

FRACTIONAL AND INTEGER MATCHINGS IN UNIFORM HYPERGRAPHS

DANIELA KÜHN, DERYK OSTHUS AND TIMOTHY TOWNSEND

ABSTRACT. Our main result improves bounds of Markström and Ruciński on the minimum d -degree which forces a perfect matching in a k -uniform hypergraph on n vertices. We also extend bounds of Bollobás, Daykin and Erdős by asymptotically determining the minimum vertex degree which forces a matching of size $t < n/2(k-1)$ in a k -uniform hypergraph on n vertices. Further asymptotically tight results on d -degrees which force large matchings are also obtained. Our approach is to prove fractional versions of the above results and then translate these into integer versions.

1. INTRODUCTION

1.1. Large matchings in hypergraphs with large degrees. A k -uniform hypergraph is a pair $G = (V, E)$ where V is a finite set of vertices and the edge set E consists of unordered k -tuples of elements of V . A *matching* (or *integer matching*) M in G is a set of disjoint edges of G . The *size* of M is the number of edges in M . We say M is *perfect* if it has size $|V|/k$. Given $S \in \binom{V}{d}$, where $0 \leq d \leq k-1$, let $\deg_G(S) = |\{e \in E : S \subseteq e\}|$ be the *degree* of S in G . Let $\delta_d(G) = \min_{S \in \binom{V}{d}} \{\deg_G(S)\}$ be the *minimum d -degree* of G . When $d = 1$, we refer to $\delta_1(G)$ as the *minimum vertex degree* of G . Note that $\delta_0(G) = |E|$.

For integers n, k, d, s satisfying $0 \leq d \leq k-1$ and $0 \leq s \leq n/k$, we let $m_d^s(k, n)$ denote the minimum integer m such that every k -uniform hypergraph G on n vertices with $\delta_d(G) \geq m$ has a matching of size s . We write $o(1)$ to denote some function that tends to 0 as n tends to infinity. The following degree condition for forcing perfect matchings has been conjectured in [10, 15] and has received much attention recently.

Conjecture 1.1. *Let n and $1 \leq d \leq k-1$ be such that $n, d, k, n/k \in \mathbb{N}$. Then*

$$m_d^{n/k}(k, n) = \left(\max \left\{ \frac{1}{2}, 1 - \left(\frac{k-1}{k} \right)^{k-d} \right\} + o(1) \right) \binom{n-d}{k-d}.$$

The first term in the lower bound here is given by the following parity-based construction from [14]. For any integers n, k , let H' be a k -uniform hypergraph on n vertices with vertex partition $A \cup B = V(H')$, such that $||A| - |B|| \leq 2$ and $|A|$ and n/k have different parity. Let H' have edge set consisting of all k -element subsets of $V(H')$ that intersect A in an odd number of vertices. Observe that H' has no perfect matching, and that for every $1 \leq d \leq k-1$ we have that $\delta_d(H') = (1/2 + o(1)) \binom{n-d}{k-d}$. The second term in the lower bound is given by the hypergraph $H(n/k)$ defined as follows. Let $H(s)$ be the k -uniform hypergraph on n vertices with edge set

Date: November 22, 2013.

The research leading to these results was partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreement no. 258345 (D. Kühn) and 306349 (D. Osthus).

consisting of all k -element subsets of $V(H(s))$ intersecting a given (fixed) subset of $V(H(s))$ of size $s - 1$, that is $H(s) = K_n^{(k)} - K_{n-s+1}^{(k)}$.

For $d = k - 1$, $m_{k-1}^{n/k}(k, n)$ was determined exactly for large n by Rödl, Ruciński and Szeemerédi [20]. This was generalized by Treglown and Zhao [23], who determined the extremal families for all $d \geq k/2$. The extremal constructions are similar to the parity based one of H' above. This improves asymptotic bounds in [18, 19, 20]. Recently, Keevash, Knox and Mycroft [11] investigated the structure of hypergraphs whose minimum $(k - 1)$ -degree lies below the threshold and which have no perfect matching.

For $d < k/2$ less is known. In [1] Conjecture 1.1 was proved for $k - 4 \leq d \leq k - 1$, by reducing it to a probabilistic conjecture of Samuels. In particular, this implies Conjecture 1.1 for $k \leq 5$. Khan [12], and independently Kühn, Osthus and Treglown [16], determined $m_1^{n/k}(k, n)$ exactly for $k = 3$. Khan [13] also determined $m_1^{n/k}(k, n)$ exactly for $k = 4$. It was shown by Hàn, Person and Schacht [10] that for $k \geq 3$, $1 \leq d < k/2$ we have $m_d^{n/k}(k, n) \leq ((k-d)/k + o(1)) \binom{n-d}{k-d}$. (The case $d = 1$ of this is already due to Daykin and Häggkvist [5].) These bounds were improved by Markström and Ruciński [17], using similar techniques, to

$$m_d^{n/k}(k, n) \leq \left(\frac{k-d}{k} - \frac{1}{k^{k-d}} + o(1) \right) \binom{n-d}{k-d}.$$

Our main result improves on this bound, using quite different techniques.

Theorem 1.2. *Let n and $1 \leq d < k/2$ be such that $n, k, d, n/k \in \mathbb{N}$. Then*

$$m_d^{n/k}(k, n) \leq \left(\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1) \right) \binom{n-d}{k-d}.$$

We also consider degree conditions that force smaller matchings. As a consequence of the results of Kühn, Osthus and Treglown as well as those of Khan mentioned above, $m_1^s(k, n)$ is determined exactly whenever $s \leq n/k$ and $k \leq 4$ (for details see the concluding remarks in [16]). More generally, we propose the following version of Conjecture 1.1 for non-perfect matchings.

Conjecture 1.3. *For all $\varepsilon > 0$ and all integers n, d, k, s with $1 \leq d \leq k - 1$ and $0 \leq s \leq (1 - \varepsilon)n/k$ we have*

$$m_d^s(k, n) = \left(1 - \left(1 - \frac{s}{n} \right)^{k-d} + o(1) \right) \binom{n-d}{k-d}.$$

In fact it may be that the bound holds for all $s \leq n - C$, for some C depending only on d and k . The lower bound here is given by $H(s)$. The case $d = k - 1$ of Conjecture 1.3 follows easily from the determination of $m_{k-1}^s(k, n)$ for s close to n/k in [20]. Bollobás, Daykin and Erdős [3] determined $m_1^s(k, n)$ for small s , i.e. whenever $s < n/2k^3$. For $1 \leq d \leq k - 2$ we are able to determine $m_d^s(k, n)$ asymptotically for non-perfect matchings of any size at most $n/2(k - d)$. Note that this proves Conjecture 1.3 in the case $k/2 \leq d \leq k - 2$, say.

Theorem 1.4. *Let $\varepsilon > 0$ and let n, k, d be integers with $1 \leq d \leq k - 2$, and let $0 \leq a < \min\{1/2(k - d), (1 - \varepsilon)/k\}$ be such that $an \in \mathbb{N}$. Then*

$$m_d^{an}(k, n) = \left(1 - (1 - a)^{k-d} + o(1) \right) \binom{n-d}{k-d}.$$

1.2. Large matchings in hypergraphs with many edges. In proving Theorem 1.4 it will be useful for us to consider the following related problem. A classical theorem of Erdős and Gallai [7] determines the number of edges in a graph which forces a matching of a given size. In 1965, Erdős [6] made a conjecture which would generalize this to k -uniform hypergraphs.

Conjecture 1.5. *Let $n, k \geq 2$ and $1 \leq s \leq n/k$ be integers. Then*

$$m_0^s(k, n) = \max \left\{ \binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k} \right\} + 1.$$

For $k = 3$ this conjecture was verified by Frankl [8]. For the case $k = 4$, Conjecture 1.5 was verified asymptotically by Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1]. Recently, Frankl confirmed the conjecture exactly for $s \leq n/2k$, i.e. when the aim is to cover at most half of the vertices of the hypergraph.

Theorem 1.6. [9] *Let $n, k, s \in \mathbb{N}$ be such that $n, k \geq 2$ and $n \geq (2s-1)k - s + 1$. Then*

$$m_0^s(k, n) = \binom{n}{k} - \binom{n-s+1}{k} + 1.$$

1.3. Large fractional matchings. Our approach to proving our results uses the concepts of fractional matchings and fractional vertex covers. A *fractional matching* in a k -uniform hypergraph $G = (V, E)$ is a function $w : E \rightarrow [0, 1]$ of weights of edges, such that for each $v \in V$ we have $\sum_{e \in E: v \in e} w(e) \leq 1$. The *size* of w is $\sum_{e \in E} w(e)$. We say w is *perfect* if it has size $|V|/k$. A *fractional vertex cover* in G is a function $w : V \rightarrow [0, 1]$ of weights of vertices, such that for each $e \in E$ we have $\sum_{v \in e} w(v) \geq 1$. The *size* of w is $\sum_{v \in V} w(v)$.

A key idea (already used e.g. in [1, 19]) is that we can switch between considering the largest fractional matching and the smallest fractional vertex cover of a hypergraph. The determination of these quantities are dual linear programming problems, and hence by the Duality Theorem they have the same size.

For $s \in \mathbb{R}$ we let $f_d^s(k, n)$ denote the minimum integer m such that every k -uniform hypergraph G on n vertices with $\delta_d(G) \geq m$ has a fractional matching of size s . It was shown in [19] that $f_{k-1}^{n/k}(k, n) = \lceil n/k \rceil$.

To prove Theorem 1.4, we use Theorem 1.6, along with methods similar to those developed in [1], to convert the edge-density conditions for the existence of matchings into corresponding minimum degree conditions for the existence of fractional matchings (see Proposition 4.1). We then use the Weak Hypergraph Regularity Lemma to prove Theorem 1.4 by converting our fractional matchings into integer ones. Our argument also gives the following theorem which, for $1 \leq d \leq k-2$, asymptotically determines $f_d^s(k, n)$ for fractional matchings of any size up to $n/2(k-d)$. Note that this determines $f_d^s(k, n)$ asymptotically for all $s \in (0, n/k)$ whenever $d \geq k/2$.

Theorem 1.7. *Let $n, k \geq 3$, and $1 \leq d \leq k-2$ be integers and let $0 \leq a \leq \min\{1/2(k-d), 1/k\}$. Then*

$$f_d^{an}(k, n) = \left(1 - (1-a)^{k-d} + o(1)\right) \binom{n-d}{k-d}.$$

We prove Theorem 1.2 in a similar fashion, via the following two theorems.

Theorem 1.8. *Let $n, k \geq 2, d \geq 1$ be integers. Then*

$$f_0^{n/(k+d)}(k, n) \leq \left(\frac{k}{k+d} - \frac{k-1}{(k+d)^k} + o(1)\right) \binom{n}{k}.$$

Theorem 1.9. *Let $n, k \geq 3$, $1 \leq d \leq k - 2$ be integers. Then*

$$f_d^{n/k}(k, n) \leq \left(\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1) \right) \binom{n-d}{k-d}.$$

The rest of the paper is organised as follows. In Section 2 we lay out some notation, set out some useful tools, and prove some preliminary results. Section 3 is the heart of the paper, in which we prove Theorem 1.8. In Section 4 we derive Theorems 1.7 and 1.9, and in Section 5 we derive Theorems 1.2 and 1.4. In Section 6 we give some concluding remarks.

2. NOTATION, TOOLS AND PRELIMINARY RESULTS

2.1. Notation. Since in many of the proofs in this paper we often consider vertex degrees, when $S = \{v\}$ is a set containing only one vertex we write $d_G(v)$ to denote $\deg_G(S)$ and we refer to $d_G(v)$ as the degree of v (in G). We let $e(G)$ denote the number of edges in a hypergraph G , and let $|G|$ denote the number of its vertices. For a set V and a positive integer k we let $\binom{V}{k}$ denote the set of all k -element subsets of V . For $m \in \mathbb{N}$ we let $[m]$ denote the set $\{1, \dots, m\}$. Whenever we refer to a k -tuple, we assume that it is unordered. Given a hypergraph $G = (V, E)$ and a set $S \subseteq V$, we refer to the pair $(V \setminus S, \{e \subseteq V : S \cap e = \emptyset, e \cup S \in E\})$ as the *neighbourhood hypergraph* of S (in G). If $S = \{v\}$ has just one element then we may refer to this pair as the neighbourhood hypergraph of v . For $U \subseteq V$ we denote by $G[U]$ the hypergraph induced by U on G , that is the hypergraph with vertex set U and edge set $\{e \in E : e \subseteq U\}$.

2.2. Tools and preliminary results. In proving some of our results we will use the lower bound given by the earlier construction $H(s)$, for all integers n, d, k, s with $k \geq 2$ and $0 \leq d \leq k - 1$ and $0 \leq s \leq n/k$:

$$(2.1) \quad m_d^s(k, n) \geq f_d^s(k, n) \geq \left(1 - (1 - s/n)^{k-d} + o(1)\right) \binom{n-d}{k-d}.$$

Now, as mentioned in Section 1, a key tool in this paper is that the determination of the size of the largest fractional matching of a k -uniform hypergraph is a linear programming problem, and its dual problem is to determine the size of the smallest fractional vertex cover of the hypergraph. The following proposition, which follows by the Duality Theorem, will be very useful to us.

Proposition 2.1. *Let $k \geq 2$ and let G be a k -uniform hypergraph. The size of the largest fractional matching of G is equal to the size of the smallest fractional vertex cover of G .*

In the rest of this section we collect some preliminary results.

Proposition 2.2. *Let $G = (V, E)$ be a hypergraph, $E' \subseteq E$, $S \subseteq V$, and let w be a fractional vertex cover of G . Then*

$$e(G) \leq \sum_{e \in E} \sum_{v \in e \setminus S} w(v) + \sum_{e \in E'} \sum_{v \in e \cap S} w(v) + |E \setminus E'|.$$

Proof. As w is a fractional vertex cover of G ,

$$e(G) = |E'| + |E \setminus E'| \leq \sum_{e \in E'} \sum_{v \in e} w(v) + |E \setminus E'| \leq \sum_{e \in E} \sum_{v \in e \setminus S} w(v) + \sum_{e \in E'} \sum_{v \in e \cap S} w(v) + |E \setminus E'|.$$

□

The following crude bound will sometimes be useful. The proof is immediate from the definitions.

Proposition 2.3. *Suppose that $k \geq 2$ and $0 < a, c < 1$ are fixed. Then for every $\varepsilon > 0$ there exists $n_0 = n_0(k, \varepsilon)$ such that if $n \geq n_0$ and $f_0^{an}(k, n) \leq c \binom{n}{k}$ then $f_0^{an+1}(k, n) \leq (c + \varepsilon) \binom{n}{k}$.*

In the next section we will prove Theorem 1.8 by induction. For this we will need Theorem 2.4, which will establish the base case of this induction. Theorem 2.4 is an easy consequence of the Erdős-Gallai Theorem from [7].

Theorem 2.4. *For $k = 2$ and $x \leq 1/3$ we have*

$$f_0^{xn}(k, n) = \left(1 - (1 - x)^k + o(1)\right) \binom{n}{k}.$$

The next proposition will also be needed in the proof of Theorem 1.8. To prove this proposition we will need a well-known theorem of Baranyai [2] from 1975.

Theorem 2.5 (Baranyai's Theorem). *If $n \in \ell\mathbb{N}$ then the complete ℓ -uniform hypergraph on n vertices decomposes into edge-disjoint perfect matchings.*

Proposition 2.6. *Let n, k, ℓ be integers with $k \geq 2$ and $1 \leq \ell \leq k$, and let $\eta \in [0, 1)$. Let V be a set of size n . Suppose $S \subseteq V$, with $|S| \in \ell\mathbb{N}$. Then there exists $\tilde{E} \subseteq \{e \in \binom{V}{k} : |e \cap S| = \ell\}$ such that for every $v \in S$,*

$$(2.2) \quad |\{e \in \tilde{E} : v \in e\}| = \left\lfloor \eta \binom{|S|}{\ell - 1} \binom{n - |S|}{k - \ell} \right\rfloor.$$

Proof. The cases where $\ell = 1$ or $\eta = 0$ are trivial. So suppose that $2 \leq \ell \leq k$ and $\eta \in (0, 1)$. Apply Theorem 2.5 to find a decomposition of the complete ℓ -uniform hypergraph on S into edge-disjoint perfect matchings $M_1, \dots, M_{\lfloor |S|/\ell \rfloor}$.

We now construct \tilde{E} by adding k -tuples from $\{e \in \binom{V}{k} : |e \cap S| = \ell\}$ greedily, under the following constraints:

- (i) for all $i \in \{1, \dots, \lfloor |S|/\ell \rfloor\}$, we do not add any k -tuples in $\{e \in \binom{V}{k} : e \cap S \in M_{i+1}\}$ unless we have already added all k -tuples in $\{e \in \binom{V}{k} : e \cap S \in M_i\}$;
- (ii) for every $v \in S$,

$$|\{e \in \tilde{E} : v \in e\}| \leq \eta \binom{|S|}{\ell - 1} \binom{n - |S|}{k - \ell}.$$

It is clear that (i) and (ii) ensure that the set \tilde{E} obtained in this way satisfies (2.2) for every $v \in S$. \square

3. MINIMUM EDGE-DENSITY CONDITIONS FOR FRACTIONAL MATCHINGS

We will use the following lemma to prove Theorem 1.8 inductively.

Lemma 3.1. *Let $k \geq 3$ be fixed. Suppose that $a \in (0, 1/(k + 1)]$, $c \in (0, 1)$ and that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have*

$$(3.1) \quad f_0^{an/(1-a)}(k - 1, n) \leq c \binom{n}{k - 1}.$$

Then for all $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ any k -uniform hypergraph G on n vertices with at least an vertices of degree at least

$$D := \left(c(1-a)^{k-1} + \left(1 - (1-a)^{k-1} \right) + \varepsilon \right) \binom{n-1}{k-1}$$

has a fractional matching of size an .

Proof. Let $\varepsilon > 0$ and choose n_1 sufficiently large. Consider a k -uniform hypergraph $G = (V, E)$ on n vertices with at least an vertices of degree at least D . Let $Y \subseteq V$ be the set of $\lceil an \rceil$ vertices of highest degree. Let w be a fractional vertex cover of G of least size. Consider the vertex $v_0 \in Y$ with the lowest weight $w(v_0)$. Let H be the neighbourhood hypergraph of v_0 in G . So

$$e(H) = d_G(v_0) \geq D = \left(c(1-a)^{k-1} + \left(1 - (1-a)^{k-1} \right) + \varepsilon \right) \binom{n-1}{k-1}.$$

Let $H' := H[V \setminus Y]$. Since the number of edges in H with at least one vertex in Y is at most $(1 - (1-a)^{k-1} + o(1)) \binom{n-1}{k-1}$, it follows that

$$\begin{aligned} e(H') &\geq e(H) - \left(1 - (1-a)^{k-1} + o(1) \right) \binom{n-1}{k-1} \geq \left(c(1-a)^{k-1} + \varepsilon/2 \right) \binom{n-1}{k-1} \\ &\geq (c + \varepsilon/3) \binom{|H'|}{k-1}, \end{aligned}$$

where in the last two inequalities we use that n_1 was chosen sufficiently large. Note that $|H'| \geq n/2$, so we may assume that $|H'| \geq n_0$. Now, (3.1) and Proposition 2.3 together imply that H' has a fractional matching of size

$$a|H'|/(1-a) + 1 = a(n - \lceil an \rceil)/(1-a) + 1 \geq an.$$

So let M be a fractional matching of H' of size an . Note that for all $v \in V \setminus Y$,

$$\sum_{e \in E(H'): v \in e} M(e) \leq 1.$$

So we have that

$$\sum_{v \in V} w(v) \geq \sum_{v \in Y} w(v) + \sum_{e \in E(H')} \sum_{v \in e} M(e)w(v).$$

By the minimality of $w(v_0)$, this implies that

$$\begin{aligned} \sum_{v \in V} w(v) &\geq anw(v_0) + \sum_{e \in E(H')} \sum_{v \in e} M(e)w(v) = \sum_{e \in E(H')} M(e)w(v_0) + \sum_{e \in E(H')} \sum_{v \in e} M(e)w(v) \\ &= \sum_{e \in E(H')} M(e) \left(w(v_0) + \sum_{v \in e} w(v) \right) \geq \sum_{e \in E(H')} M(e) = an. \end{aligned}$$

The last inequality holds because by definition of H' we have $e \cup \{v_0\} \in E$ for all $e \in E(H')$, and so $w(v_0) + \sum_{v \in e} w(v) \geq 1$.

Hence the size of w is at least an , so by Proposition 2.1 the largest fractional matching in G has size at least an . \square

The proof of Theorem 1.8 proceeds as follows. Suppose G has no fractional matching of size $n/(k+d)$. Then we use Lemma 3.1 and induction to show that G contains few vertices of high degree. Moreover, by duality we show that G has a small fractional vertex cover. We combine these two facts to show that the number of edges of G does not exceed the expression stated in Theorem 1.8.

Proof of Theorem 1.8. The proof will proceed by induction on k . The base step, $k = 2$, follows by Theorem 2.4, setting $x := 1/(2+d)$.

Now consider some $k > 2$ and suppose that the theorem holds for all smaller values of k . Fix $d \geq 1$. Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be sufficiently large compared to $1/\varepsilon$, k and d . For convenience let us define

$$\xi := \left(\frac{k-1}{k+d-1} - \frac{k-2}{(k+d-1)^{k-1}} \right) \left(\frac{k+d-1}{k+d} \right)^{k-1} + \left(1 - \left(\frac{k+d-1}{k+d} \right)^{k-1} \right) < 1.$$

Consider any k -uniform hypergraph $G = (V, E)$ on $n \geq n_0$ vertices, and suppose that the largest fractional matching of G is of size less than $n/(k+d)$. Then by Proposition 2.1 there exists a fractional vertex cover, *w* say, of G with size less than $n/(k+d)$. Let $a := 1/(k+d)$. So $a/(1-a) = 1/(k+d-1)$. Let

$$c := \frac{k-1}{k+d-1} - \frac{k-2}{(k+d-1)^{k-1}} + \varepsilon/4.$$

Then by induction,

$$f_0^{n'/(k+d-1)}(k-1, n) \leq c \binom{n'}{k-1},$$

for all sufficiently large n' . Thus, as n_0 is sufficiently large, Lemma 3.1 implies that there are less than $n/(k+d)$ vertices of G with degree at least $(\xi + \varepsilon/2) \binom{n-1}{k-1}$.

Let S be the set of $|S|$ vertices of G with highest degree, where $|S| \in k!\mathbb{N}$ is minimal such that $|S| \geq n/(k+d)$. So $d_G(v) < (\xi + \varepsilon/2) \binom{n-1}{k-1}$ for all $v \in V \setminus S$. For every $i \in \{0, \dots, k\}$ let $S_i := \{e \in \binom{V}{k} : |e \cap S| = i\}$. Given $X \subseteq \binom{V}{k}$, for all $v \in V$ let $t_X(v) := |\{e \in X : v \in e\}|$. Note that for all $v \in S$ the value of $t_{S_i}(v)$ is the same and $t_{S_0}(v) = 0$. Let $\ell \in \{0, \dots, k\}$ be maximal such that for any $v \in S$ we have $\sum_{i=0}^{\ell-1} t_{S_i}(v) \leq \xi \binom{n-1}{k-1}$. Let $E''' := \binom{V}{k} \setminus S_k$. Then for each $v \in S$,

$$(3.2) \quad t_{E'''}(v) = \left(1 - \frac{1}{(k+d)^{k-1}} + o(1) \right) \binom{n-1}{k-1} > \xi \binom{n-1}{k-1}.$$

The final inequality holds here for sufficiently large n_0 , as it rearranges to $d(k+d-1)^{k-2} + (k-2) + o(1) > 1$. This shows that $\ell \leq k-1$. Let

$$\eta := \left(\xi \binom{n-1}{k-1} - \sum_{i=1}^{\ell-1} t_{S_i}(v) \right) / \binom{|S|}{\ell-1} \binom{n-|S|}{k-\ell}.$$

So $\eta \in [0, 1)$. Apply Proposition 2.6 with parameters n , k , ℓ , η to obtain a set $\tilde{E} \subseteq S_\ell$ such that for every $v \in S$,

$$t_{\tilde{E}}(v) = \left\lfloor \eta \binom{|S|}{\ell-1} \binom{n-|S|}{k-\ell} \right\rfloor.$$

Let $E'' := \bigcup_{i=0}^{\ell-1} S_i \cup \tilde{E}$. Then each $v \in S$ satisfies

$$(3.3) \quad t_{E''}(v) = \left\lfloor \xi \binom{n-1}{k-1} \right\rfloor.$$

We can now give a lower bound on the size of E'' as follows: for each vertex $v \in S$ we count the number of k -tuples in E'' that contain v , and then adjust for the k -tuples that contain several vertices of S and were thus counted several times as a result. Since $S_0 \subseteq E''$ this yields

$$|E''| \stackrel{(3.3)}{\geq} \left[\xi \binom{n-1}{k-1} \right] \frac{n}{k+d} + |S_0| - \sum_{j=1}^{k-1} (j-1) |S_j|.$$

Note that since $E'' \subseteq E'''$ we only need to consider values of j up to $k-1$ in the summation, rather than k . Now, note that

$$\begin{aligned} |S_0| - \sum_{j=1}^k (j-1) |S_j| &= \binom{n}{k} - \sum_{v \in S} \binom{n-1}{k-1} = \binom{n}{k} - \left(\frac{n}{k+d} + o(1) \right) \binom{n-1}{k-1} \\ &= \left(1 - \frac{k}{k+d} + o(1) \right) \binom{n}{k}. \end{aligned}$$

Hence, as $(k-1)|S_k| = ((k-1)/(k+d)^k + o(1)) \binom{n}{k}$,

$$(3.4) \quad |E''| \geq (\xi + o(1)) \binom{n-1}{k-1} \frac{n}{k+d} + \left(1 - \frac{k}{k+d} + \frac{k-1}{(k+d)^k} + o(1) \right) \binom{n}{k}.$$

Now, let $E' := E \cap E''$. Also, note that by Proposition 2.2,

$$e(G) \leq \sum_{e \in E} \sum_{v \in e \setminus S} w(v) + \sum_{e \in E'} \sum_{v \in e \cap S} w(v) + |E \setminus E'|.$$

Recall that $d_G(v) < (\xi + \varepsilon/2) \binom{n-1}{k-1}$ for all $v \in V \setminus S$ and that by (3.3) the number of edges in E' incident to v is at most $\xi \binom{n-1}{k-1}$ for all $v \in S$. So

$$e(G) \leq \sum_{v \in V} (\xi + \varepsilon/2) \binom{n-1}{k-1} w(v) + |E \setminus E'|.$$

Now note that $|E \setminus E'| \leq \binom{V}{k} \setminus E'' = \binom{n}{k} - |E''|$ and recall that the size of w is less than $n/(k+d)$. So

$$\begin{aligned} e(G) &< (\xi + \varepsilon/2) \binom{n-1}{k-1} \frac{n}{k+d} + \binom{n}{k} - |E''| \\ &\stackrel{(3.4)}{\leq} (\xi + \varepsilon/2) \binom{n-1}{k-1} \frac{n}{k+d} - (\xi + o(1)) \binom{n-1}{k-1} \frac{n}{k+d} \\ &\quad + \left(\frac{k}{k+d} - \frac{k-1}{(k+d)^k} + o(1) \right) \binom{n}{k} \\ &\leq \left(\frac{k}{k+d} - \frac{k-1}{(k+d)^k} + \varepsilon \right) \binom{n}{k}. \end{aligned}$$

The final inequality holds since n_0 is sufficiently large. By definition, this shows that

$$f_0^{n/(k+d)}(k, n) \leq \left(\frac{k}{k+d} - \frac{k-1}{(k+d)^k} + o(1) \right) \binom{n}{k}.$$

This completes the inductive step and hence the proof. \square

4. MINIMUM DEGREE CONDITIONS FOR FRACTIONAL MATCHINGS

The following proposition generalises Proposition 1.1 in [1], with a similar proof idea. It allows us to transform bounds involving edge densities into bounds involving d -degrees.

Proposition 4.1. *Let $\varepsilon \geq 0$, let k, d, n be integers with $n \geq k \geq 3$, $1 \leq d \leq k - 2$, and $d < (1 - \varepsilon^{1/d})n$. Let $a \in [0, (1 - \varepsilon^{1/d})/k]$. Suppose H is a k -uniform hypergraph on n vertices, such that for at least $(1 - \varepsilon)\binom{n}{d}$ d -tuples of vertices $L \in \binom{V(H)}{d}$ we have*

$$\deg_H(L) \geq f_0^{an}(k - d, n - d).$$

Then H has a fractional matching of size an .

Proof. The outline of the proof goes as follows. We will assume that there is no fractional matching of size an in a k -uniform hypergraph $H = (V, E)$ on n vertices and then show that for more than $\varepsilon\binom{n}{d}$ d -tuples of vertices $L \in \binom{V}{d}$, the neighbourhood hypergraph $H(L)$ of L in H has no fractional matching of size an . This will imply that for more than $\varepsilon\binom{n}{d}$ d -tuples of vertices L , $\deg_H(L) = e(H(L)) < f_0^{an}(k - d, n - d)$. This will prove the result in contrapositive.

So suppose $H = (V, E)$ is an n -vertex k -uniform hypergraph, with no fractional matching of size an . Then by Proposition 2.1, H has a fractional vertex cover, w say, of size less than an . Let

$$E_w := \left\{ e \in \binom{V}{k} : \sum_{v \in e} w(v) \geq 1 \right\},$$

and let $H_w := (V, E_w)$. Since $H \subseteq H_w$ we can, without loss of generality, replace H with H_w . Let $U \subseteq V$ be the set of $\lfloor \varepsilon^{1/d}n \rfloor + d$ vertices of smallest weights. Let $\mathcal{L} := \binom{U}{d}$. Note that

$$|\mathcal{L}| = \binom{\lfloor \varepsilon^{1/d}n \rfloor + d}{d} > \frac{(\varepsilon^{1/d}n)^d}{d!} = \varepsilon \frac{n^d}{d!} \geq \varepsilon \binom{n}{d}.$$

Consider any $L \in \mathcal{L}$. Let $H_w(L)$ be the neighbourhood hypergraph of L in H_w . We will show that $H_w(L)$ has no fractional matching of size an . Without loss of generality we may assume that the elements of L all have equal weights, $w(L)$ say. (If not, we could replace these weights by their average, which would alter neither $\sum_{v \in V} w(v)$ nor $\sum_{v \in e} w(v)$ for any $e \supseteq L$. These are the only two quantities involving weights that we will consider in what follows.) Observe that $w(L) < 1/k$, else the size of w would be at least

$$\frac{n(1 - \varepsilon^{1/d})}{k} \geq an.$$

We now define a new weight function $w'(v)$ on the vertices in V :

$$w'(v) := \min \{ \max \{ 0, w^*(v) \}, 1 \}, \quad \text{where} \quad w^*(v) := \frac{w(v) - w(L)}{1 - kw(L)}.$$

Note that only for vertices $u \in U \setminus L$ can it be that $w^*(u) < 0$. Note also that since $w(v) \geq 0$ for all $v \in V$, we have that $w^*(u) \geq -w(L)/(1 - kw(L))$ for such vertices u . Hence,

$$\begin{aligned} \sum_{v \in V} w'(v) &\leq \left(\sum_{v \in V} w^*(v) \right) + |U \setminus L| \frac{w(L)}{1 - kw(L)} < \frac{an - nw(L) + \varepsilon^{1/d}nw(L)}{1 - kw(L)} \\ &= an \frac{1 - (1/a)(1 - \varepsilon^{1/d})w(L)}{1 - kw(L)} \leq an, \end{aligned}$$

and for any given $e \in \{e' \in E_w : e' \supseteq L\}$ we have that

$$\sum_{v \in e} w'(v) \geq \min \left\{ \frac{\sum_{v \in e} w(v) - kw(L)}{1 - kw(L)}, 1 \right\} \geq \min \left\{ \frac{1 - kw(L)}{1 - kw(L)}, 1 \right\} = 1.$$

Moreover, $\sum_{v \in L} w'(v) = 0$. It follows that the function w' restricted to $V \setminus L$ is a fractional vertex cover of $H_w(L)$ of size less than an , and so by Proposition 2.1, $H_w(L)$ has no fractional matching of size an , which completes the proof. \square

We can now derive Theorems 1.7 and 1.9.

Proof of Theorem 1.7. Let $k' := k - d$ and $n' := n - d$. Note that Theorem 1.6 implies that

$$(4.1) \quad m_0^{an}(k, n) = \left(1 - (1 - a)^k + o(1)\right) \binom{n}{k}$$

for all $a \leq 1/2k$. Now Proposition 2.3 implies that for all $0 \leq a \leq \min\{1/2(k - d), 1/k\}$,

$$\begin{aligned} f_0^{an}(k - d, n - d) &= f_0^{a(n'+d)}(k', n') \leq f_0^{an'+1}(k', n') \leq m_0^{an'}(k', n') + o(1) \binom{n'}{k'} \\ &\stackrel{(4.1)}{=} \left(1 - (1 - a)^{k'} + o(1)\right) \binom{n'}{k'} = \left(1 - (1 - a)^{k-d} + o(1)\right) \binom{n-d}{k-d}. \end{aligned}$$

The upper bound in Theorem 1.7 follows now from Proposition 4.1 applied with $\varepsilon = 0$. The lower bound follows from (2.1). \square

Proof of Theorem 1.9. Let $k' := k - d$ and $n' := n - d$. Then Theorem 1.8 and Proposition 2.3 together imply that

$$\begin{aligned} f_0^{n/k}(k - d, n - d) &= f_0^{(n'+d)/(k'+d)}(k', n') \leq f_0^{n'/(k'+d)+1}(k', n') \\ &\leq \left(\frac{k'}{k'+d} - \frac{k'-1}{(k'+d)^{k'}} + o(1)\right) \binom{n'}{k'} \\ &= \left(\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1)\right) \binom{n-d}{k-d}. \end{aligned}$$

So Theorem 1.9 follows now from Proposition 4.1 applied with $\varepsilon = 0$. \square

The case $\varepsilon > 0$ of Proposition 4.1 will be used in the next section.

5. CONSTRUCTING INTEGER MATCHINGS FROM FRACTIONAL ONES

We will construct integer matchings from fractional ones using the Weak Hypergraph Regularity Lemma. Before stating this we will need the following definitions.

Given a k -tuple (V_1, \dots, V_k) of disjoint subsets of the vertices of a k -uniform hypergraph $G = (V, E)$, we define $(V_1, \dots, V_k)_G$ to be the k -partite subhypergraph with vertex classes V_1, \dots, V_k induced on G . We let

$$d_G(V_1, \dots, V_k) = \frac{e((V_1, \dots, V_k)_G)}{\prod_{i \in \{1, \dots, k\}} |V_i|}$$

denote the *density* of $(V_1, \dots, V_k)_G$.

Definition 5.1 (ε -regularity). *Let $\varepsilon > 0$, let $G = (V, E)$ be a k -uniform hypergraph, and let $V_1, \dots, V_k \subseteq V$ be disjoint. We say that $(V_1, \dots, V_k)_G$ is ε -regular if for every subhypergraph $(V'_1, \dots, V'_k)_G$ with $V'_i \subseteq V_i$ and $|V'_i| \geq \varepsilon|V_i|$ for each $i \in \{1, \dots, k\}$, we have that*

$$|d_G(V'_1, \dots, V'_k) - d_G(V_1, \dots, V_k)| < \varepsilon.$$

The following result was proved by Chung [4]. The proof follows the lines of that of the original Regularity Lemma for graphs [21].

Lemma 5.2 (Weak Hypergraph Regularity Lemma). *For all integers $k \geq 2$, $L_0 \geq 1$, and every $\varepsilon > 0$ there exists $N = N(\varepsilon, L_0, k)$ such that if $G = (V, E)$ is a k -uniform hypergraph on $n \geq N$ vertices, then V has a partition V_0, \dots, V_L such that the following properties hold:*

- (i) $L_0 \leq L \leq N$ and $|V_0| \leq \varepsilon n$,
- (ii) $|V_1| = \dots = |V_L|$,
- (iii) for all but at most $\varepsilon \binom{L}{k}$ k -tuples $\{i_1, \dots, i_k\} \in \binom{[L]}{k}$, we have that $(V_{i_1}, \dots, V_{i_k})_G$ is ε -regular.

We call the partition classes V_1, \dots, V_L *clusters*, and V_0 the *exceptional set*. For our purposes we will in fact use the degree form of the Weak Hypergraph Regularity Lemma.

Lemma 5.3 (Degree Form of the Weak Hypergraph Regularity Lemma). *For all integers $k \geq 2$, $L_0 \geq 1$ and every $\varepsilon > 0$, there is an $N = N(\varepsilon, L_0, k)$ such that for every $d \in [0, 1)$ and for every hypergraph $G = (V, E)$ on $n \geq N$ vertices there exists a partition of V into V_0, V_1, \dots, V_L and a spanning subhypergraph G' of G such that the following properties hold:*

- (i) $L_0 \leq L \leq N$ and $|V_0| \leq \varepsilon n$,
- (ii) $|V_1| = \dots = |V_L| =: m$,
- (iii) $d_{G'}(v) > d_G(v) - (d + \varepsilon)n^{k-1}$ for all $v \in V$,
- (iv) every edge of G' with more than one vertex in a single cluster V_i , for some $i \in \{1, \dots, L\}$, has at least one vertex in V_0 ,
- (v) for all k -tuples $\{i_1, \dots, i_k\} \in \binom{[L]}{k}$, we have that $(V_{i_1}, \dots, V_{i_k})_{G'}$ is ε -regular and has density either 0 or greater than d .

The proof is very similar to that of the degree form of the Regularity Lemma for graphs, so we omit it here; for details see [22].

We now define a type of hypergraph that will be essential in our application of the Weak Hypergraph Regularity Lemma.

Definition 5.4 (Reduced Hypergraph). *Let $G = (V, E)$ be a k -uniform hypergraph. Given parameters $\varepsilon > 0$, $d \in [0, 1)$ and $L_0 \geq 1$ we define the reduced hypergraph $R = R(\varepsilon, d, L_0)$ of G as follows. Apply the degree form of the Weak Hypergraph Regularity Lemma to G , with parameters ε , d , L_0 to obtain a spanning subhypergraph G' and a partition V_0, \dots, V_L of V , with exceptional set V_0 and clusters V_1, \dots, V_L . Then R has vertices V_1, \dots, V_L , and there exists an edge between V_{i_1}, \dots, V_{i_k} precisely when $(V_{i_1}, \dots, V_{i_k})_{G'}$ is ε -regular with density greater than d .*

The following lemma tells us that this reduced hypergraph (almost) inherits the minimum degree properties of the original hypergraph. The proof is similar to that of the well known version for graphs, but we include it here for completeness.

Lemma 5.5. *Suppose $c > 0$, $k \geq 2$, $1 \leq \ell \leq k - 1$, $L_0 \geq 1$, and $0 < \varepsilon \leq d \leq c^3/64$. Let G be a k -uniform hypergraph with $\delta_\ell(G) \geq c|G|^{k-\ell}$. Let $R = R(\varepsilon, d, L_0)$ be the reduced hypergraph of G . Then at least $\binom{|R|}{\ell} - d^{1/3}(2k)^\ell|R|^\ell$ of the ℓ -tuples of vertices of R have degree at least $(c - 4d^{1/3})|R|^{k-\ell}$.*

Proof. Let G' be the spanning subhypergraph of G obtained by applying the degree form of the Weak Hypergraph Regularity Lemma to G with parameters ε , d , L_0 ; let V_1, \dots, V_L denote the vertices of R , and let m denote the size of these clusters.

First recall that given any vertex $x \in V(G')$ we know that $d_{G'}(x) > d_G(x) - (d + \varepsilon)|G|^{k-1}$. Note that since $|V_0| \leq \varepsilon|G|$, we have that the number of edges incident to x that contain a vertex in V_0 is at most $\varepsilon|G| \binom{|G|}{k-2} \leq \varepsilon|G|^{k-1}$. Hence for all $v \in V(G' - V_0)$, we have that

$$d_{G'-V_0}(v) > d_G(v) - (d + 2\varepsilon)|G|^{k-1} \geq d_G(v) - 3d|G|^{k-1}.$$

We call an ℓ -tuple A of vertices of $G' - V_0$ *bad* if $\deg_{G'-V_0}(A) \leq \deg_G(A) - 3d^{1/3}|G|^{k-\ell}$. So for each $v \in V(G' - V_0)$ there are at most $\binom{k-1}{\ell-1}d^{2/3}|G|^{\ell-1}$ bad ℓ -tuples A with $v \in A$. (This follows by double-counting the number of pairs (A, e) where A is a bad ℓ -tuple with $v \in A$ and $e \in E(G) \setminus E(G' - V_0)$ is an edge containing A .) This in turn implies that in total at most $\binom{k-1}{\ell-1}d^{2/3}|G|^\ell$ of the ℓ -tuples A are bad. Given $1 \leq s \leq k$ and an s -tuple $(V_{i_1}, \dots, V_{i_s})$ of clusters of R , we say that an s -tuple A of vertices of $G' - V_0$ *lies in* $(V_{i_1}, \dots, V_{i_s})$ if $|A \cap V_{i_\alpha}| = 1$ for all $\alpha \in \{1, \dots, s\}$. We call an ℓ -tuple $(V_{i_1}, \dots, V_{i_\ell})$ of clusters of R *nice* if there are less than $d^{1/3}m^\ell$ bad ℓ -tuples A of vertices of $G' - V_0$ which lie in $(V_{i_1}, \dots, V_{i_\ell})$. So less than $\binom{k-1}{\ell-1}d^{1/3}|G|^\ell/m^\ell \leq d^{1/3}(2k)^\ell|R|^\ell$ of the ℓ -tuples of clusters of R are not nice. Hence it suffices to show that any nice ℓ -tuple of clusters of R has degree at least $(c - 4d^{1/3})|R|^{k-\ell}$ in R .

Consider any nice ℓ -tuple of clusters of R , say $(V_{i_1}, \dots, V_{i_\ell})$. Let \mathcal{A} denote the set of all ℓ -tuples A of vertices of $G' - V_0$ which lie in $(V_{i_1}, \dots, V_{i_\ell})$ and are not bad. So $|\mathcal{A}| \geq (1 - d^{1/3})m^\ell$. Moreover, the number of edges e of $G' - V_0$ with $|e \cap V_{i_\alpha}| = 1$ for all $\alpha \in \{1, \dots, \ell\}$ is at least

$$(5.1) \quad \sum_{A \in \mathcal{A}} \deg_{G'-V_0}(A) \geq |\mathcal{A}| \left(c - 3d^{1/3} \right) |G|^{k-\ell} \geq \left(c - 4d^{1/3} \right) |G|^{k-\ell} m^\ell.$$

Now suppose that the degree in R of $(V_{i_1}, \dots, V_{i_\ell})$ is less than $(c - 4d^{1/3})|R|^{k-\ell}$. Then the number of $(k - \ell)$ -tuples $\{j_1, \dots, j_{k-\ell}\} \in \binom{[L]}{k-\ell}$ for which $(V_{i_1}, \dots, V_{i_\ell}, V_{j_1}, \dots, V_{j_{k-\ell}})_{G'}$ is ε -regular with density greater than d is less than $(c - 4d^{1/3})|R|^{k-\ell}$. Note that at most m^k edges of $G' - V_0$ lie in such a subhypergraph. So the number of edges e of $G' - V_0$ with $|e \cap V_{i_\alpha}| = 1$ for all $\alpha \in \{1, \dots, \ell\}$ is less than

$$(c - 4d^{1/3})|R|^{k-\ell} m^k \leq (c - 4d^{1/3})|G|^{k-\ell} m^\ell,$$

contradicting (5.1). This completes the proof. \square

The following lemma uses all of the previous results of this section to allow us to convert our fractional matchings into integer ones. We will use the notation $a \ll b$ to mean that we can find an increasing function f for which all of the conditions in the proof are satisfied whenever $a \leq f(b)$.

Lemma 5.6. *Let $k \geq 2$ and $1 \leq \ell \leq k - 1$ be integers, and let $\varepsilon > 0$. Suppose that for some $b, c \in (0, 1)$ and some integer n_0 , any k -uniform hypergraph on $n \geq n_0$ vertices with at least $(1 - \varepsilon) \binom{n}{\ell}$ ℓ -tuples of vertices of degree at least $cn^{k-\ell}$ has a fractional matching of size $(b + \varepsilon)n$. Then there exists an integer n'_0 such that any k -uniform hypergraph G on $n \geq n'_0$ vertices with $\delta_\ell(G) \geq (c + \varepsilon)n^{k-\ell}$ has an (integer) matching of size at least bn .*

Proof. Define $n'_0 \in \mathbb{N}$ and new constants ε' and d such that $0 < 1/n'_0 \ll \varepsilon' \ll d \ll \varepsilon, c, 1/k, 1/n_0$. Let G be a k -uniform hypergraph on $n \geq n'_0$ vertices, with $\delta_\ell(G) \geq (c + \varepsilon)n^{k-1}$. Let G' be the spanning subhypergraph of G obtained by applying the degree form of the Weak

Hypergraph Regularity Lemma to G with parameters ε' , d , n_0 . Let $R := R(\varepsilon', d, n_0)$ be the corresponding reduced hypergraph, and let $L := |R|$. By Lemma 5.5 at least $(1 - \varepsilon) \binom{L}{\ell}$ ℓ -tuples of vertices of R have degree at least

$$(c + \varepsilon - 4d^{1/3})L^{k-\ell} \geq cL^{k-\ell}.$$

So by the assumption in the statement of the lemma, R has a fractional matching, F say, of size $(b + \varepsilon)L$.

For each $e \in E(R)$, let $K_e := \lceil (1 - 2\varepsilon')F(e)m \rceil$, where m is the size of each of the clusters of R . Now construct an integer matching, M say, in G by greedily adding to M edges of G' until, for each $e = \{V_{j_1}, \dots, V_{j_k}\} \in E(R)$, M contains precisely K_e edges of $(V_{j_1}, \dots, V_{j_k})_{G'}$. Note that at each stage of this process the number of vertices in each $V_i \in V(R)$ that would be covered by M is at most

$$\sum_{e: V_i \in e} K_e \leq \sum_{e: V_i \in e} ((1 - 2\varepsilon')F(e)m + 1) \leq (1 - 2\varepsilon')m + \binom{L-1}{k-1} \leq (1 - \varepsilon')m.$$

Note also that for every edge $e = \{V_{j_1}, \dots, V_{j_k}\} \in E(R)$, we have that $(V_{j_1}, \dots, V_{j_k})_{G'}$ is ε' -regular with density $d > \varepsilon'$. So indeed, by the definition of ε' -regularity, it is possible to successively add edges to M in order to obtain a matching M as desired.

Note that the size of M is

$$\sum_{e \in E(R)} K_e \geq \sum_{e \in E(R)} (1 - 2\varepsilon')F(e)m = (1 - 2\varepsilon')m(b + \varepsilon)L \geq (1 - 2\varepsilon')(b + \varepsilon)(1 - \varepsilon')n \geq bn.$$

So indeed G has an (integer) matching of size at least bn . \square

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\varepsilon' > 0$ and let $0 < \varepsilon'' \ll \varepsilon', \varepsilon, 1/k, 1/2(k-d) - a$. Let $n_0 \in \mathbb{N}$ be sufficiently large and suppose that $n \geq n_0$. Let $k' := k - d$ and $n' := n - d$. Then

$$\begin{aligned} f_0^{(a+\varepsilon'')n}(k-d, n-d) &= f_0^{(a+\varepsilon'')(n'+d)}(k', n') \leq f_0^{(a+2\varepsilon'')n'}(k', n') \leq m_0^{(a+2\varepsilon'')n'}(k', n') \\ &\stackrel{(4.1)}{\leq} \left(1 - (1-a-2\varepsilon'')^{k'} + \frac{\varepsilon'}{4}\right) \binom{n'}{k'} \leq \left(1 - (1-a)^{k-d} + \frac{\varepsilon'}{2}\right) \binom{n-d}{k-d}. \end{aligned}$$

So by Proposition 4.1, if H is a k -uniform hypergraph on n vertices such that for at least $(1 - \varepsilon'') \binom{n}{d}$ d -tuples of vertices $L \in \binom{V(H)}{d}$ we have

$$\deg_H(L) \geq \frac{1 - (1-a)^{k-d} + \varepsilon'/2}{(k-d)!} n^{k-d} \geq \left(1 - (1-a)^{k-d} + \frac{\varepsilon'}{2}\right) \binom{n-d}{k-d},$$

then H has a fractional matching of size $(a + \varepsilon'')n$. So by Lemma 5.6, any k -uniform hypergraph G on $n \geq n'_0$ vertices (where n'_0 is sufficiently large) with

$$\delta_d(G) \geq \left(1 - (1-a)^{k-d} + \varepsilon'\right) \binom{n-d}{k-d} \geq \left(\frac{1 - (1-a)^{k-d} + \varepsilon'/2}{(k-d)!} + \varepsilon''\right) n^{k-d}$$

has an (integer) matching of size at least an . This gives the upper bound in Theorem 1.4. The lower bound follows from (2.1). \square

We can prove Theorem 1.2 in a similar way, but to do so we will need to use the absorbing technique as introduced by Rödl, Ruciński and Szemerédi [20]. More precisely, we use the existence of a small and powerful matching M_{abs} in G which, by ‘absorbing’ vertices, can transform any almost perfect matching into a perfect matching. M_{abs} has the property that whenever X is a sufficiently small set of vertices of G not covered by M_{abs} (and $|X| \in k\mathbb{N}$) there exists a matching in G which covers precisely the vertices in $X \cup V(M_{abs})$. Since this part of the proof of Theorem 1.2 is very similar to the corresponding part of the proof of Theorem 1.1 in [1], we only sketch it.

Proof of Theorem 1.2 (sketch). Let $\varepsilon > 0$ and suppose that G is a k -uniform hypergraph on n vertices with minimum d -degree at least

$$(5.2) \quad \left(\frac{k-d}{k} - \frac{k-d-1}{k^{k-1}} + \varepsilon \right) \binom{n-d}{k-d} > \left(\frac{1}{2} + \varepsilon \right) \binom{n-d}{k-d}.$$

(5.2) implies that we can use the Strong Absorbing Lemma from [10] to find an absorbing matching M_{abs} in G , and set $G' := G \setminus V(M_{abs})$. Using the degree condition, Theorem 1.9 gives us a perfect fractional matching in G' for sufficiently large n . Lemma 5.6(ii) then transforms this into an almost perfect integer matching M_{alm} in G' . We then extend $M_{alm} \cup M_{abs}$ to a perfect matching of G by using the absorbing property of M_{abs} . \square

6. CONCLUDING REMARKS

Using a similar method to that employed in proving Theorem 1.2 it is possible to prove a variant of Theorem 1.6 that verifies Conjecture 1.5 asymptotically for all $s \in \mathbb{N}$ satisfying $s/n \leq a_k$, where a_k is the unique solution in $(0, 1/(k+1))$ to

$$1 = \frac{1 - (1 - 2a_k)^{k-1}}{(1 - a_k)^{k-1}}.$$

For small values of k this allows us to verify Conjecture 1.5 asymptotically for some values of s not covered by Theorem 1.6. For example for $k = 4$ this allows s to range up to $0.567n/k$. This approach also yields a slight improvement, for small values of k , to the range of matching sizes allowed in Theorem 1.4, (at the expense of lengthy calculations). But for large k Theorem 1.6 gives the better bounds on the matching sizes allowed (as a_k is close to $0.48/k$ in this case). For details see [22].

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School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {d.kuhn, d.osthus, txt238}@bham.ac.uk