EDGE-DISJOINT HAMILTON CYCLES IN GRAPHS

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Abstract. In this paper we give an approximate answer to a question of Nash-Williams from 1970: we show that for every $\alpha > 0$, every sufficiently large graph on n vertices with minimum degree at least $(1/2 + \alpha)n$ contains at least $n/8$ edge-disjoint Hamilton cycles. More generally, we give an asymptotically best possible answer for the number of edge-disjoint Hamilton cycles that a graph G with minimum degree δ must have. We also prove an approximate version of another long-standing conjecture of Nash-Williams: we show that for every $\alpha > 0$, every (almost) regular and sufficiently large graph on n vertices with minimum degree at least $(1/2 + \alpha)n$ can be almost decomposed into edge-disjoint Hamilton cycles.

1. Introduction

Dirac's theorem [2] states that every graph on $n \geqslant 3$ vertices of minimum degree at least $n/2$ contains a Hamilton cycle. The theorem is best possible since there are graphs of minimum degree at least $\lfloor (n-1)/2 \rfloor$ which do not contain any Hamilton cycle.

Nash-Williams [13] proved the surprising result that the conditions of Dirac's theorem, despite being best possible, even guarantee the existence of many edge-disjoint Hamilton cycles.

Theorem 1 ([13])**.** *Every graph on n vertices of minimum degree at least $n/2$ contains at least $\lfloor 5n/224 \rfloor$ edge-disjoint Hamilton cycles.*

Nash-Williams [12, 13, 14] asked whether the above bound on the number of Hamilton cycles can be improved. Clearly we cannot expect more than $\lfloor (n+1)/4 \rfloor$ edge-disjoint Hamilton cycles and Nash-Williams [12] initially conjectured that one might be able to achieve this. However, soon afterwards, it was pointed out by Babai (see [12]) that this conjecture is false. Babai's idea was carried further by Nash-Williams [12] who gave an example of a graph on $n = 4m$ vertices with minimum degree $2m$ having at most $\lfloor (n+4)/8 \rfloor$ edge-disjoint Hamilton cycles. Here is a similar example having at most $\lfloor (n+2)/8 \rfloor$ edge-disjoint Hamilton cycles: Let A be an empty graph on $2m$ vertices, B a graph consisting of $m+1$ disjoint edges and let G be the graph obtained from the disjoint union of A and B by adding all possible edges between A and B. So G is a graph on $4m+2$ vertices with minimum degree $2m+1$. Observe that any Hamilton cycle of G must use at least 2 edges from B and thus G has at most $\lfloor (m+1)/2 \rfloor$ edge-disjoint Hamilton cycles. We will prove that this example is asymptotically best possible.

Theorem 2. *For every $\alpha > 0$ there is an integer n_0 so that every graph on $n \geqslant n_0$ vertices of minimum degree at least $(1/2+\alpha)n$ contains at least $n/8$ edge-disjoint Hamilton cycles.*

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Nash-Williams [12, 14] pointed out that the construction described above depends heavily on the graph being non-regular. He thus conjectured [14] the following, which if true is clearly best possible.

Conjecture 3 ([14]). *Let G be a d-regular graph on at most $2d$ vertices. Then G contains $\lfloor d/2 \rfloor$ edge-disjoint Hamilton cycles.*

The conjecture was also raised independently by Jackson [5]. For complete graphs, its truth follows from a construction of Walecki (see e.g. [1, 10]). The best result towards this conjecture is the following result of Jackson [5].

Theorem 4 ([5]). *Let G be a d-regular graph on $14 \leqslant n \leqslant 2d+1$ vertices. Then G contains $\lfloor (3d-n+1)/6 \rfloor$ edge-disjoint Hamilton cycles.*

In this paper we prove an approximate version of Conjecture 3.

Theorem 5. *For every $\alpha > 0$ there is an integer n_0 so that every d-regular graph on $n \geqslant n_0$ vertices with $d \geqslant (1/2+\alpha)n$ contains at least $(d-\alpha n)/2$ edge-disjoint Hamilton cycles.*

In fact, we will prove the following more general result which states that Theorem 5 is true for almost regular graphs as well. Note that the construction showing that one cannot achieve more than $\lfloor (n+2)/8 \rfloor$ edge-disjoint Hamilton cycles under the conditions of Dirac's theorem is almost regular. However in the following result we also demand that the minimum degree is a little larger than $n/2$.

Theorem 6. *There exists $\alpha_0 > 0$ so that for every $0 < \alpha \leq \alpha_0$ there is an integer n_0 for which every graph on $n \geqslant n_0$ vertices with minimum degree $\delta \geqslant (1/2+\alpha)n$ and maximum degree $\Delta \leqslant \delta + \alpha^2 n/5$ contains at least $(\delta - \alpha n)/2$ edge-disjoint Hamilton cycles.*

Frieze and Krivelevich [3] proved that the above results hold if one also knows that the graph is quasi-random (in which case one can drop the condition on the minimum degree). So in particular, it follows that a binomial random graph $G_{n,p}$ with constant edge probability p can 'almost' be decomposed into Hamilton cycles with high probability. For such p, it is still an open question whether one can improve this to show that with high probability the number of edge-disjoint Hamilton cycles is exactly half the minimum degree – see e.g. [3] for a further discussion. Our proof makes use of the ideas in [3].

Finally, we answer the question of what happens if we have a better bound on the minimum degree than in Theorem 2. The following result approximately describes how the number of edge-disjoint Hamilton cycles guaranteed in G gradually approaches $\delta(G)/2$ as $\delta(G)$ approaches $n-1$.

Theorem 7.

(i) *For all positive integers δ, n with $n/2 < \delta < n$, there is a graph G on n vertices with minimum degree δ such that G contains at most*

$$\frac{\delta + 2 + \sqrt{n(2\delta - n)}}{4}$$

edge-disjoint Hamilton cycles.

(ii) *For every $\alpha > 0$, there is a positive integer n_0 so that every graph on $n \geqslant n_0$ vertices of minimum degree $\delta \geqslant (1/2+\alpha)n$ contains at least*

$$\frac{\delta - \alpha n + \sqrt{n(2\delta - n)}}{4} \tag{1}$$

edge-disjoint Hamilton cycles.

Observe that Theorem 2 is an immediate consequence of Theorem 7(ii). In Section 2 we will give a simple construction which proves Theorem 7(i). This construction also yields an analogue of Theorem 7 for r-factors, where r is even: Clearly, Theorem 7(ii) implies the existence of an r-factor for any even r which is at most twice the bound in (1). The construction in Section 2 shows that this is essentially best possible. The question of which conditions on a graph guarantee an r-factor has a huge literature, see the survey by Plummer for a recent overview [15].

It turns out that the proofs of Theorems 6 and 7(ii) are very similar and we will thus prove these results simultaneously. In Section 3 we give an overview of the proof. In Section 4 we introduce some notation and also some tools that we will need in the proofs of Theorems 6 and 7(ii). We prove these theorems in Section 5.

Another long-standing conjecture in the area is due to Kelly (see e.g. [11]). It states that any regular tournament can be decomposed into edge-disjoint Hamilton cycles. Very recently, an approximate version of this conjecture was proved in [8]. The basic proof strategy is common to both papers. So we hope that the proof techniques will also be useful for further decomposition problems.

2. Proof of Theorem 7(i)

If $\delta = n-1$, then K_n contains at most

$$\frac{n-1}{2} = \frac{n+(n-2)}{4} < \frac{n+1+\sqrt{n(n-2)}}{4} = \frac{\delta+2+\sqrt{n(2\delta-n)}}{4}$$

edge-disjoint Hamilton cycles. So from now on we will assume that $\delta \leqslant n-2$.

The construction of the graph G is very similar to the construction in the introduction showing that we might not have more than $\lfloor (n+2)/8 \rfloor$ edge-disjoint Hamilton cycles. Here, G will be the disjoint union of an empty graph A of size $n-\Delta$, and a $(\delta+\Delta-n)$-regular graph B on Δ vertices, together with all edges between A and B (see Figure 1). Such a graph B exists if for example Δ is even (see e.g. [9, Problem 5.2]).

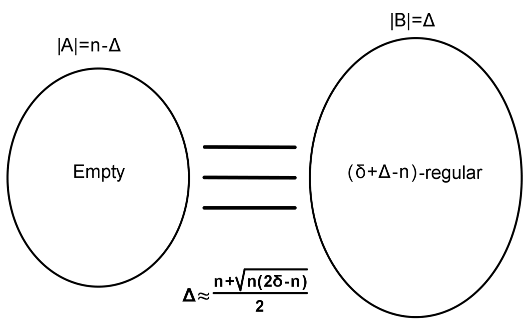

Figure 1. A graph G on n vertices with minimum degree at least $\delta > n/2$ having at most $\frac{\delta+2+\sqrt{n(2\delta-n)}}{2}$ edge-disjoint Hamilton cycles.

The value of Δ will be chosen later. At the moment we will only demand that Δ is an even integer satisfying $\delta \leqslant \Delta \leqslant n-1$. Observe that G is a graph on n vertices with minimum degree δ and maximum degree Δ. We claim that G cannot contain more than $\frac{\Delta(\delta+\Delta-n)}{2(2\Delta-n)}$ edge-disjoint Hamilton cycles. In fact, we claim that it can only contain an r-factor if $r \leq \frac{\Delta(\delta+\Delta-n)}{2\Delta-n}$. Indeed, given any r-factor H of G, since $e_H(A,B) = \sum_{v\in A} d_H(v) =$

$r(n-\Delta)$, we deduce that

$$r\Delta = \sum_{v\in B} d_H(v) \leqslant \Delta(\delta+\Delta-n) + r(n-\Delta)$$

from which our claim follows. It remains to make a judicious choice for Δ and to show that it implies the result. One can check that $\frac{n+\sqrt{n(2\delta-n)}}{2}$ minimizes $f(x) = x(\delta+x-n)/(2x-n)$ in $[\delta, n]$. (This is only used as a heuristic and it is not needed in our argument.) It can be also checked that since $\delta \leqslant n-2$ we have $\delta \leqslant \frac{n+\sqrt{n(2\delta-n)}}{2} < n-1$. Indeed, the first inequality holds if and only if $(2\delta-n)^2 \leqslant n(2\delta-n)$ which is true as $n/2 \leqslant \delta \leqslant n$ and the second inequality holds since

$$\frac{n+\sqrt{n(2\delta-n)}}{2} \leqslant \frac{n+\sqrt{n^2-4n}}{2} < \frac{n+(n-2)}{2} = n-1.$$

We define $\Delta = \frac{n+\sqrt{n(2\delta-n)}}{2} + \varepsilon$, where ε is chosen so that $|\varepsilon| \leqslant 1$ and Δ is an even integer satisfying $\delta \leqslant \Delta \leqslant n-1$. We claim that this value of Δ gives the desired bound. To see this, recall that if G contains an r-factor, then we must have

$$r \leq \frac{\Delta(\delta+\Delta-n)}{2\Delta-n} = \frac{\delta}{2} + \frac{n\delta/2}{2\Delta-n} - \frac{\Delta(n-\Delta)}{2\Delta-n}$$

and that

$$\begin{aligned}\Delta(n-\Delta) &= \left(\frac{n}{2} + \left(\frac{\sqrt{n(2\delta-n)}}{2} + \varepsilon\right)\right)\left(\frac{n}{2} - \left(\frac{\sqrt{n(2\delta-n)}}{2} + \varepsilon\right)\right) \\ &= \frac{n^2}{4} - \frac{n(2\delta-n)}{4} - \varepsilon\sqrt{n(2\delta-n)} - \varepsilon^2 = \frac{n^2-n\delta}{2} - \varepsilon\sqrt{n(2\delta-n)} - \varepsilon^2.\end{aligned}$$

Thus

$$r \leqslant \frac{\delta}{2} + \frac{n(2\delta-n) + 2\varepsilon\sqrt{n(2\delta-n)}}{2(2\Delta-n)} + \frac{\varepsilon^2}{2\Delta-n}.$$

Since also $(2\Delta-n)\sqrt{n(2\delta-n)} = n(2\delta-n) + 2\varepsilon\sqrt{n(2\delta-n)}$, we deduce that

$$r \leqslant \frac{\delta+\sqrt{n(2\delta-n)}}{2} + \frac{\varepsilon^2}{2\Delta-n} \leqslant \frac{\delta+2+\sqrt{n(2\delta-n)}}{2},$$

as required.

3. Proof overview of the main theorems

In the overview we will only discuss the case in which G is regular, say of degree λn with $\lambda > 1/2$. The other cases are similar and in fact will be treated simultaneously in the proof itself. We begin by defining additional constants such that

$$0 < \varepsilon \ll \beta \ll \gamma \ll 1.$$

By applying the Regularity Lemma to G, we obtain a partition of G into clusters $V_1, \dots, V_k$ and an exceptional set V_0. Moreover, most pairs of clusters span an ε-regular (i.e. quasi-random) bipartite graph. It turns out that for our purposes the 'standard' reduced graph defined on the clusters does not capture enough information about the original graph G. So we will instead work with the multigraph R on vertex set $\{V_1, \dots, V_k\}$ in which there are exactly $\ell_{ij} := \lfloor d(V_i, V_j)/\beta \rfloor$ multiple edges between the vertices V_i and V_j of R (provided that the pair (V_i, V_j) is ε-regular). Here $d(V_i, V_j)$ denotes the density of the bipartite subgraph induced by V_i and V_j. Then R is almost regular, with all degrees close to $\lambda k/\beta$. In particular, we can use Tutte's f-factor theorem (see Theorem 12(ii)) to deduce that R contains an r-regular submultigraph R' where r is still close to $\lambda k/\beta$. By Petersen's theorem, R' can be decomposed into 2-factors and by splitting the clusters if necessary we may assume that R' can be decomposed into 2-factors such that every cycle has even length. In particular, R' can be decomposed into r perfect matchings, say $M_1, \dots, M_r$.

We now partition (most of) the edges of G in such a way that each matching edge is assigned roughly the same number of edges of G. More precisely, given two adjacent clusters U, V of R, the edge set $E_G(U, V)$ can be decomposed into ℓ_{ij} bipartite graphs so that each is ε-regular with density close to β. These ℓ_{ij} regular pairs correspond to the ℓ_{ij} edges in R between U and V. Thus, for each matching M_i, we can define a subgraph G_i of G such that all G_i's are edge-disjoint and they consist of a union of $k' := k/2$ pairs of clusters which are ε-regular of density about β, together with the exceptional set V_0. Let m denote the size of a cluster. By moving some additional vertices to the exceptional set, we may assume that for every such pair of clusters of G_i, all vertices have degree close to βm. So for each i, we now have a set V_{0i} consisting of the exceptional set V_0 together with the vertices moved in the previous step. For each G_i we will aim to find close to $\beta m/2$ edge-disjoint Hamilton cycles consisting mostly of edges of G_i and a few further edges which do not belong to any of the G_i.

Because G may not have many edges which do not belong to any of the G_i, (in fact it may have none) before proceeding we extract random subsets of edges from each G_i to get disjoint subgraphs H_1, H_2 and H_3 of G each of density about γ which satisfy several other useful properties as well. Moreover, each pair of clusters of G_i corresponding to an edge of M_i will still be super-regular of density almost β. Each of the subgraphs H_1, H_2 and H_3 will be used for a different purpose in the proof.

H_1 will be used to connect the vertices of each V_{0i} to $G_i \setminus V_{0i}$ so that the vertices of V_{0i} have almost βm neighbours in $V(G_i) \setminus V_{0i}$. Moreover the edges added to G_i will be well spread-out in the sense that no vertex of $G_i \setminus V_{0i}$ will have large degree in V_{0i}. So every vertex of G_i now has degree close to βm.

Next, our aim is to find an s-regular spanning subgraph S_i of G_i with s close to βm. In order to achieve this, it turns out that we will first need to add some edges to G_i between pairs of clusters which do not correspond to edges of M_i. We will take these from H_2.

We may assume that the degree of S_i is even and thus by Petersen's theorem it can be decomposed into 2-factors. It will remain to use the edges of H_3 to transform each of these 2-factors into a Hamilton cycle. Several problems may arise here. Most notably, the number of edges of H_3 we will need in order to transform a given 2-factor F into a Hamilton cycle will be proportional to the number of cycles of F. So if we have a linear number of 2-factors F which have a linear number of cycles, then we will need to use a quadratic number of edges from H_3 which would destroy most of its useful properties. However, a result from [3] based on estimating the permanent of a matrix implies that the average number of cycles in a 2-factor of S_i is $o(n)$. We will apply a variant of this result proved in [7, 8]. So we can assume that our 2-factors have $o(n)$ cycles.

To complete the proof we will consider a random partition of the graph H_3 into subgraphs $H_{3,1}, \dots, H_{3,r}$, one for each graph G_i. We will use the edges of $H_{3,i}$ to transform all 2-factors of S_i into Hamilton cycles. We will achieve this by considering each 2-factor F successively. For each F, we will use the rotation-extension technique to successively merge its cycles. Roughly speaking, this means that we obtain a path P with endpoints x and y (say) by removing a suitable edge of a cycle of F. If F is not a Hamilton cycle and $H_{3,i}$ has an edge from x or y to another cycle C of F, and we can extend P to a path containing all vertices of C as well. We continue in this way until in $H_{3,i}$ both endpoints of P have all their neighbours on P. We can then use this to find a cycle C' containing precisely all vertices of P. In the final step, we make use (amongst others) of the quasi-randomness of the bipartite graphs which form $H_{3,i}$.

4. Notation and Tools

4.1. **Notation.** Given vertex sets A and B in a graph G, we write $E_G(A,B)$ for the set of all edges ab with $a \in A$ and $b \in B$ and put $e_G(A,B) = |E_G(A,B)|$. We write $(A,B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose set of edges is $E_G(A,B)$. We drop the subscripts if this is unambiguous. Given a set $E' \subseteq E_G(A,B)$, we also write $(A,B)_{E'}$ for the bipartite subgraph of G whose vertex classes are A and B and whose set of edges is E'. Given a vertex x of G and a set $A \subseteq V(G)$, we write $d_A(x)$ for the number of neighbours of x in A.

To prove Theorems 6 and 7(ii) it will be convenient to work with multigraphs instead of just (simple) graphs. All multigraphs considered in this paper will be without loops.

We write $a = b \pm c$ to mean that the real numbers a, b, c satisfy $|a-b| \leqslant c$. To avoid unnecessarily complicated calculations we will sometimes omit floor and ceiling signs and treat large numbers as if they were integers. We will also sometimes treat large numbers as if they were even integers.

4.2. **Chernoff Bounds.** Recall that a Bernoulli random variable with parameter p takes the value 1 with probability p and the value 0 with probability $1-p$. We will use the following Chernoff-type bound for a sum of independent Bernoulli random variables.

Theorem 8 (Chernoff Inequality)**.** *Let $X_1, \dots, X_n$ be independent Bernoulli random variables with parameters $p_1, \dots, p_n$ respectively and let $X = X_1 + \cdots + X_n$. Then*

$$\mathbb{P}(|X - \mathbb{E}X| \geqslant t) \leqslant 2\exp\left(-\frac{t^2}{3\mathbb{E}X}\right).$$

In particular, since a binomial random variable X with parameters n and p is a sum of n independent Bernoulli random variables, the above inequality holds for binomial random variables as well.

4.3. **Regularity Lemma.** In the proof, we will use the degree form of Szemerédi's Regularity Lemma. Before stating it, we need to introduce some notation. The *density* of a bipartite graph $G = (A,B)$ with vertex classes A and B is defined to be $d_G(A,B) := \frac{e(A,B)}{|A||B|}$. We sometimes write $d(A,B)$ for $d_G(A,B)$ if this is unambiguous. Given $\varepsilon > 0$, we say that G is *ε-regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$ we have that $|d(X,Y) - d(A,B)| < \varepsilon$. Given $d \in [0,1]$, we say that G is *(ε, d)-super-regular* if it is ε-regular and furthermore $d_G(a) \geqslant d|B|$ for all $a \in A$ and $d_G(b) \geqslant d|A|$ for all $b \in B$. We will use the following degree form of Szemerédi's Regularity Lemma:

Lemma 9 (Regularity Lemma; Degree form)**.** *For every $\varepsilon \in (0,1)$ and each positive integer M', there are positive integers M and n_0 such that if G is any graph on $n \geqslant n_0$ vertices and $d \in [0,1]$ is any real number, then there is a partition of the vertices of G into $k+1$ classes $V_0, V_1, \dots, V_k$, and a spanning subgraph G' of G with the following properties:*

- $M' \leqslant k \leqslant M$*;*
- $|V_0| \leqslant \varepsilon n, |V_1| = \cdots = |V_k| =: m$*;*
- $d_{G'}(v) \geqslant d_G(v) - (d+\varepsilon)n$ *for every* $v \in V(G)$*;*
- $G'[V_i]$ *is empty for every* $0 \leqslant i \leqslant k$*;*
- *all pairs* (V_i, V_j) *with* $1 \leqslant i < j \leqslant k$ *are ε-regular with density either 0 or at least d.*

We call $V_1, \dots, V_k$ the *clusters* of the partition and V_0 the *exceptional set*.

4.4. **Factor Theorems.** An *r-factor* of a multigraph G is an r-regular submultigraph H of G. We will use the following classical result of Petersen.

Theorem 10 (Petersen's Theorem)**.** *Every regular multigraph of positive even degree contains a 2-factor.*

Furthermore, we will use Tutte's f-factor theorem [16] which gives a necessary and sufficient condition for a multigraph to contain an f-factor. (In fact, the theorem is more general.) Before stating it we need to introduce some notation. Given a multigraph G, a positive integer r, and disjoint subsets T, U of $V(G)$, we say that a component C of $G[U]$ is *odd (with respect to r and T)* if $e(C,T) + r|C|$ is odd. We write $q(U)$ for the number of odd components of U.

Theorem 11. *A multigraph G contains an r-factor if and only if for every partition of the vertex set of G into sets S, T, U, we have*

$$\sum_{v \in T} d(v) - e(S,T) + r(|S| - |T|) \geqslant q(U). \tag{2}$$

In fact, we will only need the following consequence of Theorem 11.

Theorem 12. *Let G be a multigraph on n vertices of minimum degree $\delta \geqslant \ell n/2$, in which every pair of vertices is joined by at most ℓ edges.*

(i) *Let r be an even number such that $r \leqslant \frac{\delta+\sqrt{\ell n(2\delta-\ell n)}}{2}$. Then G contains an r-factor.*
(ii) *Let $0 < \xi < 1/9$ and suppose $(1/2+\xi)\ell n \leqslant \Delta(G) \leqslant \delta+\xi^2\ell n$. If r is an even number such that $r \leqslant \delta - \xi\ell n$ and n is sufficiently large, then G contains an r-factor.*

The case when $\ell = 1$ and r is close to $n/4$ in (i) was already proven by Katerinis [6].

Proof. By Theorem 10, in both (i) and (ii) it suffices to consider the case that r is the maximal positive even integer satisfying the conditions. Observe that since $\delta \leqslant \ell(n-1) < \ell n$ it follows that $\ell n(2\delta - \ell n) = \delta^2 - (\ell n - \delta)^2 < \delta^2$, so in case (i) we have $r < \delta$ and since both r and δ are integers we have $r \leqslant \delta - 1$. This also holds in case (ii).

By Theorem 11, it is enough to show (in both cases) that (2) holds for every partition of the vertex set of G into sets S, T and U.

Case 1. *$\ell|T| \leqslant r-1$ and $\ell|S| \leqslant \delta - r$.*

Since in this case $d_T(v) \leqslant \ell|T| \leqslant r-1$ for every $v \in V(G)$, the left hand side of (2) is

$$\sum_{v\in T}(d(v)-r) + \sum_{v\in S}(r - d_T(v)) \geqslant |T| + |S|.$$

So in this case, it is enough to show that $q(U) \leqslant |T|+|S|$. If $|T| = 0$, the result holds since in this case no component of $G[U]$ is odd, i.e $q(U) = 0$. If $|T| = 1$ and $|S| = 0$, then the degree conditions imply that $G[U]$ is connected and so $q(U) \leqslant 1 = |T| + |S|$. (Indeed, the degree conditions imply that the undirected graph obtained from G by ignoring multiple edges has minimum degree at least $n/2$ and so any subgraph of it on $n-1$ vertices must be connected.) Thus in this case, we may assume that $2 \leqslant |T|+|S| \leqslant \frac{\delta-1}{\ell}$. Observe that every vertex $v \in U$ has at most $\ell(|T| + |S|)$ neighbours in $T \cup S$ when counting multiplicity and so it has at least $\frac{\delta-\ell(|T|+|S|)}{\ell}$ distinct neighbours in U. In particular, every component of $G[U]$ contains at least $\frac{\delta+\ell-\ell(|T|+|S|)}{\ell}$ vertices and so certainly $q(U) \leqslant \frac{\ell|U|}{\delta+\ell-\ell(|T|+|S|)}$. Writing $k := |T| + |S|$, it is enough in this case to prove that $k \geqslant \frac{\ell(n-k)}{\delta+\ell-\ell k}$. But this is equivalent to proving that $k\delta + 2k\ell - \ell k^2 - \ell n \geqslant 0$, which is true since the left hand side is equal to $(k-2)(\delta - \ell k) + 2\delta - \ell n$.

Case 2. *$\ell|T| = r$.*

Since for every vertex $v \in S$ we have $d_T(v) \leqslant \ell|T| = r$, it follows that $r|S| \geq e(S,T)$. Thus the left hand side of (2) is at least $(\delta - r)|T| = (\delta - r)r/\ell$. Observe that $G[U]$ has at most $n - \delta/\ell$ components. Indeed, if C is a component of $G[U]$ and x is a vertex of C, then as x can only have neighbours in $C \cup S \cup T$ we have that $|C \cup S \cup T| \geqslant 1 + \delta/\ell$ and so U has at most $n - 1 - \delta/\ell$ other components. Thus, it is enough to show that

$(\delta - r)r \geqslant \ell n - \delta$. For case (ii) (recall that r is maximal subject to the given conditions) we have $(\delta - r)r \geqslant \xi \ell n(\delta - \xi \ell n - 2) \geq \ell n - \delta$. To see the last inequality, recall that $\delta \geq \ell n/2$ and $\xi \ell n \geq 4$ say (as we assume that n is sufficiently large). So $\xi \ell n(\delta - \xi \ell n - 2) \geq 4(\ell n(1/2 - \xi) - 2) = (2 - 4\xi)\ell n - 8 \geq \ell n$. To prove (i), note that we (always) have

$$(\delta - r)r = \frac{\delta^2}{4} - \left(r - \frac{\delta}{2}\right)^2 \geqslant \frac{\delta^2}{4} - \frac{\ell n(2\delta - \ell n)}{4} = \left(\frac{\ell n - \delta}{2}\right)^2. \tag{3}$$

So the result also holds in case (i) unless $\delta \geqslant \ell n - 3$. But if this is the case, then $(\delta - r)r \geqslant \ell n - \delta$ unless $r = 2, \delta = 3$ and $\ell n = 6$. But this violates the assumption on r in (i).

Case 3. *$|T| \geqslant \frac{r+1}{\ell}$ and $|S| \geqslant \frac{\delta - r+1}{\ell}$.*

Since $q(U) \leqslant |U| = n - |S| - |T|$, it is enough to show that in this case we have

$$(\delta - r + 1)|T| + (r + 1)|S| - \ell |S||T| \geqslant n. \tag{4}$$

By writing the left hand side of (4) as

$$\frac{(\delta - r + 1)(r + 1)}{\ell} - \ell\left(|T| - \frac{r+1}{\ell}\right)\left(|S| - \frac{\delta - r + 1}{\ell}\right), \tag{5}$$

we observe that it is minimized when $|T| + |S|$ is maximal, i.e. it is equal to n. To prove (i), observe that the left hand side of (4) is at least

$$\begin{aligned} &\frac{(\delta - r + 1)(r + 1)}{\ell} - \frac{\ell}{4}\left(n - \frac{\delta+2}{\ell}\right)^2 = \frac{(\delta - r)r}{\ell} + \frac{\delta + 1}{\ell} - \frac{\ell n^2}{4} + \frac{n(\delta+2)}{2} - \frac{(\delta+2)^2}{4\ell} \\ &\overset{(3)}{\geqslant} \frac{(\ell n - \delta)^2}{4\ell} + \frac{\delta+1}{\ell} - \frac{\ell n^2}{4} + \frac{n(\delta + 2)}{2} - \frac{(\delta+2)^2}{4\ell} = n. \end{aligned} \tag{6}$$

To prove (ii) we may assume that $\delta < (1 - \sqrt{\xi})\ell n$. Indeed if $\delta \geqslant (1 - \sqrt{\xi})\ell n$, then using that r is maximal subject to the given conditions we have

$$(\delta - r)r \geqslant \xi \ell n(\delta - \xi \ell n - 2) \geqslant \xi \ell n\left(\left(\frac{1}{2} - \xi\right)\ell n - 2\right) \geq \frac{\xi \ell^2 n^2}{4} \geqslant \frac{(\ell n - \delta)^2}{4}$$

and the result follows exactly as in case (i).

If also $|T| \leqslant \frac{\Delta}{\ell}$, then we claim that $|T| - \frac{r+1}{\ell} \leqslant |S| - \frac{\delta - r+1}{\ell}$. Indeed, this follows since

$$\begin{aligned} |T| - |S| &\leqslant \frac{2\Delta}{\ell} - n \leqslant \frac{2\delta}{\ell} + 2\xi^2 n - n \leqslant \frac{\delta}{\ell} + 2\xi^2 n - \sqrt{\xi}n \leqslant \\ &\frac{(2r - \delta) + 2\xi \ell n + 4}{\ell} + 2\xi^2 n - \sqrt{\xi}n = \frac{2r - \delta}{\ell} + (2\xi^2 + 2\xi - \sqrt{\xi})n + \frac{4}{\ell} \leq \frac{2r - \delta}{\ell}. \end{aligned}$$

This claim together with the fact that $|T| + |S| = n$ implies that (5) (and thus the left hand side of (4)) is minimized when $|T| = \Delta/\ell$ and $|S| = n - \Delta/\ell$. Note that $|T| \geqslant (1/2 + \xi)n$ in this case and so $|T| - |S| \geq 2\xi n$. Thus the left hand side of (4) is at least

$$(\delta - r)|T| + (r - \ell|T|)|S| = (\delta - r)(|T| - |S|) + (\delta - \Delta)|S| \geqslant 2\xi^2 \ell n^2 - \xi^2 \ell n^2 \geq n.$$

To complete the proof, suppose $|T| \geqslant \frac{\Delta}{\ell}$. Then $e(S, T) \leqslant \Delta|S|$ and again $|T| - |S| \geq 2\xi n$. So the left hand side of (2) is at least

$$(\delta - r)|T| + (r - \Delta)|S| = (\delta - r)(|T| - |S|) + (\delta - \Delta)|S| \geqslant 2\xi^2 \ell n^2 - \xi^2 \ell n^2 \geq n.$$

Case 4. *$|T| \geqslant \frac{r+1}{\ell}$ or $|S| \geqslant \frac{\delta - r+1}{\ell}$ but not both.*

As in Case 3 it suffices to show that (4) holds. (5) shows that in this case the left hand side of (4) is at least $(\delta - r + 1)(r + 1)/\ell$. So (i) holds since (6) implies that the left hand side of (4) is at least n. For (ii), note that

$$\frac{(\delta - r + 1)(r + 1)}{\ell} \geq \frac{\xi \ell n(r + 1)}{\ell} \geq \xi n(\delta - \xi \ell n - 1) \geq n.$$

(Here we use the maximality of r in both inequalities.) □

5. Proofs of the Main Theorems

In this section we will prove Theorems 6 and 7(ii) simultaneously. Observe that in both cases we may assume that $\alpha \ll 1$. Define additional constants such that

$$\frac{1}{n_0} \ll \zeta \ll 1/M' \ll \varepsilon \ll \beta \ll \eta \ll d \ll \gamma \ll \alpha$$

and let G be a graph on $n \geq n_0$ vertices with minimum degree $\delta \geqslant (1/2 + \alpha)n$ and maximum degree Δ.

5.1. **Applying the Regularity Lemma.** We apply the Regularity Lemma to G with parameters $\varepsilon/2$, $3d/2$ and M' to obtain a partition of G into clusters $V_1, \dots, V_k$ and an exceptional set V_0, and a spanning subgraph G' of G. Let R be the multigraph on vertex set $\{V_1, \dots, V_k\}$ obtained by adding exactly $\ell_{ij} := \lfloor d_{G'}(V_i, V_j)/\beta \rfloor$ multiple edges between the vertices V_i and V_j of R. By removing one vertex from each cluster if necessary and adding all these vertices to V_0, we may assume that $m := |V_1| = \cdots = |V_k|$ is even. So now $|V_0| \leqslant \varepsilon n/2 + k \leqslant \varepsilon n$. The next lemma shows that R inherits its minimum and maximum degree from G.

Lemma 13.

(i) $\delta(R) \geqslant \left(\frac{\delta}{n} - 2d\right) \frac{k}{\beta}$;

(ii) $\Delta(R) \leqslant \left(\frac{\Delta}{n} + 2d\right) \frac{k}{\beta}$.

Proof. For any cluster V_i of R we have

$$\sum_{x \in V_i} d_{G'}(x) \leqslant e(V_0, V_i) + \sum_{j \neq i} e_{G'}(V_i, V_j) \leqslant \varepsilon mn + \sum_{j \neq i} d_{G'}(V_i, V_j) m^2.$$

Since $d_{G'}(V_i, V_j) \leqslant \beta(\ell_{ij} + 1)$, we obtain

$$\sum_{x \in V_i} d_{G'}(x) \leqslant \varepsilon mn + (d_R(V_i) + k)\beta m^2.$$

By the definition of G' in the Regularity Lemma we also have

$$\sum_{x \in V_i} d_{G'}(x) \geqslant \sum_{x \in V_i} (d_G(x) - (3d/2 + \varepsilon)n) \geqslant \delta m - (3d/2 + \varepsilon)mn.$$

Since also $\varepsilon, \beta \ll d$, (i) follows. Similarly,

$$d_R(V_i)\beta m^2 \leqslant \sum_{j \neq i} d_{G'}(V_i, V_j) m^2 \leqslant \sum_{x \in V_i} d_{G'}(x) \leqslant \Delta m,$$

so (ii) follows. □

Since $\delta \geqslant (1/2 + \alpha)n$ and since between any two vertices of R there are at most $1/\beta$ edges, Theorem 12(i) implies that R contains an r-regular submultigraph R' for every even positive integer r satisfying

$$r \leqslant \left(\frac{\delta}{n} - 2d + \sqrt{\frac{2\delta}{n} - 4d - 1}\right) \frac{k}{2\beta} = \left(\delta - 2dn + \sqrt{2\delta n - 4dn^2 - n^2}\right) \frac{k}{2\beta n}.$$

In particular, (using the inequality $\sqrt{x-y} \geqslant \sqrt{x} - \sqrt{y}$ for $x \geq y > 0$ and the fact that $\alpha \gg d$) we may assume that

$$r = \left(\delta + \sqrt{n(2\delta - n)} - \alpha n/2\right) \frac{k}{2\beta n}. \tag{7}$$

Moreover, for the proof of Theorem 6, we have $\Delta(R) - \delta(R) \leqslant \left(\frac{\Delta - \delta}{n} + 4d\right) \frac{k}{\beta} \leqslant \alpha^2 k/4\beta$. Therefore, by taking $\xi = \alpha/2$ in Theorem 12(ii) we may even assume that

$$r = (\delta - 2\alpha n/3) \frac{k}{\beta n}. \tag{8}$$

So from now on, R' is an r-regular submultigraph of R, where r is even and is given by (7) for the proof of Theorem 7(ii) and given by (8) for the proof of Theorem 6.

By Theorem 10, R' can be decomposed into 2-factors. As mentioned in the overview, it will be more convenient to work with a matching decomposition rather than a 2-factor decomposition. If all the cycles in all the 2-factor-decompositions had even length then we could decompose them into matchings. Because this might not be the case, we will split each cluster corresponding to a vertex of R into two clusters to obtain a new multigraph R^*. More specifically, for each $1 \leqslant i \leqslant k$, we split each cluster V_i arbitrarily into two pieces V_i^1 and V_i^2 of size $m/2$. R^* is defined to be the multigraph on vertex set $V_1^1, V_1^2, \dots, V_k^1, V_k^2$ where the number of multiedges between V_i^a and V_j^b $(1 \leqslant i, j \leqslant k, 1 \leqslant a, b \leqslant 2)$ is equal to the number of multiedges of R between V_i and V_j.

Recall that by Theorem 10, R' can be decomposed into 2-factors. We claim that each cycle $v_1 \dots v_t$ of each 2-factor gives rise to two edge-disjoint even cycles in R^* each of length $2t$, which themselves give rise to a total of four matchings in R^*, each of size t. Indeed, denoting by a_i and b_i the clusters in R^* corresponding to v_i, if t is even, say $t = 2s$, then we can take the cycles $a_1a_2 \dots a_{2s}b_1b_2 \dots b_{2s}$ and $a_1b_2 \dots a_{2s-1}b_{2s}b_1a_2 \dots b_{2s-1}a_{2s}$. If t is odd, say $t = 2s+1$, then we can take the cycles $a_1b_2 \dots a_{2s-1}b_{2s}a_{2s+1}b_1b_{2s+1}a_{2s} \dots b_3a_2$ and $a_1a_{2s+1}a_{2s} \dots a_2b_1b_2 \dots b_{2s+1}$ (see Figure 2 for the cases $t = 4, 5$).

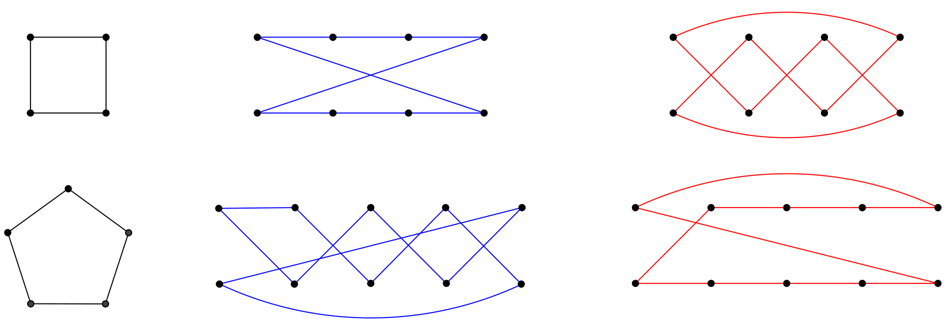

FIGURE 2. Cycles in R' and the corresponding cycles in R^*.

To simplify the notation we will now make the following relabelings: R' has served its purpose in finding a set of edge-disjoint perfect matchings in R^* and it will not be used any more, R^* is relabelled to R and the clusters $V_1^1, V_1^2, \dots, V_k^1, V_k^2$ are relabelled to $V_1, \dots, V_{k'}$. We also relabel k' back to k. Note that now each V_i has size $m' = m/2$ but we relabel m' back to m.

In particular we can now assume that we have a partition of the vertex set of G into k clusters $V_1, \dots, V_k$ and an exceptional set V_0, and a spanning subgraph G' of G satisfying the following properties:

- $|V_0| \leqslant \varepsilon n$ and $|V_1| = \dots = |V_k| =: m$;

- $d_{G'}(v) \geqslant d_G(v) - (3d/2 + \varepsilon)n$ for every $v \in V(G)$;
- $G'[V_i]$ is empty for every $0 \leqslant i \leqslant k$;
- all pairs $(V_i, V_j)_{G'}$ with $1 \leqslant i < j \leqslant k$ are ε-regular with density either 0 or at least d;
- R is a multigraph on vertex set $V_1, \dots, V_k$ having exactly $\ell_{ij} = \lfloor \frac{d_{G'}(V_i,V_j) \pm \varepsilon}{\beta} \rfloor$ edges $f_{ij}^1, \dots, f_{ij}^{\ell_{ij}}$ joining V_i and V_j;
- R has minimum degree at least $(\delta - 2dn)\frac{k}{\beta n}$ and maximum degree at most $(\Delta + 2dn)\frac{k}{\beta n}$;
- R contains a set of r edge-disjoint perfect matchings, where r satisfies (8) for Theorem 6 and (7) for Theorem 7(ii).

Later on, we will use that in both cases we have

$$k/5\beta \le r \le k/\beta \qquad \text{and} \qquad \delta \ge r\beta m + \alpha n/5. \tag{9}$$

We let $M_1, \dots, M_r$ be r edge-disjoint perfect matchings of R. We will define edge-disjoint subgraphs $G_1, \dots, G_r$ of G corresponding to the matchings $M_1, \dots, M_r$. Before doing that, for each $1 \leqslant i < j \leqslant k$ we will find ℓ_{ij} disjoint subsets $E_{ij}^1, \dots, E_{ij}^{\ell_{ij}}$ of $E_{G'}(V_i, V_j)$ corresponding to the ℓ_{ij} edges $f_{ij}^1, \dots, f_{ij}^{\ell_{ij}}$ of R between V_i and V_j. The next well known observation shows that we can choose the E_{ij}^ℓ so that each $(V_i, V_j)_{E_{ij}^\ell}$ forms a regular pair. It is e.g. a special case of Lemma 10(i) in [8]. To prove it, one considers a random partition of the edges of G' between V_i and V_j.

Lemma 14. *For each $1 \leqslant i < j \leqslant k$, there are ℓ_{ij} edge-disjoint subsets $E_{ij}^1, \dots, E_{ij}^{\ell_{ij}}$ of $E_{G'}(V_i, V_j)$ such that each $(V_i, V_j)_{E_{ij}^\ell}$ is ε-regular of density either 0 or $\beta \pm \varepsilon$.*

Given a matching M_i, we define the graph G_i on vertex set $V(G)$ as follows: Initially, the edge set of G_i is the union of the sets E_{ab}^ℓ, taken over all edges f_{ab}^ℓ of M_i. So at the moment, G_i is a disjoint union of V_0 and $k' := k/2$ pairs which are ε-regular and have density $\beta \pm \varepsilon$. For every such pair, by removing exactly $2\varepsilon m$ vertices from each cluster of the pair, we may assume that the pair is 2ε-regular and that every vertex remaining in each cluster has degree $(\beta \pm 4\varepsilon)m$ within the pair. (In particular, it is $(2\varepsilon, \beta - 4\varepsilon)$-super-regular.) We denote by V_{0i} the union of V_0, together with the set of all these removed vertices. Observe that

$$|V_{0i}| \leqslant \varepsilon n + 2\varepsilon mk \leqslant 3\varepsilon n. \tag{10}$$

Finally, we remove all edges incident to vertices of V_{0i}. We will denote the pairs of clusters of G_i corresponding to the edges of M_i by $(U_{1,i}, V_{1,i}), \dots, (U_{k',i}, V_{k',i})$ and call them the *pairs of clusters of* G_i. Observe that every cluster V of G_i is contained in a unique cluster of R, which we will denote by V^R, and each cluster V of R contains a unique cluster of G_i, which we will denote by $V(i)$. In particular we have that $|V \setminus V(i)| \leqslant 2\varepsilon m$.

So we have exactly r edge-disjoint spanning subgraphs G_i of G such that for each $1 \leqslant i \leqslant r$ the following hold:

- (a_1) G_i is a disjoint union of a set V_{0i} of size at most $3\varepsilon n$ together with k clusters $U_{1,i}, V_{1,i}, \dots, U_{k',i}, V_{k',i}$ each of size exactly $(1 - 2\varepsilon)m$;
- (a_2) For each $x \in V(G)$ the degree of x in G_i is either 0 if $x \in V_{0i}$ or $(\beta \pm 4\varepsilon)m$ otherwise;
- (a_3) For each $1 \leqslant j \leqslant k'$ the pair $(U_{j,i}, V_{j,i})$ is $(2\varepsilon, \beta - 4\varepsilon)$-super-regular;
- (a_4) Every edge of G_i lies in one of the pairs $(U_{j,i}, V_{j,i})$ for some $1 \leqslant j \leqslant k'$.

5.2. **Extracting random subgraphs.** At the moment, no G_i contains a Hamilton cycle. Our aim is to add some of the edges of G which do not belong to any of the G_i into the G_i in such a way that the graphs obtained from the G_i are still edge-disjoint and each of them contains almost $\beta m/2$ edge-disjoint Hamilton cycles. To achieve this it will be convenient however to remove some of the edges of each G_i first while still keeping most of its properties.

We will show that there are edge-disjoint subgraphs H_1, H_2 and H_3 of G satisfying the following properties:

Lemma 15. *There are edge-disjoint subgraphs H_1, H_2 and H_3 of G such that the following properties hold:*

(i) *For every vertex x of G and every $1 \leqslant j \leqslant 3$ we have $|d_{H_j}(x) - \gamma d_G(x)| \leqslant \zeta n$.*

(ii) *For every vertex x of G, every $1 \leqslant i \leqslant r$ and every $1 \leqslant j \leqslant 3$*
$$\left|d_{H_j \cap G_i}(x) - \gamma d_{G_i}(x)\right| \leqslant \zeta n.$$

(iii) *For every vertex x of G, every $1 \leqslant i \leqslant r$ and every $1 \leqslant j \leqslant 3$*
$$\left|\left|N_{H_j}(x) \cap V_{0i}\right| - \gamma\left|N_G(x) \cap V_{0i}\right|\right| \leqslant \zeta n.$$

(iv) *For every vertex x of G, every $1 \leqslant i \leqslant r$, every $1 \leqslant t \leqslant k$ and every $1 \leqslant j \leqslant 3$*
$$\left|\left|N_{H_j \cap G_i}(x) \cap V_t\right| - \gamma\left|N_{G_i}(x) \cap V_t\right|\right| \leqslant \zeta n.$$

(v) *For every vertex x of G, every $1 \leqslant t \leqslant k$ and every $1 \leqslant j \leqslant 3$*
$$\left|\left|N_{H_j}(x) \cap V_t\right| - \gamma\left|N_G(x) \cap V_t\right|\right| \leqslant \zeta n.$$

(vi) *For every $1 \leqslant i \leqslant r$, every pair of clusters (U,V) of G_i, every $A \subseteq U$ and every $B \subseteq V$ with $|A|,|B| \geq 2\varepsilon|U|$ and every $1 \leqslant j \leqslant 3$ we have*
$$\left|\left|E_{H_j \cap G_i}(A,B)\right| - \gamma\left|E_{G_i}(A,B)\right|\right| \leqslant \zeta n^2.$$

(vii) *For all clusters $U \neq V$ of R, every $A \subseteq U$ and every $B \subseteq V$ with $|A|,|B| \geq \varepsilon m$ and every $1 \leqslant j \leqslant 3$ we have*
$$\left|\left|E_{H_j \cap G'}(A,B)\right| - \gamma\left|E_{G'}(A,B)\right|\right| \leqslant \zeta n^2.$$

Proof. We construct the H_j's randomly as follows: For every edge e of G, with probability 3γ, we assign it uniformly to one of the H_j's and with probability $1-3\gamma$ to none of them. By Theorem 8, all properties hold with high probability. More specifically, the total probability of failure is at most
$$(6n + 6rn + 6rn + 6rkn + 6kn)\exp\left(-\frac{\zeta^2 n}{3\gamma}\right) + (3rk4^m + 3k^2 4^m)\exp\left(-\frac{\zeta^2 n^2}{3\gamma}\right) \ll 1. \quad \square$$

We pick subgraphs H_1, H_2 and H_3 of G as given by Lemma 15. It will be convenient for later use to split (a subgraph of) H_3 into r subgraphs called $H_{3,1}, \ldots, H_{3,r}$ satisfying the properties of the following lemma. For each i, we will add edges of $H_{3,i}$ to G_i (but not to any of the other G_j) during the final part of our proof (see Section 5.8). Roughly speaking, if (U,V) is an edge of R, then we require $H_{3,i}$ to contain some edges between U and V (but we do not need many of these edges). If (U,V) corresponds to a matching edge of M_i, then we also require the corresponding subgraph of $H_{3,i}$ to be reasonably dense. Moreover, each edge of $H_{3,i}$ will correspond to some edge of R.

Lemma 16. *There are edge-disjoint subgraphs $H_{3,1}, \ldots, H_{3,r}$ of H_3 so that the following hold:*

(i) *For every $1 \leqslant i \leqslant r$, all clusters $U \neq V$ of G_i such that U^R and V^R are adjacent in R and every $U' \subseteq U$ and $V' \subseteq V$ with $|U'|,|V'| \geqslant \varepsilon m$ there are at least $\frac{\gamma\beta\varepsilon^2 dm^2}{5k}$ edges between U' and V' in $H_{3,i}$;*

(ii) *For every $1 \leqslant i \leqslant r$ and every $1 \leqslant j \leqslant k'$, the pair $(U_{j,i}, V_{j,i})_{H_{3,i}}$ is $(5\varepsilon/2, \gamma\beta/5)$-super-regular;*

(iii) *For every $1 \leqslant i \leqslant r$, $H_{3,i}$ has maximum degree at most βm;*

(iv) *For every $1 \leqslant i \leqslant r$ and every edge e of $H_{3,i}$ there are clusters $U \neq V$ of G_i such that such that U^R and V^R are adjacent in R and e joins U to V.*

Proof. Recall that given any two adjacent vertices V_a, V_b of R, and any $1 \leqslant \ell \leqslant \ell_{ab}$, there is at most one M_i which contains the edge f_{ab}^{ℓ}. If there is no such M_i, then we assign the edges of $E_{ab}^{\ell} \cap E(H_3)$ to the $H_{3,j}$ uniformly and independently at random. If there is such an M_i, we assign every edge of $E_{ab}^{\ell} \cap E(H_3)$ to $H_{3,i}$ with probability $1/2$ or to one of the other $H_{3,j}$'s uniformly at random. Note that this means that every edge of H_3 between V_a and V_b which lies in some G_i is assigned to $H_{3,i}$ with probability $1/2$ and assigned to some other $H_{3,j}$ with probability $1/2(r-1)$.

To prove (i), observe that since $r \leqslant k/\beta$ by (9), every edge of H_3 with endpoints in U and V has probability at least $\beta/2k$ of being assigned to $H_{3,i}$. Since $(U^R, V^R)_{G'}$ is ε-regular of density at least d, there are at least $\varepsilon^2 dm^2$ edges between U' and V' in G' and so by Lemma 15(vii), H_3 contains at least $\gamma\varepsilon^2 dm^2/2$ such edges. So by Theorem 8, (i) holds with high probability.

To prove (ii), recall that before defining H_3, the pair $(U_{j,i}, V_{j,i})_{G_i}$ was $(2\varepsilon, \beta - 4\varepsilon)$-super-regular by (a_3). Thus by Lemma 15(iv) and (vi), $(U_{j,i}, V_{j,i})_{H_3 \cap G_i}$ is $(2\varepsilon, \gamma\beta/2)$-super-regular. Since every edge of $(U_{j,i}, V_{j,i})_{H_3 \cap G_i}$ has probability exactly $1/2$ of being assigned to $H_{3,i}$, another application of Theorem 8 shows that with high probability $(U_{j,i}, V_{j,i})_{H_{3,i} \cap G_i}$ is $(2\varepsilon, \gamma\beta/5)$-super-regular. On the other hand, for every edge e in $E(H_3) \setminus E(G_i)$ between $U_{j,i}$ and $V_{j,i}$ the probability that e is assigned to $H_{3,i}$ is at most $1/r \leq 5\beta/k \ll \varepsilon$ (the first inequality follows from (9)). Together with Theorem 8 this implies that with high probability $(U_{j,i}, V_{j,i})_{H_{3,i}}$ consists of $(U_{j,i}, V_{j,i})_{H_{3,i} \cap G_i}$ and at most $\varepsilon^3 m^2$ additional edges. Thus with high probability $(U_{j,i}, V_{j,i})_{H_{3,i}}$ is $(5\varepsilon/2, \gamma\beta/5)$-super-regular, i.e. (ii) holds with high probability.

To prove (iii), observe that by (a_2) and Lemma 15(ii) $(U_{j,i}, V_{j,i})_{H_3 \cap G_i}$ (and thus also $H_{3,i} \cap G_i$) has maximum degree at most $2\gamma\beta m$. Moreover, every edge in $E(H_3) \setminus E(G_i)$ has probability at most $1/r \leq 5\beta/k$ of being assigned to $H_{3,i}$. Since by Lemma 15(i) H_3 has maximum degree at most $2\gamma n$, this implies that $H_{3,i} - E(G_i)$ has maximum degree at most $10\gamma\beta n/k$. Thus (iii) follows from Theorem 8 with room to spare.

In order to satisfy (iv) we delete all the edges of $H_{3,i}$ which do not 'correspond' to an edge of R. □

We choose $H_{3,1}, \dots, H_{3,r}$ as in Lemma 16. We now redefine each G_i by removing from it every edge which belongs to one of the H_j's. Observe that each G_i still satisfies (a_1) and (a_4) and it also satisfies

(a_2') For each $x \in V(G)$ the degree of x in G_i is either 0 if $x \in V_{0i}$ or $\beta(1 \pm 4\gamma)m$ otherwise;

(a_3') For each $1 \leqslant j \leqslant k'$ the pair $(U_{j,i}, V_{j,i})$ is $(2\varepsilon, \beta(1-4\gamma))$-super-regular,

instead of (a_2) and (a_3) respectively. Indeed, (a_2') follows from (a_2) and Lemma 15(ii) while (a_3') follows from (a_3) and Lemma 15(iv),(vi). Moreover, since we have removed the edges of H_1, H_2 and H_3 from the G_i's we have

(a_5) $G_1, \dots, G_r, H_1, H_2, H_3$ are edge-disjoint.

5.3. **Adding edges between V_{0i} and $G_i \setminus V_{0i}$.** Our aim in this subsection is to add edges from $G \setminus (G_1 \cup \dots \cup G_r \cup H_2 \cup H_3)$ into the G_i's so that for each $1 \leqslant i \leqslant r$ we have the following new properties:

$(a_{2.1})$ For each $x \in V(G)$, we have $d_{G_i}(x) = (1 \pm 5\gamma)\beta m$;

$(a_{2.2})$ For each $x \in G_i \setminus V_{0i}$, we have $d_{V_{0i}}(x) \leqslant \sqrt{\varepsilon}\beta m$,

instead of (a_2'). We will also guarantee that no edge will be added to more than one of the G_i's. In particular, instead of (a_5) we will now have

(a_5') $G_1, \dots, G_r, H_2, H_3$ are edge-disjoint.

Moreover, all edges added to G_i will have one endpoint in V_{0i} and the other endpoint in $G_i \setminus V_{0i}$. In particular (a_1) and (a_3') will still be satisfied while instead of (a_4) we will have

(a_4') Every edge of G_i lies either in a pair of the form (V_{0i}, U) where U is a cluster of G_i (i.e. $U = U_{i,j}$ or $U = V_{i,j}$ for some $1 \leqslant j \leqslant k$) or in a pair of the form $(U_{j,i}, V_{j,i})$ for some $1 \leqslant j \leqslant k'$.

We add the edges as follows: Firstly, for each vertex x of G, we let $L_x = \{i : x \in V_{0i}\}$. The distribution of the new edges incident to x will depend on the size of L_x. Let us write $\ell_x = |L_x|$ and let $A = \{x : \ell_x \leqslant \gamma n/4\beta m\}$ and $B = V(G) \setminus A = \{x : \ell_x > \gamma n/4\beta m\}$.

We begin by considering the edges of H_1 incident to vertices of A. For every such edge xy, we choose one of its endpoints uniformly and independently at random. If the chosen endpoint, say x, does not belong to A, then we do nothing. If it does belong to A then we will assign xy to at most one of the G_i's for which $i \in L_x$. For each $i \in L_x$, we assign xy to G_i with probability $2\beta m/d_{H_1}(x)$. So the probability that xy is not assigned to any G_i is $1 - \frac{2\ell_x \beta m}{d_{H_1}(x)}$. (Moreover, this assignment is independent of any previous random choices.) Observe that since $\delta(G) \geqslant (1/2 + \alpha)n$, Lemma 15(i) implies that $\frac{2\ell_x\beta m}{d_{H_1}(x)} \leqslant \frac{\gamma n}{2 d_{H_1}(x)} \leqslant 1$, so this distribution is well defined. Finally, we remove all edges that lie within some V_{0i}, so that each $G_i[V_{0i}]$ becomes empty.

Lemma 17. *With probability at least* $2/3$ *the following properties hold:*

(i) *For every* i *and every* $x \in V_{0i} \cap A$ *we have* $|d_{G_i}(x) - \beta m| \leqslant 8\varepsilon\beta m$*;*

(ii) *For every* i *and every* $x \in G_i \setminus V_{0i}$ *we have* $|N_{G_i}(x) \cap (V_{0i} \cap A)| \leqslant 9\varepsilon\beta m$*.*

Proof. The results will follow by applications of Theorem 8.

(i) For every $x \in V_{0i} \cap A$ and every edge xy of H_1 with $y \notin V_{0i}$, the probability that xy is assigned to G_i is exactly $\beta m/d_{H_1}(x)$. Indeed, with probability $1/2$, the endpoint x of xy is chosen and then independently with probability $2\beta m/d_{H_1}(x)$ we assign xy to G_i. Observe that since $y \notin V_{0i}$, if the endpoint y of xy was chosen, then xy cannot be assigned to G_i. Thus, the expected size of $d_{G_i}(x)$ is $\beta m \frac{d_{H_1 \setminus V_{0i}}(x)}{d_{H_1}(x)}$, which by Lemma 15(i),(iii) is at most βm and at least
$$\beta m \left(1 - \frac{\gamma d_{V_{0i}}(x) + \zeta n}{\gamma d_G(x) - \zeta n}\right) \overset{(10)}{\geqslant} (1 - 7\varepsilon)\beta m.$$
Thus by Theorem 8, the probability that the required property fails is at most $2rn \exp\left(-\frac{\varepsilon^2\beta^2 m^2}{3\beta m}\right) \leq 1/6$.

(ii) By Lemma 15(iii) and (10), we have that $|N_{H_1}(x) \cap (V_{0i} \cap A)| \leqslant \gamma |V_{0i}| + \zeta n \leqslant 4\gamma\varepsilon n$. By Lemma 15(i), every edge xy of H_1 with $y \in V_{0i} \cap A$ has probability at most $\beta m/d_{H_1}(y) \leqslant 2\beta m/\gamma n$ of appearing in G_i. Since all such events are independent, by Theorem 8 the probability that (ii) fails is at most $2rn \exp\left(-\frac{\varepsilon^2\beta^2 m^2}{24\varepsilon\beta m}\right) \leq 1/6$. □

We now consider the edges of H_1 incident to vertices of B. Observe that on the one hand we have $\sum |V_{0i}| \geqslant |B| \frac{\gamma n}{4\beta m}$. On the other hand, (9) and (10) imply that $\sum |V_{0i}| \leqslant \frac{3\varepsilon n k}{\beta}$. Thus $|B| \leqslant 12\varepsilon n/\gamma$.

For each $x \in B$, let $E(x)$ be the set of all edges of the form xy of G such that xy does not belong to any of the G_i's or any of the H_j's and moreover $y \notin B \cup V_0$. By definition we have that all the $E(x)$ are disjoint. Moreover, using (a_2') and Lemma 15(i)
$$|E(x)| \geqslant \delta - (r - \ell_x)(1 + 4\gamma)\beta m - 3(\gamma + \zeta)n - \frac{12\varepsilon}{\gamma} n - \varepsilon n \overset{(9)}{\geqslant} \ell_x \beta m.$$

For each $x \in B$, we pick a subset $E'(x)$ of $E(x)$ of size exactly $\ell_x \beta m$. We now assign each edge in $E'(x)$ uniformly at random to the ℓ_x G_i's with $i \in L_x$. Again, we then remove from G_i any edge that lies within V_{0i}, so that $G_i[V_{0i}]$ is still empty.

Lemma 18. *With probability at least* $2/3$ *the following properties hold:*

(i) *For every* i *and every* $x \in V_{0i} \cap B$ *we have* $|d_{G_i}(x) - \beta m| \leqslant \sqrt{\varepsilon}\beta m$;

(ii) *For every* i *and every* $x \in G_i \setminus V_{0i}$ *we have* $|N_{G_i}(x) \cap (V_{0i} \cap B)| \leqslant \sqrt{\varepsilon}\beta m/2$.

Proof. The results will follow by applications of Theorem 8.

(i) For every $x \in B$ and every $y \notin V_{0i}$ with $xy \in E'(x)$, the probability that xy is assigned to G_i is exactly $1/\ell_x$. Since also $|E'(x)| - |V_{0i}| \geqslant \ell_x \beta m - 3\varepsilon n$ by (10), the expected size of $d_{G_i}(x)$ is at most βm and at least $(1-\sqrt{\varepsilon}/2)\beta m$. So by Theorem 8, the probability of failure is at most $2rn \exp\left(-\frac{\varepsilon \beta^2 m^2}{12\beta m}\right) \leq 1/6$.

(ii) We have that $|V_{0i} \cap B| \leqslant |B| \leqslant 12\varepsilon n/\gamma$ and every edge yx with $y \in V_{0i} \cap B$ has probability either $1/\ell_y$ or 0 of appearing in G_i independently of the others. So

$$\mathbb{E}\left(|N_{G_i}(x) \cap (V_{0i} \cap B)|\right) \leq \frac{|B|}{\ell_y} \leq \frac{12\varepsilon n}{\gamma} \cdot \frac{4\beta m}{\gamma n} \leq \frac{\sqrt{\varepsilon}}{4}\beta m.$$

So by Theorem 8, the probability that (ii) fails is at most $2rn \exp\left(-\frac{\sqrt{\varepsilon}\beta m}{12}\right) \leq 1/6$. □

Thus we can make a choice of edges which we add to the G_i so that both properties in Lemmas 17 and 18 hold. This in turn implies that the properties $(a_{2.1})$, $(a_{2.2})$ as well as the other properties stated at the beginning of the subsection are satisfied.

5.4. **Adding edges between the clusters of G_i.** Recall that by $(a_{2.1})$ every vertex of G_i has degree $(1 \pm 5\gamma)\beta m$. We would like to almost decompose each G_i into Hamilton cycles. This would definitely be sufficient to complete the proof of Theorems 6 and 7(ii). The first step would be to extract from G_i an s-regular spanning subgraph S_i where s is close to $(1 \pm 5\gamma)\beta m$. Observe that if G_i does not have such an S_i, then definitely it cannot be almost decomposed into Hamilton cycles. It turns out that at the moment, we cannot guarantee the existence of such an S_i. For example, consider the case when there are no edges between the vertices of V_{0i} and the vertices in clusters of the form $U_{j,i}$ (i.e. all vertices incident to V_{0i} lie in the $V_{j,i}$). This 'unbalanced' structure of G_i implies that it cannot contain any regular spanning subgraph.

Our aim in this subsection is to use edges from H_2 in order to transform the G_i's so that they have some additional properties which will guarantee the existence of S_i. We will show that adding only edges of H_2 to the G_i's we can for each $1 \leqslant i \leqslant r$ guarantee the following new properties:

$(a'_{2.1})$ For each $x \in V(G)$, $d_{G_i}(x) = (1 \pm 15\gamma)\beta m$;

(a_6) For all clusters $U \neq V$ of G_i so that U^R and V^R are adjacent in R but not in M_i, we have $|E_{G_i}(U,V)| \geqslant \beta\gamma d m^2/8k$ and moreover for every $x \in U \cup V$ we also have $|N_{G_i}(x) \cap (U \cup V)| \leqslant 10\beta\gamma m/k$.

No edge will be added to more than one of the G_i's and so (instead of (a'_5)) we will have

(a''_5) $G_1, \dots, G_r, H_3$ are edge-disjoint.

Finally, all edges added to G_i will have both endpoints in distinct clusters of G_i and moreover for each $1 \leqslant j \leqslant k'$, no edge will be added to G_i between the clusters $U_{j,i}$ and $V_{j,i}$. In particular, $(a_1), (a_{2.2})$ and (a'_3) will still hold while instead of (a'_4) we will have

(a''_4) Every edge of G_i lies in a pair of the form (V_{0i}, U), where U is a cluster of G_i, or a pair of the form (U, V), where U and V are clusters of G_i with U^R and V^R adjacent in R.

For every pair of adjacent clusters U and V of R, we will distribute the edges in $E_{H_2}(U,V)$ to the G_i so that the following lemma holds. It is then an immediate consequence that all of the above properties are satisfied.

Lemma 19. *Let U and V be adjacent clusters of R. Then we can assign some of the edges of H_2 between U and V to the G_i so that every edge is assigned to at most one G_i and moreover*

(i) *If UV is an edge of M_i, then no edge is assigned to G_i. Otherwise, at least $\beta\gamma dm^2/8k$ edges are assigned to G_i and none of these edges has an endvertex in $(U\setminus U(i))\cup(V\setminus V(i))$;*

(ii) *For every $x\in U(i)\cup V(i)$ at most $10\beta\gamma m/k$ edges incident to x are assigned to G_i.*

Proof. Given such U,V, we assign every edge of $E_{H_2}(U,V)$ independently and uniformly at random among the G_i's. If an edge assigned to G_i is incident to $(U\setminus U(i))\cup(V\setminus V(i))$ it is discarded. If moreover UV is an edge of M_i, then all edges assigned to G_i are discarded.

Since $(U,V)_{G'}$ is ε-regular of density at least d, Lemma 15(vii) implies that $|E_{H_2}(U,V)|\geqslant \gamma dm^2/2$ and so by Theorem 8, the number of edges assigned to each G_i is with high probability at least $\gamma dm^2/4r \geq \beta\gamma dm^2/4k$. (The last inequality follows from (9).) To prove (i), it is enough to show that (if UV is not an edge of M_i then) at most half of these edges are discarded. Since $|U\setminus U(i)|,|V\setminus V(i)|\leqslant 2\varepsilon m$, there are at most $4\varepsilon m^2$ such edges which are incident in G to a vertex of $(U\setminus U(i))\cup(V\setminus V(i))$. Of those, with high probability at most $5\varepsilon m^2/r\leqslant 25\varepsilon\beta m^2/k$ are assigned to G_i and are thus discarded. To complete the proof, observe that by Lemma 15(v) every vertex $x\in U$ has $|N_{H_2}(x)\cap V|\leqslant 3\gamma m/2$ (and similarly for every vertex $x\in V$), so by Theorem 8 with high probability no vertex of G_i is incident to more than $2\gamma m/r\leq 10\beta\gamma m/k$ assigned edges. □

5.5. **Finding the regular subgraph S_i.** Our aim in this subsection is to show that each G_i contains a regular spanning subgraph S_i of even degree $s:=(1-15\gamma)\,\beta m$. Moreover, for every cluster V all its vertices have most of their neighbours in the cluster that V is matched to in M_i (see Lemma 20).

To prove this lemma, we proceed as follows: A result of Frieze and Krivelevich [3] (based on the max-flow min-cut theorem) implies that every pair $(U_{j,i},V_{j,i})$ contains a regular subgraph of degree close to βm. However, the example in the previous subsection shows that it is not possible to combine these to an s-regular spanning subgraph of G_i due to the the existence of the vertices in V_{0i}. So in Lemma 21 we will first find a subgraph T_i of G_i where the vertices of V_{0i} have degree s, every non-exceptional vertex has small degree in T_i and moreover each pair $(U_{j,i},V_{j,i})$ will be balanced with respect to T_i in the following sense: the sum of the degrees of the vertices of $U_{i,j}$ in T_i is equal to the sum of the degrees of the vertices of $V_{i,j}$ in T_i. We can then use the following generalization (Lemma 22, proved in [8]) of the result in [3]: in each pair $(U_{j,i},V_{j,i})$ we can find a subgraph $\Gamma_{j,i}$ with prescribed degrees (as long as the prescribed degrees are not much smaller than βm). We then prescribe these degrees so that together with those in T_i they add up to s. So the union of the $\Gamma_{j,i}$ (over all $1\leq j\leq k'$) and T_i yields the desired s-regular subgraph S_i. Note that since S_i is regular, $(U_{j,i},V_{j,i})$ is balanced with respect to S_i in the above sense (i.e. replacing T_i with S_i). Also, the pair will clearly be balanced with respect to $\Gamma_{j,i}$. This explains why we needed to ensure that the pair is also balanced with respect to T_i.

Lemma 20. *For every $1\leqslant i\leqslant r$, G_i contains a subgraph S_i such that*

(i) *S_i is s-regular, where $s:=(1-15\gamma)\,\beta m$ is even;*

(ii) *For every $1\leqslant j\leqslant k'$ and every $x\in U_{j,i}$ we have $|N_{S_i}(x)\setminus V_{j,i}|\leqslant \eta\beta m$. Similarly, $|N_{S_i}(x)\setminus U_{j,i}|\leqslant \eta\beta m$ for every $x\in V_{j,i}$.*

As discussed above, to prove Lemma 20 we will show that every G_i contains a subgraph T_i with the following properties:

Lemma 21. *Each G_i contains a spanning subgraph T_i such that*

(i) *Every vertex x of V_{0i} has degree s;*
(ii) *Every vertex y of $G_i \setminus V_{0i}$ has degree at most $\eta\beta m$;*
(iii) *For every $1 \leqslant j \leqslant k'$, we have $\sum_{x\in U_{j,i}} d_{T_i}(x) = \sum_{x\in V_{j,i}} d_{T_i}(x)$;*
(iv) *For every $1 \leqslant j \leqslant k'$, we have $E_{T_i}(U_{j,i}, V_{j,i}) = \emptyset$.*

Having proved this lemma, we can use the following result from [8] to deduce the existence of S_i.

Lemma 22. *Let $0 < 1/m' \ll \varepsilon \ll \beta' \ll \eta \ll \eta' \ll 1$. Suppose that $\Gamma = (U, V)$ is an (ε, β')-super-regular pair where $|U| = |V| = m'$. Define $\tau := (1-\eta')\beta' m'$. Suppose we have a non-negative integer $x_u \leqslant \eta\beta' m'$ associated with each $u \in U$ and a non-negative integer $y_v \leqslant \eta\beta' m'$ associated with each $v \in V$ such that $\sum_{u\in U} x_u = \sum_{v\in V} y_v$. Then Γ contains a spanning subgraph Γ' in which $\tau - x_u$ is the degree of each $u \in U$ and $\tau - y_v$ is the degree of each $v \in V$.*

Proof of Lemma 20. To derive Lemma 20 from Lemmas 21 and 22, recall that by (a_3') for each $1 \leqslant j \leqslant k'$ the pair $(U_{j,i}, V_{j,i})$ is $(2\varepsilon, (1-4\gamma)\beta)$-super-regular. Thus we can apply Lemma 22 to $(U_{j,i}, V_{j,i})$ with 2η playing the role of η in the lemma, $\beta' := (1-4\gamma)\beta$, $\eta' := 1 - \frac{1-15\gamma}{(1-4\gamma)(1-2\varepsilon)}$, $m' := (1-2\varepsilon)m$, $x_u = d_{T_i}(u)$ for every $u \in U_{j,i}$ and $y_v = d_{T_i}(v)$ for every $v \in V_{j,i}$. Observe that with this value of η', we have $\tau = (1-15\gamma)\,\beta m = s$. Lemma 21(ii) implies that for each $u \in U_{j,i}$ and each $v \in V_{j,i}$ we have $2\eta\beta' m' = 2\eta(1-4\gamma)(1-2\varepsilon)\beta m \geqslant \eta\beta m \geqslant x_u, y_v$. Lemma 21(iii) implies that $\sum_{u\in U} x_u = \sum_{v\in V} y_v$. Thus the conditions of Lemma 22 hold and we obtain a subgraph $\Gamma_{j,i}$ of $(U_{j,i}, V_{j,i})$ in which every $u \in U_{j,i}$ has degree $s - x_u$ and every $v \in V_{j,i}$ has degree $s - y_v$. It follows from Lemma 21(i),(ii) and (iv) that $S_i = T_i \cup \left(\bigcup_{j=1}^{k'} \Gamma_{j,i}\right)$ is as required in Lemma 20. $\square$

Proof of Lemma 21. We give an algorithmic construction of T_i. We begin by arbitrarily choosing s edges (of G_i) incident to each vertex x of V_{0i}. Recall that by $(a_{2.2})$ this means that every vertex of $G_i \setminus V_{0i}$ currently has degree at most $\sqrt{\varepsilon}\beta m$. Let us write $u_{j,i} := \sum_{x\in U_{j,i}} d_{T_i}(x)$ and $v_{j,i} := \sum_{x\in V_{j,i}} d_{T_i}(x)$. Note that these values will keep changing as we add more edges from G_i into T_i and we currently have $|u_{j,i} - v_{j,i}| \leqslant \sqrt{\varepsilon}\beta m^2$.

Step 1. *By adding at most k' more edges, we may assume that for every $1 \leqslant j \leqslant k'$, $u_{j,i} - v_{j,i}$ is even.*

To prove that this is possible, take any j for which $u_{j,i} - v_{j,i}$ is odd and observe that there is a $j' \neq j$ for which $u_{j',i} - v_{j',i}$ is also odd. This holds because s is even and so there is an even number of edges between V_{0i} and $G_i \setminus V_{0i}$. Let V be a cluster of R which is a common neighbour (in R) of $U_{j,i}^R$ and $U_{j',i}^R$ and which is distinct from $V_{j,i}^R$ and $V_{j',i}^R$. The existence of V is guaranteed by the degree conditions of R (see Lemma 13(i)). Now we take an edge of G_i between $V(i)$ and $U_{j,i}$ not already added to T_i and add it to T_i. We also take an edge of G_i between $V(i)$ and $U_{j',i}$ not already added to T_i and add it to T_i. This makes the differences for j and j' even and preserves the parity of all other differences. So we can perform Step 1.

In each subsequent step, we will take two clusters U and V of G_i and add several edges between them to T_i, these edges are chosen from the edges of G_i which are not already used. The clusters U^R and V^R will be adjacent in R but not in M_i, so condition (iv) will remain true. We will only add at most $\beta\gamma d m^2/20k$ edges at each step and we will never add edges between U and V more than twice. Condition (a_6) guarantees that we have enough edges for this. (Recall that we have already added at most k' edges between each pair of clusters.) At the end of all these steps condition (iii) will hold. Moreover, we will guarantee that no cluster U is used in more than $2\eta k$ of these steps and so by (a_6) the degree of each vertex in T_i will not be increased by more than $20\beta\eta\gamma m \ll \eta\beta m$ and so condition (ii) will also be satisfied.

We call a cluster U of G_i *bad* if it is already used in more than ηk of the above steps. We will also guarantee that the number of the above steps is at most $\eta^2 k^2/2$. Since in each step we use two clusters, this will imply that at each step there are at most ηk bad clusters.

Let us now show how all the above can be achieved. Let us take a j for which $u_{j,i} \neq v_{j,i}$, say $u_{j,i} < v_{j,i}$. (The case $u_{j,i} > v_{j,i}$ is identical and will thus be omitted.) Since by Lemma 13 the minimum degree of R is at least $(\delta/n - 2d)k/\beta$ and since there are no more than $1/\beta$ parallel edges between any two vertices of R, it follows that there are at least $(\delta/n - 1/2 - 2d)k \geqslant \alpha k/2$ indices j' such that $U^R_{j,i}$ is adjacent to both $U^R_{j',i}$ and $V^R_{j',i}$ in R. Since there are at most ηk bad clusters, there are at least $\alpha k/3$ indices j' such that $U^R_{j,i}$ is adjacent to both $U^R_{j',i}$ and $V^R_{j',i}$ in R and moreover none of $U^R_{j',i}$ and $V^R_{j',i}$ is bad. As long as $v_{j,i} - u_{j,i} > \beta\gamma dm^2/10k$, we add exactly $\beta\gamma dm^2/20k$ edges between $U_{j,i}$ and $U_{j',i}$ and exactly $\beta\gamma dm^2/20k$ edges between $U_{j,i}$ and $V_{j',i}$. Note that this decreases the difference $v_{j,i} - u_{j,i}$ and leaves all other differences the same. Finally, if $0 < v_{j,i} - u_{j,i} < \beta\gamma dm^2/10k$ then we carry out the same step except that we add $(v_{j,i} - u_{j,i})/2$ edges between $U_{j,i}$ and $U_{j',i}$ and between $U_{j,i}$ and $V_{j',i}$ instead. (Recall that Step 1 guarantees that $v_{j,i} - u_{j,i}$ is even.) As observed at the beginning of the proof, the initial difference between $u_{j,i}$ and $v_{j,i}$ is at most $\sqrt{\varepsilon}\beta m^2$. This might have increased to at most $2\sqrt{\varepsilon}\beta m^2$ after performing Step 1. Thus it takes at most $20\sqrt{\varepsilon}k/\gamma d + 1 \ll \eta k$ steps to make $u_{j,i}$ and $v_{j,i}$ equal and so we may choose a different index j' in each of these steps.

We repeat this process for all $1 \leqslant j \leqslant k'$. Obviously, (iii) holds after we have considered all such j's. It remains to check that all the conditions that we claimed to be true throughout the process are indeed true. As for each j it takes at most ηk steps to make $u_{j,i}$ and $v_{j,i}$ equal, the total number of steps is at most $\eta^2 k^2/2$. Since moreover, a cluster $U_{j,i}$ or $V_{j,i}$ is used in a step only when j is considered or when it is not bad, it is never used in more than $2\eta k$ steps, as promised. □

5.6. **Choosing an almost 2-factor decomposition of S_i.** Since each S_i is regular of even degree, by Theorem 10 we can decompose it into 2-factors. Our aim will be to use the edges of $H_{3,i}$ to transform each 2-factor in this decomposition into a Hamilton cycle. To achieve this, we need each 2-factor in the decomposition to possess some additional properties. Firstly, we would like each 2-factor to contain $o(n)$ cycles. To motivate the second property, note that by Lemma 20(ii), most edges of S_i go between pairs of clusters $(U_{j,i}, V_{j,i})$. So one would expect that this is also the case for a typical 2-factor F. We will need the following stronger version of this property: for every pair $(U_{j,i}, V_{j,i})$ of clusters of G_i and every vertex $u \in U_{j,i}$, most of its S_i-neighbours in $V_{j,i}$ have both their F-neighbours in $U_{j,i}$ (and similarly for every $v \in V_{j,i}$). We will also need the analogous property with S_i replaced by $H_{3,i}$.

The following lemma tells us that we can achieve the above properties if we only demand an almost 2-factor decomposition.

Lemma 23. *S_i contains at least $\left(1 - \sqrt{\gamma}\right)\frac{\beta m}{2}$ edge-disjoint 2-factors such that for every such 2-factor F the following hold:*

(i) *F contains at most $n/(\log n)^{1/5}$ cycles;*
(ii) *For every $1 \leqslant j \leqslant k'$ and every $u \in U_{j,i}$, the number of $H_{3,i}$-neighbours of u in $V_{j,i}$ which have an F-neighbour outside $U_{j,i}$ is at most $\gamma^3\beta m$ (and similarly for the $H_{3,i}$-neighbours in $U_{j,i}$ of each $v \in V_{j,i}$).*

(iii) *For every* $1 \leqslant j \leqslant k'$ *and every* $u \in U_{j,i}$*, the number of* S_i*-neighbours of* u *in* $V_{j,i}$ *which have an* F*-neighbour outside* $U_{j,i}$ *is at most* $\gamma^3\beta m$ *(and similarly for the* S_i*-neighbours in* $U_{j,i}$ *of each* $v \in V_{j,i}$*).*

The proof of Lemma 23 will rely on the following lemma from [8]. This lemma is in turn based on a result in [7] whose proof relies on a probabilistic approach already used in [3]. A *1-factor* in an oriented graph D is a collection of disjoint directed cycles covering all the vertices of D.

Lemma 24. *Let* $0 < \theta_1, \theta_2, \theta_3 < 1/2$ *be such that* $\theta_1/\theta_3 \ll \theta_2$*. Let* D *be a* $\theta_3 n$*-regular oriented graph whose order* n *is sufficiently large. Suppose* $A_1, \dots, A_{5n}$ *are sets of vertices in* D *with* $|A_t| \geqslant n^{1/2}$*. Let* H *be an oriented subgraph of* D *such that* $d^+_H(x), d^-_H(x) \leqslant \theta_1 n$ *for all* $x \in A_t$ *and each* t*. Then* D *has a* 1*-factor* F *such that*

(i) F *contains at most* $n/(\log n)^{1/5}$ *cycles;*
(ii) *For each* t*, at most* $\theta_2|A_t|$ *edges of* $H \cap F$ *are incident to* A_t*.*

Proof of Lemma 23. We begin by choosing an arbitrary orientation D of S_i with the property that every vertex has indegree and outdegree equal to $s/2$. The existence of such an orientation follows e.g. from Theorem 10. We repeatedly extract 1-factors of D satisfying the properties of Lemma 23 as follows: Suppose we have extracted some 1-factors from D and we are left with a $\theta_3 n$-regular oriented graph D, where $\theta_3 \geqslant \sqrt{\gamma}\beta m/4n$.

For the sets A_t, we take all sets of the form $N_{H_{3,i}}(u) \cap V_{j,i}$ and all sets of the form $N_{S_i}(u) \cap V_{j,i}$ (for all $u \in U_{j,i}$ and $j = 1, \dots, k'$) as well as all sets of the form $N_{H_{3,i}}(v) \cap U_{j,i}$ and all sets of the form $N_{S_i}(v) \cap U_{j,i}$ (for all $v \in V_{j,i}$ and $j = 1, \dots, k'$). Even though the number of these sets is less than $5n$, this is not a problem as for example we might repeat each set several times. Lemmas 16(ii) and 20(ii) imply that these sets have size at least $\gamma\beta m/6 \gg n^{1/2}$.

For the subgraph H of D we take the graph consisting of all those edges of S_i which do not belong to some pair $(U_{j,i}, V_{j,i})$. Then $d^+_H(x), d^-_H(x) \leqslant \theta_1 n$ for all $x \in A_t$ (and each t), where by Lemma 20(ii) we can take $\theta_1 = \eta\beta m/n$.

Thus, taking $\theta_2 = \gamma^3$ all conditions of Lemma 24 are satisfied and so we obtain a 1-factor F of D satisfying all properties of Lemma 23. (The fact that $s \leq \beta m$ and Lemma 16(iii) imply that the A_t have size at most βm and so F satisfies Lemma 23(ii) and (iii).) It follows that we can keep extracting such 1-factors for as long as the degree of D is at least $\sqrt{\gamma}\beta m/4$ and in particular we can extract at least $(1 - \sqrt{\gamma})\beta m/2$ such 1-factors as required. □

5.7. **Transforming the 2-factors into Hamilton cycles.** To finish the proof it remains to show how we can use (for each i) the edges of $H_{3,i}$ to transform each of the 2-factors of S_i created by Lemma 23 into a Hamilton cycle. By Lemma 23, this will imply that the total number of edge-disjoint Hamilton cycles we construct is $(1 - \sqrt{\gamma})r\beta m/2$, which suffices to prove Theorems 6 and 7(ii). To achieve the transformation of each 2-factor into a Hamilton cycle, we claim that it is enough to prove the following theorem. In conditions (iv) and (v) of the theorem we say that a pair of clusters (A_i, A_j) of a graph X is *weakly* $(\varepsilon, \varepsilon')$*-regular* in a subgraph H of X if for every $U \subseteq A_i, V \subseteq A_j$ with $|U|, |V| \geqslant \varepsilon m$, there are at least $\varepsilon' m^2$ edges between U and V in H.

Roughly speaking, we will apply the following theorem successively to the 2-factors F in our almost-decomposition of S_i and where H is the union of $H_{3,i}$ together with some additional edges incident to V_{0i} . However, this does not quite work – between successive applications of the theorem we will also need to add edges to H which were removed from a previous 1-factor F when transforming F into a Hamilton cycle.

Theorem 25. *Let $1/n \ll 1/k \le \varepsilon \ll \beta \ll \gamma \ll 1$. Let m be an integer such that $(1-\varepsilon)n \le mk \le n$. Let H be a graph on n vertices and let F be a 2-factor so that F and H have the same vertices but are edge-disjoint. Let $X := F \cup H$. Let $A_1, \dots, A_k$ be disjoint subsets of X of size $(1-2\varepsilon)m$ and let $B_1, \dots, B_{k'}, D_1, \dots, D_{k'}$ be another enumeration of the $A_1, \dots, A_k$. Suppose also that the following hold:*

- (i) *F contains at most $n/(\log n)^{1/5}$ cycles;*
- (ii) *For each $1 \leqslant i \leqslant k'$ and for each vertex of B_i the number of H-neighbours in D_i having an F-neighbour outside B_i is at most $2\gamma^3\beta m$ (and similarly for the vertices in D_i);*
- (iii) *For every $1 \leqslant i \leqslant k'$, the pair $(B_i, D_i)_H$ is $(3\varepsilon, \gamma\beta/6)$-super-regular;*
- (iv) *For every $1 \leqslant i \leqslant k$ and every A_i, there are at least $(1+\alpha)k'$ distinct j's with $1 \leqslant j \leqslant k$ such that (A_i, A_j) is weakly $(\varepsilon, \varepsilon^3/k)$-regular in H;*
- (v) *For every $1 \leqslant i < j \leqslant k$, if there is an edge in X between A_i and A_j then (A_i, A_j) is weakly $(\varepsilon, \varepsilon^3/k)$-regular in H;*
- (vi) *For every vertex $x \in V(X) \setminus (A_1 \cup \ldots \cup A_k)$, both F-neighbours of x belong to $A_1 \cup \ldots \cup A_k$.*
- (vii) *Every vertex $x \in V(X) \setminus (A_1 \cup \ldots \cup A_k)$ has degree at least $\alpha n/6$ in H and every H-neighbour of x lies in $A_1 \cup \ldots \cup A_k$.*

Then there is a Hamilton cycle C of X such that $|E(C) \triangle E(F)| \leqslant 25n/(\log n)^{1/5}$.

To see that it is enough to prove the above theorem, suppose we have already transformed all 2-factors of $S_1, \dots, S_{i-1}$ guaranteed by Lemma 23 into edge-disjoint Hamilton cycles such that for each $1 \leqslant j \leqslant i-1$ the Hamilton cycles corresponding to the 2-factors of S_j lie in $G \setminus \bigcup_{j'>j} (G_{j'} \cup H_{3,j'})$. Moreover, suppose that we have also transformed ℓ of the 2-factors of S_i, say $F_1, \dots, F_\ell$, into edge-disjoint Hamilton cycles $C_1, \dots, C_\ell$ such that $C_j \subseteq G \setminus \bigcup_{i'>i} (G_{i'} \cup H_{3,i'})$ and $|E(C_j) \triangle E(F_j)| \leqslant 25n/(\log n)^{1/5}$ for all $1 \leqslant j \leqslant \ell$. Obtain H_1^* from $H_{3,i}$ as follows:

- (b_0) add all those edges of G between V_{0i} and $V(G) \setminus V_{0i}$ which do not belong to any $G_j \cup H_{3,j}$ with $j \geqslant i$ or to any Hamilton cycle already created.

Suppose that we have inductively defined graphs $H_1^*, \dots, H_\ell^*$ such that $C_j \subseteq H_j^* \cup F_j$ for all $1 \le j \le \ell$. Define $H_{\ell+1}^*$ as follows:

- (b_1) remove all edges in $E(C_\ell) \setminus E(F_\ell)$ from H_ℓ^*;
- (b_2) add all edges in $E(F_\ell) \setminus E(C_\ell)$ to H_ℓ^*.

Let $F_{\ell+1}$ be one of the 2-factors of S_i as constructed in Lemma 23 which is distinct from $F_1, \dots, F_\ell$. Finally, let $B_j = U_{j,i}$ and $D_j = V_{j,i}$ for $1 \leqslant j \leqslant k'$. We claim that all conditions of Theorem 25 hold (with $H_{\ell+1}^*$ and $F_{\ell+1}$ playing the roles of H and F). Indeed, property (i) follows from Lemma 23(i). Since $N_{H_{\ell+1}^*}(u) \cap V_{j,i} \subseteq (N_{H_{3,i}}(u) \cup N_{S_i}(u)) \cap V_{j,i}$ for every $u \in U_{j,i}$ (note that this is not necessarily true for $u \in V_{0,i}$), property (ii) follows from Lemma 23(ii) and (iii). To see that property (iii) holds, recall that by Lemma 16(ii) we have that for every $1 \leqslant j' \leqslant k'$ the pair $(B_{j'}, D_{j'})_{H_{3,i}}$ is $(5\varepsilon/2, \gamma\beta/5)$-super-regular. Since also $|E(C_j) \triangle E(F_j)| \leqslant 25n/(\log n)^{1/5}$ for each $1 \leqslant j \leqslant \ell$, we have $|E(H_{\ell+1}^* \setminus V_{0i}) \triangle E(H_{3,i} \setminus V_{0i})| \leqslant 25n^2/(\log n)^{1/5}$ and so $(B_{j'}, D_{j'})_{H_{\ell+1}^*}$ is 3ε-regular of density at least $\gamma\beta/6$. To prove that the pair is even $(3\varepsilon, \gamma\beta/6)$-super-regular, it suffices to show that for any $x \in B_{j'}$ we have

$$|N_{H_{\ell+1}^*}(x) \cap D_{j'}| \geq \gamma\beta m/6. \tag{11}$$

(A bound for the case $x \in D_{j'}$ will follow in the same way.) To prove (11), suppose that the degree of x in $(B_{j'}, D_{j'})_{H_{\ell+1}^*}$ was decreased by one compared to $(B_{j'}, D_{j'})_{H_\ell^*}$ due to (b_1). This means that an edge xy of $(B_{j'}, D_{j'})_{H_\ell^*}$ was inserted into C_ℓ. But since F_ℓ and C_ℓ are both 2-factors, this means that an edge xz from F_ℓ will be added to H_ℓ^* when forming $H_{\ell+1}^*$.

Note that $xz \in E(F_\ell) \subseteq E(S_i)$ and by our assumption on the degree of x, we have $z \notin D_{j'}$. If the degree decreases by two of x, then the argument shows that we will be adding two such edges xz_1 and xz_2 to H^*_ℓ when forming $H^*_{\ell+1}$. But since Lemma 20(ii) implies that $|N_{S_i}(x) \setminus D_{j'}| \leq \eta\beta m$, this can happen at most $\eta\beta m$ times throughout the process of constructing $C_1, \dots, C_\ell$. (Here we are also using the fact that the F_j are edge-disjoint, so we will consider such an edge xz or xz_i only once throughout.) So

$$|N_{H^*_{\ell+1}}(x) \cap D_{j'}| \geq |N_{H_{3,i}}(x) \cap D_{j'}| - \eta\beta m \geq \gamma\beta(1-2\varepsilon)m/5 - \eta\beta m \geq \gamma\beta m/6,$$

which proves (11) and thus (iii). Property (iv) follows from Lemma 16(i) together with the fact that $|E(H_{3,i} \setminus V_{0,i}) \triangle E(H^*_{\ell+1} \setminus V_{0i})| = o(n^2)$ and the fact that the minimum degree of R is at least $(1+\alpha)k/2\beta$ (see Lemma 13). Property (v) follows similarly since by (a''_4) each edge in $E(F_{\ell+1}) \subseteq E(G_i)$ between clusters corresponds to an edge of R and since by Lemma 16(iv) the analogue holds for the edges of $H_{3,i}$. Property (vi) is an immediate consequence of (a''_4). To see that (vii) holds consider a vertex $x \in V(X) \setminus (A_1 \cup \ldots \cup A_k)$. By Lemma 15(i) x has degree at most $2\gamma n$ in H_3 and thus in the union of $H_{3,j}$ with $j \geq i$. By $(a'_{2.1})$, x has degree at most $(r-i+1)(1+15\gamma)\beta m$ in the union of the G_j with $j \geq i$. The number of Hamilton cycles already constructed is at most $i(1-\sqrt{\gamma})\beta m/2$. Furthermore, x has at most $|V_{0i}| \leq 3\varepsilon n$ neighbours in V_{0i}. So altogether the number of edges of G incident to x which are not included in $H^*_{\ell+1}$ due to (b_0) and (b_1) is at most $2\gamma n + (r+1)(1+15\gamma)\beta m + 3\varepsilon n \leq \delta - \alpha n/6$, where the inequality follows from the bound on δ in (9). So the number of edges incident to x in $H^*_{\ell+1}$ is at least $\alpha n/6$. Moreover, by Lemma 16(iv) and (a''_4) no neighbour of x in $H_{3,i} \cup G_i$ lies in V_{0i} and thus the same is true for every $H^*_{\ell+1}$-neighbour of x.

5.8. **Proof of Theorem 25.** In the proof of Theorem 25 it will be convenient to use the following special case of a theorem of Ghouila-Houri [4], which is an analogue of Dirac's theorem for directed graphs.

Theorem 26 ([4]). *Let G be a directed graph on n vertices with minimum out-degree and minimum in-degree at least $n/2$. Then G contains a directed Hamilton cycle.*

We will also use the following 'rotation-extension' lemma which appears implicitly in [3] and explicitly (but for directed graphs) in [8]. The directed version implies the undirected version (and the latter is also simple to prove directly). Given a path P with endpoints in opposite clusters of an ε-regular pair, the lemma provides a cycle on the same vertex set by changing only a small number of edges.

Lemma 27. *Let $0 < 1/m \ll \varepsilon \ll \gamma' < 1$ and let G be a graph on $n \geqslant 2m$ vertices. Let U and V be disjoint subsets of $V(G)$ with $|U| = |V| = m$ such that for every $S \subseteq U$ and every $T \subseteq V$ with $|S|, |T| \geqslant \varepsilon m$ we have $e(S,T) \geqslant \gamma'|S||T|$. Let P be a path in G with endpoints x and y where $x \in U$ and $y \in V$. Let U_P be the set of vertices of P which belong to U and have all of their P-neighbours in V and let V_P be defined analogously. Suppose that $|N(x) \cap V_P|, |N(y) \cap U_P| \geqslant \gamma' m$. Then there is a cycle C in G containing precisely the vertices of P and such that C contains at most 5 edges which do not belong to P.*

Proof of Theorem 25. We will give an algorithmic construction of the Hamilton cycle. Before and after each step of our algorithm we will have a spanning subgraph H' of H and spanning subgraph F' of X which is a union of disjoint cycles and at most one path such that H' and F' are edge-disjoint. In each step we will add at most 5 edges from H' to F' and remove some edges from F' to obtain a new spanning subgraph F''. The edges added to F' will be removed from H' to obtain the new subgraph H''. It will turn out that the number of steps needed to transform F into a Hamilton cycle will be at most $5n/(\log n)^{1/5}$. This will complete the proof of Theorem 25.

To simplify the notation we will always write H and F for these subgraphs of X at each step of the algorithm. Also, let $g(n) := n/(\log n)^{1/5}$. We call all the edges of the initial F *original*. At each step of the algorithm, we will write B_i' for the set of vertices $b \in B_i$ whose neighbours in the current graph F both lie in D_i and are joined to b by original edges (for each $1 \leqslant i \leqslant k'$). We define D_i' similarly. So during the algorithm the size of each B_i' might decrease, but since we delete at most $25g(n)$ edges from the initial F during the algorithm, all but at most $50g(n)$ vertices of the initial B_i' will still belong to this set at the end of the algorithm (and similarly for each D_i').

Since at each step of the algorithm the current F differs from the initial one by at most $25g(n)$ edges (and so at most $25g(n)$ edges have been removed from the initial H), we will be able to assume that at each step of the algorithm the following conditions hold.

(a) For each $1 \leqslant i \leqslant k'$ each vertex of B_i has at most $3\gamma^3\beta m$ H-neighbours in $D_i \setminus D_i'$ (and similarly for the vertices in D_i);
(b) For every $1 \leqslant i \leqslant k'$, the pair $(B_i, D_i)_H$ is $(4\varepsilon, \gamma\beta/7)$-super-regular;
(c) For every $1 \leqslant i \leqslant k$ and every A_i, there are at least $(1+\alpha)k'$ distinct j's with $1 \leqslant j \leqslant k$ such that (A_i, A_j) is weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H;
(d) For every $1 \leqslant i < j \leqslant k$, if there is an edge in X between A_i and A_j then (A_i, A_j) is weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H;
(e) Every vertex $x \in V(X) \setminus (A_1 \cup \ldots \cup A_k)$ has degree at least $\alpha n/7$ in H and all H-neighbours of x lie in $A_1 \cup \ldots \cup A_k$.

Note that by (a) and (b) we always have

$$|B_i'|, |D_i'| \geqslant (1-\gamma)m. \tag{12}$$

To see this, suppose that initially we have $|B_i \setminus B_i'| \geq \gamma m/2$. Then by (b) there is a vertex $x \in D_i$ which has at least $\gamma^2\beta m/20 > 3\gamma^3\beta m$ H-neighbours in $B_i \setminus B_i'$, contradicting (a). So (12) follows since we have already seen that all but at most $50g(n)$ vertices of the original set B_i' still belong to B_i' at the end of the algorithm.

Claim 1. *After at most $g(n)$ steps, we may assume that F is still a 2-factor and that for each $1 \leqslant i \leqslant k'$ there is a cycle C_i of F which contains at least $\gamma\beta m/9$ vertices of B_i' and at least $\gamma\beta m/9$ vertices of D_i'.*

Note that we may have $C_i = C_j$ even if $i \neq j$ (and similarly in the later claims). To prove the claim, suppose that F does not contain such a cycle C_i for some given i. Let C be a cycle of F which contains an edge xy with $x \in B_i$ and $y \in D_i$. Note that such a cycle exists by (12). Consider the path P obtained from C by removing the edge xy. If x has an H-neighbour y' on another cycle C' of F such that y' has an F-neighbour x' with $x' \in B_i$ then we replace the path P and the cycle C' with the path $x'C'y'xPy$. (Note that x' will be one of the neighbours of y' on C'.) We view the construction of this path as carrying out one step of the algorithm. Observe that we have only used one edge from H and we have reduced the number of cycles of F by 1 when extending P. Let us relabel so that the unique path of F is called P and its endpoints x and y belong to B_i and D_i respectively. Repeating this extension step for as long as possible, we may assume that no H-neighbour of x which is not on P has an F-neighbour in B_i and similarly no H-neighbour of y which is not on P has an F-neighbour in D_i. In particular, by (a) and (b), x has at least $\gamma\beta m/8$ H-neighbours in $V(P) \cap D_i'$, and similarly y has at least $\gamma\beta m/8$ H-neighbours in $V(P) \cap B_i'$. By Lemma 27 (applied with $U := B_i$, $V := D_i$ and $G := X$) it follows that we can use at most 5 edges of H to convert P into a cycle C_i (we view this as another step of the algorithm). Note that C_i satisfies the conditions of the claim. Since the number of cycles in F is initially at most $g(n)$ and since a Hamilton cycle certainly would satisfy the claim, the number of steps can be at most $g(n)$.

Claim 2. *After at most $g(n)$ further steps, we may assume that F is still a 2-factor and that for each $1 \leqslant i \leqslant k'$ there is a cycle C'_i of F which contains all but at most $4\varepsilon m$ vertices of B'_i and all but at most $4\varepsilon m$ vertices of D'_i.*

Let C_i be a cycle of F which contains at least $\gamma\beta m/9$ vertices of B'_i and at least $\gamma\beta m/9$ vertices of D'_i. Suppose there are at least $4\varepsilon m$ vertices of B'_i not covered by C_i. Then (b) implies that there is a vertex $b \in B'_i$, which is not covered by C_i and a vertex $d \in D'_i$ which is covered by C_i such that b and d are neighbours in H. Let C' be the cycle containing b and let x be any neighbour of b on C' and y any neighbour of d on C_i. Then removing the edges bx and dy and adding the edge bd we obtain the path $xC'bdC_iy$ (see Figure 3).

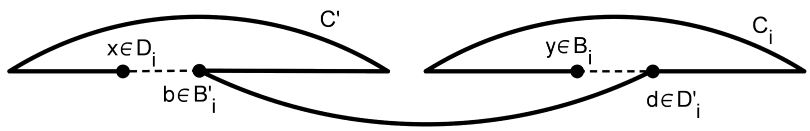

FIGURE 3. Extending C_i to include more vertices from $B'_i \cup D'_i$.

Since $x \in D_i$ and $y \in B_i$ (as $b \in B'_i$ and $d \in D'_i$) we can repeat the argument in the previous claim to extend this path into a larger path if necessary and then close it into a cycle. As long there are at least $4\varepsilon m$ vertices of B'_i not covered by the cycle or at least $4\varepsilon m$ vertices of C'_i not covered by the cycle we can repeat the above procedure to extend this into a larger cycle. Thus we can obtain a cycle C'_i with the required properties. The bound on the number of steps follows as in Claim 1.

Claim 3. *After at most $g(n)$ further steps, we may assume that F is still a 2-factor and that for each $1 \leqslant i \leqslant k'$ there is a cycle C''_i of F which contains all vertices of $B'_i \cup D'_i$.*

Let C'_i be the cycle obtained in the previous claim and suppose there is a vertex $b \in B'_i$ not covered by C'_i. By (a) and (b) it follows that b has at least $\gamma\beta m/8$ H-neighbours in $V(C'_i) \cap D'_i$. Let d be such an H-neighbour of b. Repeating the procedure in the proof of the previous claim, we can enlarge C'_i into a cycle containing b. Similarly we can extend the cycle to include any $d \in D'_i$, thus proving the claim.

Claim 4. *After at most $g(n)$ further steps, we may assume that F is still a 2-factor and that for each $1 \leqslant i \leqslant k'$ there is a cycle C'''_i of F which contains all vertices of $B_i \cup D_i$ and that there are no other cycles in F.*

Let C''_i be the cycle obtained in the previous claim and let x be a vertex in B_i not covered by C''_i. (The case when some vertex in D_i is not covered by C''_i is similar.) Let C be the cycle of F containing x and let y and z be the neighbours of x on C.

Case 1. *$y \in A_j$ for some j.*

It follows from (d) that there are at least $(1-\varepsilon)m$ vertices of A_j which have an H-neighbour in B_i. Also, y has an H-neighbour w satisfying the following:

(i) both F-neighbours of w belong to $A_j \setminus \{y\}$;
(ii) both F-neighbours of w have an H-neighbour in B'_i.

To see that we can choose such a w, suppose first that $A_j = B_{j'}$ for some j'. Then y has a set N_y of at least $\gamma\beta m/8$ H-neighbours in $D_{j'}$ by (b). By (a), at most $3\gamma^3\beta m$ vertices of N_y do not have both F-neighbours in $B_{j'}$. Note that y cannot be one of these F-neighbours in $B_{j'}$ since H and F are edge-disjoint. So N_y contains a set N^*_y of size $\gamma\beta m/9$ so that all vertices in N^*_y satisfy (i). By (d) at most $2\varepsilon m$ of these do not satisfy (ii). The argument for the case when $A_j = D_{j'}$ for some j' is identical.

The next step depends on whether w belongs to C_i'', C or some other cycle C' of F. In all cases we will find a path P from $x \in B_i$ to a vertex $y'' \in D_i$ containing all vertices of $C_i'' \cup C$. We can then proceed as before to find a cycle containing all the vertices of this path.

Case 1a. $w \in C_i''$.

Let y' be any one of the F-neighbours of w. Let x' be any H-neighbour of y' with $x' \in B_i'$ guaranteed by (ii) (so x' lies on C_i'') and let $y'' \in D_i$ be the F-neighbour of x' in the segment of C_i'' between x' and y' not containing w. Then we can replace the cycles C_i'' and C by the path $xzCywC_i''x'y'C_i''y''$ by removing the edges yx, wy' and $x'y''$ and adding the edges yw and $y'x'$.

Case 1b. $w \in C$.

Let y' be the F-neighbour of w in the segment of C between y and w not containing x. Let x' be any H-neighbour of y' with $x' \in B_i'$ and let y'' be any F-neighbour of x'. Note that x' and y'' both lie on C_i'' as $x' \in B_i'$. Then we can replace the cycles C_i'' and C by the path $xzCwyCy'x'C_i''y''$ by removing the edges yx, wy' and $x'y''$ and adding the edges yw and $y'x'$.

Case 1c. $w \in C'$ *for some* $C' \neq C, C_i''$.

Let y' be any one of the F-neighbours of w. Let x' be any H-neighbour of y' with $x' \in B_i'$ and let y'' be any F-neighbour of x'. So x' and y'' both lie on C_i''. We can replace the cycles C_i'', C and C' by the path $xzCywC'y'x'C_i''y''$ by removing the edges yx, wy' and $x'y''$ and adding the edges yw and $y'x'$.

Case 2. $y \in V(X) \setminus (A_1 \cup \cdots \cup A_k)$.

Let A be a cluster so that y has a set N_y of at least $\alpha^2 m$ H-neighbours in A' (if $A = B_j$ for some j, then A' denotes the set B_j' and similarly if $A = D_j$). Such an A exists since otherwise y would have at most $\gamma n + \alpha^2 n$ neighbours in H by (12) and the second part of (e). But this would contradict the lower bound of at least $\alpha n/7$ H-neighbours given by (e). Without loss of generality, we may assume that $A = B_j$ for some j, the argument for $A = D_j$ is identical. Then by (c) there is an index $s \neq j$ so that either (c_1) or (c_2) holds:

- (c_1) the pairs (B_s, D_j) and (D_s, B_i) are weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H;
- (c_2) the pairs (D_s, D_j) and (B_s, B_i) are weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H.

We may assume that (c_1) holds, the argument for (c_2) is identical. For convenience, we fix an orientation of each cycle of F. Given a vertex v on a cycle of F, this will enable us to refer to the successor v^+ of v and predecessor v^- of v. In particular, let N_y^+ be the successors of the vertices in N_y on C_j'' and let N_y^- be the predecessors. So $N_y^+, N_y^- \subseteq D_j$ and $|N_y^-|, |N_y^+| \geq \alpha^2 m$.

Also, let B_s'' be the subset of vertices v of B_s' so that both F-neighbours v^- and v^+ of v have at least five H-neighbours in B_i'. Since $v^-, v^+ \in D_s$, (c_1) and (12) together imply that $|B_s''| \geq m/2$. Two application of (c_1) to (B_s, D_j) now imply that there is a vertex $w \in N_y$ so that both w^+ and w^- have at least one H-neighbour in B_s'' (more precisely, apply (c_1) to the subpairs (B_s'', N_y^+) and (B_s'', N_y^-)).

Suppose first that $C \neq C_j''$. Then let $w_+ := w^+$ and we can obtain a path P_1 with the same vertex set as $C \cup C_j''$ by defining $P_1 := xzCywC_j''w_+$. If $C = C_j''$, then let w_+ be the C-neighbour of w on the segment of C between w and y which does not contain x and let $P_1 := xzCwyCw_+$.

Let v be the H-neighbour of w_+ in B_s'' (guaranteed by the definition of w). Note that $v \neq y$ and $v \neq w$ (as $s \neq j$). Suppose first that $C_s'' \neq C_j'', C$. Then we let $v_+ := v^+$ and define the path $P_2 := xP_1w_+vC_s''v_+$. If $C_s'' = C_j''$ or $C_s'' = C$, then all vertices of C_s'' already lie on P_1 and we let v_+ be the P_1-neighbour of v on the segment of P_1 towards w_+ and let $P_2 := xP_1vw_+P_1v_+$.

Now let u be an H-neighbour of v_+ in B'_i. (To see the existence of u, note that v_+ is one of th the F-neighbours of v in the definition of B''_s since $v \neq w, y$.) If $C''_i \neq C''_j$ and $C''_i \neq C''_s$, then let $u_+ := u^+$ and define the path $P_3 := xP_2v_+uC''_iu_+$. If $C''_i = C''_j$ or $C''_i = C''_s$, then all the vertices of C''_i already lie on P_2. Since at most 2 edges of C''_i do not lie on P_2 and since v_+ has at least five H-neighbours in B'_i by definition of B''_s, we can choose u in such a way that its P_2-neighbours both lie in D_i. We now let $u_+ \in D_i$ be the P_2-neighbour of u on the segment of P_2 towards v_+ and let $P_3 := xP_2uv_+P_2u_+$. Note that P_3 has endpoints $x \in B_i$ and $u_+ \in D_i$ and contains all vertices of $C''_i \cup C$, as desired. (We count the whole construction of P_3 as one step of the algorithm.) This completes Case 2.

Repeating this procedure, for each i we can find a cycle C'''_i which contains all vertices of $B_i \cup D_i$. Property (vi) of Theorem 25 and the second part of (e) together imply that no cycle in the 2-factor F thus obtained can consist entirely of vertices in $V(X)\setminus(A_1\cup\cdots\cup A_k)$ and so the C'''_i are the only cycles in F.

Claim 5. *By relabeling if necessary, we may assume that for every $1 \leqslant i \leqslant k'$, the pair (B_i, D_{i+1}) is weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H (where $D_{k'+1} := D_1$).*

For each $1 \leqslant i \leqslant k'$ we relabel B_i and D_i into D_i and B_i respectively with probability $1/2$ independently. Property (c) together with Theorem 8 imply that with high probability for each $1 \leqslant i \leqslant k'$ there are at least $(1+\alpha/2)k'/2$ indices j and least $(1+\alpha/2)k'/2$ indices j' with $1 \leqslant j, j' \leqslant k'$ and $j, j' \neq i$ such that each (B_i, D_j) and each $(B_{j'}, D_i)$ are weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H. Fix such a relabeling. Define a directed graph J on vertex set $[k']$ by joining i to j by a directed edge from i to j if and only if the pair (B_i, D_j) is weakly $(\varepsilon, \varepsilon^3/2k)$-regular in H. Then J has minimum out-degree and minimum in-degree at least $(1+\alpha/2)k'/2$ and so by Theorem 26 it contains a directed Hamilton cycle. Claim 5 now follows by reordering the indices of the B_i's and D_i's so that they comply with the ordering in the Hamilton cycle.

Claim 6. *For each $1 \leqslant j \leqslant k'$, after at most j steps, we may assume that F is a union of cycles together with a path P_j such that P_j has endpoints $x \in D_1$ and $y_j \in B_j$, where y_j has an H-neighbour in D'_{j+1}, and P_j covers all vertices of $(B_1 \cup D_1) \cup \cdots \cup (B_j \cup D_j)$. Furthermore, for every $j+1 \leqslant i \leqslant k'$, either P_j covers all vertices of C'''_i or $V(P_j) \cap V(C'''_i) = \emptyset$.*

To prove this claim we proceed by induction on j. For the case $j = 1$ observe that by Claim 5 there are at least $(1-\varepsilon)m$ vertices of B_1 which have at least one H-neighbour in D'_2. Of those, there is at least one vertex y_1 which belongs to B'_1. Let x be any F-neighbour of y_1 (so $x \in D_1$) and remove the edge xy_1 from C'''_1 to obtain the path P_1. Having obtained the path P_j, let x_{j+1} be an H-neighbour of y_j in D'_{j+1} (we count the construction of each P_j as one step of the algorithm).

Case 1. *P_j covers all vertices of C'''_{j+1}.*

In this case, let z_{j+1} be the neighbour of x_{j+1} on P_j in the segment of P_j between x_{j+1} and y_j and let Q_{j+1} be the path obtained from P_j by adding the edge y_jx_{j+1} and removing the edge $x_{j+1}z_{j+1}$. Observe that the endpoints of the path are $x \in D_1$ and $z_{j+1} \in B_{j+1}$ (but z_{j+1} need not have an H-neighbour in D'_{j+2}). By (a) and (b) z_{j+1} has at least $\gamma\beta m/8$ H-neighbours w_{j+1} in D'_{j+1}. For each such H-neighbour w_{j+1}, let w'_{j+1} be the unique neighbour of w_{j+1} on Q_{j+1} in the segment of Q_{j+1} between w_{j+1} and z_{j+1}. So $w'_{j+1} \in B_{j+1}$. Since by the previous claim at most εm vertices of B_{j+1} do not have an H-neighbour in D'_{j+2}, we can choose a w_{j+1} so that w'_{j+1} has an H-neighbour in D'_{j+2}.

We can then take $y_{j+1} := w'_{j+1}$ and obtain P_{j+1} from Q_{j+1} by adding the edge $z_{j+1}w_{j+1}$ and removing the edge $w_{j+1}w'_{j+1}$.

Case 2. $V(P_j) \cap V(C'''_{j+1}) = \emptyset$.

In this case, we let z_{j+1} be any F-neighbour of x_{j+1} and let Q_{j+1} be the path obtained from P_j and C'''_{j+1} by adding the edge y_jx_{j+1} and removing the edge $x_{j+1}z_{j+1}$. Observe that the endpoints of the path are $x \in D_1$ and $z_{j+1} \in B_{j+1}$ and so this case can be completed as the previous case.

By the case $j = k'$ of the previous claim we may assume that we now have a path $P := P_{k'}$ which covers all vertices of $A_1 \cup \dots \cup A_k$ and has endpoints $x \in D_1$ and $y := y_{k'} \in B_{k'}$ where y has an H-neighbour in D'_1. Moreover, P contains all vertices of each C'''_i and so by Claim 4 it must be a Hamilton path. Now let z be any H-neighbour of y with $z \in D'_1$ and let w be the neighbour of z in the segment of P between z and y. Let Q be the path obtained from P by removing the edge wz and adding the edge yz. So Q is a path on the same vertex set as P with endpoints $x \in D_1$ and $w \in B_1$ (we count the construction of Q as another step of the algorithm). But then we can apply Lemma 27 to transform Q into a Hamilton cycle in one more step, thus completing the proof of Theorem 25. □

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