DECOMPOSITIONS OF COMPLETE UNIFORM HYPERGRAPHS INTO HAMILTON BERGE CYCLES

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Abstract. In 1973 Bermond, Germa, Heydemann and Sotteau conjectured that if n divides $\binom{n}{k}$, then the complete k-uniform hypergraph on n vertices has a decomposition into Hamilton Berge cycles. Here a Berge cycle consists of an alternating sequence $v_1, e_1, v_2, \dots, v_n, e_n$ of distinct vertices v_i and distinct edges e_i so that each e_i contains v_i and v_{i+1}. So the divisibility condition is clearly necessary. In this note, we prove that the conjecture holds whenever $k \geq 4$ and $n \geq 30$. Our argument is based on the Kruskal-Katona theorem. The case when $k = 3$ was already solved by Verrall, building on results of Bermond.

1. Introduction

A classical result of Walecki [12] states that the complete graph K_n on n vertices has a Hamilton decomposition if and only if n is odd. (A Hamilton decomposition of a graph G is a set of edge-disjoint Hamilton cycles containing all edges of G.) Analogues of this result were proved for complete digraphs by Tillson [13] and more recently for (large) tournaments in [9]. Clearly, it is also natural to ask for a hypergraph generalisation of Walecki's theorem.

There are several notions of a hypergraph cycle, the earliest one is due Berge: A *Berge cycle* consists of an alternating sequence $v_1, e_1, v_2, \dots, v_n, e_n$ of distinct vertices v_i and distinct edges e_i so that each e_i contains v_i and v_{i+1}. A Berge cycle is a Hamilton (Berge) cycle of a hypergraph G if $\{v_1, \dots, v_n\}$ is the vertex set of G and each e_i is an edge of G. So a Hamilton Berge cycle has n edges.

Let $K_n^{(k)}$ denote the complete k-uniform hypergraph on n vertices. Clearly, a necessary condition for the existence of a decomposition of $K_n^{(k)}$ into Hamilton Berge cycles is that n divides $\binom{n}{k}$. Bermond, Germa, Heydemann and Sotteau [5] conjectured that this condition is also sufficient. For $k = 3$, this conjecture follows by combining the results of Bermond [4] and Verrall [15].

We show that as long as n is not too small, the conjecture holds for $k \geq 4$ as well.

Theorem 1. *Suppose that $4 \leq k < n$, that $n \geq 30$ and that n divides $\binom{n}{k}$. Then the complete k-uniform hypergraph $K_n^{(k)}$ on n vertices has a decomposition into Hamilton Berge cycles.*

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Walecki's theorem has a natural extension to the case when n is even: in this case, one can show that $K_n - M$ has a Hamilton decomposition, whenever M is a perfect matching. Similarly, the results of Bermond [4] and Verrall [15] together imply that for all n, either $K_n^{(3)}$ or $K_n^{(3)} - M$ have a decomposition into Hamilton Berge cycles.

We prove an analogue of this for $k \geq 4$. Note that Theorem 2 immediately implies Theorem 1.

Theorem 2. *Let $k, n \in \mathbb{N}$ be such that $3 \leq k < n$.*

(i) *Suppose that $k \geq 5$ and $n \geq 20$ or that $k = 4$ and $n \geq 30$. Let M be any set consisting of less than n edges of $K_n^{(k)}$ such that n divides $|E(K_n^{(k)}) \setminus M|$. Then $K_n^{(k)} - M$ has a decomposition into Hamilton Berge cycles.*

(ii) *Suppose that $k = 3$ and $n \geq 100$. If $\binom{n}{3}$ is not divisible by n, let M be any perfect matching in $K_n^{(k)}$, otherwise let $M := \emptyset$. Then $K_n^{(3)} - M$ has a decomposition into Hamilton Berge cycles.*

Note that if k is a prime and $\binom{n}{k}$ is not divisible by n, then k divides n and so in this case one can take the set M in (i) to be a union of perfect matchings. Also note that (ii) follows from the results of [4, 15]. However, our proof is far simpler, so we also include it in our argument.

Another popular notion of a hypergraph cycle is the following: a k-uniform hypergraph C is an *ℓ-cycle* if there exists a cyclic ordering of the vertices of C such that every edge of C consists of k consecutive vertices and such that every pair of consecutive edges (in the natural ordering of the edges) intersects in precisely ℓ vertices. If $\ell = k-1$, then C is called a *tight cycle* and if $\ell = 1$, then C is called a *loose cycle*. We conjecture an analogue of Theorem 1 for Hamilton ℓ-cycles.

Conjecture 3. *For all $k, \ell \in \mathbb{N}$ with $\ell < k$ there exists an integer n_0 such that the following holds for all $n \geq n_0$. Suppose that $k - \ell$ divides n and that $n/(k-\ell)$ divides $\binom{n}{k}$. Then $K_n^{(k)}$ has a decomposition into Hamilton ℓ-cycles.*

To see that the divisibility conditions are necessary, note that every Hamilton ℓ-cycle contains exactly $n/(k-\ell)$ edges. Moreover, it is also worth noting the following: consider the number $N := \frac{k-\ell}{n}\binom{n}{k}$ of cycles we require in the decomposition. The divisibility conditions ensure that N is not only an integer but also a multiple of $f := (k-\ell)/h$, where h is the highest common factor of k and ℓ. This is relevant as one can construct a regular hypergraph from the edge-disjoint union of t edge-disjoint Hamilton ℓ-cycles if and only if t is a multiple of f.

The 'tight' case $\ell = k-1$ of Conjecture 3 was already formulated by Bailey and Stevens [1]. In fact, if n and k are coprime, the case $\ell = k-1$ already corresponds to a conjecture made independently by Baranyai [3] and Katona on so-called 'wreath decompositions'. A k-partite analogue of the 'tight' case of Conjecture 3 was recently proved by Schroeder [14].

Conjecture 3 is known to hold 'approximately' (with some additional additional divisibility conditions on n), i.e. one can find a set of edge-disjoint Hamilton ℓ-cycles which together cover almost all the edges of $K_n^{(k)}$. This is a very special case of results in [2, 6, 7] which guarantee approximate decompositions of quasi-random uniform hypergraphs into Hamilton ℓ-cycles (again, the proofs need n to satisfy additional divisibility constraints).

2. Proof of Theorem 2

Before we can prove Theorem 2 we need to introduce some notation. Given integers $0 \leq k \leq n$, we will write $[n]^{(k)}$ for the set consisting of all k-element subsets of $[n] := \{1, \dots, n\}$. The *colexicographic order* on $[n]^{(k)}$ is the order in which $A < B$ if and only if the largest element of $(A \cup B) \setminus (A \cap B)$ lies in B (for all distinct $A, B \in [n]^{(k)}$). The *lexicographic order* on $[n]^{(k)}$ is the order in which $A < B$ if and only if the smallest element of $(A \cup B) \setminus (A \cap B)$ lies in A. Given $\ell \in \mathbb{N}$ with $\ell \leq k$ and a set $S \subseteq [n]^{(k)}$, the *ℓth lower shadow of S* is the set $\partial_\ell^-(S)$ consisting of all those $t \in [n]^{(k-\ell)}$ for which there exists $s \in S$ with $t \subseteq s$. Similarly, given $\ell \in \mathbb{N}$ with $k + \ell \leq n$ and a set $S \subseteq [n]^{(k)}$, the *ℓth upper shadow of S* is the set $\partial_\ell^+(S)$ consisting of all those $t \in [n]^{(k+\ell)}$ for which there exists $s \in S$ with $s \subseteq t$. We need the following consequence of the Kruskal-Katona theorem [8, 10].

Lemma 4.

(i) *Let $k, n \in \mathbb{N}$ be such that $3 \leq k \leq n$. Given a nonempty $S \subseteq [n]^{(k)}$, define $s \in \mathbb{R}$ by $|S| = \binom{s}{k}$. Then $|\partial_{k-2}^-(S)| \geq \binom{s}{2}$.*

(ii) *Suppose that $S' \subsetneq [n]^{(2)}$ and let $c, d \in \mathbb{N} \cup \{0\}$ be such that $c < n$, $d < n - (c+1)$ and $|S'| = cn - \binom{c+1}{2} + d$. If $n \geq 100$ and $c \leq 8$ then $|\partial_1^+(S')| \geq c\binom{n-c}{2} + 2dn/5$.*

(iii) *If $S' \subseteq [n]^{(2)}$ and $|S'| \leq n - 1$ then $|\partial_2^+(S')| \geq |S'|\binom{n-|S'|-1}{2} + \binom{|S'|}{2}(n - |S'| - 1)$.*

Proof. The Kruskal-Katona theorem states that the size of the lower shadow of a set $S \subseteq [n]^{(k)}$ is minimized if S is an initial segment of $[n]^{(k)}$ in the colexicographic order. (i) is a special case of a weaker (quantitative) version of this due to Lovász [11]. In order to prove (ii) and (iii), note that whenever $A, B \in [n]^{(k)}$ then $A < B$ in the colexicographic order if and only if $[n] \setminus A < [n] \setminus B$ in the lexicographic order on $[n]^{(n-k)}$ with the order of the ground set reversed. Thus, by considering complements, it follows from the Kruskal-Katona theorem that the size of the upper shadow of a set $S' \subseteq [n]^{(k)}$ is minimized if S' is an initial segment of $[n]^{(k)}$ in the lexicographic order. This immediately implies (iii). Moreover, if S', c and d are as in (ii), then

$$\begin{aligned} |\partial_1^+ S'| &\geq \binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{n-c}{2} + d(n-c-2) - \binom{d}{2} \\ &\geq c\binom{n-c}{2} + \frac{2}{5}dn, \end{aligned}$$

as required. $\square$

We will also use the following result of Tillson [13] on Hamilton decompositions of complete digraphs. (The *complete digraph* DK_n on n vertices has a directed edge xy between every ordered pair $x \neq y$ of vertices. So $|E(DK_n)| = n(n-1)$.)

Theorem 5. *The complete digraph DK_n on n vertices has a Hamilton decomposition if and only if $n \neq 4, 6$.*

Proof of Theorem 2. The first part of the proof for (i) and (ii) is identical. So let M be as in (i),(ii). (For (ii) note that if $\binom{n}{3}$ is not divisible by n, then 3 divides n and n divides $\binom{n}{3} - \frac{n}{3}$.) Let

$$\ell := \left\lfloor \frac{\binom{n}{k} - |M|}{n(n-1)} \right\rfloor \qquad \text{and} \qquad m := \frac{\binom{n}{k} - |M| - \ell n(n-1)}{n}.$$

Note that $m < n-1$ and $m \in \mathbb{N} \cup \{0\}$ since n divides $\binom{n}{k} - |M|$. Define an auxiliary (balanced) bipartite graph G with vertex classes A_* and B of size $\binom{n}{k} - |M|$ as follows. Let $A := E(K_n^{(k)})$ and $A_* := A \setminus M$. Let $D_1, \dots, D_\ell$ be copies of the complete digraph DK_n on n vertices. For each $i \in [\ell]$ let B_i, B_i' be a partition of $E(D_i)$ such that for every pair xy, yx of opposite directed edges, B_i contains precisely one of xy, yx. Apply Theorem 5 to find m edge-disjoint Hamilton cycles $H_1, \dots, H_m$ in DK_n. We view the sets $B_1, \dots, B_\ell$, $B_1', \dots, B_\ell'$ and $E(H_1), \dots, E(H_m)$ as being pairwise disjoint and let B denote the union of these sets. So $|B| = |A_*|$. Our auxiliary bipartite graph G contains an edge between $z \in A_*$ and $xy \in B$ if and only if $\{x, y\} \subseteq z$.

We claim that G contains a perfect matching F. Before we prove this claim, let us show how it implies Theorem 2. For each $i \in [\ell]$, apply Theorem 5 to obtain a Hamilton decomposition $H_i^1, \dots, H_i^{n-1}$ of D_i. For each $i \in [\ell]$ and each $j \in [n-1]$ let $A_i^j \subseteq A$ be the neighbourhood of $E(H_i^j)$ in F. Note that each A_i^j is the edge set of a Hamilton Berge cycle of $K_n^{(k)} - M$. Similarly, for each $i' \in [m]$ the neighbourhood $A_{i'}$ of $E(H_{i'})$ in F is the edge set of a Hamilton Berge cycle of $K_n^{(k)} - M$. Since all the sets A_i^j and $A_{i'}$ are pairwise disjoint, this gives a decomposition of $K_n^{(k)} - M$ into Hamilton Berge cycles.

Thus it remains to show that G satisfies Hall's condition. So consider any nonempty set $S \subseteq A_*$ and define $s, a \in \mathbb{R}$ with $k \leq s \leq n$ and $0 < a \leq 1$ by $|S| = a\binom{n}{k} = \binom{s}{k}$. Define b by $|N_G(S) \cap B_1| = b\binom{n}{2}$. Note that $|N_G(S) \cap B_1| \geq \binom{s}{2}$ by Lemma 4(i). But

$$\frac{b^k}{a^2} \geq \frac{\binom{s}{2}^k \binom{n}{k}^2}{\binom{n}{2}^k \binom{s}{k}^2} \geq 1,$$

and so $b \geq a^{2/k}$. Thus

$$\begin{aligned} |N_G(S)| &\geq 2\ell |N_G(S) \cap B_1| \geq 2\ell a^{2/k} \binom{n}{2} = a^{2/k}(|B| - |E(H_1) \cup \dots \cup E(H_m)|) \\ &\geq a^{2/k} \left(|A_*| - n(n-2)\right). \end{aligned}$$

Let
$$g := \frac{\binom{n}{k} - |A_*| + n(n-2)}{\binom{n}{k}}.$$
So if
$$a^{1-2/k} \leq \frac{|A_*|}{\binom{n}{k}} - \frac{n(n-2)}{\binom{n}{k}} = 1 - g, \tag{1}$$
then $|N_G(S)| \geq |S|$. We now distinguish three cases.

Case 1. $4 \leq k \leq n-3$
Since
$$|A_*| - 2n(n-1) \leq |A_*| - \left(\binom{n}{k} - |A_*|\right) - 2n(n-2) = (1-2g)\binom{n}{k} \leq (1-g)^2\binom{n}{k},$$
in this case (1) implies that $|N_G(S)| \geq |S|$ if $|S| \leq |A_*| - 2n(n-1)$. So suppose that $|S| > |A_*| - 2n(n-1)$. Note that if $k \geq 5$ then every $b \in B$ satisfies $|N_G(b)| = \binom{n-2}{k-2} - |M| \geq \binom{n-2}{3} - n \geq 2n(n-1)$ since $n \geq k+3$ and $n \geq 20$. Hence $N_G(S) = B$.

So we may assume that $k = 4$ and $S' := B \setminus N_G(S) \neq \emptyset$. Thus $S_1' := S' \cap B_1 \neq \emptyset$ and $|S'| \leq (2\ell+2)|S_1'|$. Note that $|N_G(S_1')| \leq |A_* \setminus S| < 2n(n-1)$. First suppose $|S_1'| \geq 7$. Then $|N_G(S_1')| \geq 7\binom{n-8}{2} + 21(n-8) - |M| > 2n(n-1)$ by Lemma 4(iii) and our assumption that $n \geq 30$. So we may assume that $|S_1'| \leq 6$. Apply Lemma 4(iii) again to see that
$$\begin{aligned}|N_G(S')| &\geq |S_1'|\binom{n-7}{2} - |M| \geq \frac{\binom{n-7}{2}}{2\ell+2}|S'| - n \geq \frac{6(n-7)(n-8)}{(n-2)(n-3)+24}|S'| - n \\ &\geq 2|S'| - n > |S'|.\end{aligned}$$
(Here we use that $|S'| \geq 2\ell > n$ and $n \geq 30$.) Thus $|N_G(S)| \geq |S|$, as required.

Case 2. $k = 3$
Since
$$|A_*| - 3n(n-1) \leq |A_*| - 2\left(\binom{n}{k} - |A_*|\right) - 3n(n-2) = (1-3g)\binom{n}{k} \leq (1-g)^3\binom{n}{k},$$
in this case (1) implies that $|N_G(S)| \geq |S|$ if $|S| \leq |A_*| - 3n(n-1)$. So suppose that $|S| > |A_*| - 3n(n-1)$ and that $S' := B \setminus N_G(S) \neq \emptyset$. Thus $S_1' := S' \cap B_1 \neq \emptyset$ and $|S'| \leq (2\ell+2)|S_1'| \leq ((n-2)/3+2)|S_1'|$. Let $c, d \in \mathbb{N} \cup \{0\}$ be such that $c < n$, $d < n-(c+1)$ and $|S_1'| = cn - \binom{c+1}{2} + d$. Note that $|N_G(S_1')| \leq |A_* \setminus S| < 3n(n-1)$. Thus $c < 8$ since otherwise
$$|N_G(S_1')| \geq 8\binom{n-8}{2} - |M| \geq 8\binom{n-8}{2} - \frac{n}{3} > \frac{32}{5}\binom{n}{2} > 3n(n-1)$$
by Lemma 4(ii) and our assumption that $n \geq 100$. Let $M(S_1')$ denote the set of all those edges $e \in M$ for which there is a pair $xy \in S_1'$ with $\{x, y\} \subseteq e$. Thus $M(S_1') = \partial_1^+(S_1') \cap M$. Recall that M is a matching in the case when $k = 3$. Thus

$|M(S_1')| \leq |S_1'|$. In particular $|M(S_1')| \leq d$ if $c = 0$. Apply Lemma 4(ii) again to see that

$$\begin{aligned}|N_G(S')| \geq |N_G(S_1')| &\geq c\binom{n-c}{2} + \frac{2}{5}dn - |M(S_1')| \\ &\geq \frac{4c}{5}\binom{n}{2} + \frac{2}{5}dn - \begin{cases} n/3 & \text{if } c \geq 1 \\ d & \text{if } c = 0 \end{cases} \\ &\geq (cn+d)\cdot\frac{11}{10}\cdot\frac{n-2}{3} \geq |S_1'|\left(\frac{n-2}{3}+2\right) \geq |S'|,\end{aligned}$$

where we use that $n \geq 100$. Thus $|N_G(S)| \geq |S|$, as required.

Case 3. $n-2 \leq k \leq n-1$

If $k = n-1$ then $K_n^{(k)}$ itself is a Hamilton Berge cycle, so there is nothing to show. So suppose that $k = n-2$. In this case, it helps to be more careful with the choice of the Hamilton cycles $H_1, \dots, H_m$: instead of applying Theorem 5 to find m edge-disjoint Hamilton cycles $H_1, \dots, H_m$ in DK_n, we proceed slightly differently. Note first that $\ell = 0$. Suppose that n is odd. Then $M = \emptyset$ and $m = (n-1)/2$. If n is even, then $|M| = n/2$ and $m = n/2 - 1$. In both cases we can choose $H_1, \dots, H_m$ to be m edge-disjoint Hamilton cycles of K_n. Then a perfect matching in our auxiliary graph G still corresponds to a decomposition of $K_n^{(k)} - M$ into Hamilton Berge cycles. Also, in both cases $E(H_1) \cup \dots \cup E(H_m)$ contains all but at most $n/2$ distinct elements of $[n]^{(2)}$.

Consider any $b \in B$. Then

$$|N_G(b)| \geq \binom{n-2}{k-2} - |M| = \binom{n-2}{2} - |M| \geq \binom{n}{2}\left(1 - \frac{5}{n-1}\right) \geq \frac{2}{3}\binom{n}{2} \geq \frac{2}{3}|A_*|.$$

Now consider any $a \in A_*$. Then

$$|N_G(a)| \geq \binom{k}{2} - \frac{n}{2} \geq \frac{2}{3}\binom{n}{2} \geq \frac{2}{3}|B|.$$

So Hall's condition is satisfied and so G has a perfect matching, as required. □

The lower bounds on n have been chosen so as to streamline the calculations, and could be improved by more careful calculations.

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