{i'}^j) := \left\{ xv : x \in Y_j', v \in N_{A_0^j}(x) \text{ and } uv \in E(G') \text{ for each } u \in f_{i'}(N_{i'}(x)) \right\}. \tag{5.10}$$

So for all $x \in Y_j'$,

$$N_{A_{i'}^j}(x) = N_{A_0^j}(x) \cap \bigcap_{y \in N_{i'}(x)} N_{G'}(f_{i'}(y)). \tag{5.11}$$

For each $j \in [Kr]$, we let $B_{i'}^j$ be the bipartite graph on (Y_j', U_j') with the edge set

$$E(B_{i'}^j) := \{ xv : x \in Y_j', v \in N_{B_0^j}(x) \text{ and } uv \in E(P') \text{ for each } u \in f_{i'}(N_{i'}(x)) \}. \tag{5.12}$$

Note that $A_{i'}^j \subseteq A_{i'-1}^j$ and $B_{i'}^j \subseteq B_{i'-1}^j$ for all $j \in [Kr]$. Note also that

$$A_{i'}^j = A_{i'-1}^j \text{ and } B_{i'}^j = B_{i'-1}^j \text{ if } N_{R_*}(j) \cap I_{i'} = \emptyset. \tag{5.13}$$

If for some $j \in [Kr]$, $A_{i'}^j$ is not $(\xi_{i'}, p(\vec{d}, j, i'))$-super-regular or $B_{i'}^j$ is not $(\xi_{i'}, p(\vec{\beta}, j, i'))$-super-regular, then we abort the algorithm with **failure of type 2**. If $i' < w$, then proceed to Round

$i'+1$. If $i'=w$, then we return $\phi(x):=f_w(x)$ for every $x\in V(H)$ and $F_j:=B_w^j[Y_j,U_j]$ for every $j\in[Kr]$, and we are done with success. This completes the description of the Slender graph algorithm.

Note that the description would be slightly simpler if we embed only one class in each round. However, then the number of rounds would depend on $|R_*|$ rather than on K and Δ_R, in which case our errors would accumulate too much.

A straightforward application of the Chernoff-Hoeffding bound in Lemma 3.3 shows that failure of type 1 is very unlikely. We omit the proof.

Proposition 5.3. *Suppose $\mathcal{S}=(G,P,H,R_*,A_0,\mathcal{U},\mathcal{Y},\mathcal{I},c,\varepsilon,d_0,\vec{d},\vec{\beta},K,\Delta_R,C)$ is a valid input. Then in the Slender graph algorithm, failure of type 1 occurs with probability at most $(1-3c)^{Km}$.*

For each $0\le i'\le w$ let $\mathcal{A}_{i'}$ be the event that the following two statements hold.

- (A1$_{i'}$) For each $j\in[Kr]$, $A_{i'}^j$ is $(\xi_{i'},p(\vec{d},j,i'))$-super-regular.
- (A2$_{i'}$) For each $j\in[Kr]$, $B_{i'}^j$ is $(\xi_{i'},p(\vec{\beta},j,i'))$-super-regular.

Note that $\mathcal{A}_0$ always holds for the Slender graph algorithm on $\mathcal{S}$ if $\mathcal{S}$ is a valid input (as observed after (5.6)). We also define $\mathcal{A}_0^{i'}:=\bigwedge_{j'=0}^{i'}\mathcal{A}_{j'}$.

The next lemma states that the super-regularity of the candidacy graphs is preserved throughout the algorithm and thus that properties (M1)–(M3) from Lemma 4.4 carry over into the current setting.

Lemma 5.4. *Suppose $\mathcal{S}=(G,P,H,R_*,A_0,\mathcal{U},\mathcal{Y},\mathcal{I},c,\varepsilon,d_0,\vec{d},\vec{\beta},K,\Delta_R,C)$ is a valid input. Assume that in the Slender graph algorithm on input $\mathcal{S}$, there was no failure prior to Round i'. Then the following holds.*

- (i) *For all $i\in I_{i'}$ we have $\Delta(A_{i'-1}^i-A(i))\le 4K\Delta_R\xi_{i'-1}m$. In particular, this means that $A(i)$ is $(4K\Delta_R\sqrt{\xi_{i'-1}},p(\vec{d},i,i'-1))$-super-regular for all $i\in I_{i'}$.*
- (ii) *Let $\mathcal{B}_{i'-1}$ be an event which depends only on the history of the Slender graph algorithm prior to Round i' such that $\mathbb{P}[\mathcal{A}_0^{i'-1},\mathcal{B}_{i'-1}]>0$. Then*
$$\mathbb{P}[\mathcal{A}_{i'}\mid\mathcal{A}_0^{i'-1},\mathcal{B}_{i'-1}]\ge 1-(1-3c)^{Km}.$$
- (iii) *Let $\mathcal{B}_{i'-1}$ be as in (ii), and let $h'(a)=a^{1/10}$ and $h(a)=a^{1/20}$ be as defined in (3.1). Then the following statements hold.*
  - (M′1)$_{i'}$ *For all $i\in I_{i'}$, $x\in Y_i$ and every $S\subseteq N_{A(i)}(x)\cap U_i$,*
$$\mathbb{P}[\sigma_i(x)\in S\mid\mathcal{A}_0^{i'-1},\mathcal{B}_{i'-1}]=(1\pm h(4K\Delta_R\sqrt{\xi_{i'-1}}))\frac{|S|}{p(\vec{d},i,i'-1)m}.$$
  - (M′2)$_{i'}$ *For all $i\in I_{i'}$, $v\in U_i$ and every $S\subseteq N_{A(i)}(v)\cap Y_i$,*
$$\mathbb{P}[v\in\sigma_i(S)\mid\mathcal{A}_0^{i'-1},\mathcal{B}_{i'-1}]=(1\pm h(4K\Delta_R\sqrt{\xi_{i'-1}}))\frac{|S|}{p(\vec{d},i,i'-1)m}.$$
  - (M′3)$_{i'}$ *For all $i\in I_{i'}$, all d' with $c\ll d'\le d^{K\Delta_R}/9$ and all $A'\subseteq A(i)$ with $V(A')=U_i\cup Y_i$ and $\Delta(A')\le d'm$,*
$$\mathbb{P}[|\{x\in Y_i:x\sigma_i(x)\in E(A')\}|>8d'm/p(\vec{d},i,i'-1)\mid\mathcal{A}_0^{i'-1},\mathcal{B}_{i'-1}]<(1-4Kc)^m.$$
  - (M′4)$_{i'}$ *For all $i\in I_{i'}$, all $S\subseteq Y_i'$ and all $T\subseteq U_i'$ with $|S|,|T|\ge h'(4K\Delta_R\sqrt{\xi_{i'-1}})m$,*
$$\mathbb{P}[|\sigma_i(S)\cap T|=(1\pm h'(4K\Delta_R\sqrt{\xi_{i'-1}}))|S||T|/m\mid\mathcal{A}_0^{i'-1},\mathcal{B}_{i'-1}]\ge 1-(1-4Kc)^m.$$

Proof. First, we show (i). Let us fix $i\in I_{i'}$, $v\in U_i'$ and $j\in N_{R_*}(i)$, and define
$$N_{v,j}^{\le}(G'):=\left\{x\in Y_i':|N_{A_{i'-1}^j}(x_j)\cap N_{G'}(v)|\le(p(\vec{d},j,i')-2\xi_{i'-1})m\right\},$$
where $\{x_j\}=N_{H'}(x)\cap Y_j'$. Since $G'[U_i',U_j']$ is $(2\varepsilon,d_{i,j})$-super-regular by (P2), $|N_{G'}(v)\cap U_j'|\ge(d_{i,j}-2\varepsilon)m$. Together with the fact that $A_{i'-1}^j$ is $(\xi_{i'-1},p(\vec{d},j,i'-1))$-regular (since there was

no failure prior to Round i') and $p(\vec{d},j,i'-1)(d_{i,j}-2\varepsilon)-\xi_{i'-1}>p(\vec{d},j,i')-2\xi_{i'-1}$ by (5.7), it follows that $|N^{\leq}_{v,j}(G')|\leq \xi_{i'-1}m$. Similarly we get

$$|N^{\leq}_{v,j}(P')|\leq \xi_{i'-1}m,\ |N^{\geq}_{v,j}(G')|\leq \xi_{i'-1}m \text{ and } |N^{\geq}_{v,j}(P')|\leq \xi_{i'-1}m,$$

where

$$N^{\leq}_{v,j}(P'):=\left\{x\in Y_i': |N_{B^j_{i'-1}}(x_j)\cap N_{P'}(v)|\leq (p(\vec{\beta},j,i')-2\xi_{i'-1})m\right\},$$
$$N^{\geq}_{v,j}(G'):=\left\{x\in Y_i': |N_{A^j_{i'-1}}(x_j)\cap N_{G'}(v)|\geq (p(\vec{d},j,i')+2\xi_{i'-1})m\right\},$$
$$N^{\geq}_{v,j}(P'):=\left\{x\in Y_i': |N_{B^j_{i'-1}}(x_j)\cap N_{P'}(v)|\geq (p(\vec{\beta},j,i')+2\xi_{i'-1})m\right\}.$$

When creating $A(i)$, we have removed an edge xv from $A^i_{i'-1}$ if and only if $x\in N^{\leq}_{v,j}(G')\cup N^{\geq}_{v,j}(G')\cup N^{\leq}_{v,j}(P')\cup N^{\geq}_{v,j}(P')$ for at least one j with $j\in N_{R_*}(i)$. Since $|N_{R_*}(i)|\leq K\Delta_R$ by (V3) this implies that

$$d_{A^i_{i'-1}}(v)-d_{A(i)}(v)\leq 4K\Delta_R\xi_{i'-1}m. \tag{5.14}$$

In the same way, we can show that for each $x\in Y_i'$,

$$d_{A^i_{i'-1}}(x)-d_{A(i)}(x)\leq 8K\Delta_R\varepsilon m\leq 4K\Delta_R\xi_{i'-1}m.$$

This together with (5.14) this shows that for any $z\in Y_i'\cup U_i'$,

$$d_{A^i_{i'-1}}(z)-d_{A(i)}(z)\leq 4K\Delta_R\xi_{i'-1}m.$$

Thus, by Proposition 3.9 and ($\mathrm{A1}_{i'-1}$) it follows that (i) holds.

Now we show (ii). Assume $i'\in[w]$, $i\in I_{i'}$ and that both $\mathcal{A}_0^{i'-1}$ and $\mathcal{B}_{i'-1}$ hold. Note from (V3) that $R_*[I_{i'}]$ is an empty graph and $\Delta(R_*[I_{i'},I_{j'}])\leq 1$ for $i'\neq j'$. (ii) will follow from several applications of the four graphs lemma (Lemma 5.2) with $A(i)$ playing the role of F_{12}. To prepare for this, we introduce the following graphs.

- For all $j\ell\in E(R_*)$ let $\psi_{j,\ell}:Y_j'\to Y_\ell'$ be a bijection such that $\{\psi_{j,\ell}(x)\}=N_{H'}(x)\cap Y_\ell'$ for all $x\in Y_j'$.
- For all $j\in N_{R_*}(i)$, let $F^j_{13}(i):=(Y_i',U_j';E)$, where $xv\in E$ if and only if $x\in Y_i'$, $v\in U_j'$ and $\psi_{i,j}(x)v\in E(A^j_{i'-1})$.
- For all $j\in N_{R_*}(i)$, let $F'^j_{13}(i):=(Y_i',U_j';E')$, where $xv\in E'$ if and only if $x\in Y_i'$, $v\in U_j'$ and $\psi_{i,j}(x)v\in E(B^j_{i'-1})$.
- For all $j\in N_{R_*}(i)$, let $F^j_{23}(i):=G'[U_i',U_j']$ and $F'^j_{23}(i):=P'[U_i',U_j']$.

Observe that, since $H'[Y_i',Y_j']$ is a perfect matching for each $j\in N_{R_*}(i)$, the graph $F^j_{13}(i)$ is isomorphic to $A^j_{i'-1}$ under the isomorphism $\psi'_{i,j}$ which keeps the elements of U_j' fixed while mapping x to $\psi_{i,j}(x)$ for all $x\in Y_i'$. Since we are conditioning on $\mathcal{A}_0^{i'-1}$, properties ($\mathrm{A1}_{i'-1}$) and ($\mathrm{A2}_{i'-1}$) hold and so for all $j\in N_{R_*}(i)$ we have

(a) $F^j_{13}(i)$ is $(\xi_{i'-1},p(\vec{d},j,i'-1))$-super-regular,
(a′) $F'^j_{13}(i)$ is $(\xi_{i'-1},p(\vec{\beta},j,i'-1))$-super-regular,
(b) $F^j_{23}(i)$ is $(2\varepsilon,d_{i,j})$-super-regular,
(b′) $F'^j_{23}(i)$ is $(2\varepsilon,\beta_{i,j})$-super-regular,

where (b) and (b′) follow by (P2). For $x\in Y_i'$ and $v\in U_i'$, note that

$$N_{F^j_{13}(i)}(x)\cap N_{F^j_{23}(i)}(v)=N_{A^j_{i'-1}}(\psi_{i,j}(x))\cap N_{G'}(v), \tag{5.15}$$
$$N_{F'^j_{13}(i)}(x)\cap N_{F'^j_{23}(i)}(v)=N_{B^j_{i'-1}}(\psi_{i,j}(x))\cap N_{P'}(v). \tag{5.16}$$

Moreover, by (5.7) for all $j\in N_{R_*}(i)$ we have

$$p(\vec{d},j,i')=d_{i,j}p(\vec{d},j,i'-1) \quad\text{and}\quad p(\vec{\beta},j,i')=\beta_{i,j}p(\vec{\beta},j,i'-1). \tag{5.17}$$

Together with (5.8) this means that we obtain $A(i)$ from $A^i_{i'-1}$ by deleting every edge xv which does not satisfy one of

$$|N_{F^j_{13}(i)}(x) \cap N_{F^j_{23}(i)}(v)| = (d_{i,j}p(\vec{d}, j, i'-1) \pm 2\xi_{i'-1})m,$$
$$|N_{F'^j_{13}(i)}(x) \cap N_{F'^j_{23}(i)}(v)| = (\beta_{i,j}p(\vec{\beta}, j, i'-1) \pm 2\xi_{i'-1})m \tag{5.18}$$

for some index $j \in N_{R_*}(i)$. This will be used to verify condition (R2) for Lemma 5.2. To apply Lemma 5.2 we also need to check (5.3) for the artificial vertices introduced in the preparation round. For this, note that two applications of (P3) imply the following property.

(P3′) For all $ij \in E(R_*)$, all $Q_1 = \{q_1, \dots, q_s\}$ with $|Q_1| \le K\Delta_R - 1$ where $q_\ell \in U'_{j_\ell} \setminus U_{j_\ell}$ and $j_\ell \in N_{R_*}(j)$, all $Q_2 \subseteq V(G)$ with $|Q_2| \le K\Delta_R$, all $v' \in U'_i \setminus (U_i \cup Q_1)$ and all $y \in Y'_j$, we have
- (i) $|N_{G'}(v') \cap N_{G'}(Q_1) \cap N_{G'}(Q_2) \cap N_{A'_0}(y)| = d_{i,j}|N_{G'}(Q_1) \cap N_{G'}(Q_2) \cap N_{A'_0}(y)| \pm 4\varepsilon m$,
- (ii) $|N_{P'}(v') \cap N_{P'}(Q_1) \cap N_{P'}(Q_2) \cap N_{A'_0}(y)| = \beta_{i,j}|N_{P'}(Q_1) \cap N_{P'}(Q_2) \cap N_{A'_0}(y)| \pm 4\varepsilon m$.

For all vertices $x' \in Y'_i \setminus Y_i, v' \in U'_i \setminus U_i$ with $i \in I_{i'}$ and $ij \in E(R_*)$, we apply (P3′)(i) with $Q_1 := f_{i'}(N_{i'-1}(\psi_{i,j}(x')) \setminus V(H))$, $Q_2 := f_{i'}(N_{i'-1}(\psi_{i,j}(x')) \cap V(H))$, v' and $\psi_{i,j}(x')$ to obtain

$$\begin{aligned}
|N_{F^j_{13}(i)}(x') \cap N_{F^j_{23}(i)}(v')| &\overset{(5.15)}{=} |N_{A^j_{i'-1}}(\psi_{i,j}(x')) \cap N_{G'}(v')| \\
&\overset{(5.11)}{=} |N_{A^j_0}(\psi_{i,j}(x')) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2) \cap N_{G'}(v')| \\
&\overset{(\mathrm{P3}')(i)}{=} d_{i,j}|N_{A^j_0}(\psi_{i,j}(x')) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2)| \pm 4\varepsilon m \\
&\overset{(5.11)}{=} d_{i,j}|N_{A^j_{i'-1}}(\psi_{i,j}(x'))| \pm 4\varepsilon m \\
&\overset{(\mathrm{A1})_{i'-1}}{=} d_{i,j}(p(\vec{d}, j, i'-1) \pm \xi_{i'-1})m \pm 4\varepsilon m \\
&= (d_{i,j}p(\vec{d}, j, i'-1) \pm 2\xi_{i'-1})m.
\end{aligned} \tag{5.19}$$

Similarly, we use (P3′)(ii) to obtain

$$|N_{F'^j_{13}(i)}(x') \cap N_{F'^j_{23}(i)}(v')| = |N_{B^j_{i'-1}}(\psi_{i,j}(x')) \cap N_{P'}(v')| = (\beta_{i,j}p(\vec{\beta}, j, i'-1) \pm 2\xi_{i'-1})m.$$

Note also that for all $j \in N_{R_*}(i)$, $x \in Y'_j$ and $v \in U'_j$,

$$\begin{aligned}
xv \in E(A^j_{i'}) &\overset{(5.10)}{\Leftrightarrow} xv \in E(A^j_0) \text{ and } uv \in E(G') \text{ for each } u \in f_{i'}(N_{i'}(x)) \\
&\Leftrightarrow xv \in E(A^j_{i'-1}) \text{ and } f_{i'}(\psi^{-1}_{i,j}(x)) \in N_{G'}(v) \\
&\Leftrightarrow \psi^{-1}_{i,j}(x)v \in E(F^j_{13}(i)) \text{ and } f_{i'}(\psi^{-1}_{i,j}(x))v \in E(F^j_{23}(i)).
\end{aligned} \tag{5.20}$$

We now wish to apply Lemma 5.2 for each $i \in I_{i'}$ and $j \in N_{R_*}(i)$ with the following graphs and parameters.

object/parameter	$A(i)$	$F^j_{13}(i)$	$F^j_{23}(i)$	$A^j_{i'}$	$5Kc$	m	$4K\Delta_R\sqrt{\xi_{i'-1}}$
playing the role of	F_{12}	F_{13}	F_{23}	A_{σ_i}	c	n	ε

Note that Lemma 5.4(i), (a) and (b) imply that the graphs satisfy the regularity condition (R1) stated before Lemma 5.2. Note also that (5.19) implies that (5.3) is satisfied. Recall from the description of the Slender graph algorithm that $\sigma_i : Y'_i \to U'_i$ is the bijection obtained in Round i' for $A(i)$ and $f_{i'}(\psi^{-1}_{i,j}(x)) = \sigma_i(\psi^{-1}_{i,j}(x))$ for all $x \in Y'_j$ with $j \in N_{R_*}(i)$ by (5.9). So (5.20) implies that the graph A_{σ_i} defined before Lemma 5.2 is indeed isomorphic to $A^j_{i'}$ via the isomorphism $\psi'_{i,j}$ defined by $\psi'_{i,j}(x) = \psi_{i,j}(x)$ for $x \in Y'_i$ and $\psi'_{i,j}(v) = v$ for $v \in U'_j$. Also (5.18) means that condition (R2) before Lemma 5.2 is satisfied. So altogether this means we can indeed apply Lemma 5.2. Recall that $\mathcal{A}_1^{i'-1}, \mathcal{B}_{i'-1}$ only depend on the history prior to Round i' and that

$\xi_{i'} \geq g'(4K\Delta_R\sqrt{\xi_{i'-1}})$ by (5.5). So Lemma 5.2 together with (5.17) and a union bound over all $j \in N_{R_*}(i)$ and $i \in I_{i'}$ implies that

$$\mathbb{P}[(\mathrm{A1}_{i'}) \mid \mathcal{A}_0^{i'-1}, \mathcal{B}_{i'-1}] \geq 1 - K^2 r\Delta_R(1-5Kc)^m \geq 1-(1-4c)^{Km}. \tag{5.21}$$

Similarly as in (5.20) it follows that

$$xv \in E(B_{i'}^j) \Leftrightarrow \psi_{i,j}^{-1}(x)v \in E(F_{13}'^j(i)) \text{ and } f_{i'}(\psi_{i,j}^{-1}(x))v \in E(F_{23}'^j(i)). \tag{5.22}$$

So similarly as before, by Lemma 5.2 applied to $A(i), F_{13}'^j(i), F_{23}'^j(i)$ for each $j \in N_{R_*}(i)$ with $i \in I_{i'}$, we obtain

$$\mathbb{P}[(\mathrm{A2}_{i'}) \mid \mathcal{A}_0^{i'-1}, \mathcal{B}_{i'-1}] \geq 1-(1-4c)^{Km}. \tag{5.23}$$

Recall from (5.7) and (5.13) that for all $j \notin \bigcup_{i\in I_{i'}} N_{R_*}(i)$ we have $A_{i'}^j = A_{i'-1}^j$, $B_{i'}^j = B_{i'-1}^j$, $p(\vec{d},j,i') = p(\vec{d},j,i'-1)$ and $p(\vec{\beta},j,i') = p(\vec{\beta},j,i'-1)$. So these $A_{i'}^j$ are also $(\xi_{i'}, p(\vec{d},j,i'))$-super-regular and these $B_{i'}^j$ are also $(\xi_{i'}, p(\vec{\beta},j,i'))$-super-regular. Thus, by (5.21) and (5.23), we obtain

$$\mathbb{P}[\mathcal{A}_{i'} \mid \mathcal{A}_0^{i'-1}, \mathcal{B}_{i'-1}] = \mathbb{P}[(\mathrm{A1}_{i'}), (\mathrm{A2}_{i'}) \mid \mathcal{A}_0^{i'-1}, \mathcal{B}_{i'-1}] \geq 1-2(1-4c)^{Km} \geq 1-(1-3c)^{Km}.$$

Property (iii) follows immediately from (i) as well as Lemma 4.4 and Theorem 4.1 applied to $A(i)[Y_i, U_i]$. Indeed, to check $(\mathrm{M}'3)_{i'}$, note that $p(\vec{d},i,i'-1) \geq d^{\Delta(R_*)} \geq d^{K\Delta_R} \geq 9d'$ by (5.7), (V1) and (V3). (Actually, we get an additional error arising from the existence of the artificial vertices (i.e. from considering $A(i)[Y_i,U_i]$ instead of $A(i)$) as well as the fact that we only have $m-C \leq |Y_i| = |U_i| \leq m$. However, this error is insignificant. Alternatively, one could have used that Proposition 3.9 actually implies that $A(i)$ is $(3\sqrt{K\Delta_R\xi_{i'-1}}, p(\vec{d},i,i'-1))$-super-regular.) □

We can now deduce from Proposition 5.3 and Lemma 5.4 that the Slender graph algorithm succeeds with high probability, and that the candidacy graphs F_j for the patching process have strong regularity properties.

Lemma 5.5. *Suppose $\mathcal{S} = (G,P,H,R_*,A_0,\mathcal{U},\mathcal{Y},\mathcal{I},c,\varepsilon,d_0,\vec{d},\vec{\beta},K,\Delta_R,C)$ is a valid input. Let q_* be the function defined in* (3.1)*. Then the Slender graph algorithm applied to $\mathcal{S}$ succeeds with probability at least $1-(1-2c)^{Km}$. Moreover, if it succeeds, then it returns an embedding ϕ of H into G and bipartite graphs F_j on (Y_j,U_j) (for each $j \in [Kr]$) such that $\phi(Y_i) = U_i$ for all $i \in [r]$, $\phi(x) \in N_{A_0}(x)$ for all $x \in V(H)$ and such that ϕ and the graphs F_j satisfy the following condition.*

(i) *Each F_j is a $(q_*(\varepsilon), d_0p(R_*,\vec{\beta},j))$-super-regular bipartite graph on (Y_j,U_j) such that $N_{F_j}(x) \subseteq U_j \cap N_{A_0}(x) \cap \bigcap_{u\in\phi(N_H(x))} N_P(u)$ for all $x \in Y_j$.*

Proof. To show that the Slender graph algorithm succeeds with probability at least $1-(1-2c)^{Km}$, recall from Proposition 5.3 that the failure of type 1 occurs with probability at most $(1-3c)^{Km}$. Once failure of type 1 does not occur, the event $\mathcal{A}_0^w$ is equivalent to being successful. By applying Lemma 5.4(ii) with $\mathcal{B}_{i'-1} = \mathcal{A}_0^{i'-1}$, we obtain

$$\mathbb{P}[\mathcal{A}_0^w] = \prod_{i'=1}^{w} \mathbb{P}[\mathcal{A}_{i'} \mid \mathcal{A}_0^{i'-1}] \geq \big(1-(1-3c)^{Km}\big)^w \geq 1-w(1-3c)^{Km}.$$

Thus the algorithm succeeds with probability at least $1-w(1-3c)^{Km}-(1-3c)^{Km} \geq 1-(1-2c)^{Km}$. Also $\phi(Y_i) = U_i$ and $\phi(x) \in N_{A_0^j}(x) \cap U_j \subseteq N_{A_0}(x)$ for $x \in Y_j$ are trivial from the description of the algorithm.

Now let us assume that the algorithm succeeds, i.e. $\mathcal{A}_0^w$ occurs. Note that $\xi_w = g^w(2\varepsilon)$ by (5.4). In particular, this means that for all $j \in [Kr]$ we obtain graphs B_w^j on (Y_j', U_j'), each of which is $(q_*(\varepsilon)/2, p(\vec{\beta},j,w))$-super-regular. Thus $F_j = B_w^j[Y_j,U_j]$ is $(q_*(\varepsilon), p(\vec{\beta},j,w))$-super-regular (as $|Y_j' \setminus Y_j| + |U_j' \setminus U_j| \leq 2C$). But $p(\vec{\beta},j,w) = d_0p(R_*,\vec{\beta},j)$ by (5.2), (5.6) and (5.7), so

F_j is $(q_*(\varepsilon), d_0 p(R_*, \vec{\beta}, j))$-super-regular. Also, by the definition of B^j_w, for any $x \in Y_j$

$$\begin{aligned} N_{F_j}(x) &\overset{(5.12)}{=} U_j \cap N_{B^j_0}(x) \cap \bigcap_{u \in f_w(N_{H'}(x))} N_{P'}(u) \subseteq U_j \cap N_{A_0}(x) \cap \bigcap_{u \in f_w(N_H(x))} N_P(u) \\ &= U_j \cap N_{A_0}(x) \cap \bigcap_{u \in \phi(N_H(x))} N_P(u) \end{aligned}$$

as $U_j \cap N_{P'}(u) = U_j \cap N_P(u)$ for all $u \in V(G) = V(P)$, and ϕ is the restriction of f_w to $V(H)$. Thus (i) holds. □

We will find the required embedding ϕ in Lemma 5.1 via the following random algorithm, which first preprocesses the input to satisfy the requirements of the Slender graph algorithm and then runs the Slender graph algorithm. So suppose that $(G, P, H, R, A_0, \mathcal{V}, \mathcal{X}, c, \varepsilon, d_0, \vec{d}, \vec{\beta}, k, \Delta_R, C)$ satisfies the assumption in Lemma 5.1. Recall that H^2 denotes the square of the graph H. Recall R_K denotes the K-fold blow-up of R. In Steps 1–3 below we will refine the partitions given by Lemma 5.1 in a suitable way to ensure that we have a valid input for the Slender graph algorithm satisfying (V1)–(V7). So in Step 4 we can then apply the Slender graph algorithm to obtain the desired embedding.

The Uniform embedding algorithm on $(G, P, H, R, A_0, \mathcal{V}, \mathcal{X}, c, \varepsilon, d_0, \vec{d}, \vec{\beta}, k, \Delta_R, C)$**.**

Step 1. Recall that we assume that H is $(R, \vec{k}, C)$ near-equiregular with respect to $\mathcal{X}$ (which was defined in Section 3.1). We apply the Hajnal-Szemerédi theorem (Theorem 3.5) to $H^2[X_i]$ for each $i \in [r]$ to find an equitable partition of X_i into $K := (k+1)^2\Delta_R$ sets which are independent in H^2. (Note that this is possible since $\Delta(H^2[X_i]) \leq k(k+1)\Delta_R$.) Let $\mathcal{Y} := \{Y_1, \dots, Y_{Kr}\}$ be the resulting partition of $V(H)$ such that for all $i \in [r]$

$$X_i = \bigcup_{j \in J_i} Y_j,$$

where $J_i := \{(i-1)K + 1, \dots, iK\}$ and $m - C \leq |Y_j| \leq m$ with $m := \lceil n/K \rceil$.

Since $\Delta(R) \leq \Delta_R$, we can find a vertex partition of R into independent sets $W^*_1, \dots, W^*_{\Delta_R+1}$. Let $w := (K\Delta_R)^2(\Delta_R + 1)$. We now view the K-fold blow-up R_K of R as being obtained from R by replacing each vertex i of R with the set of vertices J_i. Note that $\Delta(R_K) \leq K\Delta_R$. Let $W^{**}_1, \dots, W^{**}_{\Delta_R+1}$ be the vertex partition of $V(R_K) = [Kr]$ such that $i \in W^*_{j'}$ if and only if $J_i \subseteq W^{**}_{j'}$. For each $i' \in [\Delta_R + 1]$, we choose a vertex partition of $(R_K)^2[W^{**}_{i'}]$ into $(K\Delta_R)^2$ sets which are independent in $(R_K)^2$. Denote the classes of the resulting vertex partition of $W^{**}_{i'}$ by $I_{j'}$ where $(i'-1)(K\Delta_R)^2 + 1 \leq j' \leq i'(K\Delta_R)^2$. Note that for all $i', j' \in [w]$ we have

$$\Delta(R_K[I_{i'}, I_{j'}]) \leq 1. \tag{5.24}$$

For all $i \in [r]$, let $i'_{\max}$ be the largest index i' such that $J_i \cap I_{i'} \neq \emptyset$, and let $i'_{\min}$ be the smallest such index. Then for any $i, j \in [r]$ with $ij \in E(R)$, one of the following holds:

$$i'_{\max} < j'_{\min}, \quad \text{or} \quad j'_{\max} < i'_{\min}. \tag{5.25}$$

We let $\mathcal{I} := \{I_1, \dots, I_w\}$. Let $\vec{d}', \vec{\beta}'$ be $Kr \times Kr$ matrices such that $d'_{\ell,\ell'} := d_{i,j}$ and $\beta'_{\ell,\ell'} := \beta_{i,j}$ whenever $\ell \in J_i, \ell' \in J_j$. So we have now satisfied (V1)–(V3) in the definition of a valid input (with R_K playing the role of R_* and $d'_{\ell,\ell'}, \beta'_{\ell,\ell'}$ playing the roles of $d_{i,j}$ and $\beta_{i,j}$ respectively).

Step 2. Note that for all $1 \leq i \neq j \leq Kr$, the graph $H[Y_i, Y_j]$ is a matching if $ij \in E(R_K)$ and empty if $ij \notin E(R_K)$. For all $ij \in E(R_K)$ we add edges to $H[Y_i, Y_j]$ to obtain a graph H_* on $V(H)$ so that each $H_*[Y_i, Y_j]$ is a matching of size $\min\{|Y_i|, |Y_j|\}$. So H_* satisfies (V7) (with H_* playing the role of H).

Step 3. For each $i \in [r]$ we now choose an equitable partition $U_{(i-1)K+1}, \dots, U_{iK}$ of V_i. In the case when $A_0[X_i, V_i]$ is complete bipartite, we could simply consider an equitable partition

chosen uniformly at random. In general, there are a few vertices we need to be more careful about. More precisely, we consider

$$W_i := \{v \in V_i : \exists j \in J_i \text{ such that } |N_{A_0}(v) \cap Y_j| \neq (d_0 \pm 2\varepsilon)m\}.$$

Since $A_0[X_i, V_i]$ is (d_0, ε)-super-regular,

$$|W_i| \leq K\varepsilon n \leq K^2\varepsilon m. \tag{5.26}$$

Note that for each $v \in W_i$, there exists some $j \in J_i$ such that $|N_{A_0}(v) \cap Y_j| > (d_0 - 2\varepsilon)m$. Let $(W'_{(i-1)K+1}, \dots, W'_{iK})$ be a partition of W_i such that $v \in W'_j$ implies that $|N_{A_0}(v) \cap Y_j| > (d_0 - 2\varepsilon)m$. Now we partition $V_i \setminus W_i$ into $W''_{(i-1)K+1}, \dots, W''_{iK}$ uniformly at random subject to the condition $|W''_j| = |Y_j| - |W'_j|$. Let $U_j := W'_j \cup W''_j$. For each $j \in J_i$ and each $v \in W'_j$ we delete some arbitrary edges in A_0 between v and Y_i to obtain a spanning subgraph $A_0^\diamond$ of A_0 such that $|N_{A_0^\diamond}(u) \cap Y_j| = (d_0 \pm 2\varepsilon)m$ for all $u \in U_j$ and $j \in J_i$. Let $A_0^* := \bigcup_{j\in[Kr]} A_0^\diamond[Y_j, U_j]$. So $A_0^* \subseteq A_0$.

Let $\mathcal{U} := \{U_1, \dots, U_{Kr}\}$. Note that G and P admit vertex partition $(R_K, \mathcal{U})$ and H_* admits vertex partition $(R_K, \mathcal{Y})$. We define the following events.

- $(\mathcal{G})$ $G[U_i, U_j]$ is $(\varepsilon^{1/3}, d'_{i,j})$-super-regular for all $ij \in E(R_K)$.
- $(\mathcal{P})$ $P[U_i, U_j]$ is $(\varepsilon^{1/3}, \beta'_{i,j})$-super-regular for all $ij \in E(R_K)$.
- $(\mathcal{A}_0^*)$ $A_0^*[Y_j, U_j]$ is $(\varepsilon^{1/3}, d_0)$-super-regular for all $j \in [Kr]$.

If one of these events does not hold, we end the algorithm with **failure of type 1**.

Step 4. We apply the Slender graph algorithm on

$$\mathcal{S} = (G, P, H_*, R_K, A_0^*, \mathcal{U}, \mathcal{Y}, \mathcal{I}, c, \varepsilon^{1/3}, d_0, \vec{d}', \vec{\beta}', K, \Delta_R, C).$$

Note that if we have no failure of type 1, then $\mathcal{S}$ satisfies (V1)–(V7). Recall that in the preparation round of the Slender graph algorithm we define G', H'_*, P' and $(A_0^*)'$ by adding artificial vertices and we have $Y'_j \supseteq Y_i$ and $U'_j \supseteq U_j$ such that $|Y'_j| = |U'_j| = m$ and

- $(\mathcal{G}')$ G' is $(2\varepsilon^{1/3}, \vec{d}')$-super-regular with respect to $(R_K, U'_1, \dots, U'_{Kr})$,
- $(\mathcal{P}')$ P' is $(2\varepsilon^{1/3}, \vec{\beta}')$-super-regular with respect to $(R_K, U'_1, \dots, U'_{Kr})$,
- $(\mathcal{A}'_0)$ $(A_0^*)'[Y'_j, U'_j]$ is $(2\varepsilon^{1/3}, d_0)$-super-regular for each $j \in [Kr]$

(provided that the Slender graph algorithm does not abort with failure of type 1). If the Slender graph algorithm fails, we abort the Uniform embedding algorithm with **failure of type 2**. Otherwise, we obtain an embedding ϕ and bipartite graphs F_j on (U_j, Y_j) for each $j \in [Kr]$. For each $x \in V(H)$, let $N_x := N_{H_*}(x)$. Return $(\phi, \mathcal{Y}, \mathcal{U}, F, N)$, where $F := \bigcup_{j=1}^{Kr} F_j$ and $N := \{N_x : x \in V(H)\}$.

Claim 5.6. *If $\mathcal{S} = (G, P, H, R, A_0, \mathcal{V}, \mathcal{X}, c, \varepsilon, d_0, \vec{d}, \vec{\beta}, k, \Delta_R, C)$ satisfies the assumption in Lemma 5.1, then the Uniform embedding algorithm on $\mathcal{S}$ fails with probability at most $(1-c)^n$.*

Proof. We first consider the event $(\mathcal{A}_0^*)$. Consider any $i \in [r]$. For all $j \in J_i$ and each $u \in U_j$ we have $|N_{A_0^*}(u) \cap Y_j| = (d_0 \pm 2\varepsilon)m = (d_0 \pm \varepsilon^{1/3})m$ by construction. Also for $x \in Y_j$,

$$|N_{A_0^*}(x) \cap U_j| = |N_{A_0^*}(x) \cap W''_j| + |N_{A_0^*}(x) \cap W'_j| \overset{(5.26)}{=} |N_{A_0^*}(x) \cap W''_j| \pm K^2\varepsilon m. \tag{5.27}$$

Using the Chernoff-Hoeffding bound in Lemma 3.3, it is easy to check that with probability at least $1 - (1-3c)^n$ we have

$$\begin{aligned} |N_{A_0^*}(x) \cap W''_j| &= (1 \pm \varepsilon)\frac{|W''_j|}{|V_i \setminus W_i|}|N_{A_0}(x) \cap (V_i \setminus W_i)| \overset{(5.26)}{=} (1 \pm \varepsilon)\frac{m \pm K\varepsilon n}{n \pm K\varepsilon n}(d_0 \pm 2K\varepsilon)n \\ &= (d_0 \pm \varepsilon^{1/3}/2)m. \end{aligned}$$

Together with (5.27) this implies that

$$\mathbb{P}[|N_{A_0^*}(x) \cap U_j| = (d_0 \pm \varepsilon^{1/3})m] \geq 1 - (1-3c)^n.$$

Moreover, it is easy to check that $A_0^*[Y_j, U_j]$ is $(\varepsilon^{1/3}, d_0)$-regular. Together with a union bound over all $x \in V(H)$ this shows that $A_0^*[Y_j, U_j]$ is $(\varepsilon^{1/3}, d_0)$-super-regular for all $j \in [Kr]$ with probability at least $1 - rn(1-3c)^n$.

Second, we consider the events $(\mathcal{G})$ and $(\mathcal{P})$. For this consider $G[U_j, U_{j'}]$ for $j \in J_i$ and $j' \in J_{i'}$ with $ii' \in E(R)$. We have that for all $u \in U_j$

$$|N_G(u) \cap U_{j'}| = |N_G(u) \cap W''_{j'}| + |N_G(u) \cap W'_{j'}| \overset{(5.26)}{=} |N_G(u) \cap W''_{j'}| \pm K^2 \varepsilon m.$$

Again using Lemma 3.3, with probability at least $1 - (1-3c)^n$ we have

$$|N_G(u) \cap W''_{j'}| = (1 \pm \varepsilon) \frac{|W''_{j'}|}{|V_{i'} \setminus W_{i'}|} |N_G(u) \cap (V_{i'} \setminus W_{i'})| = (d'_{j,j'} \pm \varepsilon^{1/3}/2)m. \tag{5.28}$$

(Here we use that $d'_{j,j'} = d_{i,i'}$.) Hence, if (5.28) holds, then $|N_G(u) \cap U_{j'}| = (d'_{j,j'} \pm \varepsilon^{1/3}/2)m \pm K^2\varepsilon m = (d'_{j,j'} \pm \varepsilon^{1/3})m$. Also, $G[U_j, U_{j'}]$ is $(\varepsilon^{1/3}, d'_{j,j'})$-regular by Proposition 3.8. Together with a union bound over all $2n$ vertices of $G[U_j, U_{j'}]$ and all $|E(R_K)| \leq K^2 \Delta_R r$ edges of R_K, this shows that $(\mathcal{G})$ holds with probability at least $1 - 2nK^2\Delta_R r(1-3c)^n$. Similarly, $(\mathcal{P})$ holds with probability at least $1 - 2nK^2\Delta_R r(1-3c)^n$. Thus failure of type 1 in the uniform embedding algorithm occurs with probability at most $(1-2c)^n$.

If failure of type 1 does not occur, then $\mathcal{S}$ is a valid input for the Slender graph algorithm. Thus Lemma 5.5 implies that the Slender graph algorithm on input $\mathcal{S}$ fails with probability at most $(1-2c)^{Km}$. Therefore, the uniform embedding algorithm succeeds with probability at least $1 - (1-2c)^n - (1-2c)^{Km} \geq 1 - (1-c)^n$. □

We now proceed to the main part of the proof of Lemma 5.1, which is establishing (B1).

Claim 5.7. *(B1) holds.*

Proof. Suppose that $ij \in E(R)$, $v \in V_i$ and $S \subseteq V_j \cap N_G(v)$ satisfies $|S| > f(\varepsilon)n$. Recall from Step 1 of the Uniform embedding algorithm that $J_a = \{(a-1)K + 1, \dots, aK\}$ for each $a \in [r]$. Recall that in Step 3 of this algorithm, we take a partition of $V_a \setminus W_a$ (chosen uniformly at random) into parts W''_ℓ of size $|Y_\ell| - |W'_\ell|$ with $\ell \in J_a$ and that $|W'_\ell| \leq |W_a| \leq K\varepsilon n$. Then $U_\ell = W''_\ell \cup W'_\ell$. For every $\ell \in J_j$ let

$$S_\ell := S \cap U_\ell.$$

So S is partitioned into $S_{(j-1)K+1}, \dots, S_{jK}$. We define the following event.

(BB1) $|S_\ell| = (1 \pm \varepsilon^{1/3}) \frac{|S|}{K}$ for all $\ell \in J_j$.

Since $|S_\ell \cap W''_\ell|$ has the hypergeometric distribution, and $|S_\ell \cap W'_\ell| \leq K\varepsilon n$, Lemma 3.3 and the fact that $c \ll \varepsilon$ imply that

$$\mathbb{P}[\text{(BB1) holds }] \geq 1 - (1-2c)^n. \tag{5.29}$$

Recall that in Step 4 of the Uniform embedding algorithm we apply the Slender graph algorithm on $(G, P, H_*, R_K, A_0^*, \mathcal{U}, \mathcal{Y}, \mathcal{I}, c, \varepsilon^{1/3}, d_0, \vec{\beta'}, \vec{d'}, K, \Delta_R, C)$. In the Preparation round of the Slender graph algorithm we obtain equipartite graphs G', H'_*, P' as well as the graph $(A_0^*)'$ by adding artificial vertices. Let $U'_1, \dots, U'_{Kr}$ denote the partition classes of G' such that each $U'_{i'}$ is obtained from $U_{i'}$ by adding at most C artificial vertices $u_{i',j'}$. Analogously, let $Y'_1, \dots, Y'_{Kr}$ denote the partition classes of H'_* such that each $Y'_{i'}$ is obtained from $Y_{i'}$ by adding at most C artificial vertices $y_{i',j'}$. We define the following event. It implies that the neighbourhoods of the artificial vertices are 'well behaved' with respect to S. Recall that A_0^ℓ is defined in terms of $(A_0^*)'$ before (5.6).

(BB2) For all $\ell \in J_j$, all $Q_1 = \{q_1, \dots, q_s\} \subseteq V(G') \setminus V(G)$ with $|Q_1| \leq K\Delta_R$, $q_{\ell'} \in U'_{j_{\ell'}}$ and $j_{\ell'} \in N_{R_K}(\ell)$, for all $x \in Y'_\ell$ and all $Q_2 \subseteq V(G) \setminus U_\ell$ with $|Q_2| \leq K\Delta_R$ we have

$$|S_\ell \cap N_{A_0^\ell}(x) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2)| = \Big(\prod_{\ell'=1}^{s} d'_{\ell,j_{\ell'}}\Big) |S_\ell \cap N_{A_0^\ell}(x) \cap N_{G'}(Q_2)| \pm 2\varepsilon n.$$

So the set Q_1 consists only of artificial vertices outside U'_ℓ. Recall from the Preparation round of the Slender graph algorithm that $|N_{G'}(u_{i',j'}) \cap Q|$ has the binomial distribution for each artificial vertex $u_{i',j'} \in U'_{i'} \setminus U_{i'}$ and each $Q \subseteq U_{i''}$ with $i'i'' \in E(R_K)$. So Lemma 3.3 and the fact that $c \ll \varepsilon$ imply that

$$\mathbb{P}[\text{(BB2) holds }] \geq 1 - (1-2c)^n. \tag{5.30}$$

We define $\mathcal{B}_S := \text{(BB1)} \wedge \text{(BB2)}$, then (5.29) and (5.30) imply

$$\mathbb{P}[\mathcal{B}_S] \geq 1 - 2(1-2c)^n. \tag{5.31}$$

Note that it is easy to see that two applications of (BB2) imply the following.

(BB2′) For all $\ell \in J_j$, any $Q_1 \subseteq \bigcup_{j' \in N_{R_K}(\ell)} (U'_{j'} \setminus U_{j'})$ with $|Q_1| \leq K\Delta_R - 1$, any $Q_2 \subseteq V(G) \setminus U_\ell$ with $|Q_2| \leq K\Delta_R$, any $x \in Y'_\ell$, and any $z \in U'_{\ell'} \setminus (V(G) \cup Q_1)$ with $\ell' \in N_{R_K}(\ell)$,

$$|N_{G'}(z) \cap S_\ell \cap N_{A_0^\ell}(x) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2)| = d'_{\ell,\ell'} |S_\ell \cap N_{A_0^\ell}(x) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2)| \pm 4\varepsilon n.$$

(BB2′) will be used to check that after each round the artificial vertices have the 'right' number of neighbours in S_ℓ in the current candidacy graph. Recall that in Step 1 of the Uniform embedding algorithm we have chosen a partition $\mathcal{I} = (I_1, \dots, I_w)$ of $V(R_K)$ such that each I_b is an independent set in $(R_K)^2$. Together with Observation 3.1 this means that there are indices $a, a', b'_1 < \dots < b'_K$ and numbers $b_1, \dots, b_K$ satisfying

$$v \in U'_a, \;\; a \in I_{a'}, \;\; a \in J_i, \;\; J_j \subseteq \bigcup_{s=1}^{K} I_{b'_s} \quad \text{and} \quad J_j \cap I_{b'_s} = \{b_s\}, \tag{5.32}$$

where $v \in V_i$ is as defined in (B1). So $J_j = \{b_1, \dots, b_K\}$.

Note that by (5.4) for all $t \in [w]$ we have

$$\xi_t = g^t(2\varepsilon^{1/3}) \;\text{ and }\; \xi_0 = 2\varepsilon^{1/3}. \tag{5.33}$$

(Indeed, recall that we apply the Slender graph algorithm with $\varepsilon^{1/3}$ playing the role of ε.) Moreover,

$$m = |U'_\ell| = |Y'_\ell| = \left\lceil \frac{n}{K} \right\rceil. \tag{5.34}$$

So m is playing the role of m in the Slender graph algorithm, which is the size of a largest partition class in $\mathcal{U}$ and $\mathcal{Y}$. Let

$$\begin{aligned} &x := \phi^{-1}(v), \\ &\tau_{\ell,\ell'} : Y'_\ell \to Y'_{\ell'} \text{ with } \{\tau_{\ell,\ell'}(z)\} = N_{H'_*}(z) \cap Y'_{\ell'} \text{ for all } z \in Y'_\ell \text{ with } \ell\ell' \in E(R_K), \\ &y'_b := \tau_{a,b}(x) \;\text{ and }\; y'_{b,\ell} := \tau_{b,\ell}(y'_b) \text{ whenever } ab, b\ell \in E(R_K). \end{aligned} \tag{5.35}$$

Note that x, y'_b and $y'_{b,\ell}$ are random variables, where x is fixed in Round a', and that $\tau_{\ell,\ell'}$ is a bijection whenever $\ell\ell' \in E(R_K)$. Recall that $S \subseteq V_j$, so $S \cap U'_b \neq \emptyset$ only for $b \in J_j = \{b_1, \dots, b_K\}$ (cf. (5.32)). By (5.25), either $a' < b'_1$ or $b'_K < a'$ holds. We treat these two cases separately. Roughly speaking, in the first case we condition on an embedding of a vertex x onto v in Round a' and show that with high probability the 'right' number of vertices of $N_H(x)$ is embedded to S. In the second case we condition on an embedding of H restricted to S; (B1) is then determined by the choice of $x = \phi^{-1}(v)$.

CASE 1. $a' < b'_1$.

For $a' \leq \ell' \leq w$, and $b \in J_j$ we define the following events.

(B1$^b_{\ell'}$) $|N_{A^b_{\ell'}}(y'_b) \cap S_b| = (d_{i,j}^{-1} p(\vec{d}', b, \ell') \pm \xi_{\ell'})|S_b|$.

(B1$_{\ell'}$) $\bigwedge_{b_s \in J_j : b'_s > \ell'}$ (B1$^{b_s}_{\ell'}$) holds.

Here $p(\vec{d'}, b, \ell')$ is as defined in (5.7) with R_K playing the role of R_*, and $A^b_{\ell'}$ is as defined in (5.10). Recall that the event $\mathcal{A}_{\ell'}$ was defined before Lemma 5.4. To motivate (B1$^b_{\ell'}$), note that conditional on $\mathcal{A}_{\ell'}$, $p(\vec{d'}, b, \ell')$ is the density of $A^b_{\ell'}$. As one would expect from the algorithm, in Round $\ell' > a'$ (i.e after the pre-image x of v has been chosen), the probability that y'_b is embedded onto a vertex in S_b will turn out to be close to $|N \cap S_b|/|N|$, where $N := N_{A^b_{\ell'}}(y'_b)$. This is because the set of candidates for $\phi(y'_b)$ will be almost all of N and every candidate will be (almost) equally likely. Since $S_b \subseteq N_{G'}(v) \cap U_b$, and since $|N_{G'}(v) \cap U_b| \sim d_{i,j}m$, for $\ell' > a'$ we expect this probability to be close to $|S_b|/d_{i,j}m$, which is equivalent to (B1$^b_{\ell'}$).

We define events $\mathcal{B}_{\ell'}$ in the following way:

$$\mathcal{B}_{\ell'} := \begin{cases} \mathcal{A}_{\ell'} & \text{for } 0 \le \ell' \le a'-1, \\ \mathcal{A}_{\ell'} \wedge (\text{B1}_{\ell'}) & \text{for } a' \le \ell' \le w. \end{cases}$$

We will see that $\mathcal{B}_{\ell'}$ typically holds for any $0 \le \ell' \le w$. We let $\mathcal{B}_0^{\ell'} := \bigwedge_{j'=0}^{\ell'} \mathcal{B}_{j'}$ and let $\mathcal{B}_0^{-1}$ be the event which always occurs. Thus $\mathcal{B}_0^0 = \mathcal{B}_0^{-1}$ since $\mathcal{A}_0$ always holds. Note that $\mathcal{B}_0^{\ell'-1}$ and $\mathcal{B}_S$ are events which depend only on the history of the Slender graph algorithm prior to Round ℓ'. Thus Lemma 5.4(ii) implies that for all $0 \le \ell' \le w$,

$$\mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{B}_0^{\ell'-1}, \mathcal{B}_S] \ge 1 - (1-3c)^{Km} \ge 1 - (1-3c)^n. \tag{5.36}$$

Together with (5.31) this implies that

$$\mathbb{P}[\mathcal{B}_0^{a'-1}, \mathcal{B}_S] = \mathbb{P}[\mathcal{B}_S] \prod_{\ell'=0}^{a'-1} \mathbb{P}[\mathcal{B}_{\ell'} \mid \mathcal{B}_0^{\ell'-1}, \mathcal{B}_S] = \mathbb{P}[\mathcal{B}_S] \prod_{\ell'=0}^{a'-1} \mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{B}_0^{\ell'-1}, \mathcal{B}_S] \ge 1 - 3(1-2c)^n. \tag{5.37}$$

Subclaim 1. $\mathbb{P}[\mathcal{B}_{a'} \mid \mathcal{B}_0^{a'-1}, \mathcal{B}_S] \ge 1 - \xi_{a'}$.

To prove Subclaim 1, assume that $\mathcal{B}_S$ and $\mathcal{B}_0^{a'-1}$ hold and we are in Round a' of the Slender graph algorithm. For each $b \in J_j$ consider the graph $F^b_{13}(a) := (Y'_a, U'_b; E)$, as described in the proof of Lemma 5.4(ii), i.e. $yv \in E$ if and only if $y \in Y'_a, v \in U'_b$ and $\tau_{a,b}(y)v \in E(A^b_{a'-1})$. Let $A(a)$ be the subgraph of $A^a_{a'-1}$ as defined in (5.8) of the Slender graph algorithm.

Since $\tau_{a,b}$ is a bijection, the graph $F^b_{13}(a)$ is isomorphic to $A^b_{a'-1}$ under the isomorphism which keeps the elements of U'_b fixed while mapping Y'_a to Y'_b via $\tau_{a,b}$. Thus $F^b_{13}(a)$ is $(\xi_{a'-1}, p(\vec{d'}, b, a'-1))$-super-regular using $\mathcal{B}_{a'-1} = \mathcal{A}_{a'-1}$. Also Lemma 5.4(i) implies that

$$A(a) \text{ is } (4K\Delta_R\sqrt{\xi_{a'-1}}, p(\vec{d'}, a, a'-1))\text{-super-regular.} \tag{5.38}$$

Moreover, by Proposition 3.8,

$$F^b_{13}(a)[N_{A(a)}(v), S_b] \text{ is } (\xi^{1/3}_{a'-1}, p(\vec{d'}, b, a'-1))\text{-regular} \tag{5.39}$$

since $|S_b| > f(\varepsilon)m/2$ by $\mathcal{B}_S$. (Here we also use that $2\xi_{a'-1}/f(\varepsilon) \le \xi^{1/3}_{a'-1}$ by (5.33)). Let

$$T_b := \{y \in N_{A(a)}(v) \cap Y_a : |N_{F^b_{13}(a)}(y) \cap S_b| = p(\vec{d'}, b, a'-1)(1 \pm \xi^{1/4}_{a'-1})|S_b|\}.$$

Recall that the vertex $x = \phi^{-1}(v)$ is determined in Round a' of the Slender graph algorithm. So T_b is the set of candidates for $x = \phi^{-1}(v)$ so that $\tau_{a,b}(x)$ is 'typical' with respect to S_b. Then (5.38) and (5.39) together imply that

$$|T_b| \ge (1 - \xi^{1/4}_{a'-1})p(\vec{d'}, a, a'-1)m. \tag{5.40}$$

Since $N_{A^b_{a'}}(y'_b) = N_{A^b_{a'-1}}(y'_b) \cap N_{G'}(v)$ (see (5.10)) and $S_b \subseteq N_{G'}(v)$, it follows that $N_{A^b_{a'}}(y'_b) \cap S_b = N_{A^b_{a'-1}}(y'_b) \cap S_b$. This implies that if $x \in T_b$, then

$$\begin{aligned} |N_{A^b_{a'}}(y'_b) \cap S_b| &= |N_{A^b_{a'-1}}(y'_b) \cap S_b| = |N_{F^b_{13}(a)}(x) \cap S_b| = p(\vec{d'}, b, a'-1)(1 \pm \xi^{1/4}_{a'-1})|S_b| \\ &\overset{(5.5)}{=} (p(\vec{d'}, b, a'-1) \pm \xi_{a'})|S_b| = (d^{-1}_{i,j}p(\vec{d'}, b, a') \pm \xi_{a'})|S_b|. \end{aligned}$$

We get the final equality by (5.7) and the fact that $d'_{a,b} = d_{i,j}$ as $a \in J_i, b \in J_j$ and $N_{R_K}(b) \cap I_{a'} = \{a\}$ as $ij \in E(R)$ and so $ab \in E(R_K)$. Thus by $(\mathrm{M}'2)_{a'}$,

$$\begin{aligned}
&\mathbb{P}\left[|N_{A^b_{a'}}(y'_b) \cap S_b| = (d_{i,j}^{-1}p(\vec{d'}, b, a') \pm \xi_{a'})|S_b| \mid \mathcal{B}_0^{a'-1}, \mathcal{B}_S\right] \geq \mathbb{P}\left[\phi^{-1}(v) \in T_b \mid \mathcal{B}_0^{a'-1}, \mathcal{B}_S\right] \\
&\geq \quad (1 - h(4K\Delta_R\sqrt{\xi_{a'-1}}))\frac{|T_b|}{p(\vec{d'}, a, a'-1)m} \\
&\overset{(5.40),(3.1)}{\geq} \quad 1 - \frac{\xi_{a'}}{2K}. \qquad (5.41)
\end{aligned}$$

Thus taking a union bound over all $b \in J_j$ in (5.41), we obtain

$$\mathbb{P}[(\mathrm{B1}_{a'}) \mid \mathcal{B}_0^{a'-1}, \mathcal{B}_S] \geq 1 - \frac{\xi_{a'}}{2}. \qquad (5.42)$$

By (5.36) and (5.42) we conclude

$$\mathbb{P}[\mathcal{B}_{a'} \mid \mathcal{B}_0^{a'-1}, \mathcal{B}_S] = \mathbb{P}[\mathcal{A}_{a'}, (\mathrm{B1}_{a'}) \mid \mathcal{B}_0^{a'-1}, \mathcal{B}_S] \geq 1 - (1-3c)^n - \frac{\xi_{a'}}{2} \geq 1 - \xi_{a'}.$$

This completes the proof of Subclaim 1.

Subclaim 2. For all $a' < \ell' \leq w$ we have $\mathbb{P}[\mathcal{B}_{\ell'} \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v] \geq 1 - \xi_{\ell'}$.

To prove Subclaim 2, assume that $\mathcal{B}_S$ and $\mathcal{B}_0^{\ell'-1}$ hold and we are in Round ℓ' of the Slender graph algorithm. Recall that Round a' of the Slender graph algorithm determines $x = \phi^{-1}(v)$, which in turn determines y'_b and $y'_{b,\ell}$ for all $b \in N_{R_K}(a)$ and $\ell \in N_{R_K}(b)$. Note that for fixed $\ell' > a'$, each of $\mathcal{B}_0^{\ell'-1}$, $\mathcal{B}_S$, and $\phi(x) = v$ is an event which depends only on the history prior to Round ℓ'. Thus by Lemma 5.4(ii),

$$\mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{B}_0^{\ell'-1}, \mathcal{B}_S, \phi(x) = v] \geq 1 - (1-3c)^{Km} \geq 1 - (1-3c)^n. \qquad (5.43)$$

Consider some $b \in J_j$ with $b \in I_{b'}$ for some $b' > \ell'$. First, assume that $b \notin N_{R_K}(I_{\ell'})$. Then $A^b_{\ell'} = A^b_{\ell'-1}$ by (5.13). So $(\mathrm{B1}^b_{\ell'-1})$ implies $(\mathrm{B1}^b_{\ell'})$ since in this case $p(\vec{d'}, b, \ell'-1) = p(\vec{d'}, b, \ell')$ by (5.7) and since $\xi_{\ell'-1} \leq \xi_{\ell'}$. So $(\mathrm{B1}^b_{\ell'})$ holds in this case.

So let us next assume that $b \in N_{R_K}(I_{\ell'})$. Then by (5.24) there is a unique index $\ell \in N_{R_K}(b) \cap I_{\ell'}$.

First we verify $(\mathrm{B1}^b_{\ell'})$ in the case when $y'_{b,\ell}$ is an artificial vertex, i.e. $y'_{b,\ell} = y_{\ell,j'} \in Y'_\ell \setminus Y_\ell$ for some $j' \in [C]$. Then, as described after (5.8), we have $f_{\ell'}(y'_{b,\ell}) = u_{\ell,j'} \in V(G') \setminus V(G)$. Note that

$$N_{A^b_{\ell'}}(y'_b) \cap S_b \overset{(5.10)}{=} N_{G'}(f_{\ell'}(y'_{b,\ell})) \cap N_{A^b_{\ell'-1}}(y'_b) \cap S_b \qquad (5.44)$$

and

$$N_{A^b_{\ell'-1}}(y'_b) \cap S_b \overset{(5.11)}{=} N_{A^b_0}(y'_b) \cap N_{G'}(f_{\ell'-1}(N_{\ell'-1}(y'_b))) \cap S_b. \qquad (5.45)$$

Let

$$Q_1 := (V(G') \setminus V(G)) \cap f_{\ell'-1}(N_{\ell'-1}(y'_b)) \quad \text{and} \quad Q_2 := V(G) \cap f_{\ell'-1}(N_{\ell'-1}(y'_b)).$$

Since $f_{\ell'}(y'_{b,\ell}) \in V(G') \setminus (V(G) \cup Q_1)$, similarly as in (5.19), $(\mathrm{BB}2')$ (with b, ℓ playing the roles of ℓ, ℓ') implies that

$$\begin{aligned}
|N_{A^b_{\ell'}}(y'_b) \cap S_b| &\overset{(5.44),(5.45)}{=} |N_{G'}(f_{\ell'}(y'_{b,\ell})) \cap S_b \cap N_{A^b_0}(y'_b) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2)| \\
&= d'_{\ell,b}|S_b \cap N_{A^b_0}(y'_b) \cap N_{G'}(Q_1) \cap N_{G'}(Q_2)| \pm 4\varepsilon n \\
&\overset{(5.45)}{=} d'_{\ell,b}|N_{A^b_{\ell'-1}}(y'_b) \cap S_b| \pm 4\varepsilon n \\
&\overset{(\mathrm{B1}^b_{\ell'-1})}{=} d'_{\ell,b}(d_{i,j}^{-1}p(\vec{d'}, b, \ell'-1) \pm \xi_{\ell'-1})|S_b| \pm 4\varepsilon n \\
&\overset{(5.7)}{=} (d_{i,j}^{-1}p(\vec{d'}, b, \ell') \pm \xi_{\ell'})|S_b|. \qquad (5.46)
\end{aligned}$$

In the final equality we use that $|S_b| > f(\varepsilon)n/(2K)$ since we conditioned on $\mathcal{B}_S$. Thus (5.46) implies that

$$\mathbb{P}[(\mathrm{B1}^b_{\ell'}) \mid \mathcal{B}_S, \mathcal{B}_0^{\ell-1}, \phi(x) = v, y'_{b,\ell} = y_{\ell,j'}] = 1. \tag{5.47}$$

Now assume that $y'_{b,\ell}$ is not an artificial vertex, i.e. $y'_{b,\ell} \in V(H_*)$. Note that

$$|N_{A(\ell)}(y'_{b,\ell})| = (p(\vec{d}', \ell, \ell' - 1) \pm 4K\Delta_R\sqrt{\xi_{\ell'-1}})m \tag{5.48}$$

since $A(\ell)$ is $(4K\Delta_R\sqrt{\xi_{\ell'-1}}, p(\vec{d}', \ell, \ell' - 1))$-super-regular by Lemma 5.4(i) and $\mathcal{B}_{\ell'-1}$. Moreover, $\mathcal{B}_{\ell'-1}$ implies that

$$|N_{A^b_{\ell'-1}}(y'_b) \cap S_b| = (d_{i,j}^{-1}p(\vec{d}', b, \ell' - 1) \pm \xi_{\ell'-1})|S_b|. \tag{5.49}$$

Let

$$N^\ell(S_b) := \{u \in N_{A(\ell)}(y'_{b,\ell}) \cap U_\ell : |N_{G'}(u) \cap N_{A^b_{\ell'-1}}(y'_b) \cap S_b| = d'_{b,\ell}(d_{i,j}^{-1}p(\vec{d}', b, \ell' - 1) \pm \xi_{\ell'})|S_b|\}.$$

Note that (5.44) implies that $(\mathrm{B1}^b_{\ell'})$ holds if $\phi(y'_{b,\ell}) \in N^\ell(S_b)$ (recall that $d'_{b,\ell}p(\vec{d}', b, \ell' - 1) = p(\vec{d}', b, \ell')$). We will show that almost all current candidates $u \in N_{A(\ell)}(y'_{b,\ell})$ lie in $N^\ell(S_b)$.

Since $G'[U'_\ell, U'_b]$ is $(2\varepsilon^{1/3}, d'_{b,\ell})$-super-regular by $(\mathcal{G}')$ in Step 4,

$$|N^\ell(S_b)| \overset{(5.49)}{\geq} (1 - \varepsilon^{1/6})|N_{A(\ell)}(y'_{b,\ell})| \overset{(5.48)}{\geq} (p(\vec{d}', \ell, \ell' - 1) - 5K\Delta_R\sqrt{\xi_{\ell'-1}})m.$$

Then, by $(\mathrm{M}'1)_{\ell'}$,

$$\begin{aligned}\mathbb{P}[\phi(y'_{b,\ell}) \in N^\ell(S_b) \mid \mathcal{B}_0^{\ell'-1}, \mathcal{B}_S, \phi(x) = v, y'_{b,\ell} \in V(H_*)] &\geq (1 - h(4K\Delta_R\sqrt{\xi_{\ell'-1}}))\frac{|N_\ell(S_b)|}{p(\vec{d}', \ell, \ell' - 1)m} \\ &\geq 1 - 2h(4K\Delta_R\sqrt{\xi_{\ell'-1}}).\end{aligned} \tag{5.50}$$

So (5.50) implies that

$$\mathbb{P}[(\mathrm{B1}^b_{\ell'}) \mid \mathcal{B}_0^{\ell'-1}, \mathcal{B}_S, \phi(x) = v, y'_{b,\ell} \in V(H_*)] \geq 1 - 2h(4K\Delta_R\sqrt{\xi_{\ell'-1}}). \tag{5.51}$$

Let $B := \{b \in J_j \cap N_{R_K}(I_{\ell'}) : b \in I_{b'} \text{ with } b' > \ell'\}$ and for $b \in B$ let ℓ_b be the unique index such that $\{\ell_b\} = N_{R_K}(b) \cap I_{\ell'}$. So B keeps track of those cluster indices in J_j where the future candidates will be affected by the embeddings in the current round. Note that $|B| \leq |J_j| = K$. Thus by (5.43), (5.47), (5.51) and our previous observation that $(\mathrm{B1}^b_{\ell'})$ holds for all $b \in J_j \setminus N_{R_K}(I_{\ell'})$ with $b \in I_{b'}$ for some $b' > \ell'$, we have

$$\begin{aligned}&\mathbb{P}[\overline{\mathcal{B}_{\ell'}} \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v] \leq \mathbb{P}[\overline{\mathcal{A}_{\ell'}} \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v] \\ &+ \sum_{b\in B} \mathbb{P}[\overline{(B1^b_{\ell'})} \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v, y'_{b,\ell_b} \in V(H_*)]\mathbb{P}[y'_{b,\ell_b} \in V(H_*) \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v] \\ &+ \sum_{b\in B} \mathbb{P}[\overline{(B1^b_{\ell'})} \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v, y'_{b,\ell_b} \notin V(H_*)]\mathbb{P}[y'_{b,\ell_b} \notin V(H_*) \mid \mathcal{B}_S, \mathcal{B}_0^{\ell'-1}, \phi(x) = v] \\ &\leq (1 - 3c)^n + 2K \cdot h(4K\Delta_R\sqrt{\xi_{\ell'-1}}) + 0 \leq \xi_{\ell'},\end{aligned}$$

which proves Subclaim 2.

We are now ready to conclude the proof of Claim 5.7 in Case 1. Consider any $b \in J_j$ and let b' be such that $b \in I_{b'}$. Then $b' > a'$ since we are in Case 1. We first calculate the probability that $\phi(y'_b) \in S_b$ conditional on $\mathcal{B}_S, \mathcal{B}_0^{b'-1}, \phi(x) = v$. Note that Lemma 5.4(i) implies that

$$\begin{aligned}|N_{A(b)}(y'_b) \cap S_b| \quad &= \quad |N_{A^b_{b'-1}}(y'_b) \cap S_b| \pm 4K\Delta_R\xi_{b'-1}m \\ &\overset{(\mathrm{B1}_{b'-1})}{=} (d_{i,j}^{-1}p(\vec{d}', b, b' - 1) \pm 4K\Delta_R\sqrt{\xi_{b'-1}})|S_b|.\end{aligned} \tag{5.52}$$

(In the last equality we also use that $|S_b| \geq f(\varepsilon)m/2$ by $\mathcal{B}_S$.) Suppose that $y'_b \in V(H_*)$. Then by $(\mathrm{M}'1)_{b'}$,

$$\begin{aligned}\mathbb{P}[\phi(y'_b) \in S_b \mid \mathcal{B}_S, \mathcal{B}_0^{b'-1}, \phi(x) = v] &= (1 \pm h(4K\Delta_R\sqrt{\xi_{b'-1}}))\frac{|N_{A(b)}(y'_b) \cap S_b|}{p(\vec{d}', b, b'-1)m} \\ &\overset{(5.52),(\mathrm{BB1})}{=} (1 \pm \xi_{b'})\frac{|S|}{d_{i,j}n}. \end{aligned} \tag{5.53}$$

By Subclaim 2, (5.53), and the fact that $2w\xi_w < f(\varepsilon)^2/3$ and $\xi_{i'} < \xi_{i''}$ for $i' < i''$ (cf. (3.1) and (5.33)), we obtain

$$\begin{aligned}&\mathbb{P}[\phi(y'_b) \in S_b \mid \mathcal{B}_S, \mathcal{B}_0^{a'}, \phi(x) = v] \\ &= \mathbb{P}[\phi(y'_b) \in S_b \mid \mathcal{B}_S, \mathcal{B}_0^{b'-1}, \phi(x) = v] \prod_{i'=a'}^{b'-2} \mathbb{P}[\mathcal{B}_{i'+1} \mid \mathcal{B}_S, \mathcal{B}_0^{i'}, \phi(x) = v] \pm \sum_{i'=a'}^{b'-2} \mathbb{P}[\overline{\mathcal{B}_{i'+1}} \mid \mathcal{B}_S, \mathcal{B}_0^{i'}, \phi(x) = v] \\ &= (1 \pm \xi_{b'})\frac{|S|}{d_{i,j}n}(1 \pm \xi_w)^{b'-a'-1} \pm (b'-a'-1)\xi_w = \left(1 \pm \frac{f(\varepsilon)}{3}\right)\frac{|S|}{d_{i,j}n}. \end{aligned} \tag{5.54}$$

Note that once $x = \phi^{-1}(v)$ has been determined, the set $N_H(x) \cap X_j$ is also determined. Recall that $|N_H(x) \cap X_j| \in \{k_{i,j}, k_{i,j}+1\}$ since H is $(R, \vec{k}, C)$-near-equiregular. Let X_i^{good} be the set of vertices x in X_i that satisfy $|N_H(x) \cap X_j| = k_{i,j}$. First assume that $x \in X_i^{good}$. Write $N_H(x) \cap X_j = \{x_1, \dots, x_{k_{i,j}}\}$ and for $\ell \in [k_{i,j}]$ define b''_ℓ by $x_\ell \in Y_{b''_\ell}$. So $b''_\ell \in J_j$ and $x_\ell = y'_{b''_\ell}$. Thus by (5.54), for each $x \in X_i^{good}$ we have

$$\begin{aligned}\mathbb{E}\left[|\phi(N_H(x)) \cap S| \mid \mathcal{B}_S, \mathcal{B}_0^{a'}, \phi(x) = v\right] &= \sum_{\ell=1}^{k_{i,j}} \mathbb{P}[\phi(y_{b''_\ell}) \in S_{b''_\ell} \mid \mathcal{B}_S, \mathcal{B}_0^{a'}, \phi(x) = v] \\ &= \left(1 \pm \frac{f(\varepsilon)}{3}\right)\frac{k_{i,j}|S|}{d_{i,j}n}. \end{aligned} \tag{5.55}$$

We now show that the contribution from the vertices $x \in X_i \setminus X_i^{good}$ is insignificant (see (5.59)). Note that (5.37) and Subclaim 1 together imply that

$$\mathbb{P}[\mathcal{B}_S, \mathcal{B}_0^{a'}] \geq 1 - 2\xi_{a'}. \tag{5.56}$$

Since $A(a)$ is $(4K\Delta_R\sqrt{\xi_{a'-1}}, p(\vec{d}', a, a'-1))$-super-regular by Lemma 5.4(i) and $\mathcal{B}_0^{a'-1}$, we can apply Theorem 4.3 to see that for any $x \in Y_a$,

$$\mathbb{P}[\phi(x) = v \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}] \leq \frac{2}{p(\vec{d}', a, a'-1)m}. \tag{5.57}$$

Note that

$$\mathbb{P}[\phi(x) = v \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}] = \mathbb{P}[\phi(x) = v \mid \mathcal{B}_S, \mathcal{B}_0^{a'}]\mathbb{P}[\mathcal{B}_{a'} \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}] \pm \mathbb{P}[\overline{\mathcal{B}_{a'}} \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}].$$

Together with Subclaim 1 and the fact that $3kC\xi_{a'} < q_*(\varepsilon^{1/3})$ (by (3.1) and the assumption $\varepsilon \ll 1/k, 1/(C+1)$) this yields

$$\begin{aligned}\mathbb{P}[\phi(x) = v \mid \mathcal{B}_S, \mathcal{B}_0^{a'}] &= \frac{\mathbb{P}[\phi(x) = v \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}] \pm \mathbb{P}[\overline{\mathcal{B}_{a'}} \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}]}{\mathbb{P}[\mathcal{B}_{a'} \mid \mathcal{B}_S, \mathcal{B}_0^{a'-1}]} \\ &\overset{(5.57)}{\leq} \frac{2/(p(\vec{d}', a, a'-1)m) \pm \xi_{a'}}{1 \pm \xi_{a'}} \leq \frac{q_*(\varepsilon^{1/3})}{Ck}. \end{aligned} \tag{5.58}$$

Now (5.58) together with the fact that $|X_i \setminus X_i^{good}| \leq Ck$ implies

$$\sum_{x \in X_i \setminus X_i^{good}} \mathbb{P}[x = \phi^{-1}(v) \mid \mathcal{B}_S, \mathcal{B}_0^{a'}] \leq q_*(\varepsilon^{1/3}). \tag{5.59}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}[|N_{\phi(H)}(v)\cap S|]\\
&= \sum_{x\in X_i}\mathbb{P}[\mathcal{B}_S,\mathcal{B}_0^{a'}]\mathbb{P}[x=\phi^{-1}(v)\mid \mathcal{B}_S,\mathcal{B}_0^{a'}]\mathbb{E}[|\phi(N_H(x))\cap S|\mid \mathcal{B}_S,\mathcal{B}_0^{a'},\phi(x)=v]\\
&\quad \pm(k_{i,j}+1)\mathbb{P}[\overline{\mathcal{B}_S}\vee\overline{\mathcal{B}_0^{a'}}]\\
&\overset{(5.56)}{=} \sum_{x\in X_i}(1\pm 2\xi_{a'})\,\mathbb{P}[x=\phi^{-1}(v)\mid \mathcal{B}_S,\mathcal{B}_0^{a'}]\mathbb{E}[|\phi(N_H(x))\cap S|\mid \mathcal{B}_S,\mathcal{B}_0^{a'},\phi(x)=v]\pm 2(k_{i,j}+1)\xi_{a'}\\
&\overset{(5.55,5.59)}{=} (1\pm 2\xi_{a'})\left[\sum_{x\in X_i^{good}}\mathbb{P}[x=\phi^{-1}(v)\mid \mathcal{B}_S,\mathcal{B}_0^{a'}]\left(1\pm\frac{f(\varepsilon)}{3}\right)\frac{k_{i,j}|S|}{d_{i,j}n}\pm(k_{i,j}+1)q_*(\varepsilon^{1/3})\right]\\
&\quad \pm 2(k_{i,j}+1)\xi_{a'}\\
&\overset{(5.59)}{=} (1\pm 2\xi_{a'})\left[(1\pm q_*(\varepsilon^{1/3}))\left(1\pm\frac{f(\varepsilon)}{3}\right)\frac{k_{i,j}|S|}{d_{i,j}n}\pm(k_{i,j}+1)q_*(\varepsilon^{1/3})\right]\pm 2(k_{i,j}+1)\xi_{a'}\\
&= (1\pm f(\varepsilon))\frac{k_{i,j}|S|}{d_{i,j}n}. \qquad (5.60)
\end{aligned}$$

To obtain the final equality we used that $|S|>f(\varepsilon)n$ and $3(k_{i,j}+1)(q_*(\varepsilon^{1/3})+\xi_{a'})\le 6(k+1)\varepsilon^{(1/3)(1/300)^{w+1}}<\varepsilon^{3(1/300)^{w+2}}=(f(\varepsilon))^3$ by (3.1) and the assumption $\varepsilon\ll 1/k$. This proves Claim 5.7 in Case 1.

CASE 2. $b'_K<a'$.

So in this case $\phi^{-1}(S)$ is determined before $\phi^{-1}(v)$. For each $y\in Y_a$, we consider

$$L_y:=\{\ell\in J_j: N_H(y)\cap Y_\ell\neq\emptyset\}=\{j_1,\dots,j_{k'}\}, \qquad (5.61)$$

where $k'\in\{k_{i,j},k_{i,j}+1\}$. For each $L\in\binom{J_j}{k_{i,j}}$ define

$$X(L):=\left\{y\in Y_a\cap N_{A_0^a}(v): L_y=L\right\}. \qquad (5.62)$$

So the $X(L)$ partition most of the initial set $Y_a\cap N_{A_0^a}(v)$ of candidates y for $\phi^{-1}(v)$ according to the set L_y that records the clusters $Y_\ell\subseteq X_j$ which contain the neighbours of y. Let

$$\mathcal{L}:=\left\{L\in\binom{J_j}{k_{i,j}}:|X(L)|\ge(f(\varepsilon))^3n\right\},\ \ X':=\{y\in Y_a\cap N_{A_0^a}(v):L_y\notin\mathcal{L}\},\ \ x_L:=\frac{|X(L)|}{m}. \qquad (5.63)$$

Since $|J_j|=K$ we have $|\mathcal{L}|\le\binom{K}{k_{i,j}}\le\binom{K}{k}$ and

$$|X'|\le(f(\varepsilon))^3n\binom{K}{k}+Ck\le(f(\varepsilon))^{11/4}m. \qquad (5.64)$$

We will calculate $\mathbb{E}[|N_{\phi(H)}(v)\cap S|]$ by conditioning on $\phi^{-1}(v)\in X(L)$ for each fixed L. For this, we consider the following sets: for $b\in J_j$, let

$$N^a(S_b):=N_{H'_*}(\phi^{-1}(S_b))\cap Y'_a=\tau_{b,a}(\phi^{-1}(S_b)). \qquad (5.65)$$

These sets are relevant for the following reason: the definitions of $X(L)$ in (5.62) and $N^a(S_b)$ in (5.65) imply that under the assumption that $\phi^{-1}(v)\in X(L)$ and $b\in J_j$ we have

$$|N_{\phi(H)}(v)\cap S_b|=\begin{cases}1 & \text{if } \phi^{-1}(v)\in N^a(S_b) \text{ and } b\in L,\\ 0 & \text{else.}\end{cases} \qquad (5.66)$$

We now define events $(\mathrm{C1}_{\ell'})$ and $(\mathrm{C2}^b_{\ell'})$ for all $0\le\ell'<a'$ and $b\in(I_1\cup\dots\cup I_{\ell'})\cap J_j$ as follows.

$(\mathrm{C1}_{\ell'})$ $|N_{A^a_{\ell'}}(v)\cap X(L)|=d_0^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_Lm$ for all $L\in\mathcal{L}$.

$(\mathrm{C2}^b_{\ell'})$ $|N_{A^a_{\ell'}}(v)\cap N^a(S_b)\cap X(L)|=d_0^{-1}d_{i,j}^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_L|S_b|$ for all $L\in\mathcal{L}$.

Let $(\mathrm{C2}_{\ell'})$ be the event that $(\mathrm{C2}^b_{\ell'})$ holds for all $b \in (I_1 \cup \dots \cup I_{\ell'}) \cap J_j$. Note that if $\ell' = 0$, then $(\mathrm{C1}_{\ell'})$ holds by (5.6) and $(\mathrm{C2}_{\ell'})$ is vacuous.

Note that conditional on $(\mathrm{A1}_{\ell'})$, $p(\vec{d}', a, \ell')$ is the density of $A^a_{\ell'}$, so $(\mathrm{C1}_{\ell'})$ is exactly what one would expect (the extra d_0^{-1} arises since we assumed $X(L) \subseteq N_{A^a_0}(v)$). Similarly as in $(\mathrm{B1}^b_{\ell'})$, the extra $d_{i,j}^{-1}$ in $(\mathrm{C2}^b_{\ell'})$ comes from the fact that $S_b \subseteq S \subseteq N_{G'}(v)$.

For all $0 \le \ell' < a'$ we next define the event $\mathcal{C}_{\ell'}$ by:

$$\mathcal{C}_{\ell'} := \mathcal{A}_{\ell'} \wedge (\mathrm{C1}_{\ell'}) \wedge (\mathrm{C2}_{\ell'})$$

and let $\mathcal{C}_0^{\ell'} := \bigwedge_{j'=0}^{\ell'} \mathcal{C}_{j'}$. So in particular, $\mathcal{C}_0$ is the event which always holds.

Subclaim 3. For all $\ell' \in [a'-1]$, $\mathbb{P}[\mathcal{C}_{\ell'} \mid \mathcal{B}_S, \mathcal{C}_0^{\ell'-1}] \ge 1 - (1-2c)^n$.

To prove Subclaim 3, note that $\mathcal{B}_S$ and $\mathcal{C}_0^{\ell'-1}$ are events only depending on the history of the Slender graph algorithm prior to Round ℓ'. Thus Lemma 5.4(ii) implies that for each $\ell' \in [a'-1]$,

$$\mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{B}_S, \mathcal{C}_0^{\ell'-1}] \ge 1 - (1-3c)^{Km} \ge 1 - (1-3c)^n. \tag{5.67}$$

If $I_{\ell'} \cap N_{R_K}(a) = \emptyset$, then $A^a_{\ell'} = A^a_{\ell'-1}$ by (5.13) and $p(\vec{d}', a, \ell') = p(\vec{d}', a, \ell'-1)$ by (5.7). Moreover, $I_{\ell'} \cap J_j = \emptyset$ as $J_j \subseteq N_{R_K}(a)$. So in this case $(\mathrm{C1}_{\ell'})$ and $(\mathrm{C2}_{\ell'})$ are immediate from $\mathcal{C}_{\ell'-1}$.

So let us next assume that $I_{\ell'} \cap N_{R_K}(a) = \{\ell\}$. We now show that

$$\mathbb{P}[(\mathrm{C1}_{\ell'}) \mid \mathcal{B}_S, \mathcal{C}_0^{\ell'-1}] \ge 1 - (1-3c)^n. \tag{5.68}$$

Recall that $\sigma_\ell : Y'_\ell \to U'_\ell$ for $\ell \in I_{\ell'}$ denotes the bijection chosen in Round ℓ' of the Slender graph algorithm. By the definition of $A^a_{\ell'}$ in (5.10), (similarly as in (5.20))

$$N_{A^a_{\ell'}}(v) \cap X(L) = N_{A^a_{\ell'-1}}(v) \cap X(L) \cap \tau_{\ell,a}(\sigma_\ell^{-1}(N_{G'}(v) \cap U'_\ell)). \tag{5.69}$$

Since we assume that $(\mathrm{C1}_{\ell'-1})$ holds, for all $L \in \mathcal{L}$ we have

$$|N_{A^a_{\ell'-1}}(v) \cap X(L)| = d_0^{-1} p(\vec{d}', a, \ell'-1)(1 \pm \xi_{\ell'-1}) x_L m. \tag{5.70}$$

For each $L \in \mathcal{L}$, we let

$$Q^\ell_L := \tau_{a,\ell}(N_{A^a_{\ell'-1}}(v) \cap X(L)). \tag{5.71}$$

So $Q^\ell_L \subseteq Y'_\ell$ and

$$|Q^\ell_L| \overset{(5.70)}{=} d_0^{-1} p(\vec{d}', a, \ell'-1)(1 \pm \xi_{\ell'-1}) x_L m. \tag{5.72}$$

Using that $\sigma_\ell \circ \tau_{a,\ell} : Y'_a \to U'_\ell$ is a bijection, we obtain

$$\begin{aligned}
\sigma_\ell(Q^\ell_L) \cap N_{G'}(v) &\overset{(5.71)}{=} \sigma_\ell\left(\tau_{a,\ell}\left(N_{A^a_{\ell'-1}}(v) \cap X(L)\right)\right) \cap (N_{G'}(v) \cap U'_\ell) \\
&= \sigma_\ell \circ \tau_{a,\ell}\left(N_{A^a_{\ell'-1}}(v) \cap X(L)\right) \cap \sigma_\ell \circ \tau_{a,\ell}\left(\tau_{\ell,a}(\sigma_\ell^{-1}(N_{G'}(v) \cap U'_\ell))\right) \\
&= \sigma_\ell \circ \tau_{a,\ell}\left(N_{A^a_{\ell'-1}}(v) \cap X(L) \cap \tau_{\ell,a}(\sigma_\ell^{-1}(N_{G'}(v) \cap U'_\ell))\right) \\
&\overset{(5.69)}{=} \sigma_\ell \circ \tau_{a,\ell}(N_{A^a_{\ell'}}(v) \cap X(L)).
\end{aligned}$$

Since $\sigma_\ell \circ \tau_{a,\ell}$ is a bijection, this implies that

$$|\sigma_\ell(Q^\ell_L) \cap N_{G'}(v)| = |N_{A^a_{\ell'}}(v) \cap X(L)|. \tag{5.73}$$

We will now apply $(\mathrm{M}'4)_{\ell'}$ with Q^ℓ_L, $N_{G'}(v) \cap U'_\ell$ playing the roles of S, T. Note that $|N_{G'}(v) \cap U'_\ell| = (1 \pm 2\varepsilon^{1/3}) d'_{a,\ell} m$ by $(\mathcal{G}')$ in Step 4 and also that

$$|Q^\ell_L| \ge d^w x_L m \ge d^w (f(\varepsilon))^3 m \ge h'(4K\Delta_R \sqrt{\xi_{\ell'-1}}) m \tag{5.74}$$

by (5.63) and (5.72), so the conditions of $(\mathrm{M}'4)_{\ell'}$ are satisfied. Moreover

$(1 \pm h'(4K\Delta_R \sqrt{\xi_{\ell'-1}}))|N_{G'}(v) \cap U'_\ell||Q^\ell_L|/m$

$$\begin{aligned} &\overset{(5.72)}{=} && (1\pm h'(4K\Delta_R\sqrt{\xi_{\ell'-1}}))(1\pm 2\varepsilon^{1/3})d'_{a,\ell}d_0^{-1}p(\vec{d'},a,\ell'-1)(1\pm\xi_{\ell'-1})x_Lm\\ &\overset{(5.7),(5.33)}{=} && d_0^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_Lm. \end{aligned}$$

So $(\mathrm{M}'4)_{\ell'}$ implies

$$\begin{aligned} \mathbb{P}[(\overline{\mathrm{C1}_{\ell'}})\mid \mathcal{B}_S,\mathcal{C}_0^{\ell'-1}] &\overset{(5.73)}{\leq} \sum_{L\in\mathcal{L}}\mathbb{P}[|\sigma_\ell(Q_L^\ell)\cap N_{G'}(v)|\neq d_0^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_Lm\mid \mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\\ &\leq |\mathcal{L}|(1-4Kc)^m\leq(1-3c)^n, \end{aligned}$$

and so (5.68) holds.

Our next aim is to show that for all $\ell'\in[a'-1]$ we have

$$\mathbb{P}[(\mathrm{C2}_{\ell'})\mid\mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\geq 1-K(1-3c)^n. \tag{5.75}$$

Consider any $b\in(I_1\cup\dots\cup I_{\ell'})\cap J_j$. We first consider the case when $b\in I_{\ell'}$. Then $N_{R_K}(a)\cap I_{\ell'}=\{b\}$. Note

$$\begin{aligned} (1\pm h'(4K\Delta_R\sqrt{\xi_{\ell'-1}}))|Q_L^b||S_b|/m &\overset{(5.72)}{=} d_0^{-1}p(\vec{d'},a,\ell'-1)(1\pm\xi_{\ell'})x_L|S_b|\\ &= d_0^{-1}d_{i,j}^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_L|S_b|, \end{aligned}$$

because $d_{i,j}=d'_{a,b}$. This shows that, for each $L\in\mathcal{L}$, we can apply $(\mathrm{M}'4)_{\ell'}$ with Q_L^b, S_b playing the roles of S, T to see that

$$\begin{aligned} \mathbb{P}[|\sigma_b(Q_L^b)\cap S_b|=d_0^{-1}d_{i,j}^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_L|S_b|\mid\mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\\ \geq 1-(1-4Kc)^m\geq 1-(1-3c)^n. \end{aligned} \tag{5.76}$$

(Here we also use (5.74) and that $|S_b|=(1\pm\varepsilon^{1/3})\frac{|S|}{K}$ by (BB1) to verify the conditions of $(\mathrm{M}'4)_{\ell'}$.) We now consider $\tau_{b,a}(\sigma_b^{-1}(\sigma_b(Q_L^b)\cap S_b))\subseteq Y'_a$. Since $\tau_{b,a},\sigma_b$ are both bijections,

$$|\tau_{b,a}(\sigma_b^{-1}(\sigma_b(Q_L^b)\cap S_b))|=|\sigma_b(Q_L^b)\cap S_b|. \tag{5.77}$$

Moreover, using that $S_b\subseteq N_{G'}(v)\cap U'_b$,

$$\begin{aligned} \tau_{b,a}(\sigma_b^{-1}(\sigma_b(Q_L^b)\cap S_b)) &= \tau_{b,a}(Q_L^b)\cap\tau_{b,a}(\sigma_b^{-1}(S_b))\\ &\overset{(5.71)}{=} N_{A^a_{\ell'-1}}(v)\cap X(L)\cap\tau_{b,a}(\sigma_b^{-1}(S_b))\\ &= N_{A^a_{\ell'-1}}(v)\cap X(L)\cap\tau_{b,a}(\sigma_b^{-1}(S_b))\cap\tau_{b,a}(\sigma_b^{-1}(N_{G'}(v)\cap U'_b))\\ &\overset{(5.65),(5.69)}{=} N_{A^a_{\ell'}}(v)\cap X(L)\cap N^a(S_b). \end{aligned} \tag{5.78}$$

So (5.76), (5.77) and (5.78) together imply that for each $b\in I_{\ell'}\cap J_j$

$$\mathbb{P}[(\mathrm{C2}^b_{\ell'})\mid\mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\geq 1-(1-3c)^n. \tag{5.79}$$

So assume next that $b\in(I_1\cup\dots\cup I_{\ell'-1})\cap J_j$. (In particular, this means that $\ell'\geq 2$.) Then $\mathcal{C}_{\ell'-1}$ implies that for all $L\in\mathcal{L}$ we have

$$|N_{A^a_{\ell'-1}}(v)\cap N^a(S_b)\cap X(L)|=d_0^{-1}d_{i,j}^{-1}p(\vec{d'},a,\ell'-1)(1\pm\xi_{\ell'-1})x_L|S_b|.$$

Thus, this time we consider $\ell\in I_{\ell'}\cap N_{R_K}(a)$ and for each $L\in\mathcal{L}$ we let

$$Q'^\ell_L:=\tau_{a,\ell}(N_{A^a_{\ell'-1}}(v)\cap N^a(S_b)\cap X(L)).$$

Note $Q'^\ell_L\subseteq Y'_\ell$. Similarly as in (5.76) (but with $N_{G'}(v)\cap U'_\ell$ playing the role of T) one can use $(\mathrm{M}'4)_{\ell'}$ to see that

$$\mathbb{P}[|\sigma_\ell(Q'^\ell_L)\cap(N_{G'}(v)\cap U'_\ell)|=d_0^{-1}d_{i,j}^{-1}p(\vec{d'},a,\ell')(1\pm\xi_{\ell'})x_L|S_b|\mid\mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\geq 1-(1-3c)^n. \tag{5.80}$$

Similarly as in (5.78),

$$\begin{aligned}\tau_{\ell,a}\left(\sigma_\ell^{-1}\left(\sigma_\ell(Q'^{\ell}_L)\cap(N_{G'}(v)\cap U'_\ell)\right)\right) &= N_{A^a_{\ell'-1}}(v)\cap N^a(S_b)\cap X(L)\cap\tau_{\ell,a}\left(\sigma_\ell^{-1}\left(N_{G'}(v)\cap U'_\ell\right)\right)\\ &\overset{(5.69)}{=} N_{A^a_{\ell'}}(v)\cap X(L)\cap N^a(S_b).\end{aligned}$$

Thus $|\sigma_\ell(Q'^{\ell}_L)\cap(N_{G'}(v)\cap U'_\ell)| = |N_{A^a_{\ell'}}(v)\cap X(L)\cap N^a(S_b)|$. So if $b\in(I_1\cup\dots\cup I_{\ell'-1})\cap J_j$, we have

$$\mathbb{P}[(\mathrm{C}2^b_{\ell'})\mid\mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\geq 1-(1-3c)^n.$$

Together with (5.79) and a union bound taken over all $b\in(I_1\cup\dots\cup I_{\ell'})\cap J_j$, this implies (5.75). Thus, by (5.67), (5.68), (5.75) we obtain

$$\mathbb{P}[\mathcal{C}_{\ell'}\mid\mathcal{B}_S,\mathcal{C}_0^{\ell'-1}]\geq 1-(1-2c)^n.$$

This completes the proof of Subclaim 3.

We now proceed with the proof of Claim 5.7 in Case 2. Subclaim 3, (5.31) and the fact that $\mathcal{C}_0$ always holds together imply that

$$\mathbb{P}[\mathcal{B}_S,\mathcal{C}_0^{a'-1}]=\mathbb{P}[\mathcal{B}_S]\prod_{\ell'=1}^{a'-1}\mathbb{P}[\mathcal{C}_{\ell'}\mid\mathcal{C}_0^{\ell'-1},\mathcal{B}_S]\geq 1-(1-c)^n. \tag{5.81}$$

Consider any $L\in\mathcal{L}$. We now compute the expectation of $|N_{\phi(H)}(v)\cap S|$ conditional on $\mathcal{B}_S,\mathcal{C}_0^{a'-1}$, and $\phi^{-1}(v)\in X(L)$. Since $\mathcal{C}_{a'-1}$ holds we have $\mathcal{A}_{a'-1}$, $(\mathrm{C}1_{a'-1})$ and $(\mathrm{C}2_{a'-1})$. By Lemma 5.4(i), $|N_{A(a)}(v)|=|N_{A^a_{a'-1}}(v)|\pm 4K\Delta_R\xi_{a'-1}m$. Together with $(\mathrm{C}1_{a'-1})$, $(\mathrm{C}2_{a'-1})$ this implies that for any $b\in J_j$

$$|N_{A(a)}(v)\cap X(L)| = d_0^{-1}p(\vec{d'},a,a'-1)x_Lm\pm 5K\Delta_R\xi_{a'-1}m, \tag{5.82}$$

$$|N_{A(a)}(v)\cap X(L)\cap N^a(S_b)| = d_0^{-1}d_{i,j}^{-1}p(\vec{d'},a,a'-1)x_L|S_b|\pm 5K\Delta_R\xi_{a'-1}m. \tag{5.83}$$

(Here we also use that $J_j\subseteq I_1\cup\dots\cup I_{a'-1}$ by (5.32) and since we are in Case 2.) $(\mathrm{M}'2)_{a'}$ and (3.1) together imply that for every $L\in\mathcal{L}$ and every $b\in J_j$ we have

$$\begin{aligned}\mathbb{P}[\phi^{-1}(v)\in X(L)\cap N^a(S_b)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}] &= (1\pm h(4K\Delta_R\sqrt{\xi_{a'-1}}))\frac{|N_{A(a)}(v)\cap X(L)\cap N^a(S_b)|}{p(\vec{d'},a,a'-1)m}\\ &\overset{(5.83)}{=} (1\pm q_*(\varepsilon^{1/3})/2)\frac{x_L|S_b|}{d_0d_{i,j}m}\overset{(\mathrm{BB}1)}{=}(1\pm q_*(\varepsilon^{1/3}))\frac{x_L|S|}{d_0d_{i,j}n}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[\phi^{-1}(v)\in X(L)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}] &= (1\pm h(4K\Delta_R\sqrt{\xi_{a'-1}}))\frac{|N_{A(a)}(v)\cap X(L)|}{p(\vec{d'},a,a'-1)m}\\ &\overset{(5.82)}{=}(1\pm q_*(\varepsilon^{1/3}))\frac{x_L}{d_0}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{P}[\phi^{-1}(v)\in N^a(S_b)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1},\phi^{-1}(v)\in X(L)] &= \frac{\mathbb{P}[\phi^{-1}(v)\in X(L)\cap N^a(S_b)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}]}{\mathbb{P}[\phi^{-1}(v)\in X(L)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}]}\\ &=(1\pm 3q_*(\varepsilon^{1/3}))\frac{|S|}{d_{i,j}n}.\end{aligned} \tag{5.84}$$

Thus for any $L\in\mathcal{L}$

$$\mathbb{E}[|N_{\phi(H)}(v)\cap S|\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1},\phi^{-1}(v)\in X(L)]$$

$$\overset{(5.66)}{=} \sum_{b\in L}\mathbb{P}[\phi^{-1}(v)\in N^a(S_b)\mid \mathcal{B}_S,\mathcal{C}_0^{a'-1},\phi^{-1}(v)\in X(L)]$$
$$\overset{(5.84)}{=} \sum_{b\in L}(1\pm 3q_*(\varepsilon^{1/3}))\frac{|S|}{d_{i,j}n}\overset{(5.63)}{=}\left(1\pm\frac{f(\varepsilon)}{3}\right)\frac{k_{i,j}|S|}{d_{i,j}n}. \tag{5.85}$$

Moreover, by $(\mathrm{M}'2)_{a'}$ and (5.64),

$$\mathbb{P}[\phi^{-1}(v)\in X'\mid \mathcal{B}_S,\mathcal{C}_0^{a'-1}]\le 2\frac{|X'|}{p(\vec{d}',a,a'-1)m}\le\frac{2f(\varepsilon)^{11/4}}{d_0d^w}\le f(\varepsilon)^{8/3}. \tag{5.86}$$

Furthermore, note that $|\phi(N_H(\phi^{-1}(v)))\cap S|\le\Delta(H)\le(k+1)\Delta_R$ always holds. (Recall $k_{i,j}\le k$ for all $ij\in E(R)$.) Thus,

$$\begin{aligned}
&\mathbb{E}\left[|N_{\phi(H)}(v)\cap S|\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}\right]\\
&= \sum_{L\in\mathcal{L}}\mathbb{E}\left[|N_{\phi(H)}(v)\cap S|\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1},\phi^{-1}(v)\in X(L)\right]\cdot\mathbb{P}[\phi^{-1}(v)\in X(L)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}]\\
&\quad\pm(k+1)\Delta_R\mathbb{P}[\phi^{-1}(v)\in X'\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}]\\
&\overset{(5.85,5.86)}{=} \sum_{L\in\mathcal{L}}\left(1\pm\frac{f(\varepsilon)}{3}\right)\frac{k_{i,j}|S|}{d_{i,j}n}\mathbb{P}[\phi^{-1}(v)\in X(L)\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}]\pm(k+1)\Delta_Rf(\varepsilon)^{8/3}\\
&= \left(1\pm\frac{f(\varepsilon)}{3}\right)\frac{k_{i,j}|S|}{d_{i,j}n}\mathbb{P}[\phi^{-1}(v)\notin X'\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}]\pm f(\varepsilon)^{7/3}\\
&\overset{(5.86)}{=} \left(1\pm\frac{f(\varepsilon)}{2}\right)\frac{k_{i,j}|S|}{d_{i,j}n}.
\end{aligned}\tag{5.87}$$

Therefore, by Subclaim 3, (5.31), (5.81) and (5.87)

$$\begin{aligned}
\mathbb{E}\left[|N_{\phi(H)}(v)\cap S|\right] &= \mathbb{E}\left[|N_{\phi(H)}(v)\cap S|\mid\mathcal{B}_S,\mathcal{C}_0^{a'-1}\right]\mathbb{P}[\mathcal{B}_S,\mathcal{C}_0^{a'-1}]\\
&\quad\pm(k+1)\Delta_R\mathbb{P}[\overline{\mathcal{B}_S}]\pm(k+1)\Delta_R\sum_{\ell'=1}^{a'-1}\mathbb{P}[\overline{\mathcal{C}_{\ell'}}\mid\mathcal{C}_0^{\ell'-1},\mathcal{B}_S]\\
&= \left(1\pm\frac{f(\varepsilon)}{2}\right)\frac{k_{i,j}|S|}{d_{i,j}n}(1\pm(1-c)^n)\pm(k+1)\Delta_R\left(2(1-2c)^n+w(1-2c)^n\right)\\
&= (1\pm f(\varepsilon))\frac{k_{i,j}|S|}{d_{i,j}n}.
\end{aligned}$$

This completes the proof of Case 2 of Claim 5.7. □

Claim 5.8. *(B2) holds.*

Proof. Note that if the Slender graph algorithm applied to $\mathcal{S}$ as defined in Step 4 does not fail, the graphs F_i satisfy property (i) in Lemma 5.5. Since we defined $N_x=N_{H_*}(x)$ in Step 4, we have $x\in N_y$ if only if $y\in N_x$ for $x,y\in V(H)$. Also, since $H_*[Y_i,Y_j]$ is a matching if $ij\in E(R_K)$ and $H_*[Y_i,Y_j]$ is empty otherwise, for every $x\in Y_i$ we have $|N_x\cap Y_j|\le 1$ for all $j\in[Kr]$, and $|N_x\cap Y_j|=0$ if $j\notin N_{R_K}(i)$. Thus for all $x\in V(H), |N_x|\le\Delta(R_K)\le K\Delta_R$ and so (B2.1) holds. Properties (B2.2) and (B2.4) are immediate consequences of property (i) in Lemma 5.5 (note that H_*, A_0^* play the roles of H, A_0 in Lemma 5.5 and $A_0^*\subseteq A_0$) and the fact that $q_*(\varepsilon^{1/3})=\varepsilon^{(1/3)(1/300)^{w+1}}\le\varepsilon^{(1/300)^{w+2}}=f(\varepsilon)$ by (3.1) with room to spare. Finally, (B2.3) holds since we assume that the Uniform embedding algorithm does not fail (and thus $(\mathcal{P})$ in Step 3 holds). □

Claim 5.9. *(B3) holds.*

Proof. Assume $u \in U_i, v \in U_j$ and so $\phi^{-1}(u) \in Y'_i$ and $\phi^{-1}(v) \in Y'_j$. If $i,j \in I_{\ell'}$, then the fact that $(R_K)^2[I_{\ell'}]$ is an independent set implies that

$$\mathbb{P}\left[N_H(\phi^{-1}(u)) \cap N_H(\phi^{-1}(v)) \neq \emptyset\right] = 0.$$

Therefore, without loss of generality, we may assume that $i \in I_{\ell'}, j \in I_{\ell''}$ with $\ell' < \ell''$. Then during Round ℓ'' of the Slender graph algorithm, when we are about to determine how to map Y'_j to U'_j, $x := \phi^{-1}(u)$ is already determined. Let $Y''_j \subseteq Y'_j$ be the set of all those vertices which are at distance two from x in H. So $|Y''_j| \leq (\Delta(H))^2 \leq ((k+1)\Delta_R)^2$. Recall that $\mathcal{A}_0^{\ell''-1}$ was defined before Lemma 5.4. By $(M'2)_{\ell''}$ we have

$$\mathbb{P}[\phi^{-1}(v) \in Y''_j \mid \mathcal{A}_0^{\ell''-1}] \leq \frac{2((k+1)\Delta_R)^2}{p(\vec{d}', j, \ell''-1)m} \leq \frac{1}{2\sqrt{n}}.$$

Moreover, Lemma 5.4(ii) implies that $\mathbb{P}[\mathcal{A}_0^{\ell''-1}] \geq 1-(1-c)^n$. Thus

$$\begin{aligned}\mathbb{P}[N_H(\phi^{-1}(u)) \cap N_H(\phi^{-1}(v)) \neq \emptyset] &\leq \mathbb{P}[N_H(\phi^{-1}(u)) \cap N_H(\phi^{-1}(v)) \neq \emptyset \mid \mathcal{A}_0^{\ell''-1}]\mathbb{P}[\mathcal{A}_0^{\ell''-1}] + \mathbb{P}[\overline{\mathcal{A}_0^{\ell''-1}}] \\ &\leq \mathbb{P}[\phi^{-1}(v) \in Y''_j \mid \mathcal{A}_0^{\ell''-1}] + (1-c)^n < \frac{1}{\sqrt{n}}.\end{aligned}$$

□

Claim 5.10. *(B4) holds.*

Proof. Given $v \in V(G)$, as before, let a denote the index such that $v \in U'_a$, and a' be such that $a \in I_{a'}$. Let $f_{\ell'}$ be the partial embedding we obtain after Round ℓ' in the Slender graph algorithm. For all $\ell \in [Kr]$ let $W_{\ell'}^{\ell}$ be the set of vertices in U'_ℓ which are incident to an edge in $E(G'') \cap f_{\ell'}(E(H))$. (So $W_{\ell'}^{\ell} \subseteq U_\ell$.) Let $\tau_{i,j} : Y'_i \to Y'_j$ be as defined in (5.35).

Recall the number of rounds w was defined in (V2). For each $0 \leq i' \leq w$ we define $I_1^{i'} := I_1 \cup \cdots \cup I_{i'}$. For $i \in [Kr]$ and $i' \in [w]$ we let $R_K^{i,i'} := |N_{R_K}(i) \cap I_1^{i'}|$. So $R_K^{i,i'} \leq K\Delta_R$. For each $0 \leq \ell' \leq w$ we define the following event:

(D1$_{\ell'}$) $|W_{\ell'}^{\ell}| \leq R_K^{\ell,\ell'}\gamma^{4/5}n$ for all $\ell \in N_{R_K}(a) \cap I_1^{\ell'}$.

Note that (D1$_0$) is vacuously true. For $a' \leq \ell \leq w$ we define the following event:

(D2$_{\ell'}$) $\phi(\tau_{a,\ell}(\phi^{-1}(v))) \notin W_{\ell'}^{\ell}$ for all $\ell \in N_{R_K}(a) \cap I_1^{\ell'}$.

Recall that $\mathcal{A}_{\ell'}$ was defined before Lemma 5.4. For each $0 \leq \ell' \leq w$ we define $\mathcal{D}_{\ell'}$ and $\mathcal{D}^*_{\ell'}$ as follows:

$$\mathcal{D}_{\ell'} := \begin{cases} \mathcal{A}_{\ell'} \wedge (\text{D1}_{\ell'}) & \text{if } \ell' < a', \\ \mathcal{A}_{\ell'} \wedge (\text{D1}_{\ell'}) \wedge (\text{D2}_{\ell'}) & \text{if } \ell' \geq a', \end{cases} \tag{5.88}$$

$$\mathcal{D}^*_{\ell'} := \mathcal{A}_{\ell'} \wedge (\text{D1}_{\ell'}) \tag{5.89}$$

We denote $\mathcal{D}_0^{j'} := \bigwedge_{\ell'=0}^{j'} \mathcal{D}_{\ell'}$ and $\mathcal{D}_0^{*,j'} = \bigwedge_{\ell'=0}^{j'} \mathcal{D}^*_{\ell'}$. Note that (D2$_w$) implies that there is no $y \in V(H)$ with $\phi^{-1}(v)y \in E(H)$ such that $\phi(y)$ is incident to an edge in $\phi(E(H)) \cap E(G'')$. Thus $v \notin \phi_2(H, G, G'')$. Accordingly, our aim is to show that both $\mathbb{P}[\mathcal{D}_{\ell'} \mid \mathcal{D}_0^{\ell'-1}]$ and $\mathbb{P}[\mathcal{D}^*_{\ell'} \mid \mathcal{D}_0^{*,\ell'-1}]$ are close to 1 (see (5.102) and (5.103)).

Consider any $\ell \in I_{\ell'}$ and let $A(\ell) \subseteq A^{\ell}_{\ell'-1}$ be as defined in (5.8). Since $\mathcal{D}_0^{\ell'-1}$ and $\mathcal{D}_0^{*,\ell'-1}$ are events only depending on the history prior to Round ℓ' and are contained in $\mathcal{A}_{\ell'-1}$, Lemma 5.4(ii) together with the fact that $(1-3c)^{Km} \leq (1-3c)^n$ implies that

$$\mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{D}_0^{*,\ell'-1}] \geq 1-(1-3c)^n \quad \text{and} \quad \mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{D}_0^{\ell'-1}] \geq 1-(1-3c)^n. \tag{5.90}$$

We now show that $\mathbb{P}[(\text{D1}_{\ell'}) \mid \mathcal{D}_0^{\ell'-1}]$ is close to 1 (see (5.95)). We say that $y \in Y'_\ell$ is *dangerous* for $u \in U'_\ell$ if there exists $q \in N_{R_K}(\ell) \cap I_1^{\ell'-1}$ such that

$$f_{\ell'-1}(\tau_{\ell,q}(y))u \in E(G''). \tag{5.91}$$

Note that if y is dangerous for u then $u \in U_\ell$, and choosing $\phi(y) = u$ would mean $u \in W^\ell_{\ell'}$, unless $y\tau_{\ell,q}(y) \in E(H'_*) \setminus E(H)$. Let A'_ℓ be the subgraph of $A(\ell)$ such that $yu \in E(A'_\ell)$ if y is dangerous for $u \in U'_\ell$. Note that $E(A'_\ell) = E(A'_\ell[Y'_\ell, U_\ell])$.

We now bound $d_{A'_\ell}(u)$ for $u \in U'_\ell$. Since $d_{G''}(u) \le \gamma n$, there are at most γn vertices $y' \in V(H)$ such that $f_{\ell'-1}(y') \in N_{G''}(u)$. Thus for any fixed u, (5.91) can occur for at most γn distinct vertices y in Y'_ℓ, i.e. $d_{A'_\ell}(u) \le \gamma n$.

Next we bound $d_{A'_\ell}(y)$ for $y \in Y'_\ell$. Note that $uy \in E(A'_\ell)$ implies that

$$u \in \bigcup_{q \in N_{R_K}(\ell) \cap I_1^{\ell'-1}} N_{G''}(f_{\ell'-1}(\tau_{\ell,q}(y))) \cap U'_\ell,$$

so $d_{A'_\ell}(y) \le \Delta(R_K)\Delta(G'') \le K\Delta_R \gamma n$. Thus

$$\Delta(A'_\ell) \le K\Delta_R \gamma n \le K^2 \Delta_R \gamma m \le \gamma^{9/10} m. \tag{5.92}$$

Note that once $f_{\ell'}$ is determined, if a vertex $u \in U'_\ell$ is in $W^\ell_{\ell'}$, then there must exist an index $q \in I_1^{\ell'-1} \cap N_{R_K}(\ell)$ such that $f_{\ell'-1}(\tau_{\ell,q}(f^{-1}_{\ell'}(u)))u \in E(G'')$. Thus $f^{-1}_{\ell'}(u)u \in E(A'_\ell[Y_\ell, U_\ell])$. Since we are conditioning on $\mathcal{D}_0^{\ell'-1} \subseteq \mathcal{A}_0^{\ell'-1}$, by $(\mathrm{M}'3)_{\ell'}$ (with $\gamma^{9/10}$ playing the role of d' and $A'_\ell[Y_\ell, U_\ell]$ the role of A')

$$\mathbb{P}\left[|W^\ell_{\ell'}| \ge \gamma^{4/5} n \mid \mathcal{D}_0^{\ell'-1}\right] \le \mathbb{P}\left[|W^\ell_{\ell'}| \ge 8\gamma^{9/10} m/p(\vec{d'}, \ell, \ell'-1) \mid \mathcal{D}_0^{\ell'-1}\right] \le (1-3c)^n. \tag{5.93}$$

Assume

$$|W^\ell_{\ell'}| \le \gamma^{4/5} n \text{ for all } \ell \in I_{\ell'}. \tag{5.94}$$

We now consider $|W^q_{\ell'}|$ for $q \in I_1^{\ell'-1}$ under this assumption. Note that if $u \in W^q_{\ell'} \setminus W^q_{\ell'-1}$ then u must be incident to an edge in $E(G'') \cap f_{\ell'}(E(H)) \setminus (E(G'') \cap f_{\ell'-1}(E(H)))$. Recall that for each $q \in I_1^{\ell'-1}$, we have $|N_{R_K}(q) \cap I_{\ell'}| \le 1$. We let $q_{\ell'}$ be such that $\{q_{\ell'}\} = N_{R_K}(q) \cap I_{\ell'}$ if it exists. Then if $q_{\ell'}$ exists,

$$|W^q_{\ell'} \setminus W^q_{\ell'-1}| \le |W^{q_{\ell'}}_{\ell'}| \overset{(5.94)}{\le} \gamma^{4/5} n.$$

If $N_{R_K}(q) \cap I_{\ell'} = \emptyset$, then $W^q_{\ell'} = W^q_{\ell'-1}$. Suppose $\mathcal{D}_{\ell'-1}$ holds in addition to (5.94). Since $\mathcal{D}_{\ell'-1}$ implies $(\mathrm{D1}_{\ell'-1})$, it follows that

$$|W^q_{\ell'}| \le |W^q_{\ell'-1}| + |W^q_{\ell'} \setminus W^q_{\ell'-1}| \overset{(\mathrm{D1}_{\ell'-1})}{\le} R_K^{q,\ell'} \gamma^{4/5} n.$$

In other words, $(\mathrm{D1}_{\ell'})$ holds. Thus, for all $\ell' \in [w]$

$$\begin{aligned} \mathbb{P}[(\mathrm{D1}_{\ell'}) \mid \mathcal{D}_0^{\ell'-1}] &\ge \mathbb{P}[|W^\ell_{\ell'}| \le \gamma^{4/5} n \text{ for all } \ell \in I_{\ell'} \mid \mathcal{D}_0^{\ell'-1}] \\ &\overset{(5.93)}{\ge} 1 - |I_{\ell'}|(1-3c)^n \ge 1 - Kr(1-3c)^n. \end{aligned} \tag{5.95}$$

The same argument also shows that for any $\ell' \in [w]$,

$$\mathbb{P}[(\mathrm{D1}_{\ell'}) \mid \mathcal{D}_0^{*,\ell'-1}] \ge 1 - Kr(1-3c)^n. \tag{5.96}$$

In particular, (5.95) together with (5.90) gives us that for any $\ell' < a'$,

$$\mathbb{P}[\mathcal{D}_{\ell'} \mid \mathcal{D}_0^{\ell'-1}] \ge 1 - 2Kr(1-3c)^n. \tag{5.97}$$

Now we consider $(\mathrm{D2}_{\ell'})$ for the case when $\ell' = a'$. Let $N^a(v)$ be the set of vertices $y \in Y'_a$ which

(i) are dangerous for v (so there exists $q \in N_{R_K}(a) \cap I_1^{a'-1}$ such that $f_{a'-1}(\tau_{a,q}(y))v \in E(G'')$) or
(ii) for which there exists $q \in N_{R_K}(a) \cap I_1^{a'-1}$ such that $f_{a'-1}(\tau_{a,q}(y)) \in W^q_{a'-1}$.

Note that $f^{-1}_{a'}(v) \notin N^a(v)$ guarantees that $(\mathrm{D2}_{a'})$ holds. (Indeed, first observe that $N_{R_K}(a) \cap I_1^{a'} = N_{R_K}(a) \cap I_1^{a'-1}$ since $a \in I_{a'}$. Write $x := f^{-1}_{a'}(v)$. So if $x \notin N^a(v)$, then (ii) means that $f_{a'}(\tau_{a,\ell}(x)) \notin W^\ell_{a'-1}$ for all $\ell \in N_{R_K}(a) \cap I_1^{a'}$. But if $f_{a'}(\tau_{a,\ell}(x)) \in W^\ell_{a'} \setminus W^\ell_{a'-1}$, then $f_{a'}(\tau_{a,\ell}(x))$ is incident to an edge in $E(G'') \cap f_{a'}(E(H))) \setminus E(G'') \cap f_{a'-1}(E(H)))$. So $f_{a'}(\tau_{a,\ell}(x))v \in E(G'')$,

i.e. x is dangerous for v, which means that (i) holds, a contradiction. Thus $f_{a'}(\tau_{a,\ell}(x)) \notin W^{\ell}_{a'}$, as required.)

Now we estimate $|N^a(v)|$. By (5.92) there are at most $K\Delta_R\gamma n$ vertices y such that y is dangerous for v. Also y satisfies (ii) if and only if $y \in \tau_{q,a}(f^{-1}_{a'-1}(W^q_{a'-1}))$ for some $q \in N_{R_K}(a) \cap I_1^{a'-1}$. Suppose $\mathcal{D}_{a'-1}$ and thus $(\mathrm{D1}_{a'-1})$ holds. This shows that the number of vertices y satisfying (ii) is at most $\sum_{q\in N_{R_K}(a)\cap I_1^{a'-1}} |W^q_{a'-1}| \le K^2\Delta_R^2\gamma^{4/5}n$. Thus

$$|N^a(v)| \le K^2\Delta_R^2\gamma^{4/5}n + K\Delta_R\gamma n \le \gamma^{2/3}m.$$

So by $(\mathrm{M}'2)_{a'}$,

$$\mathbb{P}[f^{-1}_{a'}(v) \in N^a(v) \mid \mathcal{D}_0^{a'-1}] \le (1 + h(4K\Delta_R\sqrt{\xi_{a'-1}}))\frac{\gamma^{2/3}m}{p(\vec{d}', a, a'-1)m} \le \gamma^{3/5}. \tag{5.98}$$

Thus by (5.90), (5.95) and (5.98),

$$\mathbb{P}[\mathcal{D}_{a'} \mid \mathcal{D}_0^{a'-1}] = \mathbb{P}[\mathcal{A}_{a'}, (\mathrm{D1}_{a'}), (\mathrm{D2}_{a'}) \mid \mathcal{D}_0^{a'-1}] \ge 1 - (Kr+1)(1-3c)^n - \gamma^{3/5} \ge 1 - \frac{\gamma^{2/3}}{w}. \tag{5.99}$$

Finally we consider $(\mathrm{D2}_{\ell'})$ for the case when $\ell' > a'$. Let $x := \phi^{-1}(v)$. For any $q \in N_{R_K}(a)$, let $y'_q := \tau_{a,q}(x)$, and let $y'_{q,\ell} := \tau_{q,\ell}(y'_q)$ and $y'_{q,q} := y'_q$ for $\ell \in N_{R_K}(q)$. Note that $y'_{q,\ell} \in U'_\ell$. Let $N^*_{R_K}(a)$ be the set of all those $q \in N_{R_K}(a)$ for which either $|N_{R_K}(q) \cap I_{\ell'}| = 1$ or $q \in I_{\ell'}$. For $q \in N^*_{R_K}(a) \setminus I_{\ell'}$, let ℓ_q be such that $N_{R_K}(q) \cap I_{\ell'} = \{\ell_q\}$, and for $q \in N^*_{R_K}(a) \cap I_{\ell'}$, let $\ell_q := q$. Then (5.92) and $(\mathrm{M}'1)_{\ell'}$ together imply that

$$\sum_{q\in N^*_{R_K}(a)} \mathbb{P}[f_{\ell'}(y'_{q,\ell_q}) \in N_{A'_{\ell_q}}(y'_{q,\ell_q}) \mid \mathcal{D}_0^{\ell'-1}] \le \sum_{q\in N^*_{R_K}(a)} 2\frac{\gamma^{9/10}}{p(\vec{d}', \ell_q, \ell'-1)} \le \gamma^{4/5}. \tag{5.100}$$

Note that if $\mathcal{D}_0^{\ell'-1}$ (and thus $(\mathrm{D2}_{\ell'-1})$) holds, then $f_{\ell'-1}(y'_q) \notin W^q_{\ell'-1}$ for all $q \in N_{R_K}(a) \cap I_1^{\ell'-1}$. Thus if in addition we have $f_{\ell'}(y'_{q,\ell_q}) \notin N_{A'_{\ell_q}}(y'_{q,\ell_q})$ for all $q \in N^*_{R_K}(a)$, then $f_{\ell'}(y'_\ell) \notin W^\ell_{\ell'}$ for all $\ell \in N_{R_K}(a) \cap I_1^{\ell'}$, which implies $(\mathrm{D2}_{\ell'})$. Hence (5.100) gives

$$\mathbb{P}[(\mathrm{D2}_{\ell'}) \mid \mathcal{D}_0^{\ell'-1}] \ge 1 - \gamma^{4/5}. \tag{5.101}$$

Now (5.90), (5.95) and (5.101) together imply that for $\ell' > a'$

$$\mathbb{P}[\mathcal{D}_{\ell'} \mid \mathcal{D}_0^{\ell'-1}] \ge 1 - (Kr+1)(1-3c)^n - \gamma^{4/5} \ge 1 - \frac{\gamma^{2/3}}{w}. \tag{5.102}$$

Therefore, by (5.97), (5.99) and (5.102),

$$\mathbb{P}[\mathcal{D}_0^w] = \prod_{\ell'=1}^{w} \mathbb{P}[\mathcal{D}_{\ell'} \mid \mathcal{D}_0^{\ell'-1}] \ge \left(1 - \frac{\gamma^{2/3}}{w}\right)^w \ge 1 - \gamma^{1/2}.$$

Recall that $\mathcal{D}_0^w$ implies $v \notin \phi_2(H, G, G'')$. Hence, $\mathbb{P}[v \in \phi_2(H, G, G'')] \le 1 - \mathbb{P}[\mathcal{D}_0^w] \le \gamma^{1/2}$, i.e. (B4.1) holds.

Now we show that (B4.2) holds. Assume that $(\mathrm{D1}_w)$ holds. If $u \in U_j$ satisfies $u \in \phi_2(H, G, G'')$, then either $u \in W^j_w$ or there exists v, ℓ such that $v \in W^\ell_w$, $\ell \in N_{R_K}(j)$ and $uv \in \phi(E(H))$. Since $\phi(E(H))$ is a matching between U_ℓ and U_j, there can be at most $|W^\ell_w|$ vertices $u \in U_j$ such that $uv \in \phi(E(H))$ for some $v \in W^\ell_w$. Thus $(\mathrm{D1}_w)$ implies that for all $j \in [Kr]$ we have

$$\begin{aligned}|\{u : u \in U_j, u \in \phi_2(H, G, G'')\}| &\le |W^j_w| + \sum_{\ell\in N_{R_K}(j)} |W^\ell_w| \le R^{j,w}_K\gamma^{4/5}n + \sum_{\ell\in N_{R_K}(j)} R^{\ell,w}_K\gamma^{4/5}n \\ &\le (K\Delta_R + 1)(K\Delta_R)\gamma^{4/5}n \le \gamma^{3/5}n.\end{aligned}$$

Also, by (5.90) and (5.96),

$$\mathbb{P}[\mathcal{D}^*_{\ell'} \mid \mathcal{D}_0^{*,\ell'-1}] \ge 1 - 2Kr(1-3c)^n. \tag{5.103}$$

Thus we get

$$\mathbb{P}[(\mathrm{D1}_w)] \geq \mathbb{P}[\mathcal{D}_0^{*,w}] \geq \prod_{\ell'=1}^{w} \mathbb{P}[\mathcal{D}_{\ell'}^{*} \mid \mathcal{D}_0^{*,\ell'-1}] \geq (1-2Kr(1-3c)^n)^w \geq 1-(1-2c)^n.$$

Thus

$$\mathbb{P}[|\{u : u \in U_j, u \in \phi_2(H,G,G'')\}| \leq \gamma^{3/5}n \text{ for all } j \in [Kr]] \geq 1-(1-2c)^n,$$

i.e. (B4.2) holds. □

We can now deduce (B5) from (B1) and (B3).

Claim 5.11. *(B5) holds.*

Proof. First, by part (B1) of Lemma 5.1, for any $T \subseteq N_G(v_1,\dots,v_s) \cap V_j$ with $|T| \geq f(\varepsilon)n$,

$$\mathbb{E}\left[|\phi(E(H)) \cap \{v_\ell v : v \in T, \ell \in [s]\}|\right] = \sum_{\ell=1}^{s} \mathbb{E}\left[|N_{\phi(H)}(v_\ell) \cap T|\right] = (1 \pm f(\varepsilon))\frac{k_{i,j}s}{d_{i,j}n}|T|. \quad (5.104)$$

Let $X := N_G(v_1,\dots,v_s) \cap V_j$. Consider

$$T_1 := \left\{v \in X : \mathbb{P}[B_v = 1] > (1+2f(\varepsilon))\frac{k_{i,j}s}{d_{i,j}n}\right\} \text{ and } T_2 := \left\{v \in X : \mathbb{P}[B_v = 1] < (1-2f(\varepsilon))\frac{k_{i,j}s}{d_{i,j}n}\right\}.$$

First, we show that $|T_1| \leq f(\varepsilon)n$. Assume $|T_1| > f(\varepsilon)n$. Since $B_v = 1$ implies that $|\phi(E(H)) \cap \{v_\ell v : \ell \in [s]\}| \geq 1$, we get

$$\mathbb{E}\left[|\phi(E(H)) \cap \{v_\ell v : v \in T_1, \ell \in [s]\}|\right] \geq \sum_{v \in T_1} \mathbb{P}[B_v = 1] > (1+2f(\varepsilon))\frac{k_{i,j}s|T_1|}{d_{i,j}n},$$

which is a contradiction to (5.104). Thus $|T_1| \leq f(\varepsilon)n$.

Now, we show that $|T_2| \leq f(\varepsilon)n$. Assume $|T_2| > f(\varepsilon)n$. Let $A(v_1,\dots,v_s)$ be the number of pairs $v_\ell, v_{\ell'}$ such that $\phi^{-1}(v_\ell)$ and $\phi^{-1}(v'_\ell)$ share a common neighbour in H. Then by part (B3) of Lemma 5.1,

$$\mathbb{E}[A(v_1,\dots,v_s)] \leq \frac{s^2}{\sqrt{n}}. \quad (5.105)$$

Suppose $a > 0$ is defined by

$$|\phi(E(H)) \cap \{v_\ell v : v \in T_2, \ell \in [s]\}| = a + \sum_{v \in T_2} B_v.$$

By considering the bipartite subgraph of $\phi(H)$ spanned by $\{v_1,\dots,v_s\}$ and T_2, it is easy to see that there are at least $a/\Delta(H) \geq a/(k+1)$ pairs $v_\ell, v_{\ell'}$ for which there exists a vertex $v \in T_2$ such that $v_\ell v, v v_{\ell'} \in \phi(E(H))$. Thus in this case $\phi^{-1}(v_\ell)$ and $\phi^{-1}(v_{\ell'})$ share a common neighbour in H, and so $a \leq (k+1)A(v_1,\dots,v_s)$. Therefore,

$$|\phi(E(H)) \cap \{v_\ell v : v \in T_2, \ell \in [s]\}| \leq \sum_{v \in T_2} B_v + (k+1)A(v_1,\dots,v_s).$$

Thus by (5.105),

$$\begin{aligned}\mathbb{E}\left[|\phi(E(H)) \cap \{v_\ell v : v \in T_2, \ell \in [s]\}|\right] &\leq \sum_{v \in T_2} \mathbb{P}[B_v = 1] + (k+1)\mathbb{E}[A(v_1,\dots,v_s)] \\ &\leq (1-2f(\varepsilon))\frac{k_{i,j}s|T_2|}{d_{i,j}n} + \frac{(k+1)s^2}{\sqrt{n}} \\ &\leq \left(1-\frac{3}{2}f(\varepsilon)\right)\frac{k_{i,j}s|T_2|}{d_{i,j}n},\end{aligned}$$

a contradiction to (5.104). Thus $|T_1| + |T_2| \leq 2f(\varepsilon)n$, which proves (B5). □

Claim 5.12. *(B6) holds.*

Proof. We are given sets $Q \subseteq X_i, W \subseteq V_i$ with $|Q|, |W| \geq f(\varepsilon)n$. For all $\ell \in J_i$, let $Q_\ell := Q \cap Y_\ell$, and define q_ℓ by $|Q_\ell| = q_\ell n$. Then

$$\sum_{\ell \in J_i} q_\ell = \frac{|Q|}{n}. \tag{5.106}$$

Similarly for all $\ell \in J_i$, let $W_\ell := W \cap U_\ell$. We define the following event.

(BW1) $|W_\ell| = (1 \pm \varepsilon^{1/3})\frac{|W|}{K}$ for all $\ell \in J_i$.

By a similar argument as in (5.29), we have

$$\mathbb{P}[\text{(BW1) holds }] \geq 1 - (1-3c)^n. \tag{5.107}$$

Similarly as in (5.32), let $\ell'_1 < \cdots < \ell'_K$ be the indices such that $J_i \subseteq \bigcup_{j=1}^K I_{\ell'_j}$ and $J_i \cap I_{\ell'_j} = \{\ell_j\}$ for some ℓ_j. So $J_i = \{\ell_1, \ldots, \ell_K\}$. For $j \in [K]$ define the event $(\mathrm{QW}_{\ell'_j})$ by

$(\mathrm{QW}_{\ell'_j})$ $|f_{\ell'_j}(Q_{\ell_j}) \cap W_{\ell_j}| = q_{\ell_j}|W| \pm f(\varepsilon)^3 m$.

For $0 \leq \ell' \leq w$, where w is defined as in (V2) of the definition of a valid input, we define

$$\mathcal{Q}_{\ell'} := \begin{cases} \mathcal{A}_{\ell'} \wedge (\mathrm{QW}_{\ell'}) & \text{if } \ell' = \ell'_j \text{ for some } j \in [K], \\ \mathcal{A}_{\ell'} & \text{otherwise,} \end{cases}$$

and let $\mathcal{Q}_0^{\ell'} = \bigwedge_{\ell''=0}^{\ell'} \mathcal{Q}_{\ell''}$. In particular, $\mathcal{Q}_0$ is the event which always occurs. Since $\mathcal{Q}_0^{\ell'-1}$ and (BW1) are events which only depend on the history of algorithm prior to Round ℓ', Lemma 5.4(ii) implies that for all $\ell' \in [w]$

$$\mathbb{P}[\mathcal{A}_{\ell'} \mid \mathcal{Q}_0^{\ell'-1}, \text{(BW1)}] \geq 1 - (1-3c)^{Km} \geq 1 - (1-3c)^n. \tag{5.108}$$

Now we show that when $\ell' = \ell'_j$ for some $j \in [K]$, we have

$$\mathbb{P}[(\mathrm{QW}_{\ell'}) \mid \mathcal{Q}_0^{\ell'-1}, \text{(BW1)}] \geq 1 - (1-3c)^n. \tag{5.109}$$

Let $\ell := \ell_j$. If $q_\ell < f(\varepsilon)^3/K$, then we have $|f_{\ell'}(Q_\ell) \cap W_\ell| = q_\ell|W| \pm f(\varepsilon)^3 m$, so we immediately get $(\mathrm{QW}_{\ell'})$. So suppose $q_\ell \geq f(\varepsilon)^3/K$ and note that $f(\varepsilon)^3 \geq 2Kh'(4K\Delta_R\sqrt{\xi_{\ell'-1}})$. Also note that by (BW1) we have $|W_\ell| = (1 \pm \varepsilon^{1/3})|W|/K \geq h'(4K\Delta_R\sqrt{\xi_{\ell-1}})$ and

$$\Big(1 \pm h'(4K\Delta_R\sqrt{\xi_{\ell-1}})\Big)\frac{|W_\ell||Q_\ell|}{m} \overset{\text{(BW1)}}{=} \left(1 \pm \frac{f(\varepsilon)^3}{2K}\right)(1 \pm \varepsilon^{1/3})\frac{|W|}{K}Kq_\ell = q_\ell|W| \pm f(\varepsilon)^3 m.$$

Thus we can apply $(\mathrm{M}'4)_{\ell'}$ with Q_ℓ, W_ℓ playing the roles of S, T to see that

$$\begin{aligned}\mathbb{P}[(\mathrm{QW}_{\ell'}) \mid \mathcal{Q}_0^{\ell'-1}, \text{(BW1)}] &= \mathbb{P}[|f_{\ell'}(Q_\ell) \cap W_\ell| = q_\ell|W| \pm f(\varepsilon)^3 m \mid \mathcal{Q}_0^{\ell'-1}, \text{(BW1)}] \\ &\geq \mathbb{P}[|f_{\ell'}(Q_\ell) \cap W_\ell| = (1 \pm h'(4K\Delta_R\sqrt{\xi_{\ell'-1}}))|W_\ell||Q_\ell|/m \mid \mathcal{Q}_0^{\ell'-1}, \text{(BW1)}] \\ &\geq 1 - (1-4Kc)^m \geq 1 - (1-3c)^n.\end{aligned}$$

Hence, (5.109) holds. Therefore, (5.108) together with (5.109) imply

$$\mathbb{P}[\mathcal{Q}_{\ell'} \mid \mathcal{Q}_0^{\ell'-1}, \text{(BW1)}] \geq 1 - 2(1-3c)^n. \tag{5.110}$$

Thus by (5.107)

$$\mathbb{P}[\mathcal{Q}_0^w, \text{(BW1)}] \geq 1 - (2w+1)(1-3c)^n \geq 1 - (1-2c)^n.$$

Note that if $\mathcal{Q}_0^w$ holds, then

$$|\phi(Q) \cap W| = \sum_{\ell \in J_i} |\phi(Q_\ell) \cap W_\ell| = \sum_{j \in [K]} \big(q_{\ell_j}|W| \pm f(\varepsilon)^3 m\big) \overset{(5.106)}{=} (1 \pm f(\varepsilon))\frac{|Q||W|}{n}.$$

In the final equality we used that $|Q|, |W| \geq f(\varepsilon)n$. This proves (B6). □

We now deduce from Lemma 5.1 another extension of the blow-up lemma, which we shall also apply in our main algorithm in the next section. Suppose we are given an embedding ϕ of H into G and a set Z of vertices whose embedding is unsuitable (this will be the case if $\phi(H)$ overlaps with the edges of previously embedded graphs in the main packing algorithm in Section 6). Then we can re-embed these vertices using edges of a 'patching graph' P provided ϕ was well behaved with respect to P. This notion of being well behaved is captured by the candidacy bigraphs F being super-regular. Let $Z_i := Z \cap X_i$ and $W_i := \phi(Z_i) \subseteq V_i$. A candidacy bigraph F will encode the possible new images of a vertex, i.e. we may only embed $z \in Z_i$ to $w \in W_i$ if $zw \in E(F)$. So the case of Lemma 5.13 when each $F[Z_i, W_i]$ is a complete bipartite graph means that there are no constraints.

Lemma 5.13. *Suppose $0 < 1/n \leq 1/m \ll \delta \ll \beta', \beta, 1/k, 1/(C+1) \leq 1$, and $1/m \ll 1/r$. Let R be a graph on $[r]$ with $\Delta(R) \leq k$. Let $\vec{\beta}$ be a symmetric $r \times r$ matrix such that $\min_{ij \in E(R)} \beta_{i,j} = \beta$. Suppose H is a graph admitting vertex partition $(R, \mathcal{X})$ with $\mathcal{X} = (X_1, \dots, X_r)$ such that $\Delta(H) \leq k$ and $\Delta(H[X_i, X_j]) \leq 1$ for all $ij \in E[R]$. Suppose A_0 is a graph with bipartition $(V(H), V(P))$ such that $E(A_0) = \bigcup_{i \in [r]} E(A_0[X_i, V_i])$. Suppose P is a graph admitting vertex partition $(R, \mathcal{V})$ with $\mathcal{V} = (V_1, \dots, V_r)$, where $\max_{i \in [r]} |V_i| = n$ and $n - C \leq |V_i| = |X_i| \leq n$. Suppose further that $\phi : V(H) \to V(P)$ is a bijection between $V(H)$ and $V(P)$ such that $\phi(X_i) = V_i$. Suppose N is an $(H, R, \mathcal{X})$-candidacy hypergraph and F is an $(H, P, R, A_0, \phi, \mathcal{X}, \mathcal{V}, N)$-candidacy bigraph. Suppose also that $Z_i \subseteq X_i$, and $W_i = \phi(Z_i) \subseteq V_i$ are sets such that $|Z_i| = |W_i| = m$, and let $Z := \bigcup_{i=1}^r Z_i$, and $W := \bigcup_{i=1}^r W_i$. Finally, suppose the following conditions hold:*

(a) *$F[Z_i, W_i]$ is (δ, β')-super-regular for every $i \in [r]$.*
(b) *$P[W]$ is $(\delta, \vec{\beta})$-super-regular with respect to $(R, W_1, \dots, W_r)$.*

Then we can find a bijection $\phi' : V(H) \to V(P)$ with $\phi'(X_i) = V_i$ for all $i \in [r]$, and such that

(i) *$\phi'(x) = \phi(x)$ for every $x \notin Z$,*
(ii) *$\phi'(x)\phi'(y) \in E(P)$ for every edge $xy \in E(H)$ with $\{x, y\} \cap Z \neq \emptyset$,*
(iii) *$\phi'(x) \in N_F(x) \subseteq N_{A_0}(x)$ for every $x \in Z$.*

Proof. Choose an additional constants c, γ satisfying $1/m \ll c \ll \delta \ll \gamma \ll \beta', \beta, 1/k, 1/(C+1)$. Let $\mathcal{Z} := (Z_1, \dots, Z_r)$ and $\mathcal{W} := (W_1, \dots, W_r)$. Let Q be an r-partite graph admitting vertex partition $(R, \mathcal{W})$ such that $Q[W_i, W_j]$ is a complete bipartite graph for all $ij \in E(R)$. Let $\vec{\tau}$ be the $r \times r$ matrix such that $\tau_{i,j} = 1$ for $ij \in E(R)$ and $\tau_{i,j} = 0$ for $ij \notin E(R)$. Since $\Delta(H[Z_i, Z_j]) \leq 1$ for all $ij \in E(R)$, we can add edges to $H[Z]$ to obtain a graph $H' \supseteq H[Z]$ such that $H'[Z_i, Z_j]$ is a perfect matching for each $ij \in E(R)$. Apply Lemma 5.1 with the following graphs and parameters.

object/parameter	$P[W]$	Q	H'	$F[Z \cup W]$	R	$\mathcal{W}$	$\mathcal{Z}$	m	c	δ
playing the role of	G	P	H	A_0	R	$\mathcal{V}$	$\mathcal{X}$	n	c	ε
object/parameter	γ	1	β	β'	1	k	0	$\vec{\beta}$	$\vec{\tau}$	$\vec{\tau}$
playing the role of	γ	β	d	d_0	k	Δ_R	C	$\vec{d}$	$\vec{\beta}$	$\vec{k}$

Then by Lemma 5.1, with probability at least $1 - (1-c)^m$, we get an embedding $\phi' : Z \to W$ of H' into $P[W]$ (and thus also of $H[Z]$ into $P[W]$) such that for all $x \in Z$, $\phi'(x) \in N_F(x)$. Let $\phi'(x) := \phi(x)$ if $x \notin Z$. Then (i) holds by definition.

Since ϕ' is an embedding of $H[Z]$ into $P[W]$, $\phi'(x)\phi'(y) \in E(P)$ for every edge $xy \in E(H[Z])$. If $xy \in E(H)$, $x \in Z_i$ and $y \notin Z$, then $\phi'(y) = \phi(y)$. Since F is an $(H, P, R, A_0, \phi, \mathcal{X}, \mathcal{V}, N)$-candidacy bigraph, $\phi'(x) \in N_F(x)$ and (CB2) imply that

$$\begin{aligned}\phi'(x) \in N_F(x) \subseteq N_{A_0}(x) \cap \bigcap_{y' \in N_x} N_P(\phi(y')) \cap V_i \subseteq N_{A_0}(x) \cap \bigcap_{y' \in N_H(x)} N_P(\phi(y')) \cap V_i \\ \subseteq N_{A_0}(x) \cap N_P(\phi(y)) = N_{A_0}(x) \cap N_P(\phi'(y)).\end{aligned}$$

Thus (ii) and (iii) hold. (Note that we do not require properties (B1)–(B6) in this application of Lemma 5.1.) □

6. Main packing algorithm

In this section we combine the results derived in previous sections to establish Theorem 6.1, which we consider the main packing result of this paper. It guarantees an approximate decomposition of a super-regular graph G into bounded degree graphs $H_1, \dots, H_s$, provided the H_i reflect the large scale structure of G. More precisely, we assume that G has a reduced graph R of moderate degree, and for each edge ij of R, the corresponding pair is ε-regular in G, and this pair also corresponds to an almost regular bipartite graph in each H_i.

Theorem 6.1. *Suppose* $0 < 1/n \ll c \ll \varepsilon \ll \eta, \eta', \alpha, d, d_0, 1/k, 1/(C+1), 1/\Delta_R$ *and* $1/n \ll 1/r$. *Let* $s \in \mathbb{N}$ *be an integer such that* $s \le \eta^{-1}n$. *Suppose the following assertions hold.*

- (S1) R *is a graph on* $[r]$ *with* $\Delta(R) \le \Delta_R$.
- (S2) $\vec{d}$ *and* $\vec{k}^i$ *are symmetric* $r \times r$ *matrices for all* $i \in [s]$ *such that* $\min_{jj' \in E(R)} d_{j,j'} = d$, $k^i_{j,j'} \in \mathbb{N}$ *and* $k^i_{j,j'} \le k$ *for all* $jj' \in E(R)$, *and* $d_{j,j'} = k^i_{j,j'} = 0$ *if* $jj' \notin E(R)$.
- (S3) *For all* $i \in [s]$, H_i *is an* $(R, \vec{k}^i, C)$*-near-equiregular graph with respect to* $(R, X_1, \dots, X_r)$ *such that* $\max_{j\in[r]} |X_j| = n$ *and* $n - C \le |X_j| \le n$ *for all* $j \in [r]$.
- (S4) *For all* $jj' \in E(R)$, $\sum_{i=1}^s k^i_{j,j'} \le (1-\alpha) d_{j,j'} n$.
- (S5) G *is an* $(\varepsilon, \vec{d})$*-super-regular graph with respect to* $(R, V_1, \dots, V_r)$ *such that* $|X_j| = |V_j|$ *for all* $j \in [r]$.
- (S6) *For all* $i \in [s]$, A_i *is a bipartite graph with bipartition* $(V(H_i), V(G))$ *such that* $N_{A_i}(X_j) = V_j$ *and* $A_i[X_j, V_j]$ *is* (ε, d_0)*-super-regular for all* $j \in [r]$.
- (S7) *For all* $j \in [r]$, *there is a collection* $\mathcal{Q}_j$ *of subsets of* X_j *and a collection* $\mathcal{W}_j$ *of subsets of* V_j *such that* $|Q|, |W| \ge \eta' n$ *for all* $Q \in \mathcal{Q}_j$, $W \in \mathcal{W}_j$ *and such that* $|\mathcal{Q}_j||\mathcal{W}_j| \le (1+c)^n$.

Then for each $i \in [s]$ *there exists an embedding* $\phi'_i : V(H_i) \to V(G)$ *of* H_i *into* G *such that the following hold.*

- (T1) $\phi'_i(x) \in N_{A_i}(x)$ *for all* $x \in V(H_i)$.
- (T2) $\phi'_i(E(H_i)) \cap \phi'_{i'}(E(H_{i'})) = \emptyset$ *for* $i \ne i'$.
- (T3) *For all* $j \in [r]$ *and* $i \in [s]$ *and any sets* $Q \in \mathcal{Q}_j$ *and* $W \in \mathcal{W}_j$, *we have* $|\phi'_i(Q) \cap W| = (1 \pm \eta')\frac{|Q||W|}{n}$.

(T1) and (T3) will be used in further applications elsewhere. They will not be required when we apply Theorem 6.1 in Section 8. Note that in such a situation, i.e. when we apply Theorem 6.1 but conclusion (T3) is not required, then we can ignore condition (S7) along with the parameters c and η'. Similarly, if we do not require (T1), then we can ignore (S6) along with the parameter d_0 (by taking $A_i[X_j, V_j]$ to be a complete bipartite graph).

Remark 6.2. Theorem 6.1 holds even if we replace (S6) by the following.

- (S6′) For all $i \in [s]$, let $X^i_j \subseteq X_j$ be a subset with $|X^i_j| \le \varepsilon n$ and let A_i be a bipartite graph with bipartition $(V(H_i), V(G))$ such that, for all $j \in [r]$,
  - $N_{A_i}(X_j) = V_j$,
  - $d_{A_i}(x) \ge d_0 n$ for all $x \in X^i_j$, and $N_{A_i}(x) = V_j$ for all $x \in X_j \setminus X^i_j$.

Indeed, if (S6′) is given, a random subgraph A'_i of A_i such that $d_{A'_i}(x) = d_0 n$ for all $x \in X_j$ satisfies (S6) with high probability (with ε replaced by $\sqrt{\varepsilon}$, say). Thus Theorem 6.1 holds even after we replace (S6) by (S6′). In other words, (S6′) and (T1) together imply that for each $i \in [s]$ we can specify a linearly sized target set for a small linear fraction of the vertices of H_i.

Let us now briefly sketch the proof idea of Theorem 6.1. The desired packing will be constructed via a randomised algorithm, called the *Main packing algorithm*, which will be shown to succeed with high probability. The algorithm runs in T 'rounds', indexed by a parameter t. In Round t we take a collection of γn graphs H_i (where γ is a small constant and $i \in I_t$ with $|I_t| = \gamma n$) and embed them into the current remainder G^t of G. For this, we first apply Lemma 5.1 to each H_i with $i \in I_t$ independently, in order to define embeddings ϕ_i of H_i into G^t for all $i \in I_t$. We then let G^{t+1} be the graph obtained from G^t by deleting the edges in all the $\phi_i(H_i)$ with $i \in I_t$ from G^t, and proceed to the next round.

Clearly, in each round the embeddings ϕ_i with $i \in I_t$ need not be pairwise edge-disjoint. However, the overlap can be shown to be small, which will allow us to apply Lemma 5.13 in order to define slightly modified new embeddings ϕ_i', which will be pairwise edge-disjoint, as desired. The 'patching graph' $P \subseteq G$, used in Lemma 5.13 to carry out these 'patching procedures' in all rounds, is set aside at the beginning of the algorithm and its edges are not used for any ϕ_i. This procedure is repeated in every round, and in order to do so, we have to ensure that certain graphs remain sufficiently super-regular throughout the algorithm.

Since the main packing algorithm is randomised, at various steps there will be a chance of failure, and our goal is to show that the total probability of all possible failures is small.

In order to describe the main packing algorithm, we need the following definitions. Choose additional constants β, γ, δ so that

$$\frac{1}{n} \ll c \ll \varepsilon \ll \gamma \ll \delta \ll \eta, \eta', \alpha, \beta, d, d_0, \frac{1}{k}, \frac{1}{C+1}, \frac{1}{\Delta_R} \quad \text{and} \quad \beta \ll \alpha, d.$$

Let $\vec{\beta}$ be the $r \times r$ matrix with entries $\beta_{j,j'} := \beta d_{j,j'}/d$ for each $j, j' \in [r]$. So $\min_{jj' \in E(R)} \beta_{j,j'} = \beta$. Let $I_t := \{(t-1)\gamma n + 1, \dots, t\gamma n\}$. Let

$$T := \left\lceil \frac{s}{\gamma n} \right\rceil \leq \frac{1}{\gamma \eta}, \ \ \vec{d}^1 := \vec{d} - \vec{\beta}, \text{ and } K := (k+1)^2 \Delta_R, \tag{6.1}$$

$$d^{t+1}_{j,j'} := d^t_{j,j'} \prod_{i \in I_t} \left(1 - \frac{k^i_{j,j'}}{d^t_{j,j'} n}\right) \text{ for all } t \in [T] \text{ and } jj' \in E(R). \tag{6.2}$$

T will be the total number of rounds and the $d^t_{j,j'}$ track the densities of the unused leftover $G^t[V_j, V_{j'}]$ in the t-th round. Let

$$\varepsilon_1 := 2\varepsilon \quad \text{and} \quad \varepsilon_{t+1} := q(\varepsilon_t) \text{ for all } t \in [T], \tag{6.3}$$

where q is the function defined in (3.1). Note that by the choice of the functions in (3.1) (which depend only on $w := (K\Delta_R)^2(\Delta_R + 1)$ and the argument), we can assume that

$$\varepsilon_T \ll \gamma. \tag{6.4}$$

We are now ready to describe the main packing algorithm.

Main packing algorithm

Round 0. It will be convenient that in each of the T rounds we embed exactly γn graphs. For this, let H' be an arbitrary $(R, \vec{k}', C)$-near-equiregular graph with vertex partition $(R, X_1, \dots, X_r)$ where $k'_{j,j'} := 1$ for all $jj' \in E(R)$ and $k'_{j,j'} := 0$ for all $jj' \notin E(R)$. Also let A be a union of r complete bipartite graphs between X_j and V_j for $j \in [r]$. Now we let $(H_p, A_p) := (H', A)$ for $s+1 \leq p \leq T\gamma n$. Let $\mathcal{H} := \{(H_i, A_i) : i \in [T\gamma n]\}$ and $\Phi := \emptyset$. We apply Lemma 3.10 to find a graph $P \subseteq G$ such that $P^1 := P$ is $(\varepsilon_1, \vec{\beta})$-super-regular and $G^1 := G - P^1$ is $(\varepsilon_1, \vec{d}^1)$-super-regular with respect to $(R, \mathcal{V})$, where $\mathcal{V} := (V_1, \dots, V_r)$. Let $\vec{d'}^t, \vec{\beta'}$ be the $Kr \times Kr$ matrices with entries $d'^t_{\ell,\ell'} := d^t_{j,j'}, \beta'_{\ell,\ell'} := \beta_{j,j'}$ where $\lceil \ell/K \rceil = j, \lceil \ell'/K \rceil = j'$ and $j, j' \in [r]$. Recall that R_K denotes the K-fold blow-up of R. Let $\mathcal{X} := (X_1, \dots, X_r)$. Let $t := 1$, and proceed to Round 1.

Round t.

Step 1. Assume that in Round $t-1$ we have defined

(A1) a graph $G^t \subseteq G^1$, which is $(\varepsilon_t, \vec{d}^t)$-super-regular with respect to $(R, V_1, \dots, V_r)$, and
(A2) a graph $P^t \subseteq P^1$, which is $(\delta^{1/4}, \vec{\beta})$-super-regular with respect to $(R, V_1, \dots, V_r)$.

$G^t \cup P^t$ is the set of currently available edges, the main part of the embedding will be done in G^t, the patching graph P^t will be used to partially re-embed overlapping copies of the H_i.

We apply the Uniform embedding algorithm from Lemma 5.1 with the following graphs and parameters for each $i \in I_t$ one by one in the increasing order (we may do so because of (6.4), this is justified in detail in the proof of Claim 6.7).

object/parameter	G^t	P	H_i	R	A_i	$\mathcal{V}$	$\mathcal{X}$	$2c$	ε_t	$4k\Delta_R\gamma$	$\vec{\beta}$	$\vec{d}^t$	d_0	$\vec{k}^i$	k	r	Δ_R	C
playing the role of	G	P	H	R	A_0	$\mathcal{V}$	$\mathcal{X}$	c	ε	γ	$\vec{\beta}$	$\vec{d}$	d_0	$\vec{k}$	k	r	Δ_R	C

Note that we apply it with P rather than P^t; this is crucial to ensure that the regularity property of the candidacy bigraph in (F5) below is sufficiently strong. If the Uniform embedding algorithm fails for some $i \in I_t$, then end the algorithm with **failure of type 1**. Otherwise, let

$$G^{t+1} := G^t - \bigcup_{i \in I_t} \phi_i(E(H_i)). \tag{6.5}$$

We also define the following event which will be relevant for (T3).

(QW_i) For all $j \in [r]$ and any sets $Q \in \mathcal{Q}_j$ and $W \in \mathcal{W}_j$, we have $|\phi_i(Q) \cap W| = (1 \pm \gamma)\frac{|Q||W|}{n}$.

If (QW_i) fails for some $i \in I_t$, then we end the algorithm with **failure of type 2**.

The above applications of Lemma 5.1 give for every $i \in I_t$ a tuple $(\phi_i, \mathcal{X}^i, \mathcal{V}^i, F_i, N^i)$ satisfying (B1)–(B6) of Lemma 5.1. In particular, for each $i \in I_t$,

- (F0) G^t, P both admit the vertex partition $(R_K, \mathcal{V}^i)$, and H_i admits the vertex partition $(R_K, \mathcal{X}^i)$,
- (F1) $\phi_i : H_i \to G^t$ is an embedding such that $\phi_i(x) \in N_{A_i}(x)$,
- (F2) the partitions $\mathcal{X}^i = (X^i_1, \dots, X^i_{Kr})$ and $\mathcal{V}^i = (V^i_1, \dots, V^i_{Kr})$ with $\max_{j\in[Kr]} |X^i_j| = m'$ and $m' - C \le |X^i_j| = |V^i_j| \le m'$, where $m' = \lceil n/K \rceil$, act as $\mathcal{Y}$ and $\mathcal{U}$ in Lemma 5.1, respectively,
- (F3) N^i is an $(H_i, R_K, \mathcal{X}^i)$-candidacy hypergraph with $|N^i_x| \le K\Delta_R$ for all $x \in V(H_i)$,
- (F4) P is $(\varepsilon_t^{1/3}, \vec{\beta}')$-super-regular with respect to $(R_K, \mathcal{V}^i)$,
- (F5) $F_i = \bigcup_{j\in[Kr]} F_i[X^i_j, V^i_j]$, where $F_i[X^i_j, V^i_j]$ is $(f(\varepsilon_t), d_0 p(R_K, \vec{\beta}', j))$-super-regular. Moreover, each F_i is an $(H_i, P, R_K, A_i, \phi_i, \mathcal{X}^i, \mathcal{V}^i, N^i)$-candidacy bigraph. In particular, for all $x \in X^i_j$,

$$N_{F_i}(x) = N_{F_i[X^i_j, V^i_j]}(x) \subseteq N_{A_i}(x) \cap \bigcap_{y \in N^i_x} N_P(\phi_i(y)) \cap V^i_j. \tag{6.6}$$

For (6.6) recall that candidacy bigraphs are defined before Lemma 5.1. As remarked after the definition of a candidacy hypergraph, (F3) implies that $\Delta(H_i[X^i_j, X^i_{j'}]) \le 1$ for all $j \ne j' \in [Kr]$. Now for each $i \in I_t$, we define $F'_i \subseteq F_i$ by

$$E(F'_i[X^i_j, V^i_j]) := \left\{ xv \in E(F_i) : x \in X^i_j, v \in V^i_j \cap \bigcap_{y \in N^i_x} N_{P^t}(\phi_i(y)) \right\}. \tag{6.7}$$

The graph F'_i can be viewed as an update of the candidacy bigraph F_i which accounts for further restrictions imposed in the current round by the fact that the edges of $P - P^t$ are no longer available for the patching process.

If G^{t+1} as defined in (6.5) is not $(\varepsilon_{t+1}, \vec{d}^{t+1})$-super-regular with respect to $(R, V_1, \dots, V_r)$, then end the algorithm with **failure of type 3**. Otherwise, proceed to Step 2.

Step 2. Observe that in Step 1 we allowed the edge sets of different $\phi_i(H_i)$ to intersect. In Steps 3–5 we aim to resolve this issue by altering the embeddings ϕ_i in order to make them

edge-disjoint. For each $i \in I_t$, let

$$E_i := \bigcup_{j \in I_t \setminus \{i\}} (\phi_i(E(H_i)) \cap \phi_j(E(H_j))),$$
$$U_i := \{v \in e : e \in E_i\},$$
$$NU_i := \bigcup_{u \in U_i, e \in E(H_i)} \{v \in V(G) : \phi_i(e) = uv\}.$$

Note that in particular we have $U_i \subseteq NU_i$. We define the following two events.

(U1) $|\{i \in I_t : v \in NU_i\}| < \gamma^{4/3} n$ holds for all $v \in V(G)$.
(U2) $|NU_i \cap V_j^i| \le \gamma^{2/5} m'$ for all $i \in I_t$, and $j \in [Kr]$.

If (U1) or (U2) does not hold, then end the algorithm with **failure of type 4**. Otherwise, proceed to Step 3.

Step 3. Our aim is now to change the embedding ϕ_i for the vertices in U_i. It turns out that it is much easier to change the embedding on $U_i \cup Y_i$, where Y_i is randomly chosen vertex set of appropriate size. For each $i \in I_t$ we choose a set $Y_i \subseteq V(G) \setminus U_i$ uniformly at random, subject to $|(U_i \cup Y_i) \cap V_j^i| = \delta m'$ for all $j \in [Kr]$ (this is possible since, by (U2) above, we have $|U_i \cap V_j^i| \le |NU_i \cap V_j^i| \le \gamma^{2/5} m' \le \delta m'$ for all $i \in I_t$). Let

$$W_i := U_i \cup Y_i \quad \text{and} \quad Z_i := \phi_i^{-1}(W_i). \tag{6.8}$$

We define the following events.

(W1) For all $i \in I_t$ the graph $P^t[W_i]$ is $(\delta^{1/25}, \vec{\beta}')$-super-regular with respect to $(R_K, V_1^i \cap W_i, \dots, V_{Kr}^i \cap W_i)$.
(W2) For all $v \in V(G)$ we have
  (i) $|\{i \in I_t : v \in W_i\}| \le 2\delta\gamma n$,
  (ii) $\sum_{i \in I_t} |\{e \in E(H_i) : v \in \phi_i(e), \phi_i(e) \cap W_i \neq \emptyset\}| \le 3k\Delta_R \delta\gamma n$.
(W3) For all $i \in I_t$ and $j \in [Kr]$ the graph $F_i'[Z_i \cap X_j^i, W_i \cap V_j^i]$ is $(\delta^{1/20}, d_0 p(R_K, \vec{\beta}', j))$-super-regular.

So part (ii) of (W2) says that the total number of necessary changes at a given vertex in the current round will be small, and (W3) implies that ϕ_i is compatible with the structure of P^t, so we can use P^t to modify ϕ_i. If one of (W1)–(W3) fails, then end the algorithm with **failure of type 5**. Otherwise, proceed to Step 4.1.

Step 4.ℓ. Consider $i = (t-1)\gamma n + \ell$. Define

$$P_i^t := P^t - \bigcup_{j \in I_t : j < i} \phi_j'(E(H_j)). \tag{6.9}$$

Let $F_i^* \subseteq F_i'$ be defined by

$$E(F_i^*[X_j^i, V_j^i]) := \left\{ xv \in F_i' : x \in X_j^i, v \in V_j^i \cap \bigcap_{y \in N_x^i} N_{P_i^t}(\phi_i(y)) \right\}. \tag{6.10}$$

In other words, F_i^* is the maximal subgraph of F_i' that is an $(H_i, P_i^t, R_K, A_i, \phi_i, \mathcal{X}^i, \mathcal{V}^i, N^i)$-candidacy bigraph. It can be viewed as a further update of the candidacy bigraph F_i' which takes into account restrictions arising from the $\ell - 1$ embeddings ϕ_j' with $j \in I_t$, $j < i$ already made in the current round. Note that

$$P_{(t-1)\gamma n+1}^t = P^t \quad \text{and} \quad F_{(t-1)\gamma n+1}^* = F_{(t-1)\gamma n+1}'. \tag{6.11}$$

We now check whether the following conditions hold.

(a') For all $j \in [Kr]$ the graph $F_i^*[Z_i \cap X_j^i, W_i \cap V_j^i]$ is $(\delta^{1/50}, d_0 p(R_K, \vec{\beta}', j))$-super-regular.
(b') $P_i^t[W_i]$ is $(4\delta^{1/50}, \vec{\beta}')$-super-regular with respect to $(R_K, V_1^i \cap W_i, \dots, V_{Kr}^i \cap W_i)$.

If one of (a') and (b') does not hold, end the algorithm with **failure of type 6**. Otherwise, note that by (5.2), $p(R_K, \vec{\beta}', j) = p(R, \vec{\beta}, j)^K \geq \beta^{K\Delta_R}$ as $\beta = \min_{ij \in E(R)} \beta_{i,j}$. Thus Lemma 3.10 implies that we can find a spanning subgraph $F^i \subseteq F_i^*$ such that

(a'') $F^i[Z_i \cap X_j^i, W_i \cap V_j^i]$ is $(2\delta^{1/50}, d_0\beta^{K\Delta_R}/2)$-super-regular for each $j \in [Kr]$.

Let $A_i' := \bigcup_{j \in [Kr]} A_i[X_j^i, V_j^i]$. Note that since F^i is a spanning subgraph of F_i^*, F^i is an $(H_i, P_i^t, R_K, A_i', \phi_i, \mathcal{X}^i, \mathcal{V}^i, N^i)$-candidacy bigraph. We apply Lemma 5.13 with the following graphs and parameters to find a new embedding ϕ_i' of H_i into $G^t \cup P_i^t \subseteq G^t \cup P^t$ in order to make sure that the images of different ϕ_i' are edge-disjoint.

object/parameter	H_i	P_i^t	F^i	A_i'	N_i	$\mathcal{V}^i$	$\mathcal{X}^i$	R_K	$W_i \cap V_j^i$
playing the role of	H	P	F	A_0	N	$\mathcal{V}$	$\mathcal{X}$	R	W_j
object/parameter	m'	$\delta m'$	$4\delta^{1/50}$	β	$d_0\beta^{K\Delta_R}/2$	Kr	$K\Delta_R$	C	ϕ_i
playing the role of	n	m	δ	β	β'	r	k	C	ϕ

(Lemma 5.13 can be applied by (a'') and (b').) Then Lemma 5.13(i) and (iii) imply that $\phi_i'(x) = \phi_i(x)$ whenever $x \notin Z_i$ and $\phi_i'(x) \in N_{F^i}(x) \subseteq N_{A_i'}(x) \subseteq N_{A_i}(x)$ for all $x \in Z_i$. Together with (F1) this gives that

$$\phi'(x) \in N_{A_i}(x) \text{ for all } x \in V(H_i). \tag{6.12}$$

If $\ell < \gamma n$, proceed to Step 4.$(\ell+1)$. If $\ell = \gamma n$ and $t \leq T-1$, then define the graph

$$P^{t+1} := P^t - \bigcup_{i \in I_t} \phi_i'(E(H_i)), \tag{6.13}$$

and proceed to Round $t+1$. If $\ell = \gamma n$ and $t = T$, then we end the algorithm with success.

Let us now prove some properties of the above algorithm. The first one concerns the density $d_{j,j'}^t$ of $G^t[V_j, V_{j'}]$.

Claim 6.3. *For all $t \in [T]$ and all $jj' \in E(R)$ we have $d_{j,j'}^t \geq \alpha d/2$.*

Proof. Clearly, we may assume that $t \geq 2$. Then for $jj' \in E(R)$,

$$d_{j,j'}^t \overset{(6.2)}{\geq} d_{j,j'}^{t-1}\left(1 - \frac{\sum_{i \in I_{t-1}} k_{j,j'}^i}{d_{j,j'}^{t-1} n}\right) = d_{j,j'}^{t-1} - \frac{\sum_{i \in I_{t-1}} k_{j,j'}^i}{n}. \tag{6.14}$$

By (6.2) and (6.14), we get

$$\begin{aligned} d_{j,j'}^t \geq d_{j,j'}^T \geq d_{j,j'}^1 - \sum_{i=1}^{(t-1)\gamma n} \frac{k_{j,j'}^i}{n} &\overset{(S4)}{\geq} d_{j,j'}^1 - (1-\alpha)d_{j,j'} \geq (1-\beta/d)d_{j,j'} - (1-\alpha)d_{j,j'} \\ &\geq \alpha d_{j,j'}/2 \geq \alpha d/2. \end{aligned}$$

□

Part (ii) of next claim ensures that the super-regularity assumption (A2) in Step 1 is justified.

Claim 6.4. *Let $t \in [T]$. Assuming no failure prior to Round t, the following hold.*

(i) *For any vertex $v \in V(P)$, $d_{P-P^t}(v) \leq \delta^{3/5} n$.*
(ii) *P^t is $(\delta^{1/4}, \vec{\beta})$-super-regular with respect to $(R, \mathcal{V})$ and $(\delta^{1/4}, \vec{\beta}')$-super-regular with respect to $(R_K, \mathcal{V}^i)$.*
(iii) *For all $i \in I_t$ and all $jj' \in E(R_K)$, let*

$$SP_{j,j'}^t(i) := \left\{\{v, v'\} \in \binom{V_j^i}{2} : |N_{P^t}(\{v, v'\}) \cap V_{j'}^i| \neq (\beta_{j,j'}'^2 \pm 3\delta^{3/5})m'\right\}.$$

Then $|SP_{j,j'}^t(i)| \leq \gamma m'^2$.

Proof. Note that for each $v \in V(G)$

$$|\{e \in E(H_i) : v \in \phi_i'(e), \phi_i(e) \cap W_i \neq \emptyset\}| = |\{e \in E(H_i) : v \in \phi_i(e), \phi_i(e) \cap W_i \neq \emptyset\}| \pm \Delta_R,$$

where we need the $\pm\Delta_R$ only if $v \in W_i$. Moreover, since the algorithm has not ended with failure before, (W2) in Step 3 was satisfied in all Rounds t' with $t' < t$. Hence, for each $v \in V(G)$ and each $t' < t$

$$\begin{aligned} d_{P^{t'}-P^{t'+1}}(v) &\overset{(6.13)}{=} \sum_{i \in I_{t'}} |\{e \in E(H_i) : v \in \phi_i'(e), \phi_i(e) \cap W_i \neq \emptyset\}| \\ &\leq \sum_{i \in I_{t'}} |\{e \in E(H_i) : v \in \phi_i(e), \phi_i(e) \cap W_i \neq \emptyset\}| + \Delta_t |\{i \in I_{t'} : v \in W_i\}| \\ &\overset{(W2)}{\leq} 5k\Delta_R\delta\gamma n. \end{aligned} \tag{6.15}$$

Thus

$$d_{P-P^t}(v) \leq \sum_{t'=1}^{t-1} 5k\Delta_R\delta\gamma n \leq T \cdot 5k\Delta_R\delta\gamma n \overset{(6.1)}{\leq} \delta^{3/5}n. \tag{6.16}$$

The graph $P^1 = P$, defined in Round 0, is $(2\varepsilon, \vec{\beta})$-super-regular with respect to $(R, \mathcal{V})$ by construction, and so also $(\delta^{3/5}, \vec{\beta})$-super-regular since $\varepsilon \ll \delta$. Thus Proposition 3.9 implies that P^t is $(\delta^{1/4}, \vec{\beta})$-super-regular with respect to $(R, \mathcal{V})$. Similarly, (F4) and Proposition 3.9 together imply that P^t is $(\delta^{1/4}, \vec{\beta'})$-super-regular with respect to $(R_K, \mathcal{V}^i)$.

For all $i \in I_t$ and $jj' \in E(R_K)$, let

$$SP_{j,j'}(i) := \left\{ \{v, v'\} \in \binom{V_j^i}{2} : |N_P(\{v, v'\}) \cap V_{j'}^i| \neq (\beta_{j,j'}'^2 \pm 3\varepsilon_t^{1/3})m' \right\}.$$

Then (6.16) implies that $SP_{j,j'}^t(i) \subseteq SP_{j,j'}(i)$. Also (F4) and Proposition 3.12 together imply that $|SP_{j,j'}(i)| \leq \varepsilon_t^{1/3}m'^2$. Thus $|SP_{j,j'}^t(i)| \leq \varepsilon_t^{1/3}m'^2 \leq \gamma m'^2$. This completes the proof of Claim 6.4. □

The following claim implies that the embedded copies of $H_1, \dots, H_s$ are indeed edge-disjoint.

Claim 6.5. *Assume that in Round t we have constructed the embeddings ϕ_i' for all $i \in I_t$. Then the following statements hold.*

(i) $\bigcup_{i \in I_t} \phi_i'(E(H_i)) \subseteq E(G^t) \cup E(P^t)$,
(ii) $\phi_i'(E(H_i)) \cap \phi_{i'}'(E(H_{i'})) = \emptyset$ *for all distinct* $i, i' \in I_t$,
(iii) $\bigcup_{i \in I_t} \phi_i'(E(H_i)) \cap (E(G^{t+1}) \cup E(P^{t+1})) = \emptyset$.
(iv) $\phi_i'(x) \in N_{A_i}(x)$.

Proof. By Lemma 5.13, if $e = uv \in \phi_i'(E(H_i)) \setminus E(G^t)$, then $uv \in E(P_i^t) \subseteq E(P^t)$. Thus (i) holds. Statements (ii) and (iii) are immediate consequences of (6.5), (6.9) and (6.13). (iv) follows from (6.12). □

Claim 6.6. *Let $t \in [T]$. Assume that the main packing algorithm did not fail prior to Round t and that within Round t failure of type 1 did not occur. Then for all $i \in I_t$, $j \in [Kr]$ and any vertex $z \in V_j^i \cup X_j^i$,*

$$d_{F_i'}(z) = (d_0 p(R_K, \vec{\beta}', j) \pm \delta^{1/2})m'.$$

Moreover, let

$$S_j(i) := \left\{ \{x, x'\} \in \binom{X_j^i}{2} : |N_{F_i'}(\{x, x'\})| \neq ((d_0 p(R_K, \vec{\beta}', j))^2 \pm \delta^{1/2})m' \right\}.$$

Then $|S_j(i)| \leq \gamma m'^2$.

Proof. By (6.7), for all $x \in X^i_j$ and $v \in V^i_j$, if $xv \in E(F_i - F'_i)$ then there exists $y \in N^i_x$ such that $\phi_i(y)v \in E(P - P^t)$. Since $d_{P-P^t}(\phi_i(y)) \le \delta^{3/5}n$ by Claim 6.4(i),

$$d_{F_i-F'_i}(x) \le |N^i_x|\delta^{3/5}n \overset{(F3)}{\le} K\Delta_R\delta^{3/5}n. \tag{6.17}$$

Similarly, for all $x \in X^i_j$ and $v \in V^i_j$, if $xv \in E(F_i - F'_i)$ then

$$v \in \bigcup_{y\in N^i_x} N_{P-P^t}(\phi_i(y)) \Leftrightarrow v \in \bigcup_{y\colon x\in N^i_y} N_{P-P^t}(\phi_i(y)) \Leftrightarrow x \in \bigcup_{u\in N_{P-P^t}(v)} N^i_{\phi_i^{-1}(u)}. \tag{6.18}$$

Since $d_{P-P^t}(v) \le \delta^{3/5}n$, we have $d_{F_i-F'_i}(v) \le K\Delta_R\delta^{3/5}n$. Together with (6.17), this implies that

$$\Delta(F_i - F'_i) \le K\Delta_R\delta^{3/5}n. \tag{6.19}$$

By (6.4), $f(\varepsilon_t) < \gamma < \delta^{1/2}/2$. Thus for each vertex $z \in V^i_j \cup X^i_j$,

$$d_{F'_i}(z) = d_{F_i}(z) \pm K\Delta_R\delta^{3/5}n \overset{(F5)}{=} (d_0p(R_K, \vec{\beta}', j) \pm f(\varepsilon_t))m' \pm K\Delta_R\delta^{3/5}n = (d_0p(R_K, \vec{\beta}', j) \pm \delta^{1/2})m'.$$

Let

$$S_{i,j} := \left\{\{x, x'\} \in \binom{X^i_j}{2} : |N_{F_i}(\{x, x'\})| \ne ((d_0p(R_K, \vec{\beta}', j))^2 \pm 3f(\varepsilon_t))m'\right\}.$$

Then (6.19) implies that $S_j(i) \subseteq S_{i,j}$. Also (F5) together with Proposition 3.12 implies that $|S_{i,j}| \le f(\varepsilon_t)m'^2$. Thus $|S_j(i)| \le f(\varepsilon_t)m'^2 \le \gamma m'^2$. □

We will now show that each type of failure occurs with very small probability.

6.1. **Failure of type 1.**

Claim 6.7. *Failure of type 1 occurs with probability at most* $(1 - c)^n$.

Proof. Suppose $t \in [T]$ and we are in Round t. Then we can assume that there was no failure in a previous round. So by the definition of failure of type 3 (if $t > 1$) or as observed in Round 0 (if $t = 1$), the graph G^t is $(\varepsilon_t, \vec{d^t})$-super-regular with respect to $(R, V_1, \dots, V_r)$. Moreover, as stated in Round 0, P is $(\varepsilon_1, \vec{\beta})$-super-regular with respect to $(R, V_1, \dots, V_r)$. Since by Claim 6.3 and (6.4) we have $\varepsilon_t \le \varepsilon_T \ll \gamma \ll \alpha d/2 \le d^t_{j,j'}$ for all $jj' \in E(R)$, Lemma 5.1 can be applied (with $2c$ playing the role of c), and for each $i \in I_t$ the Uniform embedding algorithm fails with probability at most $(1 - 2c)^n$.

Since the number of times we apply the Uniform embedding algorithm over all rounds is at most $s \le \eta^{-1}n$, the probability that failure of type 1 ever occurs is at most $\eta^{-1}n(1 - 2c)^n \le (1 - c)^n$. □

6.2. **Failure of type 2.**

We define the following event.

(QW'_i) For all $j \in [r]$ and any sets $Q \in \mathcal{Q}_j$ and $W \in \mathcal{W}_j$, we have $|\phi'_i(Q) \cap W| = (1 \pm \eta')\frac{|Q||W|}{n}$.

Claim 6.8. *Failure of type 2 occurs with probability at most* $(1-c)^n$. *Moreover,* (QW_i) *implies* (QW'_i).

Proof. For given $Q \in \mathcal{Q}_j, W \in \mathcal{W}_j$, $i \in [s]$, let $\mathcal{E}_i(Q, W)$ be the event that $|\phi_i(Q) \cap W| = (1 \pm \gamma)\frac{|Q||W|}{n}$. Then we have

$$(\mathrm{QW}_i) = \bigwedge_{j\in[r], Q\in\mathcal{Q}_j, W\in\mathcal{W}_j} \mathcal{E}_i(Q, W).$$

Note that (B6) of Lemma 5.1 together with the fact that $f(\varepsilon_t) \ll \gamma$ imply that for given $i \in I_t$, $Q \in \mathcal{Q}_j, W \in \mathcal{W}_j$ we have

$$\mathbb{P}[\mathcal{E}_i(Q, W)] \ge 1 - (1 - 2c)^n.$$

A union bound and (S7) imply that

$$\mathbb{P}[(\mathrm{QW}_i)] \geq 1 - \sum_{j\in[r]} |\mathcal{Q}_j||\mathcal{W}_j|(1-2c)^n \geq 1 - r(1+c)^n(1-2c)^n. \tag{6.20}$$

Since $1 \leq i \leq s$, there are $s \leq \eta^{-1}n$ possible values for which the failure of type 2 can occur. Therefore, failure of type 2 ever occurs with probability at most $rs(1+c)^n(1-2c)^n < (1-c)^n$.

Now we show the 'moreover' part of Claim 6.8. Let us assume that (QW_i) holds. Note that for $j \in [r]$,

$$|\{x \in X_j : \phi_i'(x) \neq \phi_i(x)| \leq |W_i| \overset{(6.8)}{\leq} \delta n. \tag{6.21}$$

So for any sets $Q \in \mathcal{Q}_j, W \in \mathcal{W}_j$ we have

$$|\phi_i'(Q) \cap W| \overset{(6.21)}{=} |\phi_i(Q) \cap W| \pm \delta n \overset{(\mathrm{QW}_i)}{=} (1\pm\gamma)\frac{|Q||W|}{n} \pm \delta n = (1\pm\eta')\frac{|Q||W|}{n}.$$

We obtain the final identity since $|Q|, |W| \geq \eta' n$ and $\gamma \ll \delta \ll \eta'$. Thus (QW_i) implies (QW_i'). □

6.3. **Failure of type 3.** In this subsection we show that failure of type 3 occurs with small probability. We will achieve this by checking that the degrees and codegrees of G^t decrease by the expected amount when constructing G^{t+1}. Then we invoke Theorem 3.11.

Claim 6.9. *Assume that we are in Step 1 of Round $t \in [T]$ and condition on there being no failure of type 1 in this step (in particular, G^t is $(\varepsilon_t, \vec{d}^t)$-super-regular with respect to $(R, V_1, \dots, V_r)$). Then, with probability at least $1-(1-2c)^n$, G^{t+1} is $(\varepsilon_{t+1}, \vec{d}^{t+1})$-super-regular with respect to $(R, V_1, \dots, V_r)$.*

Proof. By (6.3), this means that we need to show that with high probability $G^{t+1}[V_j, V_{j'}]$ is $(q(\varepsilon_t), d_{j,j'}^{t+1})$-super-regular for all $jj' \in E(R)$.

Fix an outcome of the algorithm running for Rounds $1, \dots, t-1$. (For the proof of this claim, all probabilities and expectations are conditioned on this outcome.) We also condition on there being no failure of type 1 in Step 1. Let S be a set of s' vertices in $V_{j'}$ with $s' \leq 2$, and let $N_j(S) := N_{G^t}(S) \cap V_j$. Also we assume that S satisfies

$$|N_j(S)| = ((d_{j,j'}^t)^{s'} \pm 3\varepsilon_t)n. \tag{6.22}$$

We are applying the Uniform embedding algorithm to G^t and H_i for $i \in I_t$ to obtain embeddings ϕ_i for all $i \in I_t$. Then

$$N_{G^{t+1}}(S) \cap V_j = \{u \in N_j(S) : vu \notin \phi_i(E(H_i)) \text{ for all } i \in I_t \text{ and all } v \in S\}.$$

For $i \in I_t \cup \{(t-1)\gamma n\}$, let Q_i be the random variable defined by

$$Q_i := \mathbb{E}\left[|N_{G^{t+1}}(S) \cap V_j| \mid \phi_{(t-1)\gamma n+1}, \dots, \phi_i\right].$$

Note that this is an exposure martingale. Also, if we change one $\phi_{i'}$, then the value of $|N_{G^{t+1}}(S) \cap V_j|$ changes by at most $s'(k+1)$, thus

$$|Q_{i+1} - Q_i| \leq s'(k+1). \tag{6.23}$$

In other words, the martingale Q_i is $s'(k+1)$-Lipschitz. Our aim now is to compute $\mathbb{E}[Q_{t\gamma n}] = \mathbb{E}[|N_{G^{t+1}}(S) \cap V_j|]$ and then to apply Azuma's inequality. For each $u \in N_j(S)$ and $i \in I_t$, let B_u^i be the random variable such that

$$B_u^i := \begin{cases} 1 & \text{if } vu \in \phi_i(E(H_i)) \text{ for some } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$Q_{t\gamma n} = |N_{G^{t+1}}(S) \cap V_j| = \sum_{u\in N_j(S)} \prod_{i\in I_t} (1 - B_u^i).$$

Let $p_{i,u} := \mathbb{P}[B_u^i = 1]$. Then the fact that $\phi_i, \phi_{i'}$ are independent for $i \neq i'$ implies that

$$Q_{(t-1)\gamma n} = \mathbb{E}[Q_{t\gamma n}] = \sum_{u \in N_j(S)} \prod_{i \in I_t} (1 - p_{i,u}). \tag{6.24}$$

We will show that $p_{i,u}$ is usually very close to $s' k^i_{j,j'}/(d^t_{j,j'} n)$. For this, let $A, B \subseteq N_j(S) \times I_t$ be defined as follows.

$$A := \left\{ (u,i) : p_{i,u} > \frac{s' k^i_{j,j'}}{d^t_{j,j'} n}(1 + 2f(\varepsilon_t)) \right\} \text{ and } B := \left\{ (u,i) : p_{i,u} < \frac{s' k^i_{j,j'}}{d^t_{j,j'} n}(1 - 2f(\varepsilon_t)) \right\}. \tag{6.25}$$

Let A_i be the set of vertices $u \in N_j(S)$ such that $(u,i) \in A$, and let B_i be defined analogously. Let A^*_u be the set of indices i such that $(u,i) \in A$ and define B^*_u analogously. Then $|A_i|, |B_i| < 2f(\varepsilon_t)n$ by (B5) of Lemma 5.1. Therefore

$$|A| = \sum_{i \in I_t} |A_i| < 2f(\varepsilon_t)\gamma n^2 < f(\varepsilon_t) n^2, \tag{6.26}$$

and similarly

$$|B| < f(\varepsilon_t) n^2. \tag{6.27}$$

Moreover, we have

$$\sum_{u \in A_i} p_{i,u} \leq \frac{4s' f(\varepsilon_t) k^i_{j,j'}}{d^t_{j,j'}}. \tag{6.28}$$

Indeed, if not, consider a set $A' \subseteq N_j(S)$ of size $2f(\varepsilon_t)n$ with $A_i \subseteq A'$ and obtain a contradiction via

$$\begin{aligned} \frac{4s' f(\varepsilon_t) k^i_{j,j'}}{d^t_{j,j'}} &< \sum_{u \in A_i} p_{i,u} \leq \sum_{u \in A'} p_{i,u} = \mathbb{E}\left[|\{u \in A' : vu \in \phi_i(E(H_i)) \text{ for some } v \in S\}| \right] \\ &\leq (1 + f(\varepsilon_t)) \frac{k^i_{j,j'} |A'|}{d^t_{j,j'} n} s' < \frac{3s' f(\varepsilon_t) k^i_{j,j'}}{d^t_{j,j'}}. \end{aligned}$$

Note that the second line follows from (B1) of Lemma 5.1. We also obtain

$$\sum_{u \in B_i} p_{i,u} \leq \frac{s' k^i_{j,j'}}{d^t_{j,j'} n} |B_i| \leq \frac{2s' f(\varepsilon_t) k^i_{j,j'}}{d^t_{j,j'}}, \tag{6.29}$$

because $p_{i,u} < \frac{s' k^i_{j,j'}}{d^t_{j,j'} n}$ for $u \in B_i$ and $|B_i| < 2f(\varepsilon_t)n$. Thus

$$\begin{aligned} \sum_{u \in N_j(S)} \sum_{i \in A^*_u} p_{i,u} &= \sum_{i \in I_t} \sum_{u \in A_i} p_{i,u} \overset{(6.28)}{\leq} 4s' f(\varepsilon_t) k^i_{j,j'} \gamma n / d^t_{j,j'} < f(\varepsilon_t) n, \\ \sum_{u \in N_j(S)} \sum_{i \in B^*_u} p_{i,u} &= \sum_{i \in I_t} \sum_{u \in B_i} p_{i,u} \overset{(6.29)}{\leq} 2s' f(\varepsilon_t) k^i_{j,j'} \gamma n / d^t_{j,j'} < f(\varepsilon_t) n. \end{aligned} \tag{6.30}$$

Let C be the set of vertices $u \in N_j(S)$ satisfying at least one of $\sum_{i \in A^*_u} p_{i,u} > f(\varepsilon_t)^{1/2}$ and $\sum_{i \in B^*_u} p_{i,u} > f(\varepsilon_t)^{1/2}$. By (6.30), we have

$$|C| \leq 2f(\varepsilon_t)^{1/2} n. \tag{6.31}$$

Then $u \in N_j(S) \setminus C$ implies that

$$\sum_{i \in A^*_u} p_{i,u} \leq f(\varepsilon_t)^{1/2} \quad \text{and} \quad \sum_{i \in B^*_u} p_{i,u} \leq f(\varepsilon_t)^{1/2}. \tag{6.32}$$

Similarly, let D be the set of vertices $u \in N_j(S)$ such that $|A_u^*| > f(\varepsilon_t)^{1/2}n$ or $|B_u^*| > f(\varepsilon_t)^{1/2}n$. Then, by (6.26) and (6.27),

$$|D| \le 2f(\varepsilon_t)^{1/2}n. \tag{6.33}$$

Now, consider any vertex $u \in N_j(S) \setminus (C \cup D)$. Since $u \notin D$,

$$|A_u^* \cup B_u^*| \le 2f(\varepsilon_t)^{1/2}n. \tag{6.34}$$

Moreover, for every $u \in N_j(S) \setminus (C \cup D)$, we have (note that A_u^*, B_u^* are disjoint)

$$\begin{aligned}\prod_{i\in I_t}(1-p_{i,u}) &= \prod_{i\in I_t\setminus(A_u^*\cup B_u^*)}(1-p_{i,u})\prod_{i\in A_u^*}(1-p_{i,u})\prod_{i\in B_u^*\setminus A_u^*}(1-p_{i,u})\\ &= \prod_{i\in I_t\setminus(A_u^*\cup B_u^*)}\left(1-(1\pm 2f(\varepsilon_t))\frac{s'k_{j,j'}^i}{d_{j,j'}^t n}\right)\prod_{i\in A_u^*}(1-p_{i,u})\prod_{i\in B_u^*}(1-p_{i,u}).\end{aligned} \tag{6.35}$$

Estimating the above products by sums yields

$$\begin{aligned}\prod_{i\in A_u^*}(1-p_{i,u})\prod_{i\in B_u^*}(1-p_{i,u}) &= \left(1\pm\sum_{i\in A_u^*}p_{i,u}\right)\left(1\pm\sum_{i\in B_u^*}p_{i,u}\right) \overset{(6.32)}{=} (1\pm f(\varepsilon_t)^{1/2})^2\\ &= 1\pm f(\varepsilon_t)^{1/3}.\end{aligned} \tag{6.36}$$

Recall that $d_{j,j'}^t \ge \alpha d/2$ by Claim 6.3. Combining (6.35) and (6.36), we obtain

$$\begin{aligned}\prod_{i\in I_t}(1-p_{i,u}) &= (1\pm f(\varepsilon_t)^{1/3})\prod_{i\in I_t\setminus(A_u^*\cup B_u^*)}\left(1-(1\pm 2f(\varepsilon_t))\frac{s'k_{j,j'}^i}{d_{j,j'}^t n}\right)\\ &= (1\pm f(\varepsilon_t)^{1/4})\left(1\pm 3f(\varepsilon_t)\frac{2s'k}{\alpha dn}\right)^{\gamma n}\left(\prod_{i\in A_u^*\cup B_u^*}\left(1-\frac{s'k_{j,j'}^i}{d_{j,j'}^t n}\right)\right)^{-1}\prod_{i\in I_t}\left(1-\frac{s'k_{j,j'}^i}{d_{j,j'}^t n}\right)\\ &= (1\pm f(\varepsilon_t)^{1/4})\left(1\pm 3f(\varepsilon_t)\frac{2s'k}{\alpha dn}\right)^{\gamma n}\left(1\pm\frac{3s'k}{\alpha dn}\right)^{|A_u^*\cup B_u^*|}\prod_{i\in I_t}\left(1-\frac{s'k_{j,j'}^i}{d_{j,j'}^t n}\right)\\ &\overset{(6.34)}{=} (1\pm 2f(\varepsilon_t)^{1/4})\left(\prod_{i\in I_t}\left(1-\frac{k_{j,j'}^i}{d_{j,j'}^t n}\right)\right)^{s'}.\end{aligned}$$

Taking the sets C and D into account, we get

$$\begin{aligned}\mathbb{E}[Q_{t\gamma n}] &\overset{(6.24)}{=} (|N_j(S)|-|C|-|D|)(1\pm 2f(\varepsilon_t)^{1/4})\left(\prod_{i\in I_t}\left(1-\frac{k_{j,j'}^i}{d_{j,j'}^t n}\right)\right)^{s'} \pm (|C|+|D|)\\ &\overset{(6.31),(6.33)}{=} (1\pm 2f(\varepsilon_t)^{1/4})\left(\prod_{i\in I_t}\left(1-\frac{k_{j,j'}^i}{d_{j,j'}^t n}\right)\right)^{s'}|N_j(S)| \pm 10f(\varepsilon_t)^{1/2}n\\ &\overset{(6.22)}{=} (1\pm f(\varepsilon_t)^{1/5})\left(\prod_{i\in I_t}\left(1-\frac{k_{j,j'}^i}{d_{j,j'}^t n}\right)\right)^{s'}|N_j(S)|.\end{aligned}$$

So, applying Azuma's inequality (Theorem 3.2) to Q_i with $i \in I_t \cup \{(t-1)\gamma n\}$, which is $s'(k+1)$-Lipschitz by (6.23), we get

$$\mathbb{P}\left[Q_{t\gamma n} \ne \left(1\pm 2f(\varepsilon_t)^{1/5}\right)\left(\prod_{i\in I_t}\left(1-\frac{k_{j,j'}^i}{d_{j,j'}^t n}\right)\right)^{s'}|N_j(S)|\right] < (1-3c)^n$$

since $c \ll \varepsilon$. This holds for all sets $S \subseteq V_{j'}$ of $s' \le 2$ vertices with $|N_j(S)| = ((d_{j,j'}^t)^{s'} \pm 3\varepsilon_t)n$.

Let $D_{j,j'} := \{\{u,v\} \in \binom{V_{j'}}{2} : |N_{G^t}(\{u,v\}) \cap V_j| = ((d^t_{j,j'})^2 \pm 3\varepsilon_t)n\}$. Since $G^t[V_j, V_{j'}]$ is $(\varepsilon_t, d^t_{j,j'})$-super-regular, Proposition 3.12 implies that

$$|D_{j,j'}| \geq \binom{n}{2} - \varepsilon_t n^2 \geq \frac{1}{2}(1-2\varepsilon_t)n^2 \overset{(3.1)}{\geq} \left(\frac{1}{2} - 5q(\varepsilon_t)^6\right)n^2.$$

Then with probability at least $1 - r^2(n + |D_{j,j'}|)(1-3c)^n \geq 1 - (1-2c)^n$, for all $j, j' \in [r]$ and any $S \in D_{j,j'}$ we have

$$\begin{aligned} |N_{G^{t+1}}(S) \cap V_j| &= (1 \pm 2f(\varepsilon_t)^{1/5}) \left(\prod_{i \in I_t}\left(1 - \frac{k^i_{j,j'}}{d^t_{j,j'}n}\right)\right)^2 ((d^t_{j,j'})^2 \pm 3\varepsilon_t)n \\ &\overset{(3.1)}{=} (1 \pm q(\varepsilon_t)^6)\left(d^t_{j,j'}\prod_{i \in I_t}\left(1 - \frac{k^i_{j,j'}}{d^t_{j,j'}n}\right)\right)^2 n \\ &\overset{(6.2)}{=} (1 \pm \varepsilon^6_{t+1})(d^{t+1}_{j,j'})^2 n, \end{aligned}$$

and for any $v \in V_{j'}$

$$|N_{G^{t+1}}(v) \cap V_j| = (1 \pm q(\varepsilon_t)^6)\left(d^t_{j,j'}\prod_{i \in I_t}\left(1 - \frac{k^i_{j,j'}}{d^t_{j,j'}n}\right)\right) n \overset{(6.2)}{=} (1 \pm \varepsilon^6_{t+1})d^{t+1}_{j,j'}n.$$

This together with Theorem 3.11 implies that G^{t+1} is $(\varepsilon_{t+1}, \vec{d}^{t+1})$-super-regular with probability at least $1 - (1-2c)^n$. □

Claim 6.10. *Failure of type 3 occurs with probability at most* $(1-c)^n$.

Proof. By Claim 6.9, for every $t \in [T]$, conditioned on no previous failure, G^t is $(\varepsilon_t, \vec{d}^t)$-super-regular with probability at least $1-(1-2c)^n$. Thus failure of type 3 ever occurs with probability at most $T(1-2c)^n < (1-c)^n$. □

6.4. **Failure of type 4.**

Claim 6.11. *Failure of type* 4 *occurs with probability at most* $(1-c)^n$.

Proof. Suppose we are in Round t, Step 2 and that ϕ_i for $i \in I_t$ are the embeddings we define in Round t, Step 1. We prove the following two subclaims.

Subclaim 1. *In Round t (U1) fails with probability at most* $2rn(1-2c)^n$.

To prove Subclaim 1, fix $v \in V(G)$ and $i \in I_t$. Let $G^{t,<}(i)$ be the spanning subgraph of G^t with edge set

$$E(G^{t,<}(i)) := \bigcup_{i' \in I_t, i' < i} \phi_{i'}(E(H_{i'})),$$

and, similarly, let $G^{t,>}(i)$ be the spanning subgraph of G^t be defined by

$$E(G^{t,>}(i)) := \bigcup_{i' \in I_t, i' > i} \phi_{i'}(E(H_{i'})).$$

Let

$$I_{v,i} := \begin{cases} 1 & \text{if } v \in (\phi_i)_2(H_i, G^t, G^{t,<}(i)), \\ 0 & \text{otherwise,} \end{cases}$$

where $(\phi_i)_2(H_i, G^t, G^{t,<}(i))$ is as defined at the beginning of Section 5. Similarly, let

$$J_{v,i} := \begin{cases} 1 & \text{if } v \in (\phi_i)_2(H_i, G^t, G^{t,>}(i)), \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $\Delta(H_{i'}) \leq (k+1)\Delta_R$ for all $i \in I_t$, we have

$$\Delta(G^{t,<}(i)), \Delta(G^{t,>}(i)) \leq 2k\Delta_R\gamma n. \tag{6.37}$$

Recall that in Round t, Step 1, we apply Lemma 5.1 with $4k\Delta_R\gamma$ playing the role of γ for each $i \in I_t$ independently to obtain embeddings ϕ_i. Thus, by (B4.1) of Lemma 5.1, for any $x_{(t-1)\gamma n+1}, \dots, x_{i-1} \in \{0,1\}$ we have

$$\mathbb{P}[I_{v,i} = 1 \mid I_{v,(t-1)\gamma n+1} = x_{(t-1)\gamma n+1}, \dots, I_{v,i-1} = x_{i-1}] \le (4k\Delta_R\gamma)^{1/2} \le \gamma^{1/3}/4.$$

Let X have binomial distribution with parameters $(\gamma n, \gamma^{1/3}/4)$, so $\mathbb{E}[X] = \gamma^{4/3}n/4$. Then, by Proposition 3.4 and Lemma 3.3,

$$\mathbb{P}\left[\sum_{i\in I_t} I_{v,i} \ge \gamma^{4/3}n/2\right] \le \mathbb{P}\left[X \ge 2\mathbb{E}[X]\right] \le (1-2c)^n. \tag{6.38}$$

A symmetric argument for $J_{v,i}$ (considering the reverse-ordering of $[\gamma n]$) yields

$$\mathbb{P}\left[\sum_{i\in I_t} J_{v,i} \ge \gamma^{4/3}n/2\right] \le \mathbb{P}\left[X \ge 2\mathbb{E}[X]\right] \le (1-2c)^n. \tag{6.39}$$

Since $v \in NU_i$ implies that $I_{v,i} > 0$ or $J_{v,i} > 0$, we get

$$|\{i \in I_t : v \in NU_i\}| \le \sum_{i\in I_t}(I_{v,i} + J_{v,i}).$$

Thus, by (6.38) and (6.39), for any given $v \in V(G)$,

$$\mathbb{P}\left[|\{i \in I_t : v \in NU_i\}| > \gamma^{4/3}n\right] \le \mathbb{P}\left[\sum_{i\in I_t} I_{v,i} + \sum_{i\in I_t} J_{v,i} \ge \gamma^{4/3}n\right] \le 2(1-2c)^n.$$

Thus, with probability at least $1 - 2rn(1-2c)^n$, (U1) holds in Round t for all vertices. This completes the proof of Subclaim 1.

Subclaim 2. *(U2) fails with probability at most $\gamma n(1-2c)^n$.*

To prove Subclaim 2, let $G^{t,<}(i)$ and $G^{t,>}(i)$ be the graphs defined in Subclaim 1. Let $\mathcal{E}_i$ be the event that

$$|\{u \in V_j^i : u \in (\phi_i)_2(H_i, G^t, G^{t,<}(i) \cup G^{t,>}(i)\}| \le \gamma^{2/5}m' \text{ for all } j \in [Kr].$$

Since $\Delta(G^{t,<}(i) \cup G^{t,>}(i)) \le 4k\Delta_R\gamma n$ by (6.37) and the random variable ϕ_i is independent from the random variable $G^{t,<}(i) \cup G^{t,>}(i)$, (B4.2) of Lemma 5.1 together with a union bound imply that

$$\mathbb{P}\left[\bigwedge_{i\in I_t} \mathcal{E}_i\right] \ge 1 - \gamma n(1-2c)^n.$$

(Note that $(4k\Delta_R\gamma)^{3/5}n \le \gamma^{2/5}m'$.) If $\mathcal{E}_i$ occurs for all $i \in I_t$, then $|NU_i \cap V_j^i| \le \gamma^{2/5}m'$ holds for all $i \in I_t$ and $j \in [Kr]$. Thus (U2) fails with probability at most $\gamma n(1-2c)^n$. This completes the proof of Subclaim 2.

To summarise, in Round t, (U1) fails with probability at most $2rn(1-2c)^n$, and (U2) fails with probability $\gamma n(1-2c)^n$. Thus the probability that either one of them ever fails is at most $T(2rn(1-2c)^n + \gamma n(1-2c)^n) < (1-c)^n$. □

6.5. **Failure of type 5.**

Claim 6.12. *Failure of type 5 occurs with probability at most $(1-c)^n$.*

Proof. Suppose that the algorithm has reached Round t, Step 3, and no previous failure has occurred. In particular, (U1),(U2) of Step 2 hold.

First, we show that (W1) holds with high probability. Let $i \in I_t$ and $j \ne j' \in [Kr]$ with $jj' \in E(R_K)$ be fixed. Since by (U2) we have

$$|U_i \cap V_j^i| \le |NU_i \cap V_j^i| \le \gamma^{2/5}m', \tag{6.40}$$

it follows that

$$|Y_i \cap V_j^i| = \delta m' - |U_i \cap V_j^i| \overset{(6.40)}{\geq} \delta m' - \gamma^{2/5} m'. \tag{6.41}$$

Thus, (6.40) and (6.41) together imply that for any given vertex $v \in V_j^i \setminus U_i$ we have

$$\mathbb{P}[v \in W_i] = \mathbb{P}[v \in Y_i] = \frac{|Y_i \cap V_j^i|}{|V_j^i \setminus U_i|} = (1 \pm \gamma^{1/3})\delta. \tag{6.42}$$

Let

$$\overline{SP}^t_{j,j'}(i) := \left\{ \{v, v'\} \in \binom{V_j^i}{2} : |N_{P^t}(\{v, v'\}) \cap V_{j'}^i| = (\beta'^2_{j,j'} \pm 3\delta^{3/5})m' \right\}.$$

Let $S \subseteq V_j^i$ be such that either $|S| = 1$ or S consists of a pair of vertices in $\overline{SP}^t_{j,j'}(i)$. Then the definition of $\overline{SP}^t_{j,j'}(i)$ together with the fact that by Claim 6.4(ii) P^t is $(\delta^{1/4}, \vec{\beta}')$-super-regular with respect to $(R_K, \mathcal{V}^i)$ implies that $|N_{P^t}(S) \cap V_{j'}^i| = (\beta'^{|S|}_{j,j'} \pm 2\delta^{1/4})m'$. It follows that

$$|(N_{P^t}(S) \cap V_{j'}^i) \setminus U_i| \overset{(6.40)}{=} (\beta'^{|S|}_{j,j'} \pm 2\delta^{1/4} \pm \gamma^{2/5})m'.$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\left|N_{P^t}(S) \cap V_{j'}^i \cap Y_i\right|\right] &= (1 \pm \gamma^{1/3})\delta \left|(N_{P^t}(S) \cap V_{j'}^i) \setminus U_i\right| \\ &= (1 \pm \gamma^{1/3})\delta(\beta'^{|S|}_{j,j'} \pm 2\delta^{1/4} \pm \gamma^{2/5})m' = \left(\beta'^{|S|}_{j,j'} \pm 3\delta^{1/4}\right)\delta m'.\end{aligned}$$

By the choice of Y_i, the above random variable is hypergeometrically distributed. Thus by Lemma 3.3,

$$\begin{aligned}\mathbb{P}\left[|N_{P^t}(S) \cap V_{j'}^i \cap W_i| = (\beta'^{|S|}_{j,j'} \pm 5\delta^{1/4})\delta m'\right] &\overset{(6.40)}{\geq} \mathbb{P}\left[|N_{P^t}(S) \cap V_{j'}^i \cap Y_i| = \left(\beta'^{|S|}_{j,j'} \pm 4\delta^{1/4}\right)\delta m'\right] \\ &\geq 1 - (1-2c)^n. \end{aligned} \tag{6.43}$$

Moreover,

$$|W_i \cap V_j^i| = |W_i \cap V_{j'}^i| = \delta m', \tag{6.44}$$

and

$$\left|\binom{W_i \cap V_j^i}{2} \cap \overline{SP}^t_{j,j'}(i)\right| \geq \binom{\delta m'}{2} - \gamma m'^2 \geq (\frac{1}{2} - 25\delta^{1/4})(\delta m')^2$$

by Claim 6.4(iii). So Theorem 3.11 together with (6.43) and a union bound over all $S \in V_j^i \cup \overline{SP}^t_{j,j'}(i)$ implies that $P^t[W_i \cap V_j^i, W_i \cap V_{j'}^i]$ is $(\delta^{1/25}, \beta'_{j,j})$-regular with probability at least $1 - m'^2(1-2c)^n$.

Recall that (W1) holds if and only if $P^t[W_i \cap V_j^i, W_i \cap V_{j'}^i]$ is $(\delta^{1/25}, \beta'_{j,j'})$-super-regular for all $i \in I_t$ and $jj' \in E(R_K)$. So a union bound over all i, j, j' gives

$$\mathbb{P}[\text{(W1) fails in Round } t] \overset{(6.43)}{\leq} \gamma n (Kr)^2 m'^2 (1-2c)^n. \tag{6.45}$$

Secondly, we show that (W2)(i) holds with high probability. Fix $v \in V(G)$. Note that (6.42) together with Proposition 3.4 and Lemma 3.3 implies that with probability at least $1-(1-2c)^n$ we have $|\{i \in I_t : v \in Y_i\}| \leq 3\delta|I_t|/2 = 3\delta\gamma n/2$. Moreover, by (U1) we have

$$|\{i \in I_t : v \in U_i\}| \leq |\{i \in I_t : v \in NU_i\}| \leq \gamma^{4/3} n. \tag{6.46}$$

Since $|\{i \in I_t : v \in W_i\}| = |\{i \in I_t : v \in U_i\}| + |\{i \in I_t : v \in Y_i\}|$, together with a union bound over all $v \in V(G)$ this implies that

$$\mathbb{P}[\text{(W2)(i) fails in Round } t] \leq rn(1-2c)^n. \tag{6.47}$$

Next we show that (W2)(ii) holds with high probability. Fix $v \in V(G)$ and consider

$$C(v) := \sum_{i \in I_t} |\{e \in E(H_i) : v \in \phi_i(e), \phi_i(e) \cap W_i \neq \emptyset\}|.$$

For all integers $p \in I_t \cup \{(t-1)\gamma n\}$ define

$$X_p(v) := \mathbb{E}\left[C(v) \mid Y_{(t-1)\gamma n+1}, \dots, Y_p\right],$$

and note that $X_p(v)$ is an exposure martingale. Moreover, changing Y_p changes $C(v)$ by at most $(k+1)\Delta_R$, so the martingale is $(k+1)\Delta_R$-Lipschitz. Thus

$$\begin{aligned}
\mathbb{E}[C(v)] &\leq \sum_{i\in I_t: v\notin NU_i} \left((k+1)\Delta_R \mathbb{P}[v \in W_i] + \sum_{uv\in\phi_i(E(H_i))} \mathbb{P}[u \in W_i] \right) \\
&+ (k+1)\Delta_R |\{i \in I_t : v \in NU_i\}| \\
&\overset{(6.42,6.46)}{\leq} \sum_{i\in I_t:\, v\notin NU_i} \left((k+1)\Delta_R(1+\gamma^{1/3})\delta + \sum_{uv\in\phi_i(E(H_i))} (1+\gamma^{1/3})\delta \right) + (k+1)\Delta_R\gamma^{4/3}n \\
&\leq 2(k+1)\Delta_R(1+\gamma^{1/3})\delta\,(\gamma n - |\{i \in I_t : v \in NU_i\}|) + (k+1)\Delta_R\gamma^{4/3}n \\
&\leq \frac{5}{2}k\Delta_R\delta\gamma n.
\end{aligned}$$

So, by Azuma's inequality (Theorem 3.2), we obtain $\mathbb{P}[C(v) \geq 3k\Delta_R\delta\gamma n] \leq (1-2c)^n$. Hence, a union bound over all $v \in V(G)$ gives

$$\mathbb{P}[\text{(W2)(ii) fails in Round } t] \leq rn(1-2c)^n. \tag{6.48}$$

Finally, we show that (W3) holds with high probability. Recall that F_i' was defined in (6.7). For any $i \in I_t$ and $j \in [Kr]$, let

$$\overline{S}_j(i) := \left\{ \{x,x'\} \in \binom{X_j^i}{2} : \left| N_{F_i'}(\{x,x'\}) \right| = ((d_0 p(R_K, \vec{\beta}', j))^2 \pm \delta^{1/2})m' \right\}.$$

Then Claim 6.6 implies $|\overline{S}_j(i)| \geq \binom{|X_j^i|}{2} - \gamma m'^2$.

Let $S \subseteq X_j^i$ be such that either $|S| = 1$ or S consists of a pair of vertices in $\overline{S}_j(i)$. Then Claim 6.6 and the definition of $\overline{S}_j(i)$ imply that

$$|N_{F_i'}(S)| = ((d_0 p(R_K, \vec{\beta}', j))^{|S|} \pm \delta^{1/2})m'. \tag{6.49}$$

By the same argument as in the proof of (W1) we have

$$\begin{aligned}
\mathbb{E}\left[\left|N_{F_i'}(S) \cap Y_i\right|\right] &\overset{(6.42)}{=} (1 \pm \gamma^{1/3})\delta \left|N_{F_i'}(S) \setminus U_i\right| \\
&\overset{(6.40,6.49)}{=} (1 \pm \gamma^{1/3}) \left((d_0 p(R_K, \vec{\beta}', j))^{|S|} \pm \delta^{1/2} \pm \gamma^{2/5} \right) \delta m' \\
&= \left((d_0 p(R_K, \vec{\beta}', j))^{|S|} \pm 2\delta^{1/2} \right) \delta m'.
\end{aligned}$$

As in the proof of (W1), $|N_{F_i'}(S) \cap Y_i|$ has the hypergeometric distribution, so

$$\mathbb{P}\left[|N_{F_i'}(S) \cap Y_i| = ((d_0 p(R_K, \vec{\beta}', j))^{|S|} \pm 3\delta^{1/2})\delta m' \right] \geq 1 - (1-2c)^n.$$

Together with (6.40) this implies

$$\mathbb{P}\left[|N_{F_i'}(S) \cap W_i| = ((d_0 p(R_K, \vec{\beta}', j))^{|S|} \pm 4\delta^{1/2})\delta m' \right] \geq 1 - (1-2c)^n.$$

Together with a similar argument for vertices $v \in V_j^i$ this shows that with probability at least $1 - 2Kr(rn)^2(1-2c)^n$ for all $j \in [Kr]$, $S \subseteq X_j^i$ with $|S| = 1$ or $S \in \overline{S}_j(i)$ and all $v \in V_j^i$ we have

$$\begin{aligned}
|N_{F_i'}(S) \cap W_i| &= ((d_0 p(R_K, \vec{\beta}', j))^{|S|} \pm 4\delta^{1/2})\delta m', \\
|N_{F_i'}(v) \cap Z_i| &= (d_0 p(R_K, \vec{\beta}', j) \pm 4\delta^{1/2})\delta m'.
\end{aligned} \tag{6.50}$$

Moreover,

$$\left| \binom{Z_i \cap X_j^i}{2} \cap \overline{S}_j(i) \right| \overset{(6.44)}{\geq} \binom{\delta m'}{2} - \gamma m'^2 \geq \frac{1}{2}(1-\delta^{6/20})(\delta m')^2.$$

By Theorem 3.11, this together with (6.50) implies that $F_i'[Z_i \cap X_j^i, W_i \cap V_j^i]$ is $(\delta^{1/20}, d_0 p(R_K, \vec{\beta}', j))$-super-regular. Hence,

$$\mathbb{P}[\text{(W3) fails in Round } t] \leq 2Kr(rn)^2(1-2c)^n. \tag{6.51}$$

By (6.45), (6.47), (6.48) and (6.51), failure of type 5 occurs in some Round t with probability at most

$$T \cdot \left((Kr)^2 \gamma n m' + rn + rn + 2Kr(rn)^2\right)(1-2c)^n \leq (1-c)^n.$$

□

6.6. **Failure of type 6.** The following claim shows that failure of type 6 in fact never occurs.

Claim 6.13. *Assume that we are in Round t, Step $4.\ell$, and no failure has occurred so far, in particular, (W1)–(W3) of Step 3 hold. Then failure of type 6 does not occur.*

Proof. We need to check that $F_i^*[Z_i \cap X_j^i, W_i \cap V_j^i]$ and $P_i^t[W_i]$ satisfy (a') and (b'). Note that for every vertex $v \in V(G)$ we have

$$d_{P^t - P_i^t}(v) \leq \sum_{j \in I_t : j < i} \Delta(H_i) \leq (k+1)\Delta_R \gamma n \leq \delta^3 n. \tag{6.52}$$

First, we prove (a'). If $\ell = 1$, this is immediate from (6.11) and (W3). So assume that $\ell > 1$. The definition of F_i^* in (6.10) together with the definition of F_i' in (6.7) imply that whenever $xv \in E(F_i') \setminus E(F_i^*)$ with $x \in X_j^i, v \in V_j^i$, we can find a vertex $y \in N_x^i$ such that $v \in N_{P^t - P_i^t}(\phi_i(y))$. Hence,

$$v \in \bigcup_{y \in N_x^i} N_{P^t - P_i^t}(\phi_i(y)).$$

By (6.52) and the fact that $|N_x^i| \leq K\Delta_R$ (by (F3)) we have

$$\left| \bigcup_{y \in N_x^i} N_{P^t - P_i^t}(\phi_i(y)) \right| \leq K\Delta_R \cdot \delta^3 n \leq \delta^2 m',$$

thus x is incident to at most $\delta^2 m'$ edges in $E(F_i') \setminus E(F_i^*)$.

Similarly, for any $xv \in E(F_i') \setminus E(F_i^*)$ with $x \in X_j^i, v \in V_j^i$, an argument similar to that in (6.18) implies that

$$x \in \bigcup_{u \in N_{P^t - P_i^t}(v)} N^i_{\phi_i^{-1}(u)}.$$

As before, using (6.52), one can show that v is incident to at most $K\Delta_R \delta^3 n \leq \delta^2 m'$ edges in $E(F_i') \setminus E(F_i^*)$. So $\Delta(F_i' - F_i^*) \leq \delta^2 m' \leq 4\delta^{1/20} \cdot \delta m'$.

Moreover, by (W3), $F_i'[Z_i \cap X_j^i, W_i \cap V_j^i]$ is $(\delta^{1/20}, d_0 p(R_K, \vec{\beta}', j))$-super-regular. Since $|Z_j \cap X_j^i| = |W_j \cap V_j^i| = \delta m'$, we can apply Proposition 3.9 to deduce that $F_i^*[Z_i \cap X_j^i, W_i \cap V_j^i]$ is $(\delta^{1/50}, d_0 p(R_K, \vec{\beta}', j))$-super-regular, and (a') holds.

Next, we prove (b'). By (W1), $P^t[W_i]$ is $(\delta^{1/25}, \vec{\beta}')$-super-regular with respect to $(R_K, V_1^i \cap W_i, \dots, V_{Kr}^i \cap W_i)$. For $v \in V(P_i^t)$, (6.52) implies that

$$d_{P^t[W_i]}(v) - d_{P_i^t[W_i]}(v) \leq \delta^3 n \leq 4\delta^{1/25} \cdot \delta m'.$$

Thus Proposition 3.9 implies that $P_i^t[W_i]$ is $(4\delta^{1/50}, \beta)$-super-regular, and (b') holds. □

Proof of Lemma 6.1. Claims 6.7–6.8 and 6.10–6.13 imply that the main packing algorithm has failure probability of at most $5(1-c)^n$. Thus, the main packing algorithm succeeds with high probability, and Claim 6.5 ensures that the embedded copies $\phi_1'(H_1), \dots, \phi_s'(H_s)$ of $H_1, \dots, H_s$ are pairwise edge-disjoint subgraphs of G such that $\phi_i'(x) \in N_{A_i}(x)$ for all $x \in V(H_i)$. Thus (T1) and (T2) are satisfied for all $i \in [s]$. Claim 6.8 also implies that, provided failure of type 2 does not occur, (QW_i') holds for all $i \in [s]$. This implies (T3). □

7. Packing graphs into near-equiregular graphs

Our aim is to pack graphs $H_1, \dots, H_\ell$ of small maximum degree into a graph G which is $(\varepsilon, \vec{d})$-regular with respect to a partition $(R, V_1, \dots, V_r)$. However, one of the conditions of the blow-up lemma for approximate decompositions (Theorem 6.1) is that all the H_s are 'degree balanced' with respect to R and the corresponding densities $d_{i,j}$. If R is complete and the $d_{i,j}$ are all equal, this (roughly speaking) translates to the requirement that all the bipartite graphs $H[V_i, V_j]$ are k-regular for some constant k. More generally, the requirement is (again roughly speaking) that for each edge ij of R, the graph $H[V_i, V_j]$ is $k_{i,j}$-regular.

Suppose that we are given a family of graphs $\mathcal{H} = \{H_1, \dots, H_\ell\}$ of small maximum degree, which do not necessarily satisfy the degree balance condition. In this section we show that it is often possible to pack the graphs in $\mathcal{H}$ together in a suitable way to obtain a new family of graphs which do satisfy the degree balance condition required in Theorem 6.1.

The main result (Lemma 7.1) of Subsection 7.1 shows that this is possible if the graphs in $\mathcal{H}$ are 'edge-balanced' i.e. $e(H[V_i, V_j])$ is proportional to $d_{i,j}$. In Subsection 7.2 we will apply Lemma 7.1 in order to show that in the setting of Theorems 1.2 and 1.3, that is when R is a complete graph, we can achieve a packing of the H_s which satisfies the degree balance condition for any family $\mathcal{H}$ of bounded degree graphs. In this way we will be able to satisfy the conditions of Theorem 6.1.

7.1. Packing graphs into almost regular graphs.

The aim of this subsection is to prove the following lemma, which shows that a sufficiently large collection of 'edge-balanced' graphs of bounded maximum degree can be packed into a suitable near-equiregular graph H.

Lemma 7.1. *Suppose $0 < 1/n \ll 1/s \ll 1/\Delta, 1/\Delta_R$ and $1/n \ll 1/k, 1/(C+1)$. Let R be a graph on $[r]$ with $\Delta(R) \le \Delta_R$ and let $n \le n_1 \le \dots \le n_r \le n + C$. Let $\vec{M}$ and $\vec{k}$ be symmetric $r \times r$ matrices such that $k_{i,j} \in \mathbb{N}$ and $M_{i,j} + s^{2/3} + 1 \le k_{i,j} \le k$ for all $ij \in E(R)$.*

Suppose that $L_1, \dots, L_s$ are graphs such that each L_ℓ admits a vertex partition $(R, X_1^\ell, \dots, X_r^\ell)$ with $|X_i^\ell| = n_i$, $\Delta(L_\ell) \le \Delta$ and

$$\sum_{\ell=1}^{s} e(L_\ell[X_i^\ell, X_j^\ell]) = M_{i,j} n \tag{7.1}$$

for all $ij \in E(R)$. Then there exists an $(R, \vec{k}, C)$-near-equiregular graph H with vertex partition $(R, V_1, \dots, V_r)$ such that $|V_i| = n_i$ for all $i \in [r]$ and such that $L_1, \dots, L_s$ pack into H.

Moreover, writing $\phi(L_\ell)$ for the copy of L_ℓ in this packing of $L_1, \dots, L_s$ in H, and for each $ij \in E(R)$ writing $J_{i,j} := H[V_i, V_j] - (\phi(L_1) \cup \dots \cup \phi(L_s))$, we have $\Delta(J_{i,j}) \le k_{i,j} - M_{i,j} + 2s^{2/3}$.

In order to prove Lemma 7.1, we first show that the graphs $L_1 \dots, L_s$ pack into a graph H for which $H[V_i, V_j]$ is close to regular whenever $ij \in E(R)$ and $H[V_i, V_j]$ is empty if $ij \notin E(R)$. Note that $\Delta(L_\ell) \le \Delta$ implies that

$$M_{i,j} \le 2\Delta s. \tag{7.2}$$

Lemma 7.2. *Assume the setup of Lemma 7.1. Then there exists an r-partite graph H admitting a vertex partition $(R, V_1, \dots, V_r)$ with $|V_i| = n_i$ such that H has a decomposition into $L_1, \dots, L_s$ and $d_{H[V_i,V_j]}(v) = M_{i,j} \pm s^{2/3}$ for all $v \in V_i \cup V_j$ and $ij \in E(R)$.*

Proof. For each $\ell \in [s]$, we will now refine the partition $X_1^\ell, \dots, X_r^\ell$ of L_ℓ. The aim is that, firstly, within each refined partition class the vertex degrees are similar with respect to the other vertex classes (see Claim 7.5). Secondly, an analogue of the condition (7.1) is still preserved (with appropriate scaling and relabelling) by the subpartitions (see Claim 7.4). To achieve this, we first define a suitable 'exceptional' class $X_i^{\ell,0} \subseteq X_i^\ell$ (for each $i \in [r]$) whose removal results in partition classes whose size is a large integer multiple.

For each $i \in [r]$ let $r_i(j)$ denote the j-th smallest number in $N_R(i)$. Let $r_i(j) := 0$ if $j > |N_R(i)|$ and let $M_{i,0} := 0$. Since $\Delta(R) \le \Delta_R$, we have $r_i(\Delta_R + 1) = 0$ for each $i \in [r]$. For every $x \in X_i^\ell$ and $j \le |N_R(i)|$ we define

$$d_j^\ell(x) := |N_{L_\ell}(x) \cap X_{r_i(j)}^\ell|, \text{ which will be called the } (j,\ell)\text{-degree of } x.$$

We let $d_j^\ell(x) := 0$ if $j > |N_R(i)|$. Note that for every $x \in X_i^\ell$ and every $j \in [\Delta_R]$ we have

$$0 \le d_j^\ell(x) \le \Delta. \tag{7.3}$$

For a set $A \subseteq X_i^\ell$ we say that A is (j,ℓ)-*regular* if every vertex $v \in A$ has the same (j,ℓ)-degree, otherwise we say that A is (j,ℓ)-*irregular*. We let n' be the integer satisfying $n = (s^{\Delta_R}+1)n'+m'$ with $0 \le m' \le s^{\Delta_R}$. For each $i \in [r]$ define m_i' to satisfy $n_i = (s^{\Delta_R}+1)n' + m_i'$. So $0 \le m_i' \le s^{\Delta_R} + C$. Let

$$\hat{n}_{i,j} := \begin{cases} n_i & \text{if } j = 0, \\ (s^{\Delta_R - j}+1)n' + m_i' & \text{if } j \in [\Delta_R - 1], \\ n' + m_i' & \text{if } j = \Delta_R. \end{cases} \tag{7.4}$$

Claim 7.3. *For all $\ell \in [s]$ and $i \in [r]$, there is a set $X_i^{\ell,0} \subseteq X_i^\ell$ with $|X_i^{\ell,0}| = \hat{n}_{i,\Delta_R}$ such that for each $j \in [|N_R(i)|]$ we have*

$$\sum_{\ell=1}^{s} D_i^{\ell,j} = M_{i,r_i(j)} \pm s^{3/5}, \tag{7.5}$$

where $D_i^{\ell,j}$ is the average (j,ℓ)-degree of the vertices in $X_i^{\ell,0}$, and

$$|\{\ell \in [s] : X_i^{\ell,0} \text{ is } (j,\ell)\text{-irregular}\}| \le 3\Delta_R\Delta. \tag{7.6}$$

Proof. Let $\ell \in [s]$ and $i \in [r]$ be arbitrary but fixed. For each $0 \le j \le \Delta_R$ we define a vertex set $Y_i^{\ell,j} \subseteq X_i^\ell$ inductively as follows. Let $Y_i^{\ell,0} := X_i^\ell$. Assume that for some $0 \le j < \Delta_R$ we have defined $Y_i^{\ell,j}$ of size $\hat{n}_{i,j}$, and order the vertices in $Y_i^{\ell,j}$ as $y_1^{\ell,j}, \dots, y_{\hat{n}_{i,j}}^{\ell,j}$ so that their $(j+1,\ell)$-degrees are non-decreasing. Now we pick a number a in $[\hat{n}_{i,j}]$ uniformly at random and let

$$Y_i^{\ell,j+1} := \{y_q^{\ell,j} : a \le q < a + \hat{n}_{i,j+1}\}, \text{ where the index } q \text{ is considered modulo } \hat{n}_{i,j}.$$

We repeat this process until we obtain Y_i^{ℓ,Δ_R} with $|Y_i^{\ell,\Delta_R}| = \hat{n}_{i,\Delta_R} = n'+m_i'$. Let $X_i^{\ell,0} := Y_i^{\ell,\Delta_R}$. Consider any vertex $x \in X_i^\ell$. We have

$$\mathbb{P}[x \in X_i^{\ell,0}] = \prod_{j=1}^{\Delta_R} \mathbb{P}[x \in Y_i^{\ell,j} \mid x \in Y_i^{\ell,j-1}] = \prod_{j=1}^{\Delta_R} \frac{|Y_i^{\ell,j}|}{|Y_i^{\ell,j-1}|} = \frac{\hat{n}_{i,\Delta_R}}{n_i}.$$

Thus

$$\begin{aligned} \mathbb{E}\left[\sum_{\ell=1}^{s} D_i^{\ell,j}\right] &= \sum_{\ell=1}^{s} \sum_{x \in X_i^\ell} \mathbb{P}[x \in X_i^{\ell,0}] \frac{d_j^\ell(x)}{\hat{n}_{i,\Delta_R}} = \frac{1}{n_i} \sum_{\ell=1}^{s} \sum_{x \in X_i^\ell} d_j^\ell(x) \overset{(7.1)}{=} \frac{M_{i,r_i(j)} n}{n_i} \\ &\overset{(7.2)}{=} M_{i,r_i(j)} \pm \frac{3C\Delta s}{n}. \end{aligned}$$

Consider the exposure martingale $Z_i^\ell := \mathbb{E}[\sum_{\ell'=1}^{s} D_i^{\ell',j} \mid X_i^{1,0}, \dots, X_i^{\ell,0}]$ for $0 \le \ell \le s$. Note that $D_i^{\ell,j} \le \Delta$ and $D_i^{\ell,j}$ is determined by $X_i^{\ell,0}$. Thus Z_i^ℓ is Δ-Lipschitz. So by Azuma's inequality (Theorem 3.2) we know that

$$\mathbb{P}\left[\sum_{\ell=1}^{s} D_i^{\ell,j} \ne M_{i,r_i(j)} \pm s^{3/5}\right] \le 2e^{-\frac{s^{6/5}}{8\Delta^2 s}} < e^{-s^{1/6}}. \tag{7.7}$$

Let $Z_{i,j}^\ell$ be the indicator variable defined by

$$Z_{i,j}^\ell := \begin{cases} 1 & \text{if } X_i^{\ell,0} \text{ is } (j,\ell)\text{-irregular}, \\ 0 & \text{if } X_i^{\ell,0} \text{ is } (j,\ell)\text{-regular}. \end{cases}$$

Also we let $Z_{i,j} := \sum_{\ell=1}^{s} Z_{i,j}^\ell$. Note that since $X_i^{\ell,0} = Y_i^{\ell,\Delta_R} \subseteq Y_i^{\ell,j}$ for all $j \in [\Delta_R - 1]$ we get that if $Y_i^{\ell,j}$ is (j,ℓ)-regular, then $X_i^{\ell,0}$ is also (j,ℓ)-regular.

Note that there are at most $\Delta+1$ indices q such that $y_q^{\ell,j-1}$ and $y_{q+1}^{\ell,j-1}$ have distinct (j,ℓ)-degrees (by (7.3) and since the ordering $y_1^{\ell,j-1},\dots,y_{\hat{n}_{j,i}}^{\ell,j-1}$ of $Y_i^{\ell,j-1}$ is such that the (j,ℓ)-degrees are non-decreasing). So for a given $j\in[\Delta_R]$,

$$\mathbb{P}[Z_{i,j}^{\ell}=1]\le \mathbb{P}[Y_i^{\ell,j} \text{ is } (j,\ell)\text{-irregular}]\le \frac{(\Delta+1)\hat{n}_{i,j}}{\hat{n}_{i,j-1}} \overset{(7.4)}{\le} \frac{3\Delta}{2s}.$$

So $\mathbb{E}[Z_{i,j}]\le \frac{3\Delta s}{2s}=\frac{3\Delta}{2}$, and hence by Markov's inequality,

$$\mathbb{P}\left[Z_{i,j}\ge 3\Delta_R\Delta\right]\le \frac{1}{2\Delta_R}.$$

Together with (7.7) and a union bound over all choices of $j\in[\Delta_R]$ this implies that with probability at least $1-\Delta_R(e^{-s^{1/6}}+1/(2\Delta_R))>0$ both (7.5) and (7.6) hold for fixed $i\in[r]$, meaning that we can find sets $X_i^{\ell,0}$ as required for fixed i. Thus we may take such a choice for each $i\in[r]$, which completes the proof of the claim. □

We now proceed in a similar way for the remaining vertices. For each $\ell\in[s]$ and $i\in[r]$ let $X_i^{\ell,0}$ be as guaranteed by Claim 7.3 and let

$$X_i^{\ell,*}:=X_i^{\ell}\setminus X_i^{\ell,0}.$$

Then $|X_i^{\ell,*}|=s^{\Delta_R}n'$. We now order the vertices of $X_i^{\ell,*}$ in the order of non-decreasing $(1,\ell)$-degree and partition them into s blocks, each of size $s^{\Delta_R-1}n'$. For each $p_1\in[s]$ we let $A_i^{\ell}(p_1)$ denote the p_1-th block. We next order the vertices of each $A_i^{\ell}(p_1)$ in the order of non-decreasing $(2,\ell)$-degree and partition them into s blocks, each of size $s^{\Delta_R-2}n'$. For all $p_1,p_2\in[s]$ we let $A_i^{\ell}(p_1,p_2)$ denote the p_2-th block of $A_i^{\ell}(p_1)$. Continuing in this way, we obtain s^{Δ_R} blocks $A_i^{\ell}(p_1,p_2,\dots,p_{\Delta_R})$ of size n' each.

We now show that for fixed $\ell\in[s]$, $i\in[r]$ and $j\in[\Delta_R]$ we have

$$|\{(p_1,\dots,p_{\Delta_R}): A_i^{\ell}(p_1,\dots,p_{\Delta_R}) \text{ is } (j,\ell)\text{-irregular}\}|\le(\Delta+1)s^{\Delta_R-1}. \tag{7.8}$$

Note that, for each $j\in[\Delta_R]$, if $A_i^{\ell}(p_1,\dots,p_j)$ is (j,ℓ)-regular, then $A_i^{\ell}(p_1,\dots,p_{\Delta_R})$ is also (j,ℓ)-regular. But by (7.3) all but at most $\Delta+1$ of the blocks $A_i^{\ell}(p_1,\dots,p_j)$ are (j,ℓ)-regular. This implies (7.8).

For each $\ell\in[s]$ and $i\in[r]$ we now relabel the s^{Δ_R} blocks $A_i^{\ell}(p_1,\dots,p_{\Delta_R})$ by $X_i^{\ell,j'}$ with $j'\in[s^{\Delta_R}]$. Thus $X_i^{\ell,1},X_i^{\ell,2},\dots,X_i^{\ell,s^{\Delta_R}}$ forms a partition of $X_i^{\ell,*}$ into blocks of size n'. Let $d_j^{\ell}(i,j')$ be the average (j,ℓ)-degree of the vertices in $X_i^{\ell,j'}$ where the index j' is considered modulo s^{Δ_R}. Now we choose a random integer $j'_{i,\ell}\in[s^{\Delta_R}]$ for each $\ell\in[s]$ and $i\in[r]$.

Claim 7.4. *For any fixed $i\in[r],j\in[\Delta_R],q\in[s^{\Delta_R}]$, let $B_q(i,j)$ be the event that*

$$\sum_{\ell=1}^{s} d_j^{\ell}(i,j'_{i,\ell}+q)\neq M_{i,r_i(j)}\pm 2s^{3/5}. \tag{7.9}$$

Then $\mathbb{P}[B_q(i,j)]\le e^{-s^{1/6}}$.

Proof. For each vertex $x\in X_i^{\ell,*}$, we have $\mathbb{P}[x\in X_i^{\ell,j'_{i,\ell}+q}]=1/s^{\Delta_R}$ since there exists one index j' such that $x\in X_i^{\ell,j'}$, and the probability that $j'_{i,\ell}+q=j'$ is exactly $1/s^{\Delta_R}$. So,

$$\mathbb{E}\left[\sum_{\ell=1}^{s} d_j^{\ell}(i,j'_{i,\ell}+q)\right]=\sum_{\ell=1}^{s}\sum_{x\in X_i^{\ell,*}}\frac{d_j^{\ell}(x)}{n'}\cdot\frac{1}{s^{\Delta_R}}. \tag{7.10}$$

By the assumptions of Lemma 7.2 and (7.5),

$$\sum_{\ell=1}^{s}\sum_{x\in X_i^{\ell}} d_j^{\ell}(x)=M_{i,r_i(j)}n \quad\text{and}\quad \sum_{\ell=1}^{s}\sum_{x\in X_i^{\ell,0}} d_j^{\ell}(x)=(M_{i,r_i(j)}\pm s^{3/5})\hat{n}_{i,\Delta_R}.$$

Thus

$$\sum_{\ell=1}^{s}\sum_{x\in X_i^{\ell,*}} d_j^\ell(x) \quad = \quad \sum_{\ell=1}^{s}\sum_{x\in X_i^{\ell}\setminus X_i^{\ell,0}} d_j^\ell(x) = M_{i,r_i(j)}(n_i \pm C) - (M_{i,r_i(j)} \pm s^{3/5})\hat{n}_{i,\Delta_R}$$
$$\overset{(7.4)}{=} (M_{i,r_i(j)} \pm s^{3/5})s^{\Delta_R}n'.$$

Together with (7.10) this implies that

$$\mathbb{E}\left[\sum_{\ell=1}^{s} d_j^\ell(i, j'_{i,\ell}+q)\right] = M_{i,r_i(j)} \pm s^{3/5}.$$

For fixed i, j and all $0 \le \ell \le s$, let

$$Z_{i,j}^\ell := \mathbb{E}\left[\sum_{\ell'=1}^{s} d_j^{\ell'}(i, j'_{i,\ell'}+q) \mid j'_{i,1}, \dots, j'_{i,\ell}\right].$$

Then $Z_{i,j}^\ell$ is an exposure martingale which is Δ-Lipschitz by (7.3). Therefore, by Azuma's inequality (Theorem 3.2),

$$\mathbb{P}[B_q(i,j)] \le 2e^{-\frac{s^{6/5}}{2\Delta^2 s}} < e^{-s^{1/6}}.$$

□

Claim 7.5. *For any fixed $i \in [r], j \in [\Delta_R], q \in [s^{\Delta_R}]$, let $B'_q(i,j)$ be the event that*

$$|\{\ell \in [s] : X_i^{\ell, j'_{i,\ell}+q} \text{ is } (j,\ell)\text{-irregular}\}| \ge s^{3/5}.$$

Then $\mathbb{P}[B'_q(i,j)] \le e^{-s^{1/6}}$.

Proof. Let

$$Z(i,j,q,\ell) := \begin{cases} 1 & \text{if } X_i^{\ell, j'_{i,\ell}+q} \text{ is } (j,\ell)\text{-irregular,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z(i,j,q) := \sum_{\ell=1}^{s} Z(i,j,q,\ell)$. By (7.8),

$$\mathbb{P}\left[X_i^{\ell, j'_{i,\ell}+q} \text{ is } (j,\ell)\text{-irregular}\right] \le \frac{(\Delta+1)s^{\Delta_R-1}}{s^{\Delta_R}} \le \frac{2\Delta}{s}.$$

This implies that for all $\ell \in [s]$ and all $z_1, \dots, z_{\ell-1} \in \{0,1\}$,

$$\mathbb{P}\left[Z(i,j,q,\ell) = 1 \mid Z(i,j,q,1) = z_1, \dots, Z(i,j,q,\ell-1) = z_{\ell-1}\right] \le \frac{2\Delta}{s}.$$

Thus, by Lemma 3.3 and Proposition 3.4, $\mathbb{P}[Z(i,j,q) \ge s^{3/5}] \le 2e^{-s^{6/5}/2s} \le e^{-s^{1/6}}$. □

Claims 7.4 and 7.5 imply that for fixed $i \in [r]$, with probability at least $1 - \Delta_R s^{\Delta_R}(e^{-s^{1/6}} + e^{-s^{1/6}}) > 1/2$ for all $j \in [\Delta_R]$ and $q \in [s^{\Delta_R}]$ neither $B_q(i,j)$ nor $B'_q(i,j)$ occurs. Thus for fixed $i \in [r]$, there exists a choice of $j'_{i,\ell}$ for each $\ell \in [s]$ such that this is indeed the case. Hence we can choose such integers $j'_{i,\ell}$ for all $i \in [r]$. We now re-label, letting $X_{i,q}^\ell := X_i^{\ell, j'_{i,\ell}+q}$ for all $q \in [s^{\Delta_R}]$, and $X_{i,0}^\ell := X_i^{\ell,0}$.

In order to construct the graph H required in Lemma 7.2, we now apply the blow-up lemma. For this, we divide each V_i into $V_{i,0}$ of size $n' + m'_i$, and $V_{i,q}$ of size n' for all $q \in [s^{\Delta_R}]$. Let G_1 be a complete r-partite graph with vertex partition $V_1, \dots, V_r$. For each $\ell \in [s]$ in turn, we choose an embedding $\phi_\ell : L_\ell \to G_\ell$ of L_ℓ into G_ℓ in such a way that $\phi_\ell(X_{i,0}^\ell) = V_{i,0}$ and $\phi_\ell(X_{i,q}^\ell) = V_{i,q}$ for all $q \in [s^{\Delta_R}]$. We then let $G_{\ell+1} := G_\ell - \phi_\ell(E(L_\ell))$. We repeat this process until we obtain G_{s+1}. Note that this process succeeds because for all $x \in V(G_\ell) \setminus V_i$, and all $q \in [s^{\Delta_R}] \cup \{0\}$ we have $|N_{G_\ell}(x) \cap V_{i,q}| \ge |V_{i,q}| - \Delta s \ge (1 - \frac{\Delta s}{n'})|V_{i,q}|$. So G_ℓ is $(\frac{\Delta s}{n'}, 1)$-super-regular with respect to $(R, V_1, \dots, V_r)$ for all $\ell \in [s]$. Thus we can keep applying the Blow-up lemma (Theorem 3.6) until we obtain G_{s+1}. Let $H := \bigcup_{\ell=1}^{s} \phi_\ell(L_\ell)$.

We claim that H is as desired. Given a vertex $x \in V_{i,q}$, let $d_{H,j}(x) := |N_H(x) \cap V_{r_i(j)}|$. Note that $d_j^\ell(\phi_\ell^{-1}(x)) = d_j^\ell(i, j'_{i,\ell} + q)$ unless $X_{i,q}^\ell$ is (j,ℓ)-irregular. But $X_{i,q}^\ell$ is (j,ℓ)-irregular for at most $s^{3/5}$ indices ℓ because $B'_q(i,j)$ does not occur. Also $|d_j^\ell(\phi_\ell^{-1}(x)) - d_j^\ell(i, j'_{i,\ell} + q)| \le \Delta$ even when $X_{i,q}^\ell$ is (j,ℓ)-irregular. Thus

$$\begin{aligned} d_{H,j}(x) &= \sum_{\ell=1}^{s} d_j^\ell(\phi_\ell^{-1}(x)) = \sum_{\ell=1}^{s} d_j^\ell(i, j'_{i,\ell} + q) \pm \Delta s^{3/5} \overset{(7.9)}{=} M_{i,r_i(j)} \pm 2s^{3/5} \pm \Delta s^{3/5} \\ &= M_{i,r_i(j)} \pm s^{2/3}. \end{aligned}$$

This completes the proof of Lemma 7.2. □

Now we show that we can transform an almost regular graph H (as in the conclusion of Lemma 7.2) into an $(R, \vec{k}, C)$-near-equiregular graph by adding a small number of edges. First we show that we can achieve this if the vertex classes have equal size. We write $\overline{H}$ for the complement of H.

Lemma 7.6. *Suppose $r \in \mathbb{N}$ and $0 < 1/n \ll 1/s \ll 1/\Delta$. Let R be a graph on $[r]$. Let $\vec{M}$ and $\vec{k}$ be symmetric $r \times r$ matrices such that $k_{i,j} \in \mathbb{N}$, $M_{i,j} \le 2\Delta s$ and $M_{i,j} + s^{2/3} + 1 \le k_{i,j} < n/2$ for all $ij \in E(R)$. Suppose H is an r-partite graph with vertex partition $(R, V_1, \dots, V_r)$ such that $|V_i| = n$ for all $i \in [r]$ and that for all $ij \in E(R)$ and every vertex $x \in V_i$ we have $M_{i,j} - s^{2/3} - 1 \le d_{H[V_i,V_j]}(x) \le M_{i,j} + s^{2/3}$. Then there exists a graph H' with vertex partition $(R, V_1, \dots, V_r)$ such that $H \subseteq H'$ and $H'[V_i, V_j]$ is $k_{i,j}$-regular for all $ij \in E(R)$.*

Proof. It suffices to prove the lemma for the case when $r = 2$ and $R = K_2$, and $k_{1,2} \le M_{1,2} + 2s^{2/3}$. (Indeed, if $r \ge 3$, we proceed in the same way as when $r = 2$ for each pair V_i, V_j with $ij \in E(R)$. Also if $M_{1,2} + 2s^{2/3} < k_{1,2} < n/2$, then we can set $k'_{1,2} := M_{1,2} + 2s^{2/3}$ and find H'' with $H \subseteq H''$ such that $H''[V_1, V_2]$ is $k'_{1,2}$-regular. Since $k'_{1,2} < n/2$, we have $\delta(\overline{H''}[V_1, V_2]) \ge n/2$, and thus $\overline{H''}[V_1, V_2]$ has a perfect matching which we can add to H''. We repeat this until we obtain a $k_{1,2}$-regular graph $H'[V_1, V_2]$. This is possible since $k_{1,2} < n/2$.)

So suppose that $r = 2$ and $R = K_2$ and $k_{1,2} \le M_{1,2} + 2s^{2/3}$. We will apply the Max-flow min-cut theorem. For this sake let us define a digraph G' and an edge capacity function c as follows:

- $V(G') := \{s, t\} \cup V_1 \cup V_2$,
- $E(G') := \{\overrightarrow{vv'} : vv' \in E(\overline{H}[V_1, V_2]), v \in V_1, v' \in V_2\} \cup \{\overrightarrow{sv} : v \in V_1\} \cup \{\overrightarrow{v't} : v' \in V_2\}$,
- $c(\overrightarrow{vv'}) := 1, c(\overrightarrow{sv}) := k_{1,2} - d_{H[V_1,V_2]}(v)$, and $c(\overrightarrow{v't}) := k_{1,2} - d_{H[V_1,V_2]}(v')$ for all $v \in V_1, v' \in V_2$.

Let

$$C^* := \sum_{v \in V_1} c(\overrightarrow{sv}) = \sum_{v \in V_1} (k_{1,2} - d_{H[V_1,V_2]}(v)) = \sum_{v' \in V_2} c(\overrightarrow{v't})$$

be the capacity of the cut $\{\{s\}, V_1 \cup V_2 \cup \{t\}\}$. If the above cut is a minimum (s,t)-cut, then the Max-flow min-cut theorem ensures that there is an integer flow f from s to t with value C^*. Let E' be the set of all edges from V_1 to V_2 with value 1 in f. Then $E' \subseteq E(\overline{H})$, so letting $E(H'[V_1, V_2])) := E(H[V_1, V_2]) \cup E'$ will ensure that $H'[V_1, V_2]$ is $k_{1,2}$-regular. Thus it suffices to prove that any (s,t)-cut in G' has capacity at least C^*.

Note that for any $v \in V_1, v' \in V_2$,

$$1 \le c(\overrightarrow{sv}), c(\overrightarrow{v't}) \le k_{1,2} - (M_{1,2} - s^{2/3} - 1) \le 4s^{2/3}. \tag{7.11}$$

Assume we have an (s,t)-cut $\{U, W\}$ with $s \in U, t \in W$ which has capacity C'. Let $U_1 := U \cap V_1, U_2 := U \cap V_2, W_1 := W \cap V_1, W_2 := W \cap V_2$. Then

$$C' = \sum_{v \in W_1} c(\overrightarrow{sv}) + e_{\overline{H}}(U_1, W_2) + \sum_{v' \in U_2} c(\overrightarrow{v't}).$$

First, assume that $|W_2| \geq M_{1,2} + 5s^{2/3}$. Then for each $v \in U_1$, $e_{\overline{H}}(\{v\}, W_2) \geq |W_2| - d_{H[V_1,V_2]}(v) \geq M_{1,2} + 5s^{2/3} - (M_{1,2} + s^{2/3}) \geq c(\overrightarrow{sv})$ by (7.11), so we get

$$C' \geq \sum_{v \in W_1} c(\overrightarrow{sv}) + e_{\overline{H}}(U_1, W_2) \geq \sum_{v \in W_1} c(\overrightarrow{sv}) + \sum_{v \in U_1} c(\overrightarrow{sv}) = C^*.$$

So we may assume that $|W_2| < M_{1,2} + 5s^{2/3}$. A similar argument shows that we may assume $|U_1| < M_{1,2} + 5s^{2/3}$. Hence $|U_2|, |W_1| \geq n - M_{1,2} - 5s^{2/3} \geq n/2$ since $1/n \ll 1/(2\Delta s) \leq 1/M_{1,2}$. This implies that

$$\begin{aligned} C' &\geq \sum_{v \in W_1} c(\overrightarrow{sv}) + \sum_{v' \in U_2} c(\overrightarrow{v't}) \overset{(7.11)}{\geq} \sum_{v \in W_1} c(\overrightarrow{sv}) + \sum_{v' \in U_2} 1 \geq \sum_{v \in W_1} c(\overrightarrow{sv}) + n/2 \\ &\geq \sum_{v \in W_1} c(\overrightarrow{sv}) + |U_1| \cdot 4s^{2/3} \overset{(7.11)}{\geq} \sum_{v \in W_1} c(\overrightarrow{sv}) + \sum_{v \in U_1} c(\overrightarrow{sv}) = C^*, \end{aligned}$$

thus every (s,t)-cut has capacity at least C^*, and we are done. □

We now extend Lemma 7.6 to the setting where the vertex class sizes might not be exactly equal.

Lemma 7.7. *Suppose $0 < 1/n \ll 1/s \ll 1/\Delta$ and $1/n \ll 1/k, 1/(C+1)$. Let R be a graph on $[r]$ and let $n \leq n_1 \leq \cdots \leq n_r \leq n + C$. Let $\vec{M}$ and $\vec{k}$ be symmetric $r \times r$ matrices such that $M_{i,j} \leq 2\Delta s$, $k_{i,j} \in \mathbb{N}$, $M_{i,j} + s^{2/3} + 1 \leq k_{i,j} \leq k$ for all $ij \in E(R)$. Suppose H is an r-partite graph with vertex partition $(R, V_1, \dots, V_r)$ such that $|V_i| = n_i$ for all $i \in [r]$ and that for all $ij \in E(R)$ and every vertex $x \in V_i$ we have $d_{H[V_i,V_j]}(x) = M_{i,j} \pm s^{2/3}$. Then there exists an $(R, \vec{k}, C)$-near-equiregular graph H'' with vertex partition $(R, V_1, \dots, V_r)$ such that $H \subseteq H''$.*

Proof. Consider $ij \in E(R)$ with $|V_i| \geq |V_j|$ and let $a := |V_i| - |V_j|$. Greedily choose a set A_{ij} of a vertices in V_i such that their neighbourhoods in $H[V_i, V_j]$ are disjoint (note that we can do this as $n \geq Ck^2$). Note that $H_{ij} := H[V_i, V_j] - A_{ij}$ satisfies that for all $x \in V_i \cup V_j$, $M_{i,j} - s^{2/3} - 1 \leq d_{H_{ij}}(x) \leq M_{i,j} + s^{2/3}$. Now apply Lemma 7.6 to H_{ij} to obtain a bipartite graph H'_{ij} which is $k_{i,j}$-regular and is obtained from H_{ij} by adding edges. Obtain H''_{ij} from H'_{ij} by adding A_{ij} and connecting each $v \in A_{ij}$ to $k_{i,j}$ vertices in V_j such that $N_{H[V_i,V_j]}(v) \subseteq N_{H''_{ij}}(v)$ and such that the $N_{H''_{ij}}(v)$ are disjoint for different $v \in A_{ij}$. Repeat this for all $ij \in E(R)$ and let H'' be the union of the H''_{ij}. From the construction of H'' it is immediate that H'' is $(R, \vec{k}, C)$-near-equiregular. □

We can now combine the above results to obtain Lemma 7.1.

Proof of Lemma 7.1. First we apply Lemma 7.2 to obtain H as described there. So in particular, H can be decomposed into copies $\phi(L_1), \dots \phi(L_s)$ of $L_1, \dots, L_s$. Then we apply Lemma 7.7 to obtain H' which satisfies the requirements of Lemma 7.1.

To check the 'moreover' part of Lemma 7.1, for each $ij \in E(R)$ recall $J_{i,j} = H'[V_i, V_j] - (\phi(L_1) \cup \cdots \cup \phi(L_s)) = (H - H')[V_i, V_j]$. Lemmas 7.2 and 7.7 together imply that each $x \in V_i$ satisfies $d_{H[V_i,V_j]}(x) = M_{i,j} \pm s^{2/3}$, and $d_{H'[V_i,V_j]}(x) \in \{k_{i,j}, k_{i,j}+1\}$. Thus $\Delta(J_{i,j}) \leq k_{i,j} - M_{i,j} + 2s^{2/3}$. □

7.2. **Obtaining $(r,k,2)$-near-equiregular graphs.** In this subsection we deal with the case when the 'reduced graph' R of the partitions satisfies $R = K_r$. As described at the beginning of Section 7, we will show that we can pack any suitable family $\mathcal{L}$ of bounded degree graphs into 'edge-balanced' graphs H (see Lemma 7.9). We then can apply Lemma 7.1 to pack these graphs H together to form degree balanced graphs. We begin with a simple arithmetic observation.

Proposition 7.8. *Suppose $r, \Delta, n \in \mathbb{N}$ with $r \geq 3\Delta + 2$, and let $\overline{n} := rn + c$ for some integer c with $0 \leq c \leq r - 1$. Then there are non-negative integers a_i and n_i for all $i \in [3]$ such that*

(i) $a_2, a_3 \in \{0\} \cup \{\Delta + 1, \Delta + 2, \dots\}$,
(ii) $a_1 + a_2 + a_3 = r$,

(iii) $n_1 \in \{n-1, n\}$ *and* $n_3 = n_2 + 1 = n_1 + 2$,
(iv) $a_1 n_1 + a_2 n_2 + a_3 n_3 = \overline{n}$.

Proof. If $c = 0$, then we let $n_1 := n$, $a_1 := r$, $a_2 := a_3 := 0$. If $0 < c \leq \Delta$, then we let $n_1 := n-1$, $a_1 := \Delta + 1 - c$ and $a_3 := \Delta + 1$. If $\Delta + 1 \leq c \leq r - 1$, then we let $n_1 := n, a_1 := r - c$ and $a_3 := 0$. In all cases we let $a_2 := r - a_1 - a_3$, $n_2 := n_1 + 1$ and $n_3 := n_1 + 2$. It is easy to check that conditions (i)–(iv) are satisfied. □

Now we can state our main goal in this subsection.

Lemma 7.9. *Suppose* $0 < 1/n \ll 1/M \ll \xi \ll 1/\Delta, 1/r$ *and that* $\overline{n} = rn + c$ *for some integer* c *with* $0 \leq c \leq r - 1$. *Suppose* $s, k \in \mathbb{N}$ *and that* $(1 + \xi)M \leq k \leq (1 + 2\xi)M$. *Suppose* $L_1, \dots, L_s$ *are graphs of order* $\overline{n}$ *with* $\Delta(L_i) \leq \Delta$ *and* $e(L_i) \geq n/4$ *for all* $i \in [s]$, *and such that* $\sum_{i=1}^{s} e(L_i) = M\binom{r}{2}n$.

(i) *Suppose further that* $c = 0$ *and each* L_i *has an equitable* r*-colouring. Then there exists a graph* H *with vertex classes* $V_1, \dots, V_r$ *such that* $|V_i| = n$ *for all* $i \in [r]$, $H[V_i, V_j]$ *is* k*-regular for all* $1 \leq i < j \leq r$, *and such that* $L_1, \dots, L_s$ *pack into* H.
(ii) *Suppose alternatively that* $r \geq 3\Delta + 2$ *and* a_j, n_j *are as given by Proposition 7.8 (for all* $j \in [3]$*). Then there exists an* $(r, k, 2)$*-near-equiregular graph* H *with vertex classes* $V_1, \dots, V_r$ *and a partition of* $[r]$ *into* I_1, I_2, I_3 *where* $|I_j| = a_j$ *and* $|V_i| = n_j$ *for all* $i \in I_j$, *and such that* $L_1, \dots, L_s$ *pack into* H.

Moreover, writing $\phi(L_\ell)$ *for the copy of* L_ℓ *in this packing of* $L_1, \dots, L_s$ *in* H, *and for each* $ij \in E(R)$ *writing* $J_{i,j} := H[V_i, V_j] - (\phi(L_1) \cup \dots \cup \phi(L_s))$, *we have* $\Delta(J_{i,j}) \leq 3\xi k$.

The proof of Lemma 7.9 will be conducted via two claims. First we show how in case (i) of Lemma 7.9 the partition classes of the L_i can be individually re-arranged so that the union of the L_i is edge-balanced, i.e. the total numbers of edges in the bipartite graphs induced by each pair of partition classes is almost equal for different pairs.

Claim 7.10. *Suppose* $0 < 1/n \ll 1/M \ll 1/\Delta, 1/r$. *Suppose* $L_1, \dots, L_s$ *are graphs of order* rn *such that each* L_ℓ *satisfies* $\Delta(L_\ell) \leq \Delta$ *and has an equitable* r*-colouring with colour classes* $X_1^\ell, \dots, X_r^\ell$. *If* $e(L_\ell) \geq n/5$ *for each* $\ell \in [s]$ *and* $\sum_{\ell=1}^{s} e(L_\ell) = M\binom{r}{2}n \pm M^{3/5}n/2$, *then for each* $\ell \in [s]$ *one can permute the subscript indices of the colour classes* $X_1^\ell, \dots, X_r^\ell$ *to achieve that*

$$\sum_{\ell=1}^{s} \frac{e(L_\ell[X_j^\ell, X_{j'}^\ell])}{n} = M \pm M^{3/5} \tag{7.12}$$

for all $j, j' \in [r]$ *with* $j \neq j'$.

Note that the above constraints of $e(L_i) \geq n/5$ and $\Delta(L_i) \leq \Delta$ for all $i \in [s]$ imply

$$\frac{sn}{5} \leq \sum_{i=1}^{s} e(L_i) \leq \frac{s\Delta\overline{n}}{2}.$$

This, combined with $\sum_{i=1}^{s} e(L_i) = M\binom{r}{2}n \pm M^{3/5}n/2$, yields

$$\frac{(r-1)M}{4\Delta} \leq s \leq 10M\binom{r}{2}. \tag{7.13}$$

In other words, s is of the same order of magnitude as M in our hierarchy.

Proof of Claim 7.10. For each $\ell \in [s]$ we choose a permutation σ_ℓ on $[r]$ uniformly at random. For all $j, j' \in [r]$ with $j \neq j'$ let $D_{j,j'}$ be the random variable defined by

$$D_{j,j'} := \sum_{\ell=1}^{s} \frac{e(L_\ell[X_{\sigma_\ell(j)}^\ell, X_{\sigma_\ell(j')}^\ell])}{n}.$$

Note that

$$\mathbb{E}[e(L_\ell[X^\ell_{\sigma_\ell(j)}, X^\ell_{\sigma_\ell(j')}])] = \sum_{i\neq i'} e(L_\ell[X^\ell_i, X^\ell_{i'}]) \cdot \mathbb{P}[\sigma_\ell(j) = i, \sigma_\ell(j') = i'] = \frac{e(L_\ell)}{\binom{r}{2}}.$$

Thus

$$\mathbb{E}[D_{j,j'}] = \frac{\sum_{\ell=1}^s e(L_\ell)}{\binom{r}{2}n} = M \pm M^{3/5}/2.$$

Consider the exposure martingale $J^\ell_{j,j'} := \mathbb{E}[D_{j,j'} \mid \sigma_1, \dots, \sigma_\ell]$ for $0 \le \ell \le s$. Note that this martingale is Δ-Lipschitz since changing only one permutation σ_i changes the value of $D_{j,j'}$ by at most Δ (as $0 \le e(L_\ell[X^\ell_{\sigma_\ell(j)}, X^\ell_{\sigma_\ell(j')}]) \le \Delta n$ for any fixed ℓ, j, j'). Thus, by Azuma's inequality (Theorem 3.2), by (7.13), and by the fact that $1/M \ll 1/\Delta, 1/r$, we have

$$\mathbb{P}[D_{j,j'} \neq M \pm M^{3/5}] \le 2e^{\frac{-M^{6/5}}{8\Delta^2 s}} \le e^{-M^{1/6}}.$$

Hence with probability at least $1 - \binom{r}{2}e^{-M^{1/6}} \ge 1/2$ we can permute the subscript indices so that (7.12) holds for all $1 \le j \neq j' \le r$. □

Next we prove a similar statement for case (ii) of Lemma 7.9.

Claim 7.11. *Suppose $0 < 1/n \ll 1/M \ll 1/\Delta, 1/r$ and that $r \ge 3\Delta + 2$. Let c be an integer such that $0 \le c \le r-1$. Suppose that $L_1, \dots, L_s$ are graphs with $\Delta(L_\ell) \le \Delta$, $|L_\ell| = \overline{n} = rn + c$ and $e(L_\ell) \ge n/4$ for all $\ell \in [s]$. Suppose that $\sum_{\ell=1}^s e(L_\ell) = M\binom{r}{2}n$. For all $j \in [3]$ let a_j, n_j be as given by Proposition 7.8.*

Then there is a partition I_1, I_2, I_3 of $[r]$ with $|I_j| = a_j$ and, for each $\ell \in [s]$, there is an r-colouring of L_ℓ with colour classes $X^\ell_1, \dots, X^\ell_r$ such that $|X^\ell_i| = n_j$ for all $i \in I_j$ and

$$\sum_{\ell=1}^s \frac{e(L_\ell[X^\ell_i, X^\ell_{i'}])}{n} = M \pm M^{2/3}$$

for all $1 \le i \neq i' \le r$.

Proof. Recall from Proposition 7.8 that $n_1 \in \{n-1, n\}$. For each L_ℓ we apply the Hajnal-Szemerédi theorem (Theorem 3.5) to obtain an equitable r-colouring of L_ℓ. Take this r-colouring, and take out none, one or two vertices from each colour class to ensure that the resulting vertex classes have size n_1. Let P_ℓ be the set of those 'excess' vertices and let $Q^\ell_1, \dots, Q^\ell_r$ be the remaining colour classes. We have thus partitioned $V(L_\ell)$ into sets $Q^\ell_1, \dots, Q^\ell_r, P^\ell$ such that each $L_\ell[Q^\ell_i]$ is empty, $|Q^\ell_i| = n_1$ and $|P^\ell| = a_2 + 2a_3$.

Let $n' := rn_1$. For each $\ell \in [s]$ let $L'_\ell := L_\ell \setminus P^\ell$. So $n' = |L'_\ell|$. Since $|P^\ell| \le 2r$ by Proposition 7.8(ii), at most $2r\Delta$ edges of L_ℓ are incident to P^ℓ. Thus $e(L'_\ell) \ge n'/5$ and $\sum_{\ell=1}^s e(L_\ell) = M\binom{r}{2}n' \pm n'$. Hence by Claim 7.10 for each $\ell \in [s]$ we can permute the subscript indices of the partition classes $Q^\ell_1, \dots, Q^\ell_r$ of L'_ℓ to achieve that for all $i, i' \in [r]$ with $i \neq i'$,

$$\sum_{\ell=1}^s \frac{e(L'_\ell[Q^\ell_i, Q^\ell_{i'}])}{n'} = M \pm M^{3/5}. \tag{7.14}$$

Let $I_1 := [a_1]$, $I_2 := \{a_1 + 1, \dots, a_1 + a_2\}$ and $I_3 := \{a_1 + a_2 + 1, \dots, r\}$. Partition each P^ℓ into two sets P^ℓ_2, P^ℓ_3 of size a_2 and $2a_3$ respectively. Now define

$$Y^\ell_1 := \bigcup_{i\in I_1} Q^\ell_i, \quad Y^\ell_2 := \bigcup_{i\in I_2} Q^\ell_i \cup P^\ell_2, \quad Y^\ell_3 := \bigcup_{i\in I_3} Q^\ell_i \cup P^\ell_3.$$

Then $|Y_j^\ell| = a_j n_j$ for all $j \in [3]$ by Proposition 7.8(iii). Moreover, by (7.14) for all $j, j' \in [3]$ with $j \neq j'$ we have

$$\sum_{\ell=1}^{s} e(L_\ell[Y_j^\ell]) = (M \pm M^{3/5})\binom{a_j}{2}n' \pm \begin{cases} 0 & \text{if } j = 1, \\ 2sa_j\Delta & \text{if } j \in \{2,3\} \end{cases} \tag{7.15}$$

$$= (M \pm 2M^{3/5})\binom{a_j}{2}n,$$

$$\sum_{\ell=1}^{s} e(L_\ell[Y_j^\ell, Y_{j'}^\ell]) = (M \pm 2M^{3/5})a_j a_{j'} n. \tag{7.16}$$

Recall that $|Y_j^\ell| = a_j n_j$, and that for each $j \in \{2,3\}$, either $a_j = 0$ or $a_j \geq \Delta + 1$. Thus we can apply the Hajnal-Szemerédi theorem (Theorem 3.5) to obtain an equitable a_j-colouring of $L_\ell[Y_j^\ell]$ for each $j \in \{2,3\}$. Let $X_{a_1+1}^\ell, \dots, X_{a_1+a_2}^\ell$ be the corresponding equipartition of Y_2^ℓ into sets of size n_2 and let $X_{a_1+a_2+1}^\ell, \dots, X_r^\ell$ be the corresponding equipartition of Y_3^ℓ into sets of size n_3. Let $\{X_1^\ell, \dots, X_{a_1}^\ell\} := \{Q_i^\ell : i \in I_1\}$.

For each $\ell \in [s]$ we now choose a permutation σ_ℓ on $[r]$ such that σ_ℓ is a random permutation when restricted to each of I_1, I_2 and I_3 (choose three independent permutations on I_1, I_2 and I_3 uniformly at random, and combine them). For all $1 \leq i \neq i' \leq r$ define

$$D_{i,i'}^\ell := e(L_\ell[X_{\sigma_\ell(i)}^\ell, X_{\sigma_\ell(i')}^\ell]) \quad \text{and} \quad D_{i,i'} := \frac{1}{n}\sum_{\ell=1}^{s} D_{i,i'}^\ell.$$

Given $1 \leq i \neq i' \leq r$ and $\ell \in [s]$, let $j, j' \in [3]$ be so that $i \in I_j, i' \in I_{j'}$. Then

$$\mathbb{E}[D_{i,i'}^\ell] = \sum_{b \neq b', b \in I_j, b' \in I_{j'}} e(L_\ell[X_b^\ell, X_{b'}^\ell])\mathbb{P}\left[\sigma_\ell(i) = b, \sigma_\ell(i') = b'\right]$$

$$= \begin{cases} \sum_{b \neq b' \in I_j} \frac{e(L_\ell[X_b^\ell, X_{b'}^\ell])}{a_j(a_j-1)} = \frac{2e(L_\ell[Y_j^\ell])}{a_j(a_j-1)} & \text{if } j = j' \text{ and } a_j \geq 2, \\ \sum_{b \in I_j, b' \in I_{j'}} \frac{e(L_\ell[X_b^\ell, X_{b'}^\ell])}{a_j a_{j'}} = \frac{e(L_\ell[Y_j^\ell, Y_{j'}^\ell])}{a_j a_{j'}} & \text{if } j \neq j' \text{ and } a_j, a_{j'} \geq 1. \end{cases}$$

Together with (7.15) this implies that for every $1 \leq i \neq i' \leq r$

$$\mathbb{E}[D_{i,i'}] = M \pm 2M^{3/5}.$$

Consider the exposure martingale $J_{i,i'}^\ell := \mathbb{E}[D_{i,i'} \mid \sigma_1, \dots, \sigma_\ell]$ with $0 \leq \ell \leq s$. Note that $J_{i,i'}^\ell$ is Δ-Lipschitz. Thus by Azuma's inequality (Theorem 3.2) and the fact that $1/M \ll 1/\Delta, 1/r$ we obtain

$$\mathbb{P}[D_{i,i'} \neq M \pm M^{2/3}] \leq 2e^{\frac{-M^{4/3}/4}{2\Delta^2 s}} \overset{(7.13)}{\leq} e^{-M^{1/4}}.$$

Therefore with probability at least $1 - \binom{r}{2}e^{-M^{1/4}} \geq 1/2$ we have

$$\sum_{\ell=1}^{s} \frac{e(L_\ell[X_{\sigma_\ell(i)}^\ell, X_{\sigma_\ell(i')}^\ell])}{n} = M \pm M^{2/3} \tag{7.17}$$

for all $1 \leq i \neq i' \leq r$. The claim now follows by relabelling the subscript indices of the X_i^ℓ according to σ_ℓ. □

We now conclude the proof of Lemma 7.9.

Proof of Lemma 7.9. To prove Lemma 7.9(i), we first apply Claim 7.10. We then apply Lemma 7.1 with $C = 0$, with K_r playing the role of R and with $\vec{k}$, $\vec{M}$ being the matrices with entries $k_{i,j} := k$ and $M_{i,j} := \sum_{\ell=1}^{s} \frac{e(L_\ell[X_i^\ell, X_j^\ell])}{n} = M \pm M^{3/5}$ for all $i \neq j$ (note that $s^{2/3} + M^{3/5} + 1 \leq \xi M$ by (7.13)). To prove Lemma 7.9(ii), we proceed similarly, except that we first apply Claim 7.11, and that we apply Lemma 7.1 with $C = 2$ instead of $C = 0$.

In either case, the moreover part of Lemma 7.1 implies that $\Delta(J_{i,j}) \leq k - M_{i,j} + 2s^{2/3} \leq k - (M - M^{3/5}) + 2s^{2/3} \leq 3\xi M \leq 3\xi k$. □

8. Combining the packing results

Combining Theorem 6.1 and Lemma 7.1, we obtain the following result. It gives an approximate decomposition of G into $L_1, \dots, L_s$ if the L_i have a common vertex partition so that the densities of the resulting bipartite pairs reflect the densities of the corresponding pairs in G. It immediately implies Theorem 1.4 and generalises it to the setting where the pairs of G have different densities. Properties (T1) and (T3) of Theorem 6.1 will not be used in this section, so as remarked after Theorem 6.1, we can ignore the conditions in (S6) and (S7) whenever we apply Theorem 6.1 in this section.

Theorem 8.1. *Suppose $0 < 1/n \ll \varepsilon \ll \alpha, \eta, d, 1/(C+1), 1/\Delta, 1/\Delta_R$ and $1/n \ll 1/r$. Let $s \in \mathbb{N}$ be such that $s \le \eta^{-1}n$. Suppose that R is a graph on $[r]$ with $\Delta(R) \le \Delta_R$. Let $\vec{d}$ be a symmetric $r \times r$ matrix such that $d_{j,j'} \ge d$ for all $jj' \in E(R)$ and $d_{j,j'} = 0$ if $jj' \notin E(R)$. Suppose that $L_1, \dots, L_s$ are graphs admitting a common vertex partition $(R, X_1, \dots, X_r)$ such that $\max_{j\in[r]} |X_j| = n$ and $n - C \le |X_j| \le n$ for all $j \in [r]$, $\Delta(L_\ell) \le \Delta$ for all $\ell \in [s]$ and $\sum_{\ell=1}^{s} e(L_\ell[X_j, X_{j'}]) \le (1-\alpha)d_{j,j'}n^2$ for all $jj' \in E(R)$. Suppose finally that G is an $(\varepsilon, \vec{d})$-super-regular graph with partition $(R, V_1, \dots, V_r)$ such that $|X_i| = |V_i|$ for all $i \in [r]$. Then $L_1, \dots, L_s$ pack into G.*

Proof. Choose an integer M such that $\varepsilon \ll 1/M \ll \alpha, \eta, d, 1/\Delta, 1/\Delta_R$. Without loss of generality we may assume that $s > M^2$ and that s is a multiple of M (if not, add up to M^2 empty graphs L_ℓ). Let $t := s/M$. Then we may take an arbitrary partition of $\{L_1, \dots, L_s\}$ into $\mathcal{L}_1, \dots \mathcal{L}_t$ such that $|\mathcal{L}_i| = M$ for all $i \in [t]$.

Now for all $i \in [t]$ and $jj' \in E(R)$, let $M^i_{j,j'} := \sum_{L\in\mathcal{L}_i} \frac{e(L[X_j,X_{j'}])}{n}$. Then $M^i_{j,j'} \le \Delta M$ and $\sum_{i=1}^{t} M^i_{j,j'} \le (1-\alpha)d_{j,j'}n$. If $jj' \in E(R)$, let $k^i_{j,j'}$ be the smallest integer satisfying $k^i_{j,j'} \ge M^i_{j,j'} + 2M^{2/3}$, and if $jj' \notin E(R)$ let $k^i_{j,j'} := 0$. Apply Lemma 7.1 for all $i \in [t]$ to find an $(R, \vec{k}^i, C)$-near-equiregular graph H_i with partition $(R, V_1, \dots, V_r)$ such that the graphs in $\mathcal{L}_i$ pack into H_i. Then for all $jj' \in E(R)$,

$$\begin{aligned}\sum_{i=1}^{t} k^i_{j,j'} &\le \sum_{i=1}^{t}(M^i_{j,j'} + 2M^{2/3} + 1) \le (1-\alpha)d_{j,j'}n + t(2M^{2/3}+1)\\ &\le (1-\alpha)d_{j,j'}n + \frac{3s}{M^{1/3}} \le (1-\alpha)d_{j,j'}n + \alpha dn/2 \le (1-\alpha/2)d_{j,j'}n.\end{aligned}$$

Here, we obtain the penultimate inequality since $1/M \ll \eta, \alpha, d$ and $s \le \eta^{-1}n$. It is easy to see that $H_1, \dots, H_t$, G, R satisfy the assumptions in Theorem 6.1, where $\alpha/2$ plays the role of α. Thus $H_1, \dots, H_t$ pack into G, and so $L_1, \dots, L_s$ also pack into G. □

In the next result, the reduced graph R of the partition of G is complete. In this case, we can drop the requirement that the L_ℓ and G have a common partition.

Theorem 8.2. *Suppose $0 < 1/n \ll \varepsilon \ll \lambda, d, 1/\Delta, 1/r$ and that $r \ge 3\Delta + 2$. Let $c, \overline{n}$ be integers such that $0 \le c \le r-1$ and $\overline{n} = nr + c$. Let $a_1, a_2, a_3, n_1, n_2, n_3$ be non-negative integers such that*

(i) $a_2, a_3 \in \{0\} \cup \{\Delta+1, \Delta+2, \dots\}$,
(ii) $a_1 + a_2 + a_3 = r$,
(iii) $n_1 \in \{n-1, n\}$ *and* $n_3 = n_2 + 1 = n_1 + 2$,
(iv) $a_1n_1 + a_2n_2 + a_3n_3 = \overline{n}$.

Let I_1, I_2, I_3 be a partition of $[r]$ with $|I_j| = a_j$. Suppose that $L_1, \dots, L_s$ are graphs with $|L_\ell| = \overline{n}$ and $\Delta(L_\ell) \le \Delta$ for all $\ell \in [s]$. Suppose further that G is an $\overline{n}$-vertex (ε, d)-super-regular graph with respect to a partition $(V_1, \dots, V_r)$, where $|V_i| = n_j$ for all $i \in I_j$. Suppose finally that $(1-9\lambda)e(G) \le \sum_{\ell=1}^{s} e(L_\ell) \le (1-\lambda)e(G)$. Then $L_1, \dots, L_s$ pack into G.

Moreover, writing $\phi(L_\ell)$ for the copy of L_ℓ in this packing of $L_1, \dots, L_s$ in H, and writing $J' := G - (\phi(L_1) \cup \dots \cup \phi(L_s))$, we have $\Delta(J') \le 10\lambda d\overline{n}$.

Proof. First, if there are graphs $L_{j_1},\dots,L_{j_q}$, each having fewer than $n/4$ edges, then we can pack two of them into a single graph without increasing the maximum degree. We keep doing this until we obtain at most one graph having fewer than $n/4$ edges. Once we have at most one such graph, we add at most $n/4$ suitable new edges to assume that all graphs L_ℓ have at least $n/4$ edges and still satisfy $\Delta(L_\ell)\le\Delta$. Choose $k\in\mathbb{N}$ and ξ such that $\varepsilon\ll 1/k\ll\xi\ll\lambda,1/r,1/\Delta$. Note that we may assume that $\sum_{\ell=1}^s e(L_\ell)\ge k^2n$. (Indeed, since $\sum_{\ell=1}^s e(L_\ell)\ge(1-9\lambda)e(G)$ this is clearly true if $\lambda\le 1/10$. If $\lambda>1/10$ we simply add additional graphs L_ℓ if necessary.) Let

$$m:=\frac{\sum_{\ell=1}^s e(L_\ell)}{k\binom{r}{2}n(1-3\xi/2)}. \tag{8.1}$$

Partition $\{L_1,\dots,L_s\}$ into $\mathcal{L}_1,\dots,\mathcal{L}_m$ such that for all $i\in[m]$ we have

$$(1-7\xi/4)k\binom{r}{2}n\le\sum_{L\in\mathcal{L}_i}e(L)\le(1-\xi)k\binom{r}{2}n. \tag{8.2}$$

Such a partition exists since $\sum_{\ell=1}^s e(L_\ell)\ge k^2n$ and $1/k\ll\xi,1/\Delta,1/r$.

Now for each $i\in[m]$ use Lemma 7.9(ii) with $M_i:=\sum_{L\in\mathcal{L}_i}e(L)/(\binom{r}{2}n)$ playing the role of M to obtain an $(r,k,2)$-near-equiregular graph H_i with partition $X_1,\dots,X_r$, where $|V_j|=|X_j|$ for all $j\in[r]$, and so that the graphs in $\mathcal{L}_i$ can be packed into H_i. Moreover, writing $\tau(L)$ for the copy of $L\in\mathcal{L}_i$ in this packing of the graphs in $\mathcal{L}_i$ into H_i, and for each $i'\ne j'\in[r]$ writing $J^i_{i',j'}:=H_i[V_{i'},V_{j'}]-\bigcup_{L\in\mathcal{L}_i}\tau(L)$, we have $\Delta(J^i_{i',j'})\le 3\xi k$.

Note that

$$km\overset{(8.1)}{=}\frac{\sum_{\ell=1}^s e(L_\ell)}{(1-3\xi/2)\binom{r}{2}n}\le\frac{(1-\lambda)e(G)+n/4}{(1-3\xi/2)\binom{r}{2}n}\le(1-\lambda/2)(d+\varepsilon)n\le(1-\lambda/3)dn.$$

and

$$km\overset{(8.1)}{=}\frac{\sum_{\ell=1}^s e(L_\ell)}{(1-3\xi/2)\binom{r}{2}n}\ge\frac{(1-9\lambda)e(G)}{(1-3\xi/2)\binom{r}{2}n}\ge(1-9\lambda)dn. \tag{8.3}$$

In particular, the conditions of Theorem 6.1 are satisfied with K_r playing the role of R and $\lambda/3$ playing the role of α. So we can apply Theorem 6.1 to G and $H_1,\dots,H_m$, to find a packing of $H_1,\dots,H_m$ into G. This yields a packing of $L_1,\dots,L_s$ into G.

To prove the moreover part of Theorem 8.2, note that since $\Delta(G)\le d\overline{n}$, we may assume that $\lambda\le 1/10$ (and thus we did not add any additional graphs L_ℓ at the beginning of the proof). For each $i',j'\in[r]$

$$\begin{aligned}\Delta(J'[V_{i'},V_{j'}]) &\le \Delta(G[V_{i'},V_{j'}])-km+\sum_{i=1}^m\Delta(J^i_{i',j'})+\Delta\le(d+\varepsilon)n-km+3\xi km+\Delta\\ &\overset{(8.3)}{\le}(d+\varepsilon)n-(1-4\xi)(1-9\lambda)dn\le 10\lambda dn.\end{aligned}$$

Thus, $\Delta(J')\le 10\lambda d(r-1)n\le 10\lambda d\overline{n}$. □

Now we can deduce Theorem 1.2 from Theorem 8.2 (note that the conditions (1) and (2) needed in the proof are actually weaker than the quasi-randomness assumption of Theorem 1.2).

Proof of Theorem 1.2. Choose $r,n_0\in\mathbb{N}$ and $\varepsilon>0$ such that $1/n_0\ll\varepsilon\ll 1/r\ll\alpha,p_0,1/\Delta$. Let $n\ge n_0$, $p\ge p_0$ and assume that G is an n-vertex graph satisfying the following two assumptions.

(1) Each vertex v in G satisfies $d_G(v)=(1\pm\varepsilon)pn$.
(2) There exists $D\subseteq\binom{V(G)}{2}$ such that $|D|\ge\binom{n}{2}-\varepsilon n^2$ and all pairs $\{u,v\}\in D$ satisfy $|N_G(u,v)|=(1\pm\varepsilon)p^2n$.

Note that an (ε,p)-quasi-random graph satisfies both (1) and (2). By adding additional graphs L_ℓ if necessary, we may assume that $\sum_{\ell=1}^s e(L_\ell)\ge(1-2\alpha)\binom{n}{2}p$. Let $n':=\lfloor n/r\rfloor$ and let $c:=n-rn'$. Choose a_1,a_2,a_3,n_1,n_2,n_3 satisfying (i)–(iv) of Theorem 8.2 (with n, n' playing the roles of $\overline{n}$, n in Theorem 8.2). Such a choice is possible by Proposition 7.8. Let I_1,I_2,I_3 be a partition of $[r]$ with $|I_i|=a_i$.

Now choose a partition $\mathcal{V} = (V_1, \dots, V_r)$ of $V(G)$ such that $|V_i| = a_i$ for all $i \in I_j$ with $j \in [3]$ and such that $G[V_i, V_{i'}]$ is $(\varepsilon^{1/7}, p)$-super-regular for all $i \neq i' \in [r]$. (To see that such a partition exists, consider a partition $\mathcal{V} = (V_1, \dots, V_r)$ of $V(G)$ chosen uniformly at random among all the partitions satisfying $|V_i| = n_i$ for all $i \in I_j$ and apply Lemma 3.3 as well as Theorem 3.11.)

Now we let G' be the spanning subgraph of G such that $G'[V_i] = \emptyset$ and $G'[V_i, V_j] = G[V_i, V_j]$ for $i \neq j \in [r]$. Then G' is $(\varepsilon^{1/7}, p)$-super-regular with respect to $\mathcal{V}$. Moreover, since $1/r \ll \alpha, p_0$ we have $e(G') \geq (1 - \alpha/2)e(G) = (1 - \alpha/2)(1 \pm 2\varepsilon)\binom{n}{2}p$. Thus

$$(1 - 3\alpha)\, e(G') \leq (1 - 2\alpha) \binom{n}{2} p \leq \sum_{\ell=1}^{s} e(H_\ell) \leq (1 - \alpha) \binom{n}{2} p \leq (1 - \alpha/3)\, e(G').$$

Now we apply Theorem 8.2 with $\varepsilon^{1/7}$, $\alpha/3$, p, n, n' playing the roles of ε, λ, d, $\overline{n}$, n to find the desired packing of $H_1, \dots, H_s$ in $G' \subseteq G$. Moreover, writing $\phi(H_\ell)$ for the copy of H_ℓ in this packing of $H_1, \dots, H_s$ in G, and writing $J' := G' - (\phi(H_1) \cup \dots \cup \phi(H_s))$, we have $\Delta(J') \leq 10\alpha pn/3$. Since $J = (G - G') \cup J'$, we conclude $\Delta(J) \leq \Delta(G - G') + \Delta(J') \leq n/r + 10\alpha pn/3 \leq 4\alpha pn$. □

Similarly, we can now deduce Theorem 1.3 from Theorem 6.1.

Proof of Theorem 1.3. This follows from the same argument as in the proof of Theorem 8.2 with α playing the role of λ. Here the assumption $r \geq 3\Delta + 2$ is not required since a common partition of the graphs H_ℓ and G is given. In particular, this time we apply Lemma 7.9(i) instead of Lemma 7.9(ii). □

Next, we prove Corollary 1.5. For this, we use the following theorem by Kühn and Osthus. It a special case of Theorem 1.2 in [32], which is stated for the more general class of robustly expanding graphs (in [32], this result on Hamilton decompositions of robustly expanding graphs is in turn derived from a digraph version, which is the main result of [31]).

Theorem 8.3. [31, 32] *For every $p_0 > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $p \geq p_0$ and $n \geq n_0$ for which pn is even. Let G be any pn-regular (ε, p)-quasi-random graph on n vertices. Then G has a Hamilton decomposition.*

Proof of Corollary 1.5. Note that without loss of generality we may assume that $\beta \leq p_0/3$. Choose $\varepsilon, \alpha > 0$ and $n_0 \in \mathbb{N}$ such that $1/n_0 \ll \varepsilon \ll \alpha \ll p_0, \beta, 1/\Delta$. Let $n \geq n_0$, $p \geq p_0$, and let G and $H_1, \dots, H_s$ be as given in Corollary 1.5. Choose a subgraph J of G such that $G' := G - J$ is $(2\varepsilon, p')$-quasi-random where $p' = p - 2\beta + \alpha$ and J is $(2\varepsilon, 2\beta - \alpha)$-quasi-random. (Indeed, for each edge e of G, we include e in $E(J)$ with probability $(2\beta - \alpha)/p$. Then it is straightforward to check that J is $(2\varepsilon, 2\beta - \alpha)$-quasi-random and $G' = G - J$ is also $(2\varepsilon, p - 2\beta + \alpha)$-quasi-random with high probability.)

Since $\varepsilon \ll \alpha, p', 1/\Delta$, Theorem 1.2 implies that $H_{\beta n+1}, \dots, H_s$ pack into G'. In other words, for each $\beta n + 1 \leq \ell \leq s$ there exists a copy $\phi_\ell(H_\ell)$ of H_ℓ in G such that the $\phi_\ell(H_\ell)$ are pairwise edge-disjoint. Let

$$G^* := G - \bigcup_{\ell=\beta n+1}^{s} \phi_\ell(H_\ell).$$

Then G^* is $2\beta n$-regular because G is pn-regular and each H_ℓ is r_ℓ-regular with $\sum_{\ell=\beta n+1}^{s} r_\ell = (p - 2\beta)n$. Let $G'' := G' - \bigcup_{\ell=\beta n+1}^{s} \phi_\ell(H_\ell)$. Then any vertex $v \in V(G)$ satisfies

$$d_{G''}(v) = (1 \pm 2\varepsilon)(p - 2\beta + \alpha)n - \sum_{\ell=\beta n+1}^{s} r_\ell = (\alpha \pm 2\varepsilon)n. \tag{8.4}$$

Also recall that J is $(2\varepsilon, 2\beta - \alpha)$-quasi-random. Thus for any $S \in \binom{V(G)}{2}$, we have

$$N_{G^*}(S) = N_J(S) \pm 2\Delta(G'') \overset{(8.4)}{=} (1 \pm 2\varepsilon)(2\beta - \alpha)^2 n \pm 2(\alpha \pm 2\varepsilon)n = (1 \pm \alpha^{1/2})(2\beta)^2 n.$$

Thus G^* is a $(\alpha^{1/2}, 2\beta)$-quasi-random $2\beta n$-regular graph. This means that we can apply Theorem 8.3 to G^* to obtain a decomposition of G^* into $H_1, \dots, H_{\beta n}$. Together with the $\phi_\ell(H_\ell)$ this gives the required decomposition. □

The following bipartite version of the tree packing conjecture was formulated by Hobbs, Bourgeois and Kasiraj [22] in 1987.

Conjecture 8.4. *Let $T_1, \dots, T_n$ be trees such that, for each $i \in [n]$, T_i has i vertices. Let G be the complete bipartite graph $K_{n-1,n/2}$ if n is even, and $K_{n,(n-1)/2}$ if n is odd. Then G has a decomposition into copies of $T_1, \dots, T_n$.*

The following result implies an approximate version of this conjecture for bounded degree trees (see Corollary 8.6). We derive Theorem 8.5 from Theorem 8.1.

Theorem 8.5. *Suppose $0 < 1/n \ll \varepsilon \ll \alpha, 1/\Delta$. Let $H_1, \dots, H_s$ be n-vertex bipartite graphs with $\Delta(H_i) \le \Delta$ for all $i \in [s]$. Let G be an (ε, d)-super-regular bipartite graph with bipartition (A, B) such that $|A| = \lfloor n/2 \rfloor$ and $|B| = n - 1$. If $\sum_{i=1}^s e(H_i) \le (1-\alpha)e(G)$, then $H_1, \dots, H_s$ pack into G.*

Proof. If H_i and H_j both have at most $n/4$ edges, then we can pack them disjointly into an n-vertex bipartite graph without increasing the maximum degree. So we may assume that $e(H_i) \ge n/4$ and $\Delta(H_i) \le 2\Delta$ for all $i \in [s]$. Note that this implies that $s \le 2n$. For each H_i, we take a bipartition (A_i, B_i) such that $|A_i| \le |B_i|$. Note that $|A_i| \le \lfloor n/2 \rfloor, |B_i| \le n-1$, so we may assume $|A_i| = \lfloor n/2 \rfloor, |B_i| = n-1$ by adding isolated vertices if needed. Let $X_1^i := A_i$. Take an equipartition (X_2^i, X_3^i) of B_i with $|X_2^i| = \lceil (n-1)/2 \rceil, |X_3^i| = \lfloor (n-1)/2 \rfloor$ such that $|e(H_i[X_1^i, X_2^i]) - e(H_i[X_1^i, X_3^i])| \le \varepsilon n$. (To see that such an equipartition exists, consider a random equipartition and apply Theorem 3.2.)

Let $X_1 := A$. We also choose a partition (X_2, X_3) of B with $|X_2| = \lceil (n-1)/2 \rceil, |X_3| = \lfloor (n-1)/2 \rfloor$ such that both $G[X_1, X_2]$ and $G[X_1, X_3]$ are $(4\varepsilon, d)$-super-regular. (Again, consider a random equipartition for this.) Then it is easy to see that

$$\sum_{i=1}^s e(H_i[X_1^i, X_2^i]) \le \sum_{i=1}^s (e(H_i)/2 + \varepsilon n) \le (1 - \alpha/3) d |X_1||X_2|.$$

Similarly $\sum_{i=1}^s e(H_i[X_1^i, X_3^i]) \le (1-\alpha/3)d|X_1||X_3|$. Let R be a path on three vertices such that $V(R) = [3]$ and $d_R(1) = 2$. Then each H_i admits vertex partition (R, X_1^i, X_2^i, X_3^i). So we can now apply Theorem 8.1 to obtain the desired packing. □

The following corollary is immediate from Theorem 8.5. Note that we consider $K_{n-1,\lfloor n/2 \rfloor}$ here, which also implies the result for $K_{n,\lfloor n/2 \rfloor}$.

Corollary 8.6. *Suppose $0 < 1/n \ll \alpha, 1/\Delta$. Let $T_{\alpha n}, \dots, T_n$ be trees such that, for each $\alpha n \le i \le n$, T_i has i vertices and $\Delta(T_i) \le \Delta$. Then $T_{\alpha n}, \dots, T_n$ pack into $K_{n-1,\lfloor n/2 \rfloor}$.*

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