

Algebraic Modules for Finite Groups



David A. Craven
St John's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Michaelmas Term, 2007

Acknowledgements

I would like to thank my supervisor, Dr Michael Collins, for his continued support, and especially for suggesting that I study the Suzuki groups, which through a rather tortuous and contrived path, resulted in the study of simply generated modules for simple groups. I would also like to thank Dr Mark Wildon for taking the time to teach me some of the wonderful theory of symmetric group representations. I would finally like to thank Dr Karin Erdmann for her year of supervision.

It would be remiss of me not to mention the collection of representation theorists (and one p -group theorist) with whom I have had the pleasure of spending various quantities of time in an office, over these three years: Louise Archer; Dusko Bogdanic; Anton Evseev; Chris Gill; Sigurdur Hannesson; Aram Mikaelian; Armin Shalile; Matt Towers; Mark Wildon; and Richard Williamson.

I would also like to thank the EPSRC, for their financial support during the term of this degree.

Abstract

The main focus of this thesis is algebraic modules—modules that satisfy a polynomial equation with integer co-efficients in the Green ring—in various finite groups, as well as their general theory. In particular, we ask the question ‘when are all the simple modules for a finite group G algebraic?’ We call this the (p) -SMA property.

The first chapter introduces the topic and deals with preliminary results, together with the trivial first results. The second chapter provides the general theory of algebraic modules, with particular attention to the relationship between algebraic modules and the composition factors of a group, and between algebraic modules and the Heller operator and Auslander–Reiten quiver.

The third chapter concerns itself with indecomposable modules for dihedral and elementary abelian groups. The study of such groups is both interesting in its own right, and can be applied to studying simple modules for simple groups, such as the sporadic groups in the final chapter.

The fourth chapter analyzes the groups $\mathrm{PSL}_2(q)$; here we determine, in characteristic 2, which simple modules for $\mathrm{PSL}_2(q)$ are algebraic, for any odd q . The fifth chapter generalizes this analysis to many groups of Lie type, although most results here are in defining characteristic only. Notable exceptions include the Ree groups ${}^2G_2(q)$, which have the 2-SMA property for all q .

The sixth and final chapter focuses on the sporadic groups: for most groups we provide results on some simple modules, and some of the groups are completely analyzed in all characteristics. This is normally carried out by restricting to the Sylow p -subgroup.

This thesis develops the current state of knowledge concerning algebraic modules for finite groups, and particularly for which simple groups, and for which primes, all simple modules are algebraic.

Contents

1	Introduction and Preliminaries	1
1.1	Basic and Previous Results	4
1.2	Required Results from Representation Theory	8
1.3	Required Results from Group Theory	11
2	General Theory of Algebraic Modules	13
2.1	The Green Ring and Algebraic Modules	13
2.2	Algebraicity and Periodicity	16
2.3	The Auslander–Reiten Quiver	19
2.4	The Normal Subgroup Structure	24
3	Dihedral and Elementary Abelian Groups	31
3.1	The Group V_4	31
3.2	The Group D_{4q}	36
3.3	The Group $C_3 \times C_3$	51
3.3.1	Dimension 2	51
3.3.2	Dimension 3	51
3.3.3	Dimension 4	53
3.3.4	Dimension 5	55
3.3.5	Dimension 6	58
4	Simple Modules for PSL_2	62
4.1	Ordinary Characters for $q \equiv 1 \pmod{4}$	64
4.2	Ordinary Characters for $q \equiv 3 \pmod{4}$	68
4.3	Modules for Dihedral Groups	71
4.4	Modular Representations for $q \equiv 3 \pmod{8}$	72
4.5	Modular Representations for $q \equiv 5 \pmod{8}$	78
4.6	Modular Representations for $q \equiv 7 \pmod{8}$	84
4.7	Block Invariants Determining Algebraicity	85

5	The Groups of Lie Type	87
5.1	The Natural Module for $GL_3(p)$	87
5.2	Special Linear Groups	89
5.3	Symplectic Groups	92
5.4	Orthogonal Groups in Odd Characteristic	94
5.5	Orthogonal Groups in Characteristic 2	96
5.6	Unitary Groups	97
5.7	Ree Groups ${}^2G_2(3^n)$	100
6	The Sporadic Groups	103
6.1	The Mathieu Groups	104
6.1.1	The Group M_{11}	104
6.1.2	The Group M_{12}	106
6.1.3	The Group M_{22}	109
6.1.4	The Group M_{23}	111
6.1.5	The Group M_{24}	114
6.2	Leech Lattice Groups	117
6.2.1	The Higman–Sims Group HS	118
6.2.2	The Janko Group J_2	124
6.2.3	The Conway Groups Co_1 , Co_2 and Co_3	127
6.2.4	The McLaughlin Group McL	127
6.2.5	The Suzuki Group Suz	129
6.3	Monster Sections	133
6.3.1	The Held Group He	133
6.3.2	The Harada–Norton Group HN	138
6.3.3	The Thompson Group Th	139
6.3.4	The Fischer Group Fi_{22}	140
6.3.5	The Fischer Group Fi_{23}	144
6.3.6	The Fischer Group Fi'_{24}	146
6.3.7	The Baby Monster B and Monster M	146
6.4	The Pariahs	147
6.4.1	The Janko Group J_1	147
6.4.2	The Janko Group J_3	150
6.4.3	The Rudvalis Group Ru	152
6.4.4	The Janko Group J_4	155
6.4.5	The O’Nan Group ON and Lyons Group Ly	156

CONTENTS

6.5 Summary	156
A MAGMA Computer Programs	158
A.1 Constructions	158
A.1.1 Constructing Groups	158
A.1.2 Constructing Modules	160
A.2 Decomposing Tensor Products	165
A.3 Determining Periodicity	168
A.4 Analyzing Algebraicity	170
Bibliography	174

Chapter 1

Introduction and Preliminaries

The notion of an algebraic module originated with Jonathan Alperin: in [1], Alperin defined an algebraic module to be a module that satisfies some polynomial with coefficients in \mathbb{Z} , where addition is direct sum and multiplication is tensor product. The natural place to consider such an object is in the Green ring $a(KG)$, which here is defined to be all \mathbb{Z} -linear combinations of isomorphism types of indecomposable modules. (Of course, one can extend the ring of coefficients to either \mathbb{Q} or \mathbb{C} , and there are often good reasons to do so, although we will not need to do so here.) There is an obvious direct analogue with the case of an algebraic number. The study of algebraic numbers has resulted in a huge edifice of mathematics, but so far the concept of algebraic modules has been rarely directly used. That said, it has been of considerable indirect use, since understanding the decomposition of tensor powers of modules into their indecomposable summands has been an important aspect of representation theory for a long time. The connection between the two areas can be seen with the following result.

Lemma 1.1 Suppose that M is a module for a group G . The following conditions are equivalent:

- (i) M satisfies a polynomial with coefficients in \mathbb{Z} ; and
- (ii) there are only finitely many isomorphism types of summand lying in $M^{\otimes n}$ as n varies.

The proof is simple: if M satisfies a polynomial with coefficients in \mathbb{Z} , say of degree n , then every summand of $M^{\otimes n}$ has already appeared in $\bigoplus_{i < n} M^{\otimes i}$, and clearly the same can be said for $M^{\otimes j}$ for $j \geq n$. Conversely, if there are finitely many, say n , different indecomposable modules that appear in tensor powers of M , write $M^{\otimes i}$ as

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

a sum of these modules, for each $1 \leq i \leq n + 1$. This gives $n + 1$ simultaneous linear equations in the n indecomposable modules, yielding a dependence amongst the $M^{\otimes i}$; i.e., it produces a polynomial with coefficients in \mathbb{Z} that M satisfies.

Of particular significance are the tensor powers of naturally occurring modules, such as permutation modules, projective modules, simple modules, and so on. Since the tensor product of two projective modules is again projective, all projective modules are algebraic. More generally, the tensor product of two summands of permutation modules is a summand of a permutation module, and so they are algebraic. The situation with simple modules is considerably more complicated, however.

In the literature, the non-trivial results on when simple modules are algebraic can be grouped into two collections: those on soluble groups, or more generally, on p -soluble groups; and those on (quasi)simple groups. In the former category, we find the result of Berger ([16], [17]), proving that every simple module for every soluble group is algebraic, and its extension to p -soluble groups by Feit [35]. In the second category, we note that Alperin [3] proved that for $G \cong \mathrm{SL}_2(2^n)$, and K a splitting field of characteristic 2, all simple modules are algebraic. In addition, it is stated in [17] as well-known that the natural module for $\mathrm{GL}_3(p)$ is non-algebraic. (This is proved in Corollary 5.6.) Apart from these results, very little appears in the literature.

This thesis significantly increases the state of knowledge with respect to algebraic modules. In particular, we prove the following theorems.

Theorem A Let G be a finite group, and let M be a non-periodic, algebraic module. Then $\Omega^i(M)$ is non-algebraic for all $i \neq 0$.

Theorem B Let G be a finite group, and let M be an indecomposable algebraic module of complexity at least 3. Write Γ for the component of the stable Auslander–Reiten quiver $\Gamma_s(KG)$ containing M . Then M lies on the end of Γ , and no other module on Γ is algebraic.

Theorem C Let G denote a dihedral 2-group, and let Γ be a component of $\Gamma_s(KG)$ with non-periodic modules. Then there is at most one algebraic module on Γ .

Theorem D Let G be the group $\mathrm{PSL}_2(q)$ where q is odd, and let K be a field of characteristic 2. Then all simple modules are algebraic if and only if $q \not\equiv 7 \pmod{8}$, and if $q \equiv 3, 5 \pmod{8}$ then all tensor products of arbitrarily many simple modules are explicitly determined.

Theorem E Let G be a sporadic group, and let K be a field of characteristic p . Then all simple KG -modules are algebraic in the following cases:

- (i) $G = M_{11}$ and $p = 2$;
- (ii) $G = M_{22}$ and $p = 3$;
- (iii) $G = HS$ and $p = 3$;
- (iv) $G = J_2$ and $p = 3$ or $p = 5$;
- (v) $G = J_1$ and $p = 2$; and
- (vi) G is a sporadic group and $p^2 \nmid |G|$.

Theorem F Let G be a group with abelian Sylow 2-subgroups, and let K be a field of characteristic 2. Then all simple KG -modules are algebraic.

The paucity of results until now is not at all unexpected; decomposing tensor products of modules is a tricky prospect, hampered by the fact that tensor products run roughshod over the block structure, and that it is generally difficult to examine the internal structure of indecomposable modules given nothing more than the fact that they are summands of a particular tensor product.

In practice—that is, when dealing with specific groups—it is difficult to find a linear dependence amongst the $M^{\otimes n}$, and harder still to try to *prove* such a dependence is valid. A slightly more effective way is to perform the following procedure: firstly, decompose $M \otimes M$ into a sum of indecomposable modules M_i for $1 \leq i \leq r$; then decompose $M_1 \otimes M$, and if any of the summands are not isomorphic with M or the M_i , we append them to the end of the list as M_{r+1} , M_{r+2} , and so on. Then decompose $M \otimes M_2$, and perform the same task again. This procedure terminates if and only if M is algebraic, and assuming this procedure terminates, we will be left with a list of modules M_1, \dots, M_s of all possible indecomposable summands of $M^{\otimes n}$ for every n . Using the decompositions of $M \otimes M_i$, one can if necessary construct a polynomial that M satisfies.

Entangled with the concept of algebraic modules is that of simply generated modules. A *simply generated* module is one that is a summand of $M_1 \otimes M_2 \otimes \dots \otimes M_r$ for some (possibly equal) simple modules M_i . Obviously, every simple module is algebraic if and only if there are only finitely many simply generated modules.

This short introductory chapter will focus on the previous results that have appeared before in the literature, and those that are basic enough to be placed at the

very start, together with necessary results from both representation theory and group theory. Known results specific to a particular section will tend to be appear within that section, however.

1.1 Basic and Previous Results

Since we will be dealing with tensor powers of modules, we will introduce the notation $T(M)$ to denote the tensor module of M ; this is the infinite-dimensional \mathbb{N} -graded module

$$T(M) = M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus \dots .$$

Thus M is algebraic if and only if $T(M)$ has only finitely many different isomorphism classes of summand in its homogeneous components.

The purpose of this section is to collate the, mostly trivial, results on moving from one algebraic module to another. The first two follow from the alternate characterization of algebraic modules given in Lemma 1.1.

Lemma 1.2 Suppose that M is an algebraic module. Then any summand of M is algebraic.

Lemma 1.3 Suppose that M_1 and M_2 are algebraic modules. Then $M_1 \oplus M_2$ and $M_1 \otimes M_2$ are algebraic. In addition, a module M is algebraic if and only if $M^{\otimes n}$ is algebraic for some positive integer n .

We can also use induction and restriction to produce new algebraic modules.

Lemma 1.4 (Berger [17]) Suppose that G is a finite group, and that H is a subgroup of G . If M is an algebraic KG -module, then $M \downarrow_H$ is an algebraic KH -module. In addition, if N is a KH -module, then $N \uparrow^G$ is algebraic if and only if N is algebraic.

Proof: (Feit [36]) Suppose that

$$\bigoplus_{i=0}^n a_i M^{\otimes i} = 0$$

for some integers a_i . Then, taking restrictions to H , we have

$$\left(\bigoplus_{i=0}^n a_i M^{\otimes i} \right) \downarrow_H = \bigoplus_{i=0}^n a_i (M \downarrow_H)^{\otimes i} = 0,$$

and hence $M \downarrow_H$ is algebraic.

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

In the other direction, we firstly proceed by induction on $|H|$. Suppose that N is algebraic; then by Mackey's tensor product theorem,

$$\begin{aligned} (N \uparrow^G)^{\otimes n} &= (N \uparrow^G)^{\otimes(n-1)} \otimes N \uparrow^G \\ &= ((N \uparrow^G)^{\otimes(n-1)} \downarrow_H \otimes N) \uparrow^G \\ &= \left(\left(\bigoplus_g N^g \downarrow_{H^g \cap H} \uparrow^H \right)^{\otimes(n-1)} \otimes N \right) \uparrow^G. \end{aligned}$$

This implies that $(N \uparrow^G)^{\otimes n}$ is a sum of modules

$$\left(\bigotimes_{i=1}^n N^{g_i} \downarrow_{H^{g_i} \cap H} \uparrow^H \right) \uparrow^G.$$

The module $N^g \downarrow_{H^g \cap H}$ is algebraic and by induction hypothesis so is $(N^g \downarrow_{H^g \cap H}) \uparrow^H$. There are only finitely many different KH -modules of the form $N^g \downarrow_{H^g \cap H}$, and so there is a finite list of indecomposable summands A_1, \dots, A_m , such that for each n , every summand of

$$\bigotimes_{i=1}^n N^{g_i} \downarrow_{H^{g_i} \cap H} \uparrow^H$$

is isomorphic with one of them. Then every summand of $(N \uparrow^G)^{\otimes n}$ is isomorphic with a summand of some $A_i \uparrow^G$, and so $N \uparrow^G$ is algebraic.

Conversely, suppose that $N \uparrow^G$ is algebraic. By the Mackey decomposition theorem,

$$(N \uparrow^G) \downarrow_H = \bigoplus_{t \in T} N^t \downarrow_{H \cap H^t} \uparrow^H,$$

where T is a set of (H, H) -double coset representatives. In particular, T can be chosen so that $1 \in T$, and so $N|(N \uparrow^G) \downarrow_H$; hence if $N \uparrow^G$ is algebraic then N is algebraic. □

Lemma 1.4 can be used to prove the following important result.

Proposition 1.5 Suppose that M is an indecomposable KG -module; let Q be a vertex for M , and let S be a source of M . Then M is algebraic if and only if S is algebraic.

Proof: The module S has the property that $S|M \downarrow_Q$ and $M|S \uparrow^G$. The first statement implies that M is algebraic if S is, and the second statement provides the converse. □

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

This, combined with the fact that if P is a cyclic p -group, then every KP -module is algebraic, and the fact that if Q is a vertex of an indecomposable module M , then Q is contained in a defect group of the block containing M , yields the following.

Corollary 1.6 Suppose that B is a block of KG with cyclic defect group. Then every B -module is algebraic. More generally, let M be an indecomposable module with cyclic vertex. Then M is algebraic.

Finally for the basic results, we include for reference the contrapositive of Lemma 1.4, which will be used extensively in later chapters.

Lemma 1.7 Suppose that M is a KG -module, and let H be a subgroup of G . Suppose that N is a summand of $M \downarrow_H$, and finally suppose that N is not algebraic. Then M is not algebraic.

Moving on to more complicated results previously found in the literature, we start with results of Berger. Berger's paper [17] collects several other results not already discussed: we give them here for future use. The first implies that we need not worry about the size of the field over which we work.

Theorem 1.8 (Berger [17]) Let K be a field of characteristic p , and let F be an extension field of K . Let M be a KG -module. Then M is algebraic if and only if $M \otimes_K F$ is algebraic.

We now come to an important concept of this thesis: if K is a field of characteristic p , and all of the simple KG -modules are algebraic, we will say that G has p -SMA; a priori, this might depend on the size of the field K . However, it does not, as we shall now prove.

Let N denote the sum of every simple KG -module. If M is a simple KG -module and F is an extension field of K , then $M \otimes_K F$ is semisimple. Furthermore, every simple FG -module is a submodule of some $M \otimes_K F$, where M is a simple KG -module. Thus every simple FG -module is a summand of $N \otimes_K F$, and $N \otimes_K F$ is semisimple.

All simple KG -modules are algebraic if and only if N (or equivalently by Theorem 1.8, $N \otimes_K F$) is algebraic, and $N \otimes_K F$ is algebraic if and only if all simple FG -modules are algebraic; this proves the claim.

Corollary 1.9 Let G be a finite group, and let K and F be any two fields of characteristic p . Then all simple KG -modules are algebraic if and only if all simple FG -modules are algebraic. Hence the statement ' G has p -SMA' does not depend on the size of the field involved.

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

Earlier in the chapter we stated that M is algebraic if and only if $T(M)$ contains only finitely many different indecomposable summands; in fact, it can be shown that M is algebraic if and only if $M^{\otimes n}$ can be written as a sum of smaller tensor powers of M .

Theorem 1.10 (Berger [17]) Suppose that M is an algebraic KG -module, where G is a finite group and K is a field. Then M satisfies a monic polynomial with coefficients in \mathbb{Z} .

Let M be an algebraic KG -module. Then M satisfies a polynomial with integer coefficients, and so M satisfies a polynomial of minimal such degree. This is referred to as the *degree* of the module M , and denoted $\deg M$. Notice that by Theorem 1.20, which lies in the next section, we can easily see that the constant term of the minimal polynomial of an absolutely indecomposable module M is non-zero if and only if $\dim M$ is prime to p and M^* is a summand of $M^{\otimes i}$ for some i .

Berger's paper [17] had the following as its main result.

Theorem 1.11 (Berger [17]) Let G be a soluble group, and let K be any field. Then all simple KG -modules are algebraic.

Feit extended Berger's result to the larger class of p -soluble groups.

Theorem 1.12 (Feit [35]) Let G be a p -soluble group and let K be a field of characteristic p . Then all simple KG -modules are algebraic.

Feit's proof revolved around taking a minimal non-central normal subgroup of the p -soluble group, and proceeding by induction on the dimension of the simple module involved. The proof relies on the classification of the finite simple groups. One of the key themes in Chapter 2 is understanding how this theorem generalizes, and how the normal subgroup structure of a group affects whether the simple modules are algebraic.

Theorem 1.13 (Alperin [3]) Let K be a field, and let $G = \mathrm{SL}_2(2^n)$. Then any simple KG -module is algebraic.

This result demonstrates that there are groups that are not p -soluble, and that do not have cyclic Sylow p -subgroups, that nevertheless have p -SMA. Another central theme in this thesis will be attempting to find other groups for which this is true.

1.2 Required Results from Representation Theory

In this section we collate the results from representation theory needed for this thesis. We start this with the definition and main results on periodicity. The concept of periodicity was first introduced for modules in [44], where some of its properties were proved. Recall that if M is a KG -module then $\Omega(M)$ is defined to be the kernel of the surjective map from the projective cover $\mathcal{P}(M)$ to M . Similarly, $\Omega^{-1}(M)$ is defined to be the cokernel of the injective map from M to the injective hull of M . Write $\Omega^0(M)$ for the sum of the non-projective summands of M . Define $\Omega^i(M)$ for all other $i \in \mathbb{Z}$ inductively.

The operations Ω and Ω^{-1} are inverse in the sense that

$$\Omega(\Omega^{-1}(M)) = \Omega^0(M) = \Omega^{-1}(\Omega(M)).$$

If M is a non-projective indecomposable module, then $\Omega^0(M) = M$, and $\Omega^i(M)$ is a non-projective indecomposable module for all $i \in \mathbb{Z}$. Since Ω and Ω^{-1} are inverse to one another on the set of all non-projective indecomposable modules, Ω induces a bijection on this collection. Note that $\Omega(P) = 0$ if P is a projective module.

Lemma 1.14 Let G be a finite group and let M_1 and M_2 be KG -modules.

- (i) $\Omega(M_1 \oplus M_2) = \Omega(M_1) \oplus \Omega(M_2)$.
- (ii) $\Omega(M_1 \otimes M_2) = \Omega^0(\Omega(M_1) \otimes M_2)$.
- (iii) $\Omega^{-1}(M_1) = \Omega(M_1^*)^*$.

There is some useful interaction between the Green correspondence and the Heller operator.

Lemma 1.15 Let P be a p -subgroup of the finite group G , and let M be an indecomposable KG -module.

- (i) The p -subgroup P is a vertex of M if and only if P is a vertex of $\Omega(M)$.
- (ii) If U is the Green correspondent of M in $H \geq N_G(D)$, then $\Omega(U)$ is the Green correspondent of $\Omega(M)$.
- (iii) If S is a source of the KG -module M , then $\Omega(S)$ is a source of the module $\Omega(M)$.

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

If $\Omega^i(M) = \Omega^0(M)$ for some non-zero i , then M is called (Ω) -periodic (as first discussed in [2]), and its period is the smallest positive i for which this statement holds. This section will investigate the impact of periodicity on whether a module is algebraic.

The following is a collection of some of the most important facts on periodic modules.

Lemma 1.16 ([36, II.6.4]) Suppose that G is a finite group. If K is periodic as a KG -module then all modules are periodic.

The module K is periodic if and only if G has cyclic or quaternion Sylow p -subgroups; i.e., if G has p -rank 1 (see [24, XII.7]). In these two cases then, all modules are periodic. In the first case, where the Sylow p -subgroups of G are cyclic, since there are only finitely many isomorphism types of indecomposable module, all KG -modules are algebraic. The second case is considerably more difficult, and no description of the algebraic modules is known. Part of the problem stems from the fact that there is no good description of the indecomposable modules, unlike the other tame cases of dihedral groups and semidihedral groups. (Even in those cases the answer is not known, although for dihedral 2-groups there are some partial results, given in Chapter 3.)

Theorem 1.17 (Carlson [22], Alperin–Evens [7]) A KG -module M is periodic if and only if the restriction of M to all elementary abelian subgroups is periodic.

If P is an abelian p -group, then a periodic module has period either 1 or 2. The corresponding result for non-abelian groups is considerably more complicated, but is a natural generalization of the abelian case.

Theorem 1.18 (Carlson [22]) Let P be a finite p -group of order p^n . Write $\mathcal{A}(G)$ for the set of all maximal abelian subgroups, and let p^r be the smallest order of an element A of $\mathcal{A}(G)$. Then the period of any periodic module divides $2p^{n-r}$.

Finally, we give a result of Carlson's, conjectured by Alperin in [2], on the dimensions of periodic modules.

Theorem 1.19 (Carlson [21]) Let G be a finite group of p -rank r . Then a periodic KG -module has dimension a multiple of p^{r-1} .

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

Having dealt with periodicity, we move on to other properties of modules. In the rest of this thesis, we write $n \cdot M$ to mean the n -fold direct sum of the module M with itself. We begin with two results on summands of tensor products.

Theorem 1.20 (Benson–Carlson [15]) Let G be a finite group and M and N be absolutely indecomposable KG -modules.

- (i) $K|M \otimes N$ if and only if $p \nmid \dim M$ and $M \cong N^*$, in which case $2 \cdot K$ is not a summand of $M \otimes N$.
- (ii) If $p \mid \dim M$, then every summand of $M \otimes N$ has dimension a multiple of p .

Proposition 1.21 (Auslander–Carlson [11, Proposition 4.9]) Let G be a finite group and K be a field of characteristic p . If M is an indecomposable module of dimension a multiple of p , then $2 \cdot M$ is a direct summand of $M \otimes M^* \otimes M$.

These two results taken together prove that if M is any KG -module, then M is a summand of $M \otimes M^* \otimes M$. The next result shows that if M is faithful, then one may find a free module inside some sum of tensor powers of M .

Proposition 1.22 ([4, 7.1], and [36, III.2.18]) Let M be a KG -module. Then $T(M)$ contains a free KG -module if and only if M is faithful. In particular, if M is a faithful module then $T(M)$ has every projective module as a summand, and all projective modules are simply generated if and only if $O_p(G) = 1$.

The next result gives us some control over the summands of tensor powers, although this control is in a very real sense quite weak.

Lemma 1.23 ([36, Lemma II.2.3]) Let G be a finite group, and let H be a subgroup of G . Let M be a KG -module, and N be a KH -module. Then

$$M \otimes N \uparrow^G = (M \downarrow_H \otimes N) \uparrow^G .$$

In particular, if M_1 and M_2 are indecomposable KG -modules, and M_1 has vertex Q , then the vertex of each summand of $M_1 \otimes M_2$ is contained within Q .

It makes sense therefore to examine the vertices of simple modules with regard to the blocks, and this is the focus of the last two results.

Theorem 1.24 (Knörr [55]) Let G be a finite group and B be a block of KG , with defect group D . Let M denote a simple B -module, with vertex P . Then P can be chosen so that $C_D(P) \leq P \leq D$; in particular if D is abelian, then all simple B -modules have vertex D .

Theorem 1.25 (Erdmann [29]) Let G be a finite group, let B be a block of KG , and let M be a simple B -module. If M has cyclic vertex P , then B has defect group P .

1.3 Required Results from Group Theory

The results from group theory that we require are essentially characterizations of various finite groups. During the 1960s and 1970s several very deep results were produced about finite groups whose Sylow 2-subgroups were of a prescribed type. We begin with the characterization of groups with abelian Sylow 2-subgroups.

Theorem 1.26 (Walter [77]) Suppose that G is a finite group with abelian Sylow 2-subgroups. Then G possesses a normal subgroup H of odd index, containing $O_{2'}(G)$, such that $H/O_{2'}(G)$ is a direct product of an abelian 2-group and simple groups with abelian Sylow 2-subgroup, which are the groups

- (i) $SL_2(2^n)$ for $n \geq 3$,
- (ii) $PSL_2(q)$ for $q \geq 5$, $q \equiv 3, 5 \pmod{8}$,
- (iii) ${}^2G_2(3^{2n+1})$ for $n \geq 1$, and
- (iv) the sporadic group J_1 .

This is fairly typical of results of this type. Next, we examine groups with dihedral Sylow 2-subgroups.

Theorem 1.27 (Gorenstein–Walter [40]) Let G be a finite group with dihedral Sylow 2-subgroups. Then G has a subgroup H of odd index, containing $O_{2'}(G)$, such that $H/O_{2'}(G)$ is isomorphic to one of the following groups:

- (i) a dihedral 2-group;
- (ii) $PSL_2(q)$, $q \geq 5$ odd;
- (iii) $PGL_2(q)$, $q \geq 5$ odd; or

(iv) A_7 .

However, when it came to groups with semidihedral Sylow 2-subgroups, Alperin, Brauer and Gorenstein could only prove a classification of such simple groups, not of an arbitrary finite group.

Theorem 1.28 (Alperin–Brauer–Gorenstein [5]) Suppose that G is a simple group with semidihedral Sylow 2-subgroups. Then G is isomorphic with one of the groups

- (i) $\text{PSL}_3(q)$ with $q \equiv 3 \pmod{4}$ and $q \geq 5$;
- (ii) $\text{PSU}_3(q)$ with $q \equiv 1 \pmod{4}$; or
- (iii) the Mathieu group M_{11} .

In the case of wreathed Sylow 2-subgroups, the classification of simple groups with such Sylow 2-subgroups was started in [5], and completed in [6], using character theory developed by Brauer in [19].

Theorem 1.29 (Alperin–Brauer–Gorenstein [5],[19],[6]) Suppose that G is a simple group with wreathed Sylow 2-subgroups. Then G is isomorphic with one of the groups

- (i) $\text{PSL}_3(q)$ with $q \equiv 1 \pmod{4}$; or
- (ii) $\text{PSU}_3(q)$ with $q \equiv 3 \pmod{4}$ and $q \geq 5$.

In [6], the three authors also prove the result that if G is a finite simple group of 2-rank two, then G has either dihedral, semidihedral, wreathed, or is $\text{PSU}_3(4)$.

Before we begin, we will describe our conventions. All words should be read left-to-right, and all maps are composed in the same way. Similarly, all of our modules are right modules.

Chapter 2

General Theory of Algebraic Modules

This chapter contains, as its title suggests, the general theory of algebraic modules. This theory currently consists of two main branches: the first is the relationship between indecomposable algebraic modules and the Heller operator and Auslander–Reiten quiver; and the second is the relationship between the normal subgroup structure of a group and the algebraicity of simple modules.

2.1 The Green Ring and Algebraic Modules

In this section we will briefly consider the quotient by an ideal consisting solely of algebraic modules, and then examine the minimal polynomial of an algebraic module.

Proposition 2.1 Let \mathcal{I} be an ideal of algebraic modules in the Green ring $a(KG)$, and let M be a KG -module. Then M is algebraic in $a(KG)$ if and only if $M + \mathcal{I}$ is algebraic in $a(KG)/\mathcal{I}$. In particular, if \mathcal{P} denotes the ideal consisting of all projective modules, then a KG -module M is algebraic if and only if $M + \mathcal{P}$ is algebraic.

Proof: Suppose that M is algebraic. Then M satisfies some polynomial in the Green ring, and therefore its coset in any quotient satisfies this polynomial as well. Conversely, suppose that $M + \mathcal{I}$ satisfies some polynomial in the quotient $a(KG)/\mathcal{I}$. Thus

$$\sum \alpha_i (M + \mathcal{I})^i = \mathcal{I}.$$

This implies that, since $(M + \mathcal{I})^i = M^{\otimes i} + \mathcal{I}$, then

$$\sum \alpha_i M^{\otimes i} \in \mathcal{I},$$

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

which consists of algebraic modules. Hence there is some polynomial involving only M witnessing the algebraicity of M . \square

In fact, one can extend the ideal \mathcal{P} to one containing not only the projective modules but all modules of cyclic vertex, by Lemma 1.23.

This proposition can be used to make the results of the next section easier to follow, as we can effectively ignore all projective summands of modules when we wish to determine algebraicity.

We now move on to examine the minimal polynomial of an algebraic module, as discussed at the end of Section 1.1. The proof that induction preserves algebraic modules did not easily yield a polynomial for the induced module in terms of the polynomial of the original module. However, restriction is better.

Lemma 2.2 Suppose that M is a KG -module, and $H \leq G$. If $p(x)$ is the minimal polynomial for M , then $p(M \downarrow_H) = 0$, and so the minimal polynomial for $M \downarrow_H$ divides $p(x)$.

If $M \downarrow_H$ is indecomposable, then this gives much information about the minimal polynomial for $M \downarrow_H$; however, if $M \downarrow_H$ is not indecomposable then this gives relatively little information about the minimal polynomial for a summand of $M \downarrow_H$.

Recall that $\deg M$ denotes the degree of the minimal polynomial of M . We start by classifying modules of degree 1.

Lemma 2.3 Suppose that M is a KG -module, and that $\deg M = 1$. Then M is a (possibly decomposable) trivial module.

Proof: This is obvious: if M satisfies a polynomial $ax - b$ then $M = (a/b) \cdot K$. \square

Next, we move on to modules of degree 2.

Proposition 2.4 Let M be an indecomposable KG -module with $\deg M = 2$. Then either M is trivial, $p \neq 2$ and M is the non-trivial 1-dimensional module for C_2 viewed as a KG -module (assuming that G has a quotient isomorphic with C_2), or M is isomorphic with a p -group algebra KH , viewed as a KG -module (assuming that G has a quotient isomorphic with H).

Proof: We assume that M is faithful; if not, then we quotient out by the kernel of M first. Suppose that M satisfies the polynomial $ax^2 - bx - c$. Then

$$a \cdot M^{\otimes 2} = b \cdot M \oplus c \cdot K.$$

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

Thus $M^{\otimes 2}$ is a sum of copies of M and K , so relabelling we may assume that

$$M^{\otimes 2} = a \cdot M \oplus b \cdot K.$$

By Theorem 1.20, either $b = 0$ or $b = 1$. If $b = 0$, then $M^{\otimes 2} = a \cdot M$, and since $T(M)$ contains a free module, M itself must be free. Hence G is a p -group and $M \cong KG$.

Suppose therefore that $b = 1$. Then $M^{\otimes 2} = a \cdot M \oplus K$. Taking congruences modulo $\dim M$ implies that $\dim M = 1$, and so $M^{\otimes 2} = K$. Again $M \oplus K$ contains a free module, and so either M is free, and hence $M = K$, or $M \oplus K$ is free, and we have that $M \oplus K$ is the group algebra of C_2 , as required. \square

The main stumbling block in continuing this process, and dealing with cubics, is that there are many modules that satisfy cubics; for example, all even-dimensional indecomposable KV_4 -modules satisfy cubics, and in fact satisfy ‘the same’ cubic,

$$x^3 - (n + 2)x^2 + 2nx = 0,$$

where $n = \dim M$.

The following is an easy partial result for cubics.

Proposition 2.5 Let M be a faithful indecomposable KG -module, and write F for the free KG -module of dimension $|G|$. Suppose that $\deg M \geq 3$ and that $M^{\otimes 2}$ is a direct sum of copies of K , copies of M , and copies of F . Then $\deg M = 3$.

Proof: Suppose that $M^{\otimes 2} = \alpha \cdot M \oplus \beta \cdot F \oplus \gamma \cdot K$ (where $\alpha, \beta, \gamma \in \mathbb{Z}$), and write $n = \dim M$. Then

$$F = \frac{M^{\otimes 2} \ominus \alpha \cdot M \ominus \gamma \cdot K}{\beta},$$

and taking the tensor product of both sides with M and dividing by n , we see that

$$F = \frac{M^{\otimes 3} \ominus \alpha \cdot M^{\otimes 2} \ominus \gamma \cdot M}{n\beta},$$

and this yields a cubic, as required. \square

Proceeding in this direction becomes more difficult and complicated as the degree increases, and with little or no benefit. We will, however, note the following related question asked by Will Turner, phrased as a conjecture.

Conjecture 2.6 Let G be a finite group, and let K be a field. Then there is an integer n , dependent only on G and K , such that if M is an algebraic indecomposable KG -module, then $\deg M \leq n$.

This conjecture is obviously true if K is a field of characteristic p and G has cyclic Sylow p -subgroups. It is also true for V_4 , as a corollary of a theorem of Conlon determining the tensor product of any two KV_4 -modules, which is given in the next chapter.

Before we continue with this chapter, we pause to introduce a potentially better invariant for an algebraic module than $\deg M$, and that is the number of distinct isomorphism types of summand in $T(M)$. If N is a summand of M then this invariant cannot increase, unlike the degree, and its behaviour on restriction and induction is also controllable to a certain extent. We will not develop a theory of this invariant here, however.

2.2 Algebraicity and Periodicity

In this section we will relate the Heller operator and algebraic modules. We begin with infinitely many examples of non-algebraic modules, at least if a finite group G has a quotient whose p -rank is at least 2 (i.e., G has non-cyclic Sylow p -subgroups).

Proposition 2.7 Let G be a finite group of p -rank at least 2, and let K be a field of characteristic p . Then, for all $i \neq 0$, the module $\Omega^i(K)$ is not algebraic.

Proof: Notice that, modulo projective modules,

$$(\Omega^i(K))^{\otimes n} = \Omega^{ni}(K),$$

and so $\Omega^{ni}(K)$ appears as a summand of the n th tensor power of $\Omega^i(K)$ for all $n \geq 1$, an infinite collection of summands since K is not periodic. \square

If G is not of p -rank 2 and does not have cyclic Sylow p -subgroups, then $p = 2$ and the Sylow 2-subgroups of G are generalized quaternion. In this case, by the Brauer–Suzuki theorem, G possesses a normal subgroup $Z^*(G)$ such that $G/Z^*(G)$ has dihedral Sylow 2-subgroups, and so there are non-algebraic modules for this quotient. Alternatively, a generalized quaternion 2-group possesses a V_4 quotient, and so there are non-algebraic modules for generalized quaternion 2-groups, whence any indecomposable module for G with one of those modules as a source would be non-algebraic.

Now suppose that a KG -module M is periodic; we will determine how this affects whether M is algebraic. In the next proof, we use the fact that a module M is algebraic if and only if $M^{\otimes i}$ is algebraic for $i \geq 1$. (See Lemma 1.3.)

Proposition 2.8 Let M be an algebraic periodic module. Then $\Omega^i(M)$ is algebraic for all i .

Proof: Suppose that $\Omega^n(M) = M$. We know that

$$\Omega(M \otimes N) = \Omega^0(\Omega(M) \otimes N) = \Omega^0(M \otimes \Omega(N)).$$

Hence, $\Omega^0(\Omega^i(M)^{\otimes n}) = \Omega^{ni}(M^{\otimes n}) = \Omega^0(M^{\otimes n})$, and since $M^{\otimes n}$ is algebraic (as M is), the module $\Omega^i(M)$ is algebraic for all i (as $\Omega(M)^{\otimes n}$ is). \square

Both possibilities allowed—that the Ω -translates of M are either all algebraic modules or all non-algebraic modules—occur in the module category of the quaternion group. Firstly, the trivial module is an algebraic periodic module, and secondly, since the group V_4 has 2-rank 2, the non-trivial Heller translates of the trivial module for that group are non-algebraic by Proposition 2.7, and so those modules, viewed as modules for the quaternion group, are also non-algebraic. It should be mentioned that no examples of non-algebraic periodic modules are known if the characteristic of the field is odd.

Now we consider non-periodic modules. Since a module M is non-periodic if and only if $M \otimes M^*$ is, we firstly consider self-dual non-periodic modules, then apply this to the general case.

Proposition 2.9 Let M be a self-dual non-periodic module. If $i \neq 0$ then $\Omega^i(M)$ is not algebraic.

Proof: Consider the module

$$\Omega^0(\Omega^i(M) \otimes \Omega^i(M) \otimes \Omega^i(M)) = \Omega^{3i}(M^{\otimes 3});$$

as M is a summand of $M^{\otimes 3}$, we see that $\Omega^{3i}(M)$ is a summand of $\Omega^i(M^{\otimes 3})$. We can clearly iterate this procedure to prove that infinitely many different Ω -translates of M lie in tensor powers of $\Omega^i(M)$ (and these all contain different indecomposable summands as M is non-periodic) proving that $\Omega^i(M)$ is non-algebraic, as required. \square

The following corollary is Theorem A in the introduction.

Corollary 2.10 Let M be a non-periodic algebraic module. Then no module $\Omega^i(M)$ for $i \neq 0$ is algebraic.

Proof: Suppose that both M and $\Omega^i(M)$ are algebraic. Then so is M^* , and therefore so is

$$\Omega^0(M^* \otimes (\Omega^i(M))) = \Omega^i(M \otimes M^*).$$

Since $M \otimes M^*$ is self-dual, this module cannot be algebraic, a contradiction. \square

Hence for non-periodic modules M , either none of the modules $\Omega^i(M)$ is algebraic, or exactly one module is, and in the latter case, if one of the modules is self-dual then this is the algebraic module. In the case of the dihedral 2-groups, there are non-periodic modules M such that no $\Omega^i(M)$ are algebraic, and there are self-dual, non-periodic algebraic modules. In fact, both possibilities for non-periodic modules must occur since all simple modules for p -soluble groups are algebraic.

To end this section, we collate the results given here into a theorem.

Theorem 2.11 Let M be a KG -module.

- (i) If M is periodic, then M is algebraic if and only if all $\Omega^i(M)$ are algebraic.
- (ii) If M is non-periodic, then at most one of the modules $\Omega^i(M)$ is algebraic, and if M is self-dual and one of the $\Omega^i(M)$ is algebraic, then it is M that is algebraic.

Furthermore, all possibilities allowed by this theorem do occur.

This theorem has the following corollary, which will be put to use in the following chapter.

Corollary 2.12 Let M be a non-periodic indecomposable module, and suppose that there is some $n \geq 2$ such that $\Omega^i(M)$ or $\Omega^i(M^*)$ is a summand of $M^{\otimes n}$ for some $i \neq 0$. Then the module $\Omega^i(M)$ is non-algebraic for all $i \in \mathbb{Z}$.

Proof: Suppose that $\Omega^i(M)$ is a summand of $M^{\otimes n}$, for some $n \geq 2$ and $i \neq 0$. Then, for each $j \in \mathbb{Z}$, we have

$$\Omega^{nj+i}(M) | \Omega^j(M)^{\otimes n},$$

and since at least one of $\Omega^{nj+i}(M)$ and $\Omega^j(M)$ is non-algebraic, we see that $T(\Omega^j(M))$, the sum of the tensor powers of $\Omega^j(M)$, contains a non-algebraic summand; hence $\Omega^j(M)$ is non-algebraic, as required.

Similarly, if $\Omega^i(M^*) \cong \Omega^{-i}(M)^*$ is a summand of $M^{\otimes n}$, then

$$\Omega^{nj+i}(M^*) | \Omega^j(M)^{\otimes n},$$

and since $\Omega^{nj+i}(M^*) \cong \Omega^{-(nj+i)}(M)^*$, at least one of $\Omega^j(M)$ and $\Omega^{nj+i}(M^*)$ is non-algebraic, and so $\Omega^j(M)$ is non-algebraic. \square

2.3 The Auslander–Reiten Quiver

It is possible to prove extensions of the results on non-periodic modules to the stable Auslander–Reiten quiver, but currently only in some cases. To state the theorem, we need the notion of complexity.

Definition 2.13 Let M be a KG -module, and suppose that

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is the minimal projective resolution for M . Then the *complexity* of M , written $\text{cx}(M)$, is the smallest integer c such that there exists a constant α with

$$\dim_K P_n \leq \alpha n^{c-1}$$

for all $n > 0$.

It is not obvious, but true, that such an integer always exists. Projective modules have complexity 0, periodic modules have complexity 1, and non-periodic modules have complexity at least 2. For the basic properties of complexity, we refer to [12, Proposition 2.2.24]. One important property that we will use is that the complexities of every module on a particular component of the (stable) Auslander–Reiten quiver are the same.

The theorem we will prove here is Theorem B from the introduction.

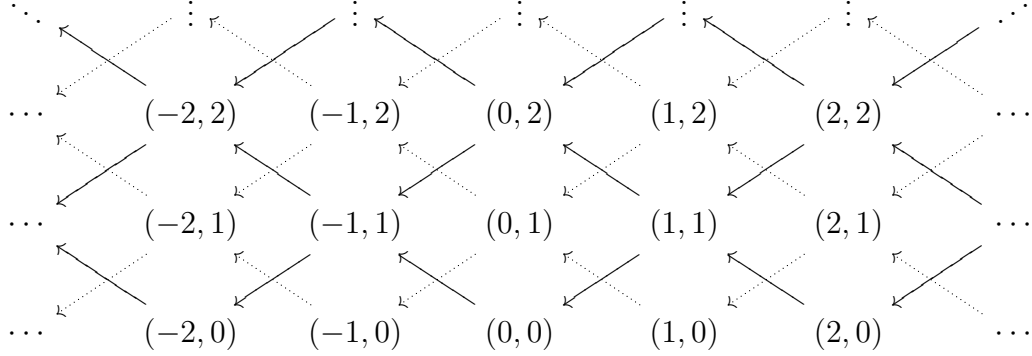
Theorem 2.14 Let G be a finite group and let K be a field of characteristic p . Let Γ be a connected component of the stable Auslander–Reiten quiver $\Gamma_s(KG)$, and suppose that modules on Γ are of complexity at least 3. Then Γ contains at most one algebraic module.

Firstly, we know that if Γ is a component of $\Gamma_s(KG)$, and the modules on Γ have complexity at least 3, then G has wild representation type, and so by a theorem of Karin Erdmann in [33], Γ has tree class A_∞ . This will be essential in what is to follow.

To prove this theorem, we first introduce the concept of an interlaced component of $\Gamma_s(KG)$. If Γ is a component and Γ consists either of non-periodic modules or of modules of even periodicity, then for each M in Γ , the module $\Omega(M)$ does not lie on Γ . An *interlaced component* is the union of the component Γ and the component consisting of the Heller translates of the modules on Γ . The reason for the name will become clear in the next paragraph.

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

We begin by co-ordinatizing an interlaced component of $\Gamma_s(KG)$, which will help immensely in this section. We co-ordinatize according to the following diagram.



[Note that this quiver consists of interlaced ‘diamonds’; when we refer to a diamond of an interlaced component, we mean such a collection of four vertices.]

For the rest of this section, Γ will denote an interlaced component of $\Gamma_s(KG)$. Write $M_{(i,j)}$ for the indecomposable module in the (i, j) position on Γ . (Of course, while j is determined, there is choice over which position on Γ is $(0, 0)$; we will assume that such a choice is made.)

We recall the following easy result.

Lemma 2.15 ([13, Proposition 4.12.10]) Let M be an indecomposable module with vertex Q , and suppose that H is a subgroup of G not containing any conjugate of Q . Then the Auslander–Reiten sequence terminating in M splits upon restriction to H .

Notice that, for our interlaced component Γ and modules $M_{(i,j)}$, this result becomes the statement that if H does not contain a vertex of $M_{(i,j)}$, then for $i > 0$,

$$M_{(i-1,j)} \downarrow_H \oplus M_{(i+1,j)} \downarrow_H \cong M_{(i,j+1)} \downarrow_H \oplus M_{(i,j-1)} \downarrow_H .$$

In particular, this implies that if the modules attached to three of the four vertices in a diamond of Γ have known restrictions to H , the fourth is uniquely determined.

We also need a slight extension to the result that the complexity of every module on the same component is the same.

Lemma 2.16 Let Γ be an interlaced component of the Auslander–Reiten quiver, and let H be a subgroup of G . Then for all M on Γ , the complexity of $M \downarrow_H$ is the same.

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

Proof: Let M be a module on Γ such that $M \downarrow_H$ has the smallest complexity, say n . Let

$$0 \rightarrow \Omega^2(M) \rightarrow N \rightarrow M \rightarrow 0$$

be the almost-split sequence terminating in M . Restricting this sequence to H yields a short exact sequence whose terms are KH -modules. For any short exact sequence, the largest two complexities of the terms are equal, and hence the complexity of $N \downarrow_H$ is equal to that of $M \downarrow_H$, by minimal choice of M . Thus if L is connected to any $\Omega^i(M)$, then $\text{cx}(L \downarrow_H) = n$. This holds for any module M such that $\text{cx}(M \downarrow_H) = n$, so the restrictions of all modules on the component of $\Gamma_s(KG)$ containing M have the same complexity. The result now follows from the fact that

$$\text{cx}(M \downarrow_H) = \text{cx}(\Omega(M) \downarrow_H).$$

□

This can be used to prove the next theorem, which is the key step in the proof of Theorem 2.14.

Theorem 2.17 Let G be a finite group and let Γ be an interlaced component of $\Gamma_s(KG)$. Suppose that P is a p -subgroup such that P does not contain a vertex of any module on Γ , and that for some M on Γ , the restriction of M to P is non-periodic. Then Γ contains at most one algebraic module and such a module lies at the end of Γ ; i.e., it is $M_{(i,0)}$ for some $i \in \mathbb{Z}$.

Proof: Since P does not contain a vertex of any module on Γ , any almost-split sequence involving terms on Γ splits upon restriction to P , and so we consider all modules $M_{(i,j)}$ to be restricted to P . For a co-ordinate (i, j) on Γ , we attach a collection $[a_1, \dots, a_n]$, which are the non-periodic summands of $M_{(i,j)} \downarrow_P$ in a decomposition of $M_{(i,j)} \downarrow_P$ into indecomposable summands. We call this collection the *signature* of the vertex (i, j) . Notice that, since $M_{(i,j)} = \Omega(M_{(i-1,j)})$, to know the signatures of all vertices in a row, it suffices to know the signature of one of them.

Note that by Lemma 2.16, all modules on Γ have non-periodic restriction to P , and so the signature of any co-ordinate is non-empty.

By the remarks after Lemma 2.15, if we know all signatures of the bottom two rows, we can uniquely determine all signatures of higher rows, since three of the four vertices on each diamond will have known signatures. Also, the second row can be determined from the first row, because of the fact that the signature of $(i, 1)$ is equal to the sum of the signatures of $(i - 1, 0)$ and $(i + 1, 0)$.

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

In order to easily express the signatures of the vertices, if x is an element of a signature, then denote by x^i the i th Heller translate of x . Let $[x_1, \dots, x_t]$ denote the signature of the vertex $(0, 0)$. Then the signature of $(i, 0)$ is $[x_1^i, x_2^i, \dots, x_t^i]$, and the signature of $(i, 1)$ is

$$[x_1^{i-1}, x_2^{i-1}, \dots, x_t^{i-1}, x_1^{i+1}, x_2^{i+1}, \dots, x_t^{i+1}],$$

since the almost-split sequence terminating in $M_{(i,0)}$ splits on restriction to P . Write X^i for the signature

$$[x_1^i, x_2^i, \dots, x_t^i],$$

and write X^A for the signature

$$\bigcup_{a \in A} X^a.$$

We claim that the signature of (i, j) is

$$X^{\{i+j, i+j-2, \dots, i-j+2, i-j\}}.$$

To prove this, we firstly note that for $j = 0$ and $j = 1$ this formula holds. Since we know that the signatures of all vertices are uniquely determined by the first two rows, we simply have to show that it obeys the rule that, for each diamond, the sum of the signatures of the top and bottom vertices equal the sum of the signatures of the left and right vertices. This is true, as the top and bottom vertices' signatures are

$$X^{\{i+(j+1), i+(j+1)-2, \dots, i-(j+1)+2, i-(j+1)\}} \cup X^{\{i+(j-1), i+(j-1)-2, \dots, i-(j-1)+2, i-(j-1)\}},$$

and the left and right vertices' signatures are

$$X^{\{(i+1)+j, (i+1)+j-2, \dots, (i+1)-j+2, (i+1)-j\}} \cup X^{\{(i-1)+j, (i-1)+j-2, \dots, (i-1)-j+2, (i-1)-j\}}.$$

These are easily seen to be the same, and so the above formula gives the signature of the vertex (i, j) .

If $j \neq 0$, then the signature contains $X^{\{i+j, i+j-2\}}$ and since it cannot be that both x_1^{i+j} and x_1^{i+j-2} are algebraic, $M_{i,j}$ is non-algebraic for all $j > 0$.

Finally, at most one of the modules $M_{(i,0)}$ can be algebraic, and so the theorem is proved. \square

Now we are in a position to quickly prove Theorem 2.14; let D denote the vertex of some module M on Γ . Since $\text{cx}(M) = n \geq 3$ and D is a vertex for M , we see that $\text{cx}(M \downarrow_D) = n$. Thus there is a subgroup P of D such that $\text{cx}(M \downarrow_P) = n - 1$. If N

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

is some other module on Γ , we see by Lemma 2.16 that $N \downarrow_P$ has complexity $(n - 1)$ as well. Hence P cannot contain a vertex for N .

We have therefore produced a subgroup P that does not contain a vertex of any module on Γ . Furthermore, since $\text{cx}(M \downarrow_P) = n - 1 \geq 2$, the module $M \downarrow_P$ is non-periodic. Having satisfied the conditions of Theorem 2.17, we get the result.

Theorem 2.17 can be used to produce similar results to Theorem 2.14, but with various conditions. One example is the following.

Theorem 2.18 Let G be a group of wild representation type whose Sylow p -subgroups are not isomorphic with $C_p \times C_p$. Let Γ be a component of $\Gamma_s(KG)$ that contains p' -dimensional modules. Then at most one module on Γ is algebraic, and it lies at the end of Γ .

To prove this, recall that a p' -dimensional module has a Sylow p -subgroup P as a vertex. If P has p -rank at least 3, then the result is true by Theorem 2.14, so G has p -rank 2. Let M denote a module on Γ . By the Alperin–Evens theorem (Theorem 1.17) there is a subgroup Q of P isomorphic with $C_p \times C_p$, such that the complexity of $M \downarrow_Q$ is 2, and so Q is a subgroup that satisfies the conditions of Theorem 2.17.

In general it appears difficult to prove a corresponding theorem to Theorem 2.14 for arbitrary A_∞ -components of complexity 2. For tame blocks the situation becomes slightly more complicated. For dihedral 2-groups, the same result—that there is at most one algebraic module on a non-periodic component—is true, and this is Corollary 3.20. For semidihedral 2-groups, it seems that it is not difficult to prove that there are at most 2 algebraic modules on each non-periodic component, but it is not clear whether this can be sharpened, or if this is really a difference to the general case.

We end with a small proposition needed for Chapter 6, although the method can be used to prove that if $G = C_p \times C_p$, and Γ is a component of $\Gamma_s(KG)$ containing modules of p' -dimension, then there are restrictions on the positions of algebraic modules.

Proposition 2.19 Suppose that p is an odd prime, and let K be a field of characteristic p . Let E denote the heart (radical modulo socle) of the projective indecomposable module for $C_p \times C_p$ over K . Then E is non-algebraic.

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

The module E lies directly above K on the interlaced component of $\Gamma_s(KG)$, and so if G is a p -group (where p is odd) that is not $C_p \times C_p$ of cyclic, then E is non-algebraic by Theorem 2.18. Thus the case of $C_p \times C_p$ is the last remaining case.

To prove Proposition 2.19, we consider tensoring short exact sequences by modules. Note that the almost-split sequence terminating in $\Omega^{-1}(K)$ is given by

$$0 \rightarrow \Omega(K) \rightarrow E \rightarrow \Omega^{-1}(K) \rightarrow 0.$$

Tensoring this sequence by a module M gives (by [11, Theorem 3.6]) the almost-split sequence terminating in $\Omega^{-1}(M)$ if $p \nmid \dim M$ and a split sequence otherwise.

Now consider the co-ordinatization of the component Γ containing K , as suggested at the start of this section. Then $K = M_{(0,0)}$ and $E = M_{(0,1)}$, and from the paragraph above, we see that

$$M_{(0,i)} \otimes E = M_{(0,i-1)} \oplus M_{(0,i+1)} \text{ (modulo projectives)}$$

if $0 < i < p - 1$. (When $i = p - 1$, the dimension of $M_{(0,i)}$ is divisible by p .) Thus $M_{(0,i)}$ lies inside $T(E)$ for $0 \leq i \leq p - 1$. However, since $p \mid \dim M_{(0,p-1)}$, we see that

$$E \otimes M_{(0,p-1)} = \Omega(M_{(0,p-1)}) \oplus \Omega^{-1}(M_{(0,p-1)})$$

modulo projectives. Thus, since $M_{(0,p-1)}$ is non-periodic, E is non-algebraic, as claimed.

2.4 The Normal Subgroup Structure

This section examines the rôle that normal subgroups have to play in determining the algebraicity of simple modules. Our results cluster around examining normal p - and p' -subgroups at the top and bottom of a finite group. This will involve both subgroups and central extensions: as such, we make the following definitions.

Definition 2.20 Suppose that G is a finite group. Then G is said to have *hereditary p -SMA* if all subgroups of G have p -SMA. G is said to have *projective p -SMA* if all simple projective representations of G are algebraic, or equivalently all p' -central extensions \hat{G} of G have p -SMA. Finally, a group has *hereditary projective p -SMA* if all subgroups of G have projective p -SMA.

These definitions are mostly independent, and some quite surprising combinations can occur, as we demonstrate by examples.

- Example 2.21** (i) Since all simple modules for p -soluble groups are algebraic, all p -soluble groups have hereditary projective p -SMA.
- (ii) The simple group A_5 , and more generally the groups $\mathrm{PSL}_2(q)$ where $q \equiv 3, 5 \pmod{8}$, have hereditary projective p -SMA for all primes p .
- (iii) The group A_6 has hereditary p -SMA for all primes p , and all proper subgroups of A_6 have hereditary projective p -SMA for all primes p , but A_6 does not have projective 2-SMA as there are two 3-dimensional non-algebraic simple modules for $3.A_6$ in characteristic 2. (A_6 has hereditary projective 3-SMA.)
- (iv) The simple group $\mathrm{PSL}_2(7)$ has two 3-dimensional non-algebraic simple modules in characteristic 2 and so does not have 2-SMA. All proper subgroups of this group have hereditary projective p -SMA for all primes p as $\mathrm{PSL}_2(7)$ is a minimal simple group.
- (v) The group A_7 has projective 2-SMA, and has hereditary 2-SMA, but since A_6 does not have projective 2-SMA, the group A_7 does not have hereditary projective 2-SMA.

A necessary prerequisite for the theory we will be developing is the classical Clifford theory of modules, as originally described in [25]. Let G be a finite group and let H be a normal subgroup of G . Suppose that M is a simple KG -module, and let N be a summand of $M \downarrow_H$. Suppose firstly that G is the inertia subgroup of N (i.e., N is G -stable). Then there are (necessarily simple) projective representations V and W of G with $M = W \otimes V$, such that $V \downarrow_H \cong N$, and W is a projective representation of G/H . Furthermore, V and W are actual representations (rather than only projective representations) if and only if N can be extended to a KG -module.

In fact, there is a p' -central extension \hat{G} of G , with central subgroup Z , such that $\hat{H} = H \times Z$, and, viewed as a $K\hat{G}$ -module, $M = V \otimes W$, where V is a $K\hat{G}$ -module with $V \downarrow_{\hat{H}} = N$ and W is a $K(\hat{G}/H)$ -module. What this means is that if we allow ourselves to pass to a p' -central extension, we may assume that N possesses an extension to G , and we therefore can deal solely with representations and not with projective representations.

If the module N above is not G -stable, then let L denote its inertia subgroup. The module M is induced from a KL -module M' , and this theory then passes to the module M' .

We start with an obvious result, allowing us to move between a finite group and its quotient by a p -subgroup, at least in the case of simple modules.

Lemma 2.22 Let G be a finite group, and let K be a field of characteristic p .

- (i) G has p -SMA if and only if $G/O_p(G)$ has p -SMA.
- (ii) G has hereditary p -SMA if and only if $G/O_p(G)$ does.
- (iii) G has projective p -SMA if and only if $G/O_p(G)$ does.
- (iv) G has hereditary projective p -SMA if and only if $G/O_p(G)$ does.

Proof: Since $O_p(G)$ acts trivially on every simple module, the tensor products of simple KG -modules are identical to those of $G/O_p(G)$. Hence G has p -SMA if and only if $G/O_p(G)$ does, proving (i). The proof of (ii) is exactly similar; if G has hereditary p -SMA then certainly $G/O_p(G)$ does, and conversely if H is a subgroup of G then $HO_p(G)/O_p(G)$, which is isomorphic with $H/H \cap O_p(G)$, has p -SMA; so H has p -SMA by (i).

Now suppose that $G/O_p(G)$ has projective p -SMA, and let \hat{G} denote a p' -central extension of G and M be a simple $K\hat{G}$ -module. Notice that $O_p(G) \cong O_p(\hat{G})$, and that $\hat{G}/O_p(\hat{G})$ is a p' -central extension of $G/O_p(G)$. Since M is a simple $K\hat{G}$ -module, it is also a simple $K(\hat{G}/O_p(\hat{G}))$ -module, and is algebraic since $G/O_p(G)$ has projective p -SMA. Conversely, any projective representation of $G/O_p(G)$ is a projective representation of G , and so this direction is obvious, proving (iii).

Finally, suppose that $G/O_p(G)$ has hereditary projective p -SMA, and let H be a subgroup of G ; let M be a simple projective KH -representation. Then M is a simple projective $K(H/L)$ -representation, where $L = O_p(G) \cap H$, and so M is isomorphic with N , a particular simple projective $K(HO_p(G)/O_p(G))$ -representation, which is algebraic since $G/O_p(G)$ has hereditary projective p -SMA. Conversely, if G has hereditary projective p -SMA, then again $G/O_p(G)$ does so, proving the final part of the lemma. \square

As an extension to (i), if M is a KG -module, then if H is the kernel of M , the module M is also a $K(G/H)$ -module, and M is algebraic as a KG -module if and only if it is algebraic as a $K(G/H)$ -module.

We can also deal with quotients by normal p' -subgroups using Clifford theory.

Lemma 2.23 Suppose that G is a finite group, and let p be a prime. Then G has projective hereditary p -SMA if and only if $G/O_{p'}(G)$ has projective hereditary p -SMA.

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

Proof: Let K be a field of characteristic p . Suppose firstly that M is a simple KG -module, and write H for the subgroup $O_{p'}(G)$. Let N be a summand of $M \downarrow_H$, and let L be the inertia subgroup of N . Then there are projective KL -representations V and W such that $V \downarrow_H = N$ and W is a simple $K(L/H)$ -module. Since G/H has hereditary projective p -SMA, the module W is algebraic.

The module V is also algebraic; let \hat{L} be the associated p' -central extension of L . The group \hat{L} is the extension of a p' -group by a p -group, and so is p -soluble. Hence V is algebraic, since all simple modules for p -soluble groups are algebraic (by Theorem 1.12). Since both V and W are algebraic, so is M .

Hence for any group X such that $X/O_{p'}(X) \cong G/O_{p'}(G)$, all simple KX -modules are algebraic. Let \hat{G} be a p' -central extension of G ; then $G/O_{p'}(G)$ and $\hat{G}/O_{p'}(\hat{G})$ are isomorphic. Hence all simple $K\hat{G}$ -modules are algebraic, and so G has projective p -SMA.

Finally, let Y be a subgroup of G ; then $YO_{p'}(G)/O_{p'}(G)$ has hereditary projective p -SMA, and this group is isomorphic with $Y/O_{p'}(G) \cap Y$. By the above result, Y has projective p -SMA, and so every subgroup of G has projective p -SMA, as required. \square

There is a converse, in the sense that, with the setup above, where $M = V \otimes W$ and W is a $K(G/H)$ -module, then M is algebraic if and only if V and W are algebraic. The above proof dealt with the ‘if’ case; to see the rest, if M is algebraic then $M \downarrow_H$ is algebraic, which is a sum of copies of $V \downarrow_H$. Thus $V \downarrow_H$ is algebraic, and so $V \downarrow_{HP}$ is algebraic. We proved before that this means that V is algebraic. Since $V \downarrow_H$ is simple, $\dim V$ and p must be coprime, whence K is a summand of $V \otimes V^*$, by Theorem 1.20. The rest is clear: as V^* and $V \otimes W$ are algebraic, so is their tensor product, of which W is a summand.

Putting the last two results together, we prove the following theorem.

Theorem 2.24 Let G be a finite group and let H be a p -soluble normal subgroup of G . Then G has hereditary projective p -SMA if and only if G/H does.

Proof: A simple induction, by repeated, alternating application of Lemmas 2.22 and 2.23. \square

This covers everything currently known regarding results for G got from results about quotients by ‘small’ subgroups. The next result allows us to move between a group and a subgroup of p' -index; in fact, we can do this for arbitrary modules.

Proposition 2.25 Let G be a finite group and let H be a subgroup with $|G : H|$ prime to p . If M is a KG -module, then M is algebraic if and only if $M \downarrow_H$ is algebraic.

Proof: It clearly suffices to prove this for indecomposable modules, so let M be an indecomposable module. Since M is relatively H -projective, M is a summand of $(M \downarrow_H) \uparrow^G$. Hence, if $M \downarrow_H$ is algebraic, then so is $(M \downarrow_H) \uparrow^G$, and subsequently the summand M of this module. In general, if M is algebraic then $M \downarrow_H$ is, and the result is proved. \square

In particular, if M is a simple module and H is a normal subgroup of index prime to p , then M is algebraic if and only if the (semisimple) module $M \downarrow_H$ is algebraic.

Next, we consider the other case: suppose that G is a finite group, and H is a normal subgroup of index p . If K is a field of characteristic p , we would like to know whether the analogous result to Proposition 2.25 holds; that is, if M is a simple KG -module, then M is algebraic if and only if $M \downarrow_H$ is. We begin with two lemmas, which analyze the two cases where $M \downarrow_H$ is simple and when it is not. This depends on the inertia subgroup of one of the summands of $M \downarrow_H$.

Lemma 2.26 ([36, III.2.11]) Let G be a finite group and H be a normal subgroup. Let M be a simple KH -module, and suppose that the inertia subgroup of M is equal to H . Then $M \uparrow^G$ is simple.

Lemma 2.27 ([36, III.2.14]) Let G be a finite group and suppose that H is a normal subgroup of G such that G/H is cyclic. Let M be a simple KH -module whose inertia subgroup is equal to G . Then

- (i) there is a (simple) KG -module V such that $V \downarrow_H = M$, and
- (ii) if W is a simple KG -module such that M is a summand of $W \downarrow_H$, then $W \downarrow_H = M$ and $W = V \otimes X$ for some simple $K(G/H)$ -module X .

Thus if $M \downarrow_H$ is not simple, then M is induced from a simple module for the normal subgroup, and so M is algebraic if and only if $M \downarrow_H$ is. However, if $M \downarrow_H$ is not simple, then $M \downarrow_H$ can be algebraic even if M is non-algebraic. An example of this is the 7-dimensional simple module for $\mathrm{SL}_2(8) \rtimes C_3$, which is proved to be non-algebraic in Section 5.7. Since $\mathrm{SL}_2(8)$ has cyclic Sylow 3-subgroups, this group has 3-SMA. (The 7-dimensional simple module for $\mathrm{SL}_2(8) \rtimes C_3$ restricts to the 7-dimensional simple module for $\mathrm{SL}_2(8)$.)

Our ultimate goal is to determine to what extent the fact that a finite group has p -SMA is determined by its composition factors.

Proposition 2.28 Let G be a finite group, and let H/N be a subnormal section; i.e., N is normal in H , and H is a subnormal subgroup of G . If G has p -SMA then so does H/N .

Proof: Certainly if G has p -SMA then so does G/N , so the proposition reduces to showing that if G has p -SMA then so does any subnormal subgroup, and by considering a normal series starting in G and terminating in H , we reduce to the case where H is normal in G . Hence suppose that $H \triangleleft G$.

By Clifford's theorem, if M is a simple KG -module then $M \downarrow_H$ is semisimple, and is algebraic since G has p -SMA. Hence if V is a simple KH -module, then V is algebraic if V is a summand of $M \downarrow_H$ for some semisimple KG -module M . This is always the case: to see this, let ϕ be the Brauer character afforded by V , and write

$$\phi \uparrow^G = \sum a_i \psi_i,$$

where the ψ_i are Brauer characters for KG . Let M denote the semisimple module whose Brauer character is $\sum a_i \psi_i$; by Mackey's theorem, the Brauer character of $M \downarrow_H$ has ϕ as a constituent, and $M \downarrow_H$ is semisimple, and so $V | M \downarrow_H$, as required. \square

This proposition obviously yields the following result.

Corollary 2.29 Let G be a finite group with p -SMA. Then every composition factor of G has p -SMA.

This corollary leads naturally to the question of which simple groups have p -SMA. Most of the rest of this thesis will be concerned with this question in various situations. We include the last result for completeness.

Proposition 2.30 Let G be the direct product of two groups with p -SMA. Then G has p -SMA.

Proof: Write $G = H_1 \times H_2$; then this result follows easily from the fact that any KG -module is a tensor product of a KH_1 -module by a KH_2 -module. \square

We now turn to the proof of Theorem F. We firstly notice that if G is a simple group with abelian Sylow 2-subgroups, then the Schur multiplier of G is either 1 or 2. (See Tables 6.1.2 and 6.1.3 from [39].) If K is a field of characteristic 2 and H is a direct product of simple groups with abelian Sylow 2-subgroup, then every simple projective KH -representation is actually a standard KH -representation.

CHAPTER 2. GENERAL THEORY OF ALGEBRAIC MODULES

Next, suppose that G is the direct product of H above and an abelian 2-group L , and let \hat{G} be an odd central extension of G . Then it is easy to see that $\hat{G} = H \times \hat{L}$, where \hat{L} is an odd central extension of L . In particular, if every simple group with abelian Sylow 2-subgroup has 2-SMA, then by Theorem 1.26, every finite group G such that

- (i) G has abelian Sylow 2-subgroups,
- (ii) $O_{2'}(G) = 1$, and
- (iii) $O^{2'}(G) = G$,

has projective 2-SMA.

Theorem 2.31 Suppose that all simple groups with abelian Sylow 2-subgroups have 2-SMA. Then all finite groups with abelian Sylow 2-subgroups have 2-SMA.

Proof: By Proposition 2.25, if the theorem is true for all such finite groups with $G = O^{2'}(G)$ then the theorem is true for all finite groups.

Write $H = O_{2'}(G)$. We proceed by induction on $|G/H|$. Let M be a simple projective KG -representation. By replacing G with an odd central extension of G (and noting that $G/O_{2'}(G)$ does not change) we may assume that M is a simple KG -module. Let N be a simple summand of $M \downarrow_H$. Let L denote the inertia subgroup of N . If $L = G$, then write V for the projective KG -representation such that $V \downarrow_H = N$. If P is a Sylow 2-subgroup of G , then $V \downarrow_{HP}$ is a simple projective $K(HP)$ -representation, which is therefore algebraic, since HP is soluble. Thus V is algebraic. Since G/H has projective 2-SMA, this means that M is algebraic, as required.

Now suppose that $L < G$. Then there is a simple KL -module M' such that $M' \uparrow^G = M$. Now L is a group with abelian Sylow 2-subgroups, and since $H \leq L < G$, we must have $|L/O_{2'}(L)| < |G/O_{2'}(G)|$, and so L has projective 2-SMA. Thus M' is algebraic, and so M is, as required. \square

Thus Theorem F will be proved if we can prove that all simple groups with abelian Sylow 2-subgroups have 2-SMA. Theorem 1.26 listed the simple groups with abelian Sylow 2-subgroups. The groups $SL_2(2^n)$ have 2-SMA, by Alperin's Theorem 1.13. The groups $PSL_2(q)$ where $q \not\equiv \pm 1 \pmod{8}$ have 2-SMA by Theorem D, (see also Chapter 4) and the Ree groups ${}^2G_2(q)$ have 2-SMA by Theorem 5.19. The last group J_1 , has 2-SMA by Theorem 6.17. Therefore, modulo the proofs of those results, Theorem F is proved.

Chapter 3

Dihedral and Elementary Abelian Groups

This chapter has two aims: to study the indecomposable representations of the dihedral 2-groups, with a view to getting more understanding on which of these are algebraic; and to develop an ‘encyclopædia’ of small-dimensional modules for $G = C_3 \times C_3$ over $K = \text{GF}(3)$. In the direction of the second statement, we organize all modules of dimension at most 6 into their conjugacy classes (under the action of $\text{Aut } G$), and then analyze whether or not they are algebraic. Although our results are incomplete, of the 324 indecomposable KG -modules of dimension at most 6, only eight have unknown algebraicity. From this information we generate a conjecture regarding the module category of $C_p \times C_p$, and consider generalizations of it to some other p -groups.

For indecomposable modules for dihedral groups, we prove in Corollary 3.20 the analogue of Theorem 2.14 given in the previous chapter. This new focus on algebraic modules also gives an equivalent formulation of a conjecture of Karin Erdmann on the Green correspondence within V_4 blocks. It also informs us where to concentrate our efforts in order to find ‘nice’ tensor product structures in the indecomposable modules for these groups.

3.1 The Group V_4

Apart from the cyclic groups, the non-cyclic group of order 4 is the only group whose tensor products have been completely analyzed. In this section we list the indecomposable modules and describe their tensor product structure. The decomposition of the product of any two indecomposable KV_4 -modules was determined by Conlon in [26], and we reproduce his table here. In the following section we provide the construction of these modules as a special case of the construction of all indecomposable

modules for dihedral 2-groups. The description of the tensor products makes clear the following result.

Theorem 3.1 Let M be an indecomposable KV_4 -module, where K is any field of characteristic 2. Then M is algebraic if and only if M is even-dimensional or trivial.

Theorem 3.1 allows us to wield a very powerful tool in analyzing modules in characteristic 2.

Corollary 3.2 (V_4 -Restriction Test) Let K be a field of characteristic 2, and let G be a finite group. Write \mathcal{P} for the collection of all subgroups of G isomorphic with V_4 . Finally, let M be a KG -module. If M is algebraic then $M \downarrow_P$ is a sum of even-dimensional and trivial modules for all $P \in \mathcal{P}$.

The following table describes the tensor products of any two non-projective indecomposable modules. (Of course, the product of D , the projective indecomposable module, with any other indecomposable module is projective, and so this row and column have been removed.) Here, A_n and B_n are the odd-dimensional indecomposable modules $\Omega^i(K)$ for $i \in \mathbb{Z}$, and $C_n(\pi)$ and $C_n(\infty)$ are even-dimensional modules; see the next section for their construction.

$n \leq n'$	A_n	B_n	$C_n(\pi)$	$C_n(\infty)$
$A_{n'}$	$nn'D \oplus A_{n+n'}$	$n(n'+1)D \oplus A_{n'-n}$	$nn'mD \oplus C_n(\pi)$	$nn'D \oplus C_n(\infty)$
$B_{n'}$	$n(n'+1)D \oplus B_{n'-n}$	$nn'D \oplus B_{n+n'}$	$nn'mD \oplus C_n(\pi)$	$nn'D \oplus C_n(\infty)$
$C_{n'}(\pi')$	$nn'mD \oplus C_{n'}(\pi')$	$nn'mD \oplus C_{n'}(\pi')$	X	$nn'mD$
$C_{n'}(\infty)$	$nn'D \oplus C_{n'}(\infty)$	$nn'D \oplus C_{n'}(\infty)$	$nn'mD$	$n(n'-1)D \oplus 2C_n(\infty)$

The remaining entry X is $nmn'm'D$ if $\pi \neq \pi'$, and $nm(n'm-1)D \oplus 2C_n(\pi)$ if $\pi = \pi'$.

We therefore see that, modulo projective modules,

$$\begin{aligned}
 A_n^{\otimes i} &= A_{in}, \\
 B_n^{\otimes i} &= B_{in}, \\
 C_n(\pi)^{\otimes 2} &= 2 \cdot C_n(\pi), \text{ and} \\
 C_n(\infty)^{\otimes 2} &= 2 \cdot C_n(\infty).
 \end{aligned}$$

This clearly demonstrates that A_n and B_n are not algebraic, and $C_n(\pi)$ and $C_n(\infty)$ are, proving Theorem 3.1.

We also notice the following result.

Proposition 3.3 The KV_4 -modules all of whose summands are even-dimensional form an ideal of algebraic modules. Consequently, if G is a finite group, K is a field of characteristic 2, and H is subgroup isomorphic with V_4 , then the subgroup I of $a(KG)$ generated by the indecomposable KG -modules with cyclic vertex, or V_4 vertex and even-dimensional source, forms an ideal consisting of algebraic modules.

Proof: That the KV_4 -modules all of whose summands are even-dimensional are algebraic and are closed under addition is obvious, and that the additive subgroup they generate forms an ideal is a consequence of Theorem 1.20. Hence the first part of the result holds.

Next, let S denote the indecomposable modules that either have cyclic vertex, or have V_4 vertex and even-dimensional source, and let N be any indecomposable module. Suppose that M is an indecomposable module from S , and write L for a source of M . If the vertex of M is cyclic then so are the vertices of all summands of $M \otimes N$, and so $M \otimes N$ lies inside I . If the vertex of M is V_4 , then all summands of $M \otimes N$ have either cyclic or V_4 vertex, and by Lemma 1.23,

$$N \otimes M | N \otimes L \uparrow^G = (N \downarrow_H \otimes L) \uparrow^G .$$

Since L is even-dimensional, so is every summand of $N \downarrow_H \otimes L$, and so all summands of $M \otimes N$ with vertex V_4 have even-dimensional source, as required.

Since the product of an indecomposable module from I and any indecomposable module lies in I , by linearity of tensor product, we are done. \square

Having seen modules with V_4 vertex, it seems natural to consider blocks of defect V_4 ; since V_4 is abelian, all simple modules in this block have vertex equal to the defect group. In fact, we slightly expand our collection of simple modules to include all simple modules of vertex V_4 .

Conjecture 3.4 (V_4 conjecture) Let G be a finite group and let K be a field of characteristic 2. Let M be a simple KG -module with vertex isomorphic with V_4 . Then M is algebraic.

This conjecture can be restated in the following way, via the classification of indecomposable algebraic modules for V_4 .

Conjecture 3.5 (V_4 conjecture) Let G be a finite group and let K be a field of characteristic 2. Let M be a simple KG -module with vertex V_4 . If M is non-periodic then M has trivial source.

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

Suppose that B is a block with defect group V_4 . The Green correspondence for the simple B -modules was calculated up to a parameter by Karin Erdmann in [32], and we will describe the theory now.

Let G be a finite group and let K be a splitting field of characteristic 2. Suppose that G has a block B of defect group $D \cong V_4$, and write b for its Brauer correspondent in $N_G(D)$. (Note that the simple b -modules are algebraic since they are projective $N_G(D)/D$ -modules.) There are three possibilities for the simple modules and the Green correspondence; in describing them, we write $f : KG\text{-}\underline{\text{ind}} \rightarrow K N_G(D)\text{-}\underline{\text{ind}}$ for the Green correspondence.

- (i) Both B and b possess one simple module; write M for the simple B -module and S for the simple b -module. Then there exists an integer $i \in \mathbb{Z}$ such that

$$f(M) = \Omega^i(S).$$

- (ii) Both B and b possess three simple modules; write M_0, M_1 and M_2 for the simple B -modules and S_0, S_1 and S_2 for the simple b -modules. Then there exists an integer $i \in \mathbb{Z}$ such that

$$f(M_j) = \Omega^i(S_j)$$

for each $j = 0, 1, 2$.

- (iii) Both B and b possess three simple modules; write M_0, M_1 and M_2 for the simple B -modules and S_0, S_1 and S_2 for the simple b -modules. Then there exists an integer $i \in \mathbb{Z}$, and there exist uniserial b -modules T_1 and T_2 of length 2, with composition factors S_1 and S_2 and $\text{soc}(T_j) = S_j$, such that

$$f(M_0) = \Omega^i(S_0), \quad f(M_1) = \Omega^i(T_1), \quad f(M_2) = \Omega^i(T_2),$$

and the modules M_1 and M_2 are periodic of period 3.

In [32], Erdmann essentially conjectured that in all cases, the integer i is equal to 0. We will call this conjecture ‘Erdmann’s conjecture’ in the next proposition.

Proposition 3.6 Erdmann’s conjecture holds if and only if the V_4 conjecture holds for simple modules lying in blocks of defect group V_4 .

Proof: Recall that the Green correspondence preserves algebraicity of modules, and also whether the module is periodic. Write B for a block of KG with defect group $D \cong V_4$, and write b for its Brauer correspondent in $N_G(D)$. Let M be a non-periodic

simple B -module; in all three cases above, its Green correspondent in b is $\Omega^i(S)$, where S is some simple b -module. Since S is algebraic and non-periodic, it must have trivial source, and so M has source $\Omega^i(K)$. Hence M is algebraic if and only if M and S are Green correspondents.

Hence Erdmann's conjecture is true if and only if all non-periodic simple modules in V_4 blocks are algebraic. Since all periodic modules in V_4 blocks are automatically algebraic, either conjecture implies the other. \square

Erdmann's conjecture, and hence the V_4 conjecture for blocks, is a strengthened form of the Puig conjecture for V_4 blocks. For completeness, we state this conjecture now.

Conjecture 3.7 (Puig conjecture) Let D be a p -group. Then there are only finitely many different isomorphism types of source algebra of blocks with defect group D .

In [58], Linckelmann proved, based upon Erdmann's work in [32], that when $D \cong V_4$ in the Puig conjecture, this conjecture is equivalent to the boundedness of the integer i in the description of the Green correspondence for V_4 blocks. Hence if the V_4 conjecture is true then a strong form of the Puig conjecture holds, at least for these blocks.

Erdmann's conjecture is easily seen to be true in the case where B is a real block with defect group V_4 .

Proposition 3.8 Let B be a block of a group G , and suppose that B has defect group V_4 . Suppose that B is a real 2-block; i.e., suppose that for a simple module M in B , the dual module M^* is also in B . Then all simple B -modules are algebraic.

Proof: Since B contains either a single non-periodic simple module or three non-periodic simple modules, there must be a self-dual, non-periodic simple module M lying in B . This must therefore have trivial source, and so Erdmann's conjecture holds for this block. Hence all simple B -modules are algebraic, as required. \square

Obviously, if B is the unique block with defect group V_4 , then B is real, and so all simple B -modules are algebraic.

Corollary 3.9 Suppose that a finite group G possesses a unique 2-block B of defect group V_4 . Then all simple B -modules are algebraic.

We should mention the case where a simple module with vertex V_4 lies in a block that is not of defect group V_4 .

Proposition 3.10 Let G be a finite group, and let K be a field of characteristic 2. Let M be a simple KG -module of vertex V_4 , and write B for the block containing M . Then a defect group of B is dihedral or semidihedral.

Proof: We will firstly prove that if a 2-group P contains a self-centralizing subgroup of order 4, then P is of maximal class. To see this, we proceed by induction on the class of P . If P has class 1 or 2, then this is obvious, so our induction starts. Suppose that P has class n , and that P contains a self-centralizing subgroup X of order 4. Then $|N_P(X) : X| = 2$, as this embeds in $\text{Aut}(X)$, and since $N_P(X)$ contains a self-centralizing subgroup of order 4, $N = N_P(X) \cong D_8$. Since X is self-centralizing, $Z(P) = Z$ must have order 2. We claim that

$$C_{P/Z}(N/Z) = N/Z.$$

If this is true, then P/Z possesses a self-centralizing subgroup of order 4, and so is of maximal class; this will prove the result.

Let x be a non-central element of X , and suppose that yZ centralizes N/Z : then yZ and xZ commute, so $y^{-1}x^{-1}yx \in Z$. Thus $y^{-1}xy \in X$ as $Z \leq X$, so that y normalizes x . Hence N/Z is self-centralizing, as it is abelian.

The 2-groups of maximal class are the generalized quaternion groups, the dihedral groups, and the semidihedral groups. Since the generalized quaternion groups are of 2-rank 1, they cannot contain a subgroup isomorphic with V_4 , and so generalized quaternion groups cannot occur. This completes the proof. \square

In a series of papers, Karin Erdmann essentially classified the structural properties of tame 2-blocks. This classification might be useful in tackling this problem, although as of yet the author has not approached this.

3.2 The Group D_{4q}

We begin this section by constructing all indecomposable KG -modules, where G is a dihedral 2-group and K is an arbitrary field of characteristic 2. (As a special case, this will include the indecomposable representations of V_4 .) The classification of indecomposable KG -modules was originally found by Ringel [71], using methods of Gelfand and Ponomarev in [38], and independently by Bondarenko in [18].

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

There are two classes of non-projective indecomposable KG -module; the string modules and band modules. The string modules are easier to describe, and we do this first. Write $G = D_{4q}$, where q is a power of 2.

Introduce symbols a and b , and let \mathscr{W} denote the set of all finite strings of symbols a, b, a^{-1} and b^{-1} , which we will call *words*, with the proviso that a symbol of the form $a^{\pm 1}$ is followed by one of the form $b^{\pm 1}$, and vice versa. Hence there are 2^n strings of length n that start with $a^{\pm 1}$. If w is a word in \mathscr{W} , then the inverse of w will be given by

$$(w_1 w_2 \dots w_n)^{-1} = w_n^{-1} w_{n-1}^{-1} \dots w_1^{-1},$$

so that for example if $w = aba^{-1}b^{-1}a$, then $w^{-1} = a^{-1}bab^{-1}a^{-1}$. If w is a word with n symbols, then let $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$ be two $(n+1)$ -dimensional matrices given by the following procedure:

- (i) set $\alpha_{ii} = \beta_{ii} = 1$, and $\alpha_{ij} = \beta_{ij} = 0$ for $j \neq i$;
- (ii) running through the symbols of w , if the i th symbol in w is an a , then set $\alpha_{i+1,i} = 1$, and if it is a^{-1} , then set $\alpha_{i,i+1} = 1$; and
- (iii) running through the symbols of w , if the i th symbol in w is an b , then set $\beta_{i+1,i} = 1$, and if it is b^{-1} , then set $\beta_{i,i+1} = 1$.

This procedure is best seen by example: if $w = ab^{-1}aba^{-1}$, then the two matrices α and β for $M(w)$ acting on the space V with basis $\{v_1, \dots, v_6\}$ are given by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This can be represented by a diagram. Write $x' = x - 1$ and $y' = y - 1$. Then if $w = ab^{-1}aba^{-1}$, as before, the elements x' and y' act as the diagram below, with all other actions on the v_i being 0.

$$v_1 \xleftarrow{x'} v_2 \xrightarrow{y'} v_3 \xleftarrow{x'} v_4 \xleftarrow{y'} v_5 \xrightarrow{x'} v_6$$

If $G = \langle x, y : x^2 = y^2 = (xy)^{2q} = 1 \rangle$, then let $M(w)$ denote the function $G \rightarrow \text{GL}_n(2)$ defined by $x \mapsto \alpha$ and $y \mapsto \beta$. This will be a representation of the dihedral group D_{4q} whenever no instance of $(ab)^q$, $(ba)^q$, $(a^{-1}b^{-1})^q$, or $(b^{-1}a^{-1})^q$ occurs. For the subset of \mathscr{W} so defined, we use the symbol \mathscr{W}_q .

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

There are two important points to be made about the representations $M(w)$: firstly, they are always indecomposable representations; and secondly, $M(w)$ and $M(w')$ are isomorphic if and only if $w' = w$ or $w' = w^{-1}$. This latter point is important, and we will often blur the distinction between the words w and w^{-1} . The modules $M(w)$ are called *string modules*. An important fact is that any odd-dimensional indecomposable module is a string module for some string of even length.

For the group V_4 , the modules A_n and B_n , and $C_n(0)$ and $C_n(\infty)$ are the string modules, of odd and even dimension respectively.

The remaining modules are the band modules: let \mathscr{W}'_q denote the subset of words, all of whose powers lie in \mathscr{W}_q , but that are not non-trivial powers of smaller words, so that for example $ab^{-1}ab^{-1}$ is not in \mathscr{W}'_q , but also ab is not in \mathscr{W}'_q because a large power of this word does not lie in \mathscr{W}_q . A consequence of this is that all words in \mathscr{W}'_q are of even length. If w is a word in \mathscr{W}'_q , we will not make the distinction between w and w^{-1} , and between w and the word got from w by moving the first letter to the end of w , so that $abab^{-1}$ is equivalent to $b^{-1}aba$. More formally, we may take equivalence classes of words in \mathscr{W}'_q under this equivalence relation.

Let w be a word of even length n , and let V denote an m -dimensional vector space, equipped with an indecomposable linear transformation ϕ . By cycling the letters of w and by inverting, we may assume that w begins with either a or b . We intend to construct matrices similar to those for string modules.

Let α' and β' denote square matrices of size n , initially equal to the zero matrix. We will associate a pair of numbers to this: if it is direct, associate $(i + 1, i)$, and if it is inverse, associate $(i, i + 1)$.

Next, we place an I in all positions (i, j) of α' where (i, j) is associated to some $a^{\pm 1}$, and in positions (i, j) of β' where (i, j) is associated to some $b^{\pm 1}$. (These entries should be taken modulo n , so that $n + 1$ becomes 1.) The exception is the position $(1, 2)$ or $(2, 1)$, which should have a ϕ placed in this position. Finally, add I to the diagonal entries of both α' and β' .

The matrices α and β are square matrices of size mn , considered as block matrices, whose n^2 blocks are given by the entries of α' and β' . For this, regard I as the $m \times m$ identity matrix and ϕ as the matrix representing the automorphism ϕ . Finally, associate x with α and y with β ; this produces a representation of G , which is denoted by $M(w, \phi)$.

In the example $w = aba^{-1}b$ and ϕ is some map on an m -dimensional vector space,

the matrices α' and β' are

$$\alpha' = \begin{pmatrix} I & 0 & 0 & 0 \\ \phi & I & 0 & 0 \\ 0 & 0 & I & I \\ 0 & 0 & 0 & I \end{pmatrix}, \quad \beta' = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & I & I & 0 \\ I & 0 & 0 & I \end{pmatrix}.$$

The matrices α and β are block matrices represented by α' and β' , where I is the $m \times m$ identity matrix.

Similarly to the string modules, the modules $M(w, \phi)$ are all indecomposable, and $M(w, \phi)$ and $M(w', \phi')$ are isomorphic if and only if w and w' are the same word modulo inverses and cycling letters, and ϕ and ϕ' are equivalent transformations. The $M(w, \phi)$ are called *band modules*.

Restricting the indecomposable modules to the subgroups generated by the two generators individually enables us to greatly restrict the structure of tensor products. We begin with an analysis of this restriction.

Lemma 3.11 Let M be an indecomposable KG -module.

- (i) If M is odd-dimensional then $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are both the sum of a trivial module and projective modules.
- (ii) If M is an even-dimensional string module then either $M \downarrow_{\langle x \rangle}$ is projective and $M \downarrow_{\langle y \rangle}$ is the direct sum of two copies of K and a projective, or vice versa.
- (iii) If M is a band module, then both $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are projective.

Proof: Let w be a word of even length $2n$, beginning with $a^{\pm 1}$ say, and let v_i denote the standard basis, for $1 \leq i \leq 2n + 1$. Then the submodules of $M \downarrow_{\langle x \rangle}$ generated by v_i and v_{i+1} for $1 \leq i < 2n + 1$ and i odd form a copy of the projective module, which therefore splits off. Hence $M \downarrow_{\langle x \rangle}$ is the sum of n projective modules and a trivial module. The same occurs for $M \downarrow_{\langle y \rangle}$, proving (i).

If M is an even-dimensional string module then it is defined by a word w of odd length $2n - 1$, with first and last letters $a^{\pm 1}$ without loss of generality. Then $M \downarrow_{\langle y \rangle}$ has $n - 1$ submodules $\langle v_i, v_{i+1} \rangle$ (for i even) isomorphic with the projective indecomposable $K\langle y \rangle$ -module, and two trivial submodules, $\langle v_1 \rangle$ and $\langle v_{2n} \rangle$. Similarly, $\langle v_i, v_{i+1} \rangle$ is a projective submodule of $M \downarrow_{\langle x \rangle}$ for each odd i , and so $M \downarrow_{\langle x \rangle}$ is projective, proving (ii).

It remains to discuss the band modules. By cycling, we may assume that the module begins with a , and then we again see easily that the matrix corresponding to

the action of y on M is a sum of projective modules, and this is true for any band module for a word beginning $a^{\pm 1}$. However, by cycling the word we find that M is isomorphic with a band module for a word beginning $b^{\pm 1}$, and hence $M \downarrow_{\langle x \rangle}$ must also be projective, as required. \square

We collect the following basic facts about odd-dimensional string modules: for this, we will need to know the length of a word, and we write $\ell(w)$ for this quantity.

Lemma 3.12 Let $w, w' \in \mathscr{W}$ be words, and suppose that $\ell(w) = 2n$ and $\ell(w') = 2m$ are even. Write $M = M(w)$ and $M' = M(w')$.

- (i) Either w or w^{-1} begins with the symbol $a^{\pm 1}$.
- (ii) The restrictions $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are the sum of the trivial module and n projective 2-dimensional modules.
- (iii) The restrictions of the tensor product $M \otimes M'$ to $\langle x \rangle$ and $\langle y \rangle$ are both the sum of a trivial module and $2nm + n + m$ projective 2-dimensional modules.
- (iv) The tensor product $M \otimes M'$ is the direct sum of a string module $M(w'')$ for some w'' of even length and various even-dimensional band modules.

Proof: The proof of (i) is obvious, as is (ii) from the description of the matrices given in the construction of string modules; (iii) follows from the fact that $(M \otimes M') \downarrow_H = M \downarrow_H \otimes M' \downarrow_H$ for any subgroup H ; and (iv) follows from the fact that at least one odd-dimensional summand must occur in $M \otimes M'$, since it has odd dimension, and each odd-dimensional summand would contribute one copy of K to $(M \otimes M') \downarrow_{\langle x \rangle}$, which only contains one trivial summand. Finally, since both $(M \otimes M') \downarrow_{\langle x \rangle}$ and $(M \otimes M') \downarrow_{\langle y \rangle}$ contain exactly one trivial summand, no even-dimensional string modules (which have two trivial summands when restricted to one of the generators) can occur. \square

Lemma 3.12(iv) yields the following corollary.

Corollary 3.13 Let w and w' be words in \mathscr{W} of even length, and let $M = M(w)$ and $M' = M(w')$. If N is a module with a unique odd-dimensional summand, write \bar{N} for this summand. Then the set of all odd-dimensional indecomposable KG -modules form a group under the operation

$$M \circ M' = \overline{M \otimes M'}.$$

Proof: That this is a binary operation comes from Lemma 3.12, so we need to check that \circ is associative, that there is an identity, and that there is an inverse. The associativity of \circ follows immediately from the associativity of \otimes ; the trivial module clearly acts as an identity; and Theorem 1.20 implies that, if M is an odd-dimensional indecomposable module, then

$$\overline{M \otimes M^*} = K,$$

and so therefore M^* is the inverse of M . \square

This group has been studied by Louise Archer in her thesis [8], in which the following was shown.

Theorem 3.14 (Archer [8]) The group of all odd-dimensional indecomposable modules is a torsion-free abelian group that is not finitely generated.

This result yields the following corollary.

Corollary 3.15 The only algebraic odd-dimensional indecomposable KG -module is trivial. Moreover, for every non-trivial odd-dimensional string module M , the iterated tensor module $T(M)$ contains infinitely many different (odd-dimensional) string modules as summands.

Now let us turn our attention to even-dimensional string modules.

Lemma 3.16 Let $w, w' \in \mathscr{W}$ be words, and suppose that $\ell(w) = 2n - 1$ and $\ell(w') = 2m - 1$ are odd. Write $M = M(w)$ and $M' = M(w')$.

- (i) The word w begins with $a^{\pm 1}$ if and only if it ends with $a^{\pm 1}$.
- (ii) If w begins with $a^{\pm 1}$, then the restriction $M \downarrow_{\langle x \rangle}$ is projective, and the restriction $M \downarrow_{\langle y \rangle}$ is the sum of a $2(m - 1)$ -dimensional projective module and a 2-dimensional trivial module.
- (iii) If w begins with $a^{\pm 1}$ and w' begins with $b^{\pm 1}$, then $M \otimes M'$ contains no summands that are string modules.
- (iv) If both w and w' begin with $a^{\pm 1}$, then $M \otimes M'$ contains exactly two even-dimensional string module summands.

Proof: (i) is obvious, and (ii) easily follows from the construction of string modules, since the only place that a trivial summand can occur is at the beginning or end of a word. The proof of (iii) comes from the fact that if $M \otimes M'$ contains a string module, there must be a trivial summand of either $(M \otimes M') \downarrow_{\langle x \rangle}$ or $(M \otimes M') \downarrow_{\langle y \rangle}$, which is impossible since both $M \downarrow_{\langle x \rangle}$ and $M' \downarrow_{\langle y \rangle}$ are projective. The proof of (iv) is similar: if M and M' both begin with $a^{\pm 1}$, then both $M \downarrow_{\langle y \rangle}$ and $M' \downarrow_{\langle y \rangle}$ contain two trivial summands, proving that $(M \otimes M') \downarrow_{\langle y \rangle}$ contains four trivial summands. Since band modules restrict to projective modules, and no odd-dimensional summand can occur by Theorem 1.20(ii), the tensor product must contain two even-dimensional string modules as summands. \square

We also need to describe the action of the Heller operator Ω , and also the (considerably easier) Auslander–Reiten translate $\Omega^2 = \tau$, on string modules. In order to describe the action of τ on string modules effectively, we introduce two operations, L_q and R_q , on the set of all words \mathscr{W}_q . Write $A = (ab)^{q-1}a$ and $B = (ba)^{q-1}b$. The operator L_q is defined by adding or removing a string at the start of the word w , and R_q is the same but at the end of the word.

If the word w starts with Ab^{-1} or Ba^{-1} , then wL_q is w with this portion removed. If neither of these are present, then we add either $A^{-1}b$ or $B^{-1}a$ to w to get wL_q , whichever gives an element of \mathscr{W}_q . Similarly, if w ends with aB^{-1} or bA^{-1} , then wR_q is w with this portion removed. If neither of these are present, then we add either $a^{-1}B$ or $b^{-1}A$ to w to get wR_q , whichever gives a word in \mathscr{W}_q . The operators L_q and R_q commute, and are bijections on \mathscr{W}_q .

The double Heller operator Ω^2 is given by

$$\Omega^2(M(w)) = M(wL_qR_q),$$

and the almost-split sequences on string modules are given by

$$0 \rightarrow M(wL_qR_q) \rightarrow M(wL_q) \oplus M(wR_q) \rightarrow M(w) \rightarrow 0,$$

unless $w = AB^{-1}$, in which case the almost-split sequence is

$$0 \rightarrow M(wL_qR_q) \rightarrow M(wL_q) \oplus M(wR_q) \oplus KG \rightarrow M(w) \rightarrow 0,$$

where KG denotes the projective indecomposable module KG , viewed as a module over itself.

The effect of $\tau = \Omega^2$ on band modules is trivially easy; every band module satisfies $M(w, \phi) = \Omega^2(M(w, \phi))$. Thus all band modules are periodic of period either 1 or

2. The only periodic string modules are those given by words $AR_q^i = (Ab^{-1})^i A$ and $BR_q^i = (Ba^{-1})^i B$.

Now we describe the action of Ω on the string modules. Let w be a word in \mathscr{W}_q ; the *generation form* of w is it written as $w = w_1 w_2^{-1} w_3 w_4^{-1} \dots w_{2m-1} w_{2m}^{-1}$, where w_1 and w_{2m} may be empty, but all other of the w_i are non-trivial and consist solely of direct letters. The integer m is called the *generating number*. We define $K(w)$ to be the word given by

$$K(w) = v_1^{-1} v_2 v_3^{-1} v_4 \dots v_{2m},$$

where all of the letters in the v_i are direct, and $v_i w_i$ is a word of length $2q$. Finally, if w is a word, then there exist direct letters c and d , where c and d are each either a or b , such that cwd^{-1} is a word as well. The Heller translate of $M(w)$ is given by

$$\Omega(M(w)) = M(K(cwd^{-1})).$$

Let w be a word in \mathscr{W}'_q , such that the first letter of w is direct and the last letter of w is inverse. (This can be done by cycling for all words in \mathscr{W}'_q .) For the band module $M(w, \phi)$, if m is the generating number of w , then

$$\Omega(M(w, \phi)) = M(K(w), (-1)^m \phi^{-1}).$$

The periodic string modules are actually of period 1; since the modules $C_n(0)$ and $C_n(\infty)$ for V_4 subgroups of G induce to periodic indecomposable modules for G , and these modules are clearly string modules by Mackey's formula and Lemma 3.11, we see that every periodic string module for G is induced from an even-dimensional string module for some V_4 subgroup (all of which are periodic). This enables us to prove the following proposition.

Proposition 3.17 Suppose that M is a periodic, indecomposable string module. Then M is algebraic, and moreover,

$$M^{\otimes 2} = M \oplus M \oplus P,$$

where P is a sum of a projective modules and modules with vertex C_2 .

Proof: Since M is a periodic string module, it is induced from an even-dimensional string module for Q , a subgroup of G isomorphic with V_4 . Write S for a source of M . Mackey's tensor product theorem implies that

$$S \uparrow^G \otimes S \uparrow^G = \bigoplus_{t \in T} (S \otimes S^t) \downarrow_{Q \cap Q^t} \uparrow^G,$$

where T is a set of (Q, Q) -double coset representatives. Since T can be chosen such that $1 \in T$, we must have $(2 \cdot S) \uparrow^G = 2 \cdot M$ as a summand of $M \otimes M$.

As M is an even-dimensional string module, $M \otimes M$ contains exactly two even-dimensional string modules by Lemma 3.16, and so all other summands are band modules or projectives. If $Q^t \neq Q$ in the decomposition above, then all summands of $(S \otimes S^t) \downarrow_{Q \cap Q^t} \uparrow^G$ will have vertex C_2 or be projective, and so it remains to consider those t that lie in $N_G(Q)$.

Since Q is self-centralizing, $N_G(Q)$ has order 8, and so since Q is one (Q, Q) -double coset, $N_G(Q) \setminus Q$ must be another. However, for this value of t , we must have that $S^t \not\cong S$, since if the conjugate module were isomorphic, we would have more string module summands, which is not possible. Hence $S^t \not\cong S$, and so their tensor product is projective, as stated in Conlon's table for tensor products of KV_4 -modules. This completes the proof. \square

We turn our attention to non-periodic string modules of even dimension. In what follows, write z for the central element of G , and $X = \langle x, z \rangle$ and $Y = \langle y, z \rangle$.

Lemma 3.18 Let $M = M(w)$ be a non-periodic string module.

- (i) If w begins with $a^{\pm 1}$, then $M \downarrow_X$ is a sum of periodic KV_4 -modules, and $M \downarrow_Y$ is the sum of two odd-dimensional KV_4 -modules and periodic modules.
- (ii) If w begins with $b^{\pm 1}$, then $M \downarrow_Y$ is a sum of periodic KV_4 -modules, and $M \downarrow_X$ is the sum of two odd-dimensional KV_4 -modules and periodic modules.

Proof: We will prove (i) only, and simply note that (ii) is the same as (i) with x and y reversed. By the Alperin–Evens theorem (Theorem 1.17), since M is non-periodic, either $M \downarrow_X$ or $M \downarrow_Y$ is non-periodic. We can see that $M \downarrow_{\langle x \rangle}$ is free, and so is a sum of 2-dimensional modules. Hence $M \downarrow_X$ is a sum of even-dimensional modules, and so periodic. Thus $M \downarrow_Y$ must be non-periodic, and so must contain odd-dimensional summands. Since there are exactly two copies of K in $M \downarrow_{\langle y \rangle}$, there must be exactly two odd-dimensional summands of $M \downarrow_Y$, as required. \square

The component of the Auslander–Reiten quiver containing a non-periodic string module $M(w)$ consists of all modules $M(wL_q^i R_q^j)$, where i and j are elements of \mathbb{Z} , and is of type A_∞^∞ . It is possible, using the construction of the string modules above, to prove an analogue of Theorem 2.14 for components of the Auslander–Reiten quiver of complexity 3. (This is Theorem C from the introduction.)

Theorem 3.19 Let G be a dihedral 2-group as constructed above, and let w be a word of odd length in \mathscr{W}_q such that $M(w)$ is non-periodic. Then at most one of the modules

$$\{M(wL_q^iR_q^j) : i, j \in \mathbb{Z}\}$$

is algebraic.

Theorems 3.14 and 3.19 yield the following corollary.

Corollary 3.20 Let M be a non-periodic indecomposable module for a dihedral 2-group, and suppose that M is algebraic. Then no other module on the component of the Auslander–Reiten quiver containing M is algebraic.

We will prove Theorem 3.19 in a sequence of lemmas. We begin with the following observation.

Lemma 3.21 Let $G = V_4$, and let x be a non-identity element of G . Let i be a non-positive integer, and let $M = \Omega^i(K)$. Then the G -fixed points of M are equal to the x -fixed points of M .

Proof: It is easy to see that the socle of M is of dimension $i + 1$. We simply note that $M \downarrow_{\langle x \rangle}$ is the sum of K and i copies of the free module, and so its socle has dimension $i + 1$ also. Thus the lemma must hold. \square

Using this lemma, we can prove a crucial result about the summands of $M(w) \downarrow_Y$ under a certain condition on w .

Lemma 3.22 Suppose that $M = M(w)$ is an even-dimensional string module, and suppose that w begins with a^{-1} or ends with a . Finally, suppose that the odd-dimensional summands of $M \downarrow_Y$ are isomorphic with $\Omega^i(K)$ and $\Omega^j(K)$, where both i and j are non-positive. Then (at least) one of i and j is 0.

Proof: Note that, since w begins with an inverse letter, the subspace $U = \langle v_i : i \geq 2 \rangle$ is a G -submodule of M (where the v_i are the standard basis used in the construction of the string modules). Thus if there exists a Y -fixed point

$$V = v_1 + \sum_{i \in I} v_i,$$

then $\langle V \rangle$ is a summand of $M \downarrow_Y$ isomorphic with K , as required. Let N_1 and N_2 denote the two odd-dimensional summands of $M \downarrow_Y$. By Lemma 3.21, it suffices to show that there is such a point V fixed by y lying inside one of the N_i .

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

We will now calculate the possibilities for a trivial summand of $M \downarrow_{\langle y \rangle}$. Since $\langle v_2, \dots, v_{n-1} \rangle \downarrow_{\langle y \rangle}$ (where $\dim M = n$) is a free module, if $\alpha = \sum_{j \in J} v_j$ is a fixed point of $M \downarrow_{\langle y \rangle}$ with a complement, then either 1 or n lies in J . Since $M \downarrow_{\langle y \rangle}$ contains a 2-dimensional trivial module, we easily see that the fixed points with complements are given by

$$v_1 + \sum_{j \in J} v_j, \quad v_n + \sum_{j \in J} v_j, \quad v_1 + v_n + \sum_{j \in J} v_j,$$

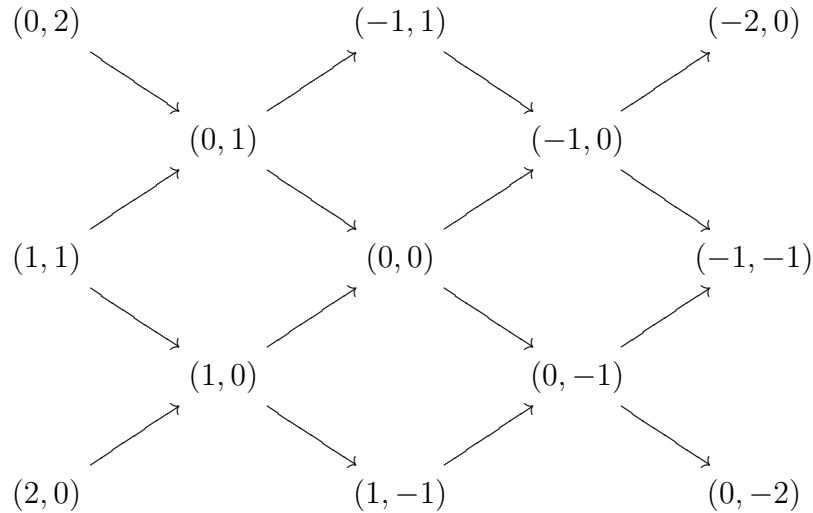
where $J \subseteq \{2, \dots, n-1\}$. Hence for some suitable choice of I , the point V given above is a y -fixed point, as required.

If w ends with a , then w^{-1} begins with a^{-1} . Since $M(w) = M(w^{-1})$, we get the result. \square

As a remark, by taking duals, one sees that if $M = M(w)$ and w begins with a or ends with a^{-1} , and the odd-dimensional summands of $M \downarrow_Y$ are isomorphic with $\Omega^i(K)$ and $\Omega^j(K)$ for $i, j \geq 0$, then (at least) one of i and j is 0.

To provide the proof of Theorem 3.19, we must analyze the components of the Auslander–Reiten quiver consisting of non-periodic, even-dimensional string modules. To do this, let M denote such an indecomposable module, and suppose without loss of generality that $M = M(w)$ where w begins with $a^{\pm 1}$. Denote by Γ the component of $\Gamma_s(KG)$ on which M lies.

We will co-ordinatize the component Γ according to a different rule from that in Chapter 2. Write $(0, 0)$ for the co-ordinates of the vertex corresponding to $M(w)$, and (i, j) for the vertex corresponding to $M(wL_q^i R_q^j)$. Then the portion of Γ around the module M is co-ordinatized as follows.



CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

[In this diagram, the Ω^2 operation is a functor moving from right to left, in the opposite direction to the previous chapter. The reversal is to make notation in this chapter slightly easier. The map $M(w) \mapsto M(wL_q)$ is a function moving down and to the left, and the map $M(w) \mapsto M(wR_q)$ moves up and to the left.]

In a similar fashion to the previous chapter, we call the *signature* of the vertex (i, j) the object $[r, s]$, where

$$\Omega^r(K) \oplus \Omega^s(K) | M_{(i,j)} \downarrow_Y .$$

[We will abuse notation slightly, and also refer to the signature of a module, as well as the signature of a vertex.] Again, as in the previous chapter, we get a ‘diamond rule’ for the diamonds of the Auslander–Reiten quiver using Lemma 2.15, so that if $M_{(i,j)}$ does not have vertex contained within Y , then

$$M_{(i,j)} \downarrow_Y \oplus M_{(i+1,j+1)} \downarrow_Y = M_{(i,j+1)} \downarrow_Y \oplus M_{(i+1,j)} \downarrow_Y .$$

Suppose that no module on Γ has vertex Y . (Since every proper subgroup of Y is cyclic, if N is a non-periodic indecomposable module with vertex contained within Y , it has vertex Y .) If the signatures are known for two adjacent rows of Γ , then they can be calculated for all rows, using the diamond rule. Since two rows (say rows α and $\alpha + 1$) are completely known, the rows $\alpha + 2$ and $\alpha - 1$ can be calculated, since every point on either of those rows lies on a diamond whose other three corners lie in the rows α and $\alpha + 1$. This process can be iterated to get the signatures for all rows.

This information makes the proof of the next proposition possible.

Proposition 3.23 Let $M = M_{(0,0)}$ be a non-periodic, even-dimensional string module, and suppose that M is algebraic. Suppose in addition that the component Γ of $\Gamma_s(KG)$ containing M contains no module with vertex Y . Let $M_{(i,j)}$ denote the indecomposable module $M(wL_q^i R_q^j)$. Write $[r, s]$ for the signature of (i, j) . Then exactly one of the following three possibilities occurs:

- (i) The signature of (i, j) is $[2i, 2j]$ (or $[2j, 2i]$);
- (ii) The signature of (i, j) is $[2i, 2i]$; and
- (iii) The signature of (i, j) is $[2j, 2j]$.

Proof: Firstly, we note that all three potential signatures satisfy the diamond rule that the sum of the signatures of (i, j) and $(i - 1, j - 1)$ is equal to the sum of the signatures of $(i - 1, j)$ and $(i, j - 1)$. We need to check that these three possibilities

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

are the only ones, and by the remarks before the proposition it suffices to check that these are the only three possibilities for the two rows with vertices (i, i) and $(i, i + 1)$ in the Auslander–Reiten quiver.

Since the signature of $(0, 0)$ is $[0, 0]$, the signature of (i, i) must be $[2i, 2i]$, since

$$M_{(i,i)} = \Omega^{2i}(M_{(0,0)}).$$

Since no module on Γ has vertex contained within Y , the diamond rule for the diamond containing $(0, 0)$ and $(1, 1)$ becomes

$$M_{(0,0)} \downarrow_Y \oplus M_{(1,1)} \downarrow_Y = M_{(0,1)} \downarrow_Y \oplus M_{(1,0)} \downarrow_Y .$$

The signatures of $(0, 0)$ and $(1, 1)$ are $[0, 0]$ and $[2, 2]$ respectively, and so the signature of $(0, 1)$ is one of $[0, 2]$ (or equivalently $[2, 0]$), $[0, 0]$ or $[2, 2]$. Thus the signatures of $(i, i + 1)$ are one of $[2i, 2i + 2]$, $[2i, 2i]$ or $[2i + 2, 2i + 2]$, which correspond to (i), (ii) and (iii) respectively in the proposition. \square

In fact, the same result holds for the two components containing non-periodic modules with vertex Y , but it requires more work.

Let M be an indecomposable module with vertex Y . If M is non-periodic, then the source S of M must also be non-periodic. Thus $S = \Omega^i(K)$ for some $i \in \mathbb{Z}$. Therefore the modules $\Omega^i(K_Y) \uparrow^G$ (where K_Y denotes the trivial module for Y) are the only non-periodic indecomposable modules with vertex Y . The module $(K_Y) \uparrow^G$ is algebraic, whereas all others are not.

We begin by considering the component containing $M_{(0,0)} = \Omega(K_Y) \uparrow^G$. This cannot contain algebraic modules, because it can have no vertex with signature $[0, 0]$. To see this, notice firstly that the signature of $(0, 0)$ is $[1, 1]$. We analyze the diamond with bottom vertex $(0, 0)$: write $[r, s]$ for the signature of the top vertex, namely $(-1, 1)$, and write $[p, q]$ for the signature of the vertex $(0, 1)$ on the left of the diamond. Then the diamond rule gives

$$[1, 1] \cup [r, s] = [p, q] \cup [p - 2, q - 2],$$

and we see that p, q, r and s are all odd. Thus all signatures of vertices $(i, i + 1)$ (i.e., the row above that containing $M_{(0,0)}$) are a pair of odd numbers. Since all diamonds not involving those modules with vertex Y obey the diamond rule, we see that all modules above the horizontal line containing $M_{(0,0)}$ have signature a pair of odd numbers. The same analysis holds for the lower half of the quiver, and so our claim holds.

The other component with modules of vertex Y , namely that containing $M_{(0,0)} = (K_Y) \uparrow^G$, does contain an algebraic module. Suppose that the signatures of the vertices on the horizontal line containing $(0, 0)$, and those on the lines directly above and below this are known. (Thus the signatures for all vertices (i, i) , $(i + 1, i)$ and $(i - 1, i)$ are known.) Then we claim that the signatures for all vertices can be deduced. This is true for the same reason as before, since all diamonds containing at most one point from the line of vertices (i, i) obey the diamond rule.

This will enable us to prove the next proposition easily.

Proposition 3.24 Let $M = M_{(0,0)}$ be the module $K_Y \uparrow^G$, where K_Y denotes the trivial module for Y . Let $M_{(i,j)}$ denote the indecomposable module $M(wL_q^i R_q^j)$. Write $[r, s]$ for the signature of (i, j) . Then exactly one of the following three possibilities occurs:

- (i) The signature of (i, j) is $[2i, 2j]$ (or $[2j, 2i]$);
- (ii) The signature of (i, j) is $[2i, 2i]$; and
- (iii) The signature of (i, j) is $[2j, 2j]$.

Proof: Firstly note that the three signature patterns obey the diamond rule everywhere, so they certainly obey it for those diamonds that split upon restriction to Y . Thus we need only show that these three possibilities are the only ones. By the preceding remarks, it suffices to show this for the horizontal lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$.

We analyze the diamond with bottom vertex $(0, 0)$: write $[r, s]$ for the signature of the top vertex, namely $(-1, 1)$, and write $[p, q]$ for the signature of the vertex $(0, 1)$ on the left of the diamond. Then the diamond rule gives

$$[0, 0] \cup [r, s] = [p, q] \cup [p - 2, q - 2],$$

and so p and q are either both 0, both 2, or one is 0 and one is 2. In any case, this uniquely determines all modules on the horizontal line containing the vertex $(0, 1)$, and they are as claimed in the proposition. We need to determine the signatures of the vertices $(i, i - 1)$ from these.

Suppose that the signature of $M_{(0,1)}$ is $[0, 0]$. Then the dual of $M_{(0,1)}$ must also have signature $[0, 0]$. The almost-split sequence terminating in $M_{(0,0)}$ is given by

$$0 \rightarrow M_{(1,1)} \rightarrow M_{(0,1)} \oplus M_{(1,0)} \rightarrow M_{(0,0)} \rightarrow 0,$$

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

and since $M_{(0,0)}$ is self-dual, the dual of this sequence is the (almost-split) sequence

$$0 \rightarrow M_{(0,0)} \rightarrow M_{(0,-1)} \oplus M_{(-1,0)} \rightarrow M_{(-1,-1)} \rightarrow 0.$$

Thus either $M_{(0,1)}^* = M_{(0,-1)}$ or $M_{(0,1)}^* = M_{(-1,0)}$. However, the second possibility cannot occur, since we know that the signature of $(-1, 0)$ is $[-2, -2]$, and thus

$$M_{(0,1)}^* = M_{(0,-1)}.$$

Hence the signature of $(0, -1)$ is $[0, 0]$, and we have proved that the three lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$ have signatures obeying possibility (ii).

Now suppose that the signature of $M_{(0,1)}$ is $[2, 2]$. Then $M_{(0,1)}^* \not\cong M_{(-1,0)}$ since the signature of $M_{(-1,0)}$ is $[0, 0]$. Thus we again have

$$M_{(0,1)}^* = M_{(0,-1)}.$$

Since the signature of $(0, 1)$ is $[2, 2]$, the signature of $(0, -1)$ is $[-2, -2]$, and so we have proved that the three lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$ have signatures obeying possibility (iii).

Finally, suppose that the signature of $(0, 1)$ is $[0, 2]$. If the signature of $M_{(0,-1)}$ is not $[0, -2]$, then its dual would have to be $M_{(0,1)}$, by the same reasoning as the previous two paragraphs. However, this is not possible, and so we have proved that the three lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$ have signatures obeying possibility (i). \square

In the first case of Propositions 3.23 and 3.24, there is a unique vertex on Γ with signature $[0, 0]$, namely the vertex $(0, 0)$, and so M is indeed the unique algebraic module on Γ . This is in accordance with Theorem 3.19.

In the second case, $K \oplus K | M(wL_q^i) \downarrow_Y$ for all $i \in \mathbb{Z}$, and

$$\Omega^{-2}(K) \oplus \Omega^{-2}(K) | M(wL_q^i R_q^{-1}) \downarrow_Y .$$

If i is a suitably large negative number, then $wL_q^i R_q^{-1}$ begins with a^{-1} . This yields a contradiction, since by Lemma 3.22, K must be a summand of $M(wL_q^i R_q^{-1}) \downarrow_Y$.

In the third case, $K \oplus K | M(wR_q^i) \downarrow_Y$ for all $i \in \mathbb{Z}$, and so

$$\Omega^2(K) \oplus \Omega^2(K) | M(wL_q^{-1} R_q^i) \downarrow_Y .$$

If i is a suitably large negative number, then $wL_q^{-1} R_q^i$ ends with a^{-1} . This yields a contradiction, since by Lemma 3.22, K must be a summand of $M(wL_q^{-1} R_q^i) \downarrow_Y$.

Thus in Propositions 3.23 and 3.24 only the first possibility can occur, and so Theorem 3.19 is proved.

3.3 The Group $C_3 \times C_3$

For this section, write P for the group $C_3 \times C_3$, and write $K = \text{GF}(3)$. There are four subgroups of order 3 in P , labelled Q_i for $1 \leq i \leq 4$. Let G denote the holomorph $P \rtimes \text{Aut}(P)$. The representation type of P is wild, and so we have no classification of the indecomposable modules. However, using a computer, we can determine the nature of the small-dimensional indecomposable modules, by taking submodules of projective modules.

Rather than determine whether each isomorphism type of indecomposable module is algebraic, we will use the fact that the module M is algebraic if and only if the conjugate module M^g is also algebraic. To find conjugate modules, we induce an indecomposable KP -module M to the group G , then restrict it back to P . By Mackey's theorem, all summands of this module are conjugates of M .

In this section, we will proceed by dimension.

3.3.1 Dimension 2

Here, there are four non-isomorphic indecomposable modules M_i , all obviously uniserial of length 2, and submodules of KP . These are the 2-dimensional modules for P/Q_i , viewed as KP -modules, and are all hence algebraic, as modules for a cyclic group. In each case,

$$M_i^{\otimes 2} = K_P \oplus \mathcal{P}(K_{P/Q_i}).$$

Since all subgroups of order 3 are conjugate in G , all of the M_i are conjugate. Thus we get the following result.

Proposition 3.25 Let M be a 2-dimensional KP -module. Then M is algebraic.

3.3.2 Dimension 3

There are several different types of indecomposable module, and we deal with each in turn. Firstly, there is the projective cover of the trivial module for P/Q_i , viewed as a KP -module: this is clearly algebraic. These modules form a G -conjugacy class of four non-isomorphic modules, uniserial of length 3. There are eight other non-isomorphic uniserial modules of length 3, all G -conjugate. Let M denote one of these; then

$$M \otimes M = M^* \oplus \Omega(M^*).$$

The tensor product $M \otimes \Omega(M^*)$ is indecomposable, but we get the decomposition

$$M \otimes M \otimes \Omega(M^*) = 2 \cdot M \oplus 2 \cdot \Omega(M) \oplus 4 \cdot \mathcal{P}(K).$$

Thus we need to examine the tensor products $M \otimes M^*$ and $M \otimes \Omega(M)$. The second is easy: we have

$$M \otimes \Omega(M) = M^* \oplus \Omega(M^*) \oplus \mathcal{P}(K).$$

This can easily be seen as $\Omega^0(M \otimes \Omega(M)) = \Omega(M \otimes M)$, along with the fact that M is periodic of period 2. Finally, although $M \otimes M^*$ is indecomposable, we have

$$M \otimes M^* \otimes M = 2 \cdot M \oplus 2 \cdot \Omega(M) \oplus \mathcal{P}(K).$$

Thus, since the module $T(M)$ contains only summands isomorphic with M , M^* , $\mathcal{P}(K)$, $\Omega(M)$ and $\Omega(M^*)$, $M \otimes M^*$ and $\Omega(M \otimes M^*)$, we have proved that M is algebraic.

There remain two other 3-dimensional modules, both with two socle layers, one with simple socle and one with simple top; they are, of course, dual to one another, and the module with simple socle is the module $\text{soc}^2(KP)$, which is the preimage in KP of the module $\text{soc}(KP/\text{soc}(KP))$. Write N for this module.

The tensor square of this module decomposes as

$$N \otimes N = N^* \oplus \Omega(N^*),$$

and so by Theorem 2.11, if N^* is not periodic then at least one of N^* and $\Omega(N^*)$ is non-algebraic, whence N is non-algebraic. Since P is abelian, we know that all periodic modules have period at most 2 (Theorem 1.18), whereas $\Omega(N)$ has dimension 15 and $\Omega^{-1}(N)$ has dimension 6, so N is not periodic. (See Proposition 5.4 for a generalization; it proves that $\text{soc}^2(KP)$ is non-algebraic for $P = C_p \times C_p$.)

Proposition 3.26 Let M be an indecomposable 3-dimensional module for P over $\text{GF}(3)$. Then M is algebraic if and only if M is periodic.

In particular, we have the following corollary.

Corollary 3.27 The natural module for $\text{GL}_3(3)$ is not algebraic, and so $\text{GL}_3(3)$ does not have 3-SMA.

Proof: If the 3-dimensional natural module for $\text{GL}_3(p)$ were algebraic, then all restrictions of this module would be algebraic. Moreover, the restrictions to this module form all possible 3-dimensional representations of all groups over $\text{GF}(3)$, one of which is non-algebraic by Proposition 3.26. \square

We provide the first reference table for modules. Included is the number of modules in the G -conjugacy class; the dimensions of the socle layers of the module; whether the dual of the module M is M itself—in which case we write ‘Module’—is G -conjugate to M —in which case we write ‘Class’—or neither of these cases—in which case we write ‘Neither’; the dimensions of the indecomposable summands of $M \otimes M$; and whether the module is algebraic. When the dimension is a multiple of 3, we also include whether the module is periodic. It will emerge that periodicity appears to be the determining factor for algebraicity of KP -modules of dimension a multiple of 3. Although this table is small in this case, it will become considerably larger later.

Class	Size	Socle Layers	Self-dual	$M \otimes M$	Algebraic?	Periodic?
A	1	1,2	Neither	3,6	No	No
A*	1	2,1	Neither	3,6	No	No
B	8	1,1,1	Class	3,6	Yes	Yes
C	4	1,1,1	Module	3,3,3	Yes	Yes

3.3.3 Dimension 4

There are even more different types of 4-dimensional indecomposable KP -module; we firstly describe those with a simple socle. There are thirteen non-isomorphic modules, nine of which are self-dual and four of which are not. The self-dual modules split into two G -conjugacy classes, one of length 6—a representative of which we will denote by M_1 —and one of length 3—a representative of which we will denote by M_2 . All four of the remaining modules, which are not self-dual, are G -conjugate, and we denote by M_3 a representative of this G -conjugacy class.

Consider the tensor square $M_1 \otimes M_1$: this decomposes as

$$M_1 \otimes M_1 = K \oplus \mathcal{P}(K_{P/Q_a}) \oplus \mathcal{P}(K_{P/Q_b}) \oplus \mathcal{P}(K),$$

and so M_1 is algebraic.

Next, the module $M_2 \otimes M_2$ decomposes as

$$M_2 \otimes M_2 = K \oplus A_1 \oplus \mathcal{P}(K),$$

where A_1 is a 6-dimensional module. The module $A_2 = M_2 \otimes A_1$ is a 24-dimensional indecomposable module, and

$$M_2 \otimes A_2 = A_1 \oplus A_1 \otimes A_1 \oplus 6 \cdot \mathcal{P}(K) = 2 \cdot A_1 \oplus 8 \cdot \mathcal{P}(K) \oplus A_3,$$

where A_3 is a 12-dimensional indecomposable module. Lastly,

$$M_2 \otimes A_3 = A_1 \oplus 2 \cdot \mathcal{P}(K) \oplus A_2.$$

Hence M_2 is algebraic.

Finally, consider the tensor square $M_3 \otimes M_3$: this decomposes as

$$M_3 \otimes M_3 = K \oplus N_1 \oplus N_2,$$

where N_1 is a 6-dimensional module and N_2 is a 7-dimensional module. The module N_2 is, in fact, the Heller translate of a 2-dimensional indecomposable module. All 2-dimensional modules are algebraic by Proposition 3.25, and so by Theorem 2.11, N_2 is not algebraic since it is non-periodic; therefore neither is M_3 .

The second class of modules are those that consist of two socle layers, each of dimension 2. There are seven non-isomorphic 4-dimensional modules with 2-dimensional socle. These fall into two G -conjugacy classes, one of length 4 and one of length 3. Write L_1 and L_2 for representatives of the respective conjugacy classes. Note that both L_1 and L_2 are self-dual.

It is not known whether L_1 is algebraic or not, although it appears that it is not. To see some evidence of this, let us calculate the first few tensor powers of L_1 .

Firstly, we have the decomposition

$$L_1 \otimes L_1 = K \oplus B_1 \oplus B_2,$$

where B_1 is a self-dual 5-dimensional indecomposable module and B_2 is a self-dual 10-dimensional indecomposable module. While the module B_2 is algebraic, the module B_1 does not appear to be so. We need to examine both $L_1 \otimes B_1$ and $L_1 \otimes B_2$ in turn. Decomposing these tensor products gives the equations

$$\begin{aligned} L_1 \otimes B_1 &= L_1 \oplus B_3, \\ L_1 \otimes B_2 &= L_1 \oplus B_4 \oplus B_4^*, \end{aligned}$$

where B_3 is a (self-dual) 16-dimensional module and B_4 is an 18-dimensional indecomposable module. Since L_1 is self-dual, we need only consider $L_1 \otimes B_4$, and not $L_1 \otimes B_4^*$ as well. This tensor product is given by

$$L_1 \otimes B_4 = 2 \cdot \mathcal{P}(K) \oplus 2 \cdot B_4 \oplus B_4^*.$$

This shows that the module B_2 is algebraic. Having dealt with the module B_2 , via the modules B_4 and B_4^* , we turn our attention to the 16-dimensional module B_3 . We have the decomposition

$$L_1 \otimes B_3 = B_1 \oplus \mathcal{P}(K) \oplus B_5 \oplus B_4 \oplus B_4^*.$$

In this equation, B_5 is a (self-dual) 14-dimensional indecomposable module. The next step is to calculate the tensor product $L_1 \otimes B_5$, which is given by

$$L_1 \otimes B_5 = 2 \cdot \mathcal{P}(K) \oplus B_3 \oplus B_6,$$

with B_6 a 22-dimensional self-dual indecomposable module. We continue for a few more iterations:

$$\begin{aligned} L_1 \otimes B_6 &= 5 \cdot \mathcal{P}(K) \oplus B_5 \oplus B_7; \\ L_1 \otimes B_7 &= 4 \cdot \mathcal{P}(K) \oplus B_4 \oplus B_4^* \oplus B_6 \oplus B_8; \\ L_1 \otimes B_8 &= 4 \cdot \mathcal{P}(K) \oplus B_9 \oplus B_7; \text{ and} \\ L_1 \otimes B_9 &= 4 \cdot \mathcal{P}(K) \oplus B_8 \oplus B_{10} \end{aligned}$$

Here, the dimensions of the B_i for $7 \leq i \leq 10$ are 29, 22, 23 and 34. While this is not proof, it is an obvious indication that this module is not algebraic.

The final case to consider is the tensor product $L_2 \otimes L_2$; this is given by

$$L_2 \otimes L_2 = 2 \cdot K \oplus 2 \cdot M_2^g \oplus A_1^h,$$

where M_2^g and A_1^h are conjugates of the modules given earlier in this subsection. Hence L_2 is algebraic, since M_2 and A_1 are.

Proposition 3.28 Let M be a 4-dimensional indecomposable module. If M has a simple socle, then M is algebraic if and only if M is self-dual. If M has a 2-dimensional socle, then M is algebraic if $M \otimes M$ contains $K \oplus K$ as a summand.

While this result is unsatisfactory, it suffices for our purposes.

Class	Size	Socle Layers	Self-dual	$M \otimes M$	Algebraic?
A	6	1,2,1	Module	1,3,3,9	Yes
B	4	1,2,1	Neither	3,6,7	No
B*	4	2,1,1	Neither	3,6,7	No
C	3	1,2,1	Module	1,6,9	Yes
D	4	2,2	Module	1,5,10	?
E	3	2,2	Module	1,1,4,4,6	Yes

3.3.4 Dimension 5

We first examine the 5-dimensional submodules of $\mathcal{P}(K)$. There are thirteen non-isomorphic 5-dimensional modules with simple socle, split up into three G -conjugacy classes, of lengths 3, 4, and 6 respectively, with representatives given by M_1 , M_2 and

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

M_3 respectively. None of these three modules is algebraic, and to see this, we take tensor products:

$$\begin{aligned} M_1 \otimes M_1 &= A_1 \oplus \Omega^2(K) \oplus \mathcal{P}(K); \\ M_2 \otimes M_2 &= A_2 \oplus B_1 \oplus \mathcal{P}(K_{P/Q_a}) \oplus \mathcal{P}(K); \text{ and} \\ M_3 \otimes M_3 &= \Omega^2(K) \oplus \mathcal{P}(K_{P/Q_a}) \oplus \mathcal{P}(K_{P/Q_b}) \oplus \mathcal{P}(K). \end{aligned}$$

Here, A_1 and A_2 are 6-dimensional modules, and are not important, and B_1 is a 7-dimensional module, and is important. The presence of $\Omega^2(K)$ implies that M_1 and M_3 are non-algebraic, and the decomposition

$$B_1 \otimes B_1 = \mathcal{P}(K_{P/Q_a}) \oplus \Omega^2(K) \oplus 4 \cdot \mathcal{P}(K)$$

proves that B_1 , and hence M_2 , is not algebraic.

Turning to the indecomposable modules with 2-dimensional socle, there are twenty-one non-isomorphic indecomposable modules with 2-dimensional socle and with top at least 2-dimensional. (Those with simple top are the duals of the modules given earlier.) These fall into five G -conjugacy classes, of lengths 1, 4, 4, 4, and 8. We will give each class a name, as in the previous subsections. The table is as below.

Class	Size	Socle Layers	Self-dual	$M \otimes M$	Algebraic?
A	6	1,2,2	Neither	3,3,9,10	No
A*	6	2,2,1	Neither	3,3,9,10	No
B	4	1,2,2	Neither	3,6,7,9	No
B*	4	2,2,1	Neither	3,6,7,9	No
C	3	1,2,2	Neither	6,9,10	No
C*	3	2,2,1	Neither	6,9,10	No
D	1	2,3	Neither	10,15	No
D*	1	3,2	Neither	10,15	No
E	4	2,2,1	Module	1,10,14	?
F	4	2,2,1	Neither	3,4,6,12	No
F*	4	2,2,1	Neither	3,4,6,12	No
G	8	2,2,1	Class	10,15	Yes

- (i) Class D consists of non-algebraic modules, because if M is the module contained within Class D, then the 15-dimensional summand of $M^{\otimes 2}$ is $\Omega(N)$, where N is a self-dual 12-dimensional module. Theorem 2.11 proves that this 15-dimensional summand is non-algebraic.
- (ii) Class E contains the 5-dimensional self-dual module B_1 given in the previous subsection. In that section, we proved that modules from Class D in dimension

CHAPTER 3. DIHEDRAL AND ELEMENTARY ABELIAN GROUPS

4 are algebraic if and only if the 5-dimensional summand of their tensor square is algebraic. This 5-dimensional summand is from Class E, and as such its algebraicity is unknown.

- (iii) Class F consists of non-algebraic modules; if M denotes a representative from Class F, then the 4-dimensional module that is a summand of $M \otimes M$ is from Class B, as given in Section 3.3.3, which is non-algebraic.
- (iv) Class G consists of algebraic modules. The proof of this simply involves decomposing tensor products: let M be a representative from Class G. Then there are indecomposable modules A_1, \dots, A_7 such that

$$\begin{aligned}
 M \otimes M &= A_1 \oplus A_2, \\
 M \otimes A_1 &= M \oplus \mathcal{P}(K) \oplus A_3 \oplus A_3^*, \\
 M \otimes A_2 &= 4 \cdot \mathcal{P}(K) \oplus A_4, \\
 M \otimes A_3 &= 4 \cdot \mathcal{P}(K) \oplus A_3 \oplus 2 \cdot A_3^*, \\
 M \otimes A_3^* &= 4 \cdot \mathcal{P}(K) \oplus A_3^* \oplus 2 \cdot A_3, \\
 M \otimes A_4 &= 9 \cdot \mathcal{P}(K) \oplus A_3 \oplus A_3^* \oplus 2 \cdot A_5 \oplus A_6, \\
 M \otimes A_5 &= 5 \cdot \mathcal{P}(K) \oplus A_3 \oplus A_3^* \oplus A_4^*, \\
 M \otimes A_6 &= 8 \cdot \mathcal{P}(K) \oplus A_2 \oplus A_5 \oplus A_4^*, \\
 M \otimes A_4^* &= 9 \cdot \mathcal{P}(K) \oplus 2 \cdot A_2 \oplus A_3 \oplus A_3^* \oplus A_7, \text{ and} \\
 M \otimes A_7 &= 10 \cdot \mathcal{P}(K) \oplus A_2 \oplus 2 \cdot A_3 \oplus 2 \cdot A_3^* \oplus A_5 \oplus A_4.
 \end{aligned}$$

The dimensions of the indecomposable modules are given below.

Module	Dimension
A_1	10
A_2	15
A_3	18
A_4	39
A_5	24
A_6	30
A_7	48

Class E is therefore the only class whose modules have unknown algebraicity: this lack of knowledge does not appear to be easily rectified, however. We therefore have the following partial result.

Proposition 3.29 Suppose that M is a 5-dimensional indecomposable module and that M^* is not G -conjugate to M . Then M is not algebraic.

3.3.5 Dimension 6

Due to the large number of indecomposable KP -modules of dimension 6, we will not describe explicitly the tensor product structure of those that are algebraic. We will give reasons why the non-algebraic modules are non-algebraic, however, since this cannot be checked easily on a computer.

We begin with the table of all indecomposable modules.

Class	Size	Socle Layers	Self-dual	$M \otimes M$	Algebraic?	Periodic?
A	8	1,2,2,1	Class	3,6,9,9,9	Yes	Yes
B	4	1,2,2,1	Module	3,3,3,9,9,9	Yes	Yes
C	1	1,2,3	Neither	9,12,15	No	No
C*	1	3,2,1	Neither	9,12,15	No	No
D	24	2,2,2	Class	6,9,9,12	Yes	Yes
E	16	2,2,2	Class	3,9,12,12	Yes	Yes
F	12	2,2,2	Neither	9,12,15	No	No
F*	12	2,3,1	Neither	9,12,15	No	No
G	12	2,2,2	Neither	3,9,9,15	No	No
G*	12	2,3,1	Neither	3,9,9,15	No	No
H	12	2,2,2	Neither	9,12,15	No	No
H*	12	2,3,1	Neither	9,12,15	No	No
I	8	2,3,1	Neither	3,9,12,12	No	No
I*	8	2,2,2	Neither	3,9,12,12	No	No
J	8	2,2,2	Class	3,3,3,6,6,6,9	Yes	Yes
K	8	2,2,2	Class	3,6,9,18	Yes	Yes
L	8	2,2,2	Class	3,3,6,6,6,12	Yes	Yes
M	8	2,2,2	Class	3,9,12,12	Yes	Yes
N	4	2,2,2	Neither	3,3,3,6,6,6,9	No	No
N*	4	2,3,1	Neither	3,3,3,6,6,6,9	No	No
P	4	2,2,2	Neither	3,9,12,12	No	No
P*	4	2,3,1	Neither	3,9,12,12	No	No
Q	4	2,3,1	Neither	3,6,9,18	No	No
Q*	4	3,2,1	Neither	3,6,9,18	No	No
R	4	2,3,1	Neither	15,21	No	No
R*	4	3,2,1	Neither	15,21	No	No
S	4	2,2,2	Module	6,9,9,12	Yes	Yes
T	4	2,2,2	Module	3,6,6,9,12	Yes	Yes
U	4	2,2,2	Module	3,3,3,3,6,6,12	Yes	Yes
V	3	2,2,2	Module	6,9,9,12	Yes	Yes
W	8	3,3	Module	1,1,1,9,12,12	Yes	No
X	4	3,3	Module	15,21	No	No

There are thirteen isomorphism classes of module whose socle is 1-dimensional: these are all Heller translates of 3-dimensional modules with 1-dimensional top. The

twelve periodic such modules are therefore algebraic by Theorem 2.11, and the remaining module is the translate of $KP/\text{rad}^2(KP)$, and so is non-algebraic, by Corollary 2.12.

Moving on to the isomorphism classes of module whose socle has dimension 2, we find that there are 199 such modules, arranged into various G -conjugacy classes. Using techniques similar to those above, we can deduce whether or not the modules are algebraic. For algebraic modules, we can check all summands of tensor powers one by one.

For the non-algebraic modules, it is fairly easy to prove that each of them is non-algebraic; like the non-algebraic 3-dimensional module given above, in most cases a Heller translate of either M or M^* lies inside the tensor square of M . The non-algebraic 6-dimensional modules are labelled by class names, as with previous dimensions. We will outline the reasons why the classes of non-algebraic modules are non-algebraic now.

- (i) Class N is non-algebraic because if M comes from Class N, then $M \otimes M$ contains the indecomposable module from Class C* in dimension 6 (which consists of a non-algebraic module) as a summand.
- (ii) Classes F, H, I, and P are non-algebraic because in each case, if M is an element from the class, then $\Omega(M^*)$ is a summand of $M \otimes M$.
- (iii) Class G is non-algebraic because if M is a representative from Class G then $\Omega^{-2}(M^*)$ is a summand of $M \otimes M$.
- (iv) Class Q is not algebraic; let M be a representative from Class Q. The module $M \otimes M$ decomposes as the sum of a 3-dimensional module, a 6-dimensional module, a 9-dimensional module N , and an 18-dimensional module. The tensor product $M \otimes N$ has an 18-dimensional summand N' , and the tensor product $M \otimes N'$ has the module $\Omega^{-1}(M)$ as a summand. Hence M is not algebraic.
- (v) Class R is not algebraic; let M be a representative from Class R. The module $M \otimes M$ decomposes as the sum of a 15-dimensional module N_1 and a 21-dimensional module N_2 . Then $\Lambda^2(N_1)$, which is a summand of $N_1 \otimes N_1$, contains the module $\Omega(N_2^*)$ as a summand. Thus at least one of N_1 and N_2 is not algebraic, and so M is non-algebraic.

There are twenty-one indecomposable KP -modules with socle of dimension 3. Nine of these are duals of modules with 2-dimensional socle, which leaves twelve to

consider. There are eight modules which are the direct sum of three 2-dimensional FP -modules, where $F = \text{GF}(27)$. When the field is extended to $\text{GF}(27)$ (which has no effect on algebraicity by Theorem 1.8), one can easily check that the 2-dimensional modules are algebraic, so this leaves the four other modules, which are all G -conjugate.

The remaining four modules, which form a single G -conjugacy class, are non-algebraic; let M denote one of these modules, and note that $M \otimes M$ is the sum of a 15-dimensional module N and a 21-dimensional module. Finally, the module $\Omega(M)$ is a summand of $N \otimes N$, proving that M is non-algebraic.

From the table, the following result is clear.

Proposition 3.30 Let M be an absolutely indecomposable KG -module of dimension 6, where $K = \text{GF}(3)$ and $G = C_3 \times C_3$. Then M is algebraic if and only if it is periodic.

The 3-dimensional and 6-dimensional absolutely indecomposable modules are algebraic if and only if they are periodic. The conjecture alluded to earlier in the chapter is the following.

Conjecture 3.31 Let p be a prime and let G be the group $C_p \times C_p$. Let M be an absolutely indecomposable KG -module, where K is a field of characteristic p , and suppose that $p \mid \dim M$. Then M is algebraic if and only if M is periodic.

For elementary abelian groups of larger rank, this conjecture is definitely false. Any extension of this conjecture will have to take account of the following two observations.

- (i) Let G be the elementary abelian group of order p^n and Q be a subgroup of order p^2 . Let M be an indecomposable KQ -module of dimension prime to p , and suppose that M is algebraic. For example, any 2-dimensional module for Q over $\text{GF}(p)$ is algebraic, since $\text{GL}_2(p)$ has cyclic Sylow p -subgroups. Then the module $M \uparrow^G$ is an algebraic non-periodic KG -module of dimension a multiple of p .
- (ii) Let M be a periodic module for G/P , where G/P has order p^2 . Then M , viewed as a KG -module, is a non-periodic algebraic module for G . Its complexity is at most $r - 1$.

To take account of these two possibilities, we make the following, rather speculative, conjecture.

Conjecture 3.32 Let G be an elementary abelian group of order p^r , and let K be a field of characteristic p . Suppose that M is an absolutely indecomposable KG -module such that $p^{r-1} \mid \dim M$.

- (i) If M is periodic then M is algebraic.
- (ii) If $\text{cx } M = r$ then M is non-algebraic.

Theorem 2.14 tells us that if $r \geq 3$ in the conjecture above, so that we are not in the case of $C_p \times C_p$, then algebraic modules of complexity r are few and far between. This conjecture states that they are non-existent for certain dimensions.

The results of this section can be used when dealing with sporadic groups, and the field involved is $\text{GF}(3)$. The philosophy is to take a particular finite group G , and then enumerate all (conjugacy classes of) subgroups isomorphic with $C_3 \times C_3$. If M is some module for G , then we restrict M to a representative from each of the classes of $C_3 \times C_3$ subgroup, and decompose it into summands. If a non-algebraic module appears as a summand in one of these decompositions, we conclude that the module M itself is non-algebraic. The low-dimensional results above are invaluable in reducing the amount of work required to prove that various simple modules for sporadic groups are non-algebraic over $\text{GF}(3)$.

Chapter 4

Simple Modules for PSL_2

In this chapter we shall examine the simple modules for the groups $\mathrm{PSL}_2(q)$ (or equivalently $\mathrm{SL}_2(q)$, where q is odd and K is a field of characteristic 2. As we have mentioned, the groups $\mathrm{SL}_2(2^n)$ were analyzed by Alperin, and this is why we restrict ourselves to the ‘odd q ’ case.

We will prove the following theorem, which is Theorem D from the introduction.

Theorem 4.1 Let $G = \mathrm{PSL}_2(q)$, and K be a field of characteristic 2. Then G has 2-SMA if and only if $q \not\equiv 7 \pmod{8}$.

This theorem, as it stands, can be proved quite easily using Erdmann’s determination of the sources of simple modules for these groups in [30]. What is done here, however, is to decompose, into indecomposable summands, the tensor product of an arbitrary number of simple modules, in the cases where $q \equiv 3 \pmod{8}$ and $q \equiv 5 \pmod{8}$. In the case where $q \equiv 7 \pmod{8}$, an alternative proof of this theorem is given from that of considering the sources of simple modules. In the remaining case, where $q \equiv 1 \pmod{8}$, the proof here relies on Erdmann’s work. We give Erdmann’s result on the sources of simple modules now.

Theorem 4.2 (Erdmann [30]) Let $G = \mathrm{PSL}_2(q)$ where q is odd, with Sylow 2-subgroup P , and let M be a non-trivial simple module lying in the principal block of KG .

- (i) If $q \equiv 3 \pmod{4}$, then M has vertex P and the source of M is of dimension $|P|/2 - 1$.
- (ii) If $q \equiv 1 \pmod{4}$, then M has a Klein four-group as vertex, and 2-dimensional source.

If $q \equiv 1 \pmod{8}$, then the non-trivial simple modules in the principal block of $\mathrm{PSL}_2(q)$ have vertex V_4 and have 2-dimensional source, and are hence algebraic, by Theorem 3.1. All non-principal blocks of $\mathrm{PSL}_2(q)$ are of either cyclic defect or defect 0, and so for $q \equiv 1 \pmod{8}$, all simple modules are algebraic. The remaining congruences modulo 8 will be dealt with below.

In the first two sections we determine the decomposition into ordinary characters of the tensor products of various ordinary characters. In the following short section, we examine modules for dihedral groups of twice and four times odd order: these are the centralizers of involutions in $\mathrm{PSL}_2(q)$, where $q \equiv 3, 5 \pmod{8}$. In the four succeeding sections we consider each congruence class modulo 8.

We begin by stating, without proof, the conjugacy classes and irreducible characters of the special linear groups $\mathrm{SL}_2(q)$. This information can be found, for example, in [47] and [28]. The conjugacy classes of $\mathrm{SL}_2(q)$ are relatively easy to describe: firstly, write z for the matrix corresponding to -1 . Obviously $\{1\}$ and $\{z\}$ are conjugacy classes, since they form the centre. Write v for a generator of the multiplicative group of $\mathrm{GF}(q)$, and write

$$a = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Finally, let b denote an element of order $q+1$, which exists in $\mathrm{SL}_2(q)$. Then the other conjugacy classes are labelled as follows, together with the sizes of the conjugacy classes, and the orders of the elements.

x	1	z	a^ℓ	b^m	c	cz	d	dz
$ x^G $	1	1	$q(q+1)$	$q(q-1)$	$(q^2-1)/2$	$(q^2-1)/2$	$(q^2-1)/2$	$(q^2-1)/2$
$o(x)$	1	2	$\frac{(q-1)}{\gcd(\ell, q-1)}$	$\frac{q+1}{\gcd(m, q+1)}$	q	$2q$	q	$2q$

In this table, $1 \leq \ell \leq (q-3)/2$ and $1 \leq m \leq (q-1)/2$. In particular, G has exactly $q+4$ conjugacy classes. Denote by τ a primitive $(q-1)$ th root of 1, and by σ a primitive $(q+1)$ th root of 1. Write $\varepsilon = (-1)^{(q-1)/2}$. Then the character table for $\mathrm{SL}_2(q)$ is given below.

	1	z	a^ℓ	b^m	c	d
1_G	1	1	1	1	1	1
ψ	q	q	1	-1	0	0
ζ_i	$q+1$	$(-1)^i(q+1)$	$\tau^{i\ell} + \tau^{-i\ell}$	0	1	1
θ_j	$q-1$	$(-1)^j(q-1)$	0	$-(\sigma^{jm} + \sigma^{-jm})$	-1	-1
η_1	$(q-1)/2$	$-\varepsilon(q-1)/2$	0	$(-1)^{m+1}$	$(-1 + \sqrt{\varepsilon q})/2$	$(-1 - \sqrt{\varepsilon q})/2$
η_2	$(q-1)/2$	$-\varepsilon(q-1)/2$	0	$(-1)^{m+1}$	$(-1 - \sqrt{\varepsilon q})/2$	$(-1 + \sqrt{\varepsilon q})/2$
ξ_1	$(q+1)/2$	$\varepsilon(q+1)/2$	$(-1)^\ell$	0	$(1 + \sqrt{\varepsilon q})/2$	$(1 - \sqrt{\varepsilon q})/2$
ξ_2	$(q+1)/2$	$\varepsilon(q+1)/2$	$(-1)^\ell$	0	$(1 - \sqrt{\varepsilon q})/2$	$(1 + \sqrt{\varepsilon q})/2$

The other conjugacy classes, zc and zd , can be computed using the formula

$$\chi(zg) = \frac{\chi(z)}{\chi(1)}\chi(g).$$

Now let us consider the conjugacy classes of $\mathrm{PSL}_2(q)$; these are labelled by sets of the conjugacy class representatives of $\mathrm{SL}_2(q)$ given above. Obviously $\{1, z\}$ is a coset, as are $\{c, zc\}$ and $\{d, zd\}$. It remains to discuss the a^ℓ and the b^m . Clearly $a^{(q-1)/2} = z$, and so the cosets are $\{a^\ell, a^{\ell+(q-1)/2}\}$. Now, a and a^{-1} are conjugate, and so the cosets containing a^ℓ and $a^{(q-1)/2-\ell}$ are conjugate. If $q \equiv 3 \pmod{4}$, then it cannot be true that $\ell = (q-1)/2 - \ell$ for $1 \leq \ell \leq (q-3)/2$, whereas if $q \equiv 1 \pmod{4}$, then this is possible. Thus a collection of conjugacy class representatives are the cosets containing a^ℓ , for $1 \leq \ell \leq (q-3)/4$ if $q \equiv 3 \pmod{4}$ and $1 \leq \ell \leq (q-1)/4$ if $q \equiv 1 \pmod{4}$.

Now consider the elements b^m ; again, we have that $b^{(q+1)/2} = z$, and so the sets $\{b^m, b^{m+(q+1)/2}\}$ are cosets. Again, b and b^{-1} are conjugate, and so again the cosets containing b^m and $b^{(q+1)/2-m}$ are conjugate. If $q \equiv 3 \pmod{4}$, then $m = (q+1)/2 - m$ if and only if $m = (q+1)/4$, and if $q \equiv 1 \pmod{4}$, it is not possible that $m = (q+1)/2 - m$. Thus a collection of conjugacy class representatives are the cosets containing b^m , for $1 \leq m \leq (q+1)/4$ if $q \equiv 3 \pmod{4}$ and $1 \leq m \leq (q-1)/4$ if $q \equiv 1 \pmod{4}$. In our discussion of $\mathrm{PSL}_2(q)$, although the conjugacy classes are technically cosets of $\mathrm{SL}_2(q)$, we will label them by elements of $\mathrm{SL}_2(q)$.

4.1 Ordinary Characters for $q \equiv 1 \pmod{4}$

Using the calculations above, the conjugacy class representatives are $1, c, d, a^\ell$ and b^m for $1 \leq \ell \leq (q-1)/4$ and $1 \leq m \leq (q-1)/4$.

The character table of $G \cong \mathrm{PSL}_2(q)$ is given below; this is taken from the character table for $\mathrm{SL}_2(q)$, which appeared at the start of the chapter. As above, let τ denote a primitive $(q-1)$ th root of unity, and σ denote a primitive $(q+1)$ th root of unity (both in \mathbb{C}).

G	1	a^ℓ	b^m	c	d
1_G	1	1	1	1	1
ψ	q	1	-1	0	0
ζ_i	$q+1$	$\tau^{i\ell} + \tau^{-i\ell}$	0	1	1
θ_j	$q-1$	0	$-\sigma^{jm} - \sigma^{-jm}$	-1	-1
ξ_1	$(q+1)/2$	$(-1)^\ell$	0	$(1 + \sqrt{q})/2$	$(1 - \sqrt{q})/2$
ξ_2	$(q+1)/2$	$(-1)^\ell$	0	$(1 - \sqrt{q})/2$	$(1 + \sqrt{q})/2$

In this table, the variables i and j are both even and satisfy $2 \leq i \leq (q-5)/2$ and $2 \leq j \leq (q-1)/2$. Thus there are $(q-1)/4$ ordinary characters θ_j and $(q-5)/4$ ordinary characters ζ_i . Any sum of the ζ_i or the θ_j will be over these ranges.

In order to simplify our discussion, we introduce four reducible characters, namely $\xi_1 + \xi_2$, $\sum_i \zeta_i$, $\sum_j \theta_j$ and the character

$$\xi_1 + \xi_2 + 2\psi + 2 \sum_i \zeta_i + 2 \sum_j \theta_j.$$

These four characters will appear frequently when we decompose tensor products of ordinary characters.

Proposition 4.3 The four ordinary characters described above have the values given in the following table.

	1	a^ℓ	b^m	c	d
$X = \xi_1 + \xi_2$	$q+1$	$2(-1)^\ell$	0	1	1
$Z = \sum \zeta_i$	$(q+1)(q-5)/4$	$-1 - (-1)^\ell$	0	$(q-5)/4$	$(q-5)/4$
$\Theta = \sum \theta_j$	$(q-1)^2/4$	0	1	$-(q-1)/4$	$-(q-1)/4$
$\Phi = 2\Theta + 2Z + 2\psi + X$	$(q-1)(q+1)$	0	0	-1	-1

Proof: Nearly all character values in the table are easily derived from those of their irreducible constituents; the exceptions are the value of Z on a^ℓ and the value of Θ on b^m . Firstly, we deal with the value of Z on a^ℓ ; write $n = (q-1)/2$, and consider the sum $\sum_i \zeta_i$. Since i is even, we may replace i by 2α , and so the sought-after character value on a^ℓ is

$$\sum_{\alpha=1}^{(n-2)/2} \tau^{2\alpha\ell} + \tau^{-2\alpha\ell}.$$

Write $\rho = \tau^2$, a primitive n th root of unity, and also notice that $\rho^{-\alpha\ell} = (\rho^{n-\alpha})^\ell$. Hence the character value on a^ℓ is

$$\sum_{\alpha=1}^{(n-2)/2} (\rho^\alpha)^\ell + (\rho^{n-\alpha})^\ell = \sum_{\alpha=1}^{n-1} (\rho^\ell)^\alpha - (\rho^{n/2})^\ell.$$

For any (not necessarily primitive, but not equal to 1) n th root of unity λ , we have

$$\sum_{i=1}^{n-1} \lambda^i = -1.$$

Since ρ^ℓ is an n th root of unity, and is not equal to 1 for ℓ in the range specified, the first term in this final expression for $Z(a^\ell)$ is equal to -1 . The second term, $(\rho^{n/2})^\ell$, is clearly $(-1)^\ell$, and so the value that Z takes on a^ℓ is as claimed.

Secondly, we consider the value of $-\Theta$ (note the sign) on b^m . Now write $\rho = \sigma^2$, and this time write $n = (q+1)/2$. Then, the value of $-\Theta$ on b^m becomes

$$\sum_{\alpha=1}^{(n-1)/2} \rho^{\alpha m} + \rho^{(n-\alpha)m} = \sum_{\alpha=1}^{n-1} (\rho^m)^\alpha = -1.$$

Hence the value of Θ on b^m is 1, as required. \square

We will now calculate the character decompositions of various products of characters, chosen specifically for the following sections, when we use the decomposition matrices to determine the decompositions of the products of the irreducible Brauer characters. In the modular setting, we have the equation $\xi_1 + \xi_2 = 1_G + \psi$, and so we do not need separate calculations for ψ . Thus we only concern ourselves with ξ_1 , ξ_2 , the θ_α and the ζ_γ .

Lemma 4.4 We have the following formulae involving products of ξ_1 and ξ_2 :

- (i) $\xi_1^2 = \xi_1 + Z + \psi + 1_G$;
- (ii) $\xi_2^2 = \xi_2 + Z + \psi + 1_G$; and
- (iii) $\xi_1 \xi_2 = \Theta + \psi$.

Proof: All three of these formulae follow easily from the character table of G and Proposition 4.3. \square

We now move onto the θ_α , and its products both with other θ_β and with the ξ -characters. For α even and in the range $(q+3)/2 \leq \alpha \leq q-1$, write θ_α for the class function whose values on the conjugacy classes are those in the character table for G , namely $q-1$, 0 , $-(\sigma^{m\alpha} + \sigma^{-m\alpha})$, -1 and -1 . We have that $\theta_\alpha = \theta_{(q+1)-\alpha}$ for $\alpha \neq (q+1)/2$; of course, since $q \equiv 1 \pmod{4}$, if α is even then $\alpha \neq (q+1)/2$.

Proposition 4.5 We have the following involving products of θ_α and ξ_1 , ξ_2 and θ_β , for any α and $\beta \neq \alpha$:

- (i) $\theta_\alpha^2 + \theta_{2\alpha} = 2\Theta + 2Z + X + \psi + 1_G$;
- (ii) $\theta_\alpha \theta_\beta + \theta_{\alpha+\beta} + \theta_{\alpha-\beta} = \Phi$;
- (iii) $\theta_\alpha \xi_1 = \xi_2 + \psi + \Theta + Z$; and
- (iv) $\theta_\alpha \xi_2 = \xi_1 + \psi + \Theta + Z$.

Proof: The first two equations follow immediately from calculating the character values of both sides of the equation. For the final two formulae, note firstly that

$$\theta_\alpha \xi_1 - \xi_2 = \theta_\alpha \xi_2 - \xi_1.$$

From this it becomes a simple calculation that

$$\theta_\alpha \xi_1 - \xi_2 = \Theta + Z + \psi,$$

which completes the proof. \square

Finally, we need to decompose products of ζ_γ with the other irreducible ordinary characters. Again, we extend our notation, and for γ even with $(q-1)/2 \leq \gamma \leq q-3$, write ζ_γ for the class function with the values as given in the character table, so that it takes the values $q+1$, $\tau^{\ell\gamma} + \tau^{-\ell\gamma}$, 0 , 1 and 1 . In this case, we have that $\zeta_{(q-1)/2} = X$, and $\zeta_\gamma = \zeta_{(q-1)-\gamma}$; for the ζ -characters, it is possible that γ is even and yet $\gamma = (q-1)/2$.

Proposition 4.6 We have the following equations involving products of ζ_γ and the various irreducible ordinary characters:

- (i) $\zeta_\gamma^2 - \zeta_{2\gamma} = 2\Theta + 2Z + X + 3\psi + 1_G$;
- (ii) $\zeta_\gamma \zeta_\delta - \zeta_{\gamma+\delta} - \zeta_{\gamma-\delta} = \Phi$ for $\delta < \gamma$;
- (iii) $\zeta_\gamma \theta_\alpha = \Phi$;
- (iv) $\zeta_\gamma \xi_1 = \xi_1 + \Theta + Z + \psi + \zeta_{(q-1)/2-\gamma}$; and
- (v) $\zeta_\gamma \xi_2 = \xi_2 + \Theta + Z + \psi + \zeta_{(q-1)/2-\gamma}$.

Proof: Unlike the previous proposition, there is a non-trivial calculation in this proof. Again, the first three calculations are simple addition and multiplication, whereas the final two require some manipulation. Firstly, note that

$$\zeta_\gamma \xi_1 - \xi_1 = \zeta_\gamma \xi_2 - \xi_2.$$

Next, we construct the characters given in the table below.

	1	a^ℓ	b^m	c	d
$A = \zeta_\gamma \xi_1 - \xi_1$	$q(q+1)/2$	$(-1)^\ell(\tau^{\gamma\ell} - 1 + \tau^{-\gamma\ell})$	0	0	0
$B = Z + \Theta + \psi$	$(q-2)(q+1)/2$	$-(-1)^\ell$	0	-1	-1
$A - B$	$(q+1)/2$	$(-1)^\ell(\tau^{\gamma\ell} + \tau^{-\gamma\ell})$	0	1	1

To prove the final two equations, it suffices to show that the character whose values are the same as $A - B$ is $\zeta_{(q-1)/2-\gamma}$.

Since $(-1)^\ell = \tau^{\ell(q-1)/2}$, we have

$$(-1)^\ell(\tau^{\gamma\ell} + \tau^{-\gamma\ell}) = \tau^{\ell((q-1)/2-\gamma)} + \tau^{-\ell((q-1)/2-\gamma)}.$$

Thus $A - B = \zeta_{(q-1)/2-\gamma}$, as claimed. \square

4.2 Ordinary Characters for $q \equiv 3 \pmod{4}$

We stick to the same conventions in this section as we did in the previous one, so that conjugacy classes of $\mathrm{PSL}_2(q)$ will be represented by elements of $\mathrm{SL}_2(q)$. Using the calculations above, the conjugacy class representatives are $1, c, d, a^\ell$ and b^m for $1 \leq \ell \leq (q-3)/4$ and $1 \leq m \leq (q+1)/4$.

The character table of $G \cong \mathrm{PSL}_2(q)$ is given below. As above, let τ denote a primitive $(q-1)$ th root of unity, and σ denote a primitive $(q+1)$ th root of unity (both in \mathbb{C}).

G	1	a^ℓ	b^m	c	d
1_G	1	1	1	1	1
ψ	q	1	-1	0	0
ζ_i	$q+1$	$\tau^{i\ell} + \tau^{-i\ell}$	0	1	1
θ_j	$q-1$	0	$-\sigma^{jm} - \sigma^{-jm}$	-1	-1
η_1	$(q-1)/2$	0	$(-1)^{m+1}$	$(-1 + \sqrt{-q})/2$	$(-1 - \sqrt{-q})/2$
η_2	$(q-1)/2$	0	$(-1)^{m+1}$	$(-1 - \sqrt{-q})/2$	$(-1 + \sqrt{-q})/2$

Here, i and j are both even, and satisfy $2 \leq i, j \leq (q-3)/2$. In particular, there are $(q-3)/4$ characters θ_j and the same number of the characters ζ_i . The variables ℓ and m take the values described in the first paragraph of this section.

In calculating these ordinary character decompositions, we will need four more (reducible) ordinary characters, the characters $\eta_1 + \eta_2, \sum \zeta_i, \sum \theta_j$ and Φ .

Proposition 4.7 The four ordinary characters described above have the character values given in the following table.

	1	a^ℓ	b^m	c	d
$E = \eta_1 + \eta_2$	$q-1$	0	$2(-1)^{m+1}$	-1	-1
$Z = \sum \zeta_i$	$(q+1)(q-3)/4$	-1	0	$(q-3)/4$	$(q-3)/4$
$\Theta = \sum \theta_j$	$(q-1)(q-3)/4$	0	$1 - (-1)^{m+1}$	$-(q-3)/4$	$-(q-3)/4$
$\Phi = 2\Theta + 2Z + 2\psi + E$	$(q-1)(q+1)$	0	0	-1	-1

Proof: That E has the value given in the table is obvious, as is Φ , once one accepts the character values for Z and Θ . The non-trivial calculations involved are proving that Z takes the value -1 on a^ℓ , and that Θ takes the value $1 - (-1)^{m+1}$ on b^m . Write $n = (q - 1)/2$.

Consider the sum $\sum \zeta_i$. Since i is even, we may replace i by 2α , and so the sought-after character value on a^ℓ is

$$\sum_{\alpha=1}^{(n-1)/2} \tau^{2\alpha\ell} + \tau^{-2\alpha\ell}.$$

Write $\rho = \tau^2$, a primitive n th root of unity, and also notice that $\rho^{-\alpha\ell} = (\rho^{n-\alpha})^\ell$. Hence the character value on a^ℓ is

$$\sum_{\alpha=1}^{(n-1)/2} (\rho^\alpha)^\ell + (\rho^{n-\alpha})^\ell = \sum_{\alpha=1}^{n-1} (\rho^\ell)^\alpha.$$

Now, for any (not necessarily primitive, but not equal to 1) n th root of unity λ , we have

$$\sum_{i=1}^{n-1} \lambda^i = -1.$$

Since ρ^ℓ is an n th root of unity, and is not equal to 1 for ℓ in the range specified, we see that Z takes the value -1 on a^ℓ .

Finally, we consider the value of $-\Theta$ (note the sign) on b^m . Now write $\rho = \sigma^2$, and this time write $n = (q + 1)/2$. Then, the value of $-\Theta$ on b^m becomes

$$\sum_{\alpha=1}^{n/2-1} \rho^{\alpha m} + \rho^{(n-\alpha)m} = \sum_{\alpha=1}^{n-1} (\rho^m)^\alpha - (\rho^{mn/2}).$$

Now $\rho^{mn/2}$ takes the value 1 when m is even, and takes the value -1 when m is odd. The sum in the final expression takes the value -1 , as it did in the previous case of the character Z , and so the value of $-\Theta$ on b^m becomes $-1 + (-1)^{m+1}$, as required. \square

We will now calculate the character decompositions of various products of characters, chosen specifically for the following sections, when we use the decomposition matrices to determine the decompositions of the products of the irreducible Brauer characters. In the modular setting, we have the equation $1_G + \eta_1 + \eta_2 = \psi$, and so we do not need separate calculations for ψ . Thus we only concern ourselves with η_1 , η_2 , the θ_α and the ζ_γ .

Lemma 4.8 We have the following formulae involving products of η_1 and η_2 :

- (i) $\eta_1^2 = \eta_2 + \Theta$;
- (ii) $\eta_2^2 = \eta_1 + \Theta$; and
- (iii) $\eta_1\eta_2 = Z + 1_G$.

Proof: All three of these formulae follow easily from the character table of G and Proposition 4.7. \square

We now move onto the θ_α , and their products both with other θ_β and with the η -characters. For α even and in the range $(q+1)/2 \leq \alpha \leq q-1$, write θ_α for the class function whose values on the conjugacy classes are those in the character table for G , namely $q-1, 0, -(\sigma^{m\alpha} + \sigma^{-m\alpha}), -1$ and -1 . Then we have that $\theta_{(q+1)/2} = E$, and $\theta_\alpha = \theta_{(q+1)-\alpha}$ for $\alpha \neq (q+1)/2$. These come quite easily from the fact that $\sigma^{(q+1)-\alpha} = \sigma^{-\alpha}$ and $\sigma^{-((q+1)-\alpha)} = \sigma^\alpha$.

Proposition 4.9 We have the following equations involving products of θ_α , and η_1, η_2 , and θ_β , for any α and $\beta \neq \alpha$:

- (i) $\theta_\alpha^2 + \theta_{2\alpha} = 2\Theta + 2Z + E + \psi + 1_G$;
- (ii) $\theta_\alpha\theta_\beta + \theta_{\alpha+\beta} + \theta_{\alpha-\beta} = \Phi$ for $\alpha > \beta$;
- (iii) $\theta_\alpha\eta_1 + \theta_{(q+1)/2-\alpha} = Z + \Theta + \eta_2 + \psi$; and
- (iv) $\theta_\alpha\eta_2 + \theta_{(q+1)/2-\alpha} = Z + \Theta + \eta_1 + \psi$.

Proof: The first two equations are easily shown by proving that their character values are equal. We focus on the final two parts. Firstly notice that $\theta_\alpha\eta_1 - \eta_2 = \theta_\alpha\eta_2 - \eta_1$, and the character is given below, along with the character of $Z + \Theta + \psi$.

	1	a^ℓ	b^m	c	d
$A = \theta_\alpha\eta_1 - \eta_2$	$(q-1)(q-2)/2$	0	$(-1)^m(\sigma^{\alpha m} + 1 + \sigma^{-\alpha m})$	1	1
$B = Z + \Theta + \psi$	$q(q-1)/2$	0	$(-1)^m$	0	0
$B - A$	$q-1$	0	$-(-1)^m(\sigma^{\alpha m} + \sigma^{-\alpha m})$	-1	-1

Now $(-1)^m = \sigma^{m(q+1)/2}$, and so

$$-(-1)^m(\sigma^{\alpha m} + \sigma^{-\alpha m}) = -(\sigma^{m((q+1)/2-\alpha)} + \sigma^{-m(q+1)/2-\alpha}).$$

Thus $B - A = \theta_{(q+1)/2-\alpha}$, as claimed. \square

Finally, we need to decompose products of ζ_γ with the other irreducible ordinary characters. Again, we extend our notation, and for $(q+1)/2 \leq \gamma \leq q-3$, write ζ_γ for the class function with the values as given in the character table, so that it takes the values $q+1$, $\tau^{\ell\gamma} + \tau^{-\ell\gamma}$, 0 , 1 and 1 . Again, we have that $\zeta_\gamma = \zeta_{(q-1)-\gamma}$.

Proposition 4.10 We have the following involving products of ζ_γ and the various irreducible ordinary characters:

- (i) $\zeta_\gamma^2 - \zeta_{2\gamma} = 2\Theta + 2Z + E + 3\psi + 1_G$;
- (ii) $\zeta_\gamma\zeta_\delta - \zeta_{\gamma+\delta} - \zeta_{\gamma-\delta} = \Phi$ for $\delta < \gamma$;
- (iii) $\zeta_\gamma\theta_\alpha = \Phi$;
- (iv) $\zeta_\gamma\eta_1 - \eta_1 = \Theta + Z + \psi$; and
- (v) $\zeta_\gamma\eta_2 - \eta_2 = \Theta + Z + \psi$.

Proof: All of these formulae are simply exercises in multiplication and addition, and there are no non-trivial calculations to perform this time. \square

4.3 Modules for Dihedral Groups

Let x be an involution in $G = \mathrm{PSL}_2(q)$, where $q \equiv 3, 5 \pmod{8}$. Then $C_G(x)$ is a dihedral group of four times odd order. In the sequel, we will need to use the Green correspondence to isolate modules of vertex C_2 in G , and hence this section will briefly describe the indecomposable modules for D_{4n} with vertex $Z = Z(D_{4n})$, where n is odd. Of course, there are only finitely many such modules. Suppose that an indecomposable KG -module M has vertex Z . Then M has trivial source, and so M can be viewed as a projective $C_G(x)/Z$ -module. This quotient group is a dihedral group of twice odd order, and it is these groups whose modules we now describe.

Let L be the dihedral group of order $2n$, where n is odd, and let K be a splitting field of characteristic 2. It is well-known that there are two linear ordinary characters of L and $(n-1)/2$ different 2-dimensional ordinary characters of L . The simple KL -modules are also easy to compute: each of the 2-dimensional $\mathbb{C}L$ -modules reduces modulo 2 to a 2-dimensional projective simple module, and the remaining two 1-dimensional modules both reduce to the same (trivial) module. The projective cover of the trivial module is clearly uniserial and 2-dimensional. Thus there are the projective simple modules, the trivial module, and the projective cover of the trivial module.

Now let H denote the dihedral group of order n , where n is odd, as above. Then the modules of vertex C_2 are just those modules discussed above, viewed as KH -modules. Note also that the tensor product of two of these modules has dimension 4, and splits up as the sum of two other modules of vertex C_2 , since all of these modules are projective for the quotient $K(H/Z(H))$. This information will be invaluable when we start decomposing tensor products of modules.

4.4 Modular Representations for $q \equiv 3 \pmod{8}$

In the case where $q \equiv 3 \pmod{8}$, the decomposition matrix of the principal block, as given in [20], is as follows.

	K	S_1	S_2
1_G	1	0	0
η_1	0	1	0
η_2	0	0	1
ψ	1	1	1

The $\mathbb{C}G$ -modules corresponding to the characters ζ_γ reduce modulo 2 to projective modules Z_γ , and the remaining modules, those corresponding to the θ_α , lie in pairs in blocks of defect 1. Write T_α for the reduction modulo 2 of the module corresponding to the character θ_α . We need to know which of the θ_j lie in the same block. Since b has twice odd order, we should examine the values of the θ_j on the characters b^{2x} for various x ; in fact, $x = 1$ is enough.

Thus consider the complex number $\theta_j(b^2)$, which is $-(\sigma^{2j} + \sigma^{-2j})$. Then

$$\sigma^{2\alpha} + \sigma^{-2\alpha} = \sigma^{2((q+1)/2-\alpha)} + \sigma^{-2((q+1)/2-\alpha)},$$

and these are the only two θ_j that are equal to $\sigma^{2\alpha} + \sigma^{-2\alpha}$. Hence $T_\alpha = T_{(q+1)/2-\alpha}$, and the T_j exist and are uniquely determined for $2 \leq j \leq (q-3)/4$, and j even. Thus there are $(q-3)/4$ modules Z_i and there are $(q-3)/8$ modules T_j .

The decomposition matrix above implies that the projective covers of the three modules in the principal block have socle layers

$$\begin{array}{ccc} S_1 & S_2 & K \\ S_2 \oplus K & S_1 \oplus K & S_1 \oplus S_2 \\ S_1 & S_2 & K \end{array}$$

The T_α have projective covers that consist of two socle layers, each isomorphic with T_α . The structures of these projective modules easily yield the dimensions of the

Ext^1 -spaces between the simple modules. The structures of the projective modules also implies that any KG -module has at most three socle layers, and that if a module has three socle layers, it contains a projective.

The structure of this section is to consider firstly tensor products between the principal block modules S_i , and then to consider tensor products with the modules T_α and both other T_β and those modules from the principal block. Finally, we consider any tensor product involving Z_γ , which will be projective. Any other indecomposable modules we will meet will also have their tensor products with the various simple modules analyzed. In this section, it will transpire that there is exactly one non-simple, non-projective, simply generated module.

We consider tensor products between modules in the principal block. Lemma 4.8(iii) implies that

$$S_1 \otimes S_2 = \bigoplus_i Z_i \oplus K.$$

Since $K \subseteq S_1 \otimes S_2$, we have that $S_2 = S_1^*$. Thus $S_1 \otimes S_1^* = \mathrm{End}_K(S_1)$ is the sum of a projective module and the trivial module; S_1 is an endo-trivial module. By the general theory of endo-trivial modules (see, for example, [23]), we know that the tensor product of any two endo-trivial modules is endo-trivial, and this, together with Lemma 4.8(i)–(ii) implies that we have the decomposition

$$S_1 \otimes S_1 = \bigoplus_j \mathcal{P}(T_j) \oplus S_2,$$

and similarly for $S_2 \otimes S_2$.

Thus we get the following result.

Lemma 4.11 Let M be an indecomposable summand of $S_i^{\otimes j}$ for any j and $i = 1, 2$. Then M is one of S_1 , S_2 , or K , or a projective indecomposable module.

Our next goal is to consider tensor products with the T_α , and so we now examine the structure of $T_\alpha \otimes X$, for X one of the T_β or S_i . Our first aim is to identify the summand of these tensor products that lies in the principal block. From this, we can determine the structure of the rest of the module.

The formulae in Proposition 4.9 severely restrict the structure of the possible indecomposable modules appearing as summands of $T_\alpha \otimes S_1$. Write $\mathrm{cf}(M)$ for the multiset of composition factors of a module M . Write \mathcal{T} and \mathcal{Z} for the multisets

$$\mathcal{T} = \{T_j : 2 \leq j \leq (q-3)/4, q \text{ even}\} \text{ and } \mathcal{Z} = \{Z_i : 2 \leq i \leq (q-3)/2, q \text{ even}\}.$$

Proposition 4.9 then yields the statements

$$\begin{aligned} \mathrm{cf}(T_\alpha \otimes S_1) &= \{2 \cdot S_2, S_1, K\} \cup 2 \cdot \mathcal{T} \cup \mathcal{Z}, \\ \mathrm{cf}(T_\alpha \otimes T_\alpha) \cup \{T_{2\alpha}\} &= \{2 \cdot S_1, 2 \cdot S_2, 2 \cdot K\} \cup 4 \cdot \mathcal{T} \cup 2 \cdot \mathcal{Z}, \text{ and} \\ \mathrm{cf}(T_\alpha \otimes T_\beta) \cup \{T_{\alpha+\beta}\} \cup \{T_{\alpha-\beta}\} &= \{3 \cdot S_1, 3 \cdot S_2, 2 \cdot K\} \cup 4 \cdot \mathcal{T} \cup 2 \cdot \mathcal{Z}, \end{aligned}$$

presuming that $\alpha \neq \beta$. (In these formulae, if \mathcal{X} is a multiset, then $n \cdot \mathcal{X}$ is the multiset containing the same elements as \mathcal{X} , with multiplicity n times the original multiplicity.)

The structure of M , the summand of $T_\alpha \otimes S_i$ that lies in the principal block, is easy to isolate: notice that the composition factors match those of $\mathcal{P}(S_{3-i})$, and by simple manipulation of Hom-spaces, we have the formulae

$$\mathrm{Hom}_{KG}(T_\alpha \otimes S_i, K) = \mathrm{Hom}_{KG}(T_\alpha \otimes S_i, S_i) = 0, \quad \mathrm{Hom}_{KG}(T_\alpha \otimes S_i, S_{3-i}) = K,$$

proving that M is a quotient of $\mathcal{P}(S_{3-i})$; we therefore see that $M \cong \mathcal{P}(S_{3-i})$.

Considering the module $T_\alpha \otimes T_\alpha$, we again label by M the summand of this tensor product lying in the principal block. Suppose firstly that T_α is not self-dual: then K is neither a summand nor a quotient of $T_\alpha \otimes T_\alpha$, and so both copies must lie in the heart (radical modulo socle) of M . Since M has three socle layers, it contains projective summands, and since both copies of K lie in the heart, they must lie in these projective summands. However, since both $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ contain exactly one copy of K in their hearts, there must be two projective summands in M , a contradiction since M contains only six composition factors.

By Theorem 1.20, K is not a summand of $T_\alpha \otimes T_\alpha$, and since S_1 and S_2 have vertex V_4 , they cannot be summands of $T_\alpha \otimes T_\alpha$ either, because T_α has cyclic vertex. However, since T_α is a composition factor of $T_\alpha \otimes S_i$ by Proposition 4.9,

$$\dim_K \mathrm{Hom}_{KG}(T_\alpha \otimes T_\alpha, S_i) = \dim_K \mathrm{Hom}_{KG}(T_\alpha \otimes S_{3-i}, T_\alpha) \geq 1.$$

Thus $\mathrm{soc} M$ contains a copy of each simple module from the principal block, as does $M/\mathrm{rad} M$. Since none of the simple modules is a summand of M , we see that $\mathrm{soc}^2 M = M$; that is, M has no projective summands. Hence all summands of M have vertex C_2 .

Finally, by the results of Section 4.3, we see that G possesses exactly one indecomposable module with vertex C_2 that is not one of the T_α , whence this must be the module M .

We will, from now on, refer to this self-dual indecomposable module as U . This module is the Green correspondent of the unique non-simple module of vertex C_2

in the centralizer of an involution, and has two socle layers, each with a single copy of each simple module from the principal block as factors. Naturally, it is uniquely determined up to isomorphism.

Let M denote the summand of $T_\alpha \otimes T_\beta$ lying in the principal block, where $\alpha \neq \beta$. Since T_α is self-dual,

$$\mathrm{Hom}_{KG}(T_\alpha \otimes T_\beta, K) = 0 = \mathrm{Hom}_{KG}(K, T_\alpha \otimes T_\beta).$$

Thus the two copies of K in this module M must both lie in the heart of M . This implies that M contains two projective summands, consuming eight composition factors. Therefore $M \cong \mathcal{P}(S_1) \oplus \mathcal{P}(S_2)$.

Since the Z_γ are all projective, their structure in the tensor products is obvious, and so it remains to discuss the T_α . They all have vertex isomorphic with C_2 , and since G contains only one class of involutions (which is easily seen from the character table), all of the T_α have the same vertex.

Consider $\mathrm{Hom}_{KG}(T_\alpha \otimes S_i, T_\beta)$: this is 1-dimensional by manipulation of Hom-spaces, and so we get the following result.

Proposition 4.12 Let α be an even number between 2 and $(q-3)/4$. Then

$$\begin{aligned} T_\alpha \otimes S_1 &= T_\alpha \oplus \mathcal{P}(S_2) \oplus \bigoplus_i Z_i \oplus \bigoplus_{\beta \neq \alpha} \mathcal{P}(T_\beta), \text{ and} \\ T_\alpha \otimes S_2 &= T_\alpha \oplus \mathcal{P}(S_1) \oplus \bigoplus_i Z_i \oplus \bigoplus_{\beta \neq \alpha} \mathcal{P}(T_\beta). \end{aligned}$$

We now discuss the remaining summands of $T_\alpha^{\otimes 2}$ and $T_\alpha \otimes T_\beta$. To do this, we need the Green correspondence. Let x be an arbitrary involution; then $L = C_G(x)$ is dihedral of order $q+1$, and has centre $\langle x \rangle$.

Let f denote the Green correspondence. If M and N are two modules with vertex P , then $f(M \otimes N) \equiv f(M) \otimes f(N)$ modulo modules of strictly smaller vertex (see [36, III.5.7]). In our case, this means that

$$f(T_\alpha^{\otimes 2}) \equiv f(T_\alpha)^{\otimes 2} \text{ modulo projectives.}$$

In our case, there are exactly two summands of $T_\alpha \otimes T_\alpha$ with cyclic vertex and the rest are projective. This follows from the similar statement at the end of Section 4.3, that the product of two indecomposable KL -modules with vertex C_2 is the sum of two indecomposable KL -modules with vertex C_2 .

There are already two indecomposable summands of cyclic vertex in $T_\alpha^{\otimes 2}$, namely the modules U and $T_{2\alpha}$, the second appearing because there are an odd number

of copies of that module as a composition factor of $T_\alpha \otimes T_\alpha$. Similarly, there are already two indecomposable modules of cyclic vertex in $T_\alpha \otimes T_\beta$ —the modules $T_{\alpha+\beta}$ and $T_{\alpha-\beta}$ —and so all other summands must be projective. This yields the following proposition.

Proposition 4.13 Let $\alpha > \beta$ be different even numbers between 2 and $(q-3)/4$. Then

$$T_\alpha \otimes T_\alpha = U \oplus 2 \cdot \bigoplus_i Z_i \oplus \bigoplus_j \mathcal{P}(T_j) \oplus \bigoplus_{j \neq \alpha} \mathcal{P}(T_j) \oplus T_{2\alpha},$$

and

$$T_\alpha \otimes T_\beta = \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_i Z_i \oplus \bigoplus_j \mathcal{P}(T_j) \oplus \bigoplus_{j \neq \alpha+\beta, \alpha-\beta} \mathcal{P}(T_j) \oplus T_{\alpha+\beta} \oplus T_{\alpha-\beta}.$$

Lastly, we consider tensor products involving Z_γ . Since this module is projective, any tensor product will be projective, and this fact, together with the formulae in Proposition 4.10, yields the following proposition.

Proposition 4.14 We have the following tensor product decompositions:

$$(i) \quad Z_\gamma \otimes Z_\gamma = Z_{2\gamma} \oplus 2 \cdot \bigoplus_i Z_i \oplus 2 \cdot \bigoplus_j \mathcal{P}(T_j) \oplus \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus \mathcal{P}(K);$$

(ii) for $\delta \neq \gamma$, we have

$$Z_\gamma \otimes Z_\delta = Z_{\gamma+\delta} \oplus Z_{\gamma-\delta} \oplus 2 \cdot \bigoplus_i Z_i \oplus 2 \cdot \bigoplus_j \mathcal{P}(T_j) \oplus \mathcal{P}(S_1) \oplus \mathcal{P}(S_2);$$

$$(iii) \quad Z_\gamma \otimes T_\alpha = 2 \cdot \bigoplus_i Z_i \oplus 2 \cdot \bigoplus_j \mathcal{P}(T_j) \oplus \mathcal{P}(S_1) \oplus \mathcal{P}(S_2);$$

$$(iv) \quad Z_\gamma \otimes S_1 = \mathcal{P}(S_1) \oplus \bigoplus_i Z_i \oplus \bigoplus_j \mathcal{P}(T_j); \text{ and}$$

$$(v) \quad Z_\gamma \otimes S_2 = \mathcal{P}(S_2) \oplus \bigoplus_i Z_i \oplus \bigoplus_j \mathcal{P}(T_j).$$

Proof: The decompositions are obvious outside the principal block; for this block, essentially we have to solve three linear equations in three unknowns, to match up the composition factors. Write x , y and z for the quantities of the projective covers of K , S_1 , and S_2 respectively. From the sets of composition factors— $\mathcal{P}(M)$ contains two copies of M and one each of the other two—we get three equations

$$2x + y + z = a, \quad x + 2y + z = b, \quad x + y + 2z = c,$$

where a , b and c are the number of composition factors isomorphic with K , S_1 , and S_2 respectively. These three equations are linearly independent, and so have a unique solution for each triple (a, b, c) . The solutions given above are easily verified, yielding the result. \square

We have thus determined the structure of $M \otimes N$ for all simple modules M and N . To complete the determination of an arbitrary tensor product of simple modules, we need to determine the tensor product of the indecomposable module U with the simple modules. (Technically, we should also consider the tensor product of the other projective indecomposable modules with the simple modules, but this is easily computable from the decompositions we have given here, together with the fact that the resulting module is projective.)

Proposition 4.15 We have the following tensor product decompositions:

- (i) $U \otimes S_i = U \oplus 2 \cdot \bigoplus_j \mathcal{P}(T_j) \oplus \bigoplus_i Z_i$;
- (ii) $U \otimes T_\alpha = 2 \cdot T_\alpha \oplus 2 \cdot \mathcal{P}(T_\alpha) \oplus 4 \cdot \bigoplus_j \mathcal{P}(T_j) \oplus 4 \cdot \bigoplus_i Z_i \oplus 2 \cdot \mathcal{P}(S_1) \oplus 2 \cdot \mathcal{P}(S_2)$.
- (iii) $U \otimes Z_\gamma = 2 \cdot Z_\gamma \oplus 4 \cdot \bigoplus_j \mathcal{P}(T_j) \oplus 4 \cdot \bigoplus_i Z_i \oplus 2 \cdot \mathcal{P}(S_1) \oplus 2 \cdot \mathcal{P}(S_2)$.

Proof: We begin with the final formula, which comes from the fact that

$$U \otimes Z_\gamma = 2 \cdot (K \otimes Z_\gamma \oplus S_1 \otimes Z_\gamma \oplus S_2 \otimes Z_\gamma).$$

The second formula can be proved using the Green correspondence: as before, in the tensor product $U \otimes T_\alpha$, exactly two summands have vertex C_2 . Examining the tensor products of T_α with the composition factors of U , one sees that all summands are projective except for six copies of T_α . Since $T_\alpha \otimes U$ must possess exactly two summands of vertex C_2 , these six composition factors isomorphic with T_α must be distributed as given in the formula.

Consider the module $U \otimes S_i$: as U has vertex C_2 , all summands of this product are either projective or have vertex C_2 . Examining the decompositions of the tensor products of S_i with the composition factors of U , we see that all of the projective modules given in the formula above must appear. The remaining composition factors are two copies each of K , S_1 , and S_2 . Since there are only four indecomposable modules in the principal block— $\mathcal{P}(K)$, $\mathcal{P}(S_1)$, $\mathcal{P}(S_2)$ and U —whose vertex is C_2 or trivial, we immediately see that these six composition factors must form a copy of U , as required. \square

This implies the following theorem.

Theorem 4.16 Let M be an indecomposable simply generated KG -module, where $G = \mathrm{PSL}_2(q)$, for $q \equiv 3 \pmod{8}$. Then M is isomorphic with a simple module, a projective indecomposable module, or the self-dual module U defined above, with two socle layers, each possessing a copy of K , S_1 , and S_2 . Furthermore, all tensor products of arbitrarily many simple modules can be decomposed using the decompositions given in this section.

4.5 Modular Representations for $q \equiv 5 \pmod{8}$

This case is similar to $q \equiv 3 \pmod{8}$, where only four ordinary irreducible characters lie in the principal block. Thus let $G \cong \mathrm{PSL}_2(q)$, with $q \equiv 5 \pmod{8}$. We again take the information on the decomposition matrices from [20].

	K	S_1	S_2
1_G	1	0	0
ξ_1	1	1	0
ξ_2	1	0	1
ψ	1	1	1

The $\mathbb{C}G$ -modules corresponding to the characters θ_α reduce modulo 2 to projective modules T_α , and the remaining modules, those corresponding to the ζ_γ , lie in pairs in blocks of defect 1. Write Z_γ for the reduction modulo 2 of the module corresponding to the character ζ_γ . We need to know which of the ζ_i lie in the same block. Since a has twice odd order, we should examine the values of the ζ_i on the characters a^{2x} for various x ; in fact, $x = 1$ is enough.

Thus consider the complex number $\zeta_i(a^2)$, which is $\tau^{2i} + \tau^{-2i}$. Then

$$\tau^{2\alpha} + \tau^{-2\alpha} = \tau^{2((q-1)/2-\alpha)} + \tau^{-2((q-1)/2-\alpha)},$$

and these are the only two ζ_i whose character value is equal to $\tau^{2\alpha} + \tau^{-2\alpha}$. Hence $Z_\gamma = Z_{(q-1)/2-\gamma}$, and the Z_i exist and are uniquely determined for $2 \leq i \leq (q-5)/4$, and i even.

As calculated in [32], the projective covers of the modules K , S_1 , and S_2 are given by

$$\mathcal{P}(S_1) = \begin{array}{c} S_1 \\ K \\ S_2 \\ K \\ S_1 \end{array} \quad \mathcal{P}(S_2) = \begin{array}{c} S_2 \\ K \\ S_1 \\ K \\ S_2 \end{array} \quad \mathcal{P}(S_K) = \begin{array}{c} K \\ S_1 \\ K \\ S_2 \\ K \end{array} \oplus \begin{array}{c} S_2 \\ K \\ S_1 \end{array}.$$

In particular, $\mathrm{Ext}_{KG}^1(K, K) = \mathrm{Ext}_{KG}^1(S_i, S_j) = 0$ for all i and j .

Again, we will start with the modules in the principal block. The equations

$$\begin{aligned}(\xi_1 - 1_G)^2 &= Z + (\xi_2 - 1_G) + 2 \cdot 1_G, \\(\xi_2 - 1_G)^2 &= Z + (\xi_1 - 1_G) + 2 \cdot 1_G, \text{ and} \\(\xi_1 - 1_G)(\xi_2 - 1_G) &= \Theta,\end{aligned}$$

yield the composition factors of the modules $S_i \otimes S_j$. The decomposition of $S_1 \otimes S_2$ is immediate.

Lemma 4.17 $S_1 \otimes S_2 = \bigoplus_j T_j$.

Proof: The modules T_α are projective simple modules, and so the right-hand side is the only possible module with character Θ . \square

Slightly more difficult is decomposing $S_i \otimes S_i$. The lemma just given proves that the S_i are self-dual, and so $\mathrm{Hom}_{KG}(S_1 \otimes S_1, K) = K$. The composition factors of $S_i \otimes S_i$ are two copies of K , one copy of S_{3-i} , and two copies of each Z_γ . Clearly

$$\mathrm{Hom}_{KG}(S_i \otimes S_i, S_{3-i}) = \mathrm{Hom}_{KG}(S_i, S_i \otimes S_{3-i}) = 0,$$

and so M , the summand of $S_i \otimes S_i$ lying in the principal block, is a quotient of $\mathcal{P}(K)$. This implies that M is uniserial, with socle layers containing K , S_{3-i} and K respectively.

It remains to discuss the summands of $S_1 \otimes S_1$ lying outside the principal block; these are either $Z_\gamma \oplus Z_\gamma$ or $\mathcal{P}(Z_\gamma)$ for each γ .

Lemma 4.18 We have

$$S_1 \otimes S_1 = \begin{array}{c} K \\ S_2 \\ K \end{array} \oplus \bigoplus_i \mathcal{P}(Z_i),$$

and similarly

$$S_2 \otimes S_2 = \begin{array}{c} K \\ S_1 \\ K \end{array} \oplus \bigoplus_i \mathcal{P}(Z_i).$$

Proof: We need to prove that $\mathrm{Hom}_{KG}(S_1 \otimes S_1, Z_\gamma) = K$. Note firstly that $Z_\gamma \otimes S_1$ contains exactly two copies of S_1 , by Proposition 4.6. If $\dim_K \mathrm{Hom}_{KG}(S_1 \otimes S_1, Z_\gamma) = 2$, then there must be a summand of $Z_\gamma \otimes S_1$ isomorphic with S_1 . However, S_1 has vertex V_4 by Theorem 1.24, whereas all summands of $S_1 \otimes Z_\gamma$ have vertices contained within C_2 , a contradiction. The result now follows. \square

This uniserial module lying in the principal block, with heart S_i , will be referred to as S'_i . We are interested in its products with S_1 and S_2 . The one is easy to understand: $S'_1 \otimes S_2 = 2 \cdot S_2 \oplus \bigoplus_j T_j$ is the only possibility, since $S_1 \otimes S_2$ is projective, and S_2 has no self-extensions. The other is not significantly more difficult: we have

$$\begin{aligned} \mathrm{Hom}_{KG}(S'_1 \otimes S_1, K) &= \mathrm{Hom}_{KG}(S'_1 \otimes S_1, S_2) = 0, \text{ and} \\ \mathrm{Hom}_{KG}(S'_1 \otimes S_1, S_1) &= \mathrm{Hom}_{KG}(S'_1, S'_2) = K. \end{aligned}$$

Hence the summand of $S'_1 \otimes S_1$ lying in the principal block is a quotient of $\mathcal{P}(S_1)$, and since they have the same number of composition factors, we get the following result.

Lemma 4.19 We have the equations

$$S'_i \otimes S_i = \mathcal{P}(S_i) \oplus \bigoplus_i \mathcal{P}(Z_i),$$

and

$$S'_i \otimes S_{3-i} = 2 \cdot S_{3-i} \oplus \bigoplus_j T_j.$$

This yields the result that all of the simple modules in the principal block are algebraic, and in particular, it determines the summands of $T(S_i)$.

Proposition 4.20 The modules S_i are algebraic, and the module $T(S_i)$ contains the non-isomorphic indecomposable summands K , S_i , S'_{3-i} and all projective indecomposable modules.

Next we consider the Z_γ , and their products with the S_i and the other Z_δ and to facilitate this discussion, we write \mathcal{T} and \mathcal{Z} for the multisets

$$\mathcal{T} = \{T_j : 2 \leq j \leq (q-1)/2, q \text{ even}\} \text{ and } \mathcal{Z} = \{Z_i : 2 \leq i \leq (q-5)/4, q \text{ even}\}.$$

Recall that $\mathrm{cf}(M)$ denotes the composition factors of the module M . The character calculations of Proposition 4.6 yield the composition factor equations

$$\begin{aligned} \mathrm{cf}(Z_\gamma \otimes S_1) &= \{2 \cdot S_1, 2 \cdot K, S_2\} \cup \mathcal{T} \cup 2 \cdot \mathcal{Z}, \\ \mathrm{cf}(Z_\gamma \otimes Z_\gamma) &= \{6 \cdot K, 4 \cdot S_1, 4 \cdot S_2\} \cup \{Z_{2\gamma}\} \cup 2 \cdot \mathcal{T} \cup 4 \cdot \mathcal{Z}, \text{ and} \\ \mathrm{cf}(Z_\gamma \otimes Z_\delta) &= \{4 \cdot K, 3 \cdot S_1, 3 \cdot S_2\} \cup \{Z_{\gamma+\delta}, Z_{\gamma-\delta}\} \cup 2 \cdot \mathcal{T} \cup 4 \cdot \mathcal{Z}. \end{aligned}$$

We aim to understand the module $Z_\gamma \otimes Z_\gamma$; similarly to the previous section, the centralizer of an involution in G is a dihedral group of four times odd order, and in Section 4.3 we determined the indecomposable modules with vertex C_2 . Each of them

has the property that the tensor square is the sum of two indecomposable modules, each with vertex C_2 . Since Z_γ has vertex C_2 , there are exactly two non-projective summands of $Z_\gamma^{\otimes 2}$, and these have vertex C_2 , by the Green correspondence.

Since there are five copies of $Z_{2\gamma}$, at least one of these is a summand. Thus it remains to find one other non-projective summand. There is exactly one remaining non-projective summand, and so all of the composition factors Z_δ must lie in projective modules. Thus the non-projective summand must come from the principal block. The number of modules of the centralizer of an involution with vertex C_2 is one more than the number of the Z_γ : this summand is uniquely determined by being the only non-simple indecomposable module with vertex C_2 .

Recall that G has a permutation representation on $q+1$ points. Let V denote the associated permutation module. This module is known to be indecomposable (see, for example, [30]), with socle structure

$$\begin{array}{c} K \\ S_1 \oplus S_2. \\ K \end{array}$$

Since $q+1$ is even but not a multiple of 4, the module V has vertex C_2 , and so this must be the remaining indecomposable module with vertex C_2 . Hence $V \oplus Z_{2\gamma} | Z_\gamma \otimes Z_\gamma$, and all remaining summands are projective. By considering composition factors, it is easy to see that

$$Z_\gamma \otimes Z_\gamma = V \oplus \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i) \oplus Z_{2\gamma}.$$

Consider $Z_\gamma \otimes Z_\delta$, where $\gamma > \delta$; this behaves in a similar way to $Z_\gamma \otimes Z_\gamma$. Since there are five composition factors isomorphic with $Z_{\gamma+\delta}$ and five with $Z_{\gamma-\delta}$, the two indecomposable non-projective summands are $Z_{\gamma+\delta}$ and $Z_{\gamma-\delta}$, and all remaining summands must be projective. This uniquely determines the decomposition, as

$$Z_\gamma \otimes Z_\delta = \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i) \oplus Z_{\gamma+\delta} \oplus Z_{\gamma-\delta}.$$

The last module decomposition in this group is $Z_\gamma \otimes S_i$. We have the easy equations

$$\mathrm{Hom}_{KG}(Z_\gamma \otimes S_i, K) = \mathrm{Hom}_{KG}(Z_\gamma \otimes S_i, S_{3-i}) = 0, \quad \mathrm{Hom}_{KG}(Z_\gamma \otimes S_i, S_i) = K,$$

and since the composition factors of $Z_\gamma \otimes S_i$ lying in the principal block are exactly those of $\mathcal{P}(S_i)$, we see that $\mathcal{P}(S_i)$ is the only summand of $Z_\gamma \otimes S_i$ lying in the principal block.

Note that for all δ ,

$$\mathrm{Hom}_{KG}(Z_\gamma \otimes S_i, Z_\delta) = \mathrm{Hom}_{KG}(Z_\gamma \otimes Z_\delta, S_i) = K,$$

and so it is $\mathcal{P}(Z_\delta)$, not $Z_\delta \oplus Z_\delta$, that is a summand of $Z_\gamma \otimes S_i$. We collate these results.

Proposition 4.21 Let Z_γ be one of the $(q+1)$ -dimensional simple modules. Then:

- (i) $Z_\gamma \otimes S_1 = \mathcal{P}(S_1) \oplus \bigoplus_j T_j \oplus \bigoplus_i \mathcal{P}(Z_i)$;
- (ii) $Z_\gamma \otimes S_2 = \mathcal{P}(S_2) \oplus \bigoplus_j T_j \oplus \bigoplus_i \mathcal{P}(Z_i)$;
- (iii) $Z_\gamma \otimes Z_\gamma = V \oplus \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i) \oplus Z_{2\gamma}$; and
- (iv) $Z_\gamma \otimes Z_\delta = \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i) \oplus Z_{\gamma+\delta} \oplus Z_{\gamma-\delta}$.

We next consider the projective modules $T_\alpha \otimes M$ for the simple modules M . In a similar way to the case where $q \equiv 3 \pmod{8}$, these are essentially linear equations in the composition factors, and their proofs are suppressed.

Proposition 4.22 We have the following tensor product decompositions:

(i)

$$T_\alpha \otimes T_\alpha = \mathcal{P}(K) \oplus 2 \cdot \bigoplus_{j \neq 2\alpha} T_j \oplus T_{2\alpha} \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i);$$

(ii) for $\beta < \alpha$, we have

$$T_\alpha \otimes T_\beta = \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_{j \neq \alpha+\beta, \alpha-\beta} T_j \oplus T_{\alpha+\beta} \oplus T_{\alpha-\beta} \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i);$$

(iii) for all α and γ ,

$$T_\alpha \otimes Z_\gamma = \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i);$$

(iv) $T_\alpha \otimes S_1 = \mathcal{P}(S_2) \oplus \bigoplus_{j \neq \alpha} T_j \oplus \bigoplus_i \mathcal{P}(Z_i)$; and

(v) $T_\alpha \otimes S_2 = \mathcal{P}(S_1) \oplus \bigoplus_{j \neq \alpha} T_j \oplus \bigoplus_i \mathcal{P}(Z_i)$.

In particular, $T(T_\alpha)$ contains exactly the projective indecomposable modules, and T_α is algebraic.

We have determined the structure of $M \otimes N$ for any two simple modules M and N ; to complete the determination of the structure of the tensor product of any collection of simple modules, we need to understand the tensor products of the non-simple indecomposable modules S'_i and V with all simple modules.

We begin with the two modules S'_i .

Proposition 4.23 We have the following decompositions:

- (i) $S'_i \otimes S_i = \mathcal{P}(S_i) \oplus \bigoplus_i \mathcal{P}(Z_i)$;
- (ii) $S'_i \otimes S_{3-i} = 2 \cdot S_{3-i} \oplus \bigoplus_j T_j$;
- (iii) $S'_i \otimes Z_\gamma = \mathcal{P}(S_i) \oplus \bigoplus_j T_j \oplus \bigoplus_i \mathcal{P}(Z_i) \oplus \mathcal{P}(Z_\gamma)$; and
- (iv) $S'_i \otimes T_\alpha = \mathcal{P}(S_{3-i}) \oplus \bigoplus_j T_j \oplus T_\alpha \oplus \bigoplus_i \mathcal{P}(Z_i)$.

Proof: The first two parts of the result are Lemma 4.19. The final part can be seen by combining the expressions for the tensor product of T_α with the composition factors of S'_i given above, remembering that T_α is projective. The third equation holds since $S'_i \otimes Z_\gamma$ is projective; to see this, note that $S'_i|_{S_{3-i} \otimes S_{3-i}}$, and $S_{3-i} \otimes Z_\gamma$ is projective. \square

Finally, we consider decompositions of tensor products containing V .

Proposition 4.24 We have the following decompositions:

- (i) $V \otimes S_i = \mathcal{P}(S_i) \oplus \bigoplus_j T_j \oplus \bigoplus_i \mathcal{P}(Z_i)$;
- (ii) $V \otimes Z_\gamma = 2 \cdot Z_\gamma \oplus \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i)$; and
- (iii) $V \otimes T_\alpha = \mathcal{P}(S_1) \oplus \mathcal{P}(S_2) \oplus 2 \cdot \bigoplus_j T_j \oplus 2 \cdot \bigoplus_i \mathcal{P}(Z_i)$.

Proof: The module V is a summand of $Z_\gamma \otimes Z_\gamma$, and so $V \otimes S_i$ lies inside the triple tensor product $Z_\gamma \otimes Z_\gamma \otimes S_i$. This tensor product is projective since $S_i \otimes Z_\gamma$ is, and so $V \otimes S_i$ is projective. The first result now follows by comparing composition factors; there is a single possibility for the structure of $V \otimes S_i$ given that it is projective.

To see the second part, note that $V \otimes Z_\gamma$ must have two non-projective summands, because of the Green correspondence. Then, the tensor product of Z_γ and S_i is projective, and so the only remaining composition factors of $V \otimes Z_\gamma$ not already definitely inside a projective module are the two copies of Z_γ got from tensoring Z_γ with the two copies of K . Hence these must be summands.

The third part of this result follows again since it is projective, and combining the decompositions given above for tensor products of T_α with the composition factors of V . All parts of the proposition have now been proved. \square

This implies the following theorem.

Theorem 4.25 Let M be an indecomposable simply generated KG -module, where $G = \mathrm{PSL}_2(q)$, for $q \equiv 5 \pmod{8}$. Then M is isomorphic with a simple module, a projective indecomposable module, one of the two uniserial modules S'_i , or the self-dual module V defined above, the permutation module got from the permutation representation on the projective line. Furthermore, all tensor products of arbitrarily many simple modules can be decomposed using the decompositions given in this section.

4.6 Modular Representations for $q \equiv 7 \pmod{8}$

In this short section we will deal with the groups $\mathrm{PSL}_2(q)$ where $q \equiv 7 \pmod{8}$. Since this is the only negative result in this article, we will present another proof that the non-trivial modules in the principal block of $G = \mathrm{PSL}_2(q)$ are not algebraic when $q \equiv 7 \pmod{8}$. We firstly reproduce the decomposition matrix for the principal block of KG .

	K	S_1	S_2
1_G	1	0	0
η_1	0	1	0
η_2	0	0	1
ψ	1	1	1
θ_j	0	1	1

Using Lemma 4.8, we again see that all of the simple modules in the principal block are indeed endo-trivial. However, S_1 and S_2 are no longer of finite order; that is, there is no integer $n \neq 0$ such that

$$S_1^{\otimes n} = K \oplus P,$$

where P is projective. (This is also true for $S_2 = S_1^*$.)

To see this, write Q for a Sylow 2-subgroup. We firstly note that $S_1 \downarrow_Q$ is not trivial modulo projectives. This is simply because S_1 has dimension $(q-1)/2$, which is not congruent to 1 modulo $|Q|$, as $|Q|$ has order at least 8. Hence $S_1 \downarrow_Q$, modulo projectives, is not the trivial element of the group of endo-trivial modules. In [23],

Carlson and Thévenaz prove (although they note that it is well-known) that the dihedral groups do not have any non-trivial, torsion endo-trivial modules, thus proving that $S_1 \downarrow_Q^{\otimes i}$ contains infinitely many indecomposable summands as i ranges over all positive integers, namely the elements of the subgroup of the group of endo-trivial modules generated by $S_1 \downarrow_Q$. Hence S_1 is not algebraic.

However, since the modules lying outside the principal block have cyclic (or trivial) vertex, the T_α and Z_γ are still algebraic. Thus we get the following result.

Proposition 4.26 Let $q \equiv 7 \pmod{8}$, and $G \cong \mathrm{PSL}_2(q)$. Then all but two simple KG -modules are algebraic, the remaining two being non-algebraic endo-trivial modules of dimension $(q - 1)/2$.

4.7 Block Invariants Determining Algebraicity

The question of how the structure of a block B of KG affects the algebraicity of the simple B -modules is a subtle one, as these examples demonstrate. The defect group D of the block B plays an important rôle: if D is cyclic, then all simple B -modules are algebraic, and if $D \cong V_4$, then, conjecturally at least, all simple B -modules are algebraic. If D is dihedral of order 8, however, non-algebraic simple B -modules can be found. If $G \cong \mathrm{PSL}_2(9)$, and B is the principal block, then all simple B -modules are algebraic, whereas if $H \cong \mathrm{PSL}_2(7)$, and B' is the principal block, then two of the three simple B' -modules are non-algebraic. Thus the defect group alone is not enough to determine algebraicity of simple B -modules.

Several deep block-theoretic conjectures about modules can be expressed in terms of the fusion system of the block involved, and so perhaps this can be of help. However, since all involutions are conjugate in both G and H , the fusion systems $\mathcal{F}_D(G)$ and $\mathcal{F}_D(H)$ are isomorphic, and so the fusion systems of the principal blocks B and B' are also isomorphic.

Delving deeper into the block-theoretic structure of B and B' , we note that the decomposition matrices of the two blocks are different. Furthermore, the principal block of A_7 , whose simple modules are also algebraic, has the same decomposition numbers as B . On a related note, if L is a soluble group and b is a block with dihedral defect group of order at least 8, then b has at most two simple modules (see [34]), so that b cannot have the same decomposition matrix as the principal block of H . We see the following result.

Theorem 4.27 Let K be a field of characteristic 2, and let G be a simple group with dihedral Sylow 2-subgroups of order at least 8, and denote by B the principal block of KG . Then B possesses three simple modules, labelled S_1 , S_2 , and the trivial module K . Furthermore, S_1 and S_2 are algebraic if and only if they are not the reduction modulo 2 of simple $\mathbb{C}G$ -modules. Thus the algebraicity of the simple B -modules is determined by the decomposition numbers of the principal block.

Now consider the central extension $3 \cdot A_6 \cong 3 \cdot \mathrm{PSL}_2(9)$, and its modules over $\mathrm{GF}(4)$. There are two (dual) non-principal blocks with Sylow 2-subgroups as defect groups, and these two blocks each contain three simple modules. Two of these modules have dimension 3 and are non-algebraic, but the 9-dimensional module has trivial source and is hence algebraic. This block has the same decomposition matrix as the principal block of $\mathrm{PSL}_2(7)$, and the bijection between the simple modules that realizes this decomposition matrix equality maps the unique algebraic module in the one block to that of the other.

Finally, consider the central extension $3 \cdot A_7$, which also possesses a non-principal 2-block with Sylow 2-subgroups as defect groups. This block contains a 6-dimensional module and a 15-dimensional module, both of which are algebraic. The following is perhaps a rash conjecture.

Conjecture 4.28 Let B be a block of a group algebra with dihedral defect group. Then

- (i) if $\ell(B) \leq 2$, then the simple B -modules are algebraic, and
- (ii) if $\ell(B) = 3$, then if B' is some other block of a finite group with dihedral defect group, then B and B' have the same number of algebraic simple modules if B is Morita equivalent to B' . In particular, B contains at least one algebraic simple module.

This conjecture has now been verified for all soluble groups and all quasisimple groups with dihedral Sylow 2-subgroup.

Since the Puig equivalence class of a block determines the sources of the simple modules, the algebraicity of modules is determined by the Puig equivalence type. Thus the algebraicity of modules is controlled by some block invariant. However, Puig equivalence is very strict, and it might be of interest to see whether less strict equivalence relations on the blocks still preserve algebraicity of simple modules.

Chapter 5

The Groups of Lie Type

For most simple groups, the Green ring structure is far too complicated, and the simple modules far too numerous, to expect the SMA property to hold. However, the possibility of finding an example of a simple group with SMA where the Sylow p -subgroup is still quite complicated, is tempting enough to attempt to answer the SMA question for arbitrary simple groups.

An idea of just how quickly the Green ring structure becomes unmanageable is the fact that $\mathrm{GL}_3(p)$ does not possess the SMA property in defining characteristic. We can use this fact to prove that the natural module is non-algebraic for all groups of Lie type for all sufficiently large ranks. Here we will only consider the classical groups and the Ree groups ${}^2G_2(q)$, although similar calculations have been performed for the other Ree groups, and can in principle be performed for the other exceptional groups.

Some discussion of the non-defining characteristic representation theory of these groups is provided, in the cases of $\mathrm{PSL}_3(q)$, $\mathrm{PSU}_3(q)$ and ${}^2G_2(q)$.

5.1 The Natural Module for $\mathrm{GL}_3(p)$

As we said in the introduction, in [17], Berger stated that the natural module for $\mathrm{GL}_3(p)$ was non-algebraic, and that this was ‘well-known’. However, the author has been unable to find a proof of this fact in the literature. Consequently, the author has developed his own proof, based on the methods in this thesis. We begin by recalling Jennings’ theorem on the group algebras of p -groups.

Let P be a finite p -group. Define the *dimension subgroups*

$$\Delta_i(P) = [P, \Delta_{i-1}(P)]\Delta_{\lceil i/p \rceil}(P)^p.$$

This is the fastest-decreasing central series whose quotients $\Delta_i(P)/\Delta_{i+1}(P)$ are elementary abelian p -groups.

Theorem 5.1 (Jennings, [54]) Let P be a p -group and $K = \text{GF}(p)$. Denote by KP the group algebra of P over K .

(i) Let $A_i(P)$ be defined by

$$A_i(P) = \{g \in P : g - 1 \in \text{rad}^i(KP)\}.$$

Then $A_i(P) = \Delta_i(P)$.

(ii) Suppose that we choose $x_{i,j} \in P$ such that $x_{i,j}\Delta_{i+1}$ form a basis of Δ_i/Δ_{i+1} . Write $X_{i,j} = x_{i,j} - 1$. Then

$$\prod_{i,j} X_{i,j}^{\alpha_{i,j}}, \quad 0 \leq \alpha_{i,j} \leq p - 1$$

generate KP . Furthermore, if the weight of such a product is defined to be $\sum_{i,j} i\alpha_{i,j}$, then all products of weight i form a basis of $\text{rad}^{i-1}(KP)/\text{rad}^i(KP)$, and all products of weight at most i form a basis for $KP/\text{rad}^i(KP)$.

Now let $P = C_p \times C_p$ be the elementary abelian group of order p^2 , and let $K = \text{GF}(p)$. Write $M_i = KP/\text{rad}^i(KP)$. Then Jennings' theorem immediately implies the following result.

Proposition 5.2 (i) The module KP has $2p - 1$ radical layers.

(ii) The module M_i has dimension $i(i + 1)/2$ if $i \leq p$.

(iii) The module M_i is spanned by all monomials in X and Y of degree at most $i - 1$, for $i \leq p$.

In particular, (iii) of this proposition implies the next lemma.

Lemma 5.3 Let $1 \leq i \leq p - 1$ be an integer. Then $S^i(M_2) = M_{i+1}$.

Proof: This is obvious if one remembers that $S^i(M_2)$ is spanned by all monomials of degree i in the basis elements of M_2 , which are $1, X$ and Y . Thus $S^i(M_2)$ is spanned by all monomials in X and Y of degree at most i . \square

Finally, recall that if $i < p$ and K is a field of characteristic p , then for any KG -module M , the module $S^i(M)$ is a summand of $M^{\otimes i}$.

Proposition 5.4 The 3-dimensional KP -module M_2 is non-algebraic.

Proof: Firstly, notice that both $S_1 = S^{p-2}(M_2)$ and $S_2 = S^{p-1}(M_2)$ are summands of $M_2^{\otimes(p-2)}$ and $M_2^{\otimes(p-1)}$ respectively, and hence if at least one of S_1 and S_2 is non-algebraic, then M_2 is non-algebraic and the proposition follows.

To see that not both of S_1 and S_2 are algebraic, we simply note that

$$S_2 = \Omega(S_1)^*.$$

Thus by Theorem 2.11, either S_1 or S_2 , and consequently M_2 , is non-algebraic. \square

This proposition has two obvious corollaries.

Corollary 5.5 The elementary abelian group P has non-algebraic modules of every dimension at least 3 over $\text{GF}(p)$.

Corollary 5.6 The natural module for $\text{GL}_3(p)$ is non-algebraic.

5.2 Special Linear Groups

In defining characteristic, the behaviour of the natural module is relatively easy to state.

Proposition 5.7 Let q be a power of p , and let G be the group $\text{SL}_n(q)$, where $n \geq 3$. Then the natural module for G is non-algebraic.

Proof: The natural module for $\text{SL}_n(q)$ restricts to the subgroup $\text{SL}_n(p)$ as the natural module for this group, so it suffices to prove the result for this group. Since the natural module for $\text{SL}_n(p)$ is algebraic if and only if the natural module for $\text{GL}_n(p)$ is algebraic (by Proposition 2.25), it suffices to find a non-algebraic module of dimension n over $\text{GF}(p)$. This is assured by Corollary 5.5. \square

If $n = 2$, then the natural module, and indeed all simple modules, are algebraic. This is obvious for $\text{SL}_2(p)$ (since a Sylow p -subgroup of this group is cyclic) but for other groups $\text{SL}_2(q)$, it is a more difficult result. The author will release details of the proof in a forthcoming paper.

In non-defining characteristic, there is no obvious method by which one can perform induction. The groups $\text{SL}_2(q)$ were analyzed in Chapter 4, and so here we consider $n \geq 3$.

The order of $G = \mathrm{SL}_3(q)$ is $q^3(q-1)^2(q+1)(q^2+q+1)$. Let p be a prime that does not divide q . We claim that if P is a Sylow p -subgroup of G and p is neither 2 nor 3, then either $p|(q-1)$ and P is $C_{p^a} \times C_{p^a}$, where p^a is the highest power of p dividing $(q-1)$, or P is cyclic.

To see this, first note that there are elements in G of orders $q-1$, $q+1$ and q^2+q+1 . Then, if p divides both $q-1$ and $q+1$ then $p=2$, and if p divides both $q+1$ and q^2+q+1 then $p|q$ and so $p|1$, a contradiction. If p divides both $q-1$ and q^2+q+1 then $p|3$, and so we get that either $p=2$, $p=3$, $p|(q-1)$, or P is cyclic. Finally, a maximal torus of G has the form $C_{q-1} \times C_{q-1}$, and the result follows.

Thus we focus our attention on the primes 2 and 3, and those that divide $q-1$. We will deal with the prime 2 first, at least for those groups for which $q \equiv 3 \pmod{4}$. In this case, the Sylow 2-subgroup P (of order 2^n) is semidihedral (see Theorem 1.28), and in [31], Erdmann calculates the vertices and sources of all modules in blocks of full defect. Indeed, there are three simple modules in the principal block, and two modules in all non-principal blocks. In Erdmann's notation, these are modules F , S and E in the principal block, and F_i and S_i in the non-principal blocks of full defect.

The modules F and F_i have P as vertex, and are trivial source. The module E has a V_4 subgroup as vertex and again, trivial source. The modules S and S_i have source a uniserial module of dimension $2^{n-3} - 1$, and vertex the quaternion 2-group of order 2^{n-1} .

If $q \equiv 3 \pmod{8}$, then 4 is the highest power of 2 that divides $(q-1)^2$ and the same is true for $q+1$, and so $n=4$ in this case. Hence all simple modules in blocks of full defect have trivial source, and so are algebraic. There may be blocks of V_4 defect group, and the author has not conclusively studied this possibility, although they appear to all be real 2-blocks. The remaining blocks are all projective.

Now suppose that $q \equiv 7 \pmod{8}$. In this case, the integer n is at least 5, and the simple module S has a non-trivial source for a quaternion group Q . Although the author has not proved this for all possible q , for $q=7$ it is true that this module is acted upon trivially by the centre of its vertex, and so it is really an odd-dimensional indecomposable module for the dihedral group $Q/Z(Q)$. It appears, therefore, as though the following is true.

Conjecture 5.8 Let $G = \mathrm{PSL}_3(q)$, where $q \equiv 3 \pmod{4}$.

- (i) If $q \equiv 3 \pmod{8}$, then all simple modules are algebraic.
- (ii) If $q \equiv 7 \pmod{8}$, then there is a non-algebraic simple module lying in each 2-block of full defect.

This fits neatly with the case of the groups $\mathrm{PSL}_2(q)$ where $q \equiv 3 \pmod{4}$. We will see in Section 5.6 that the result for unitary groups $\mathrm{PSU}_3(q)$ with $q \equiv 1 \pmod{4}$ fits with the case of the groups $\mathrm{PSL}_2(q)$ where $q \equiv 1 \pmod{4}$.

If $q \equiv 1 \pmod{4}$, then the Sylow 2-subgroup is of wreathed type (i.e., it has the form $C_{2^n} \wr C_2$). In this case, the vertices and sources are not known, and this appears to be a subject for future research.

The next prime on our list is $p = 3$.

Proposition 5.9 Let G be the group $\mathrm{PSL}_3(q)$, and let P denote the Sylow 3-subgroup of G .

- (i) If $q \equiv 2 \pmod{3}$, then P is cyclic.
- (ii) If $q \equiv 1 \pmod{3}$ and $q \not\equiv 1 \pmod{9}$, then $P = C_3 \times C_3$.

The smallest group not considered so far is $\mathrm{PSL}_3(4)$ in characteristic 3. This group has five simple modules in the principal block, and all of them are algebraic. Apart from the trivial module, there are three 15-dimensional modules, labelled M_i , and a 19-dimensional module labelled N . If P denotes a Sylow 3-subgroup of G , then

$$N \downarrow_P = A \oplus \mathcal{P}(K),$$

where A is a 10-dimensional module. The tensor square of A is easy to describe, and it is

$$A^{\otimes 2} = K \oplus B_1 \oplus B_2 \oplus B_3 \oplus 9 \cdot \mathcal{P}(K),$$

where B_1 , B_2 and B_3 are 6-dimensional indecomposable modules, and are the three conjugates from Class V in Section 3.3.5. Thus $A^{\otimes 2}$ is algebraic, and so N is algebraic.

In fact, this proves that the M_i are algebraic, since each M_i is a summand of $\Lambda^2(N)$. (The sources of these three modules M_i are the three modules B_i .) All modules lying outside the principal block are in blocks of defect 0, and so obviously algebraic. Thus $\mathrm{PSL}_3(4)$ has 3-SMA.

The same outcome occurs when $q = 7$ and $q = 13$: again, there are five simple modules in the principal block, and they have the same vertices and sources as the previous case. Again, the exterior square of the simple module with 10-dimensional source contains as summands the other non-trivial simple modules in the principal block.

It appears as though the following is true.

Conjecture 5.10 Suppose that $G = \mathrm{PSL}_3(q)$, where $q \equiv 4, 7 \pmod{9}$. Then there are five simple modules in the principal block, and all of them are algebraic. Indeed, apart from the trivial module, there is a simple module M with a 10-dimensional source, and three simple modules with 6-dimensional sources, which are summands of $\Lambda^2(M)$.¹

Finally, we consider the case where $p|(q-1)$. The smallest case where this happens is $q = 8$ and $p = 7$. In this case, there are three simple modules in the principal block, the non-trivial modules being of dimension 72 and 512. The 73-dimensional permutation representation is semisimple, with constituents the trivial module K and the simple module of dimension 72, and so this module is algebraic. Likewise, the other, 512-dimensional, simple module in the principal block has trivial source. Thus all simple modules in the principal block are algebraic. Other than the principal block, there are four simple modules lying in four blocks of defect 1, and twenty-four blocks of defect 0. Therefore $G = \mathrm{PSL}_3(8)$ has 7-SMA.

The same is true in the case of $\mathrm{PSL}_3(11)$ in characteristic 5. The permutation representation on 133 points is semisimple, and so the one simple module in the principal block is algebraic. The other simple module is similarly of trivial source. All other simple modules lie in blocks of smaller defect and so $\mathrm{PSL}_3(11)$ has 5-SMA.

It seems more difficult to offer a general strategy in this situation. An obvious first step is to organize the modules into p -blocks, using the character tables for $\mathrm{PSL}_3(q)$ given in [73] and then analyze the structure of the permutation module of dimension $q^2 + q + 1$, which is of dimension prime to p . This should be the direct sum of the smallest non-trivial simple module and the trivial module. Where to find the other simple module for the principal block (if such a module exists for all q) is more difficult.

5.3 Symplectic Groups

Write Q_m for the $m \times m$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

¹This conjecture has since been proved by the author.

and R_m for the matrix

$$\begin{pmatrix} 0 & Q_m \\ -Q_m & 0 \end{pmatrix}.$$

then the symplectic group $\mathrm{Sp}_{2n}(q)$ is the group of all $2n$ -dimensional matrices A that satisfy $A^T R_n A = R_n$. We will find a nicely embedded subgroup of $\mathrm{Sp}_{2n}(q)$, for $n \geq 3$, that is isomorphic with $\mathrm{SL}_3(q)$, hence showing that the natural module for $\mathrm{Sp}_{2n}(q)$ is not algebraic.

Proposition 5.11 Suppose that $G = \mathrm{Sp}_{2n}(q)$, where $n \geq 3$, and write $q = p^a$, where p is a prime. Write M for the natural module for G over $\mathrm{GF}(q)$. Then M is not algebraic.

Proof: Firstly, suppose that $n = 3$, and for $a \in \mathrm{SL}_3(q)$, write $b = Q_3^{-1}(a^T)^{-1}Q_3$. Then we claim that

$$g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

lies in $\mathrm{Sp}_6(q)$. To see this, we calculate:

$$\begin{aligned} g^T R_3 g &= \begin{pmatrix} a^T & 0 \\ 0 & b^T \end{pmatrix} \begin{pmatrix} 0 & Q_3 \\ -Q_3 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} a^T & 0 \\ 0 & b^T \end{pmatrix} \begin{pmatrix} 0 & Q_3 b \\ -Q_3 a & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a^T Q_3 b \\ -b^T Q_3 a & 0 \end{pmatrix}, \end{aligned}$$

and since $b = Q_3^{-1}(a^T)^{-1}Q_3$, we easily see that this last matrix is equal to R_3 , as required.

Write H for the collection of such matrices. We can clearly see that $M \downarrow_H$ has the natural KH -module as a summand (and $M \downarrow_H$ in fact is the sum of this natural module and its dual), and so M is not algebraic.

Finally, let $G = \mathrm{Sp}_{2n}(q)$, where $n > 3$. Then we can embed $H = \mathrm{Sp}_6(q)$ into G in an obvious way: for $a \in \mathrm{Sp}_6(q)$, write

$$a' = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix},$$

where I_m is the m -dimensional identity matrix. Then $a' \in \mathrm{Sp}_6(q)$, and if N denotes the natural module for $\mathrm{Sp}_6(q)$, then

$$M \downarrow_H \cong N \oplus 2(n-3) \cdot K,$$

where K denotes the trivial module. Again, M is not algebraic. \square

This does leave the case of $\mathrm{Sp}_4(q)$, where we have no subgroup isomorphic with $\mathrm{SL}_3(p)$, even badly embedded ones. The author has not considered this case, except to note that $\mathrm{Sp}_4(2) = S_6$ does indeed have 2-SMA.

5.4 Orthogonal Groups in Odd Characteristic

The orthogonal groups have a similar definition to the symplectic groups, at least in odd characteristic. Again, like symplectic groups, there is one more isomorphism type of such group in even dimension as odd. Indeed, here there are two types of orthogonal group in even dimension, called ‘plus-type’ and ‘minus-type’. We begin with plus-type.

Let Q_m be defined as before, and set

$$\mathrm{SO}_{2n}^+(q) = \{A \in \mathrm{SL}_{2n}(q) : A^T Q_{2n} A = Q_{2n}\}.$$

This group $\mathrm{SO}_{2n}^+(q)$ contains a quasisimple subgroup of index 2, denoted by $\Omega_{2n}^+(q)$. This is a new simple group if $n \geq 4$. The natural module for such a group $\Omega_{2n}^+(q)$ is non-algebraic for $n \geq 3$, as we prove now.

Proposition 5.12 Let p be an odd prime. Let G be the group $\Omega_{2n}^+(p^a)$, and let K be a field of characteristic p containing $\mathrm{GF}(p^a)$. Write M for the natural $2n$ -dimensional simple KG -module. Then M is non-algebraic.

Proof: Let a be an element of $\mathrm{SL}_3(q)$, and write $b = Q_3(a^T)^{-1}Q_3$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

lies in $\Omega_6^+(q)$. To see this, note that

$$\begin{aligned} g^T Q_6 g &= \begin{pmatrix} a^T & 0 \\ 0 & b^T \end{pmatrix} \begin{pmatrix} 0 & Q_3 \\ Q_3 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} a^T & 0 \\ 0 & b^T \end{pmatrix} \begin{pmatrix} 0 & Q_3 b \\ Q_3 a & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a^T Q_3 b \\ b^T Q_3 a & 0 \end{pmatrix}, \end{aligned}$$

and since $b = Q_3(a^T)^{-1}Q_3$, we see that this last matrix is Q_6 again. Thus $g \in \mathrm{SO}_6^+(q)$. Let H denote the subgroup of all matrices of this form. Since $H \cong \mathrm{SL}_3(q)$, a perfect group, we must have that $H \leq \Omega_6^+(q)$. Then clearly the natural module for $\Omega_6^+(q)$

restricts to the sum of two 3-dimensional simple modules for H , one of which is the natural module. Hence the natural module for $\Omega_6^+(q)$ is non-algebraic.

To prove the same result for $G = \Omega_{2n}^+(q)$, we simply note that if a is an element of $\Omega_6^+(q)$, then the matrix

$$\begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}$$

lies in $\text{SO}_{2n}^+(q)$, and again the set H of all such matrices is a perfect group, and so $H \leq G$. Clearly, the natural module for G restricts to the sum of the natural module for H and a $2(n-3)$ -dimensional trivial module. Thus the natural module for G is non-algebraic, proving the proposition. \square

Having dealt with plus-type, we will go on to define minus-type. Let ε denote a non-square in the field $K = \text{GF}(q)$, and let T_{2m} denote the $2m$ -dimensional matrix

$$T_{2m} = \begin{pmatrix} Q_{2m-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}.$$

Then we define

$$\text{SO}_{2n}^-(q) = \{A \in \text{SL}_{2n}(q) : A^T T_{2n} A = T_{2n}\}.$$

This group $\text{SO}_{2n}^-(q)$ again possesses a subgroup of index 2, denoted $\Omega_{2n}^-(q)$, which is again quasisimple.

Proposition 5.13 Let p be an odd prime, and let G be the group $\Omega_{2n}^-(q)$ for $n \geq 4$ and q is a power of p . Write K for a field of characteristic p containing $\text{GF}(q)$, and let M denote the natural KG -module. Then M is non-algebraic.

Proof: This easily follows, since one may embed $\Omega_{2n-2}^+(q)$ in G : if $a \in \Omega_{2n-2}^+(q)$, then

$$g = \begin{pmatrix} a & 0 \\ 0 & I_2 \end{pmatrix}$$

has the property that $g^T T_{2n} g = T_{2n}$. The set H of all such g has the property that $M \downarrow_H$ is the sum of natural module for H and a 2-dimensional trivial module. Hence M is non-algebraic. \square

In the odd-dimensional case, we only have the one isomorphism type of orthogonal groups

$$\text{SO}_{2n+1}(q) = \{A \in \text{SL}_{2n+1}(q) : A^T Q_{2n+1} A = Q_{2n+1}\}.$$

Again, this is not perfect, and its derived subgroup $\Omega_{2n+1}(q)$ is quasisimple. For $n \geq 3$, this gives us a new (quasi)simple group. We clearly have the following result.

Proposition 5.14 Let G be the group $\Omega_{2n+1}(q)$, for $n \geq 3$. Then the $(2n + 1)$ -dimensional natural module is non-algebraic.

Proof: Let a be an element of $\Omega_{2n}^+(q)$. Then the element

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

is a $(2n + 1)$ -dimensional matrix that clearly lies in $\text{SO}_{2n+1}(q)$. Hence the subgroup H of all such matrices lies in $\Omega_{2n+1}(q)$ (since H is perfect) and the natural module for G restricts to the $2n$ -dimensional natural module for H , together with a copy of the trivial module. Hence the natural module for G is non-algebraic. \square

5.5 Orthogonal Groups in Characteristic 2

Let f be a quadratic form on a vector space V over $\text{GF}(q)$, for q a power of 2. The *orthogonal group* of the form f is

$$\text{O}(V, f) = \{g \in \text{GL}(V) : Q(vg) = Q(v) \text{ for all } v \in V\}.$$

In odd dimension, all orthogonal groups of dimension $2n + 1$ are isomorphic, and indeed are isomorphic with $\text{Sp}_{2n}(q)$. In even dimension, there are up to isomorphism two different orthogonal groups: write e_1, \dots, e_{2n} for a basis of V , and let $x = \sum x_i e_i$ be an element of V . Let $\alpha \in K$ be an element such that the polynomial $\alpha t^2 + t + \alpha$ is irreducible in K . Then the two groups are given by the quadratic forms

$$f_1(x) = \sum_{i=1}^n x_{2i-1} x_{2i}$$

and

$$f_2(x) = \sum_{i=1}^n x_{2i-1} x_{2i} + \alpha x_{2n-1}^2 + \alpha x_{2n}^2,$$

and the groups are referred to as $\text{O}_{2n}^+(q)$ and $\text{O}_{2n}^-(q)$ respectively. Since the characteristic of the field is 2, these are also the special orthogonal versions. Again, we denote the derived subgroup by $\Omega_{2n}^\epsilon(q)$.

Proposition 5.15 Let q be a power of 2, and let K be a field containing $\text{GF}(q)$. Then the natural module for $\Omega_{2n}^+(q)$ and $\Omega_{2n}^-(q)$ are non-algebraic.

We first prove that the natural module M for $H = \Omega_6^+(2)$ is non-algebraic, which begins our induction. The group $\Omega_6^+(2)$ is isomorphic with A_8 , and the permutation module on 8 points is uniserial, with socle layers isomorphic with the trivial module, M , and the trivial module again.

Let J be a transitive subgroup of H isomorphic with $SL_3(2)$ (which exists since $SL_3(2)$ contains a subgroup of order 21), so let N be the permutation module for J . Since J has (apart from the trivial module) two dual simple modules A and A^* of dimension 3 and one of dimension 8 (see Section 4.6), the composition factors of $M \downarrow_J$ must be A and A^* , and so the composition factors of N are two copies of K and the modules A and A^* . The module N has at most three socle layers, since the permutation module on H does. Also, since the trivial module is not a summand (as p divides the number of elements being permuted) and K has no self-extensions (see [30] for example), there must be exactly three socle layers. Finally, neither A nor A^* can be a summand since N is self-dual. Hence $M \downarrow_J = A \oplus A^*$, and since A is non-algebraic, neither is M . Thus the natural module for $\Omega_6^+(q)$ is non-algebraic as well.

Now let V be a $2n$ -dimensional vector space with $n \geq 4$, as above, equipped with either of the forms f_i . In both cases, if $W = \langle e_1, \dots, e_6 \rangle$, then the restriction of f_i to W is simply the form f_1 on W . Hence, if a is an element of $H = \Omega_6^+(q)$, then the matrix

$$g = \begin{pmatrix} a & 0 \\ 0 & I_{2(n-3)} \end{pmatrix}$$

is an element of $G = O(V, f_i)$. Since the set H of all such g forms a perfect subgroup of G , in fact $H \leq G'$, and so H is a subgroup of the groups $\Omega_{2n}^\varepsilon(q)$. In both cases, the restriction of the natural module for $O(V, f_i)$ to H is clearly the sum of the 6-dimensional natural module for H and a $2(n-3)$ -dimensional trivial module. Hence the natural module for $\Omega_{2n}^\varepsilon(q)$ is non-algebraic, as required.

5.6 Unitary Groups

With the unitary groups, the situation is similar to that of the symplectic groups, in the sense that there are small-dimensional cases where we do not know whether the natural module for $SU_n(q)$ is algebraic or not. In fact, we can only determine this for $n \geq 6$.

First we recall the definition of the unitary groups. Let q be a power of a prime p , and write σ for the automorphism on $\mathrm{GL}_n(q^2)$ given by

$$\sigma : (a_{ij}) \mapsto (a_{ij}^q).$$

This automorphism, a power of the Frobenius morphism, has order 2. Then the unitary group is the automorphism group of a sesquilinear form, and can be written as

$$\mathrm{SU}_n(q) = \{A \in \mathrm{SL}_n(q^2) : (A^T)^\sigma = A^{-1}\}.$$

There are alternative definitions of $\mathrm{SO}_6^+(p)$ and $\mathrm{Sp}_6(2)$ that are helpful here. If p is odd, then $\mathrm{SO}_6^+(p)$ can be expressed as

$$\mathrm{SO}_6^+(p) = \{A \in \mathrm{SL}_6(p) : A^T = A^{-1}\},$$

and $\mathrm{Sp}_6(2)$ as

$$\mathrm{Sp}_6(2) = \{A \in \mathrm{SL}_6(2) : A^T = A^{-1}\}.$$

Using this, we may prove the following proposition.

Proposition 5.16 Let q be a power of a prime p , and let M denote the natural module for the group $\mathrm{SU}_n(q)$, where $n \geq 6$. Then M is non-algebraic.

Proof: We begin by proving that for all primes p , the natural module for $\mathrm{SU}_6(p)$ is non-algebraic. Firstly, embed $\mathrm{SL}_6(p)$ inside $\mathrm{SL}_6(p^2)$ in the obvious manner. Then, we notice that

$$H = \mathrm{SU}_6(p) \cap \mathrm{SL}_6(p) = \begin{cases} \mathrm{SO}_6^+(p) & p \text{ is odd} \\ \mathrm{Sp}_6(2) & p = 2 \end{cases}.$$

Since the natural module for $\mathrm{SU}_6(p)$ clearly restricts to the natural module for this subgroup H , we see that the natural module for $\mathrm{SU}_6(p)$ is non-algebraic. This proves also that the natural module for $\mathrm{SU}_6(q)$ is non-algebraic.

To prove the general case, simply embed $\mathrm{SU}_6(q)$ inside $G = \mathrm{SU}_n(q)$ in the obvious way: if $a \in \mathrm{SU}_6(q)$, then

$$g = \begin{pmatrix} a & 0 \\ 0 & I_{n-6} \end{pmatrix}.$$

The set of all such g forms a copy H of the subgroup $\mathrm{SU}_6(q)$, and the natural module for G restricts to the natural module for H and an $(n-6)$ -dimensional trivial module, proving that the natural module for G is non-algebraic. \square

We now turn our attention to non-defining characteristic representations. The order of $G = \mathrm{SU}_3(q)$ is $q^3(q-1)(q+1)^2(q^2-q+1)$. Let p be a prime that does not divide q . We claim that if p is neither 2 nor 3, then either $p|(q+1)$, and P is $C_{p^a} \times C_{p^a}$, where p^a is the highest power of p dividing $(q+1)$, or P is cyclic.

To see this, first note that there are elements in G of orders $q+1$, $q-1$ and q^2-q+1 . Then, if p divides both $q-1$ and $q+1$ then $p=2$, and if p divides both $q-1$ and q^2-q+1 then $p|q^2$ and so $p|1$, a contradiction. If p divides both $q+1$ and q^2-q+1 then $p|3$, and so we get that either $p=2$, $p=3$, $p|(q+1)$, or P is cyclic. Finally, a maximal torus of G has the form $C_{q+1} \times C_{q+1}$, and the result follows.

The Sylow 2-subgroup of $\mathrm{PSU}_3(q)$ is either semidihedral or wreathed, just as with $\mathrm{PSL}_3(q)$. When $q \equiv 1 \pmod{4}$, the group has semidihedral Sylow 2-subgroups, and in [31] Erdmann calculates the vertices and sources of the simple modules in the blocks of full defect. Indeed, there are three simple modules in the principal block, and two modules in all non-principal blocks. In Erdmann's notation, these are modules F , S and E in the principal block, and F_i and S_i in the non-principal blocks of full defect.

The modules F and F_i have P as vertex, and are trivial source. The module E has a V_4 subgroup as vertex and a 2-dimensional uniserial source. The modules S and S_i have source a uniserial module of dimension 2, and vertex the quaternion 2-group of order 8. Since all 2-dimensional modules are algebraic (as a corollary of Theorem 1.13 of Alperin, which proves that the natural module for $\mathrm{SL}_2(2^n)$ is algebraic), these are also algebraic.

Proposition 5.17 Let K be a field of characteristic 2, and let G be the group $\mathrm{PSU}_3(q)$ where $q \equiv 1 \pmod{4}$. Let M be a simple module from a 2-block of full defect. Then M is algebraic.

In the case of $q \equiv 3 \pmod{4}$, the answer is not known, although the non-trivial simple modules in the principal 2-block of $G = \mathrm{PSU}_3(3)$ are non-algebraic. There are three simple modules in the principal 2-block of G : the trivial module, a 6-dimensional simple module A , and a 14-dimensional simple module B . These have vertex a Sylow 2-subgroup P and their sources are $A \downarrow_P$ and $B \downarrow_P$ respectively. Both are seen to be non-algebraic by restricting to V_4 subgroups.

In fact, these vertices and sources are identical for the non-trivial simple modules in the principal block of $H = \mathrm{PSU}_3(11)$, with the simple modules having dimensions 110 and 1110 respectively. However, H has three other interesting 2-blocks, and we examine each of them. It has one with three simple 370-dimensional modules, and the block has defect group $C_4 \times C_4$. Each of these simple modules has the

same 11-dimensional source, whose tensor square is the sum of a trivial module and permutation modules. Hence this block consists of non-algebraic modules. The other two are (conjugate) blocks of cyclic defect group C_8 , whose single module is a trivial source module. (It also possesses twelve 1440-dimensional projective simple modules.)

We make the following conjecture regarding the principal 2-blocks of certain unitary groups $\text{PSU}_3(q)$.

Conjecture 5.18 Let G denote the group $\text{PSU}_3(q)$, where $q \equiv 3 \pmod{8}$. Then the principal 2-block contains three simple modules, whose sources are of dimensions 1, 6 and 14. Furthermore, as q varies, these three sources remain constant. These 6- and 14-dimensional indecomposable modules for $C_4 \wr C_2$ are non-algebraic.

It should be possible to perform an analysis for the other non-defining characteristics along a similar vein to that of $\text{PSL}_3(q)$.

5.7 Ree Groups ${}^2G_2(3^n)$

The Ree groups of type G_2 were first discovered by Ree and presented in [69] and [70]. For each q an odd power of 3, there is a group $G_q = {}^2G_2(q)$ associated to it, of order $q^3(q^3 + 1)(q - 1)$. The groups G_q all have elementary abelian Sylow 2-subgroups of order 8, and have centralizer of involution isomorphic with $C_2 \times \text{PSL}_2(q)$. In [78], Ward almost completely determines the character table, with the unknown quantities due to the fact that (at the time) the ‘groups of Ree type’ (defined by five conditions on their normalizer and centralizer structure) were not known to only consist of the groups G_q .

We will examine the characteristic 2 representations first. There are eight ordinary characters and five modular characters in the principal block, and there are other blocks of defect group V_4 , C_2 and the trivial group. All of the simple modules in the blocks of defect 0 and 1 are algebraic, and so we are left with considering the principal block and the blocks of defect 2. In [57], Landrock and Michler determined the Green correspondents of the five simple modules in the principal block for G_q , where $q = 3^{2n+1}$ and $n \geq 1$.

It turns out that the normalizer of a Sylow 2-subgroup P has order 168, and is the split extension of the group P by the non-abelian group of order 21 in $\text{Aut}(P)$. This group N has five simple modules: three of dimension 1, denoted, as in [57], by I , 1, and 1^* ; and two 3-dimensional modules, denoted by 3 and 3^* .

As n varies, the Green correspondents of the simple modules in the principal block are actually always the same: the three 1-dimensional (simple) modules I , 1 and 1^* ; a (self-dual) 6-dimensional module with socle 3 and top 3^* (assuming correct choice of 3 and 3^*); and a (self-dual) 12-dimensional module, with socle 3^* , top 3 , and heart (radical modulo socle) $3 \oplus 3^*$.

Although Landrock and Michler do not explicitly state it, these are also the Green correspondents for the five simple modules in the principal block of G_3 , which is also known as $\mathrm{SL}_2(8) \rtimes C_3$. Alperin has already proved that all simple modules of $\mathrm{SL}_2(8)$ are algebraic, and by Proposition 2.25, all simple modules of $\mathrm{SL}_2(8) \rtimes C_3$ are algebraic. Thus their Green correspondents in N , the five modules described above, are algebraic. This means that all simple modules in the principal block of G_q are algebraic, for any q .

It remains to discuss the blocks of defect 2. At the end of II.7 of [78], Ward notes that the ordinary characters of defect 2 are all real. By Corollary 3.8, all of the simple modules in real 2-blocks of defect group V_4 are algebraic. Therefore we have proved the following theorem.

Theorem 5.19 Let G be a finite group of type ${}^2G_2(3^{2n+1})$, and let K be a field of characteristic 2. Then all KG -modules are algebraic.

Having dealt with characteristic 2, we will turn our attention to characteristic 3; we will see that the natural 7-dimension module is non-algebraic.

Let G be the group ${}^2G_2(3) = \mathrm{SL}_2(8) \rtimes C_3$. This group has a 9-point permutation representation, which extends the 9-point permutation representation of $\mathrm{PSL}_2(8) = \mathrm{SL}_2(8)$. The Sylow 3-subgroup of $\mathrm{SL}_2(8)$ is well-known to be cyclic, and so is generated by x , a 9-cycle in this permutation representation. The outer automorphism y of order 3 acts non-trivially on the Sylow 3-subgroup, and must clearly fix the subgroup $\langle x^3 \rangle$. Thus $Q = \langle x^3, y \rangle$ is a subgroup of G isomorphic with $C_3 \times C_3$. Furthermore, its representation on the 9 points can easily be seen to be transitive, and so this is the regular representation of Q . Hence the KQ -module got from this permutation representation over $K = \mathrm{GF}(3)$ is isomorphic with the projective indecomposable module $\mathcal{P}(K)$.

Next, recall that $\mathrm{SL}_2(8)$ has five simple modules over a field of characteristic 3: the trivial module, a 7-dimensional simple module, and three 9-dimensional projective simple modules. It is not hard to see that the 9-dimensional permutation module of G given above is uniserial, with socle layers consisting of the trivial module, the 7-dimensional simple module, and the trivial module. The restriction of

this 9-dimensional indecomposable module to Q is the projective indecomposable, and so the restriction of the heart of the permutation module to Q is the heart of the projective indecomposable. This 7-dimensional indecomposable module for Q is non-algebraic by Proposition 2.19.

We have therefore proved that the 7-dimensional simple module for G is non-algebraic. Clearly the 7-dimensional simple module for ${}^2G_2(q)$ restricts to G as this simple module, and so we have proved the following.

Proposition 5.20 Let G be the Ree group ${}^2G_2(q)$. Then the 7-dimensional natural module for G is non-algebraic.

Chapter 6

The Sporadic Groups

The twenty-six sporadic groups are split into four families: the Mathieu groups; the Leech lattice groups; the Monster sections; and the pariahs. The five Mathieu groups have been almost completely analyzed, and so we know, with a few exceptions, whether a given simple module is algebraic for any of the five groups, and any prime p . No other family has been so comprehensively analyzed like the Mathieu groups. In this family, we find that M_{11} has algebraic simple modules in characteristic 2, and M_{22} has algebraic simple modules in characteristic 3, but other than those two examples, the rule for these groups is that if the group has non-cyclic Sylow p -subgroups, then G has at least one non-algebraic simple module. This also gives us the only two examples known of simple groups with abelian Sylow p -subgroups yet which have non-algebraic simple modules, namely M_{11} and M_{23} , both in characteristic 3.

The second section deals with the Leech lattice groups; that is, it contains all groups involved in the Conway group Co_1 that are not Mathieu groups. In this section, it is proved that the Higman–Sims group in characteristic 3, and the Janko group J_2 in characteristics 3 and 5, have SMA.

The third section involves the Monster sections: these are all groups not already considered that are involved in the Monster. The Held and Harada–Norton groups are quite well-understood, and some results are available for the Fischer groups, but understandably the Baby Monster and Monster are not considered.

The final section involves the Pariahs. These are the six sporadic simple groups not involved in the Monster. The group J_1 has been completely calculated, and has p -SMA for all primes p . The other Janko groups and the Rudvalis group have results on them, but the remaining two groups—The O’Nan group and the Lyons group—remain unanalyzed.

In this chapter, various assertions are made about simple modules for the various sporadic groups, particular concerning their restrictions to subgroups. Much of the

information about the simple modules for simple groups came from the websites [10] and [63]. All of the calculations in this chapter were performed using MAGMA, and more detail on those calculations is given in Appendix A. This includes all algorithms developed by the author to complement MAGMA's internal procedures for decomposing tensor products.

Before we begin, we will give a word on notation. For each group G , we will examine each prime in turn, and will list the p -blocks of G and give the simple modules belonging to each. We will denote the smallest non-trivial module in the principal block by S_1 , and number consecutively, so that the third-smallest simple module in the principal block is S_3 . For the second block, the smallest simple module is denoted by T_1 , and so forth, so that the second-smallest module in the fourth block is denoted by V_2 . However, we shall rarely need to go to the fourth block.

6.1 The Mathieu Groups

The five Mathieu groups, detailed by Émile Mathieu in two papers in 1860 and 1873, have been extensively studied. They appeared in two papers—[60] and [61]—in the nineteenth century, and remained the only sporadic finite simple groups known for nearly a century, until the flurry of activity following the Feit–Thompson theorem.

6.1.1 The Group M_{11}

The Mathieu group M_{11} is of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, and is generated by the two permutations $x = (1, 4, 3, 8)(2, 5, 6, 9)$ and $y = (2, 10)(4, 11)(5, 7)(8, 9)$. The group was originally constructed in [60], and by a well-known theorem of Jordan is the only sharply 4-transitive group on eleven points. Let us first note that the only important primes for considering algebraicity of modules are $p = 2$ and $p = 3$, as for other primes M_{11} possesses a cyclic or trivial Sylow p -subgroup.

Theorem 6.1 Let $G \cong M_{11}$, the Mathieu group on eleven points. Let K be a field of characteristic p , and let M be a simple KG -module.

- (i) If $p = 2$, then M is algebraic.
- (ii) If $p = 3$, then M is algebraic if and only if M is self-dual; i.e., M is algebraic if and only if M has dimension 1, 24 or 45, or if M has dimension 10 and M is self-dual.

We will first analyze the case where $p = 2$, and then deal with the case $p = 3$. The dimensions of the simple modules in characteristics 2 and 3 are given in the table below. As usual, the dual module is denoted by an asterisk. If two or more modules of the same dimension exist up to duality, then they are labelled with a subscript. For example, if M is the dual of the third module of dimension 8, then we would write 8_3^* . For each prime, the blocks are given with the defect group listed in the right-hand column.

p	Block	Simple Modules	Defect Group
2	1	{1, 10, 44}	Sylow
	2,3	{16}, {16*}	Defect 0
3	1	{1, 5, 5*, 10 ₁ , 10 ₂ , 10 ₂ *, 24}	Sylow
	2	{45}	Defect 0

We firstly focus on $p = 2$, and write $K = \text{GF}(4)$. We know from the table above that G has three blocks: the first, the principal block, contains three simple modules, all realizable over $\text{GF}(2)$; and the other two blocks are of defect 0, each with a simple module of dimension 16, realizable over $\text{GF}(4)$. The two simple modules of dimension 16 are obviously algebraic, as is the trivial module, and so the discussion rests with the two non-trivial simple modules in the principal block. Recall our convention of writing S_1 for the 10-dimensional module and S_2 for the 44-dimensional module.

Both S_1 and S_2 have trivial source; the 11-dimensional permutation representation given above is semisimple, and isomorphic with $K \oplus S_1$. However, this is not needed since M_{11} contains a subgroup isomorphic with a non-split extension of A_6 by C_2 . This subgroup has index 55, and the permutation representation on the cosets of one of these subgroups is semisimple; it is the sum of the three simples in the principal block. This proves that all simple modules are algebraic.

Now consider the prime $p = 3$: in this case, all modules are realizable over $\text{GF}(3)$. We again use our standard convention, so that the two 5-dimensional modules are denoted by S_1 and S_2 , and so on.

Firstly, if we take the 11-dimensional permutation representation, it is again semisimple, with one constituent the trivial module, and the other a 10-dimensional, self-dual, simple module. This must be S_3 , and hence S_3 is algebraic. We will deal with the modules S_1, S_2, S_4 and S_5 —the modules that are not self-dual—next. It is true that $\Lambda^2(S_1) = S_4$ and $\Lambda^2(S_2) = S_5$, upon a suitable labelling of the S_i , where Λ^2 denotes the usual exterior square. Thus, since in odd characteristic,

$$M \otimes M = \Lambda^2(M) \oplus S^2(M),$$

if S_4 is non-algebraic then S_1 and S_2 are also non-algebraic.

Let P denote a Sylow 3-subgroup of M_{11} , which is elementary abelian of order 9, and consider the restriction $S_4 \downarrow_P$. This is indecomposable, and so $S_4 \downarrow_P$ is a source of S_4 . This KP -module is actually isomorphic with the module $\Omega^2(K)$, which is non-algebraic by Proposition 2.7. Since $S_4 \downarrow_P$ is non-algebraic, so is S_4 , and so S_1, S_2, S_4 and S_5 are non-algebraic. [We do not need to know that the source is non-algebraic, merely that there is some subgroup H such that $S_4 \downarrow_H$ contains a non-algebraic summand. As the order of the group increases, this will become the dominant method for proving modules are non-algebraic.] Although it is not required, the restriction $S_1 \downarrow_P$ is indecomposable, and is in Class C (or Class C*) as given in Section 3.3.4, proving again that S_1 is non-algebraic.

Since the 45-dimensional projective simple module T_1 is clearly algebraic, it remains to discuss the module S_6 . Restricting this module to the Sylow 3-subgroup P , we get

$$S_6 \downarrow_P = 2 \cdot \mathcal{P}(K) \oplus M,$$

where $\mathcal{P}(K)$ is the (only) projective indecomposable $K(C_3 \times C_3)$ -module, and M is a 6-dimensional module, the source of S_6 . This source is from Class V of Section 3.3.5, and is hence algebraic; therefore so is S_6 .

We have therefore proved both parts of the theorem.

6.1.2 The Group M_{12}

The Mathieu group M_{12} is of order $95040 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, and is generated by the two permutations $x = (1, 4)(3, 10)(5, 11)(6, 12)$ and $y = (1, 8, 9)(2, 3, 4)(5, 12, 11)(6, 10, 7)$. It can also be generated by the permutations $a = (2, 3)(5, 6)(8, 9)(11, 12)$ and $b = (1, 2, 4)(3, 5, 7)(6, 8, 10)(9, 11, 12)$, and this produces a conjugate subgroup of S_{12} , where this conjugation induces the outer automorphism of M_{12} . This group was also originally constructed in [60], and the same theorem of Jordan proves that it is the only sharply 5-transitive group on 12 points. Again, the only important primes are 2 and 3.

Theorem 6.2 Let $G = M_{12}$, the Mathieu group on 12 points, and let K be a field of characteristic p . Let M be a simple KG -module.

- (i) If $p = 2$, then M is algebraic if and only if M is trivial or lies outside the principal block.

- (ii) If $p = 3$, then M is algebraic if and only if M is trivial or lies outside the principal block.

As with the previous group, we organize the information regarding the simple modules into a table, with the same conventions. This format of table will be used throughout the chapter.

p	Block	Simple Modules	Defect Group
2	1	{1, 10, 44}	Sylow
	2	{16, 16*, 144}	Defect 2
3	1	{1, 10 ₁ , 10 ₂ , 15, 15*, 34, 45 ₁ , 45 ₂ }	Sylow
	2	{45 ₃ , 99}	Defect 1
	3	{54}	Defect 0

We begin with characteristic 2. The modules in the principal block, together with the 144-dimensional module in the second block, can be realized over $\text{GF}(2)$, whereas the two 16-dimensional modules require $\text{GF}(4)$. Write S_1 and S_2 for the 10-dimensional and 44-dimensional modules respectively, and T_1 , T_2 and T_3 for the two 16-dimensional modules and the 144-dimensional module. We claim that the modules S_1 and S_2 are not algebraic, but the modules T_1 , T_2 and T_3 are.

That the T_i are algebraic follows immediately from Corollary 3.9, and so we focus on the principal block. Unfortunately, we found no non-algebraic modules for M_{11} , and so cannot prove that any of the simple modules for M_{12} are non-algebraic by restricting to a subgroup. In fact, the only way that we can easily prove that the modules in the principal block are non-algebraic is by Corollary 3.2, the V_4 Restriction Test; there are four conjugacy classes of subgroups isomorphic with V_4 , and in the table below we collect the lengths of their conjugacy classes.

Class	Number of Conjugates
\mathcal{C}_1	495
\mathcal{C}_2	1320
\mathcal{C}_3	1980
\mathcal{C}_4	2970

The restriction of S_1 to a V_4 subgroup Q lying in the conjugacy class \mathcal{C}_1 is given by

$$S_1 \downarrow_Q = \Omega^2(K) \oplus \Omega^{-2}(K).$$

Since $\Omega^i(K)$ is non-algebraic for non-zero i , we must have that S_1 is non-algebraic. [The restrictions to representatives from the other \mathcal{C}_i are algebraic.] The restriction of S_2 to the same subgroup Q is given by

$$S_2 \downarrow_Q = 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus 8 \cdot \mathcal{P}(K),$$

whereas the restrictions of S_2 to representatives from the other \mathcal{C}_i are algebraic. Since $\Omega^{\pm 1}(K)$ is non-algebraic, we see that S_2 is non-algebraic as well.

Having dealt with the characteristic 2 case, let us move on to when $p = 3$: in this case, all modules can be realized over $\text{GF}(3)$.

Firstly, consider the two (non-isomorphic) 12-dimensional permutation representations corresponding to the two conjugacy classes of maximal subgroup isomorphic with M_{11} , representatives of which are labelled H_1 and H_2 . These 12-dimensional permutation representations are both uniserial, with heart (radical modulo socle) isomorphic with the two 10-dimensional simple modules S_1 and S_2 . We choose the labelling so that the heart of the permutation module on the cosets of H_1 is S_1 , and similarly for H_2 and S_2 . Restricting S_1 to the H_i , we see that $S_1 \downarrow_{H_1}$ is simple (and self-dual) and hence algebraic, whereas $S_1 \downarrow_{H_2}$ is the direct sum of the two 5-dimensional simple KH_2 -modules, which in the last section we proved are non-algebraic. Similarly, $S_2 \downarrow_{H_1}$ is the sum of the two 5-dimensional simple KH_1 -modules, and so both S_1 and S_2 are non-algebraic.

This can be used to prove the non-algebraicity of the 45-dimensional modules S_6 and S_7 . The tensor square of S_1 decomposes as the direct sum of the trivial module, the 54-dimensional projective simple module and a 45-dimensional simple module. Since the trivial and the 54-dimensional modules are clearly algebraic, this 45-dimensional module must be non-algebraic, and so it cannot come from the block of defect 1. Without loss of generality, label this simple module S_6 . The tensor square of S_2 is not isomorphic with that of S_1 , and so the 45-dimensional simple module in the (semisimple) module $S_2^{\otimes 2}$ must be the other 45-dimensional module, S_7 . Thus we have proved that S_6 and S_7 are non-algebraic.

It remains to discuss S_3 , S_4 and S_5 . To prove that S_3 and S_4 are non-algebraic, we restrict to subgroups of the Sylow 3-subgroup. There are three conjugacy classes of subgroup of G isomorphic with $C_3 \times C_3$, as given in the table below.

Class	Number of Conjugates
\mathcal{C}_1	220
\mathcal{C}_2	220
\mathcal{C}_3	1760

Let Q be a subgroup from either \mathcal{C}_1 or \mathcal{C}_2 . The module $S_3 \downarrow_Q$ is indecomposable, and its tensor product breaks up as

$$S_3^{\otimes 2} \downarrow_Q = 21 \cdot \mathcal{P}(K) \oplus M \oplus \Omega^{-1}(M),$$

where M is a 15-dimensional non-periodic module and $\Omega^{-1}(M)$ is 21-dimensional. Since not both of M and $\Omega^{-1}(M)$ are algebraic, $S_3 \downarrow_Q$ is definitely not algebraic.

The last module, S_5 , has a 7-dimensional source M with vertex P , a Sylow 3-subgroup. Taking the restriction to Q , this module remains indecomposable, and is in fact isomorphic with the heart of the projective indecomposable KQ -module. In Proposition 2.19, we proved that this module was non-algebraic, and so neither is S_5 , completing Theorem 6.2.

6.1.3 The Group M_{22}

The Mathieu group M_{22} is of order $443520 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, and is generated by the two permutations $x = (1, 13)(2, 8)(3, 16)(4, 12)(6, 22)(7, 17)(9, 10)(11, 14)$ and $y = (1, 22, 3, 21)(2, 18, 4, 13)(5, 12)(6, 11, 7, 15)(8, 14, 20, 10)(17, 19)$. Originally defined by Mathieu in [61], it was proved to be one of the two simple groups (the other being A_{10}) with a particular centralizer of a central involution almost a hundred years later by Janko in [50].

Theorem 6.3 Let $G = M_{22}$, the Mathieu group on 22 points, and let K be a field of characteristic p . Let M be a simple KG -module.

- (i) If $p = 2$, then M is algebraic if and only if M is trivial.
- (ii) If $p = 3$, then M is algebraic.

As usual, we summarize the simple modules in the relevant characteristics.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 10, 10^*, 34, 70, 70^*, 98\}$	Sylow
3	1	$\{1, 49, 49^*, 55, 231\}$	Sylow
	2	$\{21, 210\}$	Defect 1
	3,4,5	$\{45\}, \{45^*\}, \{99\}$	Defect 0

We begin with $p = 2$; the two simple modules of dimension 70 require $\text{GF}(4)$ to be realized, whereas all other modules can be realized over $\text{GF}(2)$.

The maximal subgroups of M_{22} , although including several groups with non-algebraic simple modules, do not help us in proving that the non-trivial simple modules are non-algebraic. We therefore require the V_4 Restriction Test. There are four conjugacy classes of V_4 subgroup lying in M_{22} : the lengths of their conjugacy classes are given below.

Class	Number of Conjugates
\mathcal{C}_1	1155
\mathcal{C}_2	1540
\mathcal{C}_3	2310
\mathcal{C}_4	4620

Labelling a 10-dimensional module S_1 , we see that while the restriction of S_1 to a subgroup from \mathcal{C}_4 is algebraic, the other three conjugacy classes of subgroup provide witness for the non-algebraicity of S_1 ; for example, for a particular choice of S_1 , we have that the restriction of S_1 to a representative from \mathcal{C}_1 is given by

$$K \oplus \mathcal{P}(K) \oplus \Omega^2(K).$$

The 34-dimensional simple module S_3 has algebraic restriction to subgroups from \mathcal{C}_2 and \mathcal{C}_4 , and restrictions to subgroups from the other two classes are given by

$$\Omega^2(K) \oplus \Omega^{-2}(K) \oplus 6 \cdot \mathcal{P}(K).$$

Similarly, the 98-dimensional simple module S_6 has algebraic restriction to subgroups from \mathcal{C}_2 and \mathcal{C}_4 , and restrictions to subgroups from the other two classes are given by

$$\Omega^2(K) \oplus \Omega^{-2}(K) \oplus 22 \cdot \mathcal{P}(K).$$

Finally, we examine the two simple modules that are only realizable when the field contains a cube root of 1. Again, their restrictions to subgroups from \mathcal{C}_2 and \mathcal{C}_4 are algebraic, and the restrictions to subgroups from \mathcal{C}_1 and \mathcal{C}_3 look like

$$\Omega(K) \oplus \Omega^{-1}(K) \oplus 16 \cdot \mathcal{P}(K).$$

This proves that every non-trivial simple KG -module is non-algebraic.

Having dealt with the case where $p = 2$, we move on to the case where $p = 3$, where we have to prove that all simple modules are algebraic; the two projective simple modules of dimension 45 require $\text{GF}(9)$ to exist, but all other simple modules have realizations over $\text{GF}(3)$. For the rest of this section, the field over which we work is $K = \text{GF}(3)$.

To prove that all of the simple modules are algebraic, it suffices to consider the two 49-dimensional simple modules S_1 and S_2 , the 55-dimensional module S_3 and the 231-dimensional module S_4 . The group G has a maximal subgroup of index 77, isomorphic with a semidirect product $(C_2)^4 \rtimes A_6$, and the permutation representation on the cosets of this subgroup is semisimple, with constituents the trivial module, the

21-dimensional simple module T_1 , and the 55-dimensional module S_3 . Thus S_3 is a summand of a permutation module, so is certainly algebraic.

Since S_1 and S_2 are dual, it suffices to consider the module S_1 . Let P denote a Sylow 3-subgroup of G , an elementary abelian group of order 9. Then

$$S_1 \downarrow_P = M \oplus 5 \cdot \mathcal{P}(K),$$

where M is a self-dual 4-dimensional module, the source of S_1 , with simple socle. To identify it via Section 3.3.3, we need to know how its tensor square decomposes: in fact, its summands have dimensions 1, 6 and 9, whence M comes from the (algebraic) Class C from that section. Thus S_1 and S_2 are algebraic.

It remains to prove that S_4 is algebraic. We have

$$S_4 \downarrow_P = N \oplus 25 \cdot \mathcal{P}(K),$$

where N is a 6-dimensional periodic indecomposable module from Class V of Section 3.3.5. Therefore by Proposition 3.30, N is algebraic.

6.1.4 The Group M_{23}

The Mathieu group M_{23} is of order $10200960 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and is generated by the two permutations $x = (1, 2)(3, 4)(7, 8)(9, 10)(13, 14)(15, 16)(19, 20)(21, 22)$ and $y = (1, 16, 11, 3)(2, 9, 21, 12)(4, 5, 8, 23)(6, 22, 14, 18)(13, 20)(15, 17)$. Just as with the group M_{22} , this was defined by Mathieu in [61]; it is characterized by being the unique simple group with centralizer of a central involution a particular extension of the elementary abelian group of order 16 by the simple group $\text{PSL}_2(7)$.

This is the first group for which we do not have a complete result. In particular, there is a simple module of dimension 252 in characteristic 2 for which it is not known whether it is algebraic.

Theorem 6.4 Let $G = M_{23}$, the Mathieu group on 23 points, and let K be a splitting field of characteristic p . Let M be a simple KG -module.

- (i) If $p = 2$, then M is non-algebraic if M is not trivial and does not have dimension either 252 or 896. If M is trivial or has dimension 896, then M is algebraic.
- (ii) If $p = 3$, then M is algebraic if and only if M is not one of the two (dual) 104-dimensional simple modules.

We begin with the now-obligatory table of dimensions of simple modules.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 11, 11^*, 44, 44^*, 120, 220, 220^*, 252\}$	Sylow
	2,3	$\{896\}, \{896^*\},$	Defect 0
3	1	$\{1, 22, 104, 104^*, 253, 770, 770^*\}$	Sylow
	2	$\{231\}$	Defect 1
	3,4	$\{45\}, \{45^*\},$	Defect 0
	5,6,7	$\{990\}, \{990^*\}, \{1035\}$	Defect 0

We start with $p = 2$: all of the modules in the principal block are realizable over $\text{GF}(2)$, and the remaining two projective simple modules require a cube root of unity to be present in the field.

The point stabilizer in the usual 23-dimensional permutation representation is isomorphic with M_{22} , and by Proposition 2.25, a KG -module M is algebraic if and only if its restriction to this point stabilizer is algebraic. However, the restriction of any simple KG -module to one of these subgroups is indecomposable but not simple, and we therefore gain nothing from this. We therefore resort to the V_4 Restriction Test.

There are two conjugacy classes of subgroup isomorphic with V_4 , and the lengths of these are given below.

Class	Number of Conjugates
\mathcal{C}_1	8855
\mathcal{C}_2	53130

Let us begin with the 11-dimensional modules S_1 and S_2 . Their restrictions to subgroups from \mathcal{C}_2 are algebraic, whereas (with the correct choice of S_1 and S_2) the restriction of S_1 to a subgroup in \mathcal{C}_1 is isomorphic with

$$K \oplus 2 \cdot \Omega(K) \oplus \mathcal{P}(K),$$

and S_2 is the dual of this. Hence S_1 and S_2 are not algebraic.

Likewise, the restrictions of the 44-dimensional modules S_3 and S_4 to a subgroup from \mathcal{C}_2 are algebraic, but restricting one of them to a subgroup from \mathcal{C}_1 gives (up to duality) the decomposition

$$M \oplus N \oplus 2 \cdot \Omega(K) \oplus 7 \cdot \mathcal{P}(K),$$

where M is the sum of the three non-isomorphic 2-dimensional indecomposable modules, and N is a 4-dimensional self-dual indecomposable module with two socle layers. In any case, S_3 and S_4 are non-algebraic.

The restriction of the 120-dimensional module S_5 to a subgroup from \mathcal{C}_1 is

$$4 \cdot K \oplus 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus 26 \cdot \mathcal{P}(K),$$

and so despite the restriction to subgroups from \mathcal{C}_2 being algebraic, S_5 is still non-algebraic.

The restriction of one of the 220-dimensional modules S_6 and S_7 to a subgroup from \mathcal{C}_1 is given by

$$M \oplus 2 \cdot \Omega(K) \oplus N \oplus 51 \cdot \mathcal{P}(K),$$

where M and N are as above. The restriction of S_6 to a subgroup from \mathcal{C}_2 is algebraic. Thus the modules S_6 and S_7 are non-algebraic.

The 252-dimensional module S_8 is very difficult to analyze; this simple module has a 28-dimensional source S , and the restriction of S to representatives from the two conjugacy classes of V_4 subgroup are algebraic. Furthermore, even the elementary abelian subgroups of order 8 are of no help; there are three conjugacy classes of subgroups isomorphic with $(C_2)^3$, and any summand of the restriction of S to any one of them is either trivial, 2-dimensional, projective, or a 4-dimensional module that has a kernel of order 2. Hence these restrictions are all algebraic as well. Furthermore, there is no 2-subgroup for which the restriction of S contains a non-trivial odd-dimensional indecomposable summand. The author knows of no way to prove whether or not this module S is algebraic.

Now let us consider characteristic 3: for this group, all simple modules lying in the principal block can be realized over $\text{GF}(3)$, along with the 1035-dimensional projective simple module. The other four projective simple modules can only be realized when the field contains $\text{GF}(9)$.

The permutation module on 23 points is semisimple, with constituents K and the 22-dimensional simple module S_1 . The permutation module on 506 points (the cosets of a maximal subgroup isomorphic with A_8), while not semisimple, has a summand isomorphic with the simple module S_4 .

The restriction of the 104-dimensional module S_2 to a Sylow 3-subgroup P is given by

$$S_2 \downarrow_P = M \oplus 11 \cdot \mathcal{P}(K),$$

where M is a 5-dimensional indecomposable module from Class C (or Class C*) of Section 3.3.4 and so is non-algebraic. Its dual is similarly non-algebraic.

It remains to deal with the two 770-dimensional simple modules S_5 and S_6 . Restricting S_5 to a Sylow 3-subgroup gives

$$S_5 \downarrow_P = 2 \cdot K \oplus 84 \cdot \mathcal{P}(K) \oplus N,$$

where N is a the direct sum of the four 3-dimensional indecomposable modules that are induced from the trivial module for the four subgroups of order 3. Thus S_5 (and S_6) are trivial-source modules, and so algebraic.

6.1.5 The Group M_{24}

The Mathieu group M_{24} is of order $244823040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and is generated by the two permutations

$$\begin{aligned} x &= (1, 4)(2, 7)(3, 17)(5, 13)(6, 9)(8, 15)(10, 19)(11, 18)(12, 21)(14, 16)(20, 24)(22, 23) \\ y &= (1, 4, 6)(2, 21, 14)(3, 9, 15)(5, 18, 10)(13, 17, 16)(19, 24, 23). \end{aligned}$$

It is the largest of the groups found by Mathieu, and appears in [61]. It is the last group to be considered in this section. Again, we do not have a complete result in characteristic 2, and only have the following.

Theorem 6.5 Let $G = M_{24}$, the Mathieu group on 24 points, and let K be a splitting field of characteristic p . Let M be a simple KG -module.

- (i) If $p = 2$, then M is non-algebraic if M does not have dimension 1, 320 or 1792.
- (ii) If $p = 3$, then M is algebraic if and only if M is not one of the two (dual) 770-dimensional simple modules.

It is not know whether the 320-dimensional modules or the 1792-dimensional module are algebraic. This appears to be a difficult problem, and more will be said about it later.

We start with the necessary table of simple module dimensions.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 11, 11^*, 44, 44^*, 120, 220, 220^*, 252, 320, 320^*, 1242, 1792\}$	Sylow
3	1	$\{1, 22, 231, 483, 770, 770^*, 1243\}$	Sylow
	2,3	$\{45, 990\}, \{45^*, 990^*\}$	Defect 1
	4,5	$\{252, 5544\}, \{1035, 2277\},$	Defect 1
	6	$\{10395\}$	Defect 0

Let us first examine the case where $p = 2$, and write $K = \text{GF}(2)$. In this case, $G = M_{24}$ has only one block, which contains thirteen simple modules, all realizable over K . We will show that all non-trivial simple modules apart from those of dimensions 320 and 1792 are non-algebraic.

Write H for the point stabilizer under the 24-point permutation action of G ; then H is isomorphic with the group M_{23} . In fact, the nine smallest simple modules (all those of dimension at most 252) all remain simple when restricted to H . Since the 11-, 44-, 120- and 220-dimensional simple modules for M_{23} are non-algebraic, so are their correspondents for M_{24} . This deals with the S_i for $1 \leq i \leq 7$. Since we do not know if the 252-dimensional simple KH -module is algebraic, we need to examine the V_4 subgroups.

There are nine conjugacy classes of subgroup isomorphic with V_4 , and the lengths of these are given below.

Class	Number of Conjugates
\mathcal{C}_1	10626
\mathcal{C}_2	26565
\mathcal{C}_3	26565
\mathcal{C}_4	239085
\mathcal{C}_5	425040
\mathcal{C}_6	478170
\mathcal{C}_7	637560
\mathcal{C}_8	956340
\mathcal{C}_9	956340

Consider the simple module S_8 : the restriction of this module to all V_4 subgroups except those from \mathcal{C}_2 is algebraic, whereas the restriction to those from \mathcal{C}_2 is given by

$$6 \cdot K \oplus 2 \cdot M \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 56 \cdot \mathcal{P}(K),$$

where M is the direct sum of the three non-isomorphic 2-dimensional KV_4 -modules. Thus S_8 is non-algebraic.

The 1242-dimensional simple module S_{11} restricts to a V_4 subgroup from the classes \mathcal{C}_i , for $i = 3, 4, 5, 7, 9$ as

$$2 \cdot M \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus 306 \cdot \mathcal{P}(K),$$

where M is as above. The restriction to a subgroup from the class \mathcal{C}_2 is given by

$$4 \cdot K \oplus 4 \cdot N \oplus \Omega^3(K) \oplus \Omega^{-3}(K) \oplus 302 \cdot \mathcal{P}(K),$$

where N is a 4-dimensional indecomposable module with two socle layers. The restriction to subgroups from the classes \mathcal{C}_1 , \mathcal{C}_6 and \mathcal{C}_8 are algebraic. Hence S_{11} is non-algebraic.

The remaining two modules, of dimensions 320 and 1792, are of unknown algebraicity. This is because the restriction of either module to either conjugacy class of involutions is a free module, and so the V_4 Restriction Test cannot be used. Moreover, the restriction of either module to any dihedral subgroup will consist solely of band modules, and so results from Chapter 3 offer no help here. Like the 252-dimensional module for M_{23} in characteristic 2, the author knows of no way to determine the algebraicity of these modules.

Now consider the prime 3: the 45- and 990-dimensional modules lying in the blocks of defect 1 require $\text{GF}(9)$ to be realized, whereas all other modules can be realized over $\text{GF}(3)$. Only the modules lying in the principal block have any chance of being non-algebraic, so we restrict ourselves to these modules. We again write S_i for the i th non-trivial simple module lying in the principal block, so that for example S_3 is 483-dimensional, and S_6 is 1243-dimensional. We will prove that S_4 and S_5 are non-algebraic, and all other simple modules are algebraic.

The 759-dimensional permutation module, while not semisimple, is the direct sum of the 24-dimensional permutation representation (on the cosets of M_{23}), the 252-dimensional simple module V_1 in the fourth block, and the 483-dimensional module S_3 . Thus S_3 is a trivial-source module, and so algebraic.

Denote by P a Sylow 3-subgroup of G : this is extraspecial of type 3_+^{1+2} , so of exponent 3. The group P contains a unique normal subgroup Q of order 3. The restriction of the 22-dimensional module S_1 to P is given by

$$S_1 \downarrow_P = A \oplus B_1 \oplus B_2;$$

the B_i are permutation modules induced from subgroups of P of order 3, and the module A is the 4-dimensional source. This module has kernel Q , and so can be considered a module for $C_3 \times C_3$. Its tensor square has three summands, of dimensions 1, 6 and 9, and so this module is from Class C of Section 3.3.3. Hence this module is algebraic.

The fact that S_1 is algebraic proves that S_2 is also algebraic, since $S_2 = \Lambda^2(S_1)$. As $\Lambda^2(S_1)$ is a summand of $S_1^{\otimes 2}$, we see that S_2 is also algebraic.

Let S_4 denote one of the 770-dimensional simple modules. Restricting to P , we have

$$S_4 \downarrow_P = M \oplus 25 \cdot \mathcal{P}(K) \oplus 2 \cdot N_1 \oplus 2 \cdot N_2 \oplus N_3 \oplus N_4;$$

here, N_1 and N_2 are 18-dimensional modules induced from 2-dimensional modules for two different non-central subgroups of order 3, and N_3 and N_4 are 9-dimensional modules induced from the trivial module for the other two non-central subgroups of order 3, and M (or M^* , depending on the choice of S_4) is a 5-dimensional module with simple socle. This 5-dimensional module, the source of S_4 , has kernel Q , just as with the source for S_1 , and so can be viewed as a module for $C_3 \times C_3$. The tensor square of M has four summands, of dimensions 3, 3, 9, and 10, and so M comes from Class A (or Class A*) from Section 3.3.4. Hence S_4 and its dual S_5 are non-algebraic.

It remains to discuss the largest module in the principal block, the module S_6 of dimension 1243. We will slowly reduce the problem of algebraicity down until we reach something that can be managed. This module has a 19-dimensional source M , so we will analyze this module rather than the (rather unwieldy) 1243-dimensional module. The module $S^2(M)$ decomposes as the direct sum of K and seven copies of the projective indecomposable module KP ; thus we concern ourselves with the module $\Lambda^2(M)$. This module decomposes as five copies of the projective indecomposable module, together with two non-isomorphic 18-dimensional modules, which are induced from subgroups of order 9 in P . Hence we are interested in the 6-dimensional sources for the two 18-dimensional modules: these are from Class V in Section 3.3.5, and so are algebraic. Hence S_6 is algebraic, as required. This proves the final theorem in this section, Theorem 6.5.

6.2 Leech Lattice Groups

The Leech lattice groups are all (non-Mathieu) groups involved in the group Co_0 , the full automorphism group of the Leech lattice. They are the Higman–Sims group, the second Janko group, the three Conway groups, the McLaughlin group, and the Suzuki group. We will consider each of them in turn.

Our results for the Higman–Sims group are complete in characteristics 2 and 3. However, in characteristic 5, it seems difficult to prove anything concrete. We make some remarks in this direction at the end of Section 6.2.1. Even better, Janko’s second group J_2 is completely understood in all characteristics. The abundance of results in these two groups contrasts with the three Conway groups, about which nothing is known.

The final two groups are the McLaughlin group, for which some results are known in characteristic 3 but little else, and the Suzuki group, for which non-algebraic mod-

ules are known for $p = 2$ and $p = 3$, and two non-trivial algebraic modules have been found in characteristic 5.

6.2.1 The Higman–Sims Group HS

The Higman–Sims group HS is of order $44352000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. It has a 100-point permutation representation, as originally described in Donald Higman and Sims’ paper [45], and a 176-point permutation representation, as described soon after Higman–Sims’ original construction in Graham Higman’s paper [46], which means that a computer can study it with ease. Note that the point stabilizer under this 100-point representation is the Mathieu group M_{22} , which will be useful in this section. The primes p for which HS does not have a cyclic Sylow p -subgroup are 2, 3 and 5, and we summarize the dimensions of the simple modules, as usual, in the table below.

p	Block	Simple Modules	Defect Group
2	1	{1, 20, 56, 132, 518, 1000}	Sylow
	2	{896, 896*, 1408}	Defect 2
3	1	{1, 22, 154 ₁ , 321, 748, 1176, 1253}	Sylow
	2	{49, 49*, 77, 154 ₂ , 154 ₃ , 770, 770*}	Sylow
	3	{231, 825}	Defect 1
	4,5,6	{693}, {1386}, {2520}	Defect 0
5	1	{1, 21, 55, 98, 133 ₁ , 133 ₂ , 210, 280, 280*, 518}	Sylow
	2	{175, 650, 1275, 1925}	Defect 1
	3,4	{1750}, {2750}	Defect 0

Theorem 6.6 Let $G = HS$, the Higman–Sims sporadic simple group, and let K be a splitting field for G of characteristic p . Let M be a simple KG -module.

- (i) If $p = 2$, then M is algebraic if and only if M is trivial or lies outside the principal block.
- (ii) If $p = 3$, then M is algebraic.

After proving this theorem, we conclude the subsection with some remarks about the case where $p = 5$.

We begin with the prime 2: the six modules in the principal block and the 1408-dimensional module can be realized over $\text{GF}(2)$, whereas the two modules of dimension 896 require $\text{GF}(4)$.

To use the V_4 Restriction Test, we need information on the (conjugacy classes of) V_4 subgroups. There are five conjugacy classes of V_4 subgroup, with class lengths as given in the table.

Class	Number of Conjugates
\mathcal{C}_1	28875
\mathcal{C}_2	77000
\mathcal{C}_3	184800
\mathcal{C}_4	231000
\mathcal{C}_5	346500

The restriction of the 20-dimensional module S_1 to one of the subgroups in conjugacy class \mathcal{C}_1 is isomorphic with

$$2 \cdot K \oplus 2 \cdot \mathcal{P}(K) \oplus \Omega^2(K) \oplus \Omega^{-2}(K),$$

and so is not algebraic. The restrictions down to representatives from the other \mathcal{C}_i are algebraic.

The restriction of the 56-dimensional module S_2 to one of the subgroups in conjugacy class \mathcal{C}_1 is isomorphic with

$$2 \cdot M \oplus 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus 8 \cdot \mathcal{P}(K),$$

where M is the direct sum of the three non-isomorphic 2-dimensional indecomposable modules. This is therefore clearly not algebraic. The restrictions down to representatives from the other conjugacy classes \mathcal{C}_i are algebraic.

The restriction of the 132-dimensional module S_3 to one of the subgroups in conjugacy class \mathcal{C}_1 is isomorphic with

$$2 \cdot \Omega^2(K) \oplus 2 \cdot \Omega^{-2}(K) \oplus 28 \cdot \mathcal{P}(K),$$

and so is not algebraic. The restrictions down to representatives from the other conjugacy classes \mathcal{C}_i are algebraic.

The restriction of the 518-dimensional module S_4 to one of the subgroups in conjugacy class \mathcal{C}_1 is isomorphic with

$$120 \cdot \mathcal{P}(K) \oplus 2 \cdot \Omega^2(K) \oplus 2 \cdot \Omega^{-2}(K) \oplus \Omega^4(K) \oplus \Omega^{-4}(K),$$

whereas its restriction to one of the subgroups in conjugacy class \mathcal{C}_2 is isomorphic with

$$2 \cdot M \oplus 124 \cdot \mathcal{P}(K) \oplus \Omega^2(K) \oplus \Omega^{-2}(K),$$

where M is as above. The restrictions down to representatives from the conjugacy classes \mathcal{C}_3 , \mathcal{C}_4 and \mathcal{C}_5 are algebraic.

The restriction of the 1000-dimensional module S_5 to one of the subgroups in conjugacy class \mathcal{C}_1 is isomorphic with

$$4 \cdot K \oplus 8 \cdot M \oplus 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus 234 \cdot \mathcal{P}(K),$$

where M is as above. Hence S_5 is not algebraic. The restrictions down to representatives from the other conjugacy classes \mathcal{C}_i are algebraic. This deals with all modules from the principal block.

The second block is the unique block of defect 2, and in fact is of defect group V_4 . Therefore by Corollary 3.9, all simple modules in this block are algebraic.

We now turn our attention to characteristic 3: in this case, all of the simple modules are realizable over $\text{GF}(3)$. We will show that they are all algebraic. The group G has a maximal subgroup H_1 of index 100 and two conjugacy classes of maximal subgroup of index 1100, representatives of which will be denoted by H_2 and H_3 .

The 100-dimensional permutation representation on the cosets of H_1 is semisimple, with summands the trivial module, the 22-dimensional simple module S_1 in the principal block, and the 77-dimensional module T_3 in the second block. Since these three modules are trivial-source modules, they are algebraic. The permutation representation on the cosets of H_2 behaves very differently to that on the cosets of H_3 : the one contains a summand isomorphic with S_1 , and the other contains a summand isomorphic with the 154-dimensional simple module S_3 lying in the principal block. [One can distinguish between the three simple modules of that dimension using the fact that, if T_1 denotes one of the 49-dimensional simple modules, then $S_1 \otimes T_1$ is indecomposable and contains both 154-dimensional simple modules from the second block.]

If P denotes a Sylow 3-subgroup of G , and T_1 denotes a 49-dimensional simple module, as above, then

$$T_1 \downarrow_P = M \oplus \mathcal{P}(K),$$

where M is a 4-dimensional indecomposable $K(C_3 \times C_3)$ -module. We classified all such modules in Section 3.3.3: the tensor square of this module M decomposes as the sum of the trivial module, a projective module, and a 6-dimensional module. Hence it belongs to Class C from Section 3.3.3, and so is algebraic. The dual of T_1 , the module T_2 , has the same 4-dimensional source (as M is self-dual), and thus the modules T_1 and T_2 are algebraic. [Alternatively, one can note that the restriction of T_1 to the point stabilizer M_{22} is simple, and the 49-dimensional modules for M_{22} were proved to be algebraic. Then Proposition 2.25 proves the result.]

The exterior square of a 49-dimensional module T_1 is the 1176-dimensional simple module S_5 in the principal block. Since the 49-dimensional module is algebraic, and as the characteristic of the field is odd, we have

$$T_1^{\otimes 2} = \Lambda^2(T_1) \oplus S^2(T_1),$$

the module $S_5 = \Lambda^2(T_1)$ is algebraic.

To prove that the 1253-dimensional simple module S_6 is algebraic, we restrict to the point stabilizer H_1 under the 100-point permutation representation of HS . Recall that this group is the Mathieu group M_{22} , for which we proved that every simple module is algebraic when $K = \text{GF}(3)$. Restricting S_6 down to this subgroup H , we get

$$S_6 \downarrow_H = N_1 \oplus N_1^* \oplus N_2 \oplus N_3,$$

where N_1 is one of the 49-dimensional simple modules for M_{22} , the module N_2 is isomorphic with the 210-dimensional simple module for M_{22} , and N_3 is the projective cover of the 231-dimensional simple module. Each of the simple modules for M_{22} is algebraic, as are projective covers, and so this module $S_6 \downarrow_H$ is algebraic. Thus by Proposition 2.25, S_6 is algebraic.

The same method proves that the 321-dimensional module S_3 is algebraic: the restriction to H_1 is semisimple, and since all simple modules for M_{22} are algebraic, S_3 is algebraic since H_1 has index prime to 3.

For the rest of the modules, we will have to restrict to a Sylow 3-subgroup P , and work from there: for the 154-dimensional simple modules T_4 and T_5 in the second block, we get

$$T_4 \downarrow_P = K \oplus 17 \cdot \mathcal{P}(K), \quad T_5 \downarrow_P = K \oplus 17 \cdot \mathcal{P}(K),$$

so that both T_4 and T_5 are trivial-source modules, whence they are algebraic.

For the 748-dimensional simple module S_4 , we get

$$S_4 \downarrow_P = M \oplus 82 \cdot \mathcal{P}(K),$$

where M is a 10-dimensional indecomposable module. The module $M^{\otimes 2}$ is the direct sum of a trivial module, three 6-dimensional modules (the three non-isomorphic modules from Class V of Section 3.3.5) and nine projective summands. Thus $M^{\otimes 2}$ is algebraic, and so therefore is M . Since M is a source for S_4 , we see that S_4 is algebraic.

It remains to deal with the two 770-dimensional simple modules, T_6 and T_7 . Restricting the module T_6 to the Sylow subgroup P gives

$$T_6 \downarrow_P = 2 \cdot K \oplus M \oplus 84 \cdot \mathcal{P}(K),$$

where M is the sum of the four uniserial modules that are the projective covers of the trivial module for the four different cyclic quotients of P (i.e., the four modules that form Class C in Section 3.3.2). The module T_6 is therefore trivial source, so in particular is algebraic. Since T_7 is the dual of T_6 , it is algebraic, and we have dealt with all of the simple modules in the non-cyclic blocks.

Thus Theorem 6.6 is proved.

We now turn our attention to the prime 5. In this case, all modules are realizable over $\text{GF}(5)$. We cannot determine the algebraicity of any of the non-trivial simple modules in the principal block, although we can link the algebraicity of certain modules with others by means of decomposing tensor products.

The tensor square of the 21-dimensional module S_1 is semisimple, with

$$S_1^{\otimes 2} = K \oplus S_2 \oplus S_6 \oplus T_1,$$

where S_2 is the 55-dimensional simple module in the principal block, S_6 is the 210-dimensional simple module in the principal block, and T_1 is the 175-dimensional module in the block of defect 1.

The exterior square of the module S_2 is also semisimple, and $\Lambda^2(S_2) = S_6 \oplus T_3$. The exterior square of S_3 is similarly semisimple, and given by

$$\Lambda^2(S_3) = S_6 \oplus S_9 \oplus T_3 \oplus V_1.$$

The tensor product of S_1 and S_3 is semisimple as well, and is written

$$S_1 \otimes S_3 = S_2 \oplus S_6 \oplus S_9 \oplus T_3.$$

From these decompositions, we see that if S_1 is algebraic, then both S_2 and S_6 are algebraic, and if S_3 is algebraic then so are S_6 and S_9 .

The module S_4 has an 8-dimensional source, A , which appears to be algebraic, although this cannot be proved with the current techniques. The tensor square of A is given by

$$A \otimes A = K \oplus A \oplus B_1 \oplus B_2,$$

where B_1 is a 20-dimensional module and B_2 is a 35-dimensional module. Both are periodic, and so if, for odd p , every periodic module is algebraic, as might be possible, then S_4 and S_5 are indeed algebraic. To indicate the problem, we decompose some larger tensor powers of A : we have the first decomposition

$$A \otimes B_1 = B_1 \oplus 2 \cdot B_2 \oplus B_3,$$

where B_3 is a 70-dimensional (periodic) indecomposable module. Continuing, we get

$$A \otimes B_2 = 2 \cdot B_1 \oplus 3 \cdot B_2 \oplus B_3 \oplus B_4,$$

where B_4 is a 65-dimensional (periodic) indecomposable module. The tensor product $A \otimes B_3$ is given by

$$A \otimes B_3 = B_1 \oplus 2 \cdot B_3 \oplus B_5 \oplus B_6,$$

where B_5 is a 110-dimensional indecomposable module and B_6 is a 290-dimensional indecomposable module. The next decomposition is given by

$$A \otimes B_4 = B_2 \oplus B_3 \oplus B_6 \oplus \mathcal{P}(K).$$

The algorithms in Appendix A can be used to decompose the resulting 880-dimensional and 2320-dimensional modules $A \otimes B_5$ and $A \otimes B_6$, but the author has not pursued this yet.

The module S_7 has a 155-dimensional source. This is non-periodic, since its restriction down to one of the two conjugacy classes of elementary abelian subgroups of G is non-periodic. This is therefore unlikely to be algebraic, in light of Conjecture 3.31.

Putting all this together, we arrive at the following conjecture.

Conjecture 6.7 Let $G = HS$ and let K be a field of characteristic 5. Let M be a simple module. If M lies outside the principal block, then M is algebraic. If M lies inside the principal block, then M is algebraic if and only if M is trivial or has dimension 133.

This conjecture is far from being verified.

6.2.2 The Janko Group J_2

Janko's second sporadic simple group has order $604800 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, and is also known as the Hall–Janko group, HJ . It was considered first in [51], and existence and uniqueness are proved in [41]. It has a presentation

$$\langle a, b : a^2 = b^3 = (ab)^7 = [a, b]^{12} = (ababab^{-1}abab^{-1}ab^{-1}ababab^{-1}ab^{-1}abab^{-1})^3 = 1 \rangle,$$

although it can be generated as a permutation representation on a hundred points.

This group behaves similarly to the previous group, HS , in several respects. Firstly, the three relevant primes are 2, 3 and 5; secondly, for the prime 2, we again get two blocks, one of which has defect 2; and thirdly, for two primes we get the same result. We begin with the table of dimensions of simple modules for the relevant primes.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 6_1, 6_2, 14_1, 14_2, 36, 84\}$	Sylow
	2	$\{64_1, 64_2, 160\}$	Defect 2
3	1	$\{1, 13_1, 13_2, 21_1, 21_2, 57_1, 57_2, 133\}$	Sylow
	2,3	$\{36, 90\}, \{63, 225\}$	Defect 1
	4,5	$\{189_1\}, \{189_2\}$	Defect 0
5	1	$\{1, 14, 21, 41, 85, 189\}$	Sylow
	2	$\{70, 90\}$	Defect 1
	3,4,5	$\{175\}, \{225\}, \{300\}$	Defect 0

Theorem 6.8 Let G be the group J_2 , and let K be a splitting field for G of characteristic p . Let M be a simple KG -module.

- (i) If $p = 2$, then M is algebraic if and only if M is trivial or lies outside the principal block.
- (ii) If $p = 3$, then M is algebraic.
- (iii) If $p = 5$, then M is algebraic.

The first thing that we should note is that J_2 lies between $\text{PSU}_3(3) = G_2(2)'$ and $G_2(4)$, so that in particular,

$$\text{PSU}_3(3) < J_2.$$

Thus we can restrict the simple modules for J_2 down to modules for $\text{PSU}_3(3)$, and use the facts we know about $\text{PSU}_3(3)$ given in Section 5.6. The group $\text{PSU}_3(3)$ has non-algebraic simple modules of dimensions 6 and 14, and the 6- and 14-dimensional

simple modules for J_2 can easily be seen to restrict to these simple modules. Hence the simple modules S_i for $1 \leq i \leq 4$ are non-algebraic.

To determine the fate of the 36-dimensional simple module S_1 and the 84-dimensional simple module S_2 , we will apply the V_4 Restriction Test, which means that we need to know information about the conjugacy classes of V_4 subgroups.

Class	Number of Conjugates
\mathcal{C}_1	525
\mathcal{C}_2	840
\mathcal{C}_3	12600
\mathcal{C}_4	18900

Let P denote a subgroup from \mathcal{C}_1 . Restricting the modules S_5 and S_6 to the largest three classes results in an algebraic module, but

$$S_5 \downarrow_P = 2 \cdot K \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 6 \cdot \mathcal{P}(K)$$

and

$$S_6 \downarrow_P = 2 \cdot K \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 18 \cdot \mathcal{P}(K).$$

Hence both S_5 and S_6 are non-algebraic, completing the study of all modules in the principal block.

Of course, since the second block is the unique block of defect 2, all simple modules contained within it are algebraic by Corollary 3.9.

Moving on to characteristic 3, all of the modules in the principal block require $\text{GF}(9)$ to be realized except for K and the 133-dimensional module S_7 , which only require $\text{GF}(3)$.

Firstly, assume that S_1 is algebraic. Then

$$S_1^{\otimes 2} = K \oplus S_3 \oplus S_5 \oplus T_2,$$

so that both S_3 and S_5 are algebraic. Since S_2 , S_4 , and S_6 are the Frobenius twists of the modules S_1 , S_3 , and S_5 , these are also algebraic. Lastly, the tensor product $S_1 \otimes S_2$ is also semisimple, and is

$$S_1 \otimes S_2 = S_7 \oplus T_1,$$

so that all simple modules in the principal block are algebraic.

It remains to show that S_1 , or equivalently its source A_1 , is algebraic. The square of A_1 is given by

$$A_1 \otimes A_1 = K \oplus C_1 \oplus A_3 \oplus A_5 \oplus 5 \cdot \mathcal{P}(K),$$

where C_1 is the permutation module on the cosets of a subgroup of order 3, and A_3 and A_5 are the 21-dimensional and 3-dimensional sources of S_3 and S_5 respectively. Since C_1 has cyclic vertex, we may ignore this by Proposition 2.1. Then

$$A_1 \otimes A_3 = B_1 \oplus 9 \cdot \mathcal{P}(K)$$

and

$$A_1 \otimes A_5 = B_2 \oplus \mathcal{P}(K),$$

where B_1 is a 30-dimensional indecomposable module and B_2 is a 12-dimensional indecomposable. The modules B_1 and B_2 have tensor products with A_1 given by

$$A_1 \otimes B_1 = 2 \cdot A_3 \oplus B_3 \oplus 4 \cdot C_1 \oplus 11 \cdot \mathcal{P}(K)$$

and

$$A_1 \otimes B_2 = 2 \cdot A_5 \oplus B_4 \oplus 4 \cdot C_1 \oplus 4 \cdot \mathcal{P}(K),$$

where B_3 is a 15-dimensional indecomposable module and B_4 is a 6-dimensional indecomposable. Finally,

$$A_1 \otimes B_3 = C_1 \oplus A_3 \oplus B_1 \oplus 5 \cdot \mathcal{P}(K)$$

and

$$A_1 \otimes B_4 = A_5 \oplus B_2 \oplus C_1 \oplus \oplus 2 \cdot \mathcal{P}(K).$$

Thus A_1 , and hence S_1 , is algebraic, as required.

Now, let $p = 5$: in this case, all modules are realizable over $\text{GF}(5)$. As in the previous case, we begin by proving that if S_1 is algebraic then all simple modules are algebraic. To this end, suppose that the 14-dimensional S_1 is algebraic.

The tensor square of this module is semisimple, and is given by

$$S_1 \otimes S_1 = K \oplus S_1 \oplus S_2 \oplus T_1 \oplus T_2.$$

Thus S_2 is algebraic. The tensor square of S_2 is also semisimple, and given by

$$S_2 \otimes S_2 = K \oplus S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus S_5 \oplus T_2.$$

Thus all simple modules are algebraic, as claimed.

We will prove that the source A_1 of the simple module S_1 is algebraic, thus finishing the proof of Theorem 6.8. There are twenty-two different indecomposable modules lying in $T(A_1)$, and the exact decompositions are given in Section A.4.

6.2.3 The Conway Groups Co_1 , Co_2 and Co_3

The Conway groups are three sporadic simple groups found by John Conway in [27]. The 2-fold central extension of the largest of these is the group of automorphisms of the Leech lattice, a certain 24-dimensional unimodular lattice. Inside the simple group Co_1 are the maximal subgroups Co_2 and Co_3 .

The first Conway group has order $4157776806543360000 = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. Although it has some low-dimensional simple modules, it seems difficult to prove any algebraicity results about them.

The second Conway group has order $42305421312000 = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. This group is much smaller than Co_1 , but despite this, no algebraicity results are known either.

The smallest of the three Conway groups has order $495766656000 = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. Even for this group, no results are known. We will briefly describe why in characteristic 2. The dimensions and blocks of the simple modules in this characteristic are given below.

p	Block	Simple Modules	Defect Group
2	1	{1, 22, 230, 1496, 3520, 7084, 9372, 9372*, 38456, 88000}	Sylow
	2	{896, 896*, 19712, 73600, 131584}	Defect 3
	3	{129536}	Defect 1

All modules can be realized over $GF(2)$ except for the 896-dimensional modules, and possibly the 9372-dimensional modules, which the author has not constructed.

There are three conjugacy classes of V_4 subgroup of $G = Co_3$. For each of them, the simple modules S_i for $1 \leq i \leq 4$ have algebraic restriction to that subgroup. Consequently, it is difficult to analyze these modules.

It is hoped that in future, some results for these groups will become feasible, and in particular for the simple modules in the abelian block.

6.2.4 The McLaughlin Group McL

The McLaughlin group McL has order $898128000 = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$. The group was originally constructed in [62] as the (index 2 subgroup of the) automorphism group of a graph on 275 points, with point stabilizer $PSU_4(3)$, and uniqueness was proved by Janko and Wong in [53]. It can be characterized by the centralizer of an involution being isomorphic with the 2-fold central extension of the alternating group A_8 .

p	Block	Simple Modules	Defect Group
2	1	$\{1, 22, 230, 748, 748^*, 2124, 2124^*, 3584\}$	Sylow
	2	$\{3520\}$	Defect 1
	3,4,5,6	$\{896\}, \{896^*\}, \{9856\}, \{9856^*\}$	Defect 0
3	1	$\{1, 21, 104, 104^*, 210, 560, 605, 605^*, 1498, 2794\}$	Sylow
	2,3,4	$\{5103\}, \{8019\}, \{8019^*\}$	Defect 0
5	1	$\{1, 21, 210, 230, 560, 896, 896^*, 1200, 1200^*, 3038, 3245, 3245^*\}$	Sylow
	2,3,4,5,6	$\{1750\}, \{4500\}, \{8250\}, \{8250^*\}, \{9625\}$	Defect 0

We have the following result for this group, restricted to characteristic 3.

Proposition 6.9 Let G be the McLaughlin simple group, and let K be a field of characteristic 3. Then the smallest four non-trivial simple modules are non-algebraic.

There are four conjugacy classes of subgroup isomorphic with $C_3 \times C_3$, and their lengths are given in the following table.

Class	Number of Conjugates
\mathcal{C}_1	616000
\mathcal{C}_2	693000
\mathcal{C}_3	693000
\mathcal{C}_4	4158000

To see that the 21-dimensional simple module is non-algebraic, let P denote a subgroup from class \mathcal{C}_1 . Then

$$S_1 \downarrow_P = E \oplus M_1,$$

where M_1 is a 14-dimensional module, and E is the heart of the projective indecomposable module. This is non-algebraic by Proposition 2.19, and so S_1 is non-algebraic. This can also be used to prove that $S_4 = \Lambda^2(S_1)$ is non-algebraic.

Let E again denote the heart of the projective indecomposable module for $K(C_3 \times C_3)$. Then $S^2(E)$ is the direct sum of K and projective modules, and so is algebraic. Since E is not algebraic, $\Lambda^2(E)$ cannot be algebraic. Finally,

$$\Lambda^2(S_1 \downarrow_H) = \Lambda^2(E \oplus M_1) = \Lambda^2(E) \oplus \Lambda^2(M_1) \oplus E \otimes M_1,$$

and therefore S_4 is non-algebraic also.

Let Q be a representative from the conjugacy class \mathcal{C}_2 ; then we have the formula

$$S_2 \downarrow_Q = M_2 \oplus M_3 \oplus N \oplus \Omega(K) \oplus 8 \cdot \mathcal{P}K,$$

where M_2 and M_3 are 6-dimensional indecomposable modules from Class B in Section 3.3.5 and N is a 12-dimensional indecomposable module. Hence S_2 and the dual S_3 are non-algebraic. [The restriction of S_2 to a subgroup from class \mathcal{C}_3 is the sum of a projective module and a 32-dimensional indecomposable module.]

This proves Proposition 6.9.

In the case where $p = 2$, all modules in the principal block are realizable over $\text{GF}(2)$. However, all modules in the principal block have algebraic restrictions to the single conjugacy class of V_4 subgroups. The author has not explored other means of proving non-algebraicity for this group. There are no results for characteristic 5.

6.2.5 The Suzuki Group Suz

Suzuki's sporadic group has order $448345497600 = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. It was discovered, as the name suggests, by Suzuki in [75], and uniqueness was proved by Patterson and Wong in [68], characterizing Suz by the centralizer of a central involution, and by Yamaki in [79], characterizing Suz by its Sylow 2-subgroup. It has a 1782-dimensional permutation representation on the cosets of the maximal subgroup $G_2(4)$, which makes it relatively easy to work with on a computer.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 110_1, 110_2, 142, 572, 572^*, 638, 3432, 4510, 4928, 9328_1, 9328_2, 10504_1, 10504_2\}$	Sylow
	2	$\{66560, 79872, 102400\}$	Defect 3
3	1	$\{1, 64, 78, 286, 429, 649, 1938, 2925, 4785, 8436, 14730, 19449, 32967\}$	Sylow
	2	$\{5103_1, 5103_2, 15795, 72657, 160380\}$	Defect 2
	3	$\{18954, 189540\}$	Defect 1
5	1	$\{1, 143, 363, 1001, 3289, 11869, 16785, 18953, 41822, 75582, 85293, 116127\}$	Sylow
	2,3	$\{780, 15015\}, \{5005, 5005^*, 93555_1, 93555_2\}$	Defect 1
	4	$\{5940, 40040, 60620, 128920\}$	Defect 1
	5,6,7,8,9	$\{10725\}, \{14300\}, \{25025_1\}, \{25025_2\}, \{25025_2^*\}$	Defect 0
	10,11,12,13 14,15,16,17	$\{50050\}, \{50050^*\}, \{64350_1\}, \{64350_2\}, \{75075\}, \{100100\}, \{163800\}, \{193050\},$	Defect 0 Defect 0

This group has the following result attached.

Theorem 6.10 Let G be the Suzuki sporadic simple group.

- (i) If $p = 2$, then the nine smallest non-trivial simple modules are non-algebraic.

- (ii) If $p = 3$, then the seven smallest non-trivial simple modules are non-algebraic. The 15795-dimensional module is algebraic.
- (iii) If $p = 5$, then the 143-dimensional and 1001-dimensional simple modules are algebraic.

We analyze the case where $p = 2$ firstly: in this case, the 110-dimensional, 572-dimensional, and 9328-dimensional simple modules require $\text{GF}(4)$. The author does not know whether the 10504-dimensional modules require a cube root of unity to exist in the field. All other modules can be realized over any field.

The group $G = \text{Suz}$ contains a maximal subgroup isomorphic with $J_2 \rtimes C_2$, and the restriction of S_1 to this subgroup, while not semisimple, has an 84-dimensional simple module as a summand. This module remains simple upon restriction to the sporadic group J_2 , and from the results of Section 6.2.2, this module is non-algebraic. Hence S_1 and S_2 are non-algebraic simple modules.

We can restrict the simple module S_3 to the sporadic group J_2 firstly, and then apply the V_4 Restriction Test to prove non-algebraicity. Recall that there are four conjugacy classes of V_4 subgroup lying in J_2 , as given in Section 6.2.2. The restrictions of S_3 to subgroups from the largest three of these classes are algebraic, but the restriction of S_3 to a subgroup Q from the smallest class is

$$6 \cdot K \oplus 4 \cdot \Omega^2(K) \oplus 4 \cdot \Omega^{-2}(K) \oplus 24 \cdot \mathcal{P}(K).$$

The simple module S_4 restricts to the same V_4 subgroup Q of J_2 as

$$S_4 \downarrow_Q = 8 \cdot \Omega(K) \oplus 8 \cdot \Omega^{-1}(K) \oplus 2 \cdot \Omega^3(K) \oplus 2 \cdot \Omega^{-2}(K) \oplus 124 \cdot \mathcal{P}(K).$$

Hence both S_4 and $S_5 = S_4^*$ are non-algebraic. The subgroup Q can also be used to prove that S_6 is non-algebraic, since

$$S_6 \downarrow_Q = 4 \cdot K \oplus 4 \cdot \Omega(K) \oplus 4 \cdot \Omega^{-1}(K) \oplus \Omega^4(K) \oplus \Omega^{-4}(K) \oplus 148 \cdot \mathcal{P}(K).$$

This trend continues: in fact,

$$\begin{aligned} S_7 \downarrow_Q = & 8 \cdot K \oplus 16 \cdot \Omega(K) \oplus 16 \cdot \Omega^{-1}(K) \oplus 4 \cdot \Omega^2(K) \\ & \oplus 4 \cdot \Omega^{-2}(K) \oplus 4 \cdot \Omega^3(K) \oplus 4 \cdot \Omega^{-3}(K) \oplus 808 \cdot \mathcal{P}(K), \end{aligned}$$

and

$$S_8 \downarrow_Q = 4 \cdot K \oplus 4 \cdot \Omega(K) \oplus 4 \cdot \Omega^{-1}(K) \oplus \Omega^4(K) \oplus \Omega^{-4}(K) \oplus 1116 \cdot \mathcal{P}(K).$$

The largest module we will consider, S_9 , has decomposition

$$S_9 \downarrow_Q = 24 \cdot K \oplus 16 \cdot \Omega^2(K) \oplus 16 \cdot \Omega^{-2}(K) \oplus 4 \cdot \Omega^4(K) \oplus 4 \cdot \Omega^{-4}(K) \oplus 1184 \cdot \mathcal{P}(K).$$

Now let us consider characteristic 3. In this case, all modules are realizable over $\text{GF}(3)$, except possibly the 5103-dimensional modules, whose field of definition the author does not know.

There are eight conjugacy classes of $C_3 \times C_3$ subgroup, and their lengths are given below.

Class	Number of Conjugates	Class	Number of Conjugates
\mathcal{C}_1	3203200	\mathcal{C}_5	153753600
\mathcal{C}_2	17297280	\mathcal{C}_6	461260800
\mathcal{C}_3	38438400	\mathcal{C}_7	461260800
\mathcal{C}_4	76876800	\mathcal{C}_8	1383782400

Let P denote a subgroup from class \mathcal{C}_3 . Then the restriction of S_1 to P is

$$S_1 \downarrow_P = E \oplus \Omega(A) \oplus \Omega^{-1}(A^*) \oplus 3 \cdot \mathcal{P}(K),$$

where E is the heart of the projective indecomposable module, and A is the unique 3-dimensional module from Class A of Section 3.3.2. The module S_1 is non-algebraic either since E is non-algebraic (Proposition 2.19) or because $\Omega(A)$ is non-algebraic (Corollary 2.12).

The same subgroup proves that S_2 is non-algebraic. This restriction is given by

$$S_2 \downarrow_P = 6 \cdot \mathcal{P}(K) \oplus \Omega^2(A) \oplus \Omega^{-2}(A^*),$$

where A is as above. The module $\Omega^2(A)$ is also non-algebraic, and so S_2 is non-algebraic. This proves that S_6 is non-algebraic as well: as we stated in the previous section, $\Lambda^2(E)$ is non-algebraic, and is a 21-dimensional indecomposable module. The module $\Lambda^2(S_1)$ is semisimple, and is in fact isomorphic to $S_2 \oplus S_6$. Recall that

$$\Lambda^2 \left(\bigoplus_{i=1}^n N_i \right) = \bigoplus_{i=1}^n \Lambda^2(N_i) \oplus \bigoplus_{i < j} N_i \otimes N_j.$$

Since E is a summand of $S_1 \downarrow_P$, the non-algebraic module $\Lambda^2(E)$ must be a summand of

$$\Lambda^2(S_1 \downarrow_P) = S_2 \downarrow_P \oplus S_6 \downarrow_P.$$

However, $\Lambda^2(E)$ is not a summand of $S_2 \downarrow_P$ since we decomposed that above. Hence S_6 is non-algebraic.

The same technique proves that S_7 is non-algebraic as well: we have $\Lambda^2(S_2) = S_2 \oplus S_7$, and

$$\Lambda^2(\Omega^2(A)) = 4 \cdot \mathcal{P}(K) \oplus \Omega^{-4}(A^*),$$

and since the 39-dimensional indecomposable module $\Omega^{-4}(A^*)$ is not a summand of $S_2 \downarrow_P$, it must belong to $S_7 \downarrow_P$. Thus S_7 is non-algebraic.

Coming back to S_3 , we restrict to the same subgroup P , to get

$$S_3 \downarrow_P = K \oplus \Omega^2(A) \oplus \Omega^{-2}(A^*) \oplus M_1 \oplus 28 \cdot \mathcal{P}(K),$$

where M_1 is a self-dual, non-projective, indecomposable, 9-dimensional module. Thus S_3 is not algebraic, since $\Omega^2(A)$ is not.

Examining the restriction of S_4 to this subgroup P , we see that

$$S_4 \downarrow_P = \Omega(A) \oplus \Omega^{-1}(A^*) \oplus 41 \cdot \mathcal{P}(K) \oplus M_2 \oplus M_2^*,$$

(where M_2 is a 15-dimensional indecomposable module) and so S_4 is non-algebraic as well.

Finally, consider the module $S_5 \downarrow_P$. This has similar summands to before, and in fact,

$$S_5 \downarrow_P = E \oplus \Omega(A) \oplus \Omega^{-1}(A^*) \oplus 68 \cdot \mathcal{P}(K),$$

proving that S_5 is indeed non-algebraic, as claimed.

To prove that the 15795-dimensional module T_3 is algebraic, it suffices to show that it has trivial source. The permutation module on the cosets of the maximal subgroup of index 22,880 has a unique composition factor from the second block, namely T_3 . Hence T_3 must, in fact, be a summand of this permutation module.

Lastly, suppose that the characteristic of K is 5, and let P denote a Sylow 5-subgroup. Let S_1 denote the simple module of dimension 143. This module has a 28-dimensional source N , and this module satisfies

$$\Lambda^2(N) = N \oplus 14 \cdot \mathcal{P}(K)$$

and

$$S^2(N) = K \oplus X \oplus 15 \cdot \mathcal{P}(K),$$

where X is a sum of the six non-isomorphic permutation modules on the six subgroups of P of order 5. Hence N is algebraic, since modulo the ideal generated by modules of cyclic vertex, it satisfies the polynomial $x^2 = x + 1$. Thus S_4 and S_5 are both algebraic.

The simple module S_3 is algebraic, since it is a trivial-source module. To see this, note that the permutation representation on the 1782 points of the maximal subgroup $G_2(4)$ is semisimple, and this decomposes as

$$K \oplus S_3 \oplus T_1.$$

This proves all parts of Theorem 6.10.

6.3 Monster Sections

The last remaining sporadic groups that are sections of the Monster are given here. The sheer size of some of these groups makes anything more than cursory remarks impossible. For instance, we prove nothing about the Baby Monster or the Monster. However, some results are given for all other groups in this section, although in the case of Fi'_{24} , we analyze only one module.

6.3.1 The Held Group He

The Held group has order $4030387200 = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$. It has a 2058-dimensional permutation representation on the cosets of a maximal subgroup isomorphic with a subgroup H isomorphic with $Sp_4(4) \rtimes C_2$. It first appeared in [43], where Held examines the possible groups with centralizer of an involution isomorphic to one of those in M_{24} . The normalizer structure and conjugacy classes of such a group are described, but the group was eventually constructed in unpublished work of Graham Higman and McKay.

Theorem 6.11 Let $G = He$ be the Held sporadic simple group. Let K be a splitting field of characteristic p , and let M be a simple KG -module.

- (i) Suppose that $p = 2$. Then M is algebraic if and only if M is trivial or M lies outside the principal block.
- (ii) Suppose that $p = 3$. Then M is algebraic if M lies outside the principal block, or is one of the four smallest simple modules in the principal block.
- (iii) Suppose that $p = 5$. Then M is algebraic if M lies outside the principal block, or if M is one of the three smallest simple modules in the principal block.

We now give the table of simple modules.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 51, 51^*, 101, 101^*, 246_1, 246_2, 680, 2008, 2449, 2449^*\}$	Sylow
	2	$\{1920, 4352, 4608\}$	Defect 2
	3,4	$\{21504_1\}, \{21504_2\}$	Defect 0
3	1	$\{1, 679, 1275_1, 3673, 6172, 6272, 10879\}$	Sylow
	2	$\{51, 51^*, 1029_1, 1029_2, 1275_2, 1275_2^*, 1920\}$	Defect 2
	3,4,5	$\{153, 7497\}, \{153^*, 7497^*\}, \{7650, 14400\}$	Defect 1
	6,7	$\{11475\}, \{11475^*\}$	Defect 0
5	1	$\{1, 51, 51^*, 104, 153, 153^*, 925_1, 925_2, 3197, 3197^*, 4116, 4249, 6528, 10860\}$	Sylow
	2	$\{680, 1240, 4080, 9640\}$	Defect 1
	3,4,5,6,7	$\{1275_1\}, \{1275_2\}, \{1275_2^*\}, \{7650_1\}, \{7650_1^*\}$	Defect 0
	8,9,10,11	$\{7650_2\}, \{11475\}, \{11475^*\}, \{11900\}$	Defect 0
	12,13,14	$\{14400\}, \{20825\}, \{22050\}$	Defect 0
7	1	$\{1, 50, 153, 426, 798, 1072, 1700, 3654, 4249, 6154\}$	Sylow
	2	$\{6272, 7497, 14553\}$	Defect 1
	3,4,5,6,7	$\{1029_1\}, \{1029_2\}, \{13720\}, \{17493\}, \{23324\}$	Defect 0

We begin our analysis with characteristic 2: all simple modules from the principal block are realizable over $\text{GF}(2)$, which makes computation easier. The author does not know whether the two projective simple modules can be realized over $\text{GF}(2)$, or whether they require $\text{GF}(4)$.

There are eight conjugacy classes of subgroups isomorphic with V_4 , and their conjugacy class sizes are given in the table below.

Class	Number of Conjugates	Class	Number of Conjugates
\mathcal{C}_1	8330	\mathcal{C}_5	3935925
\mathcal{C}_2	437325	\mathcal{C}_6	5247900
\mathcal{C}_3	437325	\mathcal{C}_7	5247900
\mathcal{C}_4	999600	\mathcal{C}_8	7871850

Since both \mathcal{C}_2 and \mathcal{C}_3 , and \mathcal{C}_6 and \mathcal{C}_7 , have the same conjugacy class length, we will avoid using results based on these conjugacy classes where we can. However, this is not normally possible, and so there is a small amount of choice involved, both in the labelling of the simple modules and in the labelling of the conjugacy classes. This will not affect our results.

The 51-dimensional simple modules are algebraic upon restriction to the representatives from the conjugacy classes $\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_5$ and \mathcal{C}_8 , and so we immediately have to use the paired conjugacy classes. The labelling of the simple modules S_1 and S_2 , and that of the \mathcal{C}_i , can be chosen so that the restriction of S_1 to a subgroup from \mathcal{C}_2 is isomorphic with

$$2 \cdot K \oplus 10 \cdot \mathcal{P}(K) \oplus \Omega^4(K),$$

and the restriction to a subgroup in class \mathcal{C}_6 is given by

$$M \oplus \Omega^2(K) \oplus 10 \cdot \mathcal{P}(K),$$

where M is the direct sum of the three non-isomorphic indecomposable modules of dimension 2. Thus S_1 and $S_2 = S_1^*$ are not algebraic.

The 101-dimensional simple modules are similarly algebraic upon restriction to the representatives from the conjugacy classes $\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_5$ and \mathcal{C}_8 , and so we again have to use the paired conjugacy classes. The simple module S_3 , and the labelling of the \mathcal{C}_i , can be chosen so that the restriction of S_3 to a subgroup in class \mathcal{C}_2 is given by

$$\Omega(K) \oplus 2 \cdot \Omega^{-2}(K) \oplus 22 \cdot \mathcal{P}(K),$$

and the restriction to a subgroup in class \mathcal{C}_6 is given by

$$M \oplus \Omega(K) \oplus 23 \cdot \mathcal{P}(K),$$

where M is as above. Thus S_3 and $S_4 = S_3^*$ are not algebraic.

Next, we consider the 246-dimensional simple modules. The restriction of S_5 to subgroups from \mathcal{C}_5 and \mathcal{C}_6 are algebraic, but restrictions to the others are non-algebraic. For example restricting the 246-dimensional module to a subgroup from \mathcal{C}_4 , we get

$$60 \cdot \mathcal{P}(K) \oplus \Omega(K) \oplus \Omega^{-1}(K).$$

Keeping the same choice of conjugacy classes, the restriction of S_6 to subgroups from \mathcal{C}_5 and \mathcal{C}_7 are algebraic, but restrictions to the others are similarly non-algebraic. For instance, both S_5 and S_6 have the same restrictions to subgroups from \mathcal{C}_4 . Hence S_5 and S_6 are non-algebraic.

The 680-dimensional module S_7 is the next smallest to be analyzed. The restriction of this module to a subgroup from $\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7$ or \mathcal{C}_8 is algebraic. However, the restriction of S_7 to a subgroup from either \mathcal{C}_2 or \mathcal{C}_3 is given by

$$4 \cdot K \oplus 2 \cdot \Omega^4(K) \oplus 2 \cdot \Omega^{-4}(K) \oplus 160 \cdot \mathcal{P}(K).$$

Hence this module is non-algebraic as well.

There remain the 2008-dimensional module S_8 and the two dual 2449-dimensional modules S_9 and S_{10} . The module S_8 is non-algebraic, since its restriction to a subgroup from either class \mathcal{C}_2 or \mathcal{C}_3 is given by

$$4 \cdot K \oplus 2 \cdot \Omega^3(K) \oplus 2 \cdot \Omega^{-3}(K) \oplus 494 \cdot \mathcal{P}(K).$$

The restriction of the largest two simple modules in the principal block, the modules S_9 and S_{10} , to six of the eight conjugacy classes of V_4 subgroup are algebraic. However, the restriction of S_9 to a subgroup from \mathcal{C}_6 is given by

$$\Omega^4(K) \oplus 610 \cdot \mathcal{P}(K),$$

and S_{10} restricts to the dual of this module. Hence these two modules are also non-algebraic.

Finally, He contains a single block of defect 2 which, as in all previous occasions, consists solely of algebraic modules. The same can be said for the two projective simple modules, and hence we arrive at our result.

Let us now consider the case where $p = 3$: in this case, we will need to extend our field to $\text{GF}(9)$ in all cases where there are two modules of the same dimension, with the possible exception of the two projective simple modules, whose smallest field of definition is not known to the author.

Let P denote a Sylow 3-subgroup of G , which is extraspecial of order 27 and of exponent 3. Let Q denote the unique normal subgroup of order 3.

Several of the modules in characteristic 3 are trivial-source modules. For example, the 2058-point permutation representation is not semisimple, but has summands isomorphic with T_1 , T_2 and S_2 . (The remaining summand, which we will label X , is uniserial, with socle layers consisting of K , S_1 and K .) The exterior square of T_1 is T_5 , and similarly $\Lambda^2(T_2) = T_6$, and so these two modules are both algebraic and trivial-source modules. In fact,

$$T_1 \otimes T_1 = T_2 \oplus S_3 \oplus T_6.$$

The tensor product of the two 51-dimensional modules is not semisimple, but has summands isomorphic with T_7 and X , the module described above. Hence T_7 is also a trivial-source module.

The 2058-dimensional permutation module is on the cosets of the maximal subgroup $\text{PSp}_4(4) \rtimes C_2$. Taking the 4116-dimensional permutation module on the cosets of the subgroup $\text{PSp}_4(4)$, we get the module

$$T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus S_3 \oplus X,$$

and so T_3 and T_4 are also algebraic. This proves that all modules from outside the principal block are algebraic.

The module S_1 is algebraic, since

$$S_1 \downarrow_P = M_1 \oplus \mathcal{P}(K_{P/Q}) \oplus 22 \cdot \mathcal{P}(K) \oplus A_1 \oplus A_2,$$

where A_1 and A_2 are permutation modules on subgroups of P of order 3, and M_1 is the 4-dimensional source. The kernel of the source M_1 is Q , and so M_1 can be viewed as a module for $C_3 \times C_3$, and if viewed as such, it comes from Class C in Section 3.3.3. Hence M_1 is algebraic.

The module S_3 is also algebraic: the 19-dimensional source M_2 of S_3 has tensor square

$$K \oplus B_1 \oplus B_2 \oplus 12 \cdot \mathcal{P}(K),$$

where B_1 and B_2 are 18-dimensional modules induced from (different) subgroups of order 9. The 6-dimensional sources for these two modules are from the algebraic Class V in Section 3.3.5, and so $M_2^{\otimes 2}$ is algebraic.

The remaining simple modules are difficult to work with. Only the simple module of dimension 6272 is relatively easy to construct, lying as a composition factor in the 29155-dimensional permutation module. The author has not yet managed to construct the source of this module, and so no analysis can take place.

Finally, suppose that the characteristic is 5, and let P denote a Sylow 5-subgroup. The 2058-dimensional permutation representation is semisimple, with the two 51-dimensional modules S_1 and S_2 as composition factors.

Now consider the two (dual) 153-dimensional modules S_4 and S_5 . The module S_4 has a 28-dimensional source N , which is isomorphic with the 28-dimensional source of the smallest (non-trivial) simple module for Suz in characteristic 5. Since that module is algebraic, so is this.

The tensor product (with some choice of S_4 and S_5) of S_1 and S_4 is given by

$$S_1 \otimes S_4 = S_{13} \oplus V_1$$

where V_1 is one of the two 1275-dimensional projective simple modules. Hence S_{13} is also algebraic.

This completes the proof of Theorem 6.11.

When the characteristic of the field is 7, we cannot say very much. For example, the modules S_1 and S_2 remain indecomposable upon restriction to the Sylow 7-subgroup, and so these modules are difficult to work with. The module $\Lambda^2(S_1)$ is semisimple, and in fact

$$\Lambda^2(S_1) = S_2 \oplus S_5.$$

The symmetric square is also semisimple, and we have

$$S^2(S_1) = K \oplus S_1 \oplus S_3 \oplus S_4.$$

Thus if S_1 is algebraic, then so are S_2, S_3, S_4 and S_5 . Also,

$$\Lambda^2(S_2) = S_2 \oplus S_5 \oplus S_8 \oplus S_9,$$

and

$$S_1 \otimes S_2 = S_1 \oplus S_2 \oplus S_3 \oplus S_5 \oplus S_6 \oplus S_8.$$

Thus if S_1 is algebraic, then so are S_6, S_8 and S_9 as well.

6.3.2 The Harada–Norton Group HN

The Harada–Norton group has order $273030912000000 = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$. Harada, in [42], determined many of the properties of HN , starting from the fact that G is a simple group with centralizer of involution isomorphic with a double cover of either HS or its automorphism group. The group itself was constructed by Simon Norton in his Ph.D. thesis [64].

The smallest permutation representation of HN is on the cosets of a maximal subgroup isomorphic with the alternating group A_{12} , and this representation has dimension 1,140,000. This makes calculations within this group especially difficult. The best way appears to be to restrict the representation to a maximal subgroup, one in which calculations are easier.

The subgroup A_{12} appears ideal here: although some information is lost in restricting to this subgroup, enough is retained, at least with the few representations available, to prove that these modules are non-algebraic.

The modular characters and decomposition matrices have been calculated in all characteristics apart from 3, and their degrees are given below.

p	Block	Simple Modules	Defect Group
2	1	{1, 132 ₁ , 132 ₂ , 760, 2650, 2650*, 3344, 15904, 31086 ₁ , 31086 ₂ , 34352 ₁ , 34352 ₂ , 43416, 43416*, 177286, 217130, 1556136}	Sylow
	2	{2140161, 13619201, 29859841}	Defect 4
	3	{3424256}	Defect 0
5	1	{1, 133, 626, 2451, 6326, 8152, 9271, 54473, 69255, 84798, 131747, 145275, 170258, 335293, 638571, 784379}	Sylow
	2	{653125, 2131250, 2678125, 3200000}	Defect 1
	3,4,5,6	{656250 ₁ }, {656250 ₂ }, {2375000}, {4156250}	Defect 0

For this group, we have the following result.

Theorem 6.12 Let G be the sporadic simple group HN , and let K be a field of characteristic p .

- (i) If $p = 2$, then the six smallest non-trivial KG -modules are all non-algebraic.
- (ii) If $p = 3$, then the two simple modules of dimension 133 are non-algebraic.

Of the modules that can be constructed, the 760-dimensional and 3344-dimensional can be written over $\text{GF}(2)$, whereas the two 132-dimensional and two 2650-dimensional modules require a cube root of unity.

Let H denote a maximal subgroup of G isomorphic with A_{12} . There are thirteen conjugacy classes of V_4 subgroups of H , and the 132-dimensional simple module S_1 is algebraic upon restriction to eleven of them. We therefore make the judicious choice of P , a subgroup isomorphic with V_4 , from the conjugacy class with 103,950 elements. (This is the third smallest class.)

With this choice of subgroup, one can see that the modules S_1 , S_2 and S_3 are all non-algebraic, since

$$S_1 \downarrow_P = 2 \cdot K \oplus \Omega^4(K) \oplus \Omega^{-4}(K) \oplus 28 \cdot \mathcal{P}(K)$$

and

$$S_3 \downarrow_P = \Omega(K) \oplus \Omega^{-1}(K) \oplus 3 \cdot \Omega^3(K) \oplus 3 \cdot \Omega^{-1}(K) \oplus 178 \cdot \mathcal{P}(K).$$

In fact, this subgroup can detect the non-algebraicity of S_4 , S_5 and S_6 as well, since

$$S_4 \downarrow_P = 2 \cdot K \oplus 8 \cdot M \oplus 3 \cdot \Omega^2(K) \oplus 3 \cdot \Omega^{-2}(K) \oplus \Omega^4(K) \oplus \Omega^{-4}(K) \oplus 639 \cdot \mathcal{P}(K),$$

and

$$S_6 \downarrow_P = 8 \cdot K \oplus 4 \cdot \Omega^4(K) \oplus 4 \cdot \Omega^{-4}(K) \oplus 816 \cdot \mathcal{P}(K).$$

Now consider the prime 3: we again take restrictions to H . There are eight conjugacy classes of subgroup of A_{12} of isomorphism type $C_3 \times C_3$. Two of these have 61,600 elements in them, and the restriction of a 133-dimensional module S_1 over $\text{GF}(9)$ to either of them is

$$\Omega^3(K) \oplus \Omega^{-3}(K) \oplus 11 \cdot \mathcal{P}(K).$$

These decompositions provide the proof of Theorem 6.12.

6.3.3 The Thompson Group Th

The Thompson sporadic simple group has order $90745943887872000 = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$. The first evidence for this group appeared in [76], and existence was proved by a computer construction of Smith [74], with uniqueness coming via a paper

of Parrott [67]. The smallest permutation representation of Th is on 143,127,000 points, on the cosets of a semidirect product of the Steinberg triality group ${}^3D_4(2)$ by C_3 . As with the Harada–Norton group, we work with maximal subgroups, and hope that we do not lose too much information in passing down.

In all characteristics, there is a non-trivial simple module of dimension 248.

Proposition 6.13 For $p = 2$ or $p = 3$, the smallest non-trivial representation of the Thompson sporadic simple group over a field of characteristic p , which has dimension 248, is non-algebraic.

The Thompson group possesses maximal subgroups isomorphic with $PSU_3(8) \rtimes C_6$, so let H be such a subgroup. We will deal with the characteristic 2 case first. The group H contains two conjugacy classes of V_4 subgroup, namely the class \mathcal{C}_1 with 3591 elements and the class \mathcal{C}_2 with 344736 elements. While the restriction of S_1 to a subgroup from class \mathcal{C}_2 is algebraic, the restriction to a subgroup from class \mathcal{C}_1 is the module

$$\Omega(K) \oplus \Omega^{-1}(K) \oplus 3 \cdot \Omega^3(K) \oplus 3 \cdot \Omega^{-3}(K) \oplus 50 \cdot \mathcal{P}(K).$$

This proves the proposition for $p = 2$.

Now consider the module for $p = 3$. There are eight conjugacy classes of subgroup isomorphic with $C_3 \times C_3$ lying in H , so let P denote a representative from the smallest conjugacy class, which has length 25,536. Then the restriction of the 248-dimensional module to this subgroup P is given by

$$\Omega^3(K) \oplus \Omega^{-3}(K) \oplus 20 \cdot \mathcal{P}(K) \oplus X \oplus M \oplus M^*,$$

where X is a sum of four 3-dimensional modules and M is an 11-dimensional indecomposable module. Hence the smallest non-trivial simple module is non-algebraic in characteristic 3 as well.

6.3.4 The Fischer Group Fi_{22}

The Fischer group Fi_{22} has order $64561751654400 = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, and has a permutation representation on 3510 points. This is the smallest of Fischer’s three simple groups, generated by so-called 3-transpositions. The first details appeared in [37], although the rest of the proof of Fischer’s theorem on 3-transposition groups remained unpublished until Aschbacher’s account in [9].

The table of dimensions of simple modules is given below.

p	Block	Simple Modules	Defect Group
2	1	{1, 78, 350, 572, 1352, 5824, 5824*, 7942, 24596, 31668, 34892, 62952 ₁ , 62952 ₂ , 163084}	Sylow
	2	{2555904}	Defect 1
	3	{1441792}	Defect 0
3	1	{1, 77, 351, 924, 2651, 2926, 4823, 4823*, 7722, 12474 ₁ , 12474 ₂ , 17226, 21175, 29821, 50050, 55573, 92378, 92378*, 133925, 182897, 182897*, 193479}	Sylow
	2,3	{360855, 577368}, {852930, 1876446}	Defect 1
	4	{1791153}	Defect 0
5	1	{1, 78, 428, 1001, 3003, 31954, 47620, 80653, 223808, 254254, 322035, 492920, 618905, 906269, 1156715, 1313090}	Sylow
	2	{1430, 3080, 41965, 318890}	Defect 1
	3	{30030, 43680, 175890, 677040}	Defect 1
		35 blocks of defect 0	

We have the following result for this group.

Theorem 6.14 Let $G \cong Fi_{22}$, the smallest sporadic group of Fischer, and let K be a field of characteristic p .

- (i) If $p = 2$, then the four smallest non-trivial simple modules are non-algebraic.
- (ii) If $p = 3$, then the smallest two non-trivial simple modules are non-algebraic, as well as the 2651-dimensional and 2926-dimensional simple modules.
- (iii) If $p = 5$, then the smallest two non-trivial simple modules are algebraic, as well as the 3003-dimensional simple module.

We begin with characteristic 2. In this case the 5824-dimensional modules require $GF(4)$, and the author does not know whether the 62952-dimensional modules require $GF(4)$. All other modules obviously only require $GF(2)$.

The smallest maximal subgroup of Fi_{22} is isomorphic with the Mathieu group M_{12} . Earlier in the chapter, we stated that M_{12} possesses four conjugacy classes of V_4 subgroup, representatives of which we will denote by P_i for $1 \leq i \leq 4$, in accordance with the labelling from Section 6.1.2.

The simple module S_1 has algebraic restrictions to P_2 and P_3 , whereas its restrictions to P_1 and P_4 are given by

$$S_1 \downarrow_{P_1} = \Omega^3(K) \oplus \Omega^{-3}(K) \oplus 16 \cdot \mathcal{P}(K),$$

and

$$S_1 \downarrow_{P_4} = X \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus 12 \cdot \mathcal{P}(K),$$

where X is a sum of 2-dimensional indecomposable modules.

The simple module S_2 is also non-algebraic: the restriction of S_2 to P_3 is algebraic, whereas the other subgroups can detect the non-algebraicity of S_2 . For example,

$$S_2 \downarrow_{P_1} = 2 \cdot K \oplus X' \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus \Omega^4(K) \oplus \Omega^{-4}(K) \oplus 78 \cdot \mathcal{P}(K),$$

where again X' is a sum of 2-dimensional indecomposable modules.

The restriction of S_3 to either P_2 or P_3 is non-algebraic, whereas the restriction to P_1 is given by

$$S_3 \downarrow_{P_1} = \Omega(K) \oplus \Omega^{-1}(K) \oplus 3 \cdot \Omega^2(K) \oplus 3 \cdot \Omega^{-2}(K) \oplus 134 \cdot \mathcal{P}(K).$$

Hence S_3 is also non-algebraic.

The largest simple module present in the permutation module on 3510 points is the module S_4 . This is also non-algebraic, as for example

$$S_4 \downarrow_{P_1} = 4 \cdot K \oplus 2 \cdot \Omega^4(K) \oplus 2 \cdot \Omega^{-4}(K) \oplus 328 \cdot \mathcal{P}(K).$$

In the case of characteristic 3, we restrict to a $C_3 \times C_3$ subgroup of Fi_{22} . There are fifteen conjugacy classes of subgroup isomorphic with $C_3 \times C_3$.

Class	Number of Conjugates	Class	Number of Conjugates
\mathcal{C}_1	153753600	\mathcal{C}_9	8302694400
\mathcal{C}_2	205004800	\mathcal{C}_{10}	16605388800
\mathcal{C}_3	1230028800	\mathcal{C}_{11}	22140518400
\mathcal{C}_4	1383782400	\mathcal{C}_{12}	33210777600
\mathcal{C}_5	2460057600	\mathcal{C}_{13}	44281036800
\mathcal{C}_6	2767564800	\mathcal{C}_{14}	66421555200
$\mathcal{C}_7, \mathcal{C}_8$	5535129600	\mathcal{C}_{15}	132843110400

Let Q denote a representative from \mathcal{C}_2 . [This subgroup can be found for example inside a maximal subgroup of index 61, 776.] This will be used to prove that both S_1 and S_2 are non-algebraic. The restriction of S_1 to a subgroup from one of them is isomorphic with

$$K \oplus 2 \cdot E \oplus \mathcal{P}(K) \oplus M_1 \oplus M_2,$$

where M_1 is a 21-dimensional indecomposable module and M_2 is a 32-dimensional indecomposable module. The module E is a 7-dimensional module, and is in fact isomorphic with the heart of the projective indecomposable module. In Proposition 2.19, we prove that this module is non-algebraic, and so S_1 is non-algebraic, as required. The module M_1 will appear again in the decomposition of the restriction of S_2 to Q ; this decomposition is

$$S_2 \downarrow_Q = 3 \cdot K \oplus M_1 \oplus M_3 \oplus 2 \cdot M_4 \oplus 24 \cdot \mathcal{P}(K).$$

Here, M_1 is the 21-dimensional non-periodic module above, which will turn out to be non-algebraic, M_3 is a 23-dimensional indecomposable module, and M_4 is a 44-dimensional indecomposable module. The module M_3 satisfies

$$S^2(M_1) = 19 \cdot \mathcal{P}(K) \oplus \Omega^2(M_1) \oplus \Omega^{-2}(M_1).$$

Thus by Corollary 2.12, M_1 , and hence S_2 is non-algebraic.

One may use the fact that S_1 is non-algebraic to prove that S_4 and S_5 are non-algebraic as well. The module $\Lambda^2(S_1)$ is in fact S_4 , and the module $S^2(S_1)$ is semisimple, with

$$S^2(S_1) = K \oplus S_2 \oplus S_5.$$

Recall that

$$\Lambda^2 \left(\bigoplus_{i=1}^n N_i \right) = \bigoplus_{i=1}^n \Lambda^2(N_i) \oplus \bigoplus_{i < j} N_i \otimes N_j$$

and

$$S^2 \left(\bigoplus_{i=1}^n N_i \right) = \bigoplus_{i=1}^n S^2(N_i) \oplus \bigoplus_{i < j} N_i \otimes N_j;$$

the fact that $2 \cdot E$ is a summand of $S_1 \downarrow_Q$ implies that $E \otimes E | \Lambda^2(S_1) \downarrow_Q$, and since E is non-algebraic (by Proposition 2.19), so is S_4 .

To prove that S_5 is non-algebraic, notice that $S^2(S_1 \downarrow_Q)$ contains (as a summand) the module $E \otimes M_1$. This is (modulo projectives) the module $\Omega(M_1) \oplus \Omega^{-1}(M_1)$, a direct sum of two, clearly non-algebraic, 24-dimensional indecomposable modules. Since these do not lie in $S_2 \downarrow_Q$, they must be summands of $S_5 \downarrow_Q$, and so $S_5 \downarrow_Q$ is non-algebraic also.

Now let $p = 5$: in this case, all modules from the principal block can be realized over $\text{GF}(5)$.

The smallest non-trivial module S_1 , of dimension 78, is algebraic. This module has full vertex (since the block has abelian defect group) and a 28-dimensional source M_1 . The exterior square of this module is given by

$$\Lambda^2(M_1) = M_1 \oplus 14 \cdot \mathcal{P}(K),$$

and the symmetric square is given by

$$S^2(M_1) = K \oplus X \oplus 15 \cdot \mathcal{P}(K),$$

where X is the sum of the six 5-dimensional modules corresponding to permutation modules on the cosets of the six subgroups of index 5. Thus, modulo summands with cyclic vertex,

$$M_1 \otimes M_1 \equiv K \oplus M_1.$$

Therefore by Proposition 2.1, M_1 is algebraic.

The module S_2 behaves exactly similarly: it also has a Sylow 5-subgroup as vertex and a 28-dimensional source M_2 . (The modules M_1 and M_2 are not conjugate.) The exterior square of this module is given by

$$\Lambda^2(M_2) = M_2 \oplus 14 \cdot \mathcal{P}(K),$$

and the symmetric square by

$$S^2(M_2) = K \oplus X \oplus 15 \cdot \mathcal{P}(K),$$

where X is the sum of the six 5-dimensional modules corresponding to permutation modules on the cosets of the six subgroups of index 5. Thus, modulo summands with cyclic vertex,

$$M_2 \otimes M_2 \equiv K \oplus M_2.$$

Thus by Proposition 2.1, M_2 is algebraic.

The exterior square of S_1 is the 3003-dimensional module S_4 , and so this module is also algebraic. This completes the proof of Theorem 6.14.

6.3.5 The Fischer Group Fi_{23}

The Fischer group Fi_{23} has order $4089470473293004800 = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, and has a permutation representation on 31,671 points. It is the second of Fischer's three sporadic simple groups to come out of his study of 3-transposition groups. The modular character table is not known in characteristic 3, but all other characteristics are available.

p	Block	Simple Modules	Defect Group
2	1	{1, 782, 1494, 3588, 19940, 57408, 79442, 94588, 94588*, 583440, 724776, 979132, 1951872, 1997872 ₁ , 1997872 ₂ , 5812860, 7821240, 8280208, 17276520, 34744192}	Sylow
	2	{97976320, 166559744}	Defect 3
	3	{73531392}	Defect 1
	4,5	{504627200 ₁ }, {504627200 ₂ }	Defect 0
5	1	{1, 3588, 5083, 25806, 274482, 1948284, 7193549, 9103653, 9103653*, 10267269, 16864290, 37544142, 38625004, 46961837, 48125453, 178866469}	Sylow
	2	{782, 30106, 60996, 76637, 111826, 751893, 7954894, 11218572, 21074118, 25223342, 26324076, 48196224, 66457029, 120262284, 170042380, 269117549}	Sylow
		5 blocks of defect 1 33 blocks of defect 0	

We have the following result for this group.

Theorem 6.15 Let G denote the sporadic simple group Fi_{23} , and let K denote a splitting field of characteristic p .

- (i) If $p = 2$ then the three smallest non-trivial simple modules are non-algebraic.
- (ii) If $p = 3$ then the two smallest non-trivial simple modules are non-algebraic.
- (iii) If $p = 5$ then the smallest non-trivial simple module is algebraic.

We begin with characteristic 2. Let H denote a maximal subgroup of G isomorphic with S_{12} . The subgroup H contains many conjugacy classes of subgroup isomorphic with V_4 , two of which have 103,950 elements in the conjugacy class. Label representatives from these classes by P_1 and P_2 . For a particular choice of the P_i , we have

$$S_1 \downarrow_{P_1} = 8 \cdot K \oplus 3 \cdot \Omega^4(K) \oplus 3 \cdot \Omega^{-4}(K) \oplus 180 \cdot \mathcal{P}(K),$$

and the restriction to P_2 is algebraic. Hence S_1 is non-algebraic.

The same choice of the P_i gives

$$S_2 \downarrow_{P_1} = 5 \cdot \Omega^2(K) \oplus 5 \cdot \Omega^{-2}(K) \oplus X \oplus 352 \cdot \mathcal{P}(K),$$

and

$$S_2 \downarrow_{P_2} = \Omega(K) \oplus \Omega^{-1}(K) \oplus Y \oplus 354 \cdot \mathcal{P}(K)$$

where the module X is a direct sum of 2-dimensional modules and 4-dimensional non-projective modules, and Y is a direct sum of 2-dimensional modules. Thus S_2 is non-algebraic as well.

To prove that the simple module S_3 is non-algebraic, we note that

$$\begin{aligned} S_3 \downarrow_{P_1} = & 6 \cdot K \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus 3 \cdot \Omega^3(K) \\ & \oplus 3 \cdot \Omega^{-3}(K) \oplus 3 \cdot \Omega^4(K) \oplus 3 \cdot \Omega^{-4}(K) \oplus 870 \cdot \mathcal{P}(K). \end{aligned}$$

This completes our analysis of characteristic 2.

Moving on to characteristic 3, let S_1 denote the 253-dimensional simple module, and S_2 denote the 528-dimensional simple module. These are the two smallest non-trivial simple modules, and both are self-dual and realizable over $\text{GF}(3)$.

Let H denote the stabilizer of a point under the 31,671-point permutation action of G ; then H is the 2-fold central extension of Fi_{22} . The restrictions of S_1 and S_2 are semisimple and, in fact,

$$S_1 \downarrow_H = M_1 \oplus N_1,$$

and

$$S_2 \downarrow_H = K \oplus N_2 \oplus M_2,$$

where M_1 and M_2 are simple modules for the simple quotient Fi_{22} of H , and N_1 and N_2 are (different) 176-dimensional simple modules for H . The module M_1 is the 77-dimensional simple module for Fi_{22} , which is non-algebraic by Theorem 6.14, and M_2 is the 351-dimensional simple module for Fi_{22} , also non-algebraic by the same result. This proves (ii) of our claim.

Now consider characteristic 5: restricting the 782-dimensional module T_1 to a Sylow 5-subgroup P , we see that

$$T_1 \downarrow_P = 2 \cdot K \oplus M \oplus 30 \cdot \mathcal{P}(K),$$

where M is the direct sum of the six non-isomorphic 5-dimensional permutation modules. Since T_1 is therefore a trivial-source module, it is algebraic.

6.3.6 The Fischer Group Fi'_{24}

The third Fischer group Fi'_{24} has order $1255205709190661721292800 = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, and has a permutation representation on 306,936 points. Very little is known about the modular representation theory of this group: it has a 3774-dimensional representation in characteristic 2, and a 781-dimensional representation in characteristic 3. This latter module can be shown to be non-algebraic.

Proposition 6.16 Let G be the sporadic simple group Fi'_{24} , and let K be a field of characteristic 3. Then the smallest non-trivial simple module, of dimension 781, is not algebraic.

Let S_1 denote this simple module. The stabilizer H of a point under the 306,936-point action of G is the smaller Fischer group Fi_{23} . The simple module S_1 restricts to H as a semisimple module, and is the sum of the 253-dimensional and 528-dimensional simple modules, both of which are non-algebraic by Theorem 6.15. Hence S_1 itself is non-algebraic.

6.3.7 The Baby Monster B and Monster M

The Baby Monster is the second largest simple group, and has order

$$4154781481226426191177580544000000 = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$$

It has a 4370-dimensional representation in characteristic 2, which is within the limits of computing resources, but due to problems with generation of subgroups, the author has not studied it yet.

The Monster has order

$$808017424794512875886459904961710757005754368000000000,$$

which factorizes as $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$. The smallest non-trivial module for this group has dimension nearly 200 000, and so we cannot analyze any of the modules for this group over any field.

6.4 The Pariahs

Those six sporadic simple groups that are not involved in the Monster (although this is a highly non-trivial fact) form the so-called Pariahs. Wildly varying amounts of information is known, from complete information on the Janko group J_1 , to the rather sporadic information about the others. They range in order from the easy-to-handle group J_1 , of order 175560, to the largest group J_4 , which has order 86775571046077562880. The main result in this section is that J_1 has p -SMA for all primes p .

6.4.1 The Janko Group J_1

The first Janko group has order $175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. The group has presentation

$$\langle a, b : a^2 = b^3 = (ab)^7 = (ab(abab^{-1})^3)^5 = (ab(abab^{-1})^6 abab(ab^{-1})^2)^2 = 1 \rangle,$$

although it has a (relatively) easy permutation representation on 266 points. It can also be represented as 20-dimensional matrices over $\text{GF}(2)$, and as 7-dimensional matrices over $\text{GF}(11)$. It was first considered by Janko in [49], and is the only sporadic simple group with abelian Sylow 2-subgroups.

The complete result on this group is the following.

Theorem 6.17 The group J_1 has p -SMA for all primes p .

Firstly, note that all Sylow p -subgroups are cyclic if p is odd, and so this result is really about 2-SMA.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 20, 56_1, 56_2, 76_1\}$	Sylow
	2	$\{76_2\}$	Defect 1
	3,4	$\{56_3\}, \{56_4\}$	Defect 0
	5,6,7	$\{120_1\}, \{120_2\}, \{120_3\}$	Defect 0

The four simple modules of dimension 56 all require $\text{GF}(4)$ to exist, and the 120-dimensional modules require $\text{GF}(8)$. All other modules can be realized over any field.

Let P denote a Sylow 2-subgroup, with $N = N_G(P)$ denoting its normalizer in $G = J_1$. The group P is elementary abelian of order 8, and therefore all simple modules have vertex that of the defect group of their respective block, by Theorem 1.24. There are five blocks of defect 0, with two simple modules of dimension 56 and the remaining three of dimension 120. Another block is of defect 1, with only one simple module, of dimension 76. The principal block is the only unaccounted for block, and contains five simple modules.

Firstly, we note that the normalizer N of a Sylow 2-subgroup P of G is isomorphic with those of the Ree groups ${}^2G_2(q)$. As we proved earlier in Chapter 5, the simple modules for the Ree groups are algebraic in characteristic 2, and it can be seen easily using a computer (or by considering the module structure) that the Green correspondent of the module S_1 for J_1 also appears as the Green correspondent of the 12-dimensional simple module for ${}^2G_2(3) = \text{SL}_2(8) \rtimes C_3$. Thus S_1 is algebraic.

In the normalizer $N = N_G(P)$, there are five simple modules: the trivial module, two other dual 1-dimensional modules only realizable over $\text{GF}(4)$, and two dual 3-dimensional modules, realizable over any field.

Consider the simple module S_4 : again, we take its Green correspondent A_4 in the subgroup N . Since S_4 is realizable over $\text{GF}(2)$, we will consider this field. Therefore N has four simple modules: the trivial module K , the 2-dimensional simple (but not absolutely simple) module W_1 , and the two dual 3-dimensional simple modules W_2 and W_2^* .

The tensor square of A_4 is given by

$$A_4 \otimes A_4 = 2 \cdot C_1 \oplus C_2,$$

where C_1 is a 28-dimensional module with $\Omega(C_1) = C_1$ and C_2 is a non-periodic 88-dimensional indecomposable module. The decomposition of $A_4 \otimes C_1$ is given by

$$A_4 \otimes C_1 = 2 \cdot \mathcal{P}(K) \oplus 2 \cdot \mathcal{P}(W_1) \oplus 4 \cdot \mathcal{P}(W_2) \oplus 4 \cdot \mathcal{P}(W_2^*) \oplus 2 \cdot C_3,$$

where C_3 is a 56-dimensional module with $\Omega(C_3) = C_3$. Finally in this direction, we have the decomposition

$$A_4 \otimes C_3 = 2 \cdot \mathcal{P}(K) \oplus 3 \cdot \mathcal{P}(W_1) \oplus 8 \cdot \mathcal{P}(W_2) \oplus 8 \cdot \mathcal{P}(W_2^*) \oplus 4 \cdot C_1 \oplus 2 \cdot C_3.$$

Moving on to the product of A_4 and C_2 , we have

$$A_4 \otimes C_2 = 4 \cdot \mathcal{P}(K) \oplus 3 \cdot A_1 \oplus 4 \cdot \mathcal{P}(W_1) \oplus 14 \cdot \mathcal{P}(W_2) \oplus 14 \cdot \mathcal{P}(W_2^*) \oplus 2 \cdot C_1 \oplus 2 \cdot C_3 \oplus C_4 \oplus C_5,$$

where C_4 is a 12-dimensional indecomposable module and where C_5 is a periodic 84-dimensional indecomposable module. Decomposing more tensor products, we have

$$A_4 \otimes C_4 = \mathcal{P}(K) \oplus \mathcal{P}(W_1) \oplus 2 \cdot \mathcal{P}(W_2) \oplus 2 \cdot \mathcal{P}(W_2^*) \oplus C_6,$$

where C_6 is a 24-dimensional indecomposable module. The next decomposition is

$$A_4 \otimes C_6 = 2 \cdot C_4 \oplus 2 \cdot \mathcal{P}(K) \oplus 2 \cdot \mathcal{P}(W_1) \oplus 4 \cdot \mathcal{P}(W_2) \oplus 4 \cdot \mathcal{P}(W_2^*) \oplus C_7,$$

where C_7 is another 24-dimensional indecomposable module. Decomposing the next tensor product gives

$$A_4 \otimes C_7 = 2 \cdot \mathcal{P}(K) \oplus 2 \cdot \mathcal{P}(W_1) \oplus 4 \cdot \mathcal{P}(W_2) \oplus 4 \cdot \mathcal{P}(W_2^*) \oplus C_8,$$

where C_8 is a 48-dimensional indecomposable module. Finally, we have

$$A_4 \otimes C_8 = 2 \cdot C_4 \oplus 2 \cdot C_7 \oplus 3 \cdot \mathcal{P}(K) \oplus 3 \cdot \mathcal{P}(W_1) \oplus 9 \cdot \mathcal{P}(W_2) \oplus 9 \cdot \mathcal{P}(W_2^*).$$

Thus the remaining module to deal with is C_5 . This decomposes as

$$A_4 \otimes C_5 = 5 \cdot \mathcal{P}(K) \oplus 5 \cdot \mathcal{P}(W_1) \oplus 15 \cdot \mathcal{P}(W_2) \oplus 15 \cdot \mathcal{P}(W_2^*) \oplus C_9,$$

where C_9 is a 168-dimensional periodic module.

$$A_4 \otimes C_9 = 10 \cdot \mathcal{P}(K) \oplus 10 \cdot \mathcal{P}(W_1) \oplus 30 \cdot \mathcal{P}(W_2) \oplus 30 \cdot \mathcal{P}(W_2^*) \oplus 2 \cdot C_5 \oplus C_{10},$$

where C_{10} is another 168-dimensional periodic module. Next,

$$A_4 \otimes C_{10} = 10 \cdot \mathcal{P}(K) \oplus 10 \cdot \mathcal{P}(W_1) \oplus 30 \cdot \mathcal{P}(W_2) \oplus 30 \cdot \mathcal{P}(W_2^*) \oplus C_{11},$$

where C_{11} is a 336-dimensional periodic module, and finally

$$A_4 \otimes C_{11} = 21 \cdot \mathcal{P}(K) \oplus 21 \cdot \mathcal{P}(W_1) \oplus 63 \cdot \mathcal{P}(W_2) \oplus 63 \cdot \mathcal{P}(W_2^*) \oplus 2 \cdot C_5 \oplus 2 \cdot C_{10},$$

proving that S_4 is algebraic.

The last two modules to analyze are S_2 and S_3 . In this case, we really do need to extend our field to $K = \text{GF}(4)$, and we do so. Let X_1 denote the source of S_2 , an 8-dimensional non-periodic KP -module. The module $X_1^{\otimes 2}$ is indecomposable, but

$$X_1^{\otimes 3} = 2 \cdot X_2 \oplus \bigoplus_{i=1}^3 X_{3,i} \oplus Y \oplus 48 \cdot \mathcal{P}(K),$$

where X_2 is a periodic 8-dimensional module, the $X_{3,i}$ are periodic 28-dimensional modules, and Y is a sum of 4-dimensional permutation modules with cyclic vertex. The tensor product of X_1 and X_2 is simply a sum of modules with cyclic or trivial vertex, and so we consider the tensor product of X_1 and $X_{3,i}$. In this case,

$$X_1 \otimes X_{3,i} = X_{4,i} \oplus 21 \cdot \mathcal{P}(K),$$

where the $X_{4,i}$ are (non-isomorphic) 56-dimensional indecomposable modules. Next,

$$X_1 \otimes X_{4,i} = 2 \cdot X_{3,i} \oplus X_{5,i} \oplus 42 \cdot \mathcal{P}(K),$$

where the $X_{5,i}$ are (different) 56-dimensional indecomposable modules. The module $X_{5,i} \otimes X_1$ behaves similarly to $X_{3,i} \otimes X_1$, and indeed

$$X_1 \otimes X_{5,i} = X_{6,i} \oplus 42 \cdot \mathcal{P}(K),$$

where the $X_{6,i}$ are 112-dimensional indecomposable modules, and

$$X_1 \otimes X_{6,i} = 2 \cdot A_{3,i} \oplus 2 \cdot A_{5,i} \oplus 91 \cdot \mathcal{P}(K).$$

This implies that S_2 and S_3 are algebraic, since we have decomposed all possible tensor products.

This confirms that all simple modules in the principal block are algebraic, proving Theorem 6.17.

6.4.2 The Janko Group J_3

Janko's third sporadic group has order $50232960 = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$. It was originally considered, along with J_2 , in [51], although both existence and uniqueness are not proved there. It has a permutation representation of degree 6156, which makes it very amenable to computation. The table of dimensions of simple modules is given below.

p	Block	Simple Modules	Defect Group
2	1	$\{1, 78_1, 78_2, 80, 84, 84^*, 244, 322_1, 322_2, 966\}$	Sylow
	2,3,4,5	$\{1920_1\}, \{1920_2\}, \{1920_3\}, \{2432\}$	Defect 0
3	1	$\{1, 18_1, 18_2, 84, 84^*, 153_1, 153_2, 934\}$	Sylow
	2	$\{324, 2754\},$	Defect 1
	3,4	$\{1215_1\}, \{1215_2\}$	Defect 0

Proposition 6.18 Let G denote the sporadic group J_3 , and let K be a field of characteristic 2. Then the 78-dimensional, 244-dimensional, and 322-dimensional simple modules are non-algebraic.

In the principal block, those modules that occur in pairs (of dimensions 78, 84 and 322) all require a cube of unity to exist, whereas the other modules can be written over any field. Outside of the principal block, the 1920-dimensional modules require $\text{GF}(8)$ to exist, and the 2432-dimensional module can be realized over $\text{GF}(2)$.

We use the V_4 Restriction Test to prove non-algebraicity of some of the simple modules. There are two conjugacy classes of subgroups isomorphic with V_4 , as given in the table below.

Class	Number of Conjugates
\mathcal{C}_1	43605
\mathcal{C}_2	523260

Restricting either of the 78-dimensional simple modules S_1 and S_2 to a subgroup from \mathcal{C}_1 gives

$$\Omega(K) \oplus \Omega^{-1}(K) \oplus 18 \cdot \mathcal{P}(K),$$

whereas the restriction to a subgroup from \mathcal{C}_2 is algebraic. This means that S_1 and S_2 are non-algebraic.

Unfortunately, the simple modules S_3, S_4, S_5 and S_9 all have algebraic restrictions to V_4 subgroups, and so this simple test will not work for these modules. However, restricting the simple module S_6 to a representative from class \mathcal{C}_1 , we get the module

$$2 \cdot K \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 58 \cdot \mathcal{P}(K).$$

(The restriction to a representative from class \mathcal{C}_2 is algebraic.) Hence S_6 is non-algebraic.

Similarly, the modules S_7 and S_8 are non-algebraic: the restriction of S_7 (or equivalently S_8) to a representative from \mathcal{C}_1 is given by

$$\Omega^2(K) \oplus \Omega^{-2}(K) \oplus 78 \cdot \mathcal{P}(K).$$

(The restriction to a representative from class C_2 is algebraic.) Hence S_7 and S_8 are non-algebraic. This proves the proposition.

The 80-dimensional simple module has a 16-dimensional source, and it might be possible to prove whether this is non-algebraic, although the author has not attempted this calculation.

In characteristic 3, the exterior square of S_1 is the simple module S_5 , and so if S_1 is algebraic then S_5 is algebraic. However, this appears not to be the case. There are two conjugacy classes of subgroup of G isomorphic with $C_3 \times C_3$, and the restriction A of S_1 to a representative P from the smallest class is indecomposable. In fact, the tensor square $A^{\otimes 2}$ contains a 30-dimensional non-periodic module, and so it is probable that A is non-algebraic.

6.4.3 The Rudvalis Group Ru

The Rudvalis sporadic group has order $145926144000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$. Rudvalis announced its existence in [72], although Conway and Wales actually proved this result. Indeed, it is the unique 3510-point rank 3 extension of the group ${}^2F_4(2)$.

p	Block	Simple Modules	Defect Group
2	1	{1, 28, 376, 1246, 7280, 16036}	Sylow
	2	{8192 ₁ , 8192 ₂ , 102400}	Defect 2
3	1	{1, 406, 8440, 13310, 17836, 31030, 31060, 34944, 45094}	Sylow
	2,3	{3276, 20475}, {3654, 9135}	Defect 1
	4,...,9	{378}, {378*}, {783}, {27000 ₁ }, {27000 ₂ }, {27000 ₃ }	Defect 0
	10,...,14	{27405}, {43848 ₁ }, {43848 ₂ }, {43848 ₃ }, {712531}	Defect 0
15,...,19	{81432}, {98280 ₁ }, {98280 ₂ }, {110592 ₁ }, {110592 ₂ }	Defect 0	
5	1	{1, 133, 273, 378, 378*, 783, 2219, 3380, 3613, 8645, 12285, 15219, 18495, 22546, 24843, 30234, 43848, 58099}	Sylow
	2	{20475, 36425, 54925, 65975}	Defect 2
	3,4,5,6	{21750}, {27000 ₁ }, {27000 ₂ }, {27000 ₃ }	Defect 0
	7,8	{45500}, {76125}	Defect 0

There are results in all three non-trivial characteristics.

Theorem 6.19 Let G denote the sporadic simple group Ru , and let K be a splitting field of characteristic p .

- (i) If $p = 2$, then the simple modules of dimensions 28, 376, and 1246, are non-algebraic. The three modules outside the principal block are algebraic.
- (ii) If $p = 3$, then the smallest non-trivial simple module in the principal block is algebraic.

(iii) If $p = 5$, then the two 378-dimensional modules are non-algebraic.

We begin with characteristic 2: all of the modules from the principal block are realizable over $\text{GF}(2)$. The author does not know whether the 8192-dimensional simple modules require the presence of a cube root of unity.

MAGMA cannot produce subgroups of this group directly, and so we pass to a maximal subgroup: one that suffices for our purposes is the alternating group A_8 . This group has six conjugacy classes of V_4 subgroup, and we use for proving non-algebraicity the two conjugacy classes \mathcal{C}_1 and \mathcal{C}_2 of subgroup with 105 elements in each class. Let P_i be a representative from the class \mathcal{C}_i .

The 28-dimensional simple module S_1 has non-algebraic restrictions to both P_1 and P_2 , given by

$$S_1 \downarrow_{P_1} = 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus 4 \cdot \mathcal{P}(K),$$

and

$$S_1 \downarrow_{P_2} = \Omega(K) \oplus \Omega^{-1}(K) \oplus \Omega^3(K) \oplus \Omega^{-3}(K) \oplus 2 \cdot \mathcal{P}(K).$$

The 376-dimensional simple module S_2 also has non-algebraic restrictions to both P_1 and P_2 , which are

$$S_2 \downarrow_{P_1} = 2 \cdot K \oplus 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 4 \cdot M \oplus 82 \cdot \mathcal{P}(K),$$

and

$$\begin{aligned} S_2 \downarrow_{P_2} = & 4 \cdot \Omega(K) \oplus 4 \cdot \Omega^{-1}(K) \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \\ & \oplus \Omega^4(K) \oplus \Omega^{-4}(K) \oplus 2 \cdot M \oplus 78 \cdot \mathcal{P}(K). \end{aligned}$$

(Here M denotes the sum of the three different 2-dimensional permutation modules on the cosets of the three different subgroups of index 2.)

The 1246-dimensional module S_3 has the restrictions

$$S_3 \downarrow_{P_1} = 12 \cdot M \oplus 2 \cdot \Omega(K) \oplus 2 \cdot \Omega^{-1}(K) \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 288 \cdot \mathcal{P}(K)$$

and

$$S_3 \downarrow_{P_2} = 8 \cdot M \oplus 6 \cdot \Omega(K) \oplus 6 \cdot \Omega^{-1}(K) \oplus \Omega^2(K) \oplus \Omega^{-2}(K) \oplus 288 \cdot \mathcal{P}(K),$$

which prove that this module is also non-algebraic.

The second block is the unique block with defect 2, and so by Corollary 3.9, all modules from this block are algebraic. Thus Theorem 6.19(i) is true.

Now let us consider characteristic 3. In this case, all simple modules in the principal block can be realized over $\text{GF}(3)$.

The simple module S_1 is algebraic, since it has trivial source. To see this, recall that the largest maximal subgroup of G is ${}^2F_2(2)$, which has an index 2 subgroup (the Tits group). The permutation module of dimension 8120 on the cosets of this subgroup has S_1 as a summand. Its other summands come from non-principal blocks, all of whose summands are obviously algebraic.

Lastly, suppose that $p = 5$. In this case, all simple modules from the principal block are realizable over $\text{GF}(5)$.

Let S_3 denote one of the 378-dimensional simple modules. There are two conjugacy classes of subgroup of G isomorphic with $C_5 \times C_5$.

Class	Number of Conjugates
\mathcal{C}_1	12160512
\mathcal{C}_2	145926144

Let Q denote a subgroup from \mathcal{C}_1 . Then the restriction of S_3 to Q is (up to duality)

$$S_3 \downarrow_Q = \Omega^3(A) \oplus B \oplus 11 \cdot \mathcal{P}(K),$$

where B is a 76-dimensional module and A is a self-dual 48-dimensional indecomposable module, whose third Heller translate has dimension 27. This module $\Omega^3(A)$ cannot be algebraic, and so S_3 and S_4 are not algebraic either.

This completes the proof of Theorem 6.19.

Staying with characteristic 5 for the moment, we briefly consider the simple module S_1 . This has an 8-dimensional source, A_1 , and

$$\Lambda^2(A_1) = A_1 \oplus M_1 \oplus M_1^*,$$

where M_1 is a non-periodic 10-dimensional indecomposable module. The restriction of M_1 to a representative from \mathcal{C}_2 is periodic, whereas the restriction to a subgroup Q from \mathcal{C}_1 is non-periodic, and is, in fact, the module $\text{soc}^4(\mathcal{P}(K))$. This module should be non-algebraic, unless Conjecture 3.31 is incorrect, but the first few tensor powers of M_1 don't appear to offer an easy way to prove non-algebraicity.

6.4.4 The Janko Group J_4

Janko's fourth sporadic group has order $86775571046077562880 = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. The first information about this group appeared in [52], where in particular its character table is given. The group was constructed by a team of Cambridge researchers led by Simon Norton [65] using a computer, whereas a computer-free construction was given in [48].

Little is known about this group's modular representations: for example, the decomposition matrices of J_4 are not known for the primes 2, 3 and 11. Indeed, since the smallest permutation action of J_4 is on nearly 175 million points, this group is very difficult to study from a representation-theoretic perspective.

Proposition 6.20 Let G be the sporadic simple group J_4 , and let K be a field of characteristic 2. Then the four smallest non-trivial simple KG -modules are all non-algebraic. (These consist of the 112-dimensional simple module, two dual 1220-dimensional simple modules and the 3774-dimensional simple module.)

The group $G = J_4$ possesses a maximal subgroup which is of the form $M_{22} \rtimes C_2$, and this retains enough of the group structure of J_4 to prove that all of the non-trivial simple modules that can be easily constructed in characteristic 2 are non-algebraic. Specifically, one may restrict representations to a maximal subgroup isomorphic with $M_{22} \rtimes C_2$; Restricting to a particular V_4 subgroup of this subgroup, we let Q denote a subgroup from the (unique) conjugacy class of V_4 subgroups with 1540 elements, and will restrict our representations to this subgroup.

The 112-dimensional simple module S_1 is non-algebraic, since its restriction to Q is given by

$$S_1 \downarrow_Q = 2 \cdot K \oplus 4 \cdot M \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus 20 \cdot \mathcal{P}(K).$$

(Here, M denotes the sum of the three non-faithful 2-dimensional indecomposable modules over Q .)

The two dual 1220-dimensional simple modules S_2 and S_3 are also non-algebraic, as both restrict to the subgroup Q as

$$6 \cdot K \oplus 16 \cdot M \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus 278 \cdot \mathcal{P}(K).$$

Lastly, the 3774-dimensional module S_4 restricts to Q as

$$S_4 \downarrow_Q = 16 \cdot M \oplus \Omega(K) \oplus \Omega^{-1}(K) \oplus 918 \cdot \mathcal{P}(K).$$

Unfortunately, in characteristics 3 and 11 calculations are much more difficult to perform, and as yet there are no results in this direction.

6.4.5 The O’Nan Group ON and Lyons Group Ly

The O’Nan simple group has order $460815505920 = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$. It was first investigated by O’Nan in [66], and constructed by Sims.

It has a permutation representation on 122,760 points, and is a subgroup of $GL_{154}(3)$. Its minimal faithful degree in characteristic 2 is 10944. This is why the prime 2 has not been analyzed. Even outside of the prime 2, it seems difficult to compute with this group, and as such the author has no results for this group.

The Lyons simple group has order $51765179004000000 = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$. The group was first studied in [59], and existence was proved by Sims, with uniqueness not appearing properly in the literature until 1997.

There are not many accessible representations of this group: in characteristic 2, the only representation has degree 2480 (over $GF(4)$), and in characteristic 3 the smallest representation has degree 651. Characteristic 5 is better, with a representation of degree 111. It has not been possible, however, to analyze any of these modules.

6.5 Summary

We summarize our results in the table on the next page: in this table, a tick implies that the group has p -SMA, a cross indicates that the group does not, a question mark indicates that the answer is unknown, and no mark indicates that p does not divide the order of the group.

In particular, we have the following theorem, which is Theorem E from the introduction.

Theorem 6.21 Let G be a sporadic group, and let K be a field of characteristic p . Then G has p -SMA in the following cases:

- (i) $G = M_{11}$ and $p = 2$;
- (ii) $G = M_{22}$ and $p = 3$;
- (iii) $G = HS$ and $p = 3$;
- (iv) $G = J_2$ and $p = 3$ or $p = 5$;
- (v) $G = J_1$ and $p = 2$; and
- (vi) G is a sporadic group and $p^2 \nmid |G|$.

CHAPTER 6. THE SPORADIC GROUPS

There are in addition some likely candidates for groups with p -SMA, such as the Held group in characteristic 3.

	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	59	67	71
M_{11}	✓	X	✓		✓													
M_{12}	X	X	✓		✓													
M_{22}	X	✓	✓	✓	✓													
M_{23}	X	X	✓	✓	✓				✓									
M_{24}	X	X	✓	✓	✓				✓									
HS	X	✓	?	✓	✓													
J_2	X	✓	✓	✓														
Co_1	?	?	?	?	✓	✓			✓									
Co_2	?	?	?	✓	✓				✓									
Co_3	?	?	?	✓	✓				✓									
McL	?	X	?	✓	✓													
Suz	X	X	?	✓	✓	✓												
He	X	?	?	?			✓											
HN	X	X	?	✓	✓			✓										
Th	X	X	?	?		✓		✓			✓							
Fi_{22}	X	X	?	✓	✓	✓												
Fi_{23}	X	X	?	✓	✓	✓	✓	✓										
Fi'_{24}	?	X	?	?	✓	✓	✓	✓	✓	✓								
B	?	?	?	?	✓	✓	✓	✓	✓	✓	✓				✓			
M	?	?	?	?	?	?	✓	✓	✓	✓	✓		✓		✓	✓		✓
J_1	✓	✓	✓	✓	✓			✓										
ON	?	?	✓	?	✓			✓			✓							
J_3	X	?	✓				✓	✓										
Ru	X	?	X	✓		✓				✓								
J_4	X	?	✓	✓	?				✓	✓	✓	✓		✓				
Ly	?	?	?	✓	✓						✓	✓						✓

Appendix A

MAGMA Computer Programs

This appendix details the computer-based techniques and algorithms used in the research of the author's thesis.

A.1 Constructions

In this section we detail the constructions of groups and modules that were used in this thesis.

A.1.1 Constructing Groups

There are several ways of constructing groups in MAGMA. The most important two for us is using pre-defined functions and using permutation representations of the group. For example, to construct the symmetric group on 10 letters you use the command

```
> G:=SymmetricGroup(10);
```

which assigns to G the symmetric group on ten letters.

Alternatively, one may use the permutation representation of a group. For example, to construct the Mathieu group M_{24} , one may use the command

```
> G<x,y>:=PermutationGroup<24|  
> \[4,7,17,1,13,9,2,15,6,19,18,21,5,16,8,14,3,11,10,24,12,23,22,20] ,  
> \[4,21,9,6,18,1,7,8,15,5,11,12,17,2,3,13,16,10,24,20,14,22,19,23]>;
```

The permutation representations of most sporadic simple groups can be found at [10]. To construct a permutation representation of a group already defined in MAGMA, one must first construct the subgroup H upon whose cosets the group G will act as permutations. Given these, one uses the command

APPENDIX A. MAGMA COMPUTER PROGRAMS

```
> temp,G:=CosetAction(G,H);
```

to construct G as a permutation group. Groups defined as permutations are computationally better than groups defined by other methods, such as by matrices.

As an example, we suppose that G is the Harada–Norton group defined as matrices, and that H is a maximal subgroup isomorphic with A_{12} . To construct H in such a way that it can be used for computation, we would construct code as such. (When MAGMA returns a matrix group, it also returns the generators, which have been removed from the output below.)

```
> G;
MatrixGroup(132, GF(2^2))
> H;
MatrixGroup(132, GF(2^2))
> MaxSubs:=MaximalSubgroups(H);
> H2:=MaxSubs[#MaxSubs] ' subgroup;
> Index(H,H2);
12
> temp,G2:=CosetAction(H,H2);
> G2;
Permutation group G2 acting on a set of cardinality 12
  (1, 2)(3, 4)(5, 7)(6, 8)(9, 10)(11, 12)
  (1, 3, 5, 7)(4, 6)(8, 9)(10, 11)
```

This is how the restrictions of the modules for HN to the maximal subgroup A_{12} are achieved in Section 6.3.2.

We should also mention how certain subgroups are constructed. Sylow p -subgroups are constructed using the command

```
> P:=SylowSubgroup(G,p);
```

where G and p have the obvious meanings. To construct the collection of all normal subgroups of G , one uses the command

```
> X:=NormalSubgroups(G);
```

Note that the set X is a collection of records, and if you want to access the actual subgroup corresponding to, for instance, the second element of X , one needs to write

```
> H:=X[2] ' subgroup;
```

To construct conjugacy classes of subgroups of a particular order, say 6, one uses the command

```
> X:=Subgroups(G:OrderEqual:=6);
```

Again, this is a collection of records, not of subgroups. If one needs representatives from all conjugacy classes of subgroups of the form $C_3 \times C_3$, then one must enter the code

```
> X:=Subgroups(G:OrderEqual:=9);
> Subs:=[];
> for i in X do
for> if(Exponent(i'subgroup) eq 3) then Append(~Subs,i'subgroup);
for|if> end if;
for> end for;
```

which yields a list `Subs` with subgroups, not records.

A.1.2 Constructing Modules

This section deals with constructing the representations considered in the thesis. Simple representations can, in the main, be easily constructed. There are two ways of doing this: by constructing permutation representations; and by constructing tensor powers of known simple modules. We begin with permutation representations.

Suppose that G is a group and H is a subgroup of G . Then the permutation module of G on the cosets of H over the field K can be produced using the command `PermutationModule(G,H,K)`. (If G is defined as a permutation group to begin with, then the natural permutation module can be defined with `PermutationModule(G,K)`.) For example, let G denote the group M_{11} . Then to construct some permutation representations, we perform the following.

```
> G;
Permutation group G acting on a set of cardinality 11
      (2, 10)(4, 11)(5, 7)(8, 9)
      (1, 4, 3, 8)(2, 5, 6, 9)
> MaxSubs:=MaximalSubgroups(G);
> MaxSubs;
Conjugacy classes of subgroups
-----
```

APPENDIX A. MAGMA COMPUTER PROGRAMS

```

[1]      Order 48          Length 165
      Permutation group acting on a set of cardinality 11
      Order = 48 = 2^4 * 3
          (1, 3)(2, 7)(5, 8)(6, 11)
          (1, 7, 6)(2, 3, 11)(4, 5, 8)
          (1, 9, 11, 10)(2, 3, 7, 6)
          (1, 6, 11, 3)(2, 9, 7, 10)
          (1, 11)(2, 7)(3, 6)(9, 10)

[2]      Order 120         Length 66
      Permutation group acting on a set of cardinality 11
      Order = 120 = 2^3 * 3 * 5
          (2, 7)(4, 10)(5, 11)(6, 8)
          (1, 11)(2, 6, 4, 8, 10, 7)(3, 5, 9)

[3]      Order 660         Length 12
      Permutation group acting on a set of cardinality 11
      Order = 660 = 2^2 * 3 * 5 * 11
          (1, 7)(2, 10)(5, 6)(9, 11)
          (1, 6, 10)(2, 8, 11)(3, 4, 5)

[4]      Order 144         Length 55
      Permutation group acting on a set of cardinality 11
      Order = 144 = 2^4 * 3^2
          (2, 9)(3, 7)(5, 11)(6, 10)
          (1, 5, 4, 3)(6, 7, 10, 11)
          (1, 10, 4, 6)(3, 11, 5, 7)
          (1, 4)(3, 5)(6, 10)(7, 11)
          (1, 5, 10)(3, 4, 6)(7, 11, 8)
          (1, 8, 4)(3, 10, 11)(5, 7, 6)

[5]      Order 720         Length 11
      Permutation group acting on a set of cardinality 11
      Order = 720 = 2^4 * 3^2 * 5
          (1, 3)(2, 9)(4, 11)(8, 10)
          (2, 8, 7, 10)(3, 5, 11, 9)

> M:=PermutationModule(G,GF(2));
> CompositionFactors(M);
[

```

APPENDIX A. MAGMA COMPUTER PROGRAMS

```

    GModule of dimension 10 over GF(2),
    GModule of dimension 1 over GF(2)
]
> N:=PermutationModule(G,MaxSubs[3]‘subgroup,GF(3));
> CompositionFactors(N);
[
    GModule of dimension 1 over GF(3),
    GModule of dimension 5 over GF(3),
    GModule of dimension 5 over GF(3),
    GModule of dimension 1 over GF(3)
]

```

Now suppose that one already has modules M and N , entered on the computer as M and N . Then one may construct the modules $M^{\otimes n}$, $\Lambda^n(M)$, $S^n(M)$, and $M \otimes N$. The commands to do this will be given below in a worked example. Suppose that we have the group He entered into the computer as G , via the 2058-dimensional permutation representation.

```

> X:=CompositionFactors(PermutationModule(G,GF(5))); X;
[
    GModule of dimension 1 over GF(5),
    GModule of dimension 1275 over GF(5),
    GModule of dimension 102 over GF(5),
    GModule of dimension 680 over GF(5)
]
> SS1:=X[3];
> Bool,S1,S2:=IsIrreducible(ChangeRing(SS1,GF(25)));
> S1;
GModule S1 of dimension 51 over GF(5^2)
> Y1:=IndecomposableSummands(ExteriorSquare(S1)); Y1;
[
    GModule of dimension 1275 over GF(5^2)
]
> Y2:=CompositionFactors(TensorProduct(S1,S2));
> Y2;
[
    GModule of dimension 680 over GF(5^2),

```


APPENDIX A. MAGMA COMPUTER PROGRAMS

```

    GModule of dimension 1240 over GF(5^2),
    GModule of dimension 680 over GF(5^2),
    GModule of dimension 1 over GF(5^2)
]
> SocleFactors(SymmetricPower(S1,2));
[
    GModule of dimension 1326 over GF(5^2)
]

```

Moving away from simple groups, consider the problem of constructing all indecomposable modules for the group $C_3 \times C_3$, which was performed in Chapter 3. Since every module is a quotient of a free module, one needs to construct the free module, and to have a program that constructs submodules.

Also in that chapter, we organized these modules into conjugacy classes according to the action of the automorphism group, and so we will detail how to do that.

We begin by setting up the groups with which we will be working.

```

> H:=CyclicGroup(3);
> G:=DirectProduct(H,H);
> A:=AutomorphismGroup(G);
> Hol:=Holomorph(G,A);
> G:=NormalSubgroups(Hol)[2] 'subgroup;
> G;
Permutation group G acting on a set of cardinality 9
Order = 9 = 3^2
    (1, 9, 5)(2, 7, 6)(3, 8, 4)
    (1, 4, 7)(2, 5, 8)(3, 6, 9)

```

Now G is the group $C_3 \times C_3$, lying in the group $G \rtimes \text{Aut } G$. Construct the free module, and some of its quotients to determine all indecomposable modules of dimension 4.

```

> KG:=PermutationModule(G,sub<G|>,GF(3));
> KG2:=DirectSum(KG,KG);
> KG3:=DirectSum(KG2,KG);
> X1:=Submodules(KG:CodimensionLimit:=4);

```

APPENDIX A. MAGMA COMPUTER PROGRAMS

At this stage, X_1 consists of all submodules of the indecomposable projective module of dimension at least 5, whose quotients will provide us with the modules of dimension 4. To construct the other indecomposable modules, we look for quotients of the 18-dimensional and 27-dimensional projectives. However, we remove the top of the module, as we want to guarantee that the quotients definitely have the correct top themselves.

```
> L2:=SocleSeries(KG2)[4];
> L3:=SocleSeries(KG3)[4];
> X2:=Submodules(L2:CodimensionLimit:=2);
> X3:=Submodules(L3:CodimensionLimit:=1);
```

We begin by only considering those quotients of dimension exactly 4, then remove all those that are not indecomposable.

```
> Y1:=[]; Y2:=[]; Y3:=[];
> for i in X1 do
for> if(Dimension(i) eq 5) then Append(~Y1,KG/i); end if;
for> end for;
> for i in X2 do if(Dimension(i) eq 14) then
for|if> if(not(IsDecomposable(KG2/i))) then Append(~Y2,KG2/i); end if;
for|if> end if; end for;
> #Y2;
338
```

Similarly, the modules filling Y_3 are constructed (which in this case is the empty set). Finally, we need to remove all duplicates of isomorphism types. Since we will need this more often, we create a new function, called `IsIsomorphicToList`, which checks whether the module M lies on the list of module I .

```
function IsIsomorphicToList(M,I);
  for i in I do
    if(IsIsomorphic(M,i)) then return true;
    end if;
  end for;
  return false;
end function;
```

```

function StripDuplicates(I);
  J:=[];
  for i in I do
    if(not(IsIsomorphicToList(i,J))) then Append(~J,i);
    end if;
  end for;
  return J;
end function;

```

With these functions, we proceed.

```

> Z1:=StripDuplicates(Y1);
> Z2:=StripDuplicates(Y2);
> Z:=Z1 cat Z2;
> #Z;
24

```

Thus the list Z contains all 24 indecomposable modules for $C_3 \times C_3$ over $\text{GF}(3)$. To determine conjugacy classes, we induce and restrict.

```

> W1:=IndecomposableSummands(Induction(Z[1],Ho1));
> W2:=IndecomposableSummands(Restriction(W1[1],G));

```

All modules in Z conjugate to $Z[1]$ are present in $W2$, and this allows us to easily construct the conjugacy classes.

A.2 Decomposing Tensor Products

Suppose that we wish to write the module $M \otimes N$ as a sum of indecomposable modules. The easiest way (most applicable for small dimensions) is to use the command

```

> IndecomposableSummands(TensorProduct(M,N));

```

In large dimensions, this is not feasible. Let M be a module whose summands we wish to compute. Begin by computing the composition factors of M , using the command `CompositionFactors(M)`. The fact that $\text{Ext}^1(A, B) = 0$ if A and B lie in different blocks has a bearing on which summands are present in M . Suppose that N is a projective simple module that is a composition factor of M . Then M possesses a complement $M1$ to N , and the problem becomes constructing $M1$ without decomposing M . We can

APPENDIX A. MAGMA COMPUTER PROGRAMS

attempt to find a submodule or quotient isomorphic with N , using the command `IsIrreducible`.

In general, `IsIrreducible` only returns a submodule, but in this case one already knows that this submodule is a summand. Computationally, this is a much less expensive procedure. For another example, suppose that M is a self-dual module, and that M contains exactly one copy of the self-dual simple module S . Suppose that `IsIrreducible` returns S as a submodule. Then S is a summand, and is complemented by a module isomorphic to M/S .

In Chapter 6, much use was made of the V_4 Restriction Test. To decompose 3000-dimensional modules over $GF(2)$ for the group V_4 , one cannot use the intrinsic command `IndecomposableSummands`, and the following algorithm was developed¹.

Suppose that M is a module over $GF(2)$ for a group G that is isomorphic with V_4 . Let KG denote the projective indecomposable module for G .

```
> Dimension(M);
2340
> while(Dimension(M) gt 200) do
while> Bool,A,B:=IsIrreducible(M);
while> if(IsIsomorphic(A,KG)) then delete M; M:=B; end if;
while> delete A; delete B; delete Bool;
while> end while;
> Dimension(M);
200
```

This method requires user intervention, because as it has been given, this algorithm might never terminate, since `IsIrreducible` might stop finding projective submodules of M before its dimension drops below 200. A simple way around this is to add code to make the program display the dimension of M every hundred dimensions or so. It becomes obvious when the program has run into a problem because if M has dimension less than 500 or so, then this program runs very efficiently. Although this situation is not ideal, it suits our purpose.

To make matters worse, despite our being careful with memory, MAGMA is not, and so if M is of dimension more than about 7000, reducing all the way down to a summand of dimension 200 will use about twenty gigabytes of memory. Also, this

¹This algorithm is far from the best available. One may (probably) find a free submodule of a module M for a p -group quite easily by taking the submodule generated by a random element. This is a very effective algorithm, that however will only work for a p -group.

APPENDIX A. MAGMA COMPUTER PROGRAMS

method only removes projective summands, although as we saw in Chapter 6, almost all summands of such modules are projective.

This method will work when the field has order 4 as well, but is much slower. This is due to the fact that there are many more low-dimensional submodules when the field is $\text{GF}(4)$ than when it is $\text{GF}(2)$. This method will not work at all when the group is not V_4 , however, since in other cases, the projective indecomposable module is too big, and will not be found randomly using `IsIrreducible`.

To get around this problem, recently the author has produced a new algorithm, that works with an arbitrary p -group. Let G be a p -group, and suppose that M is a module for G . Suppose that G and M are initiated on the computer as `G` and `M` respectively. Let `KG` be assigned to the projective indecomposable module.

```
function RemoveProjectiveSummands(M,KG);
  n:=#SocleFactors(KG);
  X:=SocleFactors(M);
  if(#X lt n) then return M; end if;
  m:=Dimension(SocleFactors(M)[n]);
  nn:=Dimension(M);
  limit:=Dimension(M)-Dimension(KG)*m;
  homs:=AHom(M,KG);
  while(Dimension(M) gt limit) do
    ker:=Kernel(Random(homs));
    if(Dimension(ker)+Dimension(KG) eq nn) then
      if(Dimension(ker + M) eq nn) then
        M:=M meet ker;
      end if;
    end if;
  end while;
  return M;
end function;
```

The first few commands are there so that the program knows how many summands of `M` are projective, and sets `limit` to be the dimension of $\Omega^0(M)$. This function then constructs the space of all homomorphisms from `M` to `KG`, and continually picks homomorphisms from this space. The first `if` command checks that the homomorphism is onto, and the second `if` command checks that the kernel `ker` of this homomorphism is a partial complement to `M`. Then the second isomorphism theorem says that the

quotient of M by $M \text{ meet ker}$ is isomorphic with KG . Thus we may replace M with $M \text{ meet ker}$ and continue until all projectives are removed.

Computationally, the expensive parts of this algorithm are the two commands `SocleFactors(M)` and `AHom(M,KG)`; after that, removing the projective summands is quick. Thus for large modules, this method can be sped up by constructing the set `homs` and then manually removing a few summands until the module M becomes a reasonable size (under 1200 dimensions, say) where this algorithm can be easily applied.

This algorithm should make decomposing tensor products of modules for p -groups very easy when such products have many projective summands. This algorithm was developed only very recently, and so its full power has not been examined, although it was the primary tool in confirming Theorem 6.17, as well as the odd characteristic parts of Theorem 6.8.

A.3 Determining Periodicity

Let G be a finite group and let M be a module for G . There are two ways of determining whether M is periodic: construct the Heller translates of M directly then use Theorem 1.18 which bounds the period of a periodic module; or use more theoretical results about complexity.

Given a module M , one will need to produce $\Omega(M)$ by a computer algorithm. To perform this, suppose that M is a module, and `Proj` is a list of projective modules, constructed using

```
> Proj:=[KG];
> for i in [1..4] do
for> Append(~Proj,DirectSum(Proj[1],Proj[#Proj]));
for> end for;
```

The set `Proj` will be a list of all projective modules from which we can choose our projective cover.

```
function HellerTranslate(M,Proj);
  n:=Dimension(M)-Dimension(JacobsonRadical(M));
  if(n gt #Proj) then
    for i in [1..n-#Proj] do
      Append(~Proj,DirectSum(Proj[1],Proj[#Proj]));
    end for;
```

APPENDIX A. MAGMA COMPUTER PROGRAMS

```

end if;
homs:=AHom(Proj[n],M);
repeat
  OM:=Kernel(Random(homs));
until (Dimension(OM)+Dimension(M) eq Dimension(Proj[n]));
return OM;
end function;

```

This function automatically increases the number of projective modules in `Proj` if it proves not to be enough.

There is another way to produce both $\Omega(M)$ and $\Omega^{-1}(M)$ if M has dimension a multiple of p . In this case, denote by E the heart of the projective indecomposable module. Then $E \otimes M$ is, modulo projective modules, $\Omega(M) \oplus \Omega^{-1}(M)$. This can be used to easily check if a module is periodic when the ambient group is abelian, since in this case the period of a periodic module is either 1 or 2. In either case,

$$\Omega(M) = \Omega^{-1}(M).$$

To implement this in MAGMA, assume that M is a module of dimension a multiple of p , and that KG is, as always, the projective indecomposable.

```

> E:=JacobsonRadical(KG)/Socle(KG);
> N:=RemoveProjectiveSummands(TensorProduct(E,M),KG);
> X:=IndecomposableSummands(N);

```

The list `X` contains the two modules corresponding to $\Omega(M)$ and $\Omega^{-1}(M)$: which is which can easily be checked by comparing socle layers.

For another way to determine periodicity, suppose that G is the group $C_p \times C_p$. It is true (as a corollary of [14, Corollary 5.10.3]) that a module for G is periodic if and only if, for any two generators x and y of G , the restriction of M to at least one of $\langle x \rangle$ and $\langle y \rangle$ is free. To check this, we can use the `Subgroups` command. For instance, suppose that M is a 6-dimensional module for the group G , which is isomorphic with $C_3 \times C_3$. Then code to determine whether M is algebraic is given below.

```

> for i in Subgroups(G:OrderEqual:=3) do
for> IndecomposableSummands(Restriction(M,i'subgroup));
for> end for;
[
  GModule of dimension 3 over GF(3),

```

```

    GModule of dimension 3 over GF(3)
]
[
    GModule of dimension 1 over GF(3),
    GModule of dimension 1 over GF(3),
    GModule of dimension 2 over GF(3),
    GModule of dimension 2 over GF(3)
]
[
    GModule of dimension 3 over GF(3),
    GModule of dimension 3 over GF(3)
]
[
    GModule of dimension 3 over GF(3),
    GModule of dimension 3 over GF(3)
]
]

```

This module is therefore periodic.

A.4 Analyzing Algebraicity

In Chapter 3, we stated that 6-dimensional periodic modules were algebraic without giving decompositions of the relevant tensor products. The following program terminates if and only if the module M is algebraic. As usual, G is the group and since this works for an arbitrary group, $Proj$ is the list of all projective indecomposable modules.

```

function CheckAlgebraicity(M,Proj);
  Mods:=Proj cat [M];
  i:=#Mods;
  j:=i;
  while(i le #Mods) do
    i-j;
    W:=IndecomposableSummands(TensorProduct(M,Mods[i]));
    for i in W do
      if(not(IsIsomorphicToList(i,Mods))) then Append(~Mods,i);
    end if;
  end while;
end function;

```


APPENDIX A. MAGMA COMPUTER PROGRAMS

```

    end for;
    i:=i+1;
  end while;
return "Module is algebraic.";
end function;

```

This method prints the number of non-projective indecomposable summands of tensor powers of M that it has so far decomposed. This is meant as a ‘progress report’ on the number of summands being analyzed.

As an example, we demonstrate this algorithm working on the group J_2 . Let $p = 5$, and let P be the Sylow 5-subgroup of G , the simple group J_2 . The simple module S_1 is of dimension 14. We will use a modified version of `CheckAlgebraicity`, which will collect the decompositions of the tensor products as well as the isomorphism types of the summands themselves.

```

> A1:=Restriction(S1,G);
> KP:=PermutationModule(P,sub<P|>,GF(5));
> Mods:=[KP,A1];
> Tens:=[];
> i:=2;
> while(i le #Mods) do
while> i-1;
while> N:=TensorProduct(A1,Mods[i]);
while> N2:=RemoveProjectiveSummands(N,KP);
while> Append(~Tens,IndecomposableSummands(N2));
while> for M in Tens[#Tens] do
while|for> if(not(IsIsomorphicToList(M,Mods))) then
while|for|if> Append(~Mods,M); end if;
while|for> end for;
while> delete N; delete N2;
while> i:=i+1;
while> end while;
1
2
[This has been removed]
20
21

```

> #Mods;

22

Using the list **Tens** we are able to reconstruct the decompositions of the tensor products of members of **Mods** with **A1**. Firstly, the trivial module and projective indecomposable modules are elements of **Mods**, and so we remove them. Write A_i for the module corresponding to the i th element of **Mods**. Then the decompositions are given by:

$$A_1 \otimes A_1 = K \oplus A_1 \oplus \bigoplus_{i=2}^8 A_i \oplus 4 \cdot \mathcal{P}(K);$$

$$A_2 \otimes A_1 = 3 \cdot A_5 \oplus \mathcal{P}(K);$$

$$A_3 \otimes A_1 = 3 \cdot A_6 \oplus \mathcal{P}(K);$$

$$A_4 \otimes A_1 = 3 \cdot A_7 \oplus \mathcal{P}(K);$$

$$A_5 \otimes A_1 = 3 \cdot A_2 \oplus 3 \cdot A_5 \oplus 6 \cdot \mathcal{P}(K);$$

$$A_6 \otimes A_1 = 3 \cdot A_3 \oplus 3 \cdot A_6 \oplus 6 \cdot \mathcal{P}(K);$$

$$A_7 \otimes A_1 = 3 \cdot A_4 \oplus 3 \cdot A_7 \oplus 6 \cdot \mathcal{P}(K);$$

$$A_8 \otimes A_1 = A_1 \oplus A_8 \oplus A_9 \oplus \bigoplus_{i=5}^7 A_i \oplus 7 \cdot \mathcal{P}(K);$$

$$A_9 \otimes A_1 = A_8 \oplus A_9 \oplus A_{10} \oplus A_{11} \oplus \bigoplus_{i=2}^7 A_i \oplus 16 \cdot \mathcal{P}(K);$$

$$A_{10} \otimes A_1 = A_{10} \oplus A_{12} \oplus 2 \cdot \mathcal{P}(K);$$

$$A_{11} \otimes A_1 = A_9 \oplus A_{10} \oplus 7 \cdot \mathcal{P}(K);$$

$$A_{12} \otimes A_1 = 2 \cdot A_{10} \oplus A_{12} \oplus A_{13} \oplus 40 \cdot \mathcal{P}(K);$$

$$A_{13} \otimes A_1 = A_{13} \oplus A_{12} \oplus A_{14} \oplus 4 \cdot \mathcal{P}(K);$$

$$A_{14} \otimes A_1 = A_{13} \oplus A_{15} \oplus A_{14} \oplus 40 \cdot \mathcal{P}(K);$$

$$A_{15} \otimes A_1 = A_{15} \oplus A_{16} \oplus A_{14} \oplus 6 \cdot \mathcal{P}(K);$$

$$A_{16} \otimes A_1 = A_{16} \oplus A_{17} \oplus 14 \cdot \mathcal{P}(K);$$

$$A_{17} \otimes A_1 = 2 \cdot A_{16} \oplus A_{17} \oplus A_{18} \oplus 16 \cdot \mathcal{P}(K);$$

$$A_{18} \otimes A_1 = A_{17} \oplus A_{18} \oplus A_{19} \oplus 28 \cdot \mathcal{P}(K);$$

$$A_{19} \otimes A_1 = A_{19} \oplus A_{18} \oplus A_{20} \oplus 16 \cdot \mathcal{P}(K); \text{ and}$$

$$A_{20} \otimes A_1 = A_{10} \oplus A_{16} \oplus A_{19} \oplus A_{20} \oplus 28 \cdot \mathcal{P}(K).$$

Here, the dimensions of the modules A_i are given in the following table.

APPENDIX A. MAGMA COMPUTER PROGRAMS

Module	Dimension	Module	Dimension
A_1	14	A_{11}	16
A_2	5	A_{12}	80
A_3	5	A_{13}	20
A_4	5	A_{14}	80
A_5	15	A_{15}	20
A_6	15	A_{16}	30
A_7	15	A_{17}	40
A_8	21	A_{18}	60
A_9	39	A_{19}	40
A_{10}	10	A_{20}	60

This proves that the simple module S_1 is algebraic. In Section 6.2.2, it is shown that S_1 is algebraic if and only if J_2 has 5-SMA, and so we get Theorem 6.8(iii).

Bibliography

- [1] Jonathan Alperin, *On modules for the linear fractional groups*, International Symposium on the Theory of Finite Groups, 1974, Tokyo (1976), 157–163.
- [2] ———, *Periodicity in groups*, Illinois J. Math. **21** (1977), 776–783.
- [3] ———, *Projective modules for $SL(2, 2^n)$* , J. Pure Appl. Algebra **15** (1979), 219–234.
- [4] ———, *Local representation theory*, Cambridge, 1996.
- [5] Jonathan Alperin, Richard Brauer, and Daniel Gorenstein, *Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups*, Trans. Amer. Math. Soc. **151** (1970), 1–261.
- [6] ———, *Finite simple groups of 2-rank two*, Scripta Math. **29** (1973), 191–214.
- [7] Jonathan Alperin and Leonard Evens, *Representations, resolutions and Quillen’s dimension theorem*, J. Pure Appl. Algebra **22** (1981), 1–9.
- [8] Louise Archer, *Hall algebras and Green rings*, Ph.D. thesis, University of Oxford, 2005.
- [9] Michael Aschbacher, *3-transposition groups*, Cambridge, 1997.
- [10] ATLAS of finite group representations,
<http://brauer.maths.qmul.ac.uk/Atlas/v3/>.
- [11] Maurice Auslander and Jon Carlson, *Almost-split sequences and group rings*, J. Algebra **103** (1986), 122–140.
- [12] David Benson, *Modular representation theory: New trends and methods*, Springer–Verlag, 1984.

- [13] ———, *Representations and cohomology, I. Basic representation theory of finite groups and associative algebras*, Cambridge, 1998.
- [14] ———, *Representations and cohomology, II. Cohomology of groups and modules*, Cambridge, 1998.
- [15] David Benson and Jon Carlson, *Nilpotent elements in the Green ring*, J. Algebra **104** (1986), 329–350.
- [16] Thomas R. Berger, *Irreducible modules of solvable groups are algebraic*, Proceedings of the Conference on Finite Groups (Univ. of Utah, Park City, UT, 1975) (1976), 541–553.
- [17] ———, *Solvable groups and algebraic modules*, J. Algebra **57** (1979), 387–406.
- [18] Vitalij Bondarenko, *Representations of dihedral groups over a field of characteristic 2*, Mat. Sbornik **96** (1975), 63–74 (Russian).
- [19] Richard Brauer, *Character theory of finite groups with wreathed Sylow 2-subgroups*, J. Algebra **19** (1971), 547–592.
- [20] Rolf Burkhardt, *Die Zerlegungsmatrizen der Gruppen $\text{PSL}(2, p^f)$* , J. Algebra **40** (1976), 75–96.
- [21] Jon Carlson, *The dimensions of periodic modules over modular group algebras*, Illinois J. Math. **23** (1979), 295–306.
- [22] ———, *The structure of periodic modules over modular group algebras*, J. Pure Appl. Algebra **22** (1981), 43–56.
- [23] Jon Carlson and Jacques Thévenaz, *Torsion endo-trivial modules*, Algebr. Represent. Theory **3** (2000), 303–355.
- [24] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, 1956.
- [25] Alfred Clifford, *Representations induced in an invariant subgroup*, Ann. of Math. **38** (1937), 533–550.
- [26] Samuel B. Conlon, *The modular representation algebra of groups with Sylow 2-subgroup $Z_2 \times Z_2$* , J. Austral. Math. Soc. **6** (1966), 76–88.

- [27] John H. Conway, *A group of order 8, 315, 553, 613, 086, 720, 000*, Bull. London Math. Soc. **1** (1969), 79–88.
- [28] Larry Dornhoff, *Group representation theory, Part A: Ordinary representation theory*, Marcel Dekker, 1971.
- [29] Karin Erdmann, *Blocks and simple modules with cyclic vertices*, Bull. London Math. Soc. **9** (1977), 216–218.
- [30] ———, *Principal blocks of groups with dihedral Sylow 2-subgroups*, Comm. Algebra **5** (1977), 665–694.
- [31] ———, *On 2-blocks with semidihedral defect groups*, Trans. Amer. Math. Soc. **256** (1979), 267–287.
- [32] ———, *Blocks whose defect groups are Klein four groups: A correction*, J. Algebra **76** (1982), 505–518.
- [33] ———, *On Auslander–Reiten components for group algebras*, J. Pure Appl. Algebra **104** (1995), 149–160.
- [34] Karin Erdmann and Gerhard Michler, *Blocks with dihedral defect groups in solvable groups*, Math. Z. **154** (1977), 143–151.
- [35] Walter Feit, *Irreducible modules of p -solvable groups*, Proc. Sympos. Pure Math (Santa Cruz, 1979) **37** (1980), 405–412.
- [36] ———, *The representation theory of finite groups*, North–Holland, 1982.
- [37] Bernd Fischer, *Finite groups generated by 3-transpositions, I*, Invent. Math. **13** (1971), 232–246.
- [38] Israil M. Gelfand and V. A. Ponomarev, *Indecomposable representations of the Lorentz group*, Uspehi Mat. Nauk **23** (1968), 3–60, Russian.
- [39] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups, Number 3*, American Mathematical Society, 1998.
- [40] Daniel Gorenstein and John Walter, *The characterization of finite groups with dihedral Sylow 2-subgroups I,II,III*, J. Algebra **2** (1965), 85–151, 218–270, 334–393.

- [41] Marshall Hall Jr and David Wales, *The simple group of order 604, 800*, J. Algebra **9** (1968), 73–84.
- [42] Koichiro Harada, *On the simple group F of order $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$* , Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975), Academic Press, 1976, pp. 119–276.
- [43] Dieter Held, *The simple groups related to M_{24}* , J. Algebra **13** (1969), 253–296.
- [44] Alex Heller, *The loop-space functor in homological algebra*, Proc. Amer. Math. Soc. **12** (1961), 640–643.
- [45] Donald Higman and Charles Sims, *A simple group of order 44, 352, 000*, Math. Z. **105** (1968), 110–113.
- [46] Graham Higman, *On the simple group of D. G. Higman and C. C. Sims*, Illinois J. Math. **13** (1969), 74–80.
- [47] James E. Humphreys, *Modular representations of finite groups of Lie type*, London Mathematical Society Lecture Note Series, vol. 326, Cambridge, 1994.
- [48] Alexander Ivanov and Ulrich Meierfrankenfeld, *A computer-free construction of J_4* , J. Algebra **219** (1999), 113–172.
- [49] Zvonimir Janko, *A new finite simple group with abelian 2-Sylow subgroups*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 657–658.
- [50] ———, *A characterization of the Mathieu simple groups I, II*, J. Algebra **9** (1968), 1–19, 20–41.
- [51] ———, *Some new simple groups of finite order. I*, Symposia Mathematica (INDAM, Rome, 1967/68), Vol. 1, Academic Press, 1969, pp. 25–64.
- [52] ———, *A new finite simple group of order 86775571046077562880 which possesses M_{24} and the full covering group of M_{22} as subgroups*, J. Algebra **42** (1976), 564–596.
- [53] Zvonimir Janko and S. K. Wong, *A characterization of McLaughlin's simple group*, J. Algebra **20** (1972), 203–225.
- [54] Stephen Jennings, *The structure of the group ring of a p -group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185.

- [55] Reinhard Knörr, *On the vertices of irreducible modules*, Ann. Math. **110** (1979), 487–499.
- [56] Peter Landrock and Gerhard Michler, *Block structure of the smallest Janko group*, Math. Ann. **232** (1978), 205–238.
- [57] ———, *Principal 2-blocks of the simple groups of Ree type*, Trans. Amer. Math. Soc. **260** (1980), 83–111.
- [58] Markus Linckelmann, *The source algebras of blocks with a Klein four defect group*, J. Algebra **167** (1994), 821–854.
- [59] Richard Lyons, *Evidence for a new finite simple group*, J. Algebra **20** (1972), 540–569.
- [60] Emile Mathieu, *Mémoire sur le nombre de valeurs que peut acquérir une fonction quand on y permute ses variables de toutes les manières possibles*, J. Math. Pures Appl. **5** (1860), 9–42.
- [61] ———, *Sur la fonction cinq fois transitive de 24 quantités*, J. Math. Pures Appl. **18** (1873), 25–46.
- [62] Jack McLaughlin, *A simple group of order 898, 128, 000*, Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968), Benjamin, 1969, pp. 109–111.
- [63] Modular Atlas homepage, <http://www.math.rwth-aachen.de/~MOC/>.
- [64] Simon Norton, *F and other simple groups*, Ph.D. thesis, University of Cambridge, 1976.
- [65] ———, *The construction of J_4* , Proc. Sympos. Pure Math (Santa Cruz, 1979) **37** (1980), 271–277.
- [66] Michael O’Nan, *Some evidence for the existence of a new simple group*, Proc. London Math. Soc. **32** (1976), 421–479.
- [67] David Parrott, *On Thompson’s simple group*, J. Algebra **46** (1977), 389–404.
- [68] Nicholas Patterson and S. K. Wong, *A characterization of the Suzuki simple group of order 448, 345, 497, 600*, J. Algebra **39** (1976), 277–286.

- [69] Rimhak Ree, *A family of simple groups associated with the simple Lie algebra of type (G_2)* , Bull. Amer. Math. Soc. **66** (1960), 508–510.
- [70] ———, *A family of simple groups associated with the simple Lie algebra of type (G_2)* , Amer. J. Math. **83** (1961), 432–462.
- [71] Claus Michael Ringel, *The indecomposable representations of the dihedral 2-groups*, Math. Ann. **214** (1975), 19–34.
- [72] Arunas Rudavlis, *A new simple group of order $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$* , Notices. Amer. Math. Soc. **20** (1973), A–95.
- [73] William Simpson and J. Sutherland Frame, *The character tables for $SL(3, q)$, $SU(3, q^2)$, $PSL(3, q)$, $PSU(3, q^2)$* , Canad. J. Math. **25** (1973), 486–494.
- [74] Peter E. Smith, *A simple subgroup of $M^?$ and $E_8(3)$* , Bull. London Math. Soc. **8** (1976), 161–165.
- [75] Michio Suzuki, *A simple group of order 448, 345, 497, 600*, Theory of Finite Groups (Symposium, Harvard Univ. Cambridge, Mass., 1968), Benjamin, 1969, pp. 113–119.
- [76] John Thompson, *A simple subgroup of $E_8(3)$* , Finite Groups (N. Iwahori, ed.), Japan Soc. Promotion Science, 1976, pp. 113–116.
- [77] John Walter, *The characterization of finite groups with abelian Sylow 2-subgroups*, Ann. of Math. **89** (1969), 405–514.
- [78] Harold Ward, *On Ree's series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89.
- [79] Hiroyoshi Yamaki, *A characterization of the Suzuki simple group of order 448, 345, 497, 600*, J. Algebra **40** (1976), 229–244.