PROOF OF THE 1-FACTORIZATION AND HAMILTON DECOMPOSITION CONJECTURES I: THE TWO CLIQUES CASE

DANIELA KÜHN, ALLAN LO, DERYK OSTHUS AND ANDREW TREGLOWN

ABSTRACT. In a sequence of four papers, we prove the following results (via a unified approach) for all sufficiently large n:

(i) [*1-factorization conjecture*] Suppose that n is even and $D \geq 2\lceil n/4\rceil - 1$. Then every D-regular graph G on n vertices has a decomposition into perfect matchings. Equivalently, $\chi'(G) = D$.

(ii) [*Hamilton decomposition conjecture*] Suppose that $D \geq \lfloor n/2 \rfloor$. Then every D-regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.

(iii) [*Optimal packings of Hamilton cycles*] Suppose that G is a graph on n vertices with minimum degree $\delta \geq n/2$. Then G contains at least $\mathrm{reg}_{\mathrm{even}}(n,\delta)/2 \geq (n-2)/8$ edge-disjoint Hamilton cycles. Here $\mathrm{reg}_{\mathrm{even}}(n,\delta)$ denotes the degree of the largest even-regular spanning subgraph one can guarantee in a graph on n vertices with minimum degree δ.

According to Dirac, (i) was first raised in the 1950s. (ii) and the special case $\delta = \lceil n/2 \rceil$ of (iii) answer questions of Nash-Williams from 1970. All of the above bounds are best possible. In the current paper, we prove the above results for the case when G is close to the union of two disjoint cliques.

CONTENTS

Date: January 1, 2014.

The research leading to these results was partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement no. 258345 (D. Kühn and A. Lo), 306349 (D. Osthus) and 259385 (A. Treglown). The research was also partially supported by the EPSRC, grant no. EP/J008087/1 (D. Kühn and D. Osthus).

1. Introduction

In a sequence of four papers, we provide a unified approach towards proving three long-standing conjectures for all sufficiently large graphs. Firstly, the 1-factorization conjecture, which can be formulated as an edge-colouring problem; secondly, the Hamilton decomposition conjecture, which provides a far-reaching generalization of Walecki's result [23] that every complete graph of odd order has a Hamilton decomposition and thirdly, a best possible result on packing edge-disjoint Hamilton cycles in Dirac graphs. The latter two were raised by Nash-Williams [25, 26, 27] in 1970.

1.1. The 1-factorization conjecture.

The 1-factorization conjecture. Vizing's theorem states that for any graph G of maximum degree Δ, its edge-chromatic number $\chi'(G)$ is either Δ or $\Delta + 1$. In general, it is a very difficult problem to determine which graphs G attain the (trivial) lower bound Δ – much of the recent book [31] is devoted to the subject. For regular graphs G, $\chi'(G) = \Delta(G)$ is equivalent to the existence of a 1-factorization: a 1-*factorization* of a graph G consists of a set of edge-disjoint perfect matchings covering all edges of G. The long-standing 1-factorization conjecture states that every regular graph of sufficiently high degree has a 1-factorization. It was first stated explicitly by Chetwynd and Hilton [2, 4] (who also proved partial results). However, they state that according to Dirac, it was already discussed in the 1950s.

Theorem 1.1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D-regular graph G on n vertices has a 1-factorization. Equivalently, $\chi'(G) = D$.*

The bound on the minimum degree in Theorem 1.1 is best possible. To see this, suppose first that $n = 2 \pmod 4$. Consider the graph which is the disjoint union of two cliques of order $n/2$ (which is odd). If $n = 0 \pmod 4$, consider the graph obtained from the disjoint union of cliques of orders $n/2 - 1$ and $n/2 + 1$ (both odd) by deleting a Hamilton cycle in the larger clique.

Note that Theorem 1.1 implies that for every regular graph G on an even number of vertices, either G or its complement has a 1-factorization. Also, Theorem 1.1 has an interpretation in terms of scheduling round-robin tournaments (where n players play all of each other in $n - 1$ rounds): one can schedule the first half of the rounds arbitrarily before one needs to plan the remainder of the tournament.

The best previous result towards Theorem 1.1 is due to Perkovic and Reed [29], who proved an approximate version, i.e. they assumed that $D \geq n/2 + \varepsilon n$. This was generalized by Vaughan [32] to multigraphs of bounded multiplicity. Indeed, he proved an approximate form of the following multigraph version of the 1-factorization conjecture which was raised by Plantholt and Tipnis [30]: Let G be a regular multigraph of even order n with multiplicity at most r. If the degree of G is at least $rn/2$ then G is 1-factorizable.

In 1986, Chetwynd and Hilton [3] made the following 'overfull subgraph' conjecture. Roughly speaking, this says that a dense graph satisfies $\chi'(G) = \Delta(G)$ unless there is a trivial obstruction in the form of a dense subgraph H on an odd number of vertices. Formally, we say that a subgraph H of G is *overfull* if $e(H) > \Delta(G)\lfloor |H|/2 \rfloor$ (note this requires $|H|$ to be odd).

Conjecture 1.2. *A graph G on n vertices with $\Delta(G) \geq n/3$ satisfies $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph.*

It is easy to see that this generalizes the 1-factorization conjecture (see e.g. [1] for the details). The overfull subgraph conjecture is still wide open – partial results are discussed in [31], which also discusses further results and questions related to the 1-factorization conjecture.

1.2. **The Hamilton decomposition conjecture.** Rather than asking for a 1-factorization, Nash-Williams [25, 27] raised the more difficult problem of finding a Hamilton decomposition in an even-regular graph. Here, a *Hamilton decomposition* of a graph G consists of a set of edge-disjoint Hamilton cycles covering all edges of G. A natural extension of this to regular graphs G of odd degree is to ask for a decomposition into Hamilton cycles and one perfect matching (i.e. one perfect matching M in G together with a Hamilton decomposition of $G - M$). The following result solves the problem of Nash-Williams for all large graphs.

Theorem 1.3. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ and $D \geq \lfloor n/2 \rfloor$. Then every D-regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

Again, the bound on the degree in Theorem 1.3 is best possible. Indeed, Proposition 3.1 shows that a smaller degree bound would not even ensure connectivity. Previous results include the following: Nash-Williams [24] showed that the degree bound in Theorem 1.3 ensures a single Hamilton cycle. Jackson [12] showed that one can ensure close to $D/2-n/6$ edge-disjoint Hamilton cycles. Christofides, Kühn and Osthus [5] obtained an approximate decomposition under the assumption that $D \geq n/2+\varepsilon n$. Under the same assumption, Kühn and Osthus [21] obtained an exact decomposition (as a consequence of the main result in [20] on Hamilton decompositions of robustly expanding graphs).

Note that Theorem 1.3 does not quite imply Theorem 1.1, as the degree threshold in the former result is slightly higher.

A natural question is whether one can extend Theorem 1.3 to sparser (quasi)-random graphs. Indeed, for random regular graphs of bounded degree this was proved by Kim and Wormald [15] and for (quasi-)random regular graphs of linear degree this was proved in [21] as a consequence of the main result in [20]. However, the intermediate range remains open.

1.3. **Packing Hamilton cycles in graphs of large minimum degree.** Although Dirac's theorem is best possible in the sense that the minimum degree condition $\delta \geq n/2$ is best possible, the conclusion can be strengthened considerably: a remarkable result of Nash-Williams [26] states that every graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains $\lfloor 5n/224 \rfloor$ edge-disjoint Hamilton cycles. He raised the question of finding the best possible bound, which we answer in Corollary 1.5 below.

We actually answer a more general form of this question: what is the number of edge-disjoint Hamilton cycles one can guarantee in a graph G of minimum degree δ?

A natural upper bound is obtained by considering the largest degree of an even-regular spanning subgraph of G. Let $\mathrm{reg}_{\mathrm{even}}(G)$ be the largest degree of an even-regular spanning subgraph of G. Then let

$$\mathrm{reg}_{\mathrm{even}}(n,\delta) := \min\{\mathrm{reg}_{\mathrm{even}}(G) : |G| = n,\ \delta(G) = \delta\}.$$

Clearly, in general we cannot guarantee more than $\mathrm{reg}_{\mathrm{even}}(n,\delta)/2$ edge-disjoint Hamilton cycles in a graph of order n and minimum degree δ. The next result shows that this bound is best possible (if $\delta < n/2$, then $\mathrm{reg}_{\mathrm{even}}(n,\delta) = 0$).

Theorem 1.4. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices with minimum degree $\delta \geq n/2$. Then G contains at least $\mathrm{reg}_{\mathrm{even}}(n,\delta)/2$ edge-disjoint Hamilton cycles.*

The main result of Kühn, Lapinskas and Osthus [18] proves Theorem 1.4 unless G is close to one of the extremal graphs for Dirac's theorem. This will allow us to restrict our attention to the latter situation (i.e. when G is close to the complete balanced bipartite graph or close to the union of two disjoint copies of a clique).

An approximate version of Theorem 1.4 for $\delta \geq n/2+\varepsilon n$ was obtained earlier by Christofides, Kühn and Osthus [5]. Hartke and Seacrest [11] gave a simpler argument with improved error bounds.

Precise estimates for $\mathrm{reg}_{\mathrm{even}}(n,\delta)$ (which yield either one or two possible values for any n, δ) are proved in [5, 10] using Tutte's theorem: Suppose that $n, \delta \in \mathbb{N}$ and

$n/2 \le \delta < n$. Then the bounds in [10] imply that

$$\frac{\delta + \sqrt{n(2\delta - n) + 8}}{2} - \varepsilon \le \mathrm{reg}_{\mathrm{even}}(n, \delta) \le \frac{\delta + \sqrt{n(2\delta - n)}}{2} + 1, \tag{1.1}$$

where $0 < \varepsilon \le 2$ is chosen to make the left hand side of (1.1) an even integer. Note that (1.1) determines $\mathrm{reg}_{\mathrm{even}}(n, n/2)$ exactly (the upper bound in this case was already proved by Katerinis [14]). Moreover, (1.1) implies that if $\delta \ge n/2$ then $\mathrm{reg}_{\mathrm{even}}(n, \delta) \ge (n-2)/4$. So we obtain the following immediate corollary of Theorem 1.4, which answers a question of Nash-Williams [26, 25, 27].

Corollary 1.5. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a graph on $n \ge n_0$ vertices with minimum degree $\delta \ge n/2$. Then G contains at least $(n-2)/8$ edge-disjoint Hamilton cycles.*

The following construction (which is based on a construction of Babai, see [25]) shows that the bound in Corollary 1.5 is best possible for $n = 8k + 2$, where $k \in \mathbb{N}$. Consider the graph G consisting of one empty vertex class A of size $4k$, one vertex class B of size $4k + 2$ containing a perfect matching and no other edges, and all possible edges between A and B. Thus G has order $n = 8k+2$ and minimum degree $4k+1 = n/2$. Any Hamilton cycle in G must contain at least two edges of the perfect matching in B, so G contains at most $\lfloor |B|/4 \rfloor = k = (n-2)/8$ edge-disjoint Hamilton cycles. The lower bound on $\mathrm{reg}_{\mathrm{even}}(n, \delta)$ in (1.1) follows from a generalization of this construction.

The following conjecture from [18] would be a common generalization of both Theorems 1.3 and 1.4 (apart from the fact that the degree threshold in Theorem 1.3 is slightly lower). It would provide a result which is best possible for every graph G (rather than the class of graphs with minimum degree at least δ).

Conjecture 1.6. *Suppose that G is a graph on n vertices with minimum degree $\delta(G) \ge n/2$. Then G contains $\mathrm{reg}_{\mathrm{even}}(G)/2$ edge-disjoint Hamilton cycles.*

For $\delta \ge (2 - \sqrt{2} + \varepsilon)n$, this conjecture was proved in [21], based on the main result of [20]. Recently, Ferber, Krivelevich and Sudakov [8] were able to obtain an approximate version of Conjecture 1.6, i.e. a set of $(1-\varepsilon)\mathrm{reg}_{\mathrm{even}}(G)/2$ edge-disjoint Hamilton cycles under the assumption that $\delta(G) \ge (1+\varepsilon)n/2$. It also makes sense to consider a directed version of Conjecture 1.6. Some related questions for digraphs are discussed in [21].

It is natural to ask for which other graphs one can obtain similar results. One such instance is the binomial random graph $G_{n,p}$: for any p, asymptotically almost surely it contains $\lfloor \delta(G_{n,p})/2 \rfloor$ edge-disjoint Hamilton cycles, which is clearly optimal. This follows from the main result of Krivelevich and Samotij [17] combined with that of Knox, Kühn and Osthus [16] (which builds on a number of previous results).

1.4. **Overall structure of the argument.** For all three of our main results, we split the argument according to the structure of the graph G under consideration:

(i) G is close to the complete balanced bipartite graph $K_{n/2,n/2}$;
(ii) G is close to the union of two disjoint copies of a clique $K_{n/2}$;

(iii) G is a 'robust expander'.

Roughly speaking, G is a robust expander if for every set S of vertices, its neighbourhood is at least a little larger than $|S|$, even if we delete a small proportion of the vertices and edges of G. The main result of [20] states that every dense regular robust expander has a Hamilton decomposition (see Theorem 3.4). This immediately implies Theorems 1.1 and 1.3 in Case (iii). For Theorem 1.4, Case (iii) is proved in [18] using a more involved argument, but also based on the main result of [20] (see Theorem 3.7).

Case (i) is proved in [6]. The current paper is devoted to the proof of Case (ii). Some of the key lemmas needed for Case (ii) are proved in [19]. (These lemmas provide a suitable decomposition of the set of 'exceptional edges' – these include the edges between the two almost complete graphs induced by G). Case (ii) is by far the hardest case for Theorems 1.1 and 1.3, as the extremal examples are all close to the union of two cliques. On the other hand, the proof of Theorem 1.4 is comparatively simple in this case, as for this result, the extremal construction is close to the complete balanced bipartite graph.

The arguments in the current paper for Case (ii) as well as those in [6] for Case (i) make use of an 'approximate' decomposition result. This result is proved in [7] and is much simpler to obtain. The arguments for both (i) and (ii) use the main lemma from [20] (the 'robust decomposition lemma') when transforming this approximate decomposition into an exact one.

The main proof in [20] (but not the proof of the robust decomposition lemma) makes use of Szemerédi's regularity lemma. So due to Case (iii) the bounds on n_0 in our results are very large (of tower type). However, the case of Theorem 1.1 when both $\delta \geq n/2$ and (iii) hold was proved by Perkovic and Reed [29] using 'elementary' methods, i.e. with a much better bound on n_0. Since the arguments for Cases (i) and (ii) do not rely on the regularity lemma, this means that if we assume that $\delta \geq n/2$, we get much better bounds on n_0 in our 1-factorization result (Theorem 1.1).

In Section 3, we derive Theorems 1.1, 1.3 and 1.4 from the structural results covering Cases (i)–(iii). The remainder of the paper is then devoted to the proof of Theorem 3.9 (i.e. Case (ii) of Theorem 1.4) and of Theorem 3.3, which is a common generalization of Case (ii) of Theorems 1.1 and 1.3. In Section 4, we give a sketch of the arguments for the 'two cliques' Case (ii) (i.e. the proofs of Theorems 3.3 and 3.9). Sections 5–8 (and part of Section 9) are common to the proofs of both Theorems 3.3 and 3.9. Theorem 3.9 is proved in Section 9. All the subsequent sections are devoted to the proof of Theorem 3.3.

2. Notation

Unless stated otherwise, all the graphs and digraphs considered in this paper are simple and do not contain loops. So in a digraph G, we allow up to two edges between any two vertices, at most one edge in each direction. Given a graph or digraph G, we write $V(G)$ for its vertex set, $E(G)$ for its edge set, $e(G) := |E(G)|$ for the number of edges in G and $|G| := |V(G)|$ for the number of vertices in G. We denote the complement of G by $\overline{G}$.

Suppose that G is an undirected graph. We write $\delta(G)$ for the minimum degree of G, $\Delta(G)$ for its maximum degree and $\chi'(G)$ for the edge-chromatic number of G. Given a vertex v of G and a set $A \subseteq V(G)$, we write $d_G(v, A)$ for the number of neighbours of v in G which lie in A. Given $A, B \subseteq V(G)$, we write $E_G(A)$ for the set of edges of G which have both endvertices in A and $E_G(A, B)$ for the set of edges of G which have one endvertex in A and its other endvertex in B. We also call the edges in $E_G(A, B)$ *AB-edges* of G. We let $e_G(A) := |E_G(A)|$ and $e_G(A, B) := |E_G(A, B)|$. If $A \cap B = \emptyset$, we denote by $G[A, B]$ the bipartite subgraph of G with vertex classes A and B and edge set $E_G(A, B)$. We often omit the index G if the graph G is clear from the context. An *AB-path* in G is a path with one endpoint in A and the other in B. A spanning subgraph H of G is an *r-factor* of G if the degree of every vertex of H is r.

Given a vertex set V and two multigraphs G and H with $V(G), V(H) \subseteq V$, we write $G + H$ for the multigraph whose vertex set is $V(G) \cup V(H)$ and in which the multiplicity of xy in $G + H$ is the sum of the multiplicities of xy in G and in H (for all $x, y \in V(G) \cup V(H)$). If G and H are simple graphs, we write $G \cup H$ for the (simple) graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$. We write $G - H$ for the subgraph of G which is obtained from G by deleting all the edges in $E(G) \cap E(H)$. Given $A \subseteq V(G)$, we write $G - A$ for the graph obtained from G by deleting all vertices in A.

We say that a graph or digraph G has a *decomposition* into $H_1, \dots, H_r$ if $G = H_1 + \dots + H_r$ and the H_i are pairwise edge-disjoint.

A *path system* is a graph Q which is the union of vertex-disjoint paths (some of them might be trivial). We say that P is a *path in Q* if P is a component of Q and, abusing the notation, sometimes write $P \in Q$ for this. We often view a matching M as a graph (in which every vertex has degree precisely one).

If G is a digraph, we write xy for an edge directed from x to y. A digraph G is an *oriented graph* if there are no $x, y \in V(G)$ such that $xy, yx \in E(G)$. Unless stated otherwise, when we refer to paths and cycles in digraphs, we mean directed paths and cycles, i.e. the edges on these paths/cycles are oriented consistently. If x is a vertex of a digraph G, then $N^+_G(x)$ denotes the *outneighbourhood* of x, i.e. the set of all those vertices y for which $xy \in E(G)$. Similarly, $N^-_G(x)$ denotes the *inneighbourhood* of x, i.e. the set of all those vertices y for which $yx \in E(G)$. The *outdegree* of x is $d^+_G(x) := |N^+_G(x)|$ and the *indegree* of x is $d^-_G(x) := |N^-_G(x)|$. Whenever $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, we denote by $G[A, B]$ the bipartite subdigraph of G with vertex classes A and B whose edges are all the edges of G directed from A to B. A spanning subdigraph H of G is an *r-factor* of G if the outdegree and the indegree of every vertex of H is r.

In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph or digraph), then this means that there are non-decreasing functions $f : (0, 1] \to (0, 1]$, $g : (0, 1] \to (0, 1]$ and $h : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq g(b)$

and $1/n \leq h(a)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way. We will write $a = b \pm c$ as shorthand for $b - c \leq a \leq b + c$.

3. Derivation of Theorems 1.1, 1.3, 1.4 from the main structural results

In this section, we combine the main results of this paper together with the results of [6], [21] and [18] to derive Theorems 1.1, 1.3 and 1.4. Before this, we first show that the bound on the minimum degree in Theorem 1.3 is best possible.

Proposition 3.1. *For every $n \geq 6$, let $D^* := \lfloor n/2 \rfloor - 1$. Unless both D^* and n are odd, there is a disconnected D^*-regular graph G on n vertices. If both D^* and n are odd, there is a disconnected $(D^* - 1)$-regular graph G on n vertices.*

Note that if both D^* and n are odd, no D^*-regular graph exists.

Proof. If n is even, take G to be the disjoint union of two cliques of order $n/2$. Suppose that n is odd and D^* is even. This implies $n = 3 \pmod 4$. Let G be the graph obtained from the disjoint union of cliques of orders $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ by deleting a perfect matching in the bigger clique. Finally, suppose that n and D^* are both odd. This implies that $n = 1 \pmod 4$. In this case, take G to be the graph obtained from the disjoint union of cliques of orders $\lfloor n/2 \rfloor - 1$ and $\lceil n/2 \rceil + 1$ by deleting a 3-factor in the bigger clique. □

3.1. Deriving Theorems 1.1 and 1.3.

As indicated in Section 1, in the proofs of our main results we will distinguish the cases when our given graph G is close to the union of two disjoint copies of $K_{n/2}$, close to a complete bipartite graph $K_{n/2,n/2}$ or a robust expander. We will start by defining these concepts.

We say that a graph G on n vertices is *ε-close to the union of two disjoint copies of $K_{n/2}$* if there exists $A \subseteq V(G)$ with $|A| = \lfloor n/2 \rfloor$ and such that $e(A, V(G) \setminus A) \leq \varepsilon n^2$. We say that G is *ε-close to $K_{n/2,n/2}$* if there exists $A \subseteq V(G)$ with $|A| = \lfloor n/2 \rfloor$ and such that $e(A) \leq \varepsilon n^2$. We say that G is *ε-bipartite* if there exists $A \subseteq V(G)$ with $|A| = \lfloor n/2 \rfloor$ such that $e(A), e(V(G) \setminus A) \leq \varepsilon n^2$. So every ε-bipartite graph is ε-close to $K_{n/2,n/2}$. Conversely, if $1/n \ll \varepsilon$ and G is a regular graph on n vertices which ε-close to $K_{n/2,n/2}$, then G is 2ε-bipartite.

Given $0 < \nu \leq \tau < 1$, we say that a graph G on n vertices is a *robust (ν, τ)-expander*, if for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1 - \tau)n$ the number of vertices that have at least νn neighbours in S is at least $|S| + \nu n$.

The following result from [18] implies that we can split the proofs of Theorems 1.1 and 1.3 into three cases.

Lemma 3.2. *Suppose that $0 < 1/n \ll \kappa \ll \nu \ll \tau, \varepsilon < 1$. Let G be a graph on n vertices of minimum degree $\delta := \delta(G) \geq (1/2 - \kappa)n$. Then G satisfies one of the following properties:*

(i) *G is ε-close to $K_{n/2,n/2}$;*
(ii) *G is ε-close to the union of two disjoint copies of $K_{n/2}$;*

(iii) *G is a robust (ν, τ)-expander.*

Recall that in this paper we prove Theorems 1.1 and 1.3 in Case (ii) when our given graph G is ε-close to the union of two disjoint copies of $K_{n/2}$. The following result is sufficiently general to imply both Theorems 1.1 and 1.3 in this case. We will prove it in Section 14.

Theorem 3.3. *For every $\varepsilon_{\rm ex} > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that $D \geq n - 2\lfloor n/4 \rfloor - 1$ and that G is a D-regular graph on n vertices which is $\varepsilon_{\rm ex}$-close to the union of two disjoint copies of $K_{n/2}$. Let F be the size of a minimum cut in G. Then G can be decomposed into $\lfloor \min\{D, F\}/2 \rfloor$ Hamilton cycles and $D - 2\lfloor \min\{D, F\}/2 \rfloor$ perfect matchings.*

Note that Theorem 3.3 provides structural insight into the extremal graphs for Theorem 1.3 – they are those with a cut of size less than D.

Throughout this paper, we will use the following fact.

$$n - 2\lfloor n/4 \rfloor - 1 = \begin{cases} n/2 - 1 & \text{if } n = 0 \pmod 4, \\ (n-1)/2 & \text{if } n = 1 \pmod 4, \\ n/2 & \text{if } n = 2 \pmod 4, \\ (n+1)/2 & \text{if } n = 3 \pmod 4. \end{cases} \tag{3.1}$$

The next result from [21] (derived from the main result of [20]) shows that every even-regular robust expander of linear degree has a Hamilton decomposition. It will be used to prove Theorems 1.1 and 1.3 in the case when our given graph G is a robust expander.

Theorem 3.4. *For every $\alpha > 0$ there exists $\tau > 0$ such that for every $\nu > 0$ there exists $n_0 = n_0(\alpha, \nu, \tau)$ for which the following holds. Suppose that*

(i) *G is an r-regular graph on $n \geq n_0$ vertices, where $r \geq \alpha n$ is even;*
(ii) *G is a robust (ν, τ)-expander.*

Then G has a Hamilton decomposition.

The following result from [6] implies Theorems 1.1 and 1.3 in the case when our given graph is ε-close to $K_{n/2,n/2}$. Note that unlike the case when G is ε-close to the union of two disjoint copies of $K_{n/2}$, we have room to spare in the lower bound on D.

Theorem 3.5. *There are $\varepsilon_{\rm ex} > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let $n \geq n_0$ and suppose that $D \geq (1/2 - \varepsilon_{\rm ex})n$ is even. Suppose that G is a D-regular graph on n vertices which is $\varepsilon_{\rm ex}$-bipartite. Then G has a Hamilton decomposition.*

The following result is an easy consequence of Tutte's theorem and gives the degree threshold for a single perfect matching in a regular graph. Note the condition on D is the same as in Theorem 1.1.

Proposition 3.6. *Suppose that $D \geq 2\lceil n/4 \rceil - 1$ and n is even. Then every D-regular graph G on n vertices has a perfect matching.*

Proof. If $D \geq n/2$ then G has a Hamilton cycle (and thus a perfect matching) by Dirac's theorem. So we may assume that $D = n/2 - 1$ and so $n = 0 \pmod 4$. In this case, we will use Tutte's theorem which states that a graph G has a perfect matching if for every set $S \subseteq V(G)$ the graph $G - S$ has at most $|S|$ odd components (i.e. components on an odd number of vertices). The latter condition holds if $|S| \leq 1$ and if $|S| \geq n/2$.

If $|S| = n/2 - 1$ and $G - S$ has more than $|S|$ odd components, then $G - S$ consists of isolated vertices. But this implies that each vertex outside S is joined to all vertices in S, contradicting the $(n/2 - 1)$-regularity of G.

If $2 \leq |S| \leq n/2 - 2$, then every component of $G - S$ has at least $n/2 - |S|$ vertices and so $G - S$ has at most $\lfloor (n - |S|)/(n/2 - |S|) \rfloor$ components. But $\lfloor (n - |S|)/(n/2 - |S|) \rfloor \leq |S|$ unless $n = 8$ and $|S| = 2$. (Indeed, note that $(n - |S|)/(n/2 - |S|) \leq |S|$ if and only if $n + |S|^2 - (n/2 + 1)|S| \leq 0$. The latter holds for $|S| = 3$ and $|S| = n/2 - 2$, and so for all values in between. The case $|S| = 2$ can be checked separately.) If $n = 8$ and $|S| = 2$, it is easy to see that $G - S$ has at most two odd components. □

Proof of Theorem 1.1. Let $\tau = \tau(1/3)$ be the constant returned by Theorem 3.4 for $\alpha := 1/3$. Choose $n_0 \in \mathbb{N}$ and constants $\nu, \varepsilon_{\rm ex}$ such that $1/n_0 \ll \nu \ll \tau, \varepsilon_{\rm ex}$ and $\varepsilon_{\rm ex} \ll 1$. Let $n \geq n_0$ and let G be a D-regular graph as in Theorem 1.1. Lemma 3.2 implies that G satisfies one of the following properties:

(i) G is $\varepsilon_{\rm ex}$-close to $K_{n/2,n/2}$;
(ii) G is $\varepsilon_{\rm ex}$-close to the union of two disjoint copies of $K_{n/2}$;
(iii) G is a robust (ν, τ)-expander.

If (i) holds and D is even, then as observed at the beginning of this subsection, this implies that G is $2\varepsilon_{\rm ex}$-bipartite. So Theorem 3.5 implies that G has a Hamilton decomposition and thus also a 1-factorization (as n is even and so every Hamilton cycle can be decomposed into two perfect matchings). Suppose that (i) holds and D is odd. Then Proposition 3.6 implies that G contains a perfect matching M. Now $G - M$ is still $\varepsilon_{\rm ex}$-close to $K_{n/2,n/2}$ and so Theorem 3.5 implies that $G - M$ has a Hamilton decomposition. Thus G has a 1-factorization. If (ii) holds, then Theorem 3.3 and (3.1) imply that G has a 1-factorization. If (iii) holds and D is odd, we use Proposition 3.6 to choose a perfect matching M in G and let $G' := G - M$. If D is even, let $G' := G$. In both cases, $G' - M$ is still a robust $(\nu/2, \tau)$-expander. So Theorem 3.4 gives a Hamilton decomposition of G'. So G has a 1-factorization. □

The proof of Theorem 1.3 is similar to that of Theorem 1.1.

Proof of Theorem 1.3. Choose $n_0 \in \mathbb{N}$ and constants $\tau, \nu, \varepsilon_{\rm ex}$ as in the proof of Theorem 1.1. Let $n \geq n_0$ and let G be a D-regular graph as in Theorem 1.3. As before, Lemma 3.2 implies that G satisfies one of (i)–(iii). Suppose first that (i) holds. If D is odd, n must be even and so $D \geq n/2$. Choose a perfect matching M in G (e.g. by applying Dirac's theorem) and let $G' := G - M$. If D is even, let

$G' := G$. Note that in both cases G' is $\varepsilon_{\rm ex}$-close to $K_{n/2,n/2}$ and so $2\varepsilon_{\rm ex}$-bipartite. Thus Theorem 3.5 implies that G' has a Hamilton decomposition.

Suppose next that (ii) holds. Note that by (3.1), $D \geq n - 2\lfloor n/4 \rfloor - 1$ unless $n = 3 \pmod 4$ and $D = \lfloor n/2 \rfloor$. But the latter would mean that both n and D are odd, which is impossible. So the conditions of Theorem 3.3 are satisfied. Moreover, since $D \geq \lfloor n/2 \rfloor$, Proposition 6.1(ii) implies that the size of a minimum cut in G is at least D. Thus Theorem 3.3 implies that G has a decomposition into Hamilton cycles and at most one perfect matching.

Finally, suppose that (iii) holds. If D is odd (and thus n is even), we can apply Proposition 3.6 again to find a perfect matching M in G and let $G' := G - M$. If D is even, let $G' := G$. In both cases, G' is still a robust $(\nu/2, \tau)$-expander. So Theorem 3.4 gives a Hamilton decomposition of G'. □

3.2. **Deriving Theorem 1.4.** The derivation of Theorem 1.4 is similar to that of the previous two results. We will replace the use of Lemma 3.2 and Theorem 3.4 with the following result, which is an immediate consequence of the two main results in [18].

Theorem 3.7. *For every $\varepsilon_{\rm ex} > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/2$. Then G satisfies one of the following properties:*

- (i) *G is $\varepsilon_{\rm ex}$-close to $K_{n/2,n/2}$;*
- (ii) *G is $\varepsilon_{\rm ex}$-close to the union of two disjoint copies of $K_{n/2}$;*
- (iii) *G contains $\mathrm{reg}_{\rm even}(n,\delta)/2$ edge-disjoint Hamilton cycles.*

To deal with the near-bipartite case (i), we will apply the following result from [6].

Theorem 3.8. *For each $\alpha > 0$ there are $\varepsilon_{\rm ex} > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is an $\varepsilon_{\rm ex}$-bipartite graph on $n \geq n_0$ vertices with $\delta(G) \geq (1/2 - \varepsilon_{\rm ex})n$. Suppose that G has a D-regular spanning subgraph such that $n/100 \leq D \leq (1/2 - \alpha)n$ and D is even. Then G contains $D/2$ edge-disjoint Hamilton cycles.*

The next result immediately implies Theorem 1.4 in Case (ii) when G is ε-close to the union of two disjoint copies of $K_{n/2}$. We will prove it in Section 9 of this paper. Since G is far from extremal in this case, we obtain almost twice as many edge-disjoint Hamilton cycles as needed for Theorem 1.4.

Theorem 3.9. *For every $\varepsilon > 0$, there exist $\varepsilon_{\rm ex} > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose $n \geq n_0$ and G is a graph on n vertices such that G is $\varepsilon_{\rm ex}$-close to the union of two disjoint copies of $K_{n/2}$ and such that $\delta(G) \geq n/2$. Then G has at least $(1/4 - \varepsilon)n$ edge-disjoint Hamilton cycles.*

We will also use the following well-known result of Petersen.

Theorem 3.10. *Every regular graph of positive even degree contains a 2-factor.*

Proof of Theorem 1.4. Choose $n_0 \in \mathbb{N}$ and $\varepsilon_{\rm ex}$ such that $1/n_0 \ll \varepsilon_{\rm ex} \ll 1$. In particular, we choose $\varepsilon_{\rm ex} \leq \varepsilon^1_{\rm ex}(1/12)$, where $\varepsilon^1_{\rm ex}(1/12)$ is the constant returned by Theorem 3.9 for $\varepsilon := 1/12$, as well as $\varepsilon_{\rm ex} \leq \varepsilon^2_{\rm ex}(1/6)/2$, where $\varepsilon^2_{\rm ex}(1/6)$ is the constant returned by Theorem 3.8 for $\alpha := 1/6$. Let G be a graph on $n \geq n_0$ vertices with $\delta := \delta(G) \geq n/2$. Theorem 3.7 implies that we may assume that G satisfies either (i) or (ii). Note that in both cases it follows that $\delta(G) \leq (1/2 + 5\varepsilon_{\rm ex})n$. So (1.1) implies that $n/5 \leq {\rm reg}_{\rm even}(n, \delta) \leq 3n/10$.

Suppose first that (i) holds. As mentioned above, this implies that G is $2\varepsilon_{\rm ex}$-bipartite. Let G' be a D-regular spanning subgraph of G such that D is even and $D \geq {\rm reg}_{\rm even}(n, \delta)$. Petersen's theorem (Theorem 3.10) implies that by successively deleting 2-factors of G', if necessary, we may in addition assume that $D \leq n/3$. Then Theorem 3.8 (applied with $\alpha := 1/6$) implies that G contains at least $D/2 \geq {\rm reg}_{\rm even}(n, \delta)/2$ edge-disjoint Hamilton cycles.

Finally suppose that (ii) holds. Then Theorem 3.9 (applied with $\varepsilon := 1/12$) implies that G contains $n/6 \geq {\rm reg}_{\rm even}(n, \delta)/2$ edge-disjoint Hamilton cycles. □

4. Overview of the proofs of Theorems 3.3 and 3.9

The proof of Theorem 3.9 is much simpler than that of Theorems 3.3 (mainly because its assertion leaves some leeway – one could probably find a slightly larger set of edge-disjoint Hamilton cycles than guaranteed by Theorem 3.9). Moreover, the ideas used in the former all appear in the proof of the latter too.

4.1. **Proof overview for Theorem 3.9.** Let G be a graph on n vertices with $\delta(G) \geq n/2$ which is close to being the union of two disjoint cliques. So there is a vertex partition of G into sets A and B of roughly equal size so that $G[A]$ and $G[B]$ are almost complete. Our aim is to construct almost $n/4$ edge-disjoint Hamilton cycles.

Several techniques have recently been developed which yield approximate decompositions of dense (almost) regular graphs, i.e. a set of Hamilton cycles covering almost all the edges (see e.g. [5, 8, 9, 22, 28]). This leads to the following idea: replace $G[A]$ and $G[B]$ by multigraphs G_A and G_B so that any suitable pair of Hamilton cycles C_A and C_B of G_A and G_B respectively corresponds to a single Hamilton cycle C in the original graph G. We will construct G_A and G_B by deleting some edges of G and introducing some 'fictive edges'. (The introduction of these fictive edges is the reason why G_A and G_B are multigraphs.)

We next explain the key concept of these 'fictive edges'. The following graph G provides an instructive example: suppose that $n = 0 \pmod 4$. Let G be obtained from two disjoint cliques induced by sets A and B of size $n/2$ by adding a perfect matching M between A and B. Note that G is $n/2$-regular. Now pair up the edges of M into $n/4$ pairs (e_i, e_{i+1}) for $i = 1, 3, \dots, n/2 - 1$. Write $e_i =: x_i y_i$ with $x_i \in A$ and $y_i \in B$. Next let G_A be the multigraph obtained from $G[A]$ by adding all the edges $x_i x_{i+1}$, where i is odd. Similarly, let G_B be obtained from $G[B]$ by adding all the edges $y_i y_{i+1}$, where i is odd. We call the edges $x_i x_{i+1}$ and $y_i y_{i+1}$ fictive edges. Note that G_A and G_B are regular multigraphs. Now pair off the fictive edges in G_A with those in G_B, i.e. $x_i x_{i+1}$ is paired off with $y_i y_{i+1}$. Suppose that C_A is a Hamilton cycle in G_A which contains $x_i x_{i+1}$ (and no other fictive edges) and C_B is a Hamilton cycle in G_B which contains $y_i y_{i+1}$ (and no other fictive edges). Then together, C_A

and C_B correspond to a Hamilton cycle C in the original graph G (where fictive edges are replaced by the corresponding matching edges in M again).

So we have reduced the problem of finding many edge-disjoint Hamilton cycles in G to that of finding many edge-disjoint Hamilton cycles in the almost complete graph G_A (and G_B), with the additional requirement that each such Hamilton cycle contains a unique fictive edge. This can be achieved via the 'approximate decomposition result' in [7] (see Lemma 9.4 in current paper for the statement).

Additional difficulties arise from 'exceptional' vertices, namely those which have high degree into both A and B. (It is easy to see that there cannot be too many of these vertices.) Fictive edges also provide a natural way of 'eliminating' these exceptional vertices. Suppose for example that G' is obtained from the graph G above by adding a vertex a so that a is adjacent to half of the vertices in A and half of the vertices in B. (Note that $\delta(G')$ is a little smaller than $|G'|/2$, but G' is similar to graphs actually occurring in the proof.) Then we can pair off the neighbours of a into pairs within A and introduce a fictive edge f_i between each pair of neighbours. We also introduce fictive edges f_i between pairs of neighbours of a in B. Without loss of generality, we have fictive edges $f_1, f_3, \dots, f_{n/2-1}$ (and recall that $|G'| = n+1$). So we have $V(G'_A) = A$ and $V(G'_B) = B$ again. We then require each pair of Hamilton cycles C_A, C_B of G'_A and G'_B to contain x_ix_{i+1}, y_iy_{i+1} and a fictive edge f_i (which may lie in A or B) where i is odd, see Figure 1. Then C_A and C_B together correspond to a Hamilton cycle C in G' again. The subgraph J of G' which corresponds to three such fictive edges x_ix_{i+1}, y_iy_{i+1} and f_i of C is called a 'Hamilton exceptional system'. J will always be a path system. So in general, we will first find a sufficient number of edge-disjoint Hamilton exceptional systems J. Then we apply Lemma 9.4 to find edge-disjoint Hamilton cycles in G'_A and G'_B, where each pair of cycles contains a suitable set J^* of fictive edges (corresponding to some Hamilton exceptional system J).

For Lemma 9.4, we need each of the Hamilton exceptional systems J to be 'localized': given a partition of A and B into clusters, the endpoints of the corresponding set J^* of fictive edges need to be contained in a single cluster of A and of B. The fact that the Hamilton exceptional systems need to be localized is one reason for treating exceptional vertices differently from the others by introducing fictive edges for them.

4.2. **Proof overview for Theorem 3.3.** The main result of this paper is Theorem 3.3. Suppose that G is a D-regular graph satisfying the conditions of that theorem.

Using the approach of the previous subsection, one can obtain an approximate decomposition of G, i.e. a set of edge-disjoint Hamilton cycles covering almost all edges of G. However, one does not have any control over the 'leftover' graph H, which makes a complete decomposition seem infeasible. This problem was overcome in [20] by introducing the concept of a 'robustly decomposable graph' G^{rob}. Roughly speaking, this is a sparse regular graph with the following property: given *any* very sparse regular graph H with $V(H) = V(G^{\mathrm{rob}})$ which is edge-disjoint from G^{rob}, one

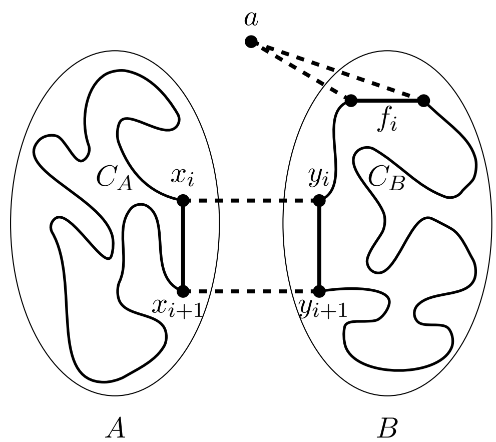

FIGURE 1. Transforming the problem of finding a Hamilton cycle on $V(G')$ into finding two Hamilton cycles C_A and C_B on A and B respectively.

can guarantee that $G^{\rm rob} \cup H$ has a Hamilton decomposition. This leads to a natural (and very general) strategy to obtain a decomposition of G:

(1) find a (sparse) robustly decomposable graph $G^{\rm rob}$ in G and let G' denote the leftover;
(2) find an approximate Hamilton decomposition of G' and let H denote the (very sparse) leftover;
(3) find a Hamilton decomposition of $G^{\rm rob} \cup H$.

It is of course far from clear that one can always find such a graph $G^{\rm rob}$. The main 'robust decomposition lemma' of [20] guarantees such a graph $G^{\rm rob}$ in any regular robustly expanding graph of linear degree. Since G is close to the disjoint union of two cliques, we are of course not in this situation. However, a regular almost complete graph is certainly a robust expander, i.e. our assumptions imply that G is close to being the disjoint union of two regular robustly expanding graphs G_A and G_B, with vertex sets A and B.

So very roughly, the strategy is to apply the robust decomposition lemma of [20] to G_A and G_B separately, to obtain a Hamilton decomposition of both G_A and G_B. Now we pair up Hamilton cycles of G_A and G_B in this decomposition, so that each such pair corresponds to a single Hamilton cycle of G and so that all edges of G are covered. It turns out that we can achieve this as in the proof of Theorem 3.9: we replace all edges of G between A and B by suitable 'fictive edges' in G_A and G_B. We then need to ensure that each Hamilton cycle in G_A and G_B contains a suitable set of fictive edges – and the set-up of the robust decomposition lemma does allow for this.

One significant difficulty compared to the proof of Theorem 3.9 is that this time we need a *decomposition* of all the 'exceptional' edges (i.e. those between A and B and those incident to the exceptional vertices) into Hamilton exceptional systems.

The nature of the decomposition depends on the structure of the bipartite subgraph $G[A', B']$ of G, where A' is obtained from A by including some subset A_0 of the exceptional vertices, and B' is obtained from B by including the remaining set B_0 of exceptional vertices. We say that G is 'critical' if many edges of $G[A', B']$ are incident to very few (exceptional) vertices. In our decomposition into Hamilton exceptional systems, we will need to distinguish between the critical and non-critical case (when in addition $G[A', B']$ contains many edges) and the case when $G[A', B']$ contains only a few edges. The lemmas guaranteeing this decomposition are stated and discussed in Section 11, but proved in [19].

Finding these localized Hamilton exceptional systems becomes more feasible if we can assume that there are no edges with both endpoints in the exceptional set A_0 or both endpoints in B_0. So in Section 10, we find and remove a set of edge-disjoint Hamilton cycles covering all edges in $G[A_0]$ and $G[B_0]$. We can then find the localized Hamilton exceptional systems in Section 11. After this, we need to extend and combine them into certain path systems and factors in Section 12, before we can use them as an 'input' for the robust decomposition lemma in Section 13. Finally, all these steps are combined in Section 14 to prove Theorem 3.3.

5. Tools

We will often use the following Chernoff bound for the hypergeometric distribution (see e.g. [13, Theorem 2.10]). The hypergeometric random variable X with parameters (n, m, k) is defined as follows. We let N be a set of size n, fix $S \subseteq N$ of size $|S| = m$, pick a uniformly random $T \subseteq N$ of size $|T| = k$, then define $X := |T \cap S|$. Note that $\mathbb{E}X = km/n$.

Proposition 5.1. *Suppose X has hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-a^2\mathbb{E}X/3}$.*

We will also need the following fact, which is a simple consequence of Vizing's theorem and was first observed by McDiarmid and independently by de Werra (see e.g. [34]).

Proposition 5.2. *Let G be a graph with $\chi'(G) \leq m$. Then G has a decomposition into m matchings $M_1, \dots, M_m$ with $|e(M_i) - e(M_j)| \leq 1$ for all $i, j \leq m$.*

If $G = (A, B)$ is an undirected bipartite graph with vertex classes A and B, then the *density* of G is defined as

$$d(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

For any $\varepsilon > 0$, we say that G is *ε-regular* if for any $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ we have $|d(A', B') - d(A, B)| < \varepsilon$. We say that G is *$(\varepsilon, \geq d)$-regular* if it is ε-regular and has density d' for some $d' \geq d - \varepsilon$.

We say that G is *$[\varepsilon, d]$-superregular* if it is ε-regular and $d_G(a) = (d \pm \varepsilon)|B|$ for every $a \in A$ and $d_G(b) = (d \pm \varepsilon)|A|$ for every $b \in B$. G is *$[\varepsilon, \geq d]$-superregular* if it is *$[\varepsilon, d']$-superregular* for some $d' \geq d$.

Given disjoint vertex sets X and Y in a digraph G, recall that $G[X,Y]$ denotes the bipartite subdigraph of G whose vertex classes are X and Y and whose edges are all the edges of G directed from X to Y. We often view $G[X,Y]$ as an undirected bipartite graph. In particular, we say $G[X,Y]$ is *ε-regular*, *$(\varepsilon, \geq d)$-regular*, *$[\varepsilon, d]$-superregular* or *$[\varepsilon, \geq d]$-superregular* if this holds when $G[X,Y]$ is viewed as an undirected graph.

The following proposition states that the graph obtained from a superregular pair by removing a small number of edges at every vertex is still superregular (with slightly worse parameters). We omit the proof which follows straightforwardly from the definition of superregularity. A similar argument is for example included in [20].

Proposition 5.3. *Suppose that $0 < 1/m \ll \varepsilon \leq d' \ll d \leq 1$. Let G be a bipartite graph with vertex classes A and B of size m. Suppose that G' is obtained from G by removing at most $d'm$ vertices from each vertex class and at most $d'm$ edges incident to each vertex from G. If G is $[\varepsilon, d]$-superregular then G' is $[2\sqrt{d'}, d]$-superregular.*

We will also use the following well-known observation, which easily follows from Hall's theorem and the definition of $[\varepsilon, d]$-superregularity.

Proposition 5.4. *Suppose that $0 < 1/m \ll \varepsilon \ll d \leq 1$. Suppose that G is an $[\varepsilon, d]$-superregular bipartite graph with vertex classes of size m. Then G contains a perfect matching.*

6. Partitions and frameworks

6.1. **Edges between partition classes.** Let A', B' be a partition of the vertex set of a graph G. The aim of this subsection is to give some useful bounds on the number $e_G(A', B')$ of edges between A' and B' in G.

Proposition 6.1. *Let G be a graph on n vertices with $\delta(G) \geq D$. Let A', B' be a partition of $V(G)$. Then the following properties hold:*

(i) $e_G(A', B') \geq (D - |B'| + 1)|B'|$.
(ii) *If $D \geq n - 2\lfloor n/4 \rfloor - 1$, then $e_G(A', B') \geq D$ unless $n = 0 \pmod 4$, $D = n/2 - 1$ and $|A'| = |B'| = n/2$.*

Proof. Since $\delta(G) \geq D$ we have $d(v, A') \geq D - |B'| + 1$ for all $v \in B'$ and so $e_G(A', B') \geq (D - |B'| + 1)|B'|$, which implies (i). (ii) follows from (3.1) and (i). □

Proposition 6.2. *Let G be a D-regular graph on n vertices together with a vertex partition A', B'. Then*

(i) *$e_G(A', B')$ is odd if and only if both $|A'|$ and D are odd.*
(ii) $e_G(A', B') = e_{\overline{G}}(A') + e_{\overline{G}}(B') + \frac{(2D+2-n)n}{4} - \frac{(|A'|-|B'|)^2}{4}$.

Proof. Note that $e_G(A', B') = \sum_{v \in A'} d(v, B') = \sum_{v \in A'} (D - d(v, A')) = |A'|D - 2e_G(A')$. Hence (i) follows.

For (ii), note that

$$e_{\overline{G}}(A') = \binom{|A'|}{2} - e_G(A') = \binom{|A'|}{2} - \frac{1}{2}\left(D|A'| - e_G(A', B')\right),$$

and similarly $e_{\overline{G}}(B') = \binom{|B'|}{2} - (D|B'| - e_G(A', B'))/2$. Since $|A'| + |B'| = n$ it follows that

$$\begin{aligned} e_G(A', B') &= e_{\overline{G}}(A') + e_{\overline{G}}(B') - \frac{1}{2}\left(|A'|^2 + |B'|^2 - n(D+1)\right) \\ &= e_{\overline{G}}(A') + e_{\overline{G}}(B') + \frac{(2D+2-n)n}{4} - \frac{(|A'| - |B'|)^2}{4}, \end{aligned}$$

as required. □

Proposition 6.3. *Let G be a D-regular graph on n vertices with $D \geq \lfloor n/2 \rfloor$. Let A', B' be a partition of $V(G)$ with $|A'|, |B'| \geq D/2$ and $\Delta(G[A', B']) \leq D/2$. Then*

$$e_{G-U}(A', B') \geq \begin{cases} D - 28 & \text{if } D \geq n/2, \\ D/2 - 28 & \text{if } D = (n-1)/2 \end{cases}$$

for every $U \subseteq V(G)$ with $|U| \leq 3$.

Proof. Without loss of generality, we may assume that $|A'| \geq |B'|$. Set $G' := G[A', B']$. If $|B'| \leq D - 4$, then $e(G') \geq (D - |B'| + 1)|B'| \geq 5D/2$ by Proposition 6.1(i). Since $\Delta(G') \leq D/2$ we have $e(G' - U) \geq e(G') - 3D/2 \geq D$. Thus we may assume that $|B'| \geq D - 3$. For every $v \in B'$, we have

$$d_{G'}(v) = d_G(v, A') = D - d_G(v, B') = D - (|B'| - d_{\overline{G}}(v, B') - 1) \leq d_{\overline{G}}(v, B') + 4,$$

and similarly $d_{G'}(v) \leq d_{\overline{G}}(v, A') + 4$ for all $v \in A'$. Thus

$$\begin{aligned} \sum_{u \in U} d_{G'}(u) &\leq 12 + \sum_{u \in U \cap A'} d_{\overline{G}}(u, A') + \sum_{u \in U \cap B'} d_{\overline{G}}(u, B') \\ &\leq 15 + e_{\overline{G}}(A') + e_{\overline{G}}(B'). \end{aligned} \tag{6.1}$$

Note that $|A'| - |B'| \leq 7$ since $|A'| \geq |B'| \geq D - 3 \geq \lfloor n/2 \rfloor - 3$. By Proposition 6.2(ii), we have

$$\begin{aligned} e(G' - U) &\geq e(G') - \sum_{u \in U} d_{G'}(u) \\ &\geq e_{\overline{G}}(A') + e_{\overline{G}}(B') + \frac{(2D+2-n)n}{4} - \frac{(|A'| - |B'|)^2}{4} - \sum_{u \in U} d_{G'}(u) \\ &\overset{(6.1)}{\geq} \frac{(2D+2-n)n}{4} - \frac{(|A'| - |B'|)^2}{4} - 15 \geq \frac{(2D+2-n)n}{4} - 28. \end{aligned}$$

Hence the proposition follows. □

The following result is an analogue of Proposition 6.3 for the case when G is $(n/2-1)$-regular with $n = 0 \pmod 4$ and $|A'| = n/2 = |B'|$.

Proposition 6.4. *Let G be an $(n/2-1)$-regular graph on n vertices with $n = 0 \pmod 4$. Let A', B' be a partition of $V(G)$ with $|A'| = n/2 = |B'|$. Then*

$$e_G(A' \setminus X, B') \geq e_G(X, B') - |X|(|X|-1)$$

for every vertex set $X \subseteq A'$. Moreover, $\Delta(G[A', B']) \leq e_G(A', B')/2$.

Proof. For every $v \in A'$, we have

$$d_G(v, B') = n/2 - 1 - d_G(v, A') = |A'| - 1 - d_G(v, A') = d_{\overline{G}}(v, A').$$

By summing over all $v \in A'$ we obtain

$$\begin{aligned} e_G(A', B') = 2e_{\overline{G}}(A') &\geq 2\left(\sum_{x \in X} d_{\overline{G}}(x, A') - \binom{|X|}{2}\right) = 2\sum_{x \in X} d_G(x, B') - |X|(|X|-1) \\ &= 2e_G(X, B') - |X|(|X|-1). \end{aligned}$$

Therefore,

$$e_G(A' \setminus X, B') = e_G(A', B') - e_G(X, B') \geq e_G(X, B') - |X|(|X|-1).$$

In particular, this implies that for each vertex $x \in A'$ we have $e_G(A' \setminus \{x\}, B') \geq e_G(\{x\}, B') = d_G(x, B')$ and so $2d_G(x, B') \leq e_G(A', B')$. By symmetry, for any $y \in B'$ we have $2d(y, A') \leq e_G(A', B')$. Therefore, $\Delta(G[A', B']) \leq e_G(A', B')/2$. □

6.2. **Frameworks.** Throughout the proof, we will consider partitions into sets A and B of equal size (which induce 'near-cliques') as well as 'exceptional sets' A_0 and B_0. The following definition formalizes this. Given a graph G, we say that (G, A, A_0, B, B_0) is an (ε_0, K)-*framework* if the following holds, where $A' := A_0 \cup A$, $B' := B_0 \cup B$ and $n := |V(G)|$:

(FR1) A, A_0, B, B_0 forms a partition of $V(G)$.
(FR2) $e(A', B') \leq \varepsilon_0 n^2$.
(FR3) $|A| = |B|$ is divisible by K, $|A_0| \geq |B_0|$ and $|A_0| + |B_0| \leq \varepsilon_0 n$.
(FR4) If $v \in A$ then $d(v, B') < \varepsilon_0 n$ and if $v \in B$ then $d(v, A') < \varepsilon_0 n$.

We often write V_0 for $A_0 \cup B_0$ and think of the vertices in V_0 as 'exceptional vertices'. Also, whenever (G, A, A_0, B, B_0) is an (ε_0, K)-framework, we will write $A' := A_0 \cup A$, $B' := B_0 \cup B$.

Proposition 6.5. *Let $0 < 1/n \ll \varepsilon_{\rm ex}, 1/K \ll 1$ and $\varepsilon_{\rm ex} \ll \varepsilon_0 \ll 1$. Let G be a graph on n vertices with $\delta(G) = D \geq n - 2\lfloor n/4 \rfloor - 1$ that is $\varepsilon_{\rm ex}$-close to the union of two disjoint copies of $K_{n/2}$. Then there is a partition A, A_0, B, B_0 of $V(G)$ such that (G, A, A_0, B, B_0) is an (ε_0, K)-framework, $d(v, A') \geq d(v)/2$ for all $v \in A'$ and $d(v, B') \geq d(v)/2$ for all $v \in B'$.*

Proof. Write $\varepsilon := \varepsilon_{\rm ex}$. Since G is ε-close to the union of two disjoint copies of $K_{n/2}$, there exists a partition A'', B'' of $V(G)$ such that $|A''| = \lfloor n/2 \rfloor$ and $e(A'', B'') \le \varepsilon n^2$. If there exists a vertex $v \in A''$ such that $d(v, A'') < d(v, B'')$, then we move v to B''. We still denote the vertex classes thus obtained by A'' and B''. Similarly, if there exists a vertex $v \in B''$ such that $d(v, B'') < d(v, A'')$, then we move v to A''. We repeat this process until $d(v, A'') \ge d(v, B'')$ for all $v \in A''$ and $d(v, B'') \ge d(v, A'')$ for all $v \in B''$. Note that this process must terminate since at each step the value of $e(A'', B'')$ decreases. Let A', B' denote the resulting partition. By relabeling the classes if necessary we may assume that $|A'| \ge |B'|$. By construction, $e(A', B') \le e(A'', B'') \le \varepsilon n^2$ and so (FR2) holds. Suppose that $|B'| < (1-5\varepsilon)n/2$. Then at some stage in the process we have that $|B''| = (1-5\varepsilon)n/2$. But then by Proposition 6.1(i),

$$e(A'', B'') \ge (D - |B''| + 1)|B''| > \varepsilon n^2,$$

a contradiction to the definition of ε-closeness (as the number of edges between the partition classes has not increased while moving the vertices). Hence, $|A'| \ge |B'| \ge (1-5\varepsilon)n/2$. Let B'_0 be the set of vertices v in B' such that $d(v, A') \ge \sqrt{\varepsilon} n$. Since $\sqrt{\varepsilon} n |B'_0| \le e(A', B') \le \varepsilon n^2$ we have $|B'_0| \le \sqrt{\varepsilon} n$. Note that

$$|B'| - |B'_0| \ge (1-5\varepsilon)n/2 - \sqrt{\varepsilon} n \ge (1 - 3\sqrt{\varepsilon})n/2. \tag{6.2}$$

Similarly, let A'_0 be the set of vertices v in A' such that $d(v, B') \ge \sqrt{\varepsilon} n$. Thus, $|A'_0| \le \sqrt{\varepsilon} n$ and $|A'| - |A'_0| \ge n/2 - |A'_0| \ge (1 - 2\sqrt{\varepsilon})n/2$. Let m be the largest integer such that $Km \le |A'| - |A'_0|, |B'| - |B'_0|$. Let A and B be Km-subsets of $A' \setminus A'_0$ and $B' \setminus B'_0$ respectively. Set $A_0 := A' \setminus A$ and $B_0 := B' \setminus B$. Note that (6.2) and its analogue for A' together imply that $|A_0| + |B_0| \le 3\sqrt{\varepsilon} n + 2K \le \varepsilon_0 n$. Therefore, (G, A, A_0, B, B_0) is an (ε_0, K)-framework. □

7. Exceptional systems and (K, m, ε_0)-partitions

The definitions and observations in this section will enable us to 'reduce' the problem of finding Hamilton cycles in G to that of finding suitable pairs C_A, C_B of cycles with $V(C_A) = A$ and $V(C_B) = B$. In particular, they will enable us to 'ignore' the exceptional set $V_0 = A_0 \cup B_0$. Roughly speaking, for each Hamilton cycle we seek, we find a certain path system J covering V_0 (called an exceptional system). From this, we derive a set J^* of edges whose endvertices lie in $A \cup B$ by replacing paths of J with 'fictive edges' in a suitable way. We can then work with J^* instead of J when constructing our Hamilton cycles (see Proposition 7.1 and the explanation preceding it).

Suppose that A, A_0, B, B_0 forms a partition of a vertex set V of size n such that $|A| = |B|$. Let $V_0 := A_0 \cup B_0$. An *exceptional cover* J is a graph which satisfies the following properties:

(EC1) J is a path system with $V_0 \subseteq V(J) \subseteq V$.
(EC2) $d_J(v) = 2$ for every $v \in V_0$ and $d_J(v) \le 1$ for every $v \in V(J) \setminus V_0$.
(EC3) $e_J(A), e_J(B) = 0$.

We say that J is an *exceptional system with parameter* ε_0, or an *ES* for short, if J satisfies the following properties:

(ES1) J is an exceptional cover.
(ES2) One of the following is satisfied:
 (HES) The number of AB-paths in J is even and positive. In this case we say J is a *Hamilton exceptional system*, or *HES* for short.
 (MES) $e_J(A', B') = 0$. In this case we say J is a *matching exceptional system*, or *MES* for short.
(ES3) J contains at most $\sqrt{\varepsilon_0}n$ AB-paths.

Note that by definition, every AB-path in J is maximal. So the number of AB-paths in J is the number of genuine 'connections' between A and B (and thus between A' and B'). If we want to extend J into a Hamilton cycle using only edges induced by A and edges induced by B, this number clearly has to be even and positive. Hamilton exceptional systems will always be extended into Hamilton cycles and matching exceptional systems will always be extended into two disjoint even cycles which together span all vertices (and thus consist of two edge-disjoint perfect matchings).

Since each maximal path in J has endpoints in $A \cup B$ and internal vertices in V_0, an exceptional system J naturally induces a matching J^*_{AB} on $A \cup B$. More precisely, if $P_1, \dots, P_{\ell'}$ are the non-trivial paths in J and x_i, y_i are the endpoints of P_i, then we define $J^*_{AB} := \{x_i y_i : i \leq \ell'\}$. Thus $e_{J^*_{AB}}(A, B)$ is equal to the number of AB-paths in J. In particular, if J is a matching exceptional system, then $e_{J^*_{AB}}(A, B) = 0$.

Let $x_1y_1, \dots, x_{2\ell}y_{2\ell}$ be a fixed enumeration of the edges of $J^*_{AB}[A, B]$ with $x_i \in A$ and $y_i \in B$. Define

$$J^*_A := J^*_{AB}[A] \cup \{x_{2i-1}x_{2i} : 1 \leq i \leq \ell\} \quad \text{and} \quad J^*_B := J^*_{AB}[B] \cup \{y_{2i}y_{2i+1} : 1 \leq i \leq \ell\}$$

(with indices considered modulo 2ℓ). Let $J^* := J^*_A + J^*_B$, see Figure 2. Note that J^* is the union of one matching induced by A and another on B, and $e(J^*) = e(J^*_{AB})$. Moreover, by (EC2) we have

$$e(J^*) = e(J^*_{AB}) \leq |V_0| + e_J(A', B') \leq 2\sqrt{\varepsilon_0}n. \tag{7.1}$$

We will call the edges in J^* *fictive* edges. Note that if J_1 and J_2 are two edge-disjoint exceptional systems, then J_1^* and J_2^* may not be edge-disjoint. However, we will always view fictive edges as being distinct from each other and from the edges in other graphs. So in particular, whenever J_1 and J_2 are two exceptional systems, we will view J_1^* and J_2^* as being edge-disjoint.

We say that a path P is *consistent with* J^*_A if P contains J^*_A and (there is an orientation of P which) visits the vertices $x_1, \dots, x_{2\ell}$ in this order. A path P is *consistent with* J^*_B if P contains J^*_B and visits the vertices $y_2, \dots, y_{2\ell}, y_1$ in this order. In a similar way we define when a cycle is consistent with J^*_A or J^*_B.

The next result shows that if J is a Hamilton exceptional system and C_A, C_B are two Hamilton cycles on A and B respectively which are consistent with J^*_A and J^*_B, then graph obtained from $C_A + C_B$ by replacing $J^* = J^*_A + J^*_B$ with J is a Hamilton cycle on V which contains J, see Figure 2. When choosing our Hamilton cycles, this property will enable us ignore all the vertices in V_0 and to consider the (almost complete) graphs induced by A and by B instead. Similarly, if J is a matching exceptional system and both $|A'|$ and $|B'|$ are even, then the graph obtained from

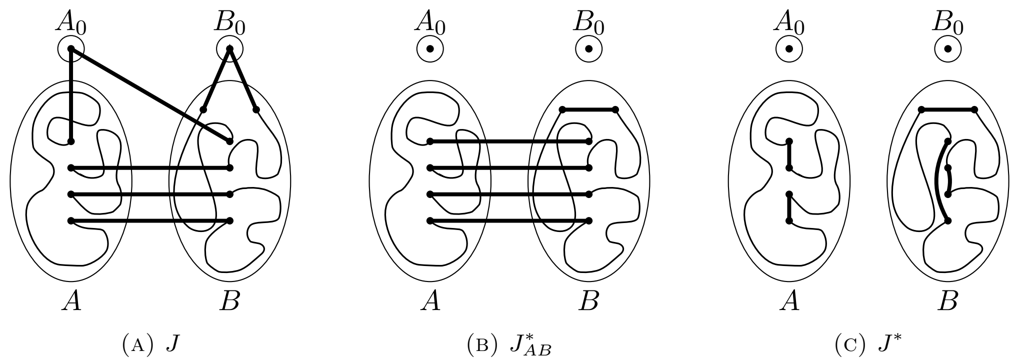

FIGURE 2. The thick lines illustrate the edges of J, J^*_{AB} and J^* respectively.

$C_A + C_B$ by replacing J^* with J is the edge-disjoint union of two perfect matchings on V.

Proposition 7.1. *Suppose that A, A_0, B, B_0 forms a partition of a vertex set V. Let J be an exceptional system. Let C_A and C_B be two cycles such that*

- *C_A is a Hamilton cycle on A that is consistent with J^*_A;*
- *C_B is a Hamilton cycle on B that is consistent with J^*_B.*

Then the following assertions hold.

(i) *If J is a Hamilton exceptional system, then $C_A + C_B - J^* + J$ is a Hamilton cycle on V.*
(ii) *If J is a matching exceptional system, then $C_A + C_B - J^* + J$ is the union of a Hamilton cycle on A' and a Hamilton cycle on B'. In particular, if both $|A'|$ and $|B'|$ are even, then $C_A + C_B - J^* + J$ is the union of two edge-disjoint perfect matchings on V.*

Proof. Suppose that J is a Hamilton exceptional system. Let $x_1y_1, \dots, x_{2\ell}y_{2\ell}$ be an enumeration of the edges of $J^*_{AB}[A, B]$ with $x_i \in A$ and $y_i \in B$ and such that $J^*_A = J^*_{AB}[A] \cup \{x_{2i-1}x_{2i} : 1 \le i \le \ell\}$ and $J^*_B = J^*_{AB}[B] \cup \{y_{2i}y_{2i+1} : 1 \le i \le \ell\}$. Let $P^A_1, \dots, P^A_\ell$ be the paths in $C_A - \{x_{2i-1}x_{2i} : 1 \le i \le \ell\}$. Since C_A is consistent with J^*_A, we may assume that P^A_i is a path from x_{2i-2} to x_{2i-1} for all $i \le \ell$. Similarly, let $P^B_1, \dots, P^B_\ell$ be the paths in $C_B - \{y_{2i}y_{2i+1} : 1 \le i \le \ell\}$. Again, we may assume that P^B_i is a path from y_{2i-1} to y_{2i} for all $i \le \ell$. Define C^* to be the 2-regular graph on $A \cup B$ obtained from concatenating $P^A_1, x_1y_1, P^B_1, y_2x_2, P^A_2, x_3y_3, \dots, P^B_\ell$ and $y_{2\ell}x_{2\ell}$. Together with (HES), the construction implies that C^* is a Hamilton cycle on $A \cup B$ and $C^* = C_A + C_B - J^* + J^*_{AB}$. Thus $C := C^* - J^*_{AB} + J$ is a Hamilton cycle on V. Since $C = C_A + C_B - J^* + J$, (i) holds.

The proof of (ii) is similar to that of (i). Indeed, the previous argument shows that C^* is the union of a Hamilton cycle on A and a Hamilton cycle on B. (MES) now implies that C is the union of a Hamilton cycle on A' and one on B'. $\square$

In general, we construct an exceptional system by first choosing an exceptional system candidate (defined below) and then extending it to an exceptional system. More precisely, suppose that A, A_0, B, B_0 forms a partition of a vertex set V. Let $V_0 := A_0 \cup B_0$. A graph F is called an *exceptional system candidate with parameter* ε_0, or an *ESC* for short, if F satisfies the following properties:

(ESC1) F is a path system with $V_0 \subseteq V(F) \subseteq V$ and such that $e_F(A), e_F(B) = 0$.
(ESC2) $d_F(v) \leq 2$ for all $v \in V_0$ and $d_F(v) = 1$ for all $v \in V(F) \setminus V_0$.
(ESC3) $e_F(A', B') \leq \sqrt{\varepsilon_0}n/2$. In particular, $|V(F) \cap A|, |V(F) \cap B| \leq 2|V_0| + \sqrt{\varepsilon_0}n/2$.
(ESC4) One of the following holds:
- (HESC) Let $b(F)$ be the number of maximal paths in F with one endpoint in A' and the other in B'. Then $b(F)$ is even and $b(F) > 0$. In this case we say that F is a *Hamilton exceptional system candidate*, or *HESC* for short.
- (MESC) $e_F(A', B') = 0$. In this case, F is called a *matching exceptional system candidate* or *MESC* for short.

Note that if $d_F(v) = 2$ for all $v \in V_0$, then F is an exceptional system. Also, if F is a Hamilton exceptional system candidate with $e(F) = 2$, then F consists of two independent $A'B'$-edges. Moreover, note that (EC2) allows an exceptional cover J (and so also an exceptional system J) to contain vertices in $A \cup B$ which are isolated in J. However, (ESC2) does not allow for this in an exceptional system candidate F.

Similarly to condition (HES), in (HESC) the parameter $b(F)$ counts the number of 'connections' between A' and B'. In order to extend a Hamilton exceptional system candidate into a Hamilton cycle without using any additional $A'B'$-edges, it is clearly necessary that $b(F)$ is positive and even.

The next result shows that we can extend an exceptional system candidate into an exceptional system by adding suitable A_0A- and B_0B-edges. In the proof of Lemma 10.1 we will use that if G is a D-regular graph with $D \geq n/100$ (say) and (G, A, A_0, B, B_0) is an (ε_0, K)-framework with $\Delta(G[A', B']) \leq D/2$, then conditions (i) and (ii) below are satisfied.

Lemma 7.2. *Suppose that* $0 < 1/n \ll \varepsilon_0 \ll 1$ *and that* $n \in \mathbb{N}$. *Let* G *be a graph on* n *vertices so that*

(i) A, A_0, B, B_0 *forms a partition of* $V(G)$ *with* $|A_0 \cup B_0| \leq \varepsilon_0 n$.
(ii) $d(v, A) \geq \sqrt{\varepsilon_0}n$ *for all* $v \in A_0$ *and* $d(v, B) \geq \sqrt{\varepsilon_0}n$ *for all* $v \in B_0$.

Let F *be an exceptional system candidate with parameter* ε_0. *Then there exists an exceptional system* J *with parameter* ε_0 *such that* $F \subseteq J \subseteq G + F$ *and such that every edge of* $J - F$ *lies in* $G[A_0, A] + G[B_0, B]$. *Moreover, if* F *is a Hamilton exceptional system candidate, then* J *is a Hamilton exceptional system. Otherwise* J *is a matching exceptional system.*

Proof. For each vertex $v \in A_0$, we select $2 - d_F(v)$ edges uv in G with $u \in A \setminus V(F)$. Since $d_G(v, A) \geq \sqrt{\varepsilon_0}n \geq |V(F) \cap A| + 2|V_0|$ by (ESC3), these edges can be chosen such that they have no common endpoint in A. Similarly, for each vertex $v \in B_0$, we select $2 - d_F(v)$ edges uv in G with $u \in B \setminus V(F)$. Again, these edges are chosen such that they have no common endpoint in B. Let J be the graph obtained from F

by adding all these edges. Note that J is an exceptional cover such that every edge of $J-F$ lies in $G[A_0,A]+G[B_0,B]$. Furthermore, the number of AB-paths in J is at most $e_F(A',B')\le\sqrt{\varepsilon_0}n/2$.

Suppose F is a Hamilton exceptional system candidate with parameter ε_0. Our construction of J implies that the number of AB-paths in J equals $b(F)$. So (HES) follows from (HESC). Now suppose F is a matching exceptional system candidate. Then (MES) is satisfied since $e_J(A',B')=e_F(A',B')=0$ by (MESC). This proves the lemma. □

Let $K,m\in\mathbb{N}$ and $\varepsilon_0>0$. A (K,m,ε_0)*-partition* $\mathcal{P}$ of a set V of vertices is a partition of V into sets $A_0,A_1,\dots,A_K$ and $B_0,B_1,\dots,B_K$ such that $|A_i|=|B_i|=m$ for all $i\ge 1$ and $|A_0\cup B_0|\le\varepsilon_0|V|$. The sets $A_1,\dots,A_K$ and $B_1,\dots,B_K$ are called *clusters* of $\mathcal{P}$ and A_0, B_0 are called *exceptional sets*. We often write V_0 for $A_0\cup B_0$ and think of the vertices in V_0 as 'exceptional vertices'. Unless stated otherwise, whenever $\mathcal{P}$ is a (K,m,ε_0)-partition, we will denote the clusters by $A_1,\dots,A_K$ and $B_1,\dots,B_K$ and the exceptional sets by A_0 and B_0. We will also write $A:=A_1\cup\dots\cup A_K$, $B:=B_1\cup\dots\cup B_K$, $A':=A_0\cup A_1\cup\dots\cup A_K$ and $B':=B_0\cup B_1\cup\dots\cup B_K$.

Given a (K,m,ε_0)-partition $\mathcal{P}$ and $1\le i,i'\le K$, we say that J is an (i,i')*-localized Hamilton exceptional system* (abbreviated as (i,i')*-HES*) if J is a Hamilton exceptional system and $V(J)\subseteq V_0\cup A_i\cup B_{i'}$. In a similar way, we define

- (i,i')*-localized matching exceptional systems* ((i,i')*-MES*),
- (i,i')*-localized exceptional systems* ((i,i')*-ES*),
- (i,i')*-localized Hamilton exceptional system candidates* ((i,i')*-HESC*),
- (i,i')*-localized matching exceptional system candidates* ((i,i')*-MESC*),
- (i,i')*-localized exceptional system candidates* ((i,i')*-ESC*).

To make clear with which partition we are working, we sometimes also say that J is an (i,i')-localized Hamilton exceptional system with respect to $\mathcal{P}$ etc.

8. Schemes and exceptional schemes

It will often be convenient to consider the 'exceptional' and 'non-exceptional' part of a graph G separately. For this, we introduce a 'scheme' (which corresponds to the non-exceptional part and also incorporates a refined partition of G) and an 'exceptional scheme' (which corresponds to the exceptional part and also incorporates a refined partition of G).

Given a graph G and a partition $\mathcal{P}$ of a vertex set V, we call $(G,\mathcal{P})$ a $(K,m,\varepsilon_0,\varepsilon)$-*scheme* if the following properties hold:

(Sch1) $\mathcal{P}$ is a (K,m,ε_0)-partition of V.
(Sch2) $V(G)=A\cup B$ and $e_G(A,B)=0$.
(Sch3) For all $1\le i\le K$ and all $v\in A$ we have $d(v,A_i)\ge(1-\varepsilon)m$. Similarly, for all $1\le i\le K$ and all $v\in B$ we have $d(v,B_i)\ge(1-\varepsilon)m$.

The next proposition shows that if $(G,\mathcal{P})$ is a scheme and G' is obtained from G by removing a small number of edges at each vertex, then $(G',\mathcal{P})$ is also a scheme with slightly worse parameters. Its proof is immediate from the definition of a scheme.

Proposition 8.1. *Suppose that $0 < 1/m \ll \varepsilon, \varepsilon' \ll 1$ and that $K, m \in \mathbb{N}$. Let $(G, \mathcal{P})$ be a $(K, m, \varepsilon_0, \varepsilon)$-scheme. Let G' be a spanning subgraph of G such that $\Delta(G - G') \leq \varepsilon' m$. Then $(G', \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon + \varepsilon')$-scheme.*

Given a graph G on n vertices and a partition $\mathcal{P}$ of $V(G)$ we call $(G, \mathcal{P})$ a $(K, m, \varepsilon_0, \varepsilon)$-*exceptional scheme* if the following properties are satisfied:

(ESch1) $\mathcal{P}$ is a (K, m, ε_0)-partition of $V(G)$.
(ESch2) $e(A), e(B) = 0$.
(ESch3) If $v \in A$ then $d(v, B') < \varepsilon_0 n$ and if $v \in B$ then $d(v, A') < \varepsilon_0 n$.
(ESch4) For all $v \in V(G)$ and all $1 \leq i \leq K$ we have $d(v, A_i) = (d(v, A) \pm \varepsilon n)/K$ and $d(v, B_i) = (d(v, B) \pm \varepsilon n)/K$.
(ESch5) For all $1 \leq i, i' \leq K$ we have

$$\begin{aligned}
e(A_0, A_i) &= (e(A_0, A) \pm \varepsilon \max\{e(A_0, A), n\})/K, \\
e(B_0, A_i) &= (e(B_0, A) \pm \varepsilon \max\{e(B_0, A), n\})/K, \\
e(A_0, B_i) &= (e(A_0, B) \pm \varepsilon \max\{e(A_0, B), n\})/K, \\
e(B_0, B_i) &= (e(B_0, B) \pm \varepsilon \max\{e(B_0, B), n\})/K, \\
e(A_i, B_{i'}) &= (e(A, B) \pm \varepsilon \max\{e(A, B), n\})/K^2.
\end{aligned}$$

The aim of this section is to show that every framework can be 'split' into a scheme and an exceptional scheme (see Lemma 8.3). In order to do this, we need the following lemma, which is a special case of Lemma 5.1 in [6]. The proof proceeds by considering a suitable random partition of $V(G)$.

Lemma 8.2. *Suppose that $0 < 1/n \ll \varepsilon, \varepsilon_1 \ll \varepsilon_2 \ll 1/K \ll 1$, that $r \leq 2K$, that $Km \geq n/4$ and that $r, K, n, m \in \mathbb{N}$. Let G be a graph on n vertices. Suppose that there is a vertex partition of $V(G)$ into $U, R_1, \dots, R_r$ with the following properties:*

- *$|U| = Km$.*
- *$\delta(G[U]) \geq \varepsilon n$.*
- *For each $j \leq r$ we either have $d_G(u, R_j) \leq \varepsilon n$ for all $u \in U$ or $d_G(x, U) \geq \varepsilon n$ for all $x \in R_j$.*

Then there exists a partition of U into K parts $U_1, \dots, U_K$ satisfying the following properties:

(i) *$|U_i| = m$ for all $i \leq K$.*
(ii) *$d_G(v, U_i) = (d_G(v, U) \pm \varepsilon_1 n)/K$ for all $v \in V(G)$ and all $i \leq K$.*
(iii) *$e_G(U_i, U_{i'}) = 2(e_G(U) \pm \varepsilon_2 \max\{n, e_G(U)\})/K^2$ for all $1 \leq i \neq i' \leq K$.*
(iv) *$e_G(U_i) = (e_G(U) \pm \varepsilon_2 \max\{n, e_G(U)\})/K^2$ for all $i \leq K$.*
(v) *$e_G(U_i, R_j) = (e_G(U, R_j) \pm \varepsilon_2 \max\{n, e_G(U, R_j)\})/K$ for all $i \leq K$ and $j \leq r$.*

Suppose that (G, A, A_0, B, B_0) is an (ε_0, K)-framework. The next lemma shows that there is a refinement of the vertex partition A, A_0, B, B_0 of $V(G)$ into a (K, m, ε_0)-partition $\mathcal{P}$ such that $(G[A] + G[B], \mathcal{P})$ is a scheme and $(G - G[A] - G[B], \mathcal{P})$ is an exceptional scheme.

Lemma 8.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll 1$, that $\varepsilon_0 \ll \varepsilon_1 \leq \varepsilon_2 \ll 1$, that $1/n \ll \mu \ll \varepsilon_2$ and that $n, K, m \in \mathbb{N}$. Let G be a graph on n vertices such that*

$\delta(G) \geq (1-\mu)n/2$. Let (G, A, A_0, B, B_0) be an (ε_0, K)-framework with $|A| = |B| = Km$. Then there are partitions $A_1, \dots, A_K$ of A and $B_1, \dots, B_K$ of B which satisfy the following properties:

(i) *The partition $\mathcal{P}$ formed by A_0, B_0 and all these $2K$ clusters is a (K, m, ε_0)-partition of $V(G)$.*
(ii) *$(G[A] + G[B], \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon_2)$-scheme.*
(iii) *$(G - G[A] - G[B], \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon_1)$-exceptional scheme.*
(iv) *For all $v \in V(G)$ and all $1 \leq i \leq K$ we have $d_G(v, A_i) = (d_G(v, A) \pm \varepsilon_0 n)/K$ and $d_G(v, B_i) = (d_G(v, B) \pm \varepsilon_0 n)/K$.*

Proof. Define a new constant ε_1' such that $\varepsilon_0 \ll \varepsilon_1' \ll \varepsilon_1, 1/K$. In order to find the required partitions $A_1, \dots, A_K$ of A and $B_1, \dots, B_K$ of B we will apply Lemma 8.2 twice, as follows.

In our first application of Lemma 8.2 we let $U := A$ and let A_0, B_0, B play the roles of R_1, R_2, R_3. Note that $\delta(G[A]) \geq \delta(G) - |A_0| - \varepsilon_0 n \geq \varepsilon_0 n$ (with room to spare) by (FR3), (FR4) and that $d(a, R_j) \leq |R_j| \leq \varepsilon_0 n$ for all $a \in A$ and $j = 1, 2$ by (FR3). Moreover, (FR4) implies that $d(a, R_3) \leq d(a, B') \leq \varepsilon_0 n$ for all $a \in A$. Thus we can apply Lemma 8.2 with $\varepsilon_0, \varepsilon_0$ and ε_1' playing the roles of $\varepsilon, \varepsilon_1$ and ε_2 to obtain a partition of A into K clusters $A_1, \dots, A_K$, each of size m. Then by Lemma 8.2(ii) for all $v \in V(G)$ and all $1 \leq i \leq K$ we have

$$d_G(v, A_i) = (d_G(v, A) \pm \varepsilon_0 n)/K. \tag{8.1}$$

Moreover, Lemma 8.2(v) implies that the first two equalities in (ESch5) hold with respect to ε_1' (for G and thus also for $G - G[A] - G[B]$). Furthermore,

$$e_G(A_i, B) = (e_G(A, B) \pm \varepsilon_1' \max\{n, e_G(A, B)\})/K. \tag{8.2}$$

For the second application of Lemma 8.2 we let $U := B$ and let $B_0, A_0, A_1, \dots, A_K$ play the roles of $R_1, \dots, R_{K+2}$. As before, $\delta(G[B]) \geq \varepsilon_0 n$ by (FR3), (FR4) and $d(b, R_j) \leq |R_j| \leq \varepsilon_0 n$ for all $b \in B$ and $j = 1, 2$ by (FR3). Moreover, (FR4) implies that $d(b, R_j) \leq d(b, A') \leq \varepsilon_0 n$ for all $b \in B$ and all $j = 3, \dots, K+2$. Thus we can apply Lemma 8.2 with $\varepsilon_0, \varepsilon_0$ and ε_1' playing the roles of $\varepsilon, \varepsilon_1$ and ε_2 to obtain a partition of B into K clusters $B_1, \dots, B_K$, each of size m. Similarly as before one can show that for all $v \in V(G)$ and all $1 \leq i \leq K$ we have

$$d_G(v, B_i) = (d_G(v, B) \pm \varepsilon_0 n)/K, \tag{8.3}$$

and that the third and the fourth equalities in (ESch5) hold with respect to ε_1' (for G and thus also for $G - G[A] - G[B]$). Moreover, Lemma 8.2(v) implies that for all $1 \leq i' \leq K$ we have

$$\begin{aligned} e_G(A_i, B_{i'}) &= (e_G(A_i, B) \pm \varepsilon_1' \max\{n, e_G(A_i, B)\})/K \\ &\overset{(8.2)}{=} (e_G(A, B) \pm \varepsilon_1' \max\{n, e_G(A, B)\} \pm K\varepsilon_1' \max\{n, e_G(A_i, B)\})/K^2 \\ &= (e_G(A, B) \pm \varepsilon_1 \max\{n, e_G(A, B)\})/K^2, \end{aligned}$$

i.e. the last equality in (ESch5) holds too. Let $\mathcal{P}$ be the partition formed by $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$. Then (i) holds.

Let us now verify (ii). Clearly $(G[A]+G[B],\mathcal{P})$ satisfies (Sch1) and (Sch2). In order to check (Sch3), let $G_1 := G[A]+G[B]$ and note that for all $v\in A$ and all $1\le i\le K$ we have

$$\begin{aligned} d_{G_1}(v,A_i) = d_G(v,A_i) &\overset{(8.1)}{\ge} (d_G(v,A)-\varepsilon_0 n)/K \overset{(\mathrm{FR4})}{\ge} (\delta(G)-|A_0|-2\varepsilon_0 n)/K \\ &\overset{(\mathrm{FR3})}{\ge} ((1-\mu)n/2-3\varepsilon_0 n)/K \ge (1-\varepsilon_2)m. \end{aligned}$$

Similarly one can use (8.3) to show that $d_{G_1}(v,B_i)\ge(1-\varepsilon_2)m$ for all $v\in B$ and all $1\le i\le K$. This implies (Sch3) and thus (ii).

Note that (iv) follows from (8.1) and (8.3). Thus it remains to check (iii). Clearly $(G-G[A]-G[B],\mathcal{P})$ satisfies (ESch1), (ESch2) and we have already verified (ESch5). (ESch3) follows from (FR4) and (ESch4) follows from (8.1) and (8.3). □

9. Proof of Theorem 3.9

The key tool in the proof of Theorem 3.9 is Lemma 9.4, which guarantees an 'approximate' Hamilton decomposition of a graph G, provided that G is close to the union of two disjoint copies of $K_{n/2}$. This yields the required number of Hamilton cycles for Theorem 3.9. As an 'input', Lemma 9.4 requires an appropriate number of localized Hamilton exceptional systems.

To find these, we proceed as follows: the next lemma (Lemma 9.1) guarantees many edge-disjoint Hamilton exceptional systems in a given framework. We will apply it to 'localized subgraphs' (obtained from Lemma 9.2) of the original graph to ensure that the exceptional systems guaranteed by Lemma 9.1 are also localized. These can then be used as the required input for Lemma 9.4.

Lemma 9.1. *Suppose that $0<1/n\ll\varepsilon_0\ll\varepsilon\ll\alpha\ll 1$ and that $n,\alpha n\in\mathbb{N}$. Let G be a graph on n vertices. Suppose that (G,A,A_0,B,B_0) is an (ε_0,K)-framework which satisfies the following conditions:*

(a) $e_G(A',B')\ge 2(\alpha+\varepsilon)n$.
(b) $e_{G-v}(A',B')\ge\alpha n$ *for all* $v\in A_0\cup B_0$.
(c) $d(v)\ge 2(\alpha+\varepsilon)n$ *for all* $v\in A_0\cup B_0$.
(d) $d(v,A')\ge d(v,B')-\varepsilon n$ *for all* $v\in A_0$ *and* $d(v,B')\ge d(v,A')-\varepsilon n$ *for all* $v\in B_0$.

Then there exist αn edge-disjoint Hamilton exceptional systems with parameter ε_0 in G.

Proof. First we will find αn edge-disjoint matchings of size 2 in $G[A',B']$. If $\Delta(G[A',B'])\le(\alpha+\varepsilon/2)n$, then by (a) and Proposition 5.2 we can find such matchings. So suppose that $\Delta(G[A',B'])\ge(\alpha+\varepsilon/2)n$ and let v be a vertex such that $d_{G[A',B']}(v)\ge(\alpha+\varepsilon/2)n$. Thus $v\in A_0\cup B_0$ by (FR4). By (b) there are αn edges $e_1,\dots,e_{\alpha n}$ in $G[A',B']-v$. Since $d_{G[A',B']}(v)\ge(\alpha+\varepsilon/2)n$, for each e_s in turn we can find an edge e'_s incident to v in $G[A',B']$ such that e'_s is vertex-disjoint from e_s and such that the e'_s are distinct for different indices $s\le\alpha n$. Then the matchings consisting of e_s and e'_s are as required. Thus in both cases we can find edge-disjoint matchings $M_1, \dots, M_{\alpha n}$ of size 2 in $G[A', B']$.

Our aim is to extend each M_s into a Hamilton exceptional system J_s such that all these J_s are pairwise edge-disjoint. Initially, we set $F_s := M_s$ for all $s \leq \alpha n$. So each F_s is a Hamilton exceptional system candidate. For each $v \in V_0$ in turn, we are going to assign at most two edges joining v to $A \cup B$ to each of $F_1, \dots, F_{\alpha n}$ in such a way that now each F_s is a Hamilton exceptional system candidate with $d_{F_s}(v) = 2$. Thus after we have carried out these assignments for all $v \in V_0$, every F_s will be a Hamilton exceptional system with parameter ε_0.

So consider any $v \in V_0$. Without loss of generality we may assume that $v \in A_0$. Moreover, by relabelling the F_s if necessary, we may assume that there exists an integer $0 \leq r \leq \alpha n$ such that $d_{F_s}(v) = 1$ for all $s \leq r$ and $d_{F_s}(v) = 0$ for $r < s \leq \alpha n$. For each $s \leq r$ our aim is to assign some edge vw_s between v and A to F_s such that $w_s \notin V(F_s)$ and such that the vertices w_s are distinct for different $s \leq r$. To check that such an assignment of edges is possible, note that $|V(F_s) \cap A|, |V(F_s) \cap B| \leq 2|V_0| + 2 \leq 3\varepsilon_0 n$. Together with (c) and (d) this implies that

$$d(v, A) \geq d(v, A') - |A_0| \geq (\alpha + \varepsilon/2 - \varepsilon_0)n > r + |V(F_s) \cap A|.$$

Thus for all $s \leq r$ we can assign an edge vw_s to F_s as required.

It remains to assign two edges at v to each of $F_{r+1}, \dots, F_{\alpha n}$. We will do this for each $s = r+1, \dots, \alpha n$ in turn and for each such s we will either assign two edges between v and A to F_s or two edges between v and B. (This will ensure that we still have $b(F_s) = 2$, where $b(F_s)$ is the number of vertex-disjoint $A'B'$-paths in the path system F_s.) So suppose that for some $r < s \leq \alpha n$ we have already assigned two edges at v to each of $F_{r+1}, \dots, F_{s-1}$. Set $G_s := G - \sum_{s'=1}^{\alpha n} F_{s'}$. The fact that v has degree at most two in each $F_{s'}$ and (c) together imply that $d_{G_s}(v) \geq d_G(v) - 2\alpha n \geq 10\varepsilon_0 n$. So either $d_{G_s}(v, A') \geq 5\varepsilon_0 n$ or $d_{G_s}(v, B') \geq 5\varepsilon_0 n$. If the former holds then

$$d_{G_s}(v, A) \geq d_{G_s}(v, A') - |A_0| \geq 4\varepsilon_0 n \geq |V(F_s) \cap A| + 2$$

and so we can assign two edges vw and vw' of G_s to F_s such that $w, w' \in A \setminus V(F_s)$. Similarly if $d_{G_s}(v, B') \geq 5\varepsilon_0 n$ then we can assign two edges vw and vw' in G_s to F_s such that $w, w' \in B \setminus V(F_s)$. This shows that to each of $F_{r+1}, \dots, F_{\alpha n}$ we can assign two suitable edges at v.

Let $J_1, \dots, J_{\alpha n}$ be the graphs obtained after carrying out these assignments for all $v \in V_0$. Then the J_s are pairwise edge-disjoint and it is easy to check that each J_s is a Hamilton exceptional system with parameter ε_0. (Note that (ES2) and (ES3) hold since $b(J_s) = 2$ and so the number of AB-paths is two.) $\square$

The next lemma guarantees a decomposition of an exceptional scheme $(G, \mathcal{P})$ into suitable 'localized slices' $G(i, i')$ whose edges are induced by A_0, B_0 and two clusters of $\mathcal{P}$. We will use it again in [19].

Lemma 9.2. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll 1/K \ll 1$ and that $n, K, m \in \mathbb{N}$. Let $(G, \mathcal{P})$ be a $(K, m, \varepsilon_0, \varepsilon)$-exceptional scheme with $|G| = n$ and $e_G(A_0), e_G(B_0) = 0$. Then G can be decomposed into edge-disjoint spanning subgraphs $H(i, i')$ and $H'(i, i')$*

of G (for all $i, i' \le K$) such that the following properties hold, where $G(i,i') := H(i,i') + H'(i,i')$:

- (a_1) *Each $H(i,i')$ contains only A_0A_i-edges and $B_0B_{i'}$-edges.*
- (a_2) *All edges of $H'(i,i')$ lie in $G[A_0 \cup A_i, B_0 \cup B_{i'}]$.*
- (a_3) *$e(H'(i,i')) = (e_G(A',B') \pm 4\varepsilon \max\{n, e_G(A',B')\})/K^2$.*
- (a_4) *$d_{H'(i,i')}(v) = (d_{G[A',B']}(v) \pm 2\varepsilon n)/K^2$ for all $v \in V_0$.*
- (a_5) *$d_{G(i,i')}(v) = (d_G(v) \pm 4\varepsilon n)/K^2$ for all $v \in V_0$.*

Proof. First we decompose G into K^2 'random' edge-disjoint spanning subgraphs $G(i,i')$ (one for all $i, i' \le K$) as follows:

- Initially set $V(G(i,i')) := V(G)$ and $E(G(i,i')) := \emptyset$ for all $i, i' \le K$.
- Add all the $A_iB_{i'}$-edges of G to $G(i,i')$.
- Choose a partition of $E(A_0, B_0)$ into K^2 sets $U_{i,i'}$ (one for all $i, i' \le K$) whose sizes are as equal as possible. Add the edges in $U_{i,i'}$ to $G(i,i')$.
- For all $i \le K$, choose a random partition of $E(A_0, A_i)$ into K sets $U'_{i'}$ of equal size (one for each $i' \le K$) and add the edges in $U'_{i'}$ to $G(i,i')$. (If $e(A_0, A_i)$ is not divisible by K, first distribute up to $K-1$ edges arbitrarily among the $U'_{i'}$ to achieve divisibility.) For all $i' \le K$ proceed similarly to distribute each edge in $E(B_0, B_{i'})$ to $G(i,i')$ for some $i \le K$.
- For all $i' \le K$, choose a random partition of $E(A_0, B_{i'})$ into K sets U''_i of equal size (one for each $i \le K$) and add the edges in U''_i to $G(i,i')$. (If $e(A_0, B_{i'})$ is not divisible by K, first distribute up to $K-1$ edges arbitrarily among the U''_i to achieve divisibility.) For all $i \le K$ proceed similarly to distribute each edge in $E(B_0, A_i)$ to $G(i,i')$ for some $i' \le K$.

Thus every edge of G is added to precisely one of the subgraphs $G(i,i')$. Set $H(i,i') := G(i,i')[A'] + G(i,i')[B']$ and $H'(i,i') := G(i,i')[A',B']$. So conditions ($a_1$) and ($a_2$) hold. Fix any $i, i' \le K$ and set $H := H(i,i')$ and $H' := H'(i,i')$. To verify (a_3), note that

$$\begin{aligned}
e(H') &= e_{H'}(A_i, B_{i'}) + e_{H'}(A_0, B_0) + e_{H'}(A_0, B_{i'}) + e_{H'}(B_0, A_i) \\
&= e_G(A_i, B_{i'}) + e_G(A_0, B_0)/K^2 + e_G(A_0, B_{i'})/K + e_G(B_0, A_i)/K \pm 3 \\
&= \frac{e_G(A,B) + e_G(A_0,B_0) + e_G(A_0,B) + e_G(B_0,A) \pm 3\varepsilon \max\{e_G(A',B'), n\}}{K^2} \pm 3 \\
&= \frac{e_G(A',B') \pm 4\varepsilon \max\{e_G(A',B'), n\}}{K^2}.
\end{aligned}$$

Here the third equality follows from (ESch5).

To prove (a_4), suppose first that $v \in A_0$. If $d_G(v, B_{i'}) \le \varepsilon n/K^2$ then clearly $0 \le d_{H'(i,i')}(v) \le \varepsilon n/K^2 + |V_0| \le 2\varepsilon n/K^2$. Further by (ESch4) we have $d_G(v,B) \le K d_G(v, B_{i'}) + \varepsilon n = \varepsilon n/K + \varepsilon n$. So $d_G(v, B') \le 2\varepsilon n$. Together this shows that (a_4) is satisfied.

So assume that $d_G(v, B_{i'}) \ge \varepsilon n/K^2$. Proposition 5.1 this implies that with probability at least $1 - \mathrm{e}^{-\sqrt{n}}$ (with room to spare) we have

$$d_{G(i,i')}(v, B_{i'}) = (d_G(v, B_{i'}) \pm \varepsilon n/2K)/K \overset{\text{(ESch4)}}{=} (d_G(v,B) \pm 3\varepsilon n/2)/K^2. \tag{9.1}$$

Since

$$\begin{aligned} d_{H'(i,i')}(v) &= d_{G(i,i')}(v, B_{i'}) + d_{G(i,i')}(v, B_0) = d_{G(i,i')}(v, B_{i'}) \pm \varepsilon_0 n \\ &\overset{(9.1)}{=} (d_G(v, B') \pm 2\varepsilon n)/K^2, \end{aligned}$$

it follows that v satisfies (a$_4$). The argument for the case when $v \in B_0$ is similar. Thus (a$_4$) holds with probability at least $1 - n\mathrm{e}^{-\sqrt{n}}$.

Similarly as (9.1) one can show that with probability at least $1 - n\mathrm{e}^{-\sqrt{n}}$ we have $d_{G(i,i')}(v, A_i) = (d_G(v, A) \pm 3\varepsilon n/2)/K^2$ for all $v \in A_0$ and $d_{G(i,i')}(v, B_{i'}) = (d_G(v, B) \pm 3\varepsilon n/2)/K^2$ for all $v \in B_0$. Together with the fact that $e_G(A_0), e_G(B_0) = 0$ and (a$_4$) this now implies (a$_5$). □

The next lemma first applies the previous one to construct localized subgraphs $G(i, i')$ and then applies Lemma 9.1 to find many Hamilton exceptional systems within each of the localized slices $G(i, i')$. Altogether, this yields many localized Hamilton exceptional systems in G.

Lemma 9.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \phi, 1/K \ll 1$ and that $n, K, m, (1/4-\phi)n/K^2 \in \mathbb{N}$. Suppose that (G, A, A_0, B, B_0) is an (ε_0, K)-framework with $|G| = n$, $\delta(G) \geq n/2$ and such that $d_G(v, A') \geq d_G(v)/2$ for all $v \in A'$ and $d_G(v, B') \geq d_G(v)/2$ for all $v \in B'$. Suppose that $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ is a refinement of the partition A, A_0, B, B_0 such that $(G-G[A]-G[B], \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon)$-exceptional scheme. Then there is a set $\mathcal{J}$ of $(1/4 - \phi)n$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in G such that, for each $i, i' \leq K$, $\mathcal{J}$ contains precisely $(1/4 - \phi)n/K^2$ (i, i')-HES.*

Proof. Let $\alpha := (1/4 - \phi)/K^2$ and choose a new constant ε' such that $\varepsilon \ll \varepsilon' \ll \phi, 1/K$. Note that (FR3) implies that $|A'| \geq |B'|$. If $|B'| < n/2$, then Proposition 6.1(i) implies that $e_G(A', B') \geq 2|B'| \geq (1 - \varepsilon_0)n \geq 3K^2\alpha n$ (where the second inequality follows from (FR3) and there is room to spare in the final inequality). Since $d_{G[A',B']}(v) \leq n/2$ for every vertex $v \in V(G)$, it follows that $e_{G-v}(A', B') \geq (1/2 - \varepsilon_0)n \geq 3K^2\alpha n/2$. If $|B'| = n/2$, then $|A'| = |B'|$ and Proposition 6.1(i) implies that $e_G(A', B') \geq |B'| = n/2 \geq 2K^2(\alpha + \varepsilon')n$. Moreover, $|A'| = |B'|$ together with the fact that $\delta(G) \geq n/2$ also implies that $d_{G[A',B']}(v) \geq 1$ for any vertex $v \in V(G)$. Hence $e_{G-v}(A', B') \geq n/2 - 1 \geq 3K^2\alpha n/2$. Thus regardless of the size of B', we always have

$$e_G(A', B') \geq 2K^2(\alpha + \varepsilon')n \tag{9.2}$$

and

$$e_{G-v}(A', B') \geq 3K^2\alpha n/2 \geq K^2(\alpha + \varepsilon')n \quad \text{for any } v \in V(G). \tag{9.3}$$

Set $G^\diamond := G - G[A] - G[B] - G[A_0] - G[B_0]$. Note that each vertex $v \in V_0$ satisfies

$$d_{G^\diamond}(v) \geq (1/2 - \varepsilon_0)n \geq 2K^2(\alpha + \varepsilon')n. \tag{9.4}$$

Moreover, both (9.2) and (9.3) also hold for $G^\diamond$, and since $(G - G[A] - G[B], \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon)$-exceptional scheme, $(G^\diamond, \mathcal{P})$ is also a $(K, m, \varepsilon_0, \varepsilon)$-exceptional scheme. Thus we can apply Lemma 9.2 to $G^\diamond$ to obtain edge-disjoint spanning subgraphs

$H(i,i')$, $H'(i,i')$ of $G^\diamond$ (for all $i,i' \le K$) which satisfy (a$_1$)–(a$_5$) of Lemma 9.2. Set $G(i,i') := H(i,i') + H'(i,i')$ for all $i,i' \le K$. We claim that each $G(i,i')$ satisfies the following properties:

(i) All edges of $G(i,i')$ lie in $G^\diamond[A_0 \cup A_i \cup B_0 \cup B_{i'}]$.
(ii) $e_{G(i,i')}(A', B') \ge 2(\alpha + \sqrt{\varepsilon})n$.
(iii) $e_{G(i,i')-v}(A', B') \ge \alpha n$ for all $v \in V_0$.
(iv) $d_{G(i,i')}(v) \ge 2(\alpha + \sqrt{\varepsilon})n$ for all $v \in V_0$.
(v) $d_{G(i,i')}(v, A') \ge d_{G(i,i')}(v, B') - \sqrt{\varepsilon} n$ for all $v \in A_0$ and $d_{G(i,i')}(v, B') \ge d_{G(i,i')}(v, A') - \sqrt{\varepsilon} n$ for all $v \in B_0$.

Indeed, (i) follows from (a$_1$) and (a$_2$). To prove (ii), note that $e_{G(i,i')}(A', B') = e(H'(i,i'))$. Now apply (a$_3$) and (9.2). For (iii), note that (a$_4$) and $\Delta(G[A', B']) \le n/2$ imply that for all $v \in V_0$,

$$d_{G(i,i')[A',B']}(v) = d_{H'(i,i')}(v) \le (d_{G[A',B']}(v) + 2\varepsilon n)/K^2 \le (1/2 + 2\varepsilon)n/K^2.$$

If $e_G(A', B') \ge n$, then (a$_3$) implies that $e_{G(i,i')}(A', B') \ge (1 - 4\varepsilon)n/K^2 \ge \alpha n + d_{G(i,i')[A',B']}(v)$ and so (iii) follows. If $e_G(A', B') < n$, then for all $v \in V_0$

$$e_{G(i,i')-v}(A', B') = e(H'(i,i')) - d_{H'(i,i')}(v) \overset{(a_3),(a_4)}{\ge} (e_{G-v}(A', B') - 6\varepsilon n)/K^2 \overset{(9.3)}{\ge} \alpha n.$$

So (iii) follows again. (iv) follows from (a$_5$) and (9.4). For (v), note that (a$_1$) and (a$_2$) imply that for $v \in A_0$,

$$\begin{aligned} d_{G(i,i')}(v, A') = d_{G(i,i')}(v) - d_{H'(i,i')}(v) &\overset{(a_4),(a_5)}{\ge} (d_G(v, A') - 6\varepsilon n)/K^2 \\ &\ge (d_G(v, B') - 6\varepsilon n)/K^2 \overset{(a_4)}{\ge} d_{H'(i,i')}(v) - 8\varepsilon n = d_{G(i,i')}(v, B') - 8\varepsilon n. \end{aligned}$$

The second part of (v) follows similarly.

Note that each $(G(i,i'), A, A_0, B, B_0)$ is an (ε_0, K)-framework since this holds for (G, A, A_0, B, B_0). Thus for all $i, i' \le K$ we can apply Lemma 9.1 (with $\sqrt{\varepsilon}$ playing the role of ε) to the (ε_0, K)-framework $(G(i,i'), A, A_0, B, B_0)$ in order to obtain αn edge-disjoint Hamilton exceptional systems with parameter ε_0 in $G(i,i')$. By (i), we may delete any vertices outside $A_0 \cup A_i \cup B_0 \cup B_{i'}$ from these systems without affecting their edges. So each of these Hamilton exceptional systems is in fact an (i,i')-HES. The set $\mathcal{J}$ consisting of all these $K^2\alpha n$ Hamilton exceptional systems is as required in the lemma. □

Given the appropriate set $\mathcal{J}$ of localized Hamilton exceptional systems, the next lemma from [7] guarantees a set of $|\mathcal{J}|$ edge-disjoint Hamilton cycles in a graph G such that each of them contains one exceptional system from $\mathcal{J}$, provided that G is sufficiently close to the union of two disjoint copies of $K_{n/2}$. The lemma also allows $\mathcal{J}$ to contain matching exceptional systems (each of these will then be extended into a perfect matching of G). Note that with a suitable $\mathcal{J}$ and an appropriate choice of parameters we can achieve that the 'uncovered' graph has density $2\rho \pm 2/K \ll 1$, i.e. we do have an approximate decomposition.

Lemma 9.4. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll \rho \ll 1$ and $0 \leq \mu \ll 1$, where $n, K \in \mathbb{N}$ and K is odd. Suppose that G is a graph on n vertices and $\mathcal{P}$ is a (K, m, ε_0)-partition of $V(G)$. Furthermore, suppose that the following conditions hold:*

(a) *$d(v, A_i) = (1 - 4\mu \pm 4/K)m$ and $d(w, B_i) = (1 - 4\mu \pm 4/K)m$ for all $v \in A$, $w \in B$ and $1 \leq i \leq K$.*
(b) *There is a set $\mathcal{J}$ which consists of at most $(1/4 - \mu - \rho)n$ edge-disjoint exceptional systems with parameter ε_0 in G.*
(c) *$\mathcal{J}$ has a partition into K^2 sets $\mathcal{J}_{i,i'}$ (one for all $1 \leq i, i' \leq K$) such that each $\mathcal{J}_{i,i'}$ consists of precisely $|\mathcal{J}|/K^2$ (i, i')-ES with respect to $\mathcal{P}$.*
(d) *If $\mathcal{J}$ contains matching exceptional systems then $|A'| = |B'|$ is even.*

Then G contains $|\mathcal{J}|$ edge-disjoint spanning subgraphs $H_1, \dots, H_{|\mathcal{J}|}$ which satisfy the following properties:

- *For each H_s there is some $J_s \in \mathcal{J}$ such that $J_s \subseteq H_s$.*
- *If J_s is a Hamilton exceptional system, then H_s is a Hamilton cycle of G. If J_s is a matching exceptional system, then H_s is the edge-disjoint union of two perfect matchings in G.*

Matching exceptional systems do no play any role in the current application to prove Theorem 3.9, but they will occur when we use Lemma 9.4 again in the proof of Theorem 3.3.

To prove Theorem 3.9, we first apply Lemma 9.3 to find suitable localized Hamilton exceptional systems and then apply Theorem 9.4 to transform these into Hamilton cycles.

Proof of Theorem 3.9. Choose new constants $\varepsilon_{\rm ex}$, ε_0, ε_1, ε_2, ϕ and an odd number $K \in \mathbb{N}$ such that

$$1/n_0 \ll \varepsilon_{\rm ex} \ll \varepsilon_0 \ll \varepsilon_1 \ll \varepsilon_2 \ll 1/K \ll \phi \ll \varepsilon.$$

Further, we may assume that $\varepsilon \ll 1$. Let $n \geq n_0$ and let G be any graph on n vertices such that $\delta(G) \geq n/2$ and such that G is $\varepsilon_{\rm ex}$-close to two disjoint copies of $K_{n/2}$. By modifying ϕ slightly, we may assume that $(1/4 - \phi)n/K^2 \in \mathbb{N}$.

Apply Proposition 6.5 to obtain a partition A, A_0, B, B_0 of $V(G)$ such that such that (G, A, A_0, B, B_0) is an (ε_0, K)-framework, $d(v, A') \geq d(v)/2$ for all $v \in A'$ and $d(v, B') \geq d(v)/2$ for all $v \in B'$. Let $m := |A|/K = |B|/K$. Apply Lemma 8.3 with ε_0 playing the role of μ to obtain partitions $A_1, \dots, A_K$ of A and $B_1, \dots, B_K$ of B which satisfy the following properties, where $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$:

- $(G[A] + G[B], \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon_2)$-scheme.
- $(G - G[A] - G[B], \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon_1)$-exceptional scheme.

Apply Lemma 9.3 to obtain a set $\mathcal{J}$ of $(1/4 - \phi)n$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in G such that, for each $i, i' \leq K$, $\mathcal{J}$ contains precisely $(1/4 - \phi)n/K^2$ (i, i')-HES. Finally, our aim is to apply Lemma 9.4 with $\mu := 1/K$ and $\rho := \phi - 1/K$. So let us check that conditions (a)–(c) of Lemma 9.4 hold (note that (d) is not relevant). Clearly (b) and (c) hold. To verify (a) note that (Sch3) implies that for all $v \in A$ we have $d(v, A_i) \geq (1 - \varepsilon_2)m \geq (1 - 1/K)m \geq (1 - 4\mu - 4/K)m$.

Similarly, for all $w \in B$ we have $d(w, B_i) \geq (1 - 4\mu - 4/K)m$. So we can apply Lemma 9.4 to obtain $|\mathcal{J}| \geq (1/4 - \varepsilon)n$ edge-disjoint Hamilton cycles. □

10. Eliminating the edges inside A_0 and B_0

This and the remaining sections are all devoted to the proof of Theorem 3.3. Suppose that G is a D-regular graph and (G, A, A_0, B, B_0) is an (ε_0, K)-framework with $\Delta(G[A', B']) \leq D/2$. The aim of this section is to construct a small number of Hamilton cycles (and perfect matchings if appropriate) which together cover all the edges of $G[A_0]$ and $G[B_0]$. The first step is to construct a small number of exceptional systems containing all the edges of $G[A_0]$ and $G[B_0]$.

Lemma 10.1. *Suppose that $0 < 1/n \ll \varepsilon_0 \leq \lambda \ll 1$ and that $n, \lambda n, D, K \in \mathbb{N}$. Let G be a D-regular graph on n vertices with $D \geq n - 2\lfloor n/4 \rfloor - 1$. Suppose that (G, A, A_0, B, B_0) is an (ε_0, K)-framework with $\Delta(G[A', B']) \leq D/2$. Let*

$$\ell := \left\lfloor \frac{\max\{0, D - e_G(A', B')\}}{2} \right\rfloor \quad \text{and} \quad \phi n := \begin{cases} 2\lambda n + 1 & \text{if } D \text{ is odd,} \\ 2\lambda n & \text{if } D \text{ is even.} \end{cases}$$

Let w_1 and w_2 be vertices of G such that $d_{G[A',B']}(w_1) \geq d_{G[A',B']}(w_2) \geq d_{G[A',B']}(v)$ for all $v \in V(G)\setminus\{w_1, w_2\}$. Then there exist $\lambda n+1$ edge-disjoint subgraphs $J_0, J_1, \dots, J_{\lambda n}$ of G which cover all the edges in $G[A_0] + G[B_0]$ and satisfy the following properties:

- (i) *If D is odd, then J_0 is a perfect matching in G with $e_{J_0}(A', B') \leq 1$. If D is even, then J_0 is empty.*
- (ii) *J_s is a matching exceptional system with parameter ε_0 for all $1 \leq s \leq \min\{\ell, \lambda n\}$.*
- (iii) *J_s is a Hamilton exceptional system with parameter ε_0 and such that $e_{J_s}(A', B') = 2$ for all $\ell < s \leq \lambda n$.*
- (iv) *Let $\mathcal{J}$ be the union of all the J_s and let $H^\diamond := G[A', B'] - \mathcal{J}$. Then $e_{\mathcal{J}}(A', B') \leq \phi n$ and $d_{\mathcal{J}}(v) = \phi n$ for all $v \in V_0$. Moreover, $e(H^\diamond)$ is even.*
- (v) *$d_{H^\diamond}(w_1) \leq (D - \phi n)/2$. Furthermore, if $D = n/2 - 1$ then $d_{H^\diamond}(w_2) \leq (D - \phi n)/2$.*
- (vi) *If $e_G(A', B') < D$, then $e(H^\diamond) \leq D - \phi n$ and $\Delta(H^\diamond) \leq e(H^\diamond)/2$.*

As indicated in Section 4, the main proof of Theorem 3.3 splits into three cases: (a) the non-critical case with $e_G(A', B') \geq D$, (b) the critical case with $e_G(A', B') \geq D$ and (c) the case with $e_G(A', B') < D$. The formal definition of 'critical' and a more detailed discussion of the different cases is given in Section 11.

The above lemma will be used in all three cases. In these different cases, we will need that the Hamilton cycles or perfect matchings produced by the lemma use appropriate edges between A' and B' (and thus the 'leftover' $H^\diamond$ has suitable properties). In particular, (v) will ensure that we can apply Lemma 11.4 in case (b). Similarly, (vi) will ensure that we can apply Lemma 11.5 in case (c). (ii) and (vi) will only be relevant in case (c).

Proof. Set $H := G[A', B']$ and $W := \{w_1, w_2\}$. First, we construct J_0. If D is even, then (i) is trivial, so we may assume that D is odd (and so n is even). We will construct J_0 such that it satisfies (i) as well as the following additional property:

- (i$'$) If w_1w_2 is an edge in $G[A'] + G[B']$, then w_1w_2 lies in J_0. Moreover, $e_{J_0}(A', B') = 1$ if $|A'|$ is odd and $e_{J_0}(A', B') = 0$ if $|A'|$ is even.

Suppose first that $|A'|$ is even (and so $|B'|$ is even as well). Since our assumptions imply that $\delta(G[A']) \geq \lceil D/2 \rceil \geq 3\varepsilon_0 n$, there exists a matching M'_A in $G[A']$ of size at most $|A_0|+2$ covering all the vertices of $A_0 \cup (A' \cap W)$. Moreover, if w_1w_2 is an edge in $G[A']$, then we can ensure that $w_1w_2 \in M'_A$. Note that $A'' := A' \setminus V(M'_A)$ is a subset of A and $|A''|$ is even. (FR4) implies that $\delta(G[A'']) \geq D - \varepsilon_0 n - 2(|A_0|+2) \geq |A''|/2$. Therefore, there exists a perfect matching M''_A in $G[A'']$ (e.g. by Dirac's theorem). Hence, $M_A := M'_A + M''_A$ is a perfect matching in $G[A']$. Similarly, there is a perfect matching M_B in $G[B']$ such that if w_1w_2 is an edge in $G[B']$, then w_1w_2 is in M_B. Set $J_0 := M_A + M_B$.

Next assume that $|A'|$ is odd. If $D \geq \lfloor n/2 \rfloor$, then Proposition 6.3 implies that $e(H - W) > 0$. If $D = n/2 - 1$, then $n = 0 \pmod 4$ and so $|B'| \leq n/2 - 1$ since $|A'|$ is odd. Together with Proposition 6.1(ii) this implies that $e(H) \geq n/2 - 1$. Since in this case we also have that $\Delta(H) \leq \lfloor D/2 \rfloor = n/4 - 1$, it follows that $e(H - W) \geq e(H) - 2\Delta(H) > 0$. Thus in both cases there exists an edge ab in $H - W$ with $a \in A'$ and $b \in B'$. Note that both $|A' \setminus \{a\}|$ and $|B' \setminus \{b\}|$ are even. Moreover, $\delta(G[A' \setminus \{a\}]) \geq \lceil D/2 \rceil - 1 \geq 3\varepsilon_0 n$ and $\delta(G[B' \setminus \{b\}]) \geq \lceil D/2 \rceil - 1 \geq 3\varepsilon_0 n$. Thus we can argue as in the case when $|A'|$ is even to find perfect matchings M_A and M_B in $G[A' \setminus \{a\}]$ and $G[B' \setminus \{b\}]$ respectively such that if w_1w_2 is an edge in $G[A'] + G[B']$ then $w_1w_2 \in M_A + M_B$. Set $J_0 := M_A + M_B + ab$.

This completes the construction of J_0. (If D is even we set $J_0 := \emptyset$.) So (i) and (i$'$) hold. Let $G' := G - J_0$ and $H' := G'[A', B']$. Since $|A_0| + |B_0| \leq \varepsilon_0 n \leq \lambda n$, Vizing's theorem implies that we can decompose $G'[A_0]+G'[B_0]$ into λn edge-disjoint (possibly empty) matchings $M_1, \dots, M_{\lambda n}$. By relabeling these matchings if necessary, we may assume that if $w_1w_2 \in E_{G'}(A_0)$ or $w_1w_2 \in E_{G'}(B_0)$, then $w_1w_2 \in M_1$.

Case 1: $e(H) \geq D$**.**

Note that in this case $\ell = 0$ and $e(H') \geq D - 1$. For each $s = 1, \dots, \lambda n$ in turn we will extend M_s into a Hamilton exceptional system J_s with $e_{J_s}(A', B') = 2$ and such that J_s and $J_{s'}$ are edge-disjoint for all $0 \leq s' < s$. In order to do this, we will first extend M_s into a Hamilton exceptional system candidate F_s by adding two independent $A'B'$-edges f_s and f'_s. We will then use Lemma 7.2 to extend F_s into a Hamilton exceptional system J_s. For all s with $1 \leq s \leq \lambda n$, we will choose these edges and sets to satisfy the following:

- (α_1) J_s is a Hamilton exceptional system with parameter ε_0 such that $e_{J_s}(A', B') = 2$.
- (α_2) Suppose that $d_H(w_1) \geq 2\lambda n$. Then w_1 is an endpoint of f_s.
- (α_3) Suppose that $d_H(w_2) \geq 2\lambda n$. Then w_2 is an endpoint of f'_s, unless both $s = 1$ and $w_1w_2 \in M_1$.
- (α_4) J_s contains M_s as well as the edges f_s and f'_s. $J_s - M_s - f_s - f'_s$ only contains A_0A-edges and B_0B-edges of G. J_s is edge-disjoint from $J_0, \dots, J_{s-1}$.

First suppose that $w_1w_2 \in M_1$. We construct J_1 satisfying the above. Our assumption means that w_1w_2 is an edge in $G[A'] + G[B']$, so D is even (or else $w_1w_2 \in J_0$ by (i$'$)). Moreover, $H' = H$ and $D \geq \lfloor n/2 \rfloor$ by (3.1) and the fact that D is even. Together with Proposition 6.3 this implies that $e(H'-W) = e(H-W) > 0$. Pick an $A'B'$-edge f'_1 in $H'-W$. Let U_1 be the connected component in $M_1+f'_1$ containing f'_1. So $|U_1| \leq 4$ and $w_1 \notin U_1$. If $d_H(w_1) \geq 2\lambda n$, we can find an $A'B'$-edge f_1 such that w_1 is one endpoint of f_1 and the other endpoint of f_1 does not lie in U_1. If $d_H(w_1) < 2\lambda n$, then the choice of w_1 implies that $\Delta(H) \leq 2\lambda n$. So there exists an $A'B'$-edge f_1 in $H' - V(U_1) = H - V(U_1)$ since $e(H - V(U_1)) \geq e(H) - |U_1|\Delta(H) \geq e(H) - 8\lambda n > 0$. Set $F_1 := M_1 + f_1 + f'_1$. Note that f_1 satisfies (α_2) and that F_1 is a Hamilton exceptional system candidate with $e_{F_1}(A', B') = 2$. By Lemma 7.2, we can extend F_1 into a Hamilton exceptional system J_1 with parameter ε_0 in G such that $F_1 \subseteq J_1$ and such that $J_1 - F_1$ only contains A_0A-edges and B_0B-edges of G.

Next, suppose that for some $1 \leq s \leq \lambda n$ we have already constructed $J_0, \dots, J_{s-1}$ satisfying (α_1)–(α_4). So $s \geq 2$ if $w_1w_2 \in M_1$. Let $G_s := G - \sum_{j=s}^{\lambda n} M_j - \sum_{j=0}^{s-1} J_j$ and $H_s := G_s[A', B']$. Note that

$$e(H_s) \geq e(H) - 2(s-1) - 1 \geq D - 2\lambda n. \tag{10.1}$$

Moreover, note that $d_{G_s}(v, A) \geq d_G(v, A) - 2(s-1) - 1 \geq \sqrt{\varepsilon_0}n$ for all $v \in A_0$ and $d_{G_s}(v, B) \geq \sqrt{\varepsilon_0}n$ for all $v \in B_0$.

We first pick the edge f'_s as follows. If $d_H(w_2) \geq 2\lambda n$, then $d_{H_s}(w_2) \geq d_H(w_2) - s \geq \lambda n$. So we can pick an $A'B'$-edge f'_s of H_s such that w_2 is an endpoint of f'_s and the connected component U_s of $M_s + f'_s$ containing f'_s does not contain w_1. If $d_H(w_2) < 2\lambda n$, then pick an $A'B'$-edge f'_s of H_s such that the connected component U_s of $M_s + f'_s$ containing f'_s does not contain w_1. To see that such an edge exists, note that in this case the neighbour w'_1 of w_1 in M_s satisfies $d_H(w'_1) \leq d_H(w_2) < 2\lambda n$ (if w'_1 exists) and that (10.1) implies that $e(H_s) \geq D - 2\lambda n > D/2 + 2\lambda n \geq d_H(w_1) + 2\lambda n$. Observe that in both cases $|U_s| \leq 4$.

We now pick the edge f_s as follows. If $d_H(w_1) \geq 2\lambda n$, then $d_{H_s}(w_1) \geq d_H(w_1) - s \geq \lambda n$. So we can find an $A'B'$-edge f_s of H_s such that w_1 is one endpoint of f_s and the other endpoint of f_s does not lie in U_s. If $d_H(w_1) < 2\lambda n$, then $\Delta(H) \leq 2\lambda n$ and thus (10.1) implies that

$$e(H_s - V(U_s)) \geq D - 2\lambda n - 2\lambda n|U_s| \geq 1.$$

So there exists an $A'B'$-edge f_s in $H_s - V(U_s)$.

In all cases the edges f_s and f'_s satisfy (α_2) and (α_3). Set $F_s := M_s + f_s + f'_s$. Clearly, F_s is a Hamilton exceptional system candidate with $e_{F_s}(A', B') = 2$. Recall that $d_{G_s}(v, A) \geq \sqrt{\varepsilon_0}n$ for all $v \in A_0$ and $d_{G_s}(v, B) \geq \sqrt{\varepsilon_0}n$ for all $v \in B_0$. Thus by Lemma 7.2, we can extend F_s into a Hamilton exceptional system J_s with parameter ε_0 such that $F_s \subseteq J_s \subseteq G_s + F_s$ and such that $J_s - F_s$ only contains A_0A-edges and B_0B-edges of G_s. Hence we have constructed $J_1, \dots, J_{\lambda n}$ satisfying (α_1)–(α_4). So (iii) holds. Note (ii) and (vi) are vacuously true.

To verify (iv), recall that $\mathcal{J} := J_0 \cup \dots \cup J_{\lambda n}$ and $H^\diamond = G[A', B'] - \mathcal{J}$. For all $1 \leq s \leq \lambda n$ we have $e_{J_s}(A', B') = 2$ by (iii). Moreover, (i) and (i$'$) together imply that $e_{J_0}(A', B') = 1$ if and only if both $|A'|$ and D are odd. Therefore, $e_{\mathcal{J}}(A', B') \leq \phi n$.

Moreover, since $e(H^\diamond) = e(H) - 2\lambda n - e_{J_0}(A', B')$, Proposition 6.2(i) implies that $e(H^\diamond)$ is even. Thus (iv) holds.

To verify (v), note that if $d_H(w_1) \le 2\lambda n$ then clearly $d_{H^\diamond}(w_1) \le 2\lambda n \le (D-\phi n)/2$. If $d_H(w_1) \ge 2\lambda n$ then (α_2) implies that $d_{J_s[A',B']}(w_1) = 1$ for all $1 \le s \le \lambda n$. Hence $d_{H^\diamond}(w_1) \le \lfloor D/2 \rfloor - \lambda n = (D - \phi n)/2$. Now suppose that $D = n/2 - 1$ and so $n = 0 \pmod 4$ by (3.1). Thus D is odd and so (i$'$) implies that if w_1w_2 is an edge in $G[A'] + G[B']$, then $w_1w_2 \in J_0$. In particular $w_1w_2 \notin M_1$. (Note that if $w_1w_2 \in G[A', B']$, then w_1w_2 is not contained in M_1 either since $M_1 \subseteq G[A_0] + G[B_0]$.) Thus in the case when $d_H(w_2) \ge 2\lambda n$, (α_3) implies that $d_{J_s[A',B']}(w_2) = 1$ for all $1 \le s \le \lambda n$. Hence $d_{H^\diamond}(w_2) \le \lfloor D/2 \rfloor - \lambda n = (D - \phi n)/2$. If $d_H(w_2) \le 2\lambda n$ then clearly $d_{H^\diamond}(w_2) \le 2\lambda n \le (D - \phi n)/2$. Therefore (v) holds.

Case 2: $e(H) < D$

Together with Proposition 6.1(ii) this implies that $n = 0 \pmod 4$, $D = n/2 - 1$ and $|A'| = n/2 = |B'|$. So D is odd and $|A'|$ is even. In particular, by Proposition 6.2(i) $e(H)$ is even and by (i) and (i$'$) J_0 is a perfect matching with $e_{J_0}(A', B') = 0$. Moreover, Proposition 6.4 implies that $\Delta(H) \le e(H)/2$ in this case (recall that $H := G[A', B']$).

Note that each M_s is a matching exceptional system candidate. By Lemma 7.2, for each $1 \le s \le \min\{\ell, \lambda n\}$ in turn, we can extend M_s into a matching exceptional system J_s with parameter ε_0 in $G' = G - J_0$ such that $M_s \subseteq J_s$, and such that J_s and $J_{s'}$ are edge-disjoint whenever $1 \le s' < s \le \min\{\ell, \lambda n\}$. Thus (ii) holds.

If $\ell \ge \lambda n$, then $e(H) \le D - 2\lambda n = D - \phi n + 1$. But since $e(H)$ is even and $D - \phi n + 1$ is odd this means that $e(H) \le D - \phi n$. Thus $\Delta(H) \le e(H)/2 \le (D - \phi n)/2$. Moreover, $d_{\mathcal{J}}(v) = 2\lambda n + d_{J_0}(v) = \phi n$ for all $v \in V_0$. Hence (iv)–(vi) hold since $H^\diamond = H$. ((iii) is vacuously true.)

Therefore, we may assume that $\ell < \lambda n$. Using a similar argument as in Case 1, for all $\ell < s \le \lambda n$ we can extend the matchings M_s into edge-disjoint Hamilton exceptional systems J_s satisfying (α_1)–(α_4) and which are edge-disjoint from $J_0, \dots, J_\ell$. Indeed, suppose that for $\ell < s \le \lambda n$ we have already constructed $J_{\ell+1}, \dots, J_{s-1}$ satisfying (α_1)–(α_4). (Note that (i$'$) implies that the exception in (α_3) is not relevant.) The fact that D is odd and $e(H)$ is even implies that $\ell = (D - e(H) - 1)/2$. Then defining H_s analogously to Case 1, we have

$$e(H_s) \ge e(H) - 2(s - 1 - \ell) = D - 2s \ge D - 2\lambda n,$$

where in the first inequality we use that $e_{J_0}(A', B') = 0$ by (i$'$). So the analogue of (10.1) holds. Hence we can proceed exactly as in Case 1 to construct J_s (the remaining calculations go through as before). Thus (iii) holds.

To verify (iv), note that $e_{\mathcal{J}}(A', B') = 2(\lambda n - \ell)$. So

$$e(H^\diamond) = e(H) - 2(\lambda n - \ell) = e(H) - 2\lambda n + (D - e(H) - 1) = D - \phi n. \tag{10.2}$$

In particular, $e(H^\diamond)$ is even and $e_{\mathcal{J}}(A', B') = e(H) - e(H^\diamond) < \phi n$. So (iv) holds.

In order to verify (vi), recall that $\Delta(H) \le e(H)/2$. Moreover, note that (α_2) implies that if $d_H(w_1) \ge 2\lambda n$, then $d_{J_s[A',B']}(w_1) = 1$ for all $\ell < s \le \lambda n$. Hence

$$\begin{aligned} d_{H^\diamond}(w_1) &= d_H(w_1) - (\lambda n - \ell) = \Delta(H) - \lambda n + \ell \\ &\le e(H)/2 - \lambda n + (D - e(H) - 1)/2 = (D - \phi n)/2 \overset{(10.2)}{=} e(H^\diamond)/2. \end{aligned}$$

Similarly if $d_H(w_2) \ge 2\lambda n$, then $d_{H^\diamond}(w_2) \le e(H^\diamond)/2$. If $d_H(w_1) \le 2\lambda n$, then $d_{H^\diamond}(w_1) \le 2\lambda n \le e(H^\diamond)/2$ by (10.2) and the analogue also holds for w_2. Thus in all cases $d_H(w_1), d_H(w_2) \le e(H^\diamond)/2$. Our choice of w_1 and w_2 implies that for all $v \in V(G) \setminus W$ we have

$$d_H(v) \le (e(H)+3)/3 \le (D+3)/3 \overset{(10.2)}{<} e(H^\diamond)/2.$$

Therefore, $\Delta(H^\diamond) \le e(H^\diamond)/2$. Together with (10.2) this implies (vi) and thus (v). □

The next lemma implies that each of the exceptional systems J_s guaranteed by Lemma 10.1 can be extended into a Hamilton cycle (if J_s is a Hamilton exceptional system) or into two perfect matchings (if J_s is a matching exceptional system and both $|A'|$ and $|B'|$ are even).

Lemma 10.2. *Suppose that $0 < 1/n \ll \varepsilon_0 \le \lambda \ll 1$ and that $n, \lambda n, K \in \mathbb{N}$. Suppose that (G, A, A_0, B, B_0) is an (ε_0, K)-framework such that $\delta(G[A]) \ge 4|A|/5$ and $\delta(G[B]) \ge 4|B|/5$. Let $J_1, \dots, J_{\lambda n}$ be exceptional systems with parameter ε_0. Suppose that G and $J_1, \dots, J_{\lambda n}$ are pairwise edge-disjoint. Then there are edge-disjoint subgraphs $H_1, \dots, H_{\lambda n}$ in $G + \sum_{s=1}^{\lambda n} J_s$ which satisfy the following properties:*

- (i) *$J_s \subseteq H_s$ and $E(H_s - J_s) \subseteq E(G[A] + G[B])$ for all $1 \le s \le \lambda n$.*
- (ii) *If J_s is a Hamilton exceptional system, then H_s is a Hamilton cycle on $V(G)$.*
- (iii) *If J_s is a matching exceptional system, then H_s is an union of a Hamilton cycle on $A' = A \cup A_0$ and a Hamilton cycle on $B' = B \cup B_0$.*

Proof. Recall that, given an exceptional system J, we have defined matchings J^*_A, J^*_B and $J^* = J^*_A + J^*_B$ in Section 7. We will write $J^*_{s,A} := (J_s)^*_A$ and $J^*_{s,B} := (J_s)^*_B$. For each $s \le \lambda n$ in turn, we will find a subgraph H^*_s of $G[A] + G[B] + J^*_s$ containing J^*_s such that H^*_s is edge-disjoint from $H^*_1, \dots, H^*_{s-1}$. Moreover, H^*_s will be the union of two cycles C_A and C_B such that C_A is a Hamilton cycle on A which is consistent with $J^*_{s,A}$ and C_B is a Hamilton cycle on B which is consistent with $J^*_{s,B}$. (Recall from Section 7 that we always view different J^*_i as being edge-disjoint from each other. So asking H^*_s to be edge-disjoint from $H^*_1, \dots, H^*_{s-1}$ is the same as asking $H^*_s - J^*_s$ to be edge-disjoint from $H^*_1 - J^*_1, \dots, H^*_{s-1} - J^*_{s-1}$.)

Suppose that for some $1 \le s \le \lambda n$ we have already found $H^*_1, \dots, H^*_{s-1}$. For all $i < s$, let $H_i := H^*_i - J^*_i + J_i$. Let $G_s := G - (H_1 \cup \dots \cup H_{s-1})$. First we construct C_A as follows. Recall from (7.1) that $J^*_{s,A}$ is a matching of size at most $2\sqrt{\varepsilon_0} n$. Note that $\delta(G_s[A]) \ge \delta(G[A]) - 2s \ge (4/5 - 5\lambda n)|A|$. So we can greedily find a path P_A of length at most $6\sqrt{\varepsilon_0} n$ in $G_s[A] + J^*_{s,A}$ such that P_A is consistent with $J^*_{s,A}$. Let u and v denote the endpoints of P_A. Let G^A_s be the graph obtained from $G_s[A] - V(P_A)$ by adding a new vertex w whose neighbourhood is precisely

$(N_{G_s}(u) \cap N_{G_s}(v)) \setminus V(P_A)$. Note that $\delta(G_s^A) \geq |G_s^A|/2$ (with room to spare). Thus G_s^A contains a Hamilton cycle C'_A by Dirac's theorem. But C'_A corresponds to a Hamilton cycle C_A of $G_s[A]+J^*_{s,A}$ that is consistent with $J^*_{s,A}$. Similarly, we can find a Hamilton cycle C_B of $G_s[B]+J^*_{s,B}$ that is consistent with $J^*_{s,B}$. Let $H^*_s = C_A + C_B$. This completes the construction of $H^*_1, \dots, H^*_{\lambda n}$.

For each $1 \leq s \leq \lambda n$ we take $H_s := H^*_s - J^*_s + J_s$. Then (i) holds. Proposition 7.1 implies (ii) and (iii). □

By combining Lemmas 10.1 and 10.2 we obtain the following result, which guarantees a set of edge-disjoint Hamilton cycles covering all edges of $G[A_0]$ and $G[B_0]$.

Lemma 10.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \phi \ll 1$ and that $D, n, (D-\phi n)/2, K \in \mathbb{N}$. Let G be a D-regular graph on n vertices with $D \geq n - 2\lfloor n/4 \rfloor - 1$. Suppose that (G, A, A_0, B, B_0) is an (ε_0, K)-framework with $\Delta(G[A', B']) \leq D/2$. Let w_1 and w_2 be (fixed) vertices of G such that $d_{G[A',B']}(w_1) \geq d_{G[A',B']}(w_2) \geq d_{G[A',B']}(v)$ for all $v \in V(G) \setminus \{w_1, w_2\}$. Then there exists a ϕn-regular spanning subgraph G_0 of G which satisfies the following properties:*

- (i) *$G[A_0] + G[B_0] \subseteq G_0$.*
- (ii) *$e_{G_0}(A', B') \leq \phi n$ and $e_{G-G_0}(A', B')$ is even.*
- (iii) *G_0 can be decomposed into $\lfloor e_{G_0}(A', B')/2 \rfloor$ Hamilton cycles and $\phi n - 2\lfloor e_{G_0}(A', B')/2 \rfloor$ perfect matchings. Moreover, if $e_G(A', B') \geq D$, then this decomposition of G_0 uses $\lfloor \phi n/2 \rfloor$ Hamilton cycles and one perfect matching if D is odd.*
- (iv) *Let $H^\diamond := G[A', B'] - G_0$. Then $d_{H^\diamond}(w_1) \leq (D-\phi n)/2$. Furthermore, if $D = n/2 - 1$ then $d_{H^\diamond}(w_2) \leq (D - \phi n)/2$.*
- (v) *If $e_G(A', B') < D$, then $\Delta(H^\diamond) \leq e(H^\diamond)/2 \leq (D - \phi n)/2$.*

Proof. Let

$$\ell := \left\lfloor \frac{\max\{0, D - e_G(A', B')\}}{2} \right\rfloor \quad \text{and} \quad \lambda n := \lfloor \phi n/2 \rfloor = \begin{cases} (\phi n - 1)/2 & \text{if } D \text{ is odd,} \\ \phi n/2 & \text{if } D \text{ is even.} \end{cases}$$

(The last equality holds since our assumption that $(D - \phi n)/2 \in \mathbb{N}$ implies that D is odd if and only if ϕn is odd.) So ℓ, ϕ and λ are as in Lemma 10.1. Thus we can apply Lemma 10.1 to G in order to obtain $\lambda n + 1$ subgraphs $J_0, \dots, J_{\lambda n}$ as described there. Let G' be the graph obtained from $G[A'] + G[B']$ by removing all the edges in $J_0 \cup \dots \cup J_{\lambda n}$. Recall that J_0 is either a perfect matching in G or empty. Since each of $J_1, \dots, J_{\lambda n}$ is an exceptional system and so by (EC3) we have $e_{J_s}(A) = 0$ for all $1 \leq s \leq \lambda n$, it follows that $\delta(G'[A]) \geq \delta(G[A]) - 1 \geq 4|A|/5$, where the final inequality follows from (FR3) and (FR4). Similarly $\delta(G'[B]) \geq 4|B|/5$. So we can apply Lemma 10.2 with G' playing the role of G in order to extend $J_1, \dots, J_{\lambda n}$ into edge-disjoint subgraphs $H_1, \dots, H_{\lambda n}$ of $G' + \sum_{s=1}^{\lambda n} J_s$ such that

- (a) H_s is a Hamilton cycle on $V(G)$ which contains precisely two $A'B'$-edges for all $\ell < s \leq \lambda n$;
- (b) H_s is the union of a Hamilton cycle on A' and a Hamilton cycle on B' for all $1 \leq s \leq \min\{\ell, \lambda n\}$.

Indeed, the property $e_{H_s}(A', B') = 2$ in (a) follows from Lemma 10.1(iii) and 10.2(i). Let $G_0 := J_0 + \sum_{s=1}^{\lambda n} H_s$. Then (i) holds since by Lemma 10.1 all the $J_0, \dots, J_{\lambda n}$ together cover all edges in $G[A_0]$ and $G[B_0]$. Let $\mathcal{J}_{\rm HC}$ be the union of all J_s with $\ell < s \le \lambda n$ and let $\mathcal{J}$ be the union of all J_s with $0 \le s \le \lambda n$. The definition of G_0, Lemma 10.1(ii),(iii) and Lemma 10.2(i) together imply that $G_0[A', B'] = \mathcal{J}[A', B'] = J_0[A', B'] + \mathcal{J}_{\rm HC}[A', B']$ and so

$$
\begin{align}
e_{G_0}(A', B') &= e_{\mathcal{J}}(A', B') \tag{10.3}\\
&= e_{J_0}(A', B') + 2(\max\{0, \lambda n - \ell\}). \tag{10.4}
\end{align}
$$

Together with Lemma 10.1(iv), (10.3) implies (ii). Moreover, the graph $H^\diamond$ defined in (iv) is the same as the graph $H^\diamond$ defined in Lemma 10.1(iv). Thus (iv) and (v) follow from Lemma 10.1(v) and (vi).

So it remains to verify (iii). Note that if $\ell > 0$ then $e_G(A', B') < D$ and so $n = 0 \pmod 4$, $D = n/2 - 1$ and $|A'| = n/2 = |B'|$ by Proposition 6.1(ii). In particular, both A' and B' are even and so for all $1 \le s \le \ell$ the graph H_s can be decomposed into two edge-disjoint perfect matchings. Recall that by Lemma 10.1(i) the graph J_0 is a perfect matching if D is odd and empty if D is even. Thus, if $\ell \le \lambda n$, then G_0 can be decomposed into $\lambda n - \ell$ edge-disjoint Hamilton cycles and $n_{\rm match}$ edge-disjoint perfect matchings, where $n_{\rm match} = 2\ell$ if D is even and $n_{\rm match} = 2\ell + 1$ if D is odd. In particular, this implies the 'moreover part' of (iii) (since $\ell = 0$ if $e_G(A', B') \ge D$). Also, (10.4) together with the fact that $e_{J_0}(A', B') \le 1$ by Lemma 10.1(i) implies that $\lambda n - \ell = \lfloor e_{G_0}(A', B')/2 \rfloor$ and so $\phi n - 2\lfloor e_{G_0}(A', B')/2 \rfloor = n_{\rm match}$. Thus (iii) holds in this case. If $\ell > \lambda n$, then (a) implies that there are no Hamilton cycles at all in the decomposition. Also (10.4) implies that $\lfloor e_{G_0}(A', B')/2 \rfloor = 0$, as required in (iii). Similarly, (b) implies that $n_{\rm match} = 2\lambda n$ if D is even and $n_{\rm match} = 2\lambda n + 1$ if D is odd, which also agrees with (iii). □

11. Constructing localized exceptional systems

Suppose that (G, A, A_0, B, B_0) is an (ε_0, K)-framework and that G_0 is the spanning subgraph of our given D-regular graph G obtained by Lemma 10.3. Set $G' := G - G_0$. (So G' has no edges inside A_0 or B_0.) Roughly speaking, the aim of this section is to decompose $G' - G'[A] - G'[B]$ into edge-disjoint exceptional systems. Each of these exceptional systems J will then be extended into a Hamilton cycle (in the case when J is a Hamilton exceptional system) or into two perfect matchings (in the case when J is a matching exceptional system). We will ensure that all but a small number of these exceptional systems are localized (with respect to some (K, m, ε_0)-partition $\mathcal{P}$ of $V(G)$ refining the partition A, A_0, B, B_0). Moreover, for all $1 \le i, i' \le K$, the number of (i, i')-localized exceptional systems in our decomposition will be the same. (Recall that (i, i')-localized exceptional systems were defined in Section 7.)

However, rather than decomposing the above 'leftover' $G' - G'[A] - G'[B]$ in a single step, we actually need to proceed in two steps: initially, we find a small number of exceptional systems J which have some additional useful properties (e.g. the number of $A'B'$-edges of J is either zero or two). These exceptional systems will be used to construct the robustly decomposable graph $G^{\rm rob}$. (Recall that the role of $G^{\rm rob}$ was discussed in Section 4.) Let $G'' := G - G_0 - G^{\rm rob}$. Some of the additional properties of the exceptional systems contained in $G^{\rm rob}$ then allow us to find the desired decomposition of $G^\diamond := G'' - G''[A] - G''[B]$. (We need to proceed in two steps rather than one as we have little control over the structure of $G^{\rm rob}$.)

In order to construct the required (localized) exceptional systems, we will distinguish three cases:

(a) the case when G is 'non-critical' and contains at least D $A'B'$-edges (see Lemma 11.3);
(b) the case when G is 'critical' and contains at least D $A'B'$-edges (see Lemma 11.4);
(c) the case when G contains less than D $A'B'$-edges (see Lemma 11.5).

Each of the three lemmas above is formulated in such a way that we can apply it twice: firstly to obtain the small number of exceptional systems needed for the robustly decomposable graph $G^{\rm rob}$ and secondly for the decomposition of the graph $G^\diamond$ into exceptional systems. Lemmas 11.3–11.5 are proved in [19].

11.1. **Critical graphs.** Roughly speaking, G is critical if most of its $A'B'$-edges are incident to only a few vertices. More precisely, given a partition A', B' of $V(G)$ and $D \in \mathbb{N}$, we say that G is *critical* (with respect to A', B' and D) if both of the following hold:

- $\Delta(G[A', B']) \geq 11D/40$;
- $e(H) \leq 41D/40$ for all subgraphs H of $G[A', B']$ with $\Delta(H) \leq 11D/40$.

Note that the property of G being critical depends only on D and the partition $A' = A \cup A_0$ and $B' = B \cup B_0$ of $V(G)$, which is fixed after we have applied Proposition 6.5 to obtain a framework (G, A, A_0, B, B_0). In particular, it does not depend on the choice of the (K, m, ε_0)-partition $\mathcal{P}$ of $V(G)$ refining A, A_0, B, B_0. (In the proof of Theorem 3.3 we will fix a framework (G, A, A_0, B, B_0), but will then choose two different partitions refining A, A_0, B, B_0.)

One example of a critical graph is the following: $G_{\rm crit}$ consists of two disjoint cliques on $(n-1)/2$ vertices with vertex set A and B respectively, where $n = 1 \pmod 4$. In addition, there is a vertex a which is adjacent to exactly half of the vertices in each of A and B. Also, add a perfect matching M between those vertices of A and those vertices in B not adjacent to a. Let $A' := A \cup \{a\}$, $B' := B$ and $D := (n-1)/2$. Then $G_{\rm crit}$ is critical, and D-regular with $e(A', B') = D$. Note that $e(M) = D/2$. To obtain a Hamilton decomposition of $G_{\rm crit}$, we will need to decompose $G_{\rm crit}[A', B']$ into $D/2$ Hamilton exceptional system candidates J_s (which need to be matchings of size exactly two in this case). In this example, this decomposition is essentially unique: every J_s has to consist of exactly one edge in M and one edge incident to a. Note that in this way, every edge between a and B yields a 'connection' (i.e. a maximal path) between A' and B' required in (ESC4).

The following lemma from [19] collects some properties of critical graphs. In particular, there is a set W consisting of between one and three vertices with many neighbours in both A and B. We will need to use $A'B'$-edges incident to one or two vertices of W to provide 'connections' between A' and B' when constructing the Hamilton exceptional system candidates in the critical case (b).

Lemma 11.1. *Suppose that $0 < 1/n \ll 1$ and that $D, n \in \mathbb{N}$ with $D \geq n-2\lfloor n/4 \rfloor -1$. Let G be a D-regular graph on n vertices and let A', B' be a partition of $V(G)$ with $|A'|, |B'| \geq D/2$ and $\Delta(G[A', B']) \leq D/2$. Suppose that G is critical. Let W be the set of vertices $w \in V(G)$ such that $d_{G[A',B']}(w) \geq 11D/40$. Then the following properties are satisfied:*

(i) $1 \leq |W| \leq 3$.
(ii) *Either $D = (n-1)/2$ and $n = 1 \pmod 4$, or $D = n/2 - 1$ and $n = 0 \pmod 4$. Furthermore, if $n = 1 \pmod 4$, then $|W| = 1$.*
(iii) $e_G(A', B') \leq 17D/10 + 5 < n$.

Recall from Proposition 6.1(ii) that we have $e_G(A', B') \geq D$ unless $D = n/2 - 1$, $n = 0 \pmod 4$ and $|A| = |B| = n/2$. Together with Lemma 11.1(ii) this shows that in order to find the decomposition into exceptional systems, we can distinguish the following three cases.

Corollary 11.2. *Suppose that $0 < 1/n \ll 1$ and that $D, n \in \mathbb{N}$ with $D \geq n - 2\lfloor n/4 \rfloor - 1$. Let G be a D-regular graph on n vertices and let A', B' be a partition of $V(G)$ with $|A'|, |B'| \geq D/2$ and $\Delta(G[A', B']) \leq D/2$. Then exactly one of the following holds:*

(a) *$e_G(A', B') \geq D$ and G is not critical.*
(b) *$e_G(A', B') \geq D$ and G is critical. In particular, $e_G(A', B') < n$ and either $D = (n-1)/2$ and $n = 1 \pmod 4$, or $D = n/2 - 1$ and $n = 0 \pmod 4$.*
(c) *$e_G(A', B') < D$. In particular, $D = n/2 - 1$, $n = 0 \pmod 4$ and $|A| = |B| = n/2$.*

11.2. **Decomposition into exceptional systems.** Recall from the beginning of Section 11 that our aim is to find a decomposition of $G - G_0 - G[A] - G[B]$ into suitable exceptional systems (in particular, most of these exceptional systems have to be localized). The following lemma states that this can be done if we are in Case (a) of Corollary 11.2, i.e. if G is not critical and $e_G(A', B') \geq D$.

Lemma 11.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \lambda, 1/K \ll 1$, that $D \geq n/3$, that $0 \leq \phi \ll 1$ and that $D, n, K, m, \lambda n/K^2, (D - \phi n)/(400K^2) \in \mathbb{N}$. Suppose that the following conditions hold:*

(i) *G is a D-regular graph on n vertices.*
(ii) *$\mathcal{P}$ is a (K, m, ε_0)-partition of $V(G)$ such that $D \leq e_G(A', B') \leq \varepsilon_0 n^2$ and $\Delta(G[A', B']) \leq D/2$. Furthermore, G is not critical.*
(iii) *G_0 is a subgraph of G such that $G[A_0] + G[B_0] \subseteq G_0$, $e_{G_0}(A', B') \leq \phi n$ and $d_{G_0}(v) = \phi n$ for all $v \in V_0$.*
(iv) *Let $G^\diamond := G - G[A] - G[B] - G_0$. $e_{G^\diamond}(A', B')$ is even and $(G^\diamond, \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon)$-exceptional scheme.*

Then there exists a set $\mathcal{J}$ consisting of $(D - \phi n)/2$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in $G^\diamond$ which satisfies the following properties:

(a) *Together all the Hamilton exceptional systems in $\mathcal{J}$ cover all edges of $G^\diamond$.*

(b) *For all $1 \leq i, i' \leq K$, the set $\mathcal{J}$ contains $(D-(\phi+2\lambda)n)/(2K^2)$ (i,i')-HES. Moreover, $\lambda n/K^2$ of these (i,i')-HES J are such that $e_J(A',B')=2$.*

Note that (b) implies that $\mathcal{J}$ contains λn Hamilton exceptional systems which might not be localized. This will make them less useful for our purposes and we extend them into Hamilton cycles in a separate step. On the other hand, the lemma is 'robust' in the sense that we can remove a sparse subgraph G_0 before we find the decomposition $\mathcal{J}$ into Hamilton exceptional systems. In our first application of Lemma 11.3 (i.e. to construct the exceptional systems for the robustly decomposable graph $G^{\rm rob}$), we will let G_0 be the graph obtained from Lemma 10.3. In the second application, G_0 also includes $G^{\rm rob}$. In our first application of Lemma 11.3, we will only use the (i,i')-HES J with $e_J(A',B')=2$.

The next lemma is an analogue of Lemma 11.3 for the case when G is critical and $e_G(A',B') \geq D$. By Corollary 11.2(b) we know that in this case $D=(n-1)/2$ or $D=n/2-1$.

Lemma 11.4. *Suppose that the assumptions of Lemma 11.3 are satisfied except that now $D \geq n-2\lfloor n/4 \rfloor -1$ and (ii) is replaced by the following condition:*

(ii) *$\mathcal{P}$ is a (K,m,ε_0)-partition of $V(G)$ such that $e_G(A',B') \geq D$ and $\Delta(G[A',B']) \leq D/2$. Furthermore, G is critical. In particular, $e_G(A',B') < n$ and $D=(n-1)/2$ or $D=n/2-1$ by Lemma 11.1(ii) and (iii).*

Suppose in addition that the following condition holds too:

(v) *Let w_1 and w_2 be (fixed) vertices such that $d_{G[A',B']}(w_1) \geq d_{G[A',B']}(w_2) \geq d_{G[A',B']}(v)$ for all $v \in V(G) \setminus \{w_1,w_2\}$. Suppose that*

$$d_{G^\diamond[A',B']}(w_1), d_{G^\diamond[A',B']}(w_2) \leq (D-\phi n)/2. \tag{11.1}$$

Then there exists a set $\mathcal{J}$ consisting of $(D-\phi n)/2$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in $G^\diamond$ which satisfies the following properties:

(a) *Together the Hamilton exceptional systems in $\mathcal{J}$ cover all edges of $G^\diamond$.*
(b) *For each $1 \leq i, i' \leq K$, the set $\mathcal{J}$ contains $(D-(\phi+2\lambda)n)/(2K^2)$ (i,i')-HES. Moreover, $\lambda n/K^2$ of these (i,i')-HES J are such that*
 (b_1) *$e_J(A',B')=2$ and*
 (b_2) *$d_{J[A',B']}(w)=1$ for all $w \in \{w_1,w_2\}$ with $d_{G[A',B']}(w) \geq 11D/40$.*

Similarly as for Lemma 11.3, (b) implies that $\mathcal{J}$ contains λn Hamilton exceptional systems which might not be localized. Another similarity is that when constructing the robustly decomposable graph $G^{\rm rob}$, we only use those Hamilton exceptional systems J which have some additional useful properties, namely (b_1) and (b_2) in this case. This guarantees that (11.1) will be satisfied in the second application of Lemma 11.4 (i.e. after the removal of $G^{\rm rob}$), by 'tracking' the degrees of the high degree vertices w_1 and w_2. Indeed, if $d_{G[A',B']}(w_2) \geq 11D/40$, then ($b_2$) will imply that $d_{G^{\rm rob}[A',B']}(w_i)$ is large for $i=1,2$. This in turn means that after removing $G^{\rm rob}$, in the leftover graph $G^\diamond$, $d_{G^\diamond[A',B']}(w_i)$ is comparatively small, i.e. condition (11.1) will hold in the second application of Lemma 11.4.

Condition (11.1) itself is natural for the following reason: suppose for example that it is violated for w_1 and that $w_1 \in A_0$. Then for some Hamilton exceptional system J returned by the lemma, both edges of J incident to w_1 will have their other endpoint in B'. So (the edges at) w_1 cannot be used as a 'connection' between A' and B' in the Hamilton cycle which will extend J, and it may be impossible to find these connections elsewhere.

The next lemma is an analogue of Lemma 11.3 for the case when $e_G(A', B') < D$. Recall that Proposition 6.1(ii) (or Corollary 11.2) implies that in this case we have $n = 0 \pmod 4$, $D = n/2 - 1$ and $|A'| = |B'| = n/2$. In particular, $|A'|$ and $|B'|$ are both even. This agrees with the fact that the decomposition may also involve matching exceptional systems in the current case: we will later extend each such system to a cycle spanning A' and one spanning B'. As $|A'|$ and $|B'|$ are both even, these cycles correspond to two edge-disjoint perfect matchings in G.

Lemma 11.5. *Suppose that* $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \lambda, 1/K \ll 1$, *that* $0 \leq \phi \ll 1$ *and that* $n/4, K, m, \lambda n/K^2, (n/2 - 1 - \phi n)/(2K^2) \in \mathbb{N}$. *Suppose that the following conditions hold:*

(i) *G is an $(n/2-1)$-regular graph on n vertices.*
(ii) *$\mathcal{P}$ is a (K, m, ε_0)-partition of $V(G)$ such that $\Delta(G[A', B']) \leq n/4$ and $|A'| = |B'| = n/2$.*
(iii) *G_0 is a subgraph of G such that $G[A_0] + G[B_0] \subseteq G_0$ and $d_{G_0}(v) = \phi n$ for all $v \in V_0$.*
(iv) *Let $G^\diamond := G - G[A] - G[B] - G_0$. $e_{G^\diamond}(A', B')$ is even and $(G^\diamond, \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon)$-exceptional scheme.*
(v) *$\Delta(G^\diamond[A', B']) \leq e_{G^\diamond}(A', B')/2 \leq (n/2 - 1 - \phi n)/2$.*

Then there exists a set $\mathcal{J}$ consisting of $(n/2 - 1 - \phi n)/2$ edge-disjoint exceptional systems in $G^\diamond$ which satisfies the following properties:

(a) *Together the exceptional systems in $\mathcal{J}$ cover all edges of $G^\diamond$. Each J in $\mathcal{J}$ is either a Hamilton exceptional system with $e_J(A', B') = 2$ or a matching exceptional system.*
(b) *For all $1 \leq i, i' \leq K$, the set $\mathcal{J}$ contains $(n/2-1-(\phi n+2\lambda))/(2K^2)$ (i, i')-ES.*

As in the other two cases, we will use the exceptional systems in (b) to construct the robustly decomposable graph $G^{\rm rob}$. Unlike the critical case with $e_G(A', B') \geq D$, there is no need to 'track' the degrees of the vertices w_i of high degree in $G[A', B']$ this time. Indeed, let $G'' := G - G_0 - G^{\rm rob}$, where G_0 is the graph defined by Lemma 10.3. Then $G''[A', B']$ is the union of all those J in $\mathcal{J}$ (from the first application of Lemma 11.5) not used in the construction of $G^{\rm rob}$. So (a) implies that $G''[A', B']$ is a union of matchings of size two. So (v) will be trivially satisfied when we apply Lemma 11.5 for the second time (i.e. with $G_0 + G^{\rm rob}$ playing the role of G_0).

12. Special factors and exceptional factors

As discussed in the proof sketch, the main proof proceeds as follows. First we remove a sparse 'robustly decomposable' graph $G^{\rm rob}$ from the original graph G. Then we find an approximate decomposition of $G - G^{\rm rob}$. Finally we find a decomposition of $G^{\rm rob} + G'$, where G' is the (very sparse) leftover from the approximate decomposition.

Both the approximate decomposition as well as the actual decomposition step assume that we work with a graph with two components, one on A and the other on B. So in both steps, we would need $A_0 \cup B_0$ to be empty, which we clearly cannot assume. We build on the ideas of Section 7 to deal with this problem. In both steps, one can choose 'exceptional path systems' in G with the following crucial property: one can replace each such exceptional path system EPS with a path system EPS^* so that

- (α_1) EPS^* can be partitioned into EPS^*_A and EPS^*_B with the vertex sets of EPS^*_A and EPS^*_B being contained in A and B respectively;
- (α_2) the union of any Hamilton cycle C^*_A in $G^*_A := G[A]-EPS+EPS^*_A$ containing EPS^*_A and any Hamilton cycle C^*_B in $G^*_B := G[B]-EPS+EPS^*_B$ containing EPS^*_B corresponds to either a Hamilton cycle of G containing EPS or to the union of two edge-disjoint perfect matchings in G containing EPS.

Each exceptional path system EPS will contain one of the exceptional systems J constructed in Section 11. EPS^* will then be obtained from EPS by replacing J by J^*. (Recall that J^* was defined in Section 7 and that we view the edges of J^* as 'fictive edges' which are different from the edges of G.) So G^*_A is obtained from $G[A]$ by adding $J^*_A = J^*[A]$. Furthermore, J determines which of the cases in (α_2) holds: If J is a Hamilton exceptional system, then (α_2) will give a Hamilton cycle of G, while in the case when J is a matching exceptional system, (α_2) will give the union of two edge-disjoint perfect matchings in G.

So, roughly speaking, this allows us to work with G^*_A and G^*_B rather than G in the two steps. A convenient way of handling these exceptional path systems is to combine many of them into an 'exceptional factor' EF (see Section 12.2 for the definition).

One complication is that the 'robust decomposition lemma' (Lemma 13.4) we use from [20] deals with digraphs rather than undirected graphs. So in order to be able to apply it, we need to suitably orient the edges of G and so we will actually consider a directed path system $EPS^*_{\rm dir}$ instead of the EPS^* above (the exceptional path system EPS itself will still be undirected). Moreover, we have to apply the robust decomposition lemma twice, once to G^*_A and once to G^*_B.

The formulation of the robust decomposition lemma is quite general and rather than guaranteeing (α_2) directly, it assumes the existence of certain directed 'special paths systems' SPS which are combined into 'special factors' SF. These are introduced in Section 12.1. Each of the Hamilton cycles produced by the lemma then contains exactly one of these special path systems. So to apply the lemma, it suffices to check that each of our exceptional path systems EPS corresponds to two path systems $EPS^*_{A,{\rm dir}}$ and $EPS^*_{B,{\rm dir}}$ which both satisfy the conditions required of a special path system.

12.1. **Special path systems and special factors.** As mentioned above, the robust decomposition lemma requires 'special path systems' and 'special factors' as an input when constructing the robustly decomposable graph. These are defined in this subsection.

Let $K, m \in \mathbb{N}$. A (K, m)-*equipartition* $\mathcal{Q}$ of a set V of vertices is a partition of V into sets $V_1, \dots, V_K$ such that $|V_i| = m$ for all $i \leq K$. The V_i are called *clusters* of $\mathcal{Q}$. Suppose that $\mathcal{Q} = \{V_1, \dots, V_K\}$ is a (K, m)-equipartition of V and $L, m/L \in \mathbb{N}$. We say that $(\mathcal{Q}, \mathcal{Q}')$ is a (K, L, m)-*equipartition of* V if $\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by partitioning each cluster V_i of $\mathcal{Q}$ into L sets $V_{i,1}, \dots, V_{i,L}$ of size m/L. So $\mathcal{Q}'$ consists of the KL clusters $V_{i,j}$.

Let $(\mathcal{Q}, \mathcal{Q}')$ be a (K, L, m)-equipartition of V. Consider a spanning cycle $C = V_1 \dots V_K$ on the clusters of $\mathcal{Q}$. Given an integer f dividing K, the *canonical interval partition* $\mathcal{I}$ of C into f intervals consists of the intervals

$$V_{(i-1)K/f+1} V_{(i-1)K/f+2} \dots V_{iK/f+1}$$

for all $i \leq f$ (with addition modulo K).

Suppose that G is a digraph on V and $h \leq L$. Let $I = V_j V_{j+1} \dots V_{j'}$ be an interval in $\mathcal{I}$. A *special path system* SPS *of style* h *in* G *spanning the interval* I consists of m/L vertex-disjoint directed paths $P_1, \dots, P_{m/L}$ such that the following conditions hold:

(SPS1) Every P_s has its initial vertex in $V_{j,h}$ and its final vertex in $V_{j',h}$.
(SPS2) SPS contains a matching $\mathrm{Fict}(SPS)$ such that all the edges in $\mathrm{Fict}(SPS)$ avoid the endclusters V_j and $V_{j'}$ of I and such that $E(P_s) \setminus \mathrm{Fict}(SPS) \subseteq E(G)$.
(SPS3) The vertex set of SPS is $V_{j,h} \cup V_{j+1,h} \cup \dots \cup V_{j',h}$.

The edges in $\mathrm{Fict}(SPS)$ are called *fictive edges of* SPS.

Let $\mathcal{I} = \{I_1, \dots, I_f\}$. A *special factor* SF *with parameters* (L, f) *in* G *(with respect to* C, $\mathcal{Q}'$*)* is a 1-regular digraph on V which is the union of Lf digraphs $SPS_{j,h}$ (one for all $j \leq f$ and $h \leq L$) such that each $SPS_{j,h}$ is a special path system of style h in G which spans I_j. We write $\mathrm{Fict}(SF)$ for the union of the sets $\mathrm{Fict}(SPS_{j,h})$ over all $j \leq f$ and $h \leq L$ and call the edges in $\mathrm{Fict}(SF)$ *fictive edges of* SF.

We will always view fictive edges as being distinct from each other and from the edges in other digraphs. So if we say that special factors $SF_1, \dots, SF_r$ are pairwise edge-disjoint from each other and from some digraph Q on V, then this means that Q and all the $SF_i - \mathrm{Fict}(SF_i)$ are pairwise edge-disjoint, but for example there could be an edge from x to y in Q as well as in $\mathrm{Fict}(SF_i)$ for several indices $i \leq r$. But these are the only instances of multiedges that we allow, i.e. if there is more than one edge from x to y, then all but at most one of these edges are fictive edges.

12.2. **Exceptional path systems and exceptional factors.** We now introduce 'exceptional path systems' which will be combined into 'exceptional factors'. These will satisfy the requirements of special path systems and special factors respectively. So they can be used as an 'input' for the robust decomposition lemma. Moreover, they will satisfy the properties (α_1) and (α_2) described at the beginning of Section 12 (see Proposition 12.1). More precisely, suppose that

$$\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$$

is a (K, m, ε_0)-partition of a vertex set V and $L, m/L \in \mathbb{N}$. We say that $(\mathcal{P}, \mathcal{P}')$ is a (K, L, m, ε_0)-*partition of* V if $\mathcal{P}'$ is obtained from $\mathcal{P}$ by partitioning each cluster A_i

of $\mathcal{P}$ into L sets $A_{i,1},\dots,A_{i,L}$ of size m/L and partitioning each cluster B_i of $\mathcal{P}$ into L sets $B_{i,1},\dots,B_{i,L}$ of size m/L. (So $\mathcal{P}'$ consists of the exceptional sets A_0, B_0, the KL clusters $A_{i,j}$ and the KL clusters $B_{i,j}$.) Set

$$\begin{aligned} &\mathcal{Q}_A := \{A_1,\dots,A_K\}, && \mathcal{Q}'_A := \{A_{1,1},\dots,A_{K,L}\},\\ &\mathcal{Q}_B := \{B_1,\dots,B_K\}, && \mathcal{Q}'_B := \{B_{1,1},\dots,B_{K,L}\}. \end{aligned} \tag{12.1}$$

Note that $(\mathcal{Q}_A,\mathcal{Q}'_A)$ and $(\mathcal{Q}_B,\mathcal{Q}'_B)$ are (K,L,m)-equipartitions of A and B respectively (where we recall that $A=\bigcup_{i=1}^K A_i$ and $B=\bigcup_{i=1}^K B_i$).

Suppose that J is a Hamilton exceptional system (for the partition A,A_0,B,B_0) with $e_J(A',B')=2$. Thus J contains precisely two AB-paths. Let $P_1=a_1\dots b_1$ and $P_2=a_2\dots b_2$ be these two paths, where $a_1,a_2\in A$ and $b_1,b_2\in B$. Recall from Section 7 that J^*_A is the matching consisting of the edge a_1a_2 and an edge between any two vertices $a,a'\in A$ for which J contains a path $P_{aa'}$ whose endvertices are a and a'. We also defined a matching J^*_B in a similar way and set $J^*:=J^*_A\cup J^*_B$. We say that an *orientation of J is good* if every path in J is oriented consistently and one of the paths P_1, P_2 is oriented towards B while the other is oriented towards A. Given a good orientation $J_{\rm dir}$ of J, the *orientation $J^*_{\rm dir}$ of J^* induced by $J_{\rm dir}$* is defined as follows:

- For every path $P_{aa'}$ in J whose endvertices a,a' both belong to A, we orient the edge aa' of J^* towards its endpoint of the (oriented) path $P_{aa'}$ in $J_{\rm dir}$.
- If in $J_{\rm dir}$ the path P_1 is oriented towards b_1 (and thus P_2 is oriented towards a_2), then we orient the edge a_1a_2 of J^* towards a_2 and the edge b_1b_2 of J^* towards b_1. The analogue holds if P_1 is oriented towards a_1 (and thus P_2 is oriented towards b_2).

If J is a matching exceptional system, we define good orientations of J and the corresponding induced orientations of J^* in a similar way.

We now define exceptional path systems. As mentioned at the beginning of Section 12, each such exceptional path system EPS will correspond to two directed path systems $EPS^*_{A,\rm dir}$ and $EPS^*_{B,\rm dir}$ satisfying the conditions of a special path system (for $(\mathcal{Q}_A,\mathcal{Q}'_A)$ and $(\mathcal{Q}_B,\mathcal{Q}'_B)$ respectively).

Let $(\mathcal{P},\mathcal{P}')$ be a (K,L,m,ε_0)-partition of a vertex set V. Suppose that $K/f\in\mathbb{N}$. The *canonical interval partition* $\mathcal{I}(f,K)$ of $[K]:=\{1,\dots,K\}$ into f intervals consists of the intervals

$$\{(i-1)K/f+1,(i-1)K/f+2,\dots,iK/f+1\}$$

for all $i\le f$ (with addition modulo K).

Suppose that G is an oriented graph on $A\cup B$ such that $G=G[A]+G[B]$. Let $h\le L$ and suppose that $I\in\mathcal{I}(f,K)$ is an interval with $I=\{j,j+1,\dots,j'\}$. An *exceptional path system EPS of style h for G spanning I* consists of $2m/L$ vertex-disjoint undirected paths $P_0,P'_0,P^A_1,\dots,P^A_{m/L-1},P^B_1,\dots,P^B_{m/L-1}$, such that the following conditions hold:

(EPS1) $V(P^A_s)\subseteq A$ and P^A_s has one endvertex in $A_{j,h}$ and its other endvertex in $A_{j',h}$ (for all $1\le s<m/L$). The analogue holds for every P^B_s.

(EPS2) Each of P_0 and P'_0 has one endvertex in $A_{j,h} \cup B_{j,h}$ and its other endvertex in $A_{j',h} \cup B_{j',h}$.

(EPS3) $J := EPS - EPS[A] - EPS[B]$ is either a Hamilton exceptional system with $e_J(A', B') = 2$ or a matching exceptional system (with respect to the partition A, A_0, B, B_0). Moreover $E(J) \subseteq E(P_0) \cup E(P'_0)$ and no edge of J has an endvertex in $A_{j,h} \cup A_{j',h} \cup B_{j,h} \cup B_{j',h}$.

(EPS4) Let $P_{0,\mathrm{dir}}$ and $P'_{0,\mathrm{dir}}$ be the paths obtained by orienting P_0 and P'_0 towards their endvertices in $A_{j',h} \cup B_{j',h}$. Then the orientation J_{dir} of J obtained in this way is good. Let J^*_{dir} be the orientation of J^* induced by J_{dir}. Then $(P_{0,\mathrm{dir}} + P'_{0,\mathrm{dir}}) - J_{\mathrm{dir}} + J^*_{\mathrm{dir}}$ consists of two vertex-disjoint paths $P^A_{0,\mathrm{dir}}$ and $P^B_{0,\mathrm{dir}}$ such that $V(P^A_{0,\mathrm{dir}}) \subseteq A$, $P^A_{0,\mathrm{dir}}$ has one endvertex in $A_{j,h}$ and its other endvertex in $A_{j',h}$ and such that the analogue holds for $P^B_{0,\mathrm{dir}}$.

(EPS5) The vertex set of EPS is $V_0 \cup A_{j,h} \cup A_{j+1,h} \cdots \cup A_{j',h} \cup B_{j,h} \cup B_{j+1,h} \cdots \cup B_{j',h}$.

(EPS6) For each $1 \leq s < m/L$, let $P^A_{s,\mathrm{dir}}$ be the path obtained by orienting P^A_s towards its endvertex in $A_{j',h}$. Define $P^B_{s,\mathrm{dir}}$ in a similar way. Then $E(P^A_{0,\mathrm{dir}}) \setminus E(J_{\mathrm{dir}}), E(P^B_{0,\mathrm{dir}}) \setminus E(J_{\mathrm{dir}}) \subseteq E(G)$ and $E(P^A_{s,\mathrm{dir}}), E(P^B_{s,\mathrm{dir}}) \subseteq E(G)$ for every $1 \leq s < m/L$.

We call EPS *a Hamilton exceptional path system* if J (as defined in (EPS3)) is a Hamilton exceptional system, and a *matching exceptional path system* otherwise. Let $EPS^*_{A,\mathrm{dir}}$ be the (directed) path system consisting of $P^A_{0,\mathrm{dir}}, P^A_{1,\mathrm{dir}}, \dots, P^A_{m/L-1,\mathrm{dir}}$. Then $EPS^*_{A,\mathrm{dir}}$ is a special path system of style h in $G[A]$ which spans the interval $A_j A_{j+1} \dots A_{j'}$ of the cycle $A_1 \dots A_K$ and satisfies $\mathrm{Fict}(EPS^*_{A,\mathrm{dir}}) = J^*_{\mathrm{dir}}[A]$. Define $EPS^*_{B,\mathrm{dir}}$ similarly and let $EPS^*_{\mathrm{dir}} := EPS^*_{A,\mathrm{dir}} + EPS^*_{B,\mathrm{dir}}$ and $\mathrm{Fict}(EPS^*_{\mathrm{dir}}) := \mathrm{Fict}(EPS^*_{A,\mathrm{dir}}) \cup \mathrm{Fict}(EPS^*_{B,\mathrm{dir}})$ (see Figure 3).

Let $\mathcal{I}(f, K) = \{I_1, \dots, I_f\}$. An *exceptional factor EF with parameters (L, f) for G (with respect to $(\mathcal{P}, \mathcal{P}')$)* is the union of Lf edge-disjoint undirected graphs $EPS_{j,h}$ (one for all $j \leq f$ and $h \leq L$) such that each $EPS_{j,h}$ is an exceptional path system of style h for G which spans I_j. We write $EF^*_{A,\mathrm{dir}}$ for the union of $EPS^*_{j,h,A,\mathrm{dir}}$ over all $j \leq f$ and $h \leq L$. Note that $EF^*_{A,\mathrm{dir}}$ is a special factor with parameters (L, f) in $G[A]$ (with respect to $C = A_1 \dots A_K$, $\mathcal{Q}'_A$) such that $\mathrm{Fict}(EF^*_{A,\mathrm{dir}})$ is the union of $J^*_{j,h,\mathrm{dir}}[A]$ over all $j \leq f$ and $h \leq L$, where $J_{j,h}$ is the exceptional system contained in $EPS_{j,h}$ (see condition (EPS3)). Define $EF^*_{B,\mathrm{dir}}$ similarly and let $EF^*_{\mathrm{dir}} := EF^*_{A,\mathrm{dir}} + EF^*_{B,\mathrm{dir}}$ and $\mathrm{Fict}(EF^*_{\mathrm{dir}}) := \mathrm{Fict}(EF^*_{A,\mathrm{dir}}) \cup \mathrm{Fict}(EF^*_{B,\mathrm{dir}})$. Note that EF^*_{dir} is a 1-regular directed graph on $A \cup B$ while in EF is an undirected graph on V with

$$d_{EF}(v) = 2 \text{ for all } v \in V \setminus V_0 \quad \text{and} \quad d_{EF}(v) = 2Lf \text{ for all } v \in V_0. \tag{12.2}$$

Given an exceptional path system EPS, let J be as in (EPS3) and let

$$EPS^* := EPS - J + J^*, \quad EPS^*_A := EPS^*[A] \text{ and } EPS^*_B := EPS^*[B].$$

(Hence EPS^*, EPS^*_A and EPS^*_B are the undirected graphs obtained from EPS^*_{dir}, $EPS^*_{A,\mathrm{dir}}$ and $EPS^*_{B,\mathrm{dir}}$ by ignoring the orientations of all edges.) The following result is an immediate consequence of (EPS3), (EPS4) and Proposition 7.1. Roughly speaking, it implies that to find a Hamilton cycle in the 'original' graph with vertex

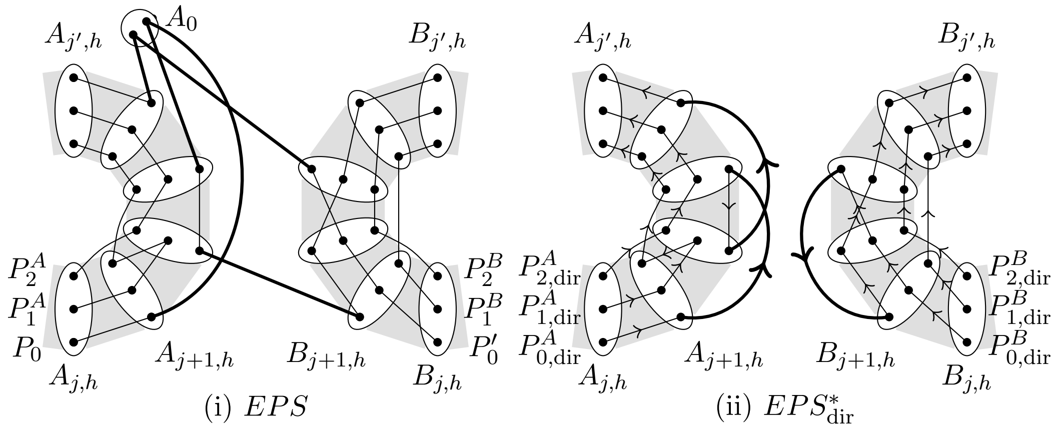

FIGURE 3. An example of an exceptional path system EPS and the corresponding directed version EPS^*_{dir} in the case when $|A_0| = 2$, $B_0 = \emptyset$, $m/L = 3$ and $|I| = 6$. The thick edges indicate J and J^*_{dir} respectively.

set V, it suffices to find a Hamilton cycle on A and one on B, containing (the edges corresponding to) an exceptional path system.

Proposition 12.1. *Let $(\mathcal{P}, \mathcal{P}')$ be a (K, L, m, ε_0)-partition of a vertex set V. Suppose that G is a graph on $V \setminus V_0$, that G_{dir} is an orientation of $G[A] + G[B]$ and that EPS is an exceptional path system for G_{dir}. Let J be as in (EPS3) and J^*_A as defined in Section 7. Let C_A and C_B be two cycles such that*

- *C_A is a Hamilton cycle on A which contains EPS^*_A;*
- *C_B is a Hamilton cycle on B which contains EPS^*_B.*

Then the following assertions hold.

(i) *If EPS is a Hamilton exceptional path system, then $C_A + C_B - EPS^* + EPS$ is a Hamilton cycle on V.*
(ii) *If EPS is a matching exceptional path system, then $C_A + C_B - EPS^* + EPS$ is the union of a Hamilton cycle on A' and a Hamilton cycle on B'. In particular, if both $|A'|$ and $|B'|$ are even, then $C_A + C_B - EPS^* + EPS$ is the union of two edge-disjoint perfect matchings on V.*

Proof. Note that $C_A + C_B - EPS^* + EPS = C_A + C_B - J^* + J$. Recall that J^*_{AB} was defined in Section 7. (EPS3) implies that $|E(J^*_A) \setminus E(J^*_{AB})| \leq 1$. Recall from Section 7 that a path P is said to consistent with J^*_A if P contains J^*_A and (there is an orientation of P which) visits the endvertices of the edges in $E(J^*_A) \setminus E(J^*_{AB})$ in a prescribed order. Since $E(J^*_A) \setminus E(J^*_{AB})$ contains at most one edge, any path containing J^*_A is also consistent with J^*_A. Therefore, C_A is consistent with J^*_A and, by a similar argument, C_B is consistent with J^*_B. So the proposition follows immediately from Proposition 7.1. □

12.3. **Finding exceptional factors in a scheme.** The next lemma (Lemma 12.2) will allow us to extend a suitable exceptional system J into an exceptional path system. In particular, we assume that J is 'localized'. This allows us to choose the path system in such a way that it spans only a few clusters. The structure within which we find the path system is called a 'scheme'. Roughly speaking, this is the structure we obtain from $G[A]+G[B]$ (i.e. the union of two almost complete graphs) by considering a random equipartition of A and B and a random orientation of its edges.

We now define this 'oriented' version of the (undirected) schemes which were introduced in Section 8. Given an oriented graph G and partitions $\mathcal{P}$ and $\mathcal{P}'$ of a vertex set V, we call $(G,\mathcal{P},\mathcal{P}')$ a $[K,L,m,\varepsilon_0,\varepsilon]$-*scheme* if the following conditions hold:

(Sch1′) $(\mathcal{P},\mathcal{P}')$ is a (K,L,m,ε_0)-partition of V.
(Sch2′) $V(G)=A\cup B$ and $e_G(A,B)=0$.
(Sch3′) $G[A_{i,j},A_{i',j'}]$ and $G[B_{i,j},B_{i',j'}]$ are $[\varepsilon,1/2]$-superregular for all $i,i'\le K$ and all $j,j'\le L$ such that $(i,j)\neq(i',j')$. Moreover, $G[A_i,A_{i'}]$ and $G[B_i,B_{i'}]$ are $[\varepsilon,1/2]$-superregular for all $i\neq i'\le K$.
(Sch4′) $|N^+_G(x)\cap N^-_G(y)\cap A_{i,j}|\ge(1/5-\varepsilon)m/L$ for all $x,y\in A$, all $i\le K$ and all $j\le L$. Similarly, $|N^+_G(x)\cap N^-_G(y)\cap B_{i,j}|\ge(1/5-\varepsilon)m/L$ for all $x,y\in B$, all $i\le K$ and all $j\le L$.

Note that if $L=1$ (and so $\mathcal{P}=\mathcal{P}'$), then (Sch1′) just says that $\mathcal{P}$ is a (K,m,ε_0)-partition of V.

Suppose that J is an (i,i')-ES with respect to $\mathcal{P}$. Given $h\le L$, we say that J has *style h (with respect to the (K,L,m,ε_0)-partition $(\mathcal{P},\mathcal{P}')$)* if all the edges of J have their endvertices in $V_0\cup A_{i,h}\cup B_{i',h}$.

Lemma 12.2. *Suppose that $K,L,n,m/L\in\mathbb{N}$, that $0<1/n\ll\varepsilon,\varepsilon_0\ll 1$ and $\varepsilon_0\ll 1/K,1/L$. Let $(G,\mathcal{P},\mathcal{P}')$ be a $[K,L,m,\varepsilon_0,\varepsilon]$-scheme with $|V(G)\cup V_0|=n$. Let $I=\{j,j+1,\dots,j'\}\subseteq[K]$ be an integer interval with $|I|\ge 4$. Let J be either an (i_1,i_2)-HES of style $h\le L$ with $e_J(A',B')=2$ or an (i_1,i_2)-MES of style $h\le L$ (with respect to $(\mathcal{P},\mathcal{P}')$), for some $i_1,i_2\in\{j+1,\dots,j'-1\}$. Then there exists an exceptional path system of style h for G which spans the interval I and contains all edges of J.*

Proof. Let $J_{\rm dir}$ be a good orientation of J and let $J^*_{\rm dir}$ be the induced orientation of J^*. Let $x_1x_2,\dots,x_{2s'-1}x_{2s'}$ be the edges of $J^*_{A,\rm dir}:=J^*_{\rm dir}[A]$. Since J is an (i_1,i_2)-ES of style h with $e_J(A',B')\le 2$ it follows that $s'=e(J^*_A)\le|V_0|+1\le 2\varepsilon_0 n$ and $x_i\in A_{i_1,h}$ for all $i\le 2s'$. Since $|I|\ge 4$ we have $i_1+1\in\{j+1,\dots,j'-1\}$ or $i_1-1\in\{j+1,\dots,j'-1\}$. We will only consider the case when $i_1+1\in\{j+1,\dots,j'-1\}$. (The argument for the other case is similar.)

Our assumption that $\varepsilon_0 \ll 1/K, 1/L$ implies that $\varepsilon_0 n \leq m/100L$ (say). Together with (Sch4$'$) this ensures that for every $1 \leq r < s'$, we can pick a vertex $w_r \in A_{i_1+1,h}$ such that $x_{2r}w_r$ and $w_r x_{2r+1}$ are (directed) edges in G and such that $w_1, \dots, w_{s'-1}$ are distinct from each other. We also pick a vertex $w_{s'} \in A_{i_1+1,h} \setminus \{w_1, \dots, w_{s'-1}\}$ such that $x_{2s'}w_{s'}$ is a (directed) edge in G. Let Q_0 be the path $x_1x_2w_1x_3x_4w_2 \dots x_{2s'-1}x_{2s'}w_{s'}$. Thus Q_0 is a directed path from $A_{i_1,h}$ to $A_{i_1+1,h}$ in $G + J^*_{\rm dir}$ which contains all edges of $J^*_{A,\rm dir}$. Note that $|V(Q_0) \cap A_{i_1,h}| = 2s'$ and $|V(Q_0) \cap A_{i_1+1,h}| = s'$. Moreover, $V(Q_0) \cap A_i = \emptyset$ for all $i \notin \{i_1, i_1+1\}$ and $V(Q_0) \cap B = \emptyset$.

Pick a vertex $w_0 \in A_{j,h}$ so that w_0x_1 is an edge of G. Find a path Q_0' from $w_{s'}$ to $A_{j',h}$ in G such that the vertex set of Q_0' consists of $w_{s'}$ and precisely one vertex in each $A_{i,h}$ for all $i \in \{j+1, \dots, j'\} \setminus \{i_1, i_1+1\}$ and no other vertices. (Sch4$'$) ensures that this can be done greedily. Define $P^A_{0,\rm dir}$ to be the concatenation of w_0x_1, Q_0 and Q_0'. Note that $P^A_{0,\rm dir}$ is a directed path from $A_{j,h}$ to $A_{j',h}$ in $G + J^*_{\rm dir}$ which contains $J^*_{A,\rm dir}$. Moreover,

$$|V(P^A_{0,\rm dir}) \cap A_{i,h}| = \begin{cases} 1 & \text{for } i \in \{j, \dots, j'\} \setminus \{i_1, i_1+1\}, \\ 2s' & \text{for } i = i_1, \\ s' & \text{for } i = i_1+1, \\ 0 & \text{otherwise,} \end{cases}$$

while $V(P^A_{0,\rm dir}) \cap B = \emptyset$ and $V(P^A_{0,\rm dir}) \cap A_{i,h'} = \emptyset$ for all $i \leq K$ and all $h' \neq h$. (Sch4$'$) ensures that we can also choose $2s'-1$ (directed) paths $P^A_{1,\rm dir}, \dots, P^A_{2s'-1,\rm dir}$ in G such that the following conditions hold:

- For all $1 \leq r < 2s'$, $P^A_{r,\rm dir}$ is a path from $A_{j,h}$ to $A_{j',h}$.
- For all $1 \leq r \leq s'$, $P^A_{r,\rm dir}$ contains precisely one vertex in $A_{i,h}$ for each $i \in \{j, \dots, j'\} \setminus \{i_1\}$ and no other vertices.
- For all $s' < r < 2s'$, $P^A_{r,\rm dir}$ contains precisely one vertex in $A_{i,h}$ for each $i \in \{j, \dots, j'\} \setminus \{i_1, i_1+1\}$ and no other vertices.
- $P^A_{0,\rm dir}, \dots, P^A_{2s'-1,\rm dir}$ are pairwise vertex-disjoint.

Let Q be the union of $P^A_{0,\rm dir}, \dots, P^A_{2s'-1,\rm dir}$. Thus Q is a path system consisting of $2s'$ vertex-disjoint directed paths from $A_{j,h}$ to $A_{j',h}$. Moreover, $V(Q)$ consists of precisely $2s'$ vertices in $A_{i,h}$ for every $j \leq i \leq j'$ and no other vertices. Set $A'_{i,h} := A_{i,h} \setminus V(Q)$ for all $i \leq K$. Note that

$$|A'_{i,h}| = \frac{m}{L} - 2s' \geq \frac{m}{L} - 4\varepsilon_0 n \geq \frac{m}{L} - 10\varepsilon_0 mK \geq (1 - \sqrt{\varepsilon_0})\frac{m}{L} \tag{12.3}$$

since $\varepsilon_0 \ll 1/K, 1/L$. Pick a new constant ε' such that $\varepsilon, \varepsilon_0 \ll \varepsilon' \ll 1$. Then Proposition 5.3, (Sch3$'$) and (12.3) together imply that $G[A'_{i,h}, A'_{i+1,h}]$ is still $[\varepsilon', 1/2]$-superregular and so by Proposition 5.4 we can find a perfect matching in $G[A'_{i,h}, A'_{i+1,h}]$ for all $j \leq i < j'$. The union Q' of all these matchings forms $m/L - 2s'$ vertex-disjoint directed paths $P^A_{2s',\mathrm{dir}},\dots,P^A_{m/L-1,\mathrm{dir}}$. Note that $P^A_{0,\mathrm{dir}},P^A_{1,\mathrm{dir}},\dots,P^A_{m/L-1,\mathrm{dir}}$ are pairwise vertex-disjoint and together cover precisely the vertices in $\bigcup_{i=j}^{j'} A_{i,h}$. Moreover, $P^A_{0,\mathrm{dir}}$ contains $J^*_{A,\mathrm{dir}}$.

Similarly, we find m/L vertex-disjoint directed paths $P^B_{0,\mathrm{dir}},P^B_{1,\mathrm{dir}},\dots,P^B_{m/L-1,\mathrm{dir}}$ from $B_{j,h}$ to $B_{j',h}$ such that $P^B_{0,\mathrm{dir}}$ contains $J^*_{B,\mathrm{dir}}$ and together the paths cover precisely the vertices in $\bigcup_{i=j}^{j'} B_{i,h}$. For each $1\le r<m/L$, let P^A_r and P^B_r be the undirected paths obtained from $P^A_{r,\mathrm{dir}}$ and $P^B_{r,\mathrm{dir}}$ by ignoring the directions of all the edges.

Since $J^*_{A,\mathrm{dir}}\subseteq P^A_{0,\mathrm{dir}}$ and $J^*_{B,\mathrm{dir}}\subseteq P^B_{0,\mathrm{dir}}$ and since J^*_{dir} is the orientation of J^* induced by J_{dir}, it follows that $P^A_{0,\mathrm{dir}}+P^B_{0,\mathrm{dir}}-J^*_{\mathrm{dir}}+J_{\mathrm{dir}}$ consists of two vertex-disjoint paths $P_{0,\mathrm{dir}}$ and $P'_{0,\mathrm{dir}}$ from $A_{j,h}\cup B_{j,h}$ to $A_{j',h}\cup B_{j',h}$ with $V(P_{0,\mathrm{dir}})\cup V(P'_{0,\mathrm{dir}})=V_0\cup V(P^A_{0,\mathrm{dir}})\cup V(P^B_{0,\mathrm{dir}})$. Let P_0 and P'_0 be the undirected paths obtained from $P_{0,\mathrm{dir}}$ and $P'_{0,\mathrm{dir}}$ by ignoring the directions of all the edges. Let EPS be the union of $P_0,P'_0,P^A_1,\dots,P^A_{m/L-1},P^B_1,\dots,P^B_{m/L-1}$. Then EPS is an exceptional path system for G, as required. To see this, note that $J=EPS-EPS[A]-EPS[B]$ since $e_J(A),e_J(B)=0$ by the definition of an exceptional system (see (EC3) in Section 7). □

The next lemma uses the previous one to show that we can obtain many edge-disjoint exceptional factors by extending exceptional systems with suitable properties.

Lemma 12.3. *Suppose that $L,f,q,n,m/L,K/f\in\mathbb{N}$, that $K/f\ge 3$, that $0<1/n\ll\varepsilon,\varepsilon_0\ll 1$, that $\varepsilon_0\ll 1/K,1/L$ and $Lq/m\ll 1$. Let $(G,\mathcal{P},\mathcal{P}')$ be a $[K,L,m,\varepsilon_0,\varepsilon]$-scheme with $|V(G)\cup V_0|=n$. Suppose that there exists a set $\mathcal{J}$ of Lfq edge-disjoint exceptional systems satisfying the following conditions:*

- (i) *Each $J\in\mathcal{J}$ is either a Hamilton exceptional system with $e_J(A',B')=2$ or a matching exceptional system.*
- (ii) *For all $i\le f$ and all $h\le L$, $\mathcal{J}$ contains precisely q (i_1,i_2)-ES of style h (with respect to $(\mathcal{P},\mathcal{P}')$) for which $i_1,i_2\in\{(i-1)K/f+2,\dots,iK/f\}$.*

Then there exist q edge-disjoint exceptional factors with parameters (L,f) for G (with respect to $(\mathcal{P},\mathcal{P}')$) covering all edges in $\bigcup\mathcal{J}$.

Recall that the canonical interval partition $\mathcal{I}(f,K)$ of $[K]$ into f intervals consists of the intervals $\{(i-1)K/f+1,\dots,iK/f+1\}$ for all $i\le f$. So (ii) ensures that for each interval $I\in\mathcal{I}(f,K)$ and each $h\le L$, the set $\mathcal{J}$ contains precisely q exceptional systems of style h whose edges are only incident to vertices in V_0 and vertices belonging to clusters A_{i_1} and B_{i_2} for which both i_1 and i_2 lie in the interior of I. We will use Lemma 12.2 to extend each such exceptional system into an exceptional path system of style h spanning I.

Proof of Lemma 12.3. Choose a new constant ε' with $\varepsilon,Lq/m\ll\varepsilon'\ll 1$. Let $\mathcal{J}_1,\dots,\mathcal{J}_q$ be a partition of $\mathcal{J}$ such that for all $j\le q$, $h\le L$ and $i\le f$, the set $\mathcal{J}_j$ contains precisely one (i_1,i_2)-ES of style h with $i_1,i_2\in\{(i-1)K/f+2,\dots,iK/f\}$.

Thus each $\mathcal{J}_j$ consists of Lf exceptional systems. For each $j \le q$ in turn, we will choose an exceptional factor EF_j with parameters (L, f) for G (with respect to $(\mathcal{P}, \mathcal{P}')$) such that EF_j and $EF_{j'}$ are edge-disjoint for all $j' < j$ and EF_j contains all edges of the exceptional systems in $\mathcal{J}_j$. Assume that for some $1 \le j \le q$ we have already constructed $EF_1, \dots, EF_{j-1}$. In order to construct EF_j, we will choose the Lf exceptional path systems forming EF_j one by one, such that each of these exceptional path systems is edge-disjoint from $EF_1, \dots, EF_{j-1}$ and contains precisely one of the exceptional systems in $\mathcal{J}_j$. Suppose that we have already chosen some of these exceptional path systems and that next we wish to choose an exceptional path system of style h which spans the interval I of the canonical interval partition $\mathcal{I}(f, K)$ and contains $J \in \mathcal{J}_j$. Let G' be the oriented graph obtained from G by deleting all the edges in the path systems already chosen for EF_j as well as deleting all the edges in $EF_1, \dots, EF_{j-1}$. Recall that $V(G) = A \cup B$. Thus $\Delta(G - G') \le 2j < 3q$ by (12.2). Together with Proposition 5.3 this implies that $(G', \mathcal{P}, \mathcal{P}')$ is still a $[K, L, m, \varepsilon_0, \varepsilon']$-scheme. (Here we use that $\Delta(G - G') < 3q = 3Lq/m \cdot m/L$ and $\varepsilon, Lq/m \ll \varepsilon' \ll 1$.) So we can apply Lemma 12.2 with ε' playing the role of ε to obtain an exceptional path system of style h for G' (and thus for G) which spans I and contains all edges of J. This completes the proof of the lemma. □

13. The robust decomposition lemma

The aim of this section is to state the robust decomposition lemma (Lemma 13.4). This is the key lemma proved in [20] and guarantees the existence of a 'robustly decomposable' digraph $G^{\rm rob}_{\rm dir}$ within a 'setup'. For our purposes, we will then derive an undirected version in Corollary 13.5 to construct a robustly decomposable graph $G^{\rm rob}$. Then $G^{\rm rob} + H$ will have a Hamilton decomposition for any sparse regular graph H which is edge-disjoint from $G^{\rm rob}$. The crucial ingredient of a setup is a 'universal walk', which we introduce in the next subsection. The (proof of the) robust decomposition lemma then uses edges guaranteed by this universal walk to 'balance out' edges of the graph H when constructing the Hamilton decomposition of $G^{\rm rob} + H$.

13.1. **Chord sequences and universal walks.** Let R be a digraph whose vertices are $V_1, \dots, V_k$ and suppose that $C = V_1 \dots V_k$ is a Hamilton cycle of R. (Later on the vertices of R will be clusters. So we denote them by capital letters.)

A *chord sequence* $CS(V_i, V_j)$ from V_i to V_j in R is an ordered sequence of edges of the form

$$CS(V_i, V_j) = (V_{i_1-1}V_{i_2}, V_{i_2-1}V_{i_3}, \dots, V_{i_t-1}V_{i_{t+1}}),$$

where $V_{i_1} = V_i$, $V_{i_{t+1}} = V_j$ and the edge $V_{i_s-1}V_{i_{s+1}}$ belongs to R for each $s \le t$.

If $i = j$ then we consider the empty set to be a chord sequence from V_i to V_j. Without loss of generality, we may assume that $CS(V_i, V_j)$ does not contain any edges of C. (Indeed, suppose that $V_{i_s-1}V_{i_{s+1}}$ is an edge of C. Then $i_s = i_{s+1}$ and so we can obtain a chord sequence from V_i to V_j with fewer edges.) For example, if $V_{i-1}V_{i+1} \in E(R)$, then the edge $V_{i-1}V_{i+1}$ is a chord sequence from V_i to V_{i+1}.

The crucial property of chord sequences is that they satisfy a 'local balance' condition. Suppose that CS is obtained by concatenating several chord sequences

$$CS(V_{i_1}, V_{i_2}), CS(V_{i_2}, V_{i_3}), \dots, CS(V_{i_{k-1}}, V_{i_k})$$

so that $V_{i_1} = V_{i_k}$. Then for every cluster V_i, the number of edges of CS leaving V_{i-1} equals the number of edges entering V_i. We will not use this property explicitly, but it underlies the proof of the robust decomposition lemma (Lemma 13.4) that we apply and appears implicitly e.g. in (U3).

A closed walk U in R is a *universal walk for C with parameter ℓ'* if the following conditions hold:

- (U1) For every $i \leq k$ there is a chord sequence $ECS(V_i, V_{i+1})$ from V_i to V_{i+1} such that U contains all edges of all these chord sequences (counted with multiplicities) and all remaining edges of U lie on C.
- (U2) Each $ECS(V_i, V_{i+1})$ consists of at most $\sqrt{\ell'}/2$ edges.
- (U3) U enters each V_i exactly ℓ' times and leaves each V_i exactly ℓ' times.

Note that condition (U1) means that if an edge $V_iV_j \in E(R) \setminus E(C)$ occurs in total 5 times (say) in $ECS(V_1, V_2), \dots, ECS(V_k, V_1)$ then it occurs precisely 5 times in U. We will identify each occurrence of V_iV_j in $ECS(V_1, V_2), \dots, ECS(V_k, V_1)$ with a (different) occurrence of V_iV_j in U. Note that the edges of $ECS(V_i, V_{i+1})$ are allowed to appear in a different order within $ECS(V_i, V_{i+1})$ and within U.

Lemma 13.1. *Let R be a digraph with vertices $V_1, \dots, V_k$. Suppose that $C = V_1 \dots V_k$ is a Hamilton cycle of R and that $V_iV_{i+2} \in E(R)$ for every $1 \leq i \leq k$. Let $\ell' \geq 4$ be an integer. Let $U_{\ell'}$ the multiset obtained from $\ell' - 1$ copies of $E(C)$ by adding $V_iV_{i+2} \in E(R)$ for every $1 \leq i \leq k$. Then the edges in $U_{\ell'}$ can be ordered so that the resulting sequence forms a universal walk for C with parameter ℓ'.*

In the remainder of the paper, we will also write $U_{\ell'}$ for the universal walk guaranteed by Lemma 13.1.

Proof. Let us first show that the edges in $U_{\ell'}$ can be ordered so that the resulting sequence forms a closed walk in R. To see this, consider the multidigraph U obtained from $U_{\ell'}$ by deleting one copy of $E(C)$. Then U is $(\ell' - 1)$-regular and thus has a decomposition into 1-factors. We order the edges of $U_{\ell'}$ as follows: We first traverse all cycles of the 1-factor decomposition of U which contain the cluster V_1. Next, we traverse the edge V_1V_2 of C. Next we traverse all those cycles of the 1-factor decomposition which contain V_2 and which have not been traversed so far. Next we traverse the edge V_2V_3 of C and so on until we reach V_1 again.

Recall that, for each $1 \leq i \leq k$, the edge $V_{i-1}V_{i+1}$ is a chord sequence from V_i to V_{i+1}. Thus we can take $ECS(V_i, V_{i+1}) := V_{i-1}V_{i+1}$. Then $U_{\ell'}$ satisfies (U1)–(U3). □

13.2. **Setups and the robust decomposition lemma.** The aim of this subsection is to state the robust decomposition lemma (Lemma 13.4, proved in [20]) and derive Corollary 13.5, which we shall use later on. The robust decomposition lemma guarantees the existence of a 'robustly decomposable' digraph $G^{\mathrm{rob}}_{\mathrm{dir}}$ within a 'setup'.

Roughly speaking, a setup is a digraph G together with its 'reduced digraph' R, which contains a Hamilton cycle C and a universal walk U. In our application, we will have two setups: $G[A]$ and $G[B]$ will play the role of G, and R will be the complete digraph in both cases. To define a setup formally, we first need to define certain 'refinements' of partitions.

Given a digraph G and a partition $\mathcal{P}$ of $V(G)$ into k clusters $V_1, \dots, V_k$ of equal size, we say that a partition $\mathcal{P}'$ of V is an *ℓ'-refinement of $\mathcal{P}$* if $\mathcal{P}'$ is obtained by splitting each V_i into ℓ' subclusters of equal size. (So $\mathcal{P}'$ consists of $\ell' k$ clusters.) $\mathcal{P}'$ is an *ε-uniform ℓ-refinement* of $\mathcal{P}$ if it is an ℓ-refinement of $\mathcal{P}$ which satisfies the following condition: Whenever x is a vertex of G, V is a cluster in $\mathcal{P}$ and $|N^+_G(x) \cap V| \geq \varepsilon|V|$ then $|N^+_G(x) \cap V'| = (1 \pm \varepsilon)|N^+_G(x) \cap V|/\ell$ for each cluster $V' \in \mathcal{P}'$ with $V' \subseteq V$. The inneighbourhoods of the vertices of G satisfy an analogous condition. We need the following simple observation from [20]. The proof proceeds by considering a random partition to obtain a uniform refinement.

Lemma 13.2. *Suppose that $0 < 1/m \ll 1/k, \varepsilon \ll \varepsilon', d, 1/\ell \leq 1$ and that $n, k, \ell, m/\ell \in \mathbb{N}$. Suppose that G is a digraph on $n = km$ vertices and that $\mathcal{P}$ is a partition of $V(G)$ into k clusters of size m. Then there exists an ε-uniform ℓ-refinement of $\mathcal{P}$. Moreover, any ε-uniform ℓ-refinement $\mathcal{P}'$ of $\mathcal{P}$ automatically satisfies the following condition:*

- *Suppose that V, W are clusters in $\mathcal{P}$ and V', W' are clusters in $\mathcal{P}'$ with $V' \subseteq V$ and $W' \subseteq W$. If $G[V, W]$ is $[\varepsilon, d']$-superregular for some $d' \geq d$ then $G[V', W']$ is $[\varepsilon', d']$-superregular.*

We will also need the following definition from [20]. $(G, \mathcal{P}, \mathcal{P}', R, C, U, U')$ is called an *$(\ell', k, m, \varepsilon, d)$-setup* if the following properties are satisfied:

(ST1) G and R are digraphs. $\mathcal{P}$ is a partition of $V(G)$ into k clusters of size m. The vertex set of R consists of these clusters.
(ST2) For every edge VW of R the corresponding pair $G[V, W]$ is $(\varepsilon, \geq d)$-regular.
(ST3) C is a Hamilton cycle of R and for every edge VW of C the corresponding pair $G[V, W]$ is $[\varepsilon, \geq d]$-superregular.
(ST4) U is a universal walk for C with parameter ℓ' and $\mathcal{P}'$ is an ε-uniform ℓ'-refinement of $\mathcal{P}$.
(ST5) Suppose that $C = V_1 \dots V_k$ and let $V_j^1, \dots, V_j^{\ell'}$ denote the clusters in $\mathcal{P}'$ which are contained in V_j (for each $1 \leq j \leq k$). Then U' is a closed walk on the clusters in $\mathcal{P}'$ which is obtained from U as follows: When U visits V_j for the ath time, we let U' visit the subcluster V_j^a (for all $1 \leq a \leq \ell'$).
(ST6) Each edge of U' corresponds to an $[\varepsilon, \geq d]$-superregular pair in G.

In [20], in a setup, the digraph G could also contain an exceptional set, but since we are only using the definition in the case when there is no such exceptional set, we have only stated it in this special case.

Suppose that $(G, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme. Recall that $A_1, \dots, A_K$ and $B_1, \dots, B_K$ denote the clusters of $\mathcal{P}$. Let $\mathcal{Q}_A := \{A_1, \dots, A_K\}$, $\mathcal{Q}_B := \{B_1, \dots, B_K\}$ and let $C_A = A_1 \dots A_K$ and $C_B = B_1 \dots B_K$ be (directed) cycles. Suppose that $\ell', m/\ell' \in \mathbb{N}$ with $\ell' \geq 4$. Let $\mathcal{Q}'_A$ be an ε-uniform ℓ'-refinement of $\mathcal{Q}_A$. Let R_A be the complete digraph whose vertices are the clusters in $\mathcal{Q}_A$. Let $U_{A,\ell'}$ be a universal walk for C_A with parameter ℓ' as defined in Lemma 13.1. Let $U'_{A,\ell'}$ be the closed walk obtained from $U_{A,\ell'}$ as described in (ST5). We will call

$$(G[A], \mathcal{Q}_A, \mathcal{Q}'_A, R_A, C_A, U_{A,\ell'}, U'_{A,\ell'})$$

the *A-setup associated to* $(G, \mathcal{P}, \mathcal{P}')$. Define $\mathcal{Q}'_B$, R_B, $U_{B,\ell'}$ and $U'_{B,\ell'}$ similarly. We will call

$$(G[B], \mathcal{Q}_B, \mathcal{Q}'_B, R_B, C_B, U_{B,\ell'}, U'_{B,\ell'})$$

the *B-setup associated to* $(G, \mathcal{P}, \mathcal{P}')$. The following lemma shows that both the A-setup and the B-setup indeed satisfy all the conditions in the definition of a setup.

Lemma 13.3. *Suppose that $1/m \ll 1/K, \varepsilon_0, \varepsilon \ll \varepsilon', 1/\ell'$ and $K, L, m/L, \ell', m/\ell' \in \mathbb{N}$ with $\ell' \geq 4$. Suppose that $(G, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme. Then each of*

$$(G[A], \mathcal{Q}_A, \mathcal{Q}'_A, R_A, C_A, U_{A,\ell'}, U'_{A,\ell'}) \quad \textit{and} \quad (G[B], \mathcal{Q}_B, \mathcal{Q}'_B, R_B, C_B, U_{B,\ell'}, U'_{B,\ell'})$$

is an $(\ell', K, m, \varepsilon', 1/2)$-setup.

Proof. It suffices to show that $(G[A], \mathcal{Q}_A, \mathcal{Q}'_A, R_A, C_A, U_{A,\ell'}, U'_{A,\ell'})$ is an $(\ell', K, m, \varepsilon', 1/2)$-setup. Clearly, (ST1) holds. (Sch3$'$) implies that (ST2) and (ST3) hold. Lemma 13.1 implies (ST4). (ST5) follows from the definition of $U'_{A,\ell'}$. (ST6) follows from Lemma 13.2 since $\mathcal{Q}'_A$ is an ε-uniform ℓ'-refinement of $\mathcal{Q}_A$. □

We now state the robust decomposition lemma from [20]. Recall that this guarantees the existence of a 'robustly decomposable' digraph $G^{\rm rob}_{\rm dir}$, whose crucial property is that $H + G^{\rm rob}_{\rm dir}$ has a Hamilton decomposition for any sparse regular digraph H which is edge-disjoint from $G^{\rm rob}_{\rm dir}$.

$G^{\rm rob}_{\rm dir}$ consists of digraphs $CA_{\rm dir}(r)$ (the 'chord absorber') and $PCA_{\rm dir}(r)$ (the 'parity extended cycle switcher') together with some special factors. $G^{\rm rob}_{\rm dir}$ is constructed in two steps: given a suitable set $\mathcal{SF}$ of special factors, the lemma first 'constructs' $CA_{\rm dir}(r)$ and then, given another suitable set $\mathcal{SF}'$ of special factors, the lemma 'constructs' $PCA_{\rm dir}(r)$. The reason for having two separate steps is that in [20], it is not clear how to construct $CA_{\rm dir}(r)$ after constructing $\mathcal{SF}'$ (rather than before), as the removal of $\mathcal{SF}'$ from the digraph under consideration affects its properties considerably.

Lemma 13.4. *Suppose that $0 < 1/m \ll 1/k \ll \varepsilon \ll 1/q \ll 1/f \ll r_1/m \ll d \ll 1/\ell', 1/g \ll 1$ and that $rk^2 \leq m$. Let*

$$r_2 := 96\ell' g^2 kr, \quad r_3 := rfk/q, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s' := rfk + 7r^\diamond$$

and suppose that $k/14, k/f, k/g, q/f, m/4\ell', fm/q, 2fk/3g(g-1) \in \mathbb{N}$. Suppose that $(G, \mathcal{P}, \mathcal{P}', R, C, U, U')$ is an $(\ell', k, m, \varepsilon, d)$-setup and $C = V_1 \dots V_k$. Suppose that $\mathcal{P}^*$ is a (q/f)-refinement of $\mathcal{P}$ and that $SF_1, \dots, SF_{r_3}$ are edge-disjoint special factors with parameters $(q/f, f)$ with respect to C, $\mathcal{P}^*$ in G. Let $\mathcal{SF} := SF_1 + \dots + SF_{r_3}$. Then there exists a digraph $CA_{\rm dir}(r)$ for which the following holds:

(i) *$CA_{\rm dir}(r)$ is an $(r_1 + r_2)$-regular spanning subdigraph of G which is edge-disjoint from $\mathcal{SF}$.*

(ii) *Suppose that $SF'_1, \dots, SF'_{r^\diamond}$ are special factors with parameters $(1,7)$ with respect to C, $\mathcal{P}$ in G which are edge-disjoint from each other and from $CA_{\rm dir}(r) + \mathcal{SF}$. Let $\mathcal{SF}' := SF'_1 + \dots + SF'_{r^\diamond}$. Then there exists a digraph $PCA_{\rm dir}(r)$ for which the following holds:*

  (a) *$PCA_{\rm dir}(r)$ is a $5r^\diamond$-regular spanning subdigraph of G which is edge-disjoint from $CA_{\rm dir}(r) + \mathcal{SF} + \mathcal{SF}'$.*

  (b) *Let $\mathcal{SPS}$ be the set consisting of all the s' special path systems contained in $\mathcal{SF} + \mathcal{SF}'$. Suppose that H is an r-regular digraph on $V(G)$ which is edge-disjoint from $G^{\rm rob}_{\rm dir} := CA_{\rm dir}(r) + PCA_{\rm dir}(r) + \mathcal{SF} + \mathcal{SF}'$. Then $H + G^{\rm rob}_{\rm dir}$ has a decomposition into s' edge-disjoint Hamilton cycles $C_1, \dots, C_{s'}$. Moreover, C_i contains one of the special path systems from $\mathcal{SPS}$, for each $i \le s'$.*

Recall from Section 12.1 that we always view fictive edges in special factors as being distinct from each other and from the edges in other graphs. So for example, saying that $CA_{\rm dir}(r)$ and $\mathcal{SF}$ are edge-disjoint in Lemma 13.4 still allows for a fictive edge xy in $\mathcal{SF}$ to occur in $CA_{\rm dir}(r)$ as well (but $CA_{\rm dir}(r)$ will avoid all non-fictive edges in $\mathcal{SF}$).

We will use the following 'undirected' consequence of Lemma 13.4.

Corollary 13.5. *Suppose that $0 < 1/m \ll \varepsilon_0, 1/K \ll \varepsilon \ll 1/L \ll 1/f \ll r_1/m \ll 1/\ell', 1/g \ll 1$ and that $rK^2 \le m$. Let*

$$r_2 := 96\ell' g^2 K r, \quad r_3 := rK/L, \quad r^\diamond := r_1 + r_2 + r - (Lf-1)r_3, \quad s' := rfK + 7r^\diamond$$

and suppose that $K/14, K/f, K/g, m/4\ell', m/L, 2fK/3g(g-1) \in \mathbb{N}$. Suppose that $(G_{\rm dir}, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme and let G' denote the underlying undirected graph of $G_{\rm dir}$. Suppose that $EF_1, \dots, EF_{r_3}$ are edge-disjoint exceptional factors with parameters (L, f) for $G_{\rm dir}$ (with respect to $(\mathcal{P}, \mathcal{P}')$). Let $\mathcal{EF} := EF_1 + \dots + EF_{r_3}$. Then there exists a graph $CA(r)$ for which the following holds:

(i) *$CA(r)$ is a $2(r_1 + r_2)$-regular spanning subgraph of G' which is edge-disjoint from $\mathcal{EF}$.*

(ii) *Suppose that $EF'_1, \dots, EF'_{r^\diamond}$ are exceptional factors with parameters $(1,7)$ for $G_{\rm dir}$ (with respect to $(\mathcal{P}, \mathcal{P})$) which are edge-disjoint from each other and from $CA(r) + \mathcal{EF}$. Let $\mathcal{EF}' := EF'_1 + \dots + EF'_{r^\diamond}$. Then there exists a graph $PCA(r)$ for which the following holds:*

  (a) *$PCA(r)$ is a $10r^\diamond$-regular spanning subgraph of G' which is edge-disjoint from $CA(r) + \mathcal{EF} + \mathcal{EF}'$.*

  (b) *Let $\mathcal{EPS}$ be the set consisting of all the s' exceptional path systems contained in $\mathcal{EF} + \mathcal{EF}'$. Suppose that H_A is a $2r$-regular graph on $A = \bigcup_{i=1}^K A_i$ and H_B is a $2r$-regular graph on $B = \bigcup_{i=1}^K B_i$. Suppose that $H := H_A + H_B$ is edge-disjoint from $G^{\rm rob} := CA(r) + PCA(r) + \mathcal{EF} + \mathcal{EF}'$. Then $H + G^{\rm rob}$ has a decomposition into s' edge-disjoint 2-factors $H_1, \dots, H_{s'}$ such that each H_i contains one of the exceptional path systems from $\mathcal{EPS}$. Moreover, for each $i \le s'$, the following assertions hold:*

(b_1) *If the exceptional path system contained in H_i is a Hamilton exceptional path system, then H_i is a Hamilton cycle on $V(G_{\rm dir}) \cup V_0$.*

(b_2) *If the exceptional path system contained in H_i is a matching exceptional path system, then H_i is the union of a Hamilton cycle on $A' = A \cup A_0$ and a Hamilton cycle on $B' = B \cup B_0$. In particular, if both $|A'|$ and $|B'|$ are even, then H_i is the union of two edge-disjoint perfect matchings on $V(G_{\rm dir}) \cup V_0$.*

We remark that, as usual, in Corollary 13.5 we write A_0 and B_0 for the exceptional sets of $\mathcal{P}$, V_0 for $A_0 \cup B_0$, and $A_1, \dots, A_K, B_1, \dots, B_K$ for the clusters in $\mathcal{P}$. Note that the vertex set of each of $\mathcal{EF}$, $\mathcal{EF}'$, $G^{\rm rob}$ includes V_0 while that of $G_{\rm dir}$, $CA(r)$, $PCA(r)$, H does not.

Moreover, note that matching exceptional systems are only constructed if both $|A'|$ and $|B'|$ are even. Indeed, we only construct matching exceptional systems in the case when $e_G(A', B') < D$. But by Proposition 6.1(ii), in this case we have that $n = 0 \pmod 4$ and $|A'| = |B'| = n/2$. Therefore, Corollary 13.5(ii)(b) implies that $H + G^{\rm rob}$ has a decomposition into Hamilton cycles and perfect matchings. The proportion of Hamilton cycles (and perfect matchings) in this decomposition is determined by $\mathcal{EF} + \mathcal{EF}'$, and does not depend on H.

Proof of Corollary 13.5. Choose new constants ε', d such that $\varepsilon \ll \varepsilon' \ll 1/L$ and $r_1/m \ll d \ll 1/\ell', 1/g$. Consider the A-setup $(G_{\rm dir}[A], \mathcal{Q}_A, \mathcal{Q}'_A, R_A, C_A, U_{A,\ell'}, U'_{A,\ell'})$ associated to $(G_{\rm dir}, \mathcal{P}, \mathcal{P}')$. By Lemma 13.3, this is an $(\ell', K, m, \varepsilon', 1/2)$-setup and thus also an $(\ell', K, m, \varepsilon', d)$-setup.

Recall that $\mathcal{P}'$ is obtained from $\mathcal{P}$ by partitioning each cluster A_i of $\mathcal{P}$ into L sets $A_{i,1}, \dots, A_{i,L}$ of equal size and partitioning each cluster B_i of $\mathcal{P}$ into L sets $B_{i,1}, \dots, B_{i,L}$ of equal size. Let $\mathcal{Q}''_A := \{A_{1,1}, \dots, A_{K,L}\}$. (So $\mathcal{Q}''_A$ plays the role of $\mathcal{Q}'_A$ in (12.1).) Let $EF^*_{i,A,{\rm dir}}$ be as defined in Section 12.2. Recall from there that, for each $i \le r_3$, $EF^*_{i,A,{\rm dir}}$ is a special factor with parameters (L, f) with respect to $C_A = A_1 \dots A_K$, $\mathcal{Q}''_A$ in $G_{\rm dir}[A]$ such that ${\rm Fict}(EF^*_{i,A,{\rm dir}})$ is the union of $J^*[A]$ over all the Lf exceptional systems J contained in EF_i. Thus we can apply Lemma 13.4 to $(G_{\rm dir}[A], \mathcal{Q}_A, \mathcal{Q}'_A, R_A, C_A, U_{A,\ell'}, U'_{A,\ell'})$ with K, Lf, ε' playing the roles of k, q, ε in order to obtain a spanning subdigraph $CA_{A,{\rm dir}}(r)$ of $G_{\rm dir}[A]$ which satisfies Lemma 13.4(i). Similarly, we obtain a spanning subdigraph $CA_{B,{\rm dir}}(r)$ of $G_{\rm dir}[B]$ which satisfies Lemma 13.4(i) (with $G_{\rm dir}[B]$ playing the role of G). Thus the underlying undirected graph $CA(r)$ of $CA_{A,{\rm dir}}(r) + CA_{B,{\rm dir}}(r)$ satisfies Corollary 13.5(i).

Now let $EF'_1, \dots, EF'_{r^\diamond}$ be exceptional factors as described in Corollary 13.5(ii). Similarly as before, for each $i \le r^\diamond$, $(EF'_i)^*_{A,{\rm dir}}$ is a special factor with parameters $(1, 7)$ with respect to C_A, $\mathcal{Q}_A$ in $G_{\rm dir}[A]$ such that ${\rm Fict}((EF'_i)^*_{A,{\rm dir}})$ is the union of $J^*[A]$ over all the 7 exceptional systems J contained in EF'_i. Thus we can apply Lemma 13.4 (with $G_{\rm dir}[A]$ playing the role of G) to obtain a spanning subdigraph $PCA_{A,{\rm dir}}(r)$ of $G_{\rm dir}[A]$ which satisfies Lemma 13.4(ii)(a) and (ii)(b). Similarly, we obtain a spanning subdigraph $PCA_{B,{\rm dir}}(r)$ of $G_{\rm dir}[B]$ which satisfies Lemma 13.4(ii)(a) and (ii)(b) (with $G_{\rm dir}[B]$ playing the role of G). Thus the underlying undirected graph $PCA(r)$ of $PCA_{A,\mathrm{dir}}(r) + PCA_{B,\mathrm{dir}}(r)$ satisfies Corollary 13.5(ii)(a).

It remains to check that Corollary 13.5(ii)(b) holds too. Thus let $H = H_A + H_B$ be as described in Corollary 13.5(ii)(b). Let $H_{A,\mathrm{dir}}$ be an r-regular orientation of H_A. (To see that such an orientation exists, apply Petersen's theorem, i.e. Theorem 3.10, to obtain a decomposition of H_A into 2-factors and then orient each 2-factor to obtain a (directed) 1-factor.) Let $\mathcal{EF}^*_{A,\mathrm{dir}} := EF^*_{1,A,\mathrm{dir}} + \dots + EF^*_{r_3,A,\mathrm{dir}}$ and let $(\mathcal{EF}')^*_{A,\mathrm{dir}} := (EF'_1)^*_{A,\mathrm{dir}} + \dots + (EF'_{r^\diamond})^*_{A,\mathrm{dir}}$. Then Lemma 13.4(ii)(b) implies that $H_{A,\mathrm{dir}} + CA_{A,\mathrm{dir}}(r) + PCA_{A,\mathrm{dir}}(r) + \mathcal{EF}^*_{A,\mathrm{dir}} + (\mathcal{EF}')^*_{A,\mathrm{dir}}$ has a decomposition into s' edge-disjoint (directed) Hamilton cycles $C'_{1,A}, \dots, C'_{s',A}$ such that each $C'_{i,A}$ contains $EPS^*_{i',A,\mathrm{dir}}$ for some exceptional path system $EPS_{i'} \in \mathcal{EPS}$. Similarly, let $H_{B,\mathrm{dir}}$ be an r-regular orientation of H_B. Then $H_{B,\mathrm{dir}} + CA_{B,\mathrm{dir}}(r) + PCA_{B,\mathrm{dir}}(r) + \mathcal{EF}^*_{B,\mathrm{dir}} + (\mathcal{EF}')^*_{B,\mathrm{dir}}$ has a decomposition into s' edge-disjoint (directed) Hamilton cycles $C'_{1,B}, \dots, C'_{s',B}$ such that each $C'_{i,B}$ contains $EPS^*_{i'',B,\mathrm{dir}}$ for some exceptional path system $EPS_{i''} \in \mathcal{EPS}$. By relabeling the $C'_{i,A}$ and $C'_{i,B}$ if necessary, we may assume that $C'_{i,A}$ contains $EPS^*_{i,A,\mathrm{dir}}$ and $C'_{i,B}$ contains $EPS^*_{i,B,\mathrm{dir}}$. Let $C_{i,A}$ and $C_{i,B}$ be the undirected cycles obtained from $C'_{i,A}$ and $C'_{i,B}$ by ignoring the directions of all the edges. So $C_{i,A}$ contains $EPS^*_{i,A}$ and $C_{i,B}$ contains $EPS^*_{i,B}$. Let $H_i := C_{i,A} + C_{i,B} - EPS^*_i + EPS_i$. Then Proposition 12.1 (applied with G' playing the role of G) implies that $H_1, \dots, H_{s'}$ is a decomposition of $H + G^{\mathrm{rob}} = H + CA(r) + PCA(r) + \mathcal{EF} + \mathcal{EF}'$ into edge-disjoint 2-factors satisfying Corollary 13.5(ii)(b$_1$) and (b$_2$). □

14. Proof of Theorem 3.3

Before we can prove Theorem 3.3, we need the following two observations. Recall that a $(K, m, \varepsilon_0, \varepsilon)$-scheme was defined in Section 8 and that a $[K, L, m, \varepsilon_0, \varepsilon']$-scheme was defined in Section 12.3.

Proposition 14.1. *Suppose that* $0 < 1/m \ll \varepsilon, \varepsilon_0 \ll \varepsilon' \ll 1/K, 1/L \ll 1$ *and that* $K, L, m/L \in \mathbb{N}$. *Suppose that* $(G, \mathcal{P}')$ *is a* $(KL, m/L, \varepsilon_0, \varepsilon)$*-scheme. Suppose that* $\mathcal{P}$ *is a* (K, m, ε_0)*-partition such that* $\mathcal{P}'$ *is an* L*-refinement of* $\mathcal{P}$. *Then there exists an orientation* G_{dir} *of* G *such that* $(G_{\mathrm{dir}}, \mathcal{P}, \mathcal{P}')$ *is a* $[K, L, m, \varepsilon_0, \varepsilon']$*-scheme.*

To prove the result, it suffices to consider a random orientation of G. For more details, see the proof of Proposition 9.1 in [6].

Proposition 14.2. *Suppose that* G *is a* D*-regular graph on* n *vertices which is* ε*-close to the union of two disjoint copies of* $K_{n/2}$. *Then* $D \le (1/2 + 4\varepsilon)n$.

Proof. Let $B \subseteq V(G)$ with $|B| = \lfloor n/2 \rfloor$ be such that $e(B, V(G) \setminus B) \le \varepsilon n^2$. Note that B exists since G is ε-close to the union of two disjoint copies of $K_{n/2}$. Let $A = V(G) \setminus B$. If $D > (1/2+4\varepsilon)n$, then Proposition 6.1(i) implies that $e(A, B) > \varepsilon n^2$, a contradiction. □

We can now put everything together and prove Theorem 3.3 in the following steps. We choose the (localized) exceptional systems needed as an 'input' for Corollary 13.5 to construct the robustly decomposable graph $G^{\rm rob}$ in Step 3. For this, we first choose appropriate constants and a suitable vertex partition in Steps 1 and 2 respectively (in Step 1, we also find some Hamilton cycles covering 'bad' edges). In Step 4, we then apply Corollary 13.5 to find $G^{\rm rob}$. Similarly, we then choose the (localized) exceptional systems needed as an 'input' for the 'approximate decomposition lemma' (Lemma 9.4) in Step 6 (in this step, we also find some Hamilton cycles which extend those exceptional systems which are not localized). For Step 6, we first choose a suitable vertex partition in Step 5. In Step 7, we find an approximate decomposition using Lemma 9.4 and in Step 8, we decompose the union of the 'leftover' and $G^{\rm rob}$ via Corollary 13.5.

Proof of Theorem 3.3.
Step 1: Choosing the constants and a framework. Choose $n_0 \in \mathbb{N}$ to be sufficiently large compared to $1/\varepsilon_{\rm ex}$. Let G and D be as in Theorem 3.3. By Proposition 14.2

$$n/2 - 1 \le D \le (1/2 + 4\varepsilon_{\rm ex})n. \tag{14.1}$$

Define new constants such that

$$\begin{aligned}0 < 1/n_0 &\ll \varepsilon_{\rm ex} \ll \varepsilon_0 \ll \phi_0 \ll \varepsilon_* \ll \varepsilon'_* \ll \varepsilon'_1 \ll \lambda_{K_2} \ll 1/K_2 \ll \gamma \ll 1/K_1\\ &\ll \varepsilon''_* \ll 1/L \ll 1/f \ll \gamma_1 \ll 1/g \ll \varepsilon'_2, \lambda_{K_1 L} \ll \varepsilon \ll 1,\end{aligned}$$

where $K_1, K_2, L, f, g \in \mathbb{N}$ and K_2 is odd. Note that we can choose the constants such that

$$\frac{D - \phi_0 n}{400(K_1 L K_2)^2}, \phi_0 n, \frac{\lambda_{K_1 L} n}{(K_1 L)^2}, \frac{\lambda_{K_2} n}{K_2^2}, \frac{K_1}{14 f g}, \frac{2 f K_1}{3 g (g-1)} \in \mathbb{N}. \tag{14.2}$$

Apply Proposition 6.5 to obtain a partition A, A_0, B, B_0 of $V(G)$ such that (G, A, A_0, B, B_0) is an $(\varepsilon_0, 4gK_1LK_2)$-framework with $\Delta(G[A', B']) \le D/2$ (where $A' := A \cup A_0$ and $B' := B \cup B_0$). Let w_1 and w_2 be two vertices of G such that $d_{G[A',B']}(w_1) \ge d_{G[A',B']}(w_2) \ge d_{G[A',B']}(v)$ for all $v \in V(G) \setminus \{w_1, w_2\}$. Note that the partition A, A_0, B, B_0 of $V(G)$ and the two vertices w_1 and w_2 are fixed throughout the proof. Moreover, in the remainder of the proof, given a graph H on $V(G)$, we will always write $H^\diamond$ for $H - H[A] - H[B]$.

Next we apply Lemma 10.3 with ϕ_0 and $4gK_1LK_2$ playing the roles of ϕ and K to find a spanning subgraph $\mathcal{H}'_1$ of G. Let $G_1 := G - \mathcal{H}'_1$. Thus the following properties are satisfied:

- (α_1) $G[A_0] + G[B_0] \subseteq \mathcal{H}'_1$ and $\mathcal{H}'_1$ is a $\phi_0 n$-regular spanning graph of G.
- (α_2) $e_{\mathcal{H}'_1}(A', B') \le \phi_0 n$ and $e_{G_1}(A', B')$ is even.
- (α_3) The edges of $\mathcal{H}'_1$ can be decomposed into $\lfloor e_{\mathcal{H}'_1}(A', B')/2 \rfloor$ Hamilton cycles and $\phi_0 n - 2\lfloor e_{\mathcal{H}'_1}(A', B')/2 \rfloor$ perfect matchings. Moreover, if $e_G(A', B') \ge D$, then this decomposition consists of $\lfloor \phi_0 n/2 \rfloor$ Hamilton cycles and one perfect matching if D is odd.

(α_4) $d_{G_1[A',B']}(w_1) \leq (D-\phi_0 n)/2$. Furthermore, if $D = n/2-1$ then $d_{G_1[A',B']}(w_2) \leq (D-\phi_0 n)/2$.

(α_5) If $e_G(A',B') < D$, then $\Delta(G_1[A',B']) \leq e(G_1[A',B'])/2 \leq (D-\phi_0 n)/2$.

Let $\mathcal{H}_1$ be the collection of Hamilton cycles and perfect matchings guaranteed by (α_3). (So $\mathcal{H}'_1 = \bigcup \mathcal{H}_1$.) Note that

$$D_1 := D - \phi_0 n \tag{14.3}$$

is even (since (14.2) implies that D and $\phi_0 n$ have the same parity) and that G_1 is D_1-regular. Moreover, (G_1, A, A_0, B, B_0) is an $(\varepsilon_0, 4gK_1LK_2)$-framework with $\Delta(G_1[A',B']) \leq D/2$. Let

$$m_1 := \frac{|A|}{K_1} = \frac{|B|}{K_1}, \qquad r := \gamma m_1, \qquad r_1 := \gamma_1 m_1, \qquad r_2 := 96g^3K_1 r,$$
$$r_3 := \frac{rK_1}{L}, \qquad r^\diamond := r_1 + r_2 + r - (Lf-1)r_3,$$
$$m_2 := \frac{|A|}{K_2} = \frac{|B|}{K_2}, \qquad D_4 := D_1 - 2(Lfr_3 + 7r^\diamond). \tag{14.4}$$

Note that (FR3) implies $m_1/L \in \mathbb{N}$. Moreover,

$$r_2, r_3 \leq \gamma^{1/2} m_1 \leq \gamma^{1/3} r_1, \qquad r_1/2 \leq r^\diamond \leq 2r_1. \tag{14.5}$$

Furthermore, by changing γ, γ_1 slightly, we may assume that $r/400LK_2^2, r_1/400K_2^2 \in \mathbb{N}$. This implies that $r_2/400K_2^2, r_3/400K_2^2, r^\diamond/400K_2^2 \in \mathbb{N}$. Together with the fact that $D_1/400K_2^2 = (D-\phi_0 n)/400K_2^2 \in \mathbb{N}$ by (14.2), this in turn implies that

$$D_4/400K_2^2 \in \mathbb{N}. \tag{14.6}$$

Step 2: Choosing a $(K_1, L, m_1, \varepsilon_0)$-partition $(\mathcal{P}_1, \mathcal{P}'_1)$. We now prepare the ground for the construction of the robustly decomposable graph G^{rob}, which we will obtain via the robust decomposition lemma (Corollary 13.5) in Step 4.

Since (G_1, A, A_0, B, B_0) is an $(\varepsilon_0, 4gK_1LK_2)$-framework, it is also an (ε_0, K_1L)-framework. Recall that G_1 is D_1-regular and $D_1 = D - \phi_0 n \geq (1-3\phi_0)n/2$ (as $D \geq n/2-1$). Apply Lemma 8.3 with G_1, m_1/L, $3\phi_0$, K_1L, ε_*, ε_* playing the roles of G, m, μ, K, ε_1, ε_2 to obtain partitions $A'_1, \dots, A'_{K_1L}$ of A and $B'_1, \dots, B'_{K_1L}$ of B into sets of size m_1/L such that the following properties are satisfied:

(S_1a) Together with A_0 and B_0 all these sets A'_i and B'_i form a $(K_1L, m_1/L, \varepsilon_0)$-partition $\mathcal{P}'_1$ of $V(G_1)$.

(S_1b) $(G_1[A] + G_1[B], \mathcal{P}'_1)$ is a $(K_1L, m_1/L, \varepsilon_0, \varepsilon_*)$-scheme.

(S_1c) $(G_1^\diamond, \mathcal{P}'_1)$ is a $(K_1L, m_1/L, \varepsilon_0, \varepsilon_*)$-exceptional scheme (where $G_1^\diamond := G_1 - G_1[A] - G_2[B]$).

Note that $(1-\varepsilon_0)n \leq n - |A_0 \cup B_0| = 2K_1m_1 \leq n$ by (FR3). For all $i \leq K_1$ and all $h \leq L$, let $A_{i,h} := A'_{(i-1)L+h}$. (So this is just a relabeling of the sets A'_i.) Define $B_{i,h}$ similarly and let $A_i := \bigcup_{h \leq L} A_{i,h}$ and $B_i := \bigcup_{h \leq L} B_{i,h}$. Let $\mathcal{P}_1 := \{A_0, B_0, A_1, \dots, A_{K_1}, B_1, \dots, B_{K_1}\}$ denote the corresponding $(K_1, m_1, \varepsilon_0)$-partition of $V(G)$. Thus $(\mathcal{P}_1, \mathcal{P}'_1)$ is a $(K_1, L, m_1, \varepsilon_0)$-partition of $V(G)$, as defined in Section 12.2.

Step 3: Exceptional systems for the robustly decomposable graph. In order to be able to apply Corollary 13.5 to obtain the robustly decomposable graph $G^{\rm rob}$, we first need to construct suitable exceptional systems with parameter ε_0. The construction of these exceptional systems depends on whether G is critical and whether $e_G(A',B') \geq D$. First we show that in each case, for all $1 \leq i'_1, i'_2 \leq K_1L$, we can always find sets $\mathcal{J}_{i'_1,i'_2}$ of $\lambda_{K_1L}n/(K_1L)^2$ (i'_1,i'_2)-ES with respect to $\mathcal{P}'_1$.

Case 1: $e_G(A',B') \geq D$ and G is not critical. Our aim is to apply Lemma 11.3 to G with $\mathcal{H}'_1$, m_1/L, K_1L, $\mathcal{P}'_1$, ε_*, ϕ_0, λ_{K_1L} playing the roles of G_0, m, K, $\mathcal{P}$, ε, ϕ, λ. First we verify that Lemma 11.3(i)–(iv) are satisfied. Lemma 11.3(i) holds trivially. (FR2) implies that $e_G(A',B') \leq \varepsilon_0 n^2$. Moreover, recall from (S_1a) that $\mathcal{P}'_1$ is a $(K_1L, m_1/L, \varepsilon_0)$-partition of $V(G)$ and that A' and B' were chosen (by Proposition 6.5) such that $\Delta(G[A',B']) \leq D/2$. Altogether this shows that Lemma 11.3(ii) holds. Lemma 11.3(iii) follows from (α_1) and (α_2). To verify Lemma 11.3(iv), note that $G^\diamond_1$ plays the role of $G^\diamond$ in Lemma 11.3 and $G^\diamond_1[A',B'] = G_1[A',B']$. So $e_{G^\diamond_1}(A',B')$ is even by (α_2). Together with the fact that $(G^\diamond_1, \mathcal{P}'_1)$ is a $(K_1L, m_1/L, \varepsilon_0, \varepsilon_*)$-exceptional scheme by ($\mathrm{S}_1$c), this implies Lemma 11.3(iv).

By Lemma 11.3, we obtain a set $\mathcal{J}$ of $\lambda_{K_1L}n$ edge-disjoint Hamilton exceptional systems J in $G^\diamond_1$ such that $e_J(A',B') = 2$ for each $J \in \mathcal{J}$ and such that for all $1 \leq i'_1, i'_2 \leq K_1L$ the set $\mathcal{J}$ contains precisely $\lambda_{K_1L}n/(K_1L)^2$ (i'_1,i'_2)-HES with respect to the partition $\mathcal{P}'_1$. For all $1 \leq i'_1, i'_2 \leq K_1L$, let $\mathcal{J}_{i'_1,i'_2}$ be the set of these $\lambda_{K_1L}n/(K_1L)^2$ (i'_1,i'_2)-HES in $\mathcal{J}$. So $\mathcal{J}$ is the union of all the sets $\mathcal{J}_{i'_1,i'_2}$. (Note that the set $\mathcal{J}$ here is a subset of the set $\mathcal{J}$ in Lemma 11.3, i.e. we do not use all the Hamilton exceptional systems constructed by Lemma 11.3. So we do not need the full strength of Lemma 11.3 at this point.)

Case 2: $e_G(A',B') \geq D$ and G is critical. Recall from Lemma 11.1(ii) that in this case we have $D = (n-1)/2$ or $D = n/2-1$. Our aim is to apply Lemma 11.4 to G with $\mathcal{H}'_1$, m_1/L, K_1L, $\mathcal{P}'_1$, ε_*, ϕ_0, λ_{K_1L} playing the roles of G_0, m, K, $\mathcal{P}$, ε, ϕ, λ. Similar arguments as in Case 1 show that Lemma 11.4(i)–(iv) hold. Recall that w_1 and w_2 are (fixed) vertices in $V(G)$ such that $d_{G[A',B']}(w_1) \geq d_{G[A',B']}(w_2) \geq d_{G[A',B']}(v)$ for all $v \in V(G) \setminus \{w_1,w_2\}$. Since $G^\diamond_1[A',B'] = G_1[A',B']$, (α_4) implies that $d_{G^\diamond_1[A',B']}(w_1) \leq (D-\phi_0 n)/2$. Moreover, if $D = n/2-1$, then $d_{G^\diamond_1[A',B']}(w_2) \leq (D-\phi_0 n)/2$. Let W be the set of vertices $w \in V(G)$ such that $d_{G[A',B']}(w) \geq 11D/40$, as defined in Lemma 11.1. If $D = (n-1)/2$, then $|W| = 1$ by Lemma 11.1(ii). This means that $w_2 \notin W$ and so $d_{G^\diamond_1[A',B']}(w_2) \leq d_{G[A',B']}(w_2) \leq 11D/40$. Thus in both cases we have that

$$d_{G^\diamond_1[A',B']}(w_1), d_{G^\diamond_1[A',B']}(w_2) \leq (D-\phi_0 n)/2. \tag{14.7}$$

Therefore, Lemma 11.4(v) holds.

By Lemma 11.4, we obtain a set $\mathcal{J}$ of $\lambda_{K_1L}n$ edge-disjoint Hamilton exceptional systems J in $G^\diamond_1$ such that, for all $1 \leq i'_1, i'_2 \leq K_1L$, the set $\mathcal{J}$ contains precisely $\lambda_{K_1L}n/(K_1L)^2$ (i'_1,i'_2)-HES with respect to the partition $\mathcal{P}'_1$. Moreover, each $J \in \mathcal{J}$ satisfies $e_J(A',B') = 2$ and $d_{J[A',B']}(w) = 1$ for all $w \in \{w_1,w_2\}$ with $d_{G[A',B']}(w) \geq$

$11D/40$. For all $1 \le i'_1, i'_2 \le K_1L$, let $\mathcal{J}_{i'_1,i'_2}$ be the set of these $\lambda_{K_1L}n/(K_1L)^2$ (i'_1, i'_2)-HES. So $\mathcal{J}$ is the union of all the sets $\mathcal{J}_{i'_1,i'_2}$. (So similarly as in Case 1, we do not use all the Hamilton exceptional systems constructed by Lemma 11.4 at this point.)

Case 3: $e_G(A', B') < D$**.** Recall from Proposition 6.1(ii) that in this case we have $D = n/2-1$, $n = 0 \pmod 4$ and $|A'| = |B'| = n/2$. Our aim is to apply Lemma 11.5 to G with $\mathcal{H}'_1$, m_1/L, K_1L, $\mathcal{P}'_1$, ε_*, ϕ_0, λ_{K_1L} playing the roles of G_0, m, K, $\mathcal{P}$, ε, ϕ, λ. Similar arguments as in Case 1 show that Lemma 11.5(i)–(iv) hold. Since $G_1^\diamond[A', B'] = G_1[A', B']$ and $D = n/2 - 1$, Lemma 11.5(v) follows from (α_5).

By Lemma 11.5, $G_1^\diamond$ can be decomposed into a set $\mathcal{J}'$ of $D_1/2$ edge-disjoint exceptional systems such that each of these exceptional systems J is either a Hamilton exceptional system with $e_J(A', B') = 2$ or a matching exceptional system. (So $\mathcal{J}'$ plays the role of the set $\mathcal{J}$ in Lemma 11.5.) Lemma 11.5(b) guarantees that we can choose a subset $\mathcal{J}$ of $\mathcal{J}'$ such that $\mathcal{J}$ consists of $\lambda_{K_1L}n$ edge-disjoint exceptional systems J in $G_1^\diamond$ such that for all $1 \le i'_1, i'_2 \le K_1L$ the set $\mathcal{J}$ contains precisely $\lambda_{K_1L}n/(K_1L)^2$ (i'_1, i'_2)-ES with respect to the partition $\mathcal{P}'_1$. For all $1 \le i'_1, i'_2 \le K_1L$, let $\mathcal{J}_{i'_1,i'_2}$ be the set of these $\lambda_{K_1L}n/(K_1L)^2$ (i'_1, i'_2)-ES. So $\mathcal{J}$ is the union of all the sets $\mathcal{J}_{i'_1,i'_2}$. (Note that to construct the robustly decomposable graph we will only use the exceptional systems in $\mathcal{J}$. However, in order to prove condition (β_5) below, we will also use the fact that $G_1^\diamond$ has a decomposition into edge-disjoint exceptional systems.)

Thus in each of the three cases, $\mathcal{J}$ is the union of all the sets $\mathcal{J}_{i'_1,i'_2}$, where for all $1 \le i'_1, i'_2 \le K_1L$, the set $\mathcal{J}$ consists of precisely $\lambda_{K_1L}n/(K_1L)^2$ (i'_1, i'_2)-ES with respect to the partition $\mathcal{P}'_1$. Moreover, all the $\lambda_{K_1L}n$ exceptional systems in $\mathcal{J}$ are edge-disjoint.

Our next aim is to choose two disjoint subsets $\mathcal{J}_{\rm CA}$ and $\mathcal{J}_{\rm PCA}$ of $\mathcal{J}$ with the following properties:

(a) In total $\mathcal{J}_{\rm CA}$ contains Lfr_3 exceptional systems. For each $i \le f$ and each $h \le L$, $\mathcal{J}_{\rm CA}$ contains precisely r_3 (i_1, i_2)-ES of style h (with respect to the $(K_1, L, m_1, \varepsilon_0)$-partition $(\mathcal{P}_1, \mathcal{P}'_1)$) such that $i_1, i_2 \in \{(i-1)K_1/f + 2, \dots, iK_1/f\}$.
(b) In total $\mathcal{J}_{\rm PCA}$ contains $7r^\diamond$ exceptional systems. For each $i \le 7$, $\mathcal{J}_{\rm PCA}$ contains precisely $r^\diamond$ (i_1, i_2)-ES (with respect to the partition $\mathcal{P}_1$) with $i_1, i_2 \in \{(i-1)K_1/7 + 2, \dots, iK_1/7\}$.
(c) Each exceptional system $J \in \mathcal{J}_{\rm CA} \cup \mathcal{J}_{\rm PCA}$ is either a Hamilton exceptional system with $e_J(A', B') = 2$ or a matching exceptional system.

(Recall that we defined in Section 12.3 when an (i_1, i_2)-ES has style h with respect to a $(K_1, L, m_1, \varepsilon_0)$-partition $(\mathcal{P}_1, \mathcal{P}'_1)$.) To see that it is possible to choose $\mathcal{J}_{\rm CA}$ and $\mathcal{J}_{\rm PCA}$, split $\mathcal{J}$ into two sets $\mathcal{J}_1$ and $\mathcal{J}_2$ such that both $\mathcal{J}_1$ and $\mathcal{J}_2$ contain at least $\lambda_{K_1L}n/3(K_1L)^2$ (i'_1, i'_2)-ES with respect to $\mathcal{P}'_1$, for all $1 \le i'_1, i'_2 \le K_1L$. Note that, for each $i \le f$, there are $(K_1/f - 1)^2$ choices of pairs (i_1, i_2) with $i_1, i_2 \in \{(i-1)K_1/f + 2, \dots, iK_1/f\}$. Moreover, for each such pair (i_1, i_2) and each $h \le L$ there is precisely one pair (i'_1, i'_2) with $1 \le i'_1, i'_2 \le K_1L$ and such that any (i'_1, i'_2)-ES with respect to $\mathcal{P}'_1$ is an (i_1, i_2)-ES of style h with respect to $(\mathcal{P}_1, \mathcal{P}'_1)$. Together with the fact that $\gamma \ll \lambda_{K_1L}, 1/L, 1/f$ and

$$\frac{(K_1/f-1)^2\lambda_{K_1L}n}{3(K_1L)^2} \geq \frac{\gamma n}{L} \geq \frac{\gamma K_1 m_1}{L} = \frac{rK_1}{L} = r_3,$$

this implies that we can choose a set $\mathcal{J}_{\rm CA} \subseteq \mathcal{J}_1$ satisfying (a).

Similarly, for each $i \leq 7$, there are $(K_1/7-1)^2$ choices of pairs (i_1, i_2) with $i_1, i_2 \in \{(i-1)K_1/7+2, \dots, iK_1/7\}$. Moreover, for each such pair (i_1,i_2) there are L^2 distinct pairs (i'_1, i'_2) with $1 \leq i'_1, i'_2 \leq K_1L$ and such that any (i'_1,i'_2)-ES with respect to $\mathcal{P}'_1$ is an (i_1,i_2)-ES with respect to $\mathcal{P}_1$. Together with the fact that $\gamma_1 \ll \lambda_{K_1L}$ and

$$\frac{(K_1/7-1)^2L^2\lambda_{K_1L}n}{3(K_1L)^2} \geq \gamma_1 n \geq 2\gamma_1 m_1 = 2r_1 \stackrel{(14.5)}{\geq} r^\diamond,$$

this implies that we can choose a set $\mathcal{J}_{\rm PCA} \subseteq \mathcal{J}_2$ satisfying (b). Our choice of $\mathcal{J} \supseteq \mathcal{J}_{\rm CA} \cup \mathcal{J}_{\rm PCA}$ guarantees that (c) holds too. Let

$$\mathcal{J}^{\rm rob} := \mathcal{J}_{\rm CA} \cup \mathcal{J}_{\rm PCA}, \quad \phi_0^{\rm rob} := (Lfr_3 + 7r^\diamond)/n \quad \text{and} \quad G_4^\diamond := G_1^\diamond - \bigcup \mathcal{J}^{\rm rob}. \tag{14.8}$$

(In Step 5 below we will define a graph G_4 which will satisfy $G_4^\diamond = G_4 - G_4[A] - G_4[B]$. So this will fit with our definition of the operator $^\diamond$.) Note that

$$\phi_0^{\rm rob} \geq \frac{7r^\diamond}{n} \stackrel{(14.5)}{\geq} \frac{3r_1}{n} = \frac{3\gamma_1 m_1}{n} \geq \frac{\gamma_1}{K_1} \geq 2\phi_0 \quad \text{and} \quad 2\phi_0^{\rm rob}n \stackrel{(14.4)}{=} D_1 - D_4. \tag{14.9}$$

Moreover, we claim that $\bigcup \mathcal{J}^{\rm rob}$ is a subgraph of $G_1^\diamond \subseteq G$ satisfying the following properties:

- (β_1) $d_{\bigcup \mathcal{J}^{\rm rob}}(v) = 2(Lfr_3 + 7r^\diamond) = 2\phi_0^{\rm rob}n$ for each $v \in V_0$.
- (β_2) $e_{\bigcup \mathcal{J}^{\rm rob}}(A', B') \leq 2\phi_0^{\rm rob}n$ is even.
- (β_3) $\mathcal{J}^{\rm rob}$ contains exactly $\phi_0^{\rm rob}n$ exceptional systems, of which precisely $e_{\bigcup \mathcal{J}^{\rm rob}}(A',B')/2$ are Hamilton exceptional systems. If $e_G(A',B') \geq D$, then $\mathcal{J}^{\rm rob}$ consists entirely of Hamilton exceptional systems. If $\mathcal{J}^{\rm rob}$ contains a matching exceptional system, then $|A'| = |B'| = n/2$ is even.
- (β_4) If $e_G(A',B') \geq D$ and G is critical, then $d_{\bigcup \mathcal{J}^{\rm rob}[A',B']}(w) = \phi_0^{\rm rob}n$ for all $w \in \{w_1, w_2\}$ with $d_{G[A',B']}(w) \geq 11D/40$. Moreover, $d_{G_4^\diamond[A',B']}(w_1), d_{G_4^\diamond[A',B']}(w_2) \leq (D - (\phi_0 + 2\phi_0^{\rm rob})n)/2$.
- (β_5) If $e_G(A',B') < D$, then $\Delta(G_4^\diamond[A',B']) \leq e(G_4^\diamond[A',B'])/2 \leq D_4/2 = (D - (\phi_0 + 2\phi_0^{\rm rob})n)/2$.

To verify the above, note that $\mathcal{J}^{\rm rob}$ consists of precisely $\phi_0^{\rm rob}n$ exceptional systems J (each of which is an exceptional cover). So (β_1) follows from (EC2). Moreover, each such J is either a Hamilton exceptional system with $e_J(A',B') = 2$ or a matching exceptional system (with $e_J(A',B') = 0$ by (MES)), which implies (β_2) and the first part of (β_3). If $e_G(A',B') \geq D$, then we are in Case 1 or 2 and so the second part of (β_3) follows from our construction of $\mathcal{J} \supseteq \mathcal{J}^{\rm rob}$. The first part of (β_4) follows from our construction of $\mathcal{J} \supseteq \mathcal{J}^{\rm rob}$ in Case 2. Since $11D/40 < (D - (\phi_0 + 2\phi_0^{\rm rob})n)/2$, we can combine the first part of (β_4) with (14.7) to obtain the 'moreover part' of (β_4). Thus it remains to verify (β_5). So suppose that $e_G(A',B') < D$. Recall from

Case 3 that $G_1^\diamond$ has a decomposition into a set $\mathcal{J}'$ of $D_1/2$ edge-disjoint exceptional systems J, each of which is either a Hamilton exceptional system with $e_J(A',B')=2$ or a matching exceptional system. This means that $J[A',B']$ is either empty or a matching of size 2. Note that $G_4^\diamond[A',B']$ is precisely the union of $J[A',B']$ over all those $D_1/2-\phi_0^{\rm rob}n=D_4/2$ exceptional systems $J\in\mathcal{J}'\setminus\mathcal{J}^{\rm rob}$. So (β_5) holds.

Step 4: Finding the robustly decomposable graph. Let $G_2:=G_1[A]+G_1[B]$. Recall from (S$_1$b) that $(G_2,\mathcal{P}_1')$ is a $(K_1L,m_1/L,\varepsilon_0,\varepsilon_*)$-scheme. Apply Proposition 14.1 with G_2, $\mathcal{P}_1$, $\mathcal{P}_1'$, K_1, m_1, ε_*, ε_*' playing the roles of G, $\mathcal{P}$, $\mathcal{P}'$, K, m, ε, ε' to obtain an orientation $G_{2,\rm dir}$ of G_2 such that $(G_{2,\rm dir},\mathcal{P}_1,\mathcal{P}_1')$ is a $[K_1,L,m_1,\varepsilon_0,\varepsilon_*']$-scheme.

Our next aim is to use Lemma 12.3 in order to extend the exceptional systems in $\mathcal{J}_{\rm CA}$ into r_3 edge-disjoint exceptional factors with parameters (L,f) for $G_{2,\rm dir}$ (with respect to $(\mathcal{P}_1,\mathcal{P}_1')$). For this, note that (a) and (c) guarantee that $\mathcal{J}_{CA}$ satisfies Lemma 12.3(i),(ii) with r_3 playing the role of q. Moreover, $Lr_3/m_1=rK_1/m_1=\gamma K_1\ll 1$. Thus we can indeed apply Lemma 12.3 to $(G_{2,\rm dir},\mathcal{P}_1,\mathcal{P}_1')$ with $\mathcal{J}_{CA}$, m_1, ε_*', K_1, r_3 playing the roles of $\mathcal{J}$, m, ε, K, q in order to obtain r_3 edge-disjoint exceptional factors $EF_1,\dots,EF_{r_3}$ with parameters (L,f) for $G_{2,\rm dir}$ (with respect to $(\mathcal{P}_1,\mathcal{P}_1')$) such that together these exceptional factors cover all edges in $\bigcup\mathcal{J}_{\rm CA}$. Let $\mathcal{EF}_{\rm CA}:=EF_1+\dots+EF_{r_3}$. Since $G_2=G_1[A]+G_1[B]$, we have $(\mathcal{EF}_{\rm CA})^\diamond=\mathcal{J}_{\rm CA}$. Moreover, each exceptional path system in $\mathcal{EF}_{\rm CA}$ contains a unique exceptional system in $\mathcal{J}_{\rm CA}$ (in particular, their numbers are equal).

Note that $m_1/4g, m_1/L\in\mathbb{N}$ since $m_1=|A|/K_1$ and $|A|$ is divisible by $4gK_1L$ as (G,A,A_0,B,B_0) is an $(\varepsilon_0,4gK_1LK_2)$-framework. Furthermore, $rK_1^2=\gamma m_1K_1^2\le\gamma^{1/2}m_1\le m_1$. Thus we can apply Corollary 13.5 to the $[K_1,L,m_1,\varepsilon_0,\varepsilon_*'']$-scheme $(G_{2,\rm dir},\mathcal{P}_1,\mathcal{P}_1')$ with K_1, m_1, ε_*'', g playing the roles of K, m, ε, ℓ' to obtain a spanning subgraph $CA(r)$ of G_2 as described there. (Note that G_2 equals the graph G' defined in Corollary 13.5.) In particular, $CA(r)$ is $2(r_1+r_2)$-regular and edge-disjoint from $\mathcal{EF}_{\rm CA}$.

Let G_3 be the graph obtained from G_2 by deleting all the edges of $CA(r)+\mathcal{EF}_{\rm CA}$. Thus G_3 is obtained from G_2 by deleting at most $2(r_1+r_2+r_3)\le 6r_1=6\gamma_1m_1$ edges at every vertex in $A\cup B$. Let $G_{3,\rm dir}$ be the orientation of G_3 in which every edge is oriented in the same way as in $G_{2,\rm dir}$. Since $(G_{2,\rm dir},\mathcal{P}_1,\mathcal{P}_1')$ is a $[K_1,L,m_1,\varepsilon_0,\varepsilon_*']$-scheme, Proposition 5.3 and the fact that $\varepsilon_*'',\gamma_1\ll\varepsilon$ imply that $(G_{3,\rm dir},\mathcal{P}_1,\mathcal{P}_1)$ is a $[K_1,1,m_1,\varepsilon_0,\varepsilon]$-scheme. Moreover,

$$\frac{r^\diamond}{m_1}\overset{(14.5)}{\le}\frac{2r_1}{m_1}=2\gamma_1\ll 1.$$

Together with (b) and (c) this ensures that we can apply Lemma 12.3 to $(G_{3,\rm dir},\mathcal{P}_1,\mathcal{P}_1)$ with $\mathcal{J}_{PCA}$, m_1, K_1, 1, 7, $r^\diamond$ playing the roles of $\mathcal{J}$, m, K, L, f, q in order to obtain $r^\diamond$ edge-disjoint exceptional factors $EF_1',\dots,EF_{r^\diamond}'$ with parameters $(1,7)$ for $G_{3,\rm dir}$ (with respect to $(\mathcal{P}_1,\mathcal{P}_1)$) such that together these exceptional factors cover all edges in $\bigcup\mathcal{J}_{\rm PCA}$. Let $\mathcal{EF}_{\rm PCA}:=EF_1'+\dots+EF_{r^\diamond}'$. Since $G_3\subseteq G_1[A]+G_1[B]$ we have $(\mathcal{EF}_{\rm PCA})^\diamond=\bigcup\mathcal{J}_{\rm PCA}$. Moreover, each exceptional path system in $\mathcal{EF}_{\rm PCA}$ contains a unique exceptional system in $\mathcal{J}_{\rm PCA}$.

Apply Corollary 13.5 to obtain a spanning subgraph $PCA(r)$ of G_2 as described there. In particular, $PCA(r)$ is $10r^\diamond$-regular and edge-disjoint from $CA(r)+\mathcal{EF}_{\rm CA}+\mathcal{EF}_{\rm PCA}$.

Let $G^{\rm rob} := CA(r) + PCA(r) + \mathcal{EF}_{\rm CA} + \mathcal{EF}_{\rm PCA}$. Note that by (12.2) all the vertices in $V_0 := A_0 \cup B_0$ have the same degree $r_0^{\rm rob} := 2(Lfr_3 + 7r^\diamond) = 2\phi_0^{\rm rob} n$ in $G^{\rm rob}$. So

$$7r_1 \overset{(14.5)}{\leq} r_0^{\rm rob} \overset{(14.5)}{\leq} 30r_1. \tag{14.10}$$

Moreover, (12.2) also implies that all the vertices in $A\cup B$ have the same degree $r^{\rm rob}$ in $G^{\rm rob}$, where $r^{\rm rob} := 2(r_1 + r_2) + 10r^\diamond + 2r_3 + 2r^\diamond = 2(r_1 + r_2 + r_3 + 6r^\diamond)$. So

$$r_0^{\rm rob} - r^{\rm rob} = 2\left(Lfr_3 + r^\diamond - (r_1 + r_2 + r_3)\right) = 2(Lfr_3 + r - (Lf-1)r_3 - r_3) = 2r.$$

Note that $(G^{\rm rob})^\diamond = \bigcup(\mathcal{J}_{\rm CA} \cup \mathcal{J}_{\rm PCA}) = \bigcup \mathcal{J}^{\rm rob}$. Recall that the number of Hamilton exceptional path systems in $\mathcal{EF}_{\rm CA}$ equals the number of Hamilton exceptional systems in $\mathcal{J}_{\rm CA}$, and that the analogue holds for $\mathcal{EF}_{\rm PCA}$. Hence, (β_1), (β_2) and (β_3) imply the follow statements:

- (β_1') $d_{G^{\rm rob}}(v) = r_0^{\rm rob} = 2\phi_0^{\rm rob}n$ for all $v \in V_0$.
- (β_2') $e_{G^{\rm rob}}(A', B') = e_{\bigcup \mathcal{J}^{\rm rob}}(A', B') \leq r_0^{\rm rob} = 2\phi_0^{\rm rob}n$ is even.
- (β_3') $\mathcal{EF}_{\rm CA} + \mathcal{EF}_{\rm PCA}$ contains exactly $\phi_0^{\rm rob}n$ exceptional path systems (and each such path system contains a unique exceptional system in $\mathcal{J}^{\rm rob}$, where $|\mathcal{J}^{\rm rob}| = \phi_0^{\rm rob}n$). Precisely $e_{\bigcup \mathcal{J}^{\rm rob}}(A', B')/2$ of these are Hamilton exceptional path systems. If $e_G(A', B') \geq D$, then every exceptional path system in $\mathcal{EF}_{\rm CA} + \mathcal{EF}_{\rm PCA}$ is a Hamilton exceptional path system. If $\mathcal{EF}_{\rm CA} + \mathcal{EF}_{\rm PCA}$ contains a matching exceptional path system, then $|A'| = |B'| = n/2$ is even.

Step 5: Choosing a $(K_2, m_2, \varepsilon_0)$-partition $\mathcal{P}_2$. We now prepare the ground for the approximate decomposition step (i.e. to apply Lemma 9.4). For this, we need to work with a finer partition of $A \cup B$ than the previous one (this will ensure that the leftover from the approximate decomposition step is sufficiently sparse compared to $G^{\rm rob}$).

So let $G_4 := G_1 - G^{\rm rob}$ (where G_1 was defined in Step 1) and note that

$$D_4 \overset{(14.4)}{=} D_1 - r_0^{\rm rob} = D_1 - r^{\rm rob} - 2r. \tag{14.11}$$

So

$$d_{G_4}(v) = D_4 + 2r \text{ for all } v \in A \cup B \qquad \text{and} \qquad d_{G_4}(v) = D_4 \text{ for all } v \in V_0. \tag{14.12}$$

Hence

$$\delta(G_4) \geq D_4 \overset{(14.9)}{=} D_1 - 2\phi_0^{\rm rob}n \overset{(14.3)}{=} D - (\phi_0 + 2\phi_0^{\rm rob})n \geq (1 - 6\phi_0^{\rm rob})n/2$$

as $\phi_0^{\rm rob} \geq 2\phi_0$ by (14.9). Moreover, note that

$$2\phi_0^{\rm rob}n = r_0^{\rm rob} \overset{(14.10)}{\leq} 30r_1 = 30\gamma_1 m_1 \leq 30\gamma_1 n/K_1,$$

so $\phi_0^{\rm rob} \ll \varepsilon_2'$. Since (G, A, A_0, B, B_0) is an $(\varepsilon_0, 4gK_1LK_2)$-framework, (G_4, A, A_0, B, B_0) is an (ε_0, K_2)-framework. Now apply Lemma 8.3 to (G_4, A, A_0, B, B_0) with K_2, m_2,

ε_1', ε_2', $6\phi_0^{\rm rob}$ playing the roles of K, m, ε_1, ε_2, μ in order to obtain partitions $A_1,\dots,A_{K_2}$ and $B_1,\dots,B_{K_2}$ of A and B satisfying the following conditions:

(S$_2$a) The vertex partition $\mathcal{P}_2 := \{A_0, B_0, A_1, \dots A_{K_2}, B_1,\dots,B_{K_2}\}$ is a $(K_2, m_2, \varepsilon_0)$-partition of $V(G)$.

(S$_2$b) $(G_4[A]+G_4[B], \mathcal{P}_2)$ is a $(K_2, m_2, \varepsilon_0, \varepsilon_2')$-scheme.

(S$_2$c) $(G_4^\diamond, \mathcal{P}_2)$ is a $(K_2, m_2, \varepsilon_0, \varepsilon_1')$-exceptional scheme.

(Recall that $G_4^\diamond := G_1^\diamond - \bigcup \mathcal{J}^{\rm rob}$ was defined towards the end of Step 3. Since $G_4 = G_1 - G^{\rm rob}$, we have $(G_4)^\diamond = G_1^\diamond - (G^{\rm rob})^\diamond = G_1^\diamond - \bigcup \mathcal{J}^{\rm rob}$, so $(G_4)^\diamond$ is indeed the same as $G_4^\diamond$.) Moreover, by Lemma 8.3(iv) we have

(14.13)
$$d_{G_4}(v, A_i) = (d_{G_4}(v,A) \pm \varepsilon_0 n)/K_2 \qquad \text{and} \qquad d_{G_4}(v, B_i) = (d_{G_4}(v,B) \pm \varepsilon_0 n)/K_2$$

for all $v \in V(G)$ and $1 \le i \le K_2$. (Note that the previous partition of A and B plays no role in the subsequent argument, so denoting the clusters in $\mathcal{P}_2$ by A_i and B_i again will cause no notational conflicts.)

Since (G_4, A, A_0, B, B_0) is an (ε_0, K_2)-framework, (FR3) and (FR4) together imply that each $v \in A$ satisfies $d_{G_4}(v, A_0) \le |V_0| \le \varepsilon_0 n$ and $d_{G_4}(v, B') \le \varepsilon_0 n$. So $d_{G_4}(v, A) = d_{G_4}(v) \pm 2\varepsilon_0 n$. Therefore, for all $v \in A$ and all $1 \le i \le K_2$ we have

$$(14.14) \quad d_{G_4}(v, A_i) \overset{(14.13)}{=} \frac{d_{G_4}(v,A) \pm \varepsilon_0 n}{K_2} = \frac{d_{G_4}(v) \pm 3\varepsilon_0 n}{K_2} = \frac{d_{G_4}(v) \pm 7\varepsilon_0 K_2 m_2}{K_2}.$$

The analogue holds for $d_{G_4}(v, B_i)$ (where $v \in B$ and $1 \le i \le K_2$).

Step 6: Exceptional systems for the approximate decomposition. In order to apply Lemma 9.4, we first need to construct suitable exceptional systems. We will show that $G_4^\diamond$ can be decomposed completely into $D_4/2$ exceptional systems with parameter ε_0. Moreover, these exceptional systems can be partitioned into sets $\mathcal{J}_0'$ and $\mathcal{J}_{i_1,i_2}'$ (one set for each pair $1 \le i_1, i_2 \le K_2$) such that the following conditions hold, where $\mathcal{J}''$ denotes the union of $\mathcal{J}_{i_1,i_2}'$ over all $1 \le i_1, i_2 \le K_2$:

(γ_1) Each $\mathcal{J}_{i_1,i_2}'$ consists of precisely $(D_4 - 2\lambda_{K_2} n)/2K_2^2$ (i_1,i_2)-ES with parameter ε_0 with respect to the partition $\mathcal{P}_2$.

(γ_2) $\mathcal{J}_0'$ contains precisely $\lambda_{K_2} n$ exceptional systems with parameter ε_0.

(γ_3) If $e_G(A', B') \ge D$, then all exceptional systems in $\mathcal{J}_0' \cup \mathcal{J}''$ are Hamilton exceptional systems.

(γ_4) If $e_G(A', B') < D$, then each exceptional system $J \in \mathcal{J}_0' \cup \mathcal{J}''$ is a Hamilton exceptional system with $e_J(A', B') = 2$ or a matching exceptional system. In particular, $\mathcal{J}_0'$ contains precisely $e_{\bigcup \mathcal{J}_0'}(A', B')/2$ Hamilton exceptional systems and $\mathcal{J}''$ contains precisely $e_{\bigcup \mathcal{J}''}(A', B')/2$ Hamilton exceptional systems.

As in Step 3, the construction of $\mathcal{J}_0'$ and the $\mathcal{J}_{i_1,i_2}'$ will depend on whether G is critical and whether $e_G(A', B') \ge D$. Recall that $G_4 = G_1 - G^{\rm rob}$ and note that

$$(14.15) \qquad \frac{D - \phi_0 n - 2\phi_0^{\rm rob} n}{400 K_2^2} = \frac{D_4}{400K_2^2} \in \mathbb{N}$$

by (14.6).

Case 1: $e_G(A', B') \geq D$ and G is not critical. Our aim is to apply Lemma 11.3 to G with $G - G_4$, m_2, K_2, $\mathcal{P}_2$, ε'_1, $\phi_0 + 2\phi_0^{\rm rob}$, λ_{K_2} playing the roles of G_0, m, K, $\mathcal{P}$, ε, ϕ, λ. (So $G_4^\diamond$ will play the role of $G^\diamond$.) First we verify that the conditions in Lemma 11.3(i)–(iv) are satisfied. Clearly, Lemma 11.3(i) and (ii) hold. Note that $G - G_4 = \mathcal{H}'_1 + G^{\rm rob}$, so (α_1), (α_2), (β'_1) and (β'_2) imply Lemma 11.3(iii). By (α_2) and (β'_2), $e_{G_4^\diamond}(A', B')$ is even. Together with the fact (S$_2$r) that $(G_4^\diamond, \mathcal{P}_2)$ is a $(K_2, m_2, \varepsilon_0, \varepsilon'_1)$-exceptional scheme, this shows that Lemma 11.3(iv) holds. Together with (14.15) this ensures that we can indeed apply Lemma 11.3 to obtain a set of $(D-(\phi_0+2\phi_0^{\rm rob})n)/2 = D_4/2$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in G_4. Moreover, these Hamilton exceptional systems can be partitioned into sets $\mathcal{J}'_0$ and $\mathcal{J}'_{i_1,i_2}$ (for all $1 \leq i_1, i_2 \leq K_2$) such that (γ_1)–(γ_3) hold.

Case 2: $e_G(A', B') \geq D$ and G is critical. Our aim is to apply Lemma 11.4 to G with $G-G_4$, m_2, K_2, $\mathcal{P}_2$, ε'_1, $\phi_0+2\phi_0^{\rm rob}$, λ_{K_2} playing the roles of G_0, m, K, $\mathcal{P}$, ε, ϕ, λ. (So as before, $G_4^\diamond$ will play the role of $G^\diamond$.) Similar arguments as in Case 1 show that Lemma 11.4(i)–(iv) hold. (β_4) implies Lemma 11.4(v). Together with (14.15) this ensures that we can indeed apply Lemma 11.4 to obtain a set of $D_4/2$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in G_4. Moreover, these Hamilton exceptional systems can be partitioned into sets $\mathcal{J}'_0$ and $\mathcal{J}'_{i_1,i_2}$ (for $1 \leq i_1, i_2 \leq K_2$) such that (γ_1)–(γ_3) hold.

Case 3: $e_G(A', B') < D$. Recall from Proposition 6.1(ii) that in this case we have $D = n/2-1$, $n = 0 \pmod 4$ and $|A'| = |B'| = n/2$. Our aim is to apply Lemma 11.5 to G with $G - G_4$, m_2, K_2, $\mathcal{P}_2$, ε'_1, $\phi_0 + 2\phi^{\rm rob}$, λ_{K_2} playing the roles of G_0, m, K, $\mathcal{P}$, ε, ϕ, λ. (So as before, $G_4^\diamond$ will play the role of $G^\diamond$.) Similar arguments as in Case 1 show that Lemma 11.5(i)–(iv) hold. (β_5) implies Lemma 11.4(v). Together with (14.15) this ensures that we can indeed apply Lemma 11.5 to obtain a set of $D_4/2$ edge-disjoint exceptional systems in G_4. Moreover, these exceptional systems can be partitioned into sets $\mathcal{J}'_0$ and $\mathcal{J}'_{i_1,i_2}$ (for all $1 \leq i_1, i_2 \leq K_2$) such that (γ_1), (γ_2) and (γ_4) hold. (In particular, (γ_4) implies that each exceptional system in these sets has parameter ε_0.)

Therefore, in each of the three cases we have constructed sets $\mathcal{J}'_0$ and $\mathcal{J}'_{i_1,i_2}$ (for all $1 \leq i_1, i_2 \leq K_2$) satisfying (γ_1)–(γ_4).

We now find Hamilton cycles and perfect matchings covering the 'non-localized' exceptional systems (i.e. the ones in $\mathcal{J}'_0$). Let $G'_4 = G_4 - G_4^\diamond$. So G'_4 is obtained from G_4 by keeping all edges inside A as well as all edges inside B, and deleting all other edges. Note that (G'_4, A, A_0, B, B_0) is an (ε_0, K_2)-framework since (G_4, A, A_0, B, B_0) is an (ε_0, K_2)-framework. Apply Lemma 10.2 to (G'_4, A, A_0, B, B_0) with K_2, λ_{K_2}, $\mathcal{J}'_0$ playing the roles of K, λ, $\{J_1, \dots, J_{\lambda n}\}$. (Recall from (S$_2$b) that $(G_4[A] + G_4[B], \mathcal{P}_2)$ is a $(K_2, m_2, \varepsilon_0, \varepsilon'_2)$-scheme, so $\delta(G'_4[A]) = \delta(G_4[A]) \geq 4|A|/5$ and $\delta(G'_4[B]) = \delta(G_4[B]) \geq 4|B|/5$ by (Sch3).) We obtain edge-disjoint subgraphs $H_1, \dots, H_{|\mathcal{J}'_0|}$ of $G'_4 + \bigcup \mathcal{J}'_0$ such that, writing $\mathcal{H}_2 := \{H_1, \dots, H_{|\mathcal{J}'_0|}\}$, the following conditions hold:

(δ_1) For each $H_s \in \mathcal{H}_2$ there is some $J_s \in \mathcal{J}'_0$ such that $J_s \subseteq H_s$.

(δ_2) If J_s is a Hamilton exceptional system, then H_s is a Hamilton cycle on $V(G)$. If J_s is a matching exceptional system, then H_s is the edge-disjoint union of two perfect matchings on $V(G)$.

(δ_3) Let $\mathcal{H}'_2 := H_1 + \cdots + H_{|\mathcal{J}'_0|}$. If $e_G(A', B') < D$, then $\mathcal{H}_2$ contains precisely $e_{\mathcal{H}'_2}(A', B')/2$ Hamilton cycles on $V(G)$.

Indeed, (δ_1) follows from Lemma 10.2(i). (δ_2) follows from Lemma 10.2(ii),(iii). (For the second part, note that (γ_3) and (γ_4) imply that $\mathcal{J}'_0$ contains matching exceptional systems only in the case when $e_G(A', B') < D$. But in this case, Proposition 6.1(ii) implies that $n = 0 \pmod 4$ and $|A'| = |B'| = n/2$, i.e. $|A'|$ and $|B'|$ are even.) For (δ_3), note that G'_4 has no $A'B'$-edges and so $e_{\bigcup \mathcal{J}'_0}(A', B') = e_{\mathcal{H}'_2}(A', B')$. Together with ($\delta_2$) and ($\gamma_4$), this now implies ($\delta_3$).

Recall that $\mathcal{J}''$ is the union of $\mathcal{J}'_{i_1,i_2}$ over all $1 \le i_1, i_2 \le K_2$. Let $G_5 := G_4 - \mathcal{H}'_2$ and $D_5 := D_4 - 2|\mathcal{H}_2| = D_4 - 2\lambda_{K_2} n$. So (14.12) implies that

$$d_{G_5}(v) = D_5 + 2r \text{ for all } v \in A \cup B \qquad \text{and} \qquad d_{G_5}(v) = D_5 \text{ for all } v \in V_0. \tag{14.16}$$

Note that

$$G_5^\diamond := G_5 - G_5[A] - G_5[B] = G_4^\diamond - \mathcal{H}'_2 = G_4^\diamond - \bigcup \mathcal{J}'_0 = \bigcup \mathcal{J}''. \tag{14.17}$$

Since $d_J(v) = 2$ for all $v \in V_0$ and all $J \in \mathcal{J}''$, it follows that

$$D_5 = 2|\mathcal{J}''|. \tag{14.18}$$

Moreover, since $(G_4[A] + G_4[B], \mathcal{P}_2)$ is a $(K_2, m_2, \varepsilon_0, \varepsilon'_2)$-scheme and $\varepsilon'_2 + 2\lambda_{K_2} \le \varepsilon$, Proposition 8.1 implies that $(G_5[A] + G_5[B], \mathcal{P}_2)$ is a $(K_2, m_2, \varepsilon_0, \varepsilon)$-scheme.

Step 7: Approximate Hamilton cycle decomposition. Our next aim is to apply Lemma 9.4 to obtain an approximate decomposition of G_5. Let

$$\mu := (r_0^{\mathrm{rob}} - 2r)/(4K_2 m_2) \qquad \text{and} \qquad \rho := \gamma/(4K_1).$$

We will apply the lemma with G_5, $\mathcal{P}_2$, K_2, m_2, $\mathcal{J}''$, ε playing the roles of G, $\mathcal{P}$, K, m, $\mathcal{J}$, ε. Clearly, conditions (c) and (d) of Lemma 9.4 hold.

In order to see that condition (a) is satisfied, recall that $m_1 K_1 = |A| = m_2 K_2$. So

$$0 \le \frac{7r_1 - 2r}{4K_2 m_2} \overset{(14.10)}{\le} \mu \overset{(14.10)}{\le} \frac{30 r_1}{4K_2 m_2} = \frac{30\gamma_1}{4K_1} \ll 1.$$

Therefore, every vertex $v \in A \cup B$ satisfies

$$\begin{aligned} d_{G_4}(v) &\overset{(14.12)}{=} D_4 + 2r \overset{(14.11)}{=} D_1 - r_0^{\mathrm{rob}} + 2r \overset{(14.3)}{=} D - \phi_0 n - 4K_2 m_2 \mu \\ &\overset{(14.1)}{=} (1/2 \pm 4\varepsilon_{\mathrm{ex}}) n - \phi_0 n - 4K_2 m_2 \mu \\ &= (1 - 4\mu \pm 3\phi_0) K_2 m_2, \end{aligned} \tag{14.19}$$

where in the last equality we recall that $(1 - \varepsilon_0) n/2 \le |A| = K_2 m_2 \le n/2$ and $\varepsilon_0, \varepsilon_{\mathrm{ex}} \ll \phi_0$. Recall that $G_5 = G_4 - \mathcal{H}'_2$ and note that

$$\Delta(\mathcal{H}'_2) = 2|\mathcal{H}_2| = 2\lambda_{K_2} n \le 5\lambda_{K_2} K_2 m_2.$$

Altogether this implies that for each $v \in A$ and for all $1 \le i \le K_2$ we have

$$\begin{aligned} d_{G_5}(v, A_i) &= d_{G_4}(v, A_i) - d_{\mathcal{H}'_2}(v, A_i) = d_{G_4}(v, A_i) \pm 5\lambda_{K_2} K_2 m_2 \\ &\overset{(14.14)}{=} (d_{G_4}(v) \pm 7\varepsilon_0 K_2 m_2)/K_2 \pm 5\lambda_{K_2} K_2 m_2 \\ &\overset{(14.19)}{=} (1 - 4\mu \pm (3\phi_0 + 7\varepsilon_0 + 5\lambda_{K_2} K_2))\, m_2. \end{aligned}$$

Since $\phi_0, \varepsilon_0, \lambda_{K_2} \ll 1/K_2$, it follows that $d_{G_5}(v, A_i) = (1 - 4\mu \pm 4/K_2)m_2$. Similarly one can show that $d_{G_5}(w, B_j) = (1 - 4\mu \pm 4/K_2)m_2$ for all $w \in B$. So Lemma 9.4(a) holds.

To check condition (b), note that $r = \gamma|A|/K_1 \ge \gamma n/3K_1$. So

$$\begin{aligned} |\mathcal{J}''| &\overset{(14.18)}{=} \frac{D_5}{2} \le \frac{D_4}{2} \overset{(14.11)}{=} \frac{D - r_0^{\rm rob}}{2} \overset{(14.1)}{\le} \frac{n}{4} + 2\varepsilon_{\rm ex} n - \frac{r_0^{\rm rob}}{2} \\ &= \frac{n}{4} + 2\varepsilon_{\rm ex} n - 2K_2 m_2 \mu - r \le \left(\frac{1}{4} + 2\varepsilon_{\rm ex} - (1-\varepsilon_0)\mu - \frac{\gamma}{3K_1}\right) n \\ &\le \left(\frac{1}{4} - \mu - \frac{\gamma}{4K_1}\right) n = \left(\frac{1}{4} - \mu - \rho\right) n. \end{aligned}$$

Thus Lemma 9.4(b) holds.

So we can indeed apply Lemma 9.4 to obtain a collection $\mathcal{H}_3$ of $|\mathcal{J}''|$ edge-disjoint spanning subgraphs $H'_1, \dots, H'_{|\mathcal{J}''|}$ of G_5 which satisfy the following properties:

- (ε_1) For each $H'_s \in \mathcal{H}_3$ there is some $J'_s \in \mathcal{J}''$ such that $J'_s \subseteq H'_s$.
- (ε_2) If J'_s is a Hamilton exceptional system then H'_s is a Hamilton cycle on $V(G)$. If J'_s is a matching exceptional system then H'_s is the edge-disjoint union of two perfect matchings on $V(G)$.
- (ε_3) Let $\mathcal{H}'_3 := H'_1 + \dots + H'_{|\mathcal{J}''|}$. If $e_G(A', B') < D$, then $\mathcal{H}_3$ contains precisely $e_{\mathcal{H}'_3}(A', B')/2$ Hamilton cycles on $V(G)$.

For (ε_3), note that (14.17) implies $G_5^\diamond = \bigcup \mathcal{J}''$ and thus $e_{\bigcup \mathcal{J}''}(A', B') = e_{\mathcal{H}'_3}(A', B')$. Together with ($\varepsilon_2$) and ($\gamma_4$), this now implies ($\varepsilon_3$).

Step 8: Decomposing the leftover and the robustly decomposable graph. Finally, we can apply the 'robust decomposition property' of $G^{\rm rob}$ guaranteed by Corollary 13.5 to obtain a decomposition of the leftover from the previous step together with $G^{\rm rob}$ into Hamilton cycles (and perfect matchings if applicable).

To achieve this, let $H' := G_5 - \mathcal{H}'_3$. Thus (14.16) and (14.18) imply that every vertex in V_0 is isolated in H' while every vertex $v \in A \cup B$ has degree $d_{G_5}(v) - 2|\mathcal{J}''| = D_5 + 2r - 2|\mathcal{J}''| = 2r$ in H' (the last equality follows from (14.18)). Moreover, $(H')^\diamond$ contains no edges. (This holds since $\bigcup \mathcal{J}'' \subseteq \mathcal{H}'_3$ and so $H' \subseteq G_5 - \bigcup \mathcal{J}'' = G_5 - G_5^\diamond$ by (14.17).) Now let $H_A := H'[A]$, $H_B := H'[B]$, $H := H_A + H_B$. Note that H is the $2r$-regular subgraph of H' obtained by removing all the vertices in V_0. Let

$$s' := rfK_1 + 7r^\diamond \overset{(14.4)}{=} Lfr_3 + 7r^\diamond \overset{(14.8)}{=} \phi_0^{\rm rob} n.$$

Recall from (β_3') that each of the s' exceptional path systems in $\mathcal{EF}_{\rm CA}+\mathcal{EF}_{\rm PCA}$ contains a unique exceptional system and $\mathcal{J}^{\rm rob}$ is the set of all these s' exceptional systems. Thus Corollary 13.5(ii)(b) implies that $H+G^{\rm rob}$ has a decomposition into edge-disjoint spanning subgraphs $H_1'',\dots,H_{s'}''$ such that, writing $\mathcal{H}_4:=\{H_1'',\dots,H_{s'}''\}$, we have:

(ζ_1) For each $H_s''\in\mathcal{H}_4$ there is some exceptional system $J_s''\in\mathcal{J}^{\rm rob}$ such that $J_s''\subseteq H_s''$.

(ζ_2) If J_s'' is a Hamilton exceptional system then H_s'' is a Hamilton cycle on $V(G)$. If J_s'' is a matching exceptional system then H_s'' is the edge-disjoint union of two perfect matchings on $V(G)$.

(ζ_3) Let $\mathcal{H}_4':=H_1''+\dots+H_{s'}''$. Then $\mathcal{H}_4$ contains precisely $e_{\mathcal{H}_4'}(A',B')/2$ Hamilton cycles on $V(G)$.

Indeed, (ζ_1) and (ζ_2) follow from Corollary 13.5(ii)(b) (recall that if $\mathcal{J}^{\rm rob}$ contains a matching exceptional system, then $|A'|=|B'|=n/2$ is even by (β_3')). For (ζ_3), note that $e_{\mathcal{H}_4'}(A',B')=e_{G^{\rm rob}}(A',B')=e_{\bigcup\mathcal{J}^{\rm rob}}(A',B')$ by (β_2'). Now (ζ_3) follows from (β_3') and (ζ_2).

Note that $\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3\cup\mathcal{H}_4$ corresponds to a decomposition of G into Hamilton cycles and perfect matchings. It remains to show that the proportion of Hamilton cycles in this decomposition is as desired.

First suppose that $e_G(A',B')\ge D$. By (α_3), $\mathcal{H}_1$ consists of Hamilton cycles and one perfect matching if D is odd. By (γ_3), (δ_2) and (ε_2), both $\mathcal{H}_2$ and $\mathcal{H}_3$ consist of Hamilton cycles. By (β_3') and (ζ_2) this also holds for $\mathcal{H}_4$. So $\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3\cup\mathcal{H}_4$ consists of Hamilton cycles and one perfect matching if D is odd.

Next suppose that $e_G(A',B')<D$. Then by (α_3), (δ_3), (ε_3) and (ζ_3) the numbers of Hamilton cycles in $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$ and $\mathcal{H}_4$ are precisely $\lfloor e_{\mathcal{H}_1'}(A',B')/2\rfloor$, $e_{\mathcal{H}_2'}(A',B')/2$, $e_{\mathcal{H}_3'}(A',B')/2$ and $e_{\mathcal{H}_4'}(A',B')/2$. Hence, $\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3\cup\mathcal{H}_4$ contains precisely

$$\left\lfloor\frac{e_{\mathcal{H}_1'\cup\mathcal{H}_2'\cup\mathcal{H}_3'\cup\mathcal{H}_4'}(A',B')}{2}\right\rfloor=\left\lfloor\frac{e_G(A',B')}{2}\right\rfloor\ge\left\lfloor\frac{F}{2}\right\rfloor$$

edge-disjoint Hamilton cycles, where F is the size of the minimum cut in G. Since clearly G cannot have more than $\lfloor F/2\rfloor$ edge-disjoint Hamilton cycles, it follows that we have equality in the final step, as required. □

15. Acknowledgement

We are very grateful to Béla Csaba for helpful discussions. We are also grateful to Katherine Staden for drawing Figure 3.

References

[1] D. Cariolaro, The 1-factorization problem and some related conjectures, Ph. D. Thesis, University of Reading, 2004.

[2] A.G. Chetwynd and A.J.W. Hilton, Regular graphs of high degree are 1-factorizable, *Proc. London Math. Soc.* **50** (1985), 193–206.

[3] A.G. Chetwynd and A.J.W. Hilton, Star multigraphs with three vertices of maximum degree, *Math. Proc. Cambridge Philosophical Soc.* **100** (1986), 303–317.
[4] A.G. Chetwynd and A.J.W. Hilton, 1-factorizing regular graphs of high degree—an improved bound, *Discrete Math.* **75** (1989), 103–112.
[5] D. Christofides, D. Kühn and D. Osthus, Edge-disjoint Hamilton cycles in graphs, *J. Combin. Theory B* **102** (2012), 1035–1060.
[6] B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown, Proof of the 1-factorization and Hamilton decomposition conjectures II: the bipartite case, preprint.
[7] B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown, Proof of the 1-factorization and Hamilton decomposition conjectures III: approximate decompositions, preprint.
[8] A. Ferber, M. Krivelevich and B. Sudakov, Counting and packing Hamilton cycles in dense graphs and oriented graphs, preprint.
[9] A. Frieze and M. Krivelevich, On packing Hamilton cycles in ε-regular graphs, *J. Combin. Theory B* **94** (2005), 159–172.
[10] S.G. Hartke, R. Martin and T. Seacrest, Relating minimum degree and the existence of a k-factor, research manuscript.
[11] S.G. Hartke and T. Seacrest, Random partitions and edge-disjoint Hamiltonian cycles, preprint.
[12] B. Jackson, Edge-disjoint Hamilton cycles in regular graphs of large degree, *J. London Math. Soc.* **19** (1979), 13–16.
[13] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, 2000.
[14] P. Katerinis, Minimum degree of a graph and the existence of k-factors. *Proc. Indian Acad. Sci. Math. Sci.* **94** (1985), 123–127.
[15] J.H. Kim and N.C. Wormald, Random matchings which induce Hamilton cycles and Hamiltonian decompositions of random regular graphs, *J. Combin. Theory B* **81** (2001), 20–44.
[16] F. Knox, D. Kühn and D. Osthus, Edge-disjoint Hamilton cycles in random graphs, *Random Structures Algorithms*, to appear.
[17] M. Krivelevich and W. Samotij, Optimal packings of Hamilton cycles in sparse random graphs, *SIAM J. Discrete Math.* **26** (2012), 964–982.
[18] D. Kühn, J. Lapinskas and D. Osthus, Optimal packings of Hamilton cycles in graphs of high minimum degree, *Combin. Probab. Comput.* **22** (2013), 394–416.
[19] D. Kühn, A. Lo and D. Osthus, Proof of the 1-factorization and Hamilton decomposition conjectures IV: exceptional systems for the two cliques case, preprint.
[20] D. Kühn and D. Osthus, Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments, *Adv. in Math.* **237** (2013), 62–146.
[21] D. Kühn and D. Osthus, Hamilton decompositions of regular expanders: applications, preprint.
[22] D. Kühn, D. Osthus and A. Treglown, Hamilton decompositions of regular tournaments, *Proc. London Math. Soc.* **101** (2010), 303–335.
[23] E. Lucas, *Récréations Mathématiques*, Vol. 2, Gautheir-Villars, 1892.
[24] C.St.J.A. Nash-Williams, Valency sequences which force graphs to have Hamiltonian circuits, University of Waterloo Research Report, Waterloo, Ontario, 1969.
[25] C.St.J.A. Nash-Williams, Hamiltonian lines in graphs whose vertices have sufficiently large valencies, in *Combinatorial theory and its applications, III (Proc. Colloq., Balatonfüred, 1969)*, North-Holland, Amsterdam (1970), 813–819.
[26] C.St.J.A. Nash-Williams, Edge-disjoint Hamiltonian circuits in graphs with vertices of large valency, in *Studies in Pure Mathematics (Presented to Richard Rado)*, Academic Press, London (1971), 157–183.
[27] C.St.J.A. Nash-Williams, Hamiltonian arcs and circuits, in *Recent Trends in Graph Theory (Proc. Conf., New York, 1970)*, Springer, Berlin (1971), 197–210.
[28] D. Osthus and K. Staden, Approximate Hamilton decompositions of regular robustly expanding digraphs, *SIAM J. Discrete Math.*, to appear.
[29] L. Perkovic and B. Reed, Edge coloring regular graphs of high degree, *Discrete Math.* **165/166** (1997), 567–578.

[30] M.J. Plantholt and S.K. Tipnis, All regular multigraphs of even order and high degree are 1-factorable, *Electron. J. Combin.* **8** (2001), R41.
[31] M. Stiebitz, D. Scheide, B. Toft and L.M. Favrholdt, *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*, Wiley 2012.
[32] E. Vaughan, An asymptotic version of the multigraph 1-factorization conjecture, *J. Graph Theory* **72** (2013), 19–29.
[33] V.G. Vizing, Critical graphs with given chromatic class, *Metody Diskret. Analiz.* **5** (1965) 9–17. (In Russian.)
[34] D.B. West, Introduction to Graph Theory (2nd Edition), Pearson 2000.

Daniela Kühn, Allan Lo, Deryk Osthus
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {d.kuhn,s.a.lo,d.osthus}@bham.ac.uk

Andrew Treglown
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London
E1 4NS
UK

E-mail addresses: a.treglown@qmul.ac.uk