Scott Modules and Lower Defect Groups

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Preface

The focus of this transfer thesis is lower defect groups, and to a greater extent, Scott modules. The concept of a lower defect group was introduced by Brauer in the late 1960s to generalize that of a defect group: they satisfy some very nice properties that the defect groups seem to lack. Several characterizations of the multiplicities of lower defect groups in a block have arisen in the literature, which help to make this concept more useful in other circumstances.

One of the most unlikely connexions is between lower defect groups and Scott modules. A Scott module is a summand of an induced module that contains the trivial module in its socle: Scott proved that this unique submodule also contains the trivial module in its top. Characteristically, Scott left this work unpublished; it was later rediscovered by Alperin, who also didn't publish. The first hint of this theory in the literature came with Burry, and in this same paper he exposed the deep fact that the multiplicities of the lower defect groups are equal to the multiplicities of the Scott modules. We will have more to say about this later. This result enables quick proofs of several of Brauer's key theorems on lower defect groups, which we perform here.

The first chapter deals with lower defect groups, giving their original definition and some of the most important results concerning them, then moving on to give the characterizations of them that have appeared so far. One of the most important for our purposes is an equivalent definition of Iizuka and Olsson (see [18]), which we will use in the second chapter.

Scott modules are the content of Chapter 2. Their existence is proven in the first section, following Burry's method in [9]. Since Scott modules are constructed in terms of induction, many of our results with Scott modules centre around induction and restriction, and this is the content of the second section. This section really exhausts most of the published results that are in the literature. The third section is devoted to a multiplicity formula of Green, which gives the multiplicity of Scott modules in terms of the rank of a bilinear form. This is the second key concept, along with Iizuka and Olsson's characterization of lower defect groups to which we alluded before, that is required in our proof of Burry's Theorem. This is completed in the fourth section. We round off with two sections describing Scott modules in particular cases: the first is for p-groups, where a result of Alperin's describes them in a way similar to that of Jennings' Theorem; and the final section deals with Scott modules over the symmetric group S_4, completely describing their structure. This rather descriptive section can be omitted on a first reading, as it is rather calculation-intensive.

In the final part of this work, we discuss the possibilities for furthering the work in this project: representation theory, due to the distinct lack of knowledge – as evidenced by the prevalence of conjectures – in the subject, is a lively area of active research. At the same time, we try to intuitively understand precisely what the relationship is between the lower defect groups and Scott modules; the author believes that it is important to understand the reasons behind the equation $M_B(P) = m_B(P)$, which embodies this relationship.

We introduce some notation here: induction and restriction are represented by $M\uparrow^H$ and $M\downarrow_H$ respectively. The symbols $\mathrm{soc}(M)$ and $\mathrm{top}(M)$ refer to the socle (largest semisimple submodule) and top (largest semisimple quotient) of a module M. We let R be either a ring or field, and reserve K exclusively for a field. The trivial KG-module is denoted by 1_G. Unless stated otherwise (particularly in the final section), K is assumed to be a splitting field for G. If M is any KG-module, then $\mathcal{P}(M)$ denotes the projective cover of M.

We assume only standard facts on modular representation theory, such as Frobenius Reciprocity, Brauer and Green Correspondences, Green's Indecomposability Criterion, and at one point the definition of relative projective covers. Other results that we require, like Mackey's Theorem, we state in the text.

Finally, the multiplication of any group elements is on the right, and all groups considered here are finite. If f is a function and x is an element in the domain, then the image is denoted by xf, not $f(x)$.

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Chapter 1

Lower Defect Groups

Lower defect groups, like defect groups before them, were introduced by Brauer. His paper [6] gives the first of several definitions of lower defect groups. In this chapter we will, in particular, survey the state of the theory of lower defect groups up until 1982, and Burry's insight into the connexion between lower defect groups and Scott modules, which we will see in Section 2.4.

The first section of this chapter will give Brauer's original definition, and some of the theorems that Brauer proved in [6]. His methods are character-theoretic, which is in contrast to the second chapter, which makes up the bulk of this work.

Brauer's original definition of a lower defect group works with the dual space of a block. This notion is reasonably difficult to work with; other equivalent definitions have been sought, the most interesting from our point of view being Burry's interpretation of the lower defect groups in terms of Scott modules. In the second section we will see other definitions of lower defect groups, due to Broué and Olsson.

This chapter is preliminary and motivational in nature; the definition of Scott modules seems to imply that they are of limited external interest outside of induced trivial modules, and transitive permutation modules in general. The connexion with lower defect groups is some evidence against this viewpoint.

1.1 Brauer's Lower Defect Groups

Let G denote a finite group, and let C denote a conjugacy class of G. Recall that a *defect group* of C is a Sylow p-subgroup of $\mathrm{C}_G(x)$, where $x \in C$. Since $\mathrm{C}_G(x^g) = \mathrm{C}_G(x)^g$, all defect groups of a conjugacy class are conjugate. Throughout this section, we will refer to a block of the **centre** of a group algebra: Brauer works inside $\mathrm{Z}(KG)$, and therefore his blocks are, in fact, the centre of the usual notion of blocks. To prevent confusion, we will denote this by $\mathrm{Z}(B)$.

Let $\bar{C} \in KG$ denote the sum of all elements of C (the so-called class sum). We say that P, a p-subgroup of G, is a *lower defect group* of B if there exists f a non-zero element of the dual space $\mathrm{Z}(B)^*$ such that:

(i) there is a conjugacy class C with defect group P with $\bar{C}f \neq 0$; and

(ii) if C is any conjugacy class with defect group D_C where $|D_C| < |P|$, then $\bar{C}f = 0$.

Let $m_B(P)$ denote the maximal dimension of a space $V \subseteq \mathrm{Z}(B)^*$ for which all elements of V satisfy the properties required for f above. Then $m_B(P)$ denotes the largest number of linearly independent functions f that satisfy the conditions given above. Notice that $m_B(P) = 0$ if and only if P is not a lower defect group. We say that $m_B(P)$ is the *multiplicity* of P as a lower defect group.

Brauer proves in [6] that the lower defect groups 'fill the group in a natural way' (to quote Burry in [9]). This result is that the sum of all of the multiplicities equals the dimension of the block; that is,

$$\sum_P m_B(P) = k(B),$$

where $k(B)$ as usual denotes the number of irreducible ordinary characters lying inside the block B.

We will state this and another similar result as a theorem in itself.

Theorem 1.1 (Brauer, [6]) Let G be a finite group and B a block of G. Let Q be a p-subgroup of G. Let N_Q denote the number of conjugacy classes of G with defect group Q. Then

$$k(B) = \sum_P m_B(P),$$

where the sum runs over the conjugacy classes of p-subgroups of G, and

$$N_Q = \sum_b m_b(Q),$$

where the sum runs over the blocks b of KG.

We will prove this later in the chapter (once we have a more tractable definition), and again vicariously through Scott modules.

This theorem is, in some way, the fundamental theorem on lower defect groups, since it shows that they are the natural extension of defect groups: whereas defect groups don't really offer much in terms of determining the dimension of a block, the lower defect groups determine it precisely. The second statement says that the lower defect groups capture all of the information about Sylow p-subgroups of centralizers.

Brauer goes on to prove several more key results regarding lower defect groups. The next in [6] shows that the p-local subgroups determine $m_B(P)$.

Theorem 1.2 (Brauer, [6]) For any block B of G, and for any p-subgroup P,

$$m_B(P) = \sum_b m_b(P),$$

where the sum ranges over all blocks b of $\mathrm{N}_G(P)$ that are Brauer correspondents of B.

The next result of Brauer's is easy to prove. It states that lower defect groups are basically smaller than defect groups.

Proposition 1.3 (Brauer, [6]) The defect group D of a block B is a maximal element in the set of all lower defect groups (under the usual partial ordering of inclusion).

Proof: Let ω_B denote the central character associated with B. Then $\omega_B(\bar{C})$ vanishes for all class sums $\bar{C}$ whose defect groups are smaller (in order) than D. However, since D is a defect group, there is a conjugacy class whose defect group is D and whose class sum does not vanish under the central character. Hence D is a lower defect group.

Now suppose that P is a lower defect group of B. There there is a block b of $\mathrm{N}_G(P)$ in Brauer correspondence with B, and for which P appears as a lower defect group. Since $P \trianglelefteq \mathrm{N}_G(P)$, P is a subgroup of all defect groups of b, and a defect group of b is (conjugate to) a subgroup of D, by the Brauer Correspondence. Hence D contains P, as required. $\square$

Finally, we give an interesting result regarding $|\mathrm{N}_G(P) : P|$ if P is a lower defect group that is not an ordinary defect group. It turns out that this index is always divisible by p; in fact, we can find a p-subgroup of G which sits above P inside $\mathrm{N}_G(P)$ with a particularly nice property.

Proposition 1.4 (Brauer, [6]) Let G be a finite group and B a block of G. If P is a lower defect group of B that is not a defect group, then there exists a p-subgroup Q of $\mathrm{N}_G(P)$ with

$$P < Q \leqslant \mathrm{N}_G(P),$$

and a block b of $N = \mathrm{N}_G(P) \cap \mathrm{N}_G(Q)$ in Brauer correspondence with B.

Proof: This proof will be related to that of Proposition 1.3. Again, there exists a block B' of $H = \mathrm{N}_G(P)$, with P as a lower defect group, and with $(B')^G = B$, by Theorem 1.2. If P were a defect group of B', then the Brauer Correspondence would immediately imply that P was a defect group of B, a contradiction. Thus P is contained in a defect group, say Q of B. Remember that $Q \leqslant H$. Then we have, again by the Brauer Correspondence, a block b of

$$\mathrm{N}_H(Q) = \mathrm{N}_G(Q) \cap H = \mathrm{N}_G(Q) \cap \mathrm{N}_G(P)$$

with Q as a defect group and in Brauer correspondence with B', which is in turn in Brauer correspondence with B. Thus $b^G = B$, as required. $\square$

1.2 Characterizing $m_B(P)$

The constants $m_B(P)$ for P a p-subgroup of G and B a block of G are essential in the sense that they completely determine the multiplicities of the lower defect groups of B. Any method of determining $m_B(P)$ without recourse to the dual space is welcome; the dual space is a difficult object to work with in terms of representation theory. In this section we will consider several other interpretations of the numbers $m_B(P)$.

Before we start this section, we will need to review some facts. We will have to use the module interpretation of block theory (as described in [4]), where blocks are viewed as $K(G \times G)$ modules. We have a diagonal map, and we restrict our 'modules' (i.e., blocks) to this diagonal map. Let Δ denote the diagonal $\{(g, g) : g \in G\}$; we can restrict B, any $K(G \times G)$-module, to Δ to get $B_\Delta = B \downarrow_\Delta$ (so that in particular $KG_\Delta = KG \downarrow_\Delta$). We need more concepts and notation though to understand this section.

Let $I_H(M)$ denote the H-invariant elements of M, and $T_{H,G}$ denote the relative trace map from $I_H(M)$ to $I_G(M)$, with its image in $I_G(M)$ denoted by $I_{H,G}(M)$. Lastly we let $I'_{H,G}(M)$ be the set

$$I'_{H,G}(M) = \sum_L I_{L,G}(M),$$

where the sum runs over all subgroups L that are G-conjugate to a *proper* subgroup of H. We have the following elementary lemma concerning invariant subspaces.

Lemma 1.5 Let G be a finite group, H be a subgroup of G and B a block of G with block idempotent $e \in \mathrm{Z}(KG)$. Let B_Δ denote the block B viewed as a $K(G \times G)$ module under the diagonal map. Then

$$I_H(B_\Delta) = I_H(KG_\Delta)e, \qquad I_{H,G}(B_\Delta) = I_{H,G}(KG_\Delta)e.$$

The proof of this lemma is obvious, and is safely left to the reader. We can now state the Iizuka–Broué–Olsson Characterization (or IBO Characterization, for short) of lower defect group multiplicities. The form of this is the following.

Theorem 1.6 (Iizuka, Broué, Olsson, [14], [7], [18]) $m_B(P) = \dim(I_{P,G}(B_\Delta)/I'_{P,G}(B_\Delta))$.

To get Broué's definition, in [7], we need to know about the Brauer morphism: this is the identity map from KG onto $K\,\mathrm{C}_G(P)$, where P is any p-subgroup of G. This is denoted by Br_P. So we have

$$x\,\mathrm{Br}_P = \begin{cases} x, & x \in \mathrm{C}_G(P) \\ 0, & x \notin \mathrm{C}_G(P) \end{cases}$$

for the basis elements of KG, which is then extended by linearity.

We can easily see that

$$\ker \mathrm{Br}_P \cap I_{P,G}(KG) = \sum_{Q<P} I_{Q,G}(KG),$$

and so we get the equation

$$(I_{P,G}(KG))\,\mathrm{Br}_P \cong I_{P,G}(KG)/\sum_{Q<P} I_{Q,G}(KG),$$

and the dimension of the right-hand side is the number $m_{KG}(P)$. Simply multiplying by the block idempotent e (of a block B) gives

$$m_B(P) = \dim_K(I_{P,G}(B))\,\mathrm{Br}_P,$$

and this is the form in which Broué originally stated this equivalence in [7].

We can also see that if B is a block with defect group D, then $B\,\mathrm{Br}_P \neq 0$ if and only if $P \leqslant D$ up to conjugacy (see, for example, [7]). Then this easily gives a proof of Proposition 1.3, which is considerably shorter than that given directly.

Notice that the spaces $I_{P,G}(B)$ and $I'_{P,G}(B)$ are spanned by all class sums whose conjugacy classes had defect group contained and properly contained within P respectively. This is an important fact, and gives us $\dim_K I_{P,G}(B)$ and $\dim_K I'_{P,G}(B)$.

The notation $I_{P,G}(B)$ is clumsy, and so we will write $J_P = I_{P,G}(B)$ and $J'_P = I'_{P,G}(B)$. Then

$$m_B(P) = \dim_K B \cap J_P - \dim_K B \cap J'_P = \dim_K \mathrm{Z}(B) \cap J_P - \dim_K \mathrm{Z}(B) \cap J'_P$$

since J_P and J'_P live inside $\mathrm{Z}(KG)$.

We now move to the concept of block splittings, as described in [18], and implied in [14]. Let Z denote the centre of the group algebra KG. Then a subspace V of Z is called *block-invariant* if whenever e is a block idempotent, $Ve \subseteq V$. Alternatively, V is block-invariant if

$$V = \sum_e Ve = \sum_{\mathrm{Z}(B)} = V \cap \mathrm{Z}(B),$$

where the first sum runs over all block idempotents e, and the second sum runs over all blocks B (and the centres of blocks $\mathrm{Z}(B)$ – remember that Brauer uses this to define lower defect groups). One of the most important results on block invariants is the following.

Proposition 1.7 (Olsson, [18]) Let $\mathcal{V}$ be a basis for a block-invariant subspace V. Then $\mathcal{V}$ is a disjoint union

$$\mathcal{V} = \bigcup_B \mathcal{V}_B,$$

where the union is taken over all blocks B, and the $\mathcal{V}_B$ have the property that if e is the block idempotent for B, then $\{ve : v \in \mathcal{V}_B\}$ is a basis for $\mathrm{Z}(B) \cap V$.

Of course, we can take $V = \mathrm{Z}(KG)$ – this is certainly a block-invariant subspace – and we already know of a basis of $\mathrm{Z}(KG)$: the class sums. Let $Cl(G)$ denote the set of conjugacy classes, and if C is a conjugacy class, denote by $\bar{C}$ the class sum of C. Then applying Proposition 1.7 to this basis gives a disjoint union

$$Cl(G) = \bigcup_B Cl(B),$$

where $Cl(B)$ is defined to be the set given in Proposition 1.7: in particular, if e is the block idempotent of B, then

$$\{\bar{C}e : C \in Cl(B)\}$$

forms a basis for $\mathrm{Z}(B)$. Now we can see the possibility of relating these to lower defect groups: the centre of a block is appearing again, which was used in the definition of lower defect groups. We call *any* decomposition of $Cl(G)$ into sets $\Psi = \{\Psi_B\}$ with the property given above a *block splitting* of $Cl(G)$. Although block splittings are not unique, they are very closely related to one another: indeed, we have the following result.

Theorem 1.8 (Olsson,[18]) Suppose that V is a block-invariant subspace of $\mathrm{Z}(KG)$. Let $\mathscr{B}$ be a subset of $Cl(G)$, whose class sums form a basis for V. If $\Psi = \{\Psi_B\}$ is any block splitting for $Cl(G)$, then the set

$$\mathcal{V}_B = \{\bar{C}e : C \in \Psi_B \cap \mathscr{B}\}$$

(where e is the block idempotent for any block B) is a basis for $V \cap \mathrm{Z}(B)$.

Proof: Let e_B denote the block idempotent of the block B. Consider the set

$$\mathcal{V} = \{\bar{C}e_B : B \text{ is a block}, \mathscr{C} \in \Psi_B \cap \mathscr{B}\}.$$

Now $\mathcal{V}$ is a linearly independent set, since $\mathcal{V}$ is a subset of the block splitting (which forms a basis itself). Also, $\mathcal{V} \subseteq V$ since V is block invariant. Now

$$\dim_K V = |\mathscr{B}|,$$

since $\mathscr{B}$ is a basis for V, and also $|\mathscr{B}| = |\mathcal{V}|$ clearly: thus $\mathcal{V}$ is a basis for V.

Now we can easily see that the result follows: Proposition 1.7, together with the fact that blocks intersect trivially, provides the rest. □

Now we can use Theorem 1.8 to find another characterization of $m_B(P)$: recall the alternative definitions of J_B and J'_B in terms of class sums.

Proposition 1.9 (Olsson, [18]) Let $\Psi = \{\Psi_B\}$ be any block splitting for $Cl(G)$. For any p-subgroup P, denote by $\Psi_{B,P}$ the subset of Ψ_B comprising all conjugacy classes with defect group P. Then $m_B(P) = |\Psi_{B,Q}|$.

Proof: Notice firstly that J_P and J'_P are block-invariant subspaces. Then the dimension of $J_P \cap \mathrm{Z}(B)$ is the number of conjugacy classes in Ψ_B with defect group contained within P, and similarly the dimension of $J'_P \cap \mathrm{Z}(B)$ is the number of conjugacy classes in Ψ_B with defect group strictly contained within P. The different in these two numbers is $|\Psi_{B,P}|$, the number of conjugacy classes in Ψ_B with defect group equal to P. Hence

$$m_B(P) = \dim_K(\mathrm{Z}(B) \cap J_P) - \dim_K(\mathrm{Z}(B) \cap J'_P) = |\Psi_{B,P}|,$$

as required. □

Remember Theorem 1.1, which we did not prove before: it is time to correct this oversight. It is now much easier with Proposition 1.9: in that theorem, N_Q was the number of conjugacy classes with defect group Q; then clearly, if $\{\Psi_B\}$ is a block splitting,

$$N_Q = \sum_b |\Psi_{b,Q}| = \sum_b m_b(Q),$$

where the sum is taken over all blocks b. For the other equality, since $k(B) = \dim \mathrm{Z}(B)$, and certainly

$$\dim_K \mathrm{Z}(B) = \dim_K B \cap \mathrm{Z}(KG) = \sum_P |\Psi_{B,P}|,$$

where the sum is taken over all conjugacy classes of P-subgroups P. The theorem is proven!

The other statement that we did not prove was Theorem 1.2, which related $m_B(P)$ with $m_b(P)$, where the blocks b were Brauer correspondents of B in $K\,\mathrm{N}_G(P)$. We will embark on a proof now. It will fairly obviously use the Brauer morphism, which we restrict to the centre of the group algebra: then we get an algebra homomorphism.

Let B be a block of KG, with block idempotent e, let P be a p-subgroup of G, and let H lie between $P\,\mathrm{C}_G(P)$ and $\mathrm{N}_G(P)$. We can write $e\,\mathrm{Br}_H$ as a sum of primitive idempotents in $\mathrm{Z}(H)$, whose corresponding blocks are Brauer correspondents of B. Denote this set by $\Omega_{B,H}$. Denote by $J_P(H)$ the space generated by all elements of $Cl(H)$ with defect group a subgroup of P, and by $J'_P(H)$ the space generated by all elements of $Cl(H)$ with defect group a *proper* subgroup of P, analogous to J_P and J'_P. Finally, write N for $\mathrm{N}_G(P)$.

We will show that Br_P induces an algebra isomorphism between J_P/J'_P and $J_P(N)/J'_P(N)$. We know that J'_P is the kernel of Br_P (as it was remarked above), and Br_P induces a bijection between the subsets of $Cl(G)$ and $Cl(N)$ of defect group P (see, for example, [16], Lemma 4.8). Then it is clear that the two algebras are isomorphic.

Now that we have a broad isomorphism between algebras, we now need to restrict our attention to blocks. Let us firstly show that J_P/J'_P is the product of its intersections with the various blocks. Let B be a block, and define a function

$$\Phi_{P,B} : J'_P + x \mapsto (J'_P + B) + xe_B,$$

where e_B is the block idempotent of B. We can form the 'product' of $\Phi_{P,B}$ for all blocks B, giving a map

$$\Phi_P : J_P/J'_P \to \prod_B \frac{J_P \cap B}{J'_P \cap B},$$

where the co-ordinate functions are simply $\Phi_{P,B}$. Suppose that $J'_P + x$ lies in the kernel of Φ_P: then $J'_P + x$ lies in the kernel of each $\Phi_{P,B}$. However, if $J'_P + x \in \ker \Phi_{P,B}$, then

$$xe_B \in J'_P \cap B \subseteq J'_P,$$

and this is true for all blocks. Thus

$$J'_P \ni \sum_B xe_B = x \sum_B e_B = x;$$

hence Φ_P is injective. Suppose that $\Psi = \{\Psi_B\}$ is a block splitting. We get surjectivity of Φ_P from the following three facts:

(i) $J_P/J'_P = N_Q$, the number of conjugacy classes of G with defect group P;

(ii) $N_Q = \sum_B m_B(P) = \sum_B |\Psi_{B,P}|$, the sum being taken over all blocks B; and

(iii) every conjugacy class lies in some block (via Ψ).

Thus we can split J_P/J'_P up into its constituent blocks.

Finally, if B is any block of G, we construct an isomorphism

$$X_B = \frac{J_P \cap B}{J'_P \cap B} \to \prod_b \frac{J_P(N) \cap b}{J'_P(N) \cap b} = Y_B,$$

where the product runs over all elements of $\Omega_{B,H}$, which was defined earlier as the blocks of $K\,\mathrm{N}_G(P)$ which feature in a decomposition of $e_B \operatorname{Br}_P$ into a sum of primitive central idempotents. Let ι denote the inclusion map from X_B into J_P/J'_P, and π denote the projection from $J_P(N)/J'_P(N)$ onto Y_B. We can construct the function $\phi = \iota \operatorname{Br}_P \pi$, which maps from X_B to Y_B, through J_P/J'_P and $J_P(N)/J'_P(N)$. We will show that ϕ is injective. This will prove that $\dim_K X_B \leqslant \dim_K Y_B$. But

$$\sum_B \dim_K X_B = \dim_K J_P/J'_P = \dim_K J_P(N)/J'_P(N) = \sum_B \dim_K Y_B,$$

and so we conclude that ϕ is an isomorphism. Let us prove that ϕ is injective then.

We want an equation for what happens to $J'_P \cap B + xe_B$ under the map ϕ: we have

$$\begin{aligned}(J'_P \cap B + xe_B)\iota \operatorname{Br}_P \pi &= (J'_P + xe_B) \operatorname{Br}_P \pi \\ &= (J'_P(N) + (x \operatorname{Br}_P)(e_B \operatorname{Br}_P))\,\pi \\ &= \prod_{b \in \Omega_{B,N}} J'_P(N) \cap b) + (x \operatorname{Br}_P)e_b.\end{aligned}$$

Suppose that $J'_P \cap B + xe_B$ lies in the kernel of ϕ. Then, given the equation above, we must have

$$(x \operatorname{Br}_P)e_B \in J'_P(N),$$

for all blocks $b \in \Omega_{B,N}$, in the same way as when we proved that Φ_P was injective. In the same way, we can sum over all blocks in $\Omega_{B,N}$ to get

$$J'_P(N) \ni \sum_{b \in \Omega_{B,N}} (x \operatorname{Br}_P)e_b = (x \operatorname{Br}_P) \sum_{b \in \Omega_{B,N}} e_b = (x \operatorname{Br}_P)(e_B \operatorname{Br}_P).$$

Now, if $(xe_B) \operatorname{Br}_P \in J'_P(N)$, then since Br_P induces an isomorphism between J_P/J'_P and $J_P(N)/J'_P(N)$, we have $xe_B \in J'_P$, and so

$$xe_B \in J'_P \cap B,$$

proving that ϕ is injective.

Theorem 1.2 is now an easy corollary of this result. Certainly $\dim_K X_B = m_B(P)$, and

$$\dim_K Y_B = \sum_{b \in \Omega_{B,N}} \dim_K \frac{J_P(N) \cap b}{J'_P(N) \cap b} = \sum_{b \in \Omega_{B,N}} m_b(P);$$

we have now proven all of the results of Brauer that we have stated here.

There is another area that we haven't covered: the notion of subpairs can also be applied to the theory of lower defect groups. This theory was started in [3], and the connexion with lower defect groups was established in [8]. We will not enter this theory here, and refer the reader to the paper [8] of Broué and Olsson above.

Chapter 2

Scott Modules

This chapter is concerned with Scott modules. These were first studied by Scott, although none of this work is published. They were independently discovered by Jonathan Alperin, who also didn't publish the results. The first mention of them in the literature was in 1982, by David Burry in [9]. The theory of Scott modules experienced a brief flourishing in the mid-1980s, although the theory petered out towards the end of that decade. In this chapter we expound most of the published results concerning Scott modules. These are spread out throughout the literature, and it is very hard to pin down the current state of the theory.

The first section deals with the existence and basic properties of Scott modules. We give the proof, due to Burry and (partially) Landrock, of the existence and uniqueness of Scott modules, and show that only conjugacy classes of p-subgroups contribute different Scott modules. (Scott modules are so-called trivial source modules – recall that trivial source modules are summands of induced trivial modules – that contain the trivial module.)

The second section is concerned with induction and restriction. Since the Scott module is a summand of an induced module, its behaviour on induction and restriction is easier to describe than that of a general indecomposable module. Restriction, in particular, is controlled tightly by Mackey's Theorem.

Theorem 2.1 (Mackey's Theorem) Let R be a local ring or field, and suppose that H and L are subgroups of the group G, let M be an RH-module, and let S be a set of double coset representatives for G with respect to H and L. Then

$$M\uparrow^G\downarrow_L = \bigoplus_{s\in S} ((M^s)\downarrow_{L\cap sHs^{-1}})\uparrow^L .$$

The third section is devoted to deriving a result of Green concerning the multiplicity of the Scott module as a summand of an arbitrary module. Green uses Fitting's Lemma to prove this.

Proposition 2.2 (Fitting's Lemma, [11]) Let M be a KG-module, and decompose it as

$$M = \bigoplus M_i,$$

where each M_i is indecomposable. For $f \in \mathrm{End}_{KG}(M)$, write f as the matrix $(f)_{i,j}$, where the i,j-th element $f_{i,j}$ of $(f)_{i,j}$ is an element of $\mathrm{Hom}_{KG}(M_i, M_j)$. Then $J(\mathrm{End}_{KG}(M))$ is the set of all endomorphisms f such that $f_{i,j}$ is not an isomorphism for all i, j.

The fourth section completes Burry's Theorem, relating Scott modules to lower defect groups. We then use this to prove some of Brauer's results on lower defect groups.

The fifth section deals with p-groups, proving in particular Alperin's characterization of the structure of transitive permutation modules over p-groups. Examples of Scott modules over V_4 and D_8 are given.

In the final section, some examples of Scott modules are given, namely all Scott modules for $G = A_4$ and $G = S_4$. These are very difficult to find in the literature, and it is mainly because of this reason that the author believes that this section is perhaps the most important. A lack of examples might well be said to defeat the object of defining something in the first place.

2.1 Existence and First Properties

Before we start this section, let us review some basic facts on vertices and sources. Let M be an indecomposable module. then P is a *vertex* of M if M is relatively P-projective but not relatively Q-projective for $Q < P$. Since all modules are relatively P-projective for P a Sylow p-subgroup of G, all vertices are p-subgroups, and they form a conjugacy class of p-subgroups. If M has vertex P, then a *source* of M is a KP-module S such that $M|S\uparrow^G$. A module is called *trivial source* if $S = 1_P$. A result that is very important for us is that M has vertex at least P if $1_P|M\downarrow_P$.

The fundamental theorem on Scott modules is the so-called Scott–Alperin Theorem, which asserts the existence and uniqueness of Scott modules. Let G be a group and H be a subgroup of G, and let 1_H denote the trivial module over KH. A Scott module is an indecomposable summand of the induced module $(1_H)\uparrow^G$ with certain nice properties.

Theorem 2.3 (Scott–Alperin Theorem) Let G be a group, and H a subgroup of G. Let P denote a Sylow p-subgroup of H. Consider the induced module $M = (1_H)\uparrow^G$. Let S be a component – i.e., an indecomposable summand – of M. Then the following are equivalent:

(i) 1_G is a summand of $\mathrm{soc}(S)$;

(ii) 1_G is a summand of $\mathrm{top}(S)$; and

(iii) S has vertex P, and if $P \neq 1$ and the Green correspondent of S in $\mathrm{N}_G(P)$ is denoted by L, then when viewed as an $\mathrm{N}_G(P)/P$-module, L is the projective cover of $1_{\mathrm{N}_G(P)/P}$, and if $P = 1$, $S \cong \mathcal{P}(1_G)$.

Furthermore, such a summand S is uniquely determined.

Proof: Let $H \leqslant G$, and P be a Sylow p-subgroup of H. If we let $M = (1_H)\uparrow^G$, then M is a transitive permutation module, with permutation basis labelled by the cosets of H in G. We know that H fixes one of the elements of the permutation basis B, say x. Thus P is a Sylow p-subgroup of the point stabilizer of x. We will normally just stick to this general setup, although will occasionally remember that $M = (1_H)\uparrow^G$ if it is useful. Notice that in this case, 1_G is always isomorphic to a submodule of M. (This submodule 1_G is unique, since

$$K \cong \mathrm{Hom}_{KH}(1_H, 1_H) = \mathrm{Hom}_{KH}((1_G)\downarrow_H, 1_H) \cong \mathrm{Hom}_{KG}(1_G, (1_H)\uparrow^G),$$

by Frobenius Reciprocity). Thus this 1_G must be a submodule of some (unique) indecomposable summand S of M. We have already shown that if the three properties (i), (ii) and (iii) are equivalent, then S is uniquely determined, although this will again be shown later in the proof. We will show that this S also has the trivial module as a quotient.

We know that this submodule 1_G lying inside M is simply the sum of the basis elements of the permutation module. Let this submodule be called I. Then $B \setminus \{x\}$ generates (as a K-module) a KP-complement N

of I. Now to see this, notice that $N \oplus I = M$ as vector spaces. Also, I is certainly P-invariant since it is G-invariant. Since P fixes x, P cannot send any element of $B \setminus \{x\}$ to it, and so N is P-invariant. Let us restrict to P. Then

$$M \downarrow_P = N \downarrow_P \oplus I \downarrow_P .$$

Now $I \downarrow_P$ is therefore a direct summand of $M \downarrow_P$. But since $I \subseteq S$, $I \downarrow_P$ must be a subset of $S \downarrow_P$, and therefore a direct summand of $S \downarrow_P$. Now $I \downarrow_P$ is simply the trivial KP-module, and therefore has vertex P; thus S must have vertex containing P. Conversely, since S is a summand of M, which is equal to $(1_H) \uparrow^G$, a vertex of S is contained within a Sylow p-subgroup of H; i.e., P. Thus P is a vertex of S.

Suppose now that $P \neq 1$. Denote by $S\Gamma$ the Green correspondent of S in $K\,\mathrm{N}_G(P)$. The Green Correspondence clearly implies

$$\mathrm{Hom}_G(1_G, S) \subseteq \mathrm{Hom}_G(1_G, (S\Gamma) \uparrow^G),$$

and since the first term is non-zero (remember 1_G is isomorphic to a submodule of S), the second term is also non-zero. Frobenius Reciprocity then gives

$$\mathrm{Hom}_G(1_G, (S\Gamma) \uparrow^G) = \mathrm{Hom}_{\mathrm{N}_G(P)}(1_{\mathrm{N}_G(P)}, S\Gamma).$$

Thus $S\Gamma$ has a submodule isomorphic with $1_{\mathrm{N}_G(P)}$ lying inside it. Now $S\Gamma$ has vertex equal to P, a normal subgroup of $\mathrm{N}_G(P)$, and has trivial source (because $S\Gamma$ is indecomposable, and is a component of $S \downarrow_{\mathrm{N}_G(P)}$, which by Mackey's Theorem is a sum of trivial source modules – note that this also shows that P acts trivially on $S\Gamma$, since $S\Gamma | (1_P) \uparrow^{\mathrm{N}_G(P)}$, and $P \trianglelefteq \mathrm{N}_G(P)$). Thus $S\Gamma$ is projective as an $\mathrm{N}_G(P)/P$-module. Hence, $S\Gamma$ is the projective cover of $1_{\mathrm{N}_G(P)/P}$. Thus (i)$\Longrightarrow$(iii).

If $P = 1$, then S has a submodule isomorphic with 1_G, has trivial source (since it is already a summand of a transitive permutation module) and has vertex 1, so is an indecomposable projective: thus $S \cong \mathcal{P}(1_G)$ in this case, and we again have (i)$\Longrightarrow$(iii).

Now, the Green Correspondence gives a bijection between (non-projective) indecomposable modules of $\mathrm{N}_G(P)$ and G, and since any module satisfying (i) satisfies (iii), we also have the converse, so (iii) implies (i).

Finally, notice that $S\Gamma$ is self-dual (since it is a projective cover of the trivial module), and recall that dualizing and the Green Correspondence commute. S has 1_G in its socle, so S^* has 1_G in its top. But

$$(S^*)\Gamma = (S\Gamma)^* = S\Gamma,$$

and since Γ is one-to-one, $S = S^*$. Thus S has 1_G in its top, and (i)$\Longrightarrow$(ii). Lastly, if S has 1_G in its top, then S^* satisfies (i), and hence (ii). Thus S^* has 1_G in its top, and so S has 1_G in its socle, and (ii)$\Longrightarrow$(i), completing the implications. $\square$

The module defined above is called the *Scott module*. It depends on only the subgroup H of G, and so is often denoted by $S(G, H)$ (see, for example, [17]). In fact, there is not so much choice of Scott module, by the following result.

Proposition 2.4 (Scott, Alperin) Let G be a group, and H and L subgroups of G. Let P and Q denote Sylow p-subgroups of H and L respectively. Then $S(G, H)$ and $S(G, L)$ are isomorphic exactly when P and Q are conjugate. In particular, $S(G, H) \cong S(G, P)$ for any Sylow p-subgroup P of H.

Proof: We will first show that $S = S(G, H) \cong S(G, P) = T$. Now let the Green Correspondence with $\mathrm{N}_G(P)$ be denoted by Γ. Then both $S\Gamma$ and $T\Gamma$ are projective covers of $1_{\mathrm{N}_G(P)/P}$, and so must be isomorphic KG-modules. Thus $S(G, H) \cong S(G, P)$.

Now we will show that P and Q are conjugate p-subgroups of G if and only if $S = S(G, P) \cong S(G, Q) = T$. Since P and Q are conjugate, Q is a vertex for S. Thus the Green correspondent of S in $\mathrm{N}_G(Q)/Q$ is the projective cover of the trivial module. However, this is also the Green correspondent of T, and so $S \cong T$. Finally, if $S \cong T$, then P and Q are both vertices of S, and so must be conjugate. This completes the proof. □

This means that, assuming G is fixed, we need only write $S(P)$ to denote the Scott module with vertex P. However, we will continue to write $S(G, P)$. We will end this section by describing two extremal cases of Scott modules.

Lemma 2.5 Let G be a group, Q a p-subgroup of G and P a Sylow p-subgroup of G.

(i) $S(G, Q)$ is self-dual;

(ii) $S(G, 1) \cong \mathcal{P}(1_G)$; and

(iii) $S(G, P) \cong 1_G$.

Proof: For (i), this was proven in the proof of the Scott–Alperin Theorem. It follows from the fact that the Green Correspondence and dualizing commute, and that when viewed as an $\mathrm{N}_G(Q)/Q$-module, the Green correspondent of the Scott module is self-dual (as it is a projective cover of a self-dual module).

For (ii), notice that the Green Correspondence Γ between G and $G = \mathrm{N}_G(1)$ is the identity, and that $S(G, 1)$ is projective as a $\mathrm{N}_G(1)/1$-module; thus $S(G, 1)$ is a projective indecomposable with 1_G in the socle. Hence $S(G, 1) \cong \mathcal{P}(1_G)$.

For (iii), notice that $S(G, P) \cong S(G, G)$, since P is a Sylow p-subgroup, and clearly $S(G, G) \cong 1_G$. □

In fact, all of the three facts given in this lemma are a consequence of a result of Thévenaz.

Proposition 2.6 (Thévenaz, [22]) Let G be a finite group, H a subgroup of G and Q a Sylow p-subgroup of H. Then $S(G, H)$ is the relative H-projective cover of 1_G, and the relative H-injective hull of 1_G.

Proof: That $S(G, H)$ is the relative H-projective cover of 1_G is obvious from the fact that the Scott module is indecomposable, is relatively Q-projective, and contains 1_G in its socle. The second statement comes from the fact that relative projective covers are relative injective hulls. □

We can easily see that this proposition implies Lemma 2.5: relative projective covers of self-dual modules are self-dual, and since 1_G is relatively P-projective for P Sylow p-subgroup of G, this implies that $S(G, P) = 1_G$. Finally, since relative 1-projectivity is the same as the usual projectivity, $S(G, 1)$ is the projective cover of the trivial module, as we saw in the lemma.

2.2 Induction and Restriction

The definition of a Scott module is as a submodule of an induced module. This suggests that the Scott module might behave well with respect to induction and restriction.

Proposition 2.7 Suppose that P is a p-subgroup of G contained in a subgroup H. Then $S(G,P)$ is a summand of the induced module $S(H,P)\uparrow^G$ and is the only summand isomorphic to a Scott module.

Proof: Since $S(H,P)$ is a summand of $(1_P)\uparrow^H$, we have that $S(H,P)\uparrow^G$ is a summand of $(1_P)\uparrow^H\uparrow^G= (1_P)\uparrow^G$. Also, $S(H,P)\uparrow^G$ contains the trivial module, so contains $S(G,P)$ as a summand by the uniqueness of the Scott module (as the only summand of $(1_P)\uparrow^G$ containing 1_G). The second statement comes from the fact that

$$\mathrm{Hom}_{KG}(1_G,(1_H)\uparrow^G)\cong K,$$

and so there is only one summand of $S(H,P)\uparrow^G$ with 1_G as a submodule. $\square$

In [9], Burry gives the converse of Proposition 2.7: namely if you induce some module and find a Scott module in what you get, then you had a Scott module to start with. However, you have to assume that your first module could have been a Scott module; that is, that your first module is a trivial source module.

Proposition 2.8 (Burry, [9]) Suppose that P is a p-subgroup of G, contained in a subgroup H. If V is a trivial source H-module and $V\uparrow^G$ contains $S(G,P)$ as a summand, then $V=S(H,P)$.

Proof: Suppose that V is a summand of $(1_X)\uparrow^H$ for some $X\leqslant H$. Then

$$S(G,P)|V\uparrow^G|(1_X)\uparrow^G,$$

and since $S(G,P)$ is the only component of $(1_X)\uparrow^G$ containing 1_G, by the Scott–Alperin Theorem P must be a Sylow p-subgroup of X. Since P is a Sylow p-subgroup of X, $(1_X)\uparrow^H$ contains $S(H,P)$, and since $S(H,P)\uparrow^G$ contains $S(G,P)$ by Proposition 2.7, this means that both V and $S(H,P)$ induce to a module containing $S(G,P)$. Since we have already said that there is only one copy of $S(G,P)$ in $(1_X)\uparrow^G$ in total, (and since V is indecomposable,) $V=S(H,P)$, as required. $\square$

Now we move on to restriction, where we find a complement to Proposition 2.7. The proof of this and a generalization, in this treatment anyway, depend on a result of Benson and Carlson.

Proposition 2.9 (Benson–Carlson, [5]) Suppose that M is a relatively P-projective KG-module. Then M contains $S(G,P)$ as a summand if and only if there is a non-zero KG-homomorphism from M to K whose restriction to P splits.

Now we can continue.

Proposition 2.10 (Burry, [9]) With the assumptions of Proposition 2.7, $S(H,P)$ is a summand of $S(G,P)$ restricted to H.

Proof: Notice that $1_P|S(G,P)\downarrow_P$, and so $1_P|V\downarrow_P$ for some indecomposable summand V of $S(G,P)\downarrow_H$. Clearly then V is relatively P-projective. By Proposition 2.9, $S(H,P)$ is a summand of V. However, V is indecomposable, and so $S(H,P)|S(G,P)\downarrow_H$, as required. $\square$

If $H\leqslant G$, then we can find Scott modules for H from Scott modules for G; more precisely, if Q is a p-subgroup of G, then $S(G,Q)\downarrow_H$ contains the Scott module $S(H,P)$, where P is a conjugate of Q that intersects maximally with H. This is a generalization of Proposition 2.10. We provide the result now.

Proposition 2.11 (Kawai, [15]) Let G be a group, and H a subgroup of G. Let S denote the Scott module of G with vertex Q. Let P denote a maximal element of the set $\{Q^g \cap H : g \in G\}$. (Note that the set $\{Q^g : g \in G\}$ is the set of all vertices of $S(G,Q)$.) Then $S\downarrow_H$ contains $S(H,P)$ as an indecomposable summand.

Proof: Notice that $1_P | S(G,Q)\downarrow_P = (S(G,Q)\downarrow_H)\downarrow_P$, since $1_Q|S(G,Q)\downarrow_Q$, so let V be a component of $S(G,Q)\downarrow_H$. Then we can use Proposition 2.9, in the same way as the previous result, to get $S(H,P) = V$, and hence $S(H,P)|S(G,Q)\downarrow_H$. □

Let us now consider the case where the restriction of a Scott module is still indecomposable. Notice that if $L \leqslant G$, and Q is a p-subgroup of L, then if $S(G,Q)\downarrow_L$ is indecomposable, it is equal to $S(L,Q)$. This is a trivial observation from Proposition 2.10.

We begin with a preparatory lemma, which is interesting in its own right.

Lemma 2.12 (Héthelyi, Szöke, Lux, [13]) Suppose that Q is a p-subgroup of another p-subgroup P of G. If $|G:P|$ is not coprime to p, then the space $\mathrm{Hom}_{KG}(S(G,Q), S(G,P))$ is at least 2-dimensional.

Proof: We will use Proposition 2.6. Notice firstly that $\rho : S(G,Q) \to S(G,P)$ sending $S(G,Q)$ to the trivial submodule of $S(G,P)$ is a (non-trivial) homomorphism. Consider the surjections $\pi : S(G,Q) \to 1_G$ and $\psi : S(G,P) \to 1_G$. Now $S(G,Q)$ is relatively Q-projective by Proposition 2.6, and is therefore relatively P-projective. Thus π factors through the relative P-projective cover of 1_G; i.e., $S(G,P)$. Thus we can factor π as

$$\pi = \alpha\psi,$$

where $\alpha : S(G,Q) \to S(G,P)$. Now p divides the index of P in G, and so the trivial submodule of $S(G,P)$ lies inside $\ker\psi$. In particular, this implies that $\rho\psi$ is the zero homomorphism. Therefore α and ρ are linearly independent in $\mathrm{Hom}_{KG}(S(G,Q), S(G,P))$, as so this space is at least 2-dimensional, as required. □

We can now prove the final result in this section.

Theorem 2.13 (Héthelyi, Szöke, Lux, [13]) Suppose that Q is a p-subgroup of G contained in L, and suppose that $S(G,Q)\downarrow_L$ is indecomposable. Then $|G:L|$ is coprime to p.

Proof: Suppose that $S(G,Q)\downarrow_L$ is indecomposable, and suppose for a contradiction that $|G:L|$ is not coprime to p. Then $S(G,Q)\downarrow_L = S(L,Q)$. Hence the space

$$\mathrm{Hom}_{KL}(S(G,Q)\downarrow_L, 1_L) \cong K$$

is 1-dimensional. But by Frobenius Reciprocity, this space is isomorphic with the space

$$\mathrm{Hom}_{KG}(S(G,Q), (1_L)\uparrow^G),$$

which contains the 2-dimensional space $\mathrm{Hom}_{KG}(S(G,Q), (S(G,L))$ by Lemma 2.12, a contradiction. □

2.3 Multiplicities of Scott Modules

In [12], Green proves a multiplicity formula for a Scott module in any KG-module N.

Theorem 2.14 (Green, [12]) Let $H \leqslant G$, and let N be a KG-module. If $a \in I_H(N)$ and $f \in I_H(N^*)$, let $\Psi : I_{H,G}(N) \times I_{H,G}(N^*) \to K$ be the bilinear form given by

$$\Psi\left(aT_{H,G}, fT_{H,G}\right) = (aT_{H,G})f = (a)(fT_{H,G}).$$

Then the multiplicity of $S(G,H)$ in N is equal to the rank of Ψ.

The rank here is equal to $\dim_K I_{H,G}(N)/X$, where X is the annihilator of the bilinear form; thus X is given by

$$X = \{x \in I_{H,G}(N) : (xT_{H,G})f = 0 \text{ for all } f \in I_H(N^*)\}.$$

We have the following corollary to this theorem.

Corollary 2.15 (Green, [12]) The multiplicity of the Scott module $S(G,H)$ in N is equal to the dimension of the space

$$I_{H,G}(N)/\{x \in I_{H,G}(N) : (xT_{H,G})f = 0 \text{ for all } f \in I_H(N^*)\}.$$

The aim of this section is to prove Theorem 2.14. This follows from another theorem given in [12], which Green calls a 'multiplicity formula'. Before we continue with this multiplicity formula, we introduce some notation. Let M and N be KG-modules. Then the indecomposable summands of M and N constitute a set $\{X_1, \ldots, X_k\}$ of finitely many indecomposable modules. Write

$$M = M_1 \oplus \cdots \oplus M_m, \qquad N = N_1 \oplus \cdots \oplus N_n;$$

we fix the notation so that the first ℓ of the modules X_i are summands of M. Write $c_i(N)$ for the multiplicity of X_i as an indecomposable summand of N, so that (as an example of the notation)

$$\dim_K N = \sum_{i=1}^{k} c_i(N) \dim_K X_i.$$

Notice that there are $c_i(N)$ numbers j from 1 to n such that the modules N_j are all isomorphic; write $n(i)$ for this set of numbers. Thus we have $X_i \cong N_j$ for $j \in n(i)$. We define $m(i)$ similarly. Notice that any element σ of $\mathrm{Hom}_{KG}(M,N)$ can be represented as a matrix, $(\sigma_{i,j})$, whose i,j-th entry lies in $\mathrm{Hom}_{KG}(M_i, N_j)$. [Now we have to depart from Green's notation in [12] significantly, since in that paper he multiplies functions on the left, whereas here we multiply on the right.]

Finally, writing A_i for top($\mathrm{End}_{KG}(M)$) and q_i for the natural quotient map, we can define a map $\mu_i : \mathrm{End}_{KG}(M) \to A_i$ by

$$\mu_i(\sigma) = \sigma_{a,b} q_i,$$

where a and b lie in $m(i)$, so that $M_a \cong M_b \cong X_i$. We will prove the following.

Theorem 2.16 (Green, [12]) Define $\Phi : \mathrm{Hom}_{KG}(M,N) \times \mathrm{Hom}_{KG}(N,M) \to A_i$ by $\Phi(f,g) = (fg)\mu_i$. Then the rank of Φ is equal to $c_i(N) \dim_K A_i$.

Proof: The rank of Φ is the dimension of the quotient $\mathrm{Hom}_{KG}(N,M)/X$, where X is the space

$$X = \{g \in \mathrm{Hom}_{KG}(N,M) : \Phi(f,g) = 0 \text{ for all } f \in \mathrm{Hom}_{KG}(M,N)\}.$$

We have a decomposition of M and N, and so we can view f and g as matrices, with the i,j-th entry of $(f)_{i,j}$ being $f_{i,j} : M_i \to N_j$. Then, since $(fg)\mu_i = (fg)_{a,b} q_i$, we want to understand $(fg)_{a,b}$, (where $a,b \in m(i)$). Then

$$(fg)_{a,b} = \sum_{j=1}^{n} f_{a,j} g_{j,b}.$$

If $j \notin n(i)$, then $N_j \ncong X_i$; since $a, b \in m(i)$, $f_{a,j} \in \mathrm{Hom}_{KG}(M_a, N_j) \cong \mathrm{Hom}_{KG}(X_i, N_j)$. Similarly, $g_{j,b} \in \mathrm{Hom}_{KG}(N_j, X_i)$. We have $(fg)_{a,b} \in \mathrm{End}_{KG}(X_i)$. Suppose that $(fg)_{a,b}$ is an automorphism. Then $f_{a,j}$ is injective, and $g_{j,b}$ is surjective, and we have

$$N_j = \mathrm{im}\, f_{a,j} \oplus \ker g_{j,b} \cong X_i \oplus \ker g_{j,b}.$$

But we know that N_j is indecomposable, and that $N_j \ncong X_i$. This contradiction yields the fact that $(fg)_{a,b}$ is *not* an isomorphism. Then by Proposition 2.2,

$$f_{a,j} g_{j,b} \in J(\mathrm{End}_{KG}(X_i)) = \ker q_i,$$

giving the equation

$$\Phi(f,g) = (fg)\mu_i = (fg)_{a,b} q_i = \left(\sum_{j \in n(i)} f_{a,j} g_{j,b} \right) q_i.$$

Now $f_{a,j}$ and $g_{j,b}$ are both elements of $\mathrm{Hom}_{KG}(X_i, X_i) \cong \mathrm{End}_{KG}(X_i)$, so we can apply q_i to these individually, to get

$$\Phi(f,g) = \sum_{j \in n(i)} (f_{a,j} q_i)(g_{j,b} q_i).$$

If $g_{j,b} q_i = 0$ for all $j \in n(i)$, then clearly $g \in X$, and also the converse holds. Thus $g_{j,b} \in X$ if and only if $g_{j,b} \in J(\mathrm{End}\, X_i)$ for all $j \in n(i)$. Since $\mathrm{End}\, X_i / J(\mathrm{End}\, X_i) \cong A_i$, we have

$$\dim_K \mathrm{Hom}_{KG}(N, M)/X = \dim_K \bigoplus_{j \in n(i)} A_i = c_i(N) \dim_K A_i,$$

as required. □

[Note that this theorem and proof work for any KG-linear function μ that maps onto a field, not just $\mathrm{top}(\mathrm{End}_{KG}(M))$.]

Now we have dispatched the multiplicity formula, we can start our attack on the main theorem. We will consider one of the modules to be the induced module $(1_H)\uparrow^G$. Then we know that this decomposes as

$$(1_H)\uparrow^G \cong S(G,H) \oplus \bigoplus_{i=2}^{m} M_i;$$

here we are trying to keep to the notation of the previous theorem by setting $(1_H)\uparrow^G = M$ and $M_1 = S(G,H)$. We need to find an analogue of μ_1 now, and since we are interested in the Scott module, we ought to make μ_1 map into K. Suppose that $\phi \in \mathrm{End}_{KG}(M)$. If we write $x_t \phi = \sum_u a_{t,u} x_u$, then we can set

$$\phi\mu_1 = \sum_u a_{1,u}.$$

This is easily KG-linear; for example, if $x_1 \psi = \sum_u b_{1,u} x_u$, then

$$(\phi + \psi)\mu_1 = \sum_u (a_{1,u} + b_{1,u}) = \sum_u a_{1,u} + \sum_u b_{1,u} = \phi\mu_1 + \psi\mu_1,$$

and the rest is as simple. Clearly μ_1 maps $\mathrm{End}_{KG}(M)$ into K. We can apply Theorem 2.16 to get

$$\Phi(f,g) = (fg)\mu_1,$$

and that the multiplicity of M_1, which is the Scott module $S(G,H)$, in N is given by the rank of Φ.

We now prove a lemma regarding the numbers $a_{t,u}$,which will be needed during the proof of this theorem.

Lemma 2.17 Let $\phi \in \mathrm{End}_{KG}(M)$, and $x_t\phi = \sum_u a_{t,u}x_u$. Then

$$\sum_t a_{t,1} = \sum_u a_{1,u}.$$

Proof: A basis for $\mathrm{End}_{KG}(M)$ can be found as follows (see for example [10]): let Δ be a non-diagonal orbital on $G \times G$. Then Δ consists of elements of the form (x, y), where x and y are elements of the permutation basis for M. To each orbital Δ there is associated an endomorphism ϕ_Δ such that if

$$x_t\phi = \sum_u a_{t,u}x_u,$$

then $a_{t,u} = 1$ if $(t, u) \in \Delta$, and $a_{t,u} = 0$ otherwise. Suppose that (t, v) and (w, u) are elements of Δ. (We can assume this since $(t, v) \in \Delta$ without loss of generality, and since G is transitive, we have $g : v \mapsto u$, that sends t to w say.) Thus

$$\begin{aligned}\sum_u a_{t,u} &= |\{\delta \in \Delta : \delta = (t, x) \text{ for some } x \in \Omega\}| \\ &= |\{(t, v)g \,:\, g \in G_t\}|\,.\end{aligned}$$

The other sum is given by

$$\begin{aligned}\sum_t a_{t,u} &= |\{\delta \in \Delta : \delta = (y, u) \text{ for some } y \in \Omega\}| \\ &= |\{(w, u)g \,:\, g \in G_u\}|\,.\end{aligned}$$

Specializing to the case given in the proposition, and assuming that $(1, v), (w, 1) \in \Delta$ (without loss of generality), we have

$$\sum_u a_{1,u} = |\mathrm{Orb}_{G_1}(v)|, \qquad \sum_t a_{t,1} = |\mathrm{Orb}_{G_1}(w)|.$$

But $(1, v)$ and $(w, 1)$ are in the same orbital, which means that there exists $h \in G$ such that $1h = w$ and $vh = 1$. Thus $G_{1,v}^h = G_{1,w}$, and so in particular $|\mathrm{Orb}_{G_1}(w)| = |\mathrm{Orb}_{G_1}(v)|$, and we are done. □

We can now continue with the proof. In order to prove Theorem 2.14 we need another form Ψ, which is related to, but slightly different to, Φ. To define it we need to define the relative trace map over other modules.

Recall that $\mathrm{Hom}_K(V_1, V_2)$ becomes a right KG-module under the map $v\phi^g = (vg^{-1})\phi g$, for $v \in V_1$ and $\phi \in \mathrm{Hom}_K(V_1, V_2)$. In this case, we can take fixed point subspaces and relative trace maps; $I_G(\mathrm{Hom}_K(A, B)) = \mathrm{Hom}_{KG}(A, B)$, and $I_H(\mathrm{Hom}_K(A, B)) = \mathrm{Hom}_{KH}(A, B)$. Again, denote by $T_{H,G}$ the relative trace map, and let $I_{H,G}(A, B)$ denote the image. Let us create a new form

$$\Theta : I_{H,G}(K, N) \times I_{H,G}(N, K) \to K$$

from our current form $\Phi : \mathrm{Hom}_{KG}(M, N) \times \mathrm{Hom}_{KG}(N, M) \to K$. To do this, we need a function from $\mathrm{Hom}_{KG}(M, N)$ to $I_{H,G}(K, N)$ and from $\mathrm{Hom}_{KG}(N, M)$ to $I_{H,G}(N, K)$. We already have one from $\mathrm{Hom}_{KG}(K, N)$ to $I_{H,G}(K, N)$ (and one from $\mathrm{Hom}_{KG}(N, K)$ to $I_{H,G}(N, K)$) given by $T_{H,G}$, and these are surjective. Thus we want functions

$$\mathrm{Hom}_{KG}(M, N) \to \mathrm{Hom}_{KH}(K, N), \qquad \mathrm{Hom}_{KG}(N, M) \to \mathrm{Hom}_{KH}(N, K).$$

To do this, recall that M is a transitive permutation module, and therefore has a permutation basis x_t labelled by a transversal τ to H in G; furthermore, H acts trivially on the basis element x_1. Let

$$\beta : M \to K, \qquad \left(\sum a_t x_t\right)\beta = a_1,$$

the co-efficient of x_1. Then this in invariant under action of H, and so is a KH-module homomorphism. This means that we have one of our desired maps, namely

$$\bar{\beta} : \mathrm{Hom}_{KG}(N, M) \to \mathrm{Hom}_{KH}(N, K), \qquad \bar{\beta} : g \mapsto g\beta.$$

We now look for another map: let $\alpha : K \to M$ be given by $k\alpha = kx_1$; that is, by letting k be the co-efficient of the basis element x_1. Then this is invariant under action of H, so again is a KH-module homomorphism. This gives us the second of our maps, namely

$$\bar{\alpha} : \mathrm{Hom}_{KG}(M, N) \to \mathrm{Hom}_{KH}(K, N), \qquad \bar{\alpha} : f \mapsto \alpha f.$$

These are actually K-isomorphisms: this is actually Frobenius Reciprocity (or at least the Nakayama Relations).

The maps defined by then hitting $\bar{\alpha}$ and $\bar{\beta}$ by relative trace maps are surjective, so we can define Θ in terms of the bilinear form given by Φ. So, if we call these composite surjections α' and β', we have

$$\alpha' : \mathrm{Hom}_{KG}(M, N) \to I_{H,G}(K, N), \qquad \beta' : \mathrm{Hom}_{KG}(N, M) \to I_{H,G}(N, K).$$

Define $\Theta(f\alpha', g\beta') = \Phi(f, g)$, where $f \in \mathrm{Hom}_{KG}(M, N)$ and $g \in \mathrm{Hom}_{KG}(N, M)$. We must show that

$$\Phi(f, g) = 0 \iff f\alpha' = 0 \text{ or } g\beta' = 0.$$

To show this, introduce two more functions: $\gamma : K \to M$, which is defined by $k \mapsto k\sum x_t$ (this is a KG-module homomorphism), and $\delta : M \to K$, which is defined by $\sum a_t x_t \mapsto \sum a_t$. Then

$$f\alpha' = \alpha f T_{H,G} = (\alpha T_{H,G})f = (x_1 T_{H,G})f = \sum x_t f = \gamma f,$$

and

$$g\beta' = g\beta T_{H,G} = g(\beta T_{H,G}) = g\left(\sum a_t\right) = g\delta.$$

Now consider $(fg)\mu$. We would like to express it as a sequence of functions which include either γf or $g\delta$: if we can do this, then if either γf or $g\delta$ is zero, we must have $\Phi(f, g) = 0$, so that Θ is well-defined. Firstly, notice that if $x_t(fg) = \sum_u a_{t,u}x_u$, then

$$(fg)\mu_1 = \sum_u a_{1,u}.$$

Now consider $1_K(\alpha fg\delta)$. Then

$$\begin{aligned} 1_K(\alpha fg\delta) &= (1_k\alpha)fg\delta \\ &= x_1(fg)\delta \\ &= \sum_u (a_{1,u}x_u)\delta \\ &= \sum_u a_{1,u}. \end{aligned}$$

Thus $(fg)\mu_1 = \alpha fg\delta$, and since $g\beta' = g\delta$, whenever $g\beta' = 0$, $\Phi(f,g) = 0$. We need another expression for $(fg)\mu_1$, so let us try $1_K(\gamma fg\beta)$. Then

$$\begin{aligned} 1_K(\gamma fg\beta) &= (1_K \sum_t x_t)fg\beta \\ &= \left(\sum_t \sum_u a_{t,u}x_u\right)\beta \\ &= \sum_t a_{t,1}. \end{aligned}$$

Now we need Lemma 2.17, which tells us that these two quantities are, in fact, equal, and so we are good; now if $\gamma f = 0$, then $\Phi(f,g) = 0$, and we can now state that Θ is well-defined. The ranks are the same, and so we have that the multiplicity of the Scott module $S(G,H)$ in a module N is equal to the rank of the bilinear form

$$\Theta : I_{H,G}(K,N) \times I_{H,G}(N,K) \to K,$$

defined by the rule that if $a \in \mathrm{Hom}_{KH}(K,N)$ and $b \in \mathrm{Hom}_{KH}(N,K)$, then

$$\Theta(aT_{H,G}, bT_{H,G}) = 1_K(bT_{H,G})a = 1_K b(aT_{H,G}).$$

This agrees with earlier work since if we write $a = \alpha f$ and $b = g\beta$, we get

$$\Phi(f,g) = 1_K(bT_{H,G})a = 1_K b(aT_{H,G}),$$

by the formulae above.

Now call $\mathrm{Hom}_K(N,K)$ the dual space N^*, and identify $\mathrm{Hom}_K(K,N)$ with N via the isomorphism ϕ, say. Then we have

$$\Psi : I_{H,G}(N) \times I_{H,G}(N^*) \to K,$$

defined by $\Psi(c,d) = \Theta(c\phi^{-1}, d)$. This finally completes the proof of Theorem 2.14.

2.4 Scott Modules and Lower Defect Groups

We have already alluded to a connexion between lower defect groups and Scott modules before. In this section we will see this connexion explained. Again, we will follow [4] in thinking of a block as a $K(G \times G)$-module.

We define $M_B(P)$ to be the multiplicity of the Scott module $S(G,P)$ as a component of B_Δ. The fundamental result of this section is the following:

Theorem 2.18 (Burry, [9]) Let G be a finite group, P a p-subgroup of G and B a block of KG. Then $M_B(P) = m_B(P)$.

Burry's original proof relates it to Broué's version of the IBO Characterization, but we will follow [12] and show that $M_B(P) = \dim(I_{P,G}(B_\Delta)/I'_{P,G}(B_\Delta))$ (see Theorem 1.6). In fact, since we have already shown that $M_B(P)$ is equal to the quantity

$$\dim_K(I_{P,G}(B_\Delta)/X),$$

where X is the subspace $\{x \in I_{P,G}(B_\Delta) : xf = 0 \text{ for all } f \in I_P(B_\Delta^*)\}$ (in Corollary 2.15), we merely have to show that this quantity is equal to $\dim(I_{P,G}(B_\Delta)/I'_{P,G}(B_\Delta))$. We will demonstrate this now.

Proposition 2.19 Let B be a block of the finite group G, and let P be a p-subgroup of G. Then

$$\{x \in I_{P,G}(B_\Delta) : xf = 0 \text{ for all } f \in I_P(B_\Delta^*)\} = \sum_{Q<P} I_{Q,G}(B_\Delta).$$

Proof: Let $Y = \{y \in I_{P,G}(KG_\Delta) : yf = 0 \text{ for all } f \in I_P(KG_\Delta^*)\}$. Now $I_{P,G}(KG_\Delta)$ has basis all class sums whose conjugacy classes have defect group lying (up to conjugacy) inside P, and $I_P(KG_\Delta^*)$ has basis all functions $f_{C'}$, where C' is a P-conjugacy class of G and if $x \in G$, $xf_{C'} = 1$ or 0 depending on whether $x \in C'$ or not. Notice that this implies that $\bar{C}f_{C'} = 0$ if $C' \nsubseteq C$, and that if $C' \subseteq C$, then

$$\bar{C}f_{\bar{C}'} = |C'| \cdot 1_K,$$

where 1_K denotes the identity of K. Since P is a p-group, and K has characteristic p, $|C'| \cdot 1_K$ is either equal to 0_K or 1_K.

Suppose that C is a conjugacy class of G with defect group $Q < P$. Then if $x \in C$, $\mathrm{C}_G(x)$ cannot contain P. Notice that $\bar{C}$ is a basis element for $I'_{P,G}(KG_\Delta)$, and that all basis elements are of this form. Consider $\bar{C}f_{C'}$, where C' is a P-conjugacy class of G. If $C' \nsubseteq C$, or $C' \subseteq C$ and $|C'| \neq 1$, then $\bar{C}f_{C'} = 0$. The remaining case is where $|C'| = 1$; i.e., $C' = \{x\}$, $x \in C$, and hence $P \leqslant \mathrm{C}_G(x)$. But this leads to a contradiction, since $\mathrm{C}_G(x)$ cannot contain P. Hence $\bar{C}f_{C'} = 0$ for all basis elements of $I_P(KG_\Delta^*)$, and so $I'_{P,G}(KG_\Delta) \subseteq Y$.

Now consider a basis element of $I_{P,G}(KG_\Delta)$ that doesn't lie within $I'_{P,G}(KG_\Delta)$; then this is a class sum whose conjugacy class C has defect group equal to P. Then in this case if $x \in C$, $P \leqslant \mathrm{C}_G(x)$, and so the P-conjugacy class of G containing x is simply $\{x\}$. Thus

$$\bar{C}f_{\{x\}} \neq 0_K;$$

that is, $\bar{C} \notin Y$. We have therefore shown that $\dim_K I'_{P,G}(KG_\Delta) = \dim_K Y$, and hence they must be equal.

Now to see the final conclusion, notice that by Lemma 1.5, we have $X = Ye$, where e is the block idempotent of B, and that

$$I'_{P,G}(KG_\Delta) = I'_{P,G}(B_\Delta).$$

Thus $X = I'_{P,G}(B_\Delta)$, as required. □

Notice that this now demonstrates Burry's Theorem, and we have found our final interpretation of lower defect groups. From this result we can prove most of the results of Section 1.1. We proceed as in [9], starting with Theorem 1.1.

Theorem 2.20 (Brauer, Burry) Let G be a finite group and B a block of G. Let Q be a p-subgroup of G. Let N_Q denote the number of conjugacy classes of G with defect group Q. Then

$$k(B) = \sum_P M_B(P),$$

where the sum runs over the conjugacy classes of p-subgroups of G, and

$$N_Q = \sum_b M_b(Q),$$

where the sum runs over the blocks b of KG.

Proof: Firstly notice that $k(B)$ is the dimension of the centre of B, so we have to prove that the sum above is equal to the dimension of the centre. But the centre of B is simply the largest trivial module lying inside B. Now each Scott module lying inside B has a trivial submodule, and they are the only such ones. Therefore the sum of all of the numbers of Scott modules is equal to the dimension of $\mathrm{Z}(B)$.

For the second part, notice that since KG is a direct sum of blocks, the quantity $\sum_b M_b(Q)$ is simply the number of times the Scott module $S(G, Q)$ appears in KG. Alternatively, if C is a conjugacy class of G, then KC is a transitive permutation module, with the Sylow p-subgroup of a point stabilizer equal to the defect group D_C of C. Thus KC contains the Scott module $S(G, D_C)$ once, and so the number of Scott modules $S(G, Q)$ in KG is equal, via the decomposition

$$KG = \sum_C KC$$

(where the sum runs over the conjugacy classes of G), to the number of conjugacy classes of defect group Q; i.e., N_Q. Thus

$$N_Q = \sum_b M_b(Q),$$

as we needed. □

Next we come to Proposition 1.3.

Proposition 2.21 (Brauer, Burry) Let G be a finite group and B be a block, with defect group D. If P is a p-subgroup of G which is not G-conjugate to a subgroup of D, then $M_B(P) = 0$.

Proof: If P is not G-conjugate to a subgroup of D, it can not possibly be a vertex of a component of B, and therefore can not be a vertex of a Scott module. □

Burry proves in [9] that the lower defect groups multiplicities can be determined from p-local subgroups, as in Theorem 1.2: we have already proven this for $m_B(P)$, and with Burry's Theorem, this is also demonstrated. We will not prove this again.

2.5 Scott Modules for p-Groups

In this section we examine in more detail the case where G is a p-group. In this case, several interesting results are easily obtainable, culminating in Alperin's result on transitive permutation modules over p-groups. In the case of a p-group, the induced module $(1_H)\uparrow^G$ is indecomposable. We can do slightly better.

Proposition 2.22 Let G be a finite group, and H a subnormal subgroup such that $|G : H|$ is a power of p. Then $(1_H)\uparrow^G$ is indecomposable.

Proof (Recall that we assume K is a splitting field.): We can refine a subnormal series linking H in G so that each factor is isomorphic with the cyclic group C_p. By Green's Indecomposability Criterion, if H/L is of order p, then an indecomposable KL-module induces to an indecomposable KH-module. We can apply Green's Indecomposability Criterion repeatedly to a refined subnormal chain of the sort above, along with the observation that $((1_H)\uparrow^L)\uparrow^G = (1_H)\uparrow^G$ for any $H \leqslant L \leqslant G$, to get the result. □

Notice that since every subgroup of a p-group is subnormal, we get that $S(G, H) = (1_H)\uparrow^G$ for any $H \leqslant G$, where G is a p-group. Either this proposition or the fact that 1_G appears only once in the socle of $(1_H)\uparrow^G$ can be used to prove the next elementary lemma.

Lemma 2.23 Let H be a subgroup of index 2 in G. Then $S(G,H)$ is uniserial of dimension 2, with a composition series consisting of two copies of 1_G.

This is a special case of the following proposition.

Proposition 2.24 Let G be a finite group, and H a (not necessarily normal) subgroup of G of index p. Then $S(G,H)$ is uniserial.

Proof: Suppose that $|G:H|=p$, and let g be an element of $G \setminus H$ of order a power of p (this is possible since the Sylow p-subgroups of G cannot be contained within H). Then the cosets of H in G can be labelled by Hg^i for $0 \leqslant i \leqslant p-1$, and g permutes these as a cycle. Let us consider $\langle g \rangle$ acting on this permutation module $S(G,H)$; i.e., consider

$$\big((1_H)\uparrow^G\big)\downarrow_{\langle g \rangle}.$$

Then g simply acts as a cycle on p vertices, and this is clearly isomorphic with the module $S(C_p,1)$, which is known to be uniserial (by, for example, consider the Brauer tree). Since even this restriction is uniserial, $S(G,H)$ must also be uniserial. $\square$

We can extend this to any subgroup H of index p^n, where the cosets of H in G are given by Hg^i for some g of prime power order.

Returning to our example of D_8, Lemma 2.23 dealt with the Scott modules from the index 2 subgroups, so we are left with the subgroups of order 2. There are, basically, i.e., up to conjugacy in $\mathrm{Aut}(D_8)$, two types of these: the centre and a non-central subgroup generated by an involution b, say, where D_8 is generated by a of order 4 and b of order 2 such that a inverts b under conjugation. [Just as $S(G,P)$ and $S(G,Q)$ are isomorphic if P and Q are conjugate, they are also basically isomorphic if P and Q are conjugate under an outer automorphism: they aren't actually isomorphic under the same G-action, but the modules are indistinguishable from one another if we are only given the abstract characterization of the group. In the case where P and Q are conjugate by an outer automorphism, $S(G,P)$ and $S(G,Q)$ are 'mirror images' of one another,k in the same that while not precisely the same, not extra information is really gained from studying both of them: all invariants like socle and radical series, dimension, and so on, are the same.] The first case, $H=\langle a^2 \rangle$ will be dealt with in the next proposition.

Proposition 2.25 Let N be a normal subgroup of the finite group G. Then $S(G,N)$ is isomorphic to $\mathcal{P}(1_{G/N})$, viewed as a KG-module.

Proof: This uses the Green Correspondence: since N is normal, and $S(G,N)$ has vertex N, the Green Correspondence f from G to $\mathrm{N}_G(N)=G$ is the identity. Now, (iii) of the Scott–Alperin Theorem states that $(S(G,N))\Gamma$ is, when viewed as an $\mathrm{N}_G(N)/N$-module (remember that N acts trivially on $S(G,N)$), isomorphic with the projective cover of $1_{\mathrm{N}_G(N)/N}$. This gives us the result. $\square$

This deals with the case of $S(D_8,\langle a^2\rangle)$, but there is still one more case to deal with. To see what is going on, let us think about the Scott modules $S(V_4,1)$ and $S(C_4,1)$. Now these are simply inflations of the projective covers of the trivial modules, and so $S(V_4,1)$ has a semisimple 2-dimensional heart, and $S(C_4,1)$ is uniserial. We want to relate the subgroup structure of V_4 and C_4 to the module structure of the Scott modules.

Let us take $G=V_4$ first. Let us suppose that $G=\langle x,y\rangle$. Then $S(G,1)$ is a transitive permutation module of degree 4, with permutation basis $\{a_i\}$ labelled by the elements of G. We can find some submodules of this naturally corresponding to the subgroups of G. For example, corresponding with $\langle x\rangle$, we have the submodule generated by $a_1 + a_x$ and $a_y + a_{xy}$. In fact, we can find the submodules

$$A = \langle a_1 + a_x + a_y + a_{xy}\rangle,$$

$$B_{\langle x\rangle} = \langle a_1 + a_x, a_y + a_{xy}\rangle,\ \ B_{\langle y\rangle} = \langle a_1 + a_y, a_x + a_{xy}\rangle,\ \ B_{\langle xy\rangle} = \langle a_1 + a_{xy}, a_x + a_y\rangle,$$

$$C = \langle a_1 + a_x, a_1 + a_y, a_1 + a_{xy}\rangle.$$

In this case, A is 1-dimensional, the B_i are all 2-dimensional, and C is 3-dimensional. We can easily determine that $S(G, 1)$ has a semisimple heart from these submodules.

Now let us take the (right) cosets of $H = \langle b\rangle$ in D_8. These are

$$\{1, b\}, \qquad \{a, a^3b\}, \qquad \{a^2, a^2b\}, \qquad \{a^3, ab\}.$$

Since each of these contains a power of a, we will label a permutation basis by $\{x_{a^i}\}$. Consider the action of a and b on the permutation basis $\{x_{a^i}\}$ for the induced module $S(G, H)$: then a cycles the x_{a^i} and b fixes x_1 and x_{a^2} and swaps the other two. It is easy to see then that we have the following submodules (generated as K-modules):

$$A = \langle x_1 + x_a + x_{a^2} + x_{a^3}\rangle,$$

$$B = \langle x_1 + x_{a^2}, x_a + x_{a^3}\rangle,$$

$$C = \langle x_1 + x_a, x_1 + x_{a^2}, x_1 + x_{a^3}\rangle.$$

Then A is 1-dimensional, B is 2-dimensional, and C is 3-dimensional; these are the only three submodules of $S(G, H)$, and so $S(G, H)$ is uniserial. Notice importantly that $L = \langle a^2, b\rangle$ is a subgroup of D_8 containing H, and that if for any module M we identify M and the submodule of any module induced from M isomorphic with M, we have $B = S(L, H)$ and $S(G, H)/B \cong S(G, L)$.

We can explain this behaviour with a result of Alperin's, in [1]. In that paper, Alperin determines the structure of all transitive permutation modules for a p-group. Since we know that, for a p-group, transitive permutation modules are the same thing as Scott modules, Alperin determines the structure of Scott modules for all p-groups, over a field of characteristic p. This generalizes Jennings' Theorem, which deals with the case of $S(G, 1) = KG$.

We let $\Gamma_i = \Gamma_i(G)$ be the dimension subgroups; that is,

$$\Gamma_1(G) = G, \qquad \Gamma_{i+1}(G) = \langle [\Gamma_i(G), G], \Gamma^p_{\lceil i+1/p\rceil}(G)\rangle.$$

[Notice that Γ_i/Γ_{i+1} is elementary abelian, and that $\Gamma_2(G) = \Phi(G)$.] Suppose that M is a transitive permutation module, with point stabilizer H. Then $M \cong (1_H)\uparrow^G = S(G, H)$. Let $J^i(G)$ denote the ith radical layer, and let

$$\Delta_i(G) = (H \cap \Gamma_i(G))\Gamma_{i+1}(G).$$

In Jennings' Theorem, we choose $x_{i,j}$ from $\Gamma_i \setminus \Gamma_{i+1}$ such that their images $\Gamma_{i+1}x_{i,j}$ formed a basis for the elementary abelian group Γ_i/Γ_{i+1}. If we let $X_{i,j} = x_{i,j} - 1$ in the group algebra, then the products

$$\prod_{i,j} X_{i,j}^{\alpha_{i,j}}, \qquad 0 \leqslant \alpha_{i,j} \leqslant p - 1$$

have particular weights, given by

$$w\left(\prod_{i,j} X_{i,j}^{\alpha_{i,j}}\right) = \sum_i i\alpha_{i,j}.$$

Then the products of weight k lie in $J^k(G)$, and form a basis of J^k/J^{k+1}.

In Alperin's Theorem, the $x_{i,j}$ are chosen more specifically. Indeed, (working with Alperin's notation,) we choose elements $y_{i,j}$ such that their images in $\Gamma_i(G)/\Delta_i(G)$ form a basis for that (elementary abelian) group, and choose $z_{i,j}$ from $H \cap \Delta_i(G)$ such that they form a basis for the (elementary abelian) group $\Delta_i(G)/\Gamma_{i+1}(G)$. Then certainly the union of the $y_{i,j}$ and the $z_{i,j}$ is a basis for $\Gamma_i(G)/\Gamma_{i+1}(G)$. We have, in essence, split the $x_{i,j}$ up into two collections: those that are present in $S(G,H)$, and those that are not. If we let $Y_{i,j}$ and $Z_{i,j}$ be defined as we did $X_{i,j}$, then Alperin proves the following.

Theorem 2.26 (Alperin's Theorem, [1]) Let H be a subgroup of G, and let ϕ denote the (right) coset map. Denote by M the Scott module with vertex H. Define the $Y_{i,j}$ as above. Then the products

$$\left(\prod_{i,j} Y_{i,j}^{\alpha_{i,j}}\right)\phi, \qquad 0 \leqslant \alpha_{i,j} \leqslant p-1$$

lexicographically ordered, form a basis of M, with the images of products of weight k forming a basis of $J^k(M)/J^{k+1}(M)$.

Proof: The strategy of this proof is to first prove a generalization of Jennings' Theorem, which shows that Jennings' result does not depend on the order of the products. Then we derive the theorem from that. To prove the generalization, we need crucially to show that the products $\prod X_{i,j}^{\alpha_{i,j}}$ span KG, and this can be deduced from the statement that $\prod x_{i,j}^{\alpha_{i,j}}$ span KG. Then we can prove, with this fact, that we have bases of J^k/J^{k+1}.

So, we will first prove that we do not have to choose one particular ordering for the $X_{i,j}^{\alpha_{i,j}}$ in Jennings' Theorem. So consider the products

$$\prod X_{i,j}^{\alpha_{i,j}}$$

in a particular, yet arbitrary ordering. We claim that the products of weight k lie in J^k, and form a basis of J^k/J^{k+1}. [The weight of a product does not depend on the ordering of the factors.] Now notice that since J^k is an ideal, certainly the products of weight k lie inside J^k. All we have to do then is show that those of weight w form a basis of J^k/J^{k+1}.

Now we show that $\prod x_{i,j}^{\alpha_{i,j}}$ form a basis of KG. We proceed by induction on the 'Loewy length', that is, the number c such that $\Gamma_c \neq 0$ but $\Gamma_{c+1} = 0$. Notice that the Γ_i are invariant under quotients. Now, Γ_c is central, and by induction we can express any element $\Gamma_c g$ of G/Γ_c as

$$\Gamma_c\left(\prod_{i<c} x_{i,j}^{\alpha_{i,j}}\right),$$

and hence

$$g = \gamma\left(\prod_{i<c} x_{i,j}^{\alpha_{i,j}}\right),$$

where $\gamma \in \Gamma_c \leqslant \mathrm{Z}(G)$. Now γ is expressible as a product of the $x_{c,j}$, all of which are central, and so can be slotted into the expression for g at the appropriate point in this new ordering.

Now we need to show that this implies that the $\prod X_{i,j}^{\alpha_{i,j}}$ span KG. This follows from the identity

$$a_1 \dots a_m - 1 = \sum (a_{j_1}-1)(a_{j_2}-1)\dots(a_{j_n}-1),$$

where the sum is taken over all subsets of $\{1,\dots,m\}$ which preserve the ordering (i.e., $j_k < j_\ell$ for $k < \ell$). This demonstrates that the $\prod x_{i,j}^{\alpha_{i,j}}$ can be expressed as products of the $X_{i,j}^{\alpha_{i,j}}$ *occurring below the highest* *element (in the ordering) of the product of the* $x_{i,j}^{\alpha_{i,j}}$, and so we can express every element of KG as a sum of various products of the $X_{i,j}^{\alpha_{i,j}}$.

Now we need to prove that the products of weight k form a basis for J^k/J^{k+1}. Suppose that $J^{k+1}=0$ but $J^k \neq 0$. Now J^k has dimension 1, since KG has a 1-dimensional socle. Now

$$\prod_{i,j} x_{i,j}^{p-1}$$

lies inside J^k as this is the element with the largest weight. This product is non-zero as J^k is non-zero, and so the result is true for k. Thus our (downward) induction is started.

Suppose that the result is true for all integers greater than ℓ, then, and consider J^ℓ. Then we have the same number of elements of weight ℓ in the new ordering as in Jennings' original ordering, so if they are linearly independent, then they are a basis for $J^\ell/J^{\ell+1}$. However, if they are linearly dependent, then one is expressible as a product of the others (modulo $J^{\ell+1}$), and so Jennings' original ordering elements wouldn't form a basis (as they have the wrong number of elements). Hence we have the result.

Now we turn to the theorem itself. Recall that ϕ is the right coset map (which is a KG-module homomorphism). Now we can order the $X_{i,j}$ arbitrarily, so let us order them as

$$\left(\prod Y_{i,j}^{\alpha_{i,j}}\right)\left(\prod Z_{i,j}^{\beta_{i,j}}\right).$$

Now ϕ annihilates the $z_{i,j}$, since they lie in H. If A is one of these products, and if $\beta_{i,j} \neq 0$ for some i and j, then we can write

$$A = aZ_{i,j},$$

where $a \in KG$. Now ϕ is a KG-module homomorphism, and so $A\phi = a(Z_{i,j}\phi) = 0$; then M is spanned by the images (under ϕ) of the products $\prod Y_{i,j}^{\alpha_{i,j}}$. These also form a basis, since there are the correct number of them: there needs to be p^{n-m} of them (where $|H| = p^m$), and we have

$$\Delta_i(G)/\Gamma_{i+1}(G) \cong (H \cap \Gamma_i(G))/(H \cap \Gamma_{i+1}(G)).$$

Notice that since $H = p^m$, the isomorphism above shows that there are m elements $z_{i,j}$, and so that there are $n-m$ elements of the form $y_{i,j}$, proving that there are p^{n-m} products $\prod Y_{i,j}^{\alpha_{i,j}}$.

All we need to do now is show that the products of weight k lie in $J^k(M)$, and form a basis for $J^k(M)/J^{k+1}(M)$. The first part has already been shown, so it remains to show that they form a basis. This proceeds as before: if this we not true, then one product could be expressed as a linear combination of the others (of that weight and higher), and so we could delete it from the basis for M, a contradiction to the fact that there are p^{n-m} elements in a basis of M. We have finally proven the theorem. □

2.6 Scott Modules of Symmetric Groups

In this section we will describe all Scott modules for $G = A_4$ and $G = S_4$. Let us first consider the case where $G = A_4$. Characteristic 3 is uninteresting (as the only Scott modules are 1_G and $\mathcal{P}(1_G)$), so we restrict our attention to characteristic 2.

In A_4, the subgroups of order 1 and 4 give uninteresting Scott modules, so let us take $H = \langle(1,2)(3,4)\rangle$, and consider $S(G,H)$. We firstly describe the simple modules over GF(4) (a splitting field for A_4). If $\mathrm{GF}(4) = \{0,1,\alpha,\alpha+1\}$ where $\alpha^2 = \alpha+1$, and $h = (1,2,3)$ and $g = (1,2)(3,4)$, then all simples are 1-dimensional, and the two non-trivial modules are given by

$$A_1 : xg = x, \quad xh = \alpha x,$$

and

$$A_2 : xg = x, \quad xh = \alpha^2 x.$$

We first describe a 2-dimensional module D that will appear in our analysis: it is indecomposable if K does not have cube roots of 1, but if K has cube roots, it splits up into $A_1 \oplus A_2$. It is given by the free K-module $\langle x, y\rangle$, with

$$xh = y, \qquad yh = x + y,$$

and upon which g acts trivially.

Let us give the cosets of H, firstly. They are

$$\begin{aligned} H &= \{1, (1,2)(3,4)\} = a, & H(1,3)(2,4) &= \{(1,3)(2,4), (1,4)(2,3)\} = f, \\ H(1,2,3) &= \{(1,2,3), (1,3,4)\} = b, & H(2,4,3) &= \{(2,4,3), (1,4,2)\} = d, \\ H(1,3,2) &= \{(1,3,2), (2,3,4)\} = c, & H(1,2,4) &= \{(1,2,4), (1,4,3)\} = e. \end{aligned}$$

Then g and h act as in the following diagram:

$$\begin{tikzcd} & a \arrow[dl, "h"'] & \\ b \arrow[rr, "h"'] \arrow[d, leftrightarrow, "g"] & & c \arrow[ul, "h"'] \arrow[d, leftrightarrow, "g"'] \\ d \arrow[rr, "h"] & & e \arrow[dl, "h"] \\ & f \arrow[ul, "h"] & \end{tikzcd}$$

[Here, g acts trivially on a and f.] We know that $\langle a+b+c+d+e+f\rangle$ is a submodule isomorphic with 1_G. The component of $(1_H)\uparrow^G$ containing N_1 is $S(G,H)$. In fact, $(1_H)\uparrow^G$ is indecomposable, as we shall see.

We search for other submodules of $M = (1_H)\uparrow^G$. We have $N_3 = \langle a+f, b+d, c+e\rangle$, which is a 3-dimensional submodule containing N_1. N_3 also contains the 2-dimensional module $N_2 = \langle a+c+e+f, a+b+d+f\rangle$. Let $x = a+c+e+f$ and $y = a+b+d+f$. Then g acts trivially on both x and y, and h sends x to y and y to $x+y$. Thus $N_2 \cong D$, as defined above, and hence N_3 is semisimple. Since M is self-dual, and is not semisimple (else M would have dimension 1), N_3 must be the socle of M.

Since we know that Scott modules are self-dual, we expect $M/N_3 \cong N_3$, but let's find the submodules of M/N_3 anyway. Notice that $M/N_3 = \langle N_3+a, N_3+b, N_3+c\rangle$ as a K-module, and in fact this is still a transitive permutation module: it has diagram

$$\begin{tikzcd} & N_3+a \arrow[dr, "h"] & \\ N_3+c \arrow[ur, "h"] & & N_3+b \arrow[ll, "h"] \end{tikzcd}$$

and g acts trivially. Clearly then $\langle a+b+c, a+f, b+d, c+e\rangle$ (i.e., $a+b+c, N_3$) is a submodule N_4, of dimension 4, with $N_4/N_3 \cong 1_G$. Also, we have a 5-dimensional module

$$N_5 = \langle a+b, b+c, a+f, b+d, c+e\rangle;$$

We would like to demonstrate that $N_5/N_3 \cong D$, which we know from the fact that Scott modules are self-dual, but it would be useful to prove it anyway. Write $x = N_3+a+b$ and $y = N_3+b+c$. Notice that

$xh = y$, $yh = x + y$, and that g acts trivially on N_5/N_3: we are done. The Scott module has socle series

$$\begin{gathered} 1_G \oplus D \\ 1_G \oplus D \end{gathered}$$

when K does not have cube roots of unity, and has socle series

$$\begin{gathered} 1_G \oplus A_1 \oplus A_2 \\ 1_G \oplus A_1 \oplus A_2 \end{gathered}$$

otherwise.

Now we turn to $G = S_4$, generated by $g = (1, 3, 2, 4)$ and $h = (1, 2, 3)$. Again, we need to describe the simple modules; in this case, there are only two simple modules, 1_G and a 2-dimensional module $D = \langle x, y\rangle$ with G-action

$$xg = y, \qquad yg = x, \qquad\qquad xh = y, \qquad yh = x + y.$$

Firstly consider $H = \langle (1, 2), (3, 4)\rangle$, and let $M = (1_H)\uparrow^G$: we have the cosets

$$\begin{aligned}
&H = \{1, (1, 2), (3, 4), (1, 2)(3, 4)\} = a, \\
&H(1, 3) = \{(1, 3), (1, 2, 3), (1, 3, 4), (1, 2, 3, 4)\} = b, \\
&H(1, 4) = \{(1, 4), (1, 4, 2, 3), (1, 3, 2, 4), (1, 2, 4, 3)\} = c, \\
&H(2, 3) = \{(2, 3), (1, 3, 2), (2, 3, 4), (1, 3, 4, 2)\} = d, \\
&H(2, 4) = \{(2, 4), (1, 4, 2), (2, 4, 3), (1, 4, 3, 2)\} = e, \\
&H(1, 3)(2, 4) = \{(1, 3)(2, 4), (1, 4, 2, 3), (1, 3, 2, 4), (1, 4)(2, 3)\} = f.
\end{aligned}$$

These are acted upon by g and h as in the following diagram:

$$\begin{tikzcd}
c \arrow[rr, dotted] \arrow[dr, "h"] & & b \arrow[ddd, "g"] \\
& a \arrow[ur, "h"] \arrow[d, leftrightarrow, "g"] & \\
& f \arrow[dl, "h"] & \\
e \arrow[uuu, "g"] & & d \arrow[ll, dotted] \arrow[ul, "h"]
\end{tikzcd}$$

Here the dotted line implies that g acts in the direction of the arrow, and h acts contrary to it. We can find a 1-dimensional submodule $N_1 = \langle a + b + c + d + e + f\rangle$, and there is also a submodule

$$N_2 = \langle a + c + d + f, a + b + e + f\rangle;$$

if we let $x = a + c + d + f$ and $y = a + b + e + f$, then $xg = xh = y$, $yg = x$ and $yh = x + y$, so $N_2 \cong D$, the 2-dimensional simple module. Since $N_1 \cap N_2 = \{0\}$, we have the submodule

$$N_3 = N_2 \oplus N_1 = \langle a + f, b + e, c + d\rangle.$$

Quotienting out by this module, we see that M/N_3 is generated by $N_3 + a$, $N_3 + b$ and $N_3 + c$, subject to the diagram

$$\begin{tikzcd}
& N_3 + a \arrow[dr, "h"] & \\
N_3 + c \arrow[ur, "h"] \arrow[rr, dotted] & & N_3 + b
\end{tikzcd}$$

(where the dotted line has the same meaning, and g acts trivially on a).

We can see a submodule $N_4/N_3 = \langle N_3 + a + b + c\rangle$, where $N_4 = \langle a + b + c, a + f, b + e, c + d\rangle$, and also a 2-dimensional quotient $\langle N_3 + a + b, N_3 + a + c\rangle$, where again we get an isomorphism between this and D via the associations $x = N_3 + a + c$ and $y = N_3 + a + b$. Thus we also have the submodule

$$N_5 = \langle a + b, a + c, a + f, b + e, c + d\rangle;$$

this gives the socle series of $S(S_4, \langle(1, 2), (3, 4)\rangle)$ as

$$\begin{array}{c} 1_G \oplus D \\ 1_G \oplus D \end{array}.$$

Next, we consider the other (non-normal) subgroup of S_4 of order 4 (up to conjugacy), namely $H = \langle(1, 3, 2, 4)\rangle$: we have the cosets

$$H = \{1, (1, 3, 2, 4), (1, 2)(3, 4), (1, 4, 2, 3)\} = a, \qquad H(1, 2) = \{(1, 2), (1, 3)(2, 4), (3, 4), (1, 4)(2, 3)\} = f,$$
$$H(2, 4) = \{(2, 4), (1, 3, 4), (1, 4, 3, 2), (1, 2, 3)\} = b, \qquad H(1, 3) = \{(1, 3), (2, 4, 3), (1, 2, 3, 4), (1, 4, 2)\} = e,$$
$$H(1, 4) = \{(1, 4), (1, 3, 2), (1, 2, 4, 3), (2, 3, 4)\} = c, \qquad H(2, 3) = \{(2, 3), (1, 2, 4), (1, 3, 4, 2), (1, 4, 3)\} = d.$$

These are acted upon by g and h as in the following diagram:

$$\begin{tikzcd}
& a \arrow[dl, "h"'] & \\
b \arrow[rr, "{g,h}"'] & & c \arrow[ul, "h"'] \arrow[d, "g"'] \\
d \arrow[u, "g"] \arrow[dr, "h"'] & & e \arrow[ll, "{g,h}"'] \\
& f \arrow[ur, "h"'] &
\end{tikzcd}$$

Here g acts trivially on a and f. Let M be the transitive permutation module $(1_H)\uparrow^G$, which has this permutation basis. As always, we have $N_1 = \langle a + b + c + d + e + f\rangle$, and we have

$$N_2 = \langle a + c + d + f, a + b + e + f\rangle;$$

then this is isomorphic with the 2-dimensional simple module D, because if we let $x = a + c + d + f$ and $y = a+b+e+f$, then all of the rules $xg = y$ and so on follow. Thus we have the module $N_3 = N_1 \oplus N_2 \cong 1_G \oplus D$ forming the socle of M. We can generate N_3 nicely using the generating set $\{a + f, b + e, c + d\}$, which easily show us that the quotient module M/N_3 is generated by $\{N_3 + a, N_3 + b, N_3 + c\}$ where g swaps $N_3 + b$ and $N_3 + c$, and h cycles the three cosets. As in the A_4 case, we have two submodules given by $N_4/N_3 = \langle N_3 + a + b + c\rangle \cong 1_G$ and $N_5/N_3 = \langle N_3 + a + c, N_3 + a + b\rangle \cong D$ via the identifications $x = N_3 + a + c$ and $y = N_3 + a + b$. By using the nice generating set for N_3, we get

$$N_4 = \langle a + b + c, a + f, b + e, c + d\rangle, \qquad N_5 = \langle a + b, a + c, a + f, b + e, c + d\rangle$$

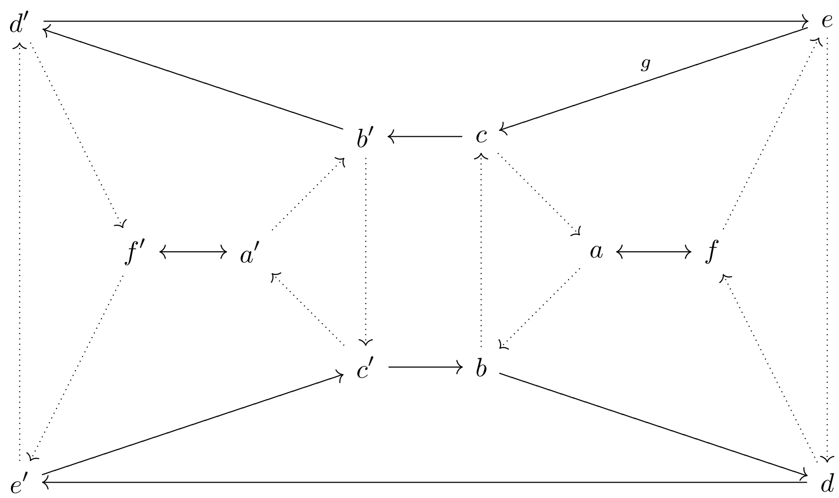
and $S(G,H)$ has socle series

$$\begin{array}{c} 1_G \oplus D \\ 1_G \oplus D \end{array}.$$

Next, suppose that $H = \langle (1,2)(3,4) \rangle$, and again let $M = S(G,H)$. The cosets of H in G are

$$\begin{aligned}
&H = \{1, (1,2)(3,4)\} = a, && H(1,2) = \{(1,2),(3,4)\} = a',\\
&H(1,2,3) = \{(1,2,3),(1,3,4)\} = b, && H(1,3) = \{(1,3),(1,2,3,4)\} = b',\\
&H(1,3,2) = \{(1,3,2),(2,3,4)\} = c, && H(2,3) = \{(2,3),(1,3,4,2)\} = c',\\
&H(1,4) = \{(1,4),(1,2,4,3)\} = d, && H(1,2,4) = \{(1,2,4),(1,4,3)\} = d',\\
&H(2,4) = \{(2,4),(1,4,3,2)\} = e, && H(1,4,2) = \{(1,4,2),(2,4,3)\} = e',\\
&H(1,3,2,4) = \{(1,3,2,4),(1,4,2,3)\} = f, && H(1,3)(2,4) = \{(1,3)(2,4),(1,4)(2,3)\} = f'.
\end{aligned}$$

These are acted upon by g and h according to the diagram



Here a solid line denotes the action of g and a dotted line denotes the action of h. We will denote by $\bar{a}$ the element $a + a'$, and so on. The labelling was chosen so that the quantities $\bar{a}$, $\bar{b}$, etc. would be of importance.

As always, we have a 1-dimensional module generated by the sum of all basis elements

$$N_1 = \langle \bar{a} + \bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f} \rangle.$$

We also have two 2-dimensional modules isomorphic with D, namely

$$D_1 = \langle \bar{a} + \bar{c} + \bar{d} + \bar{f}, \bar{a} + \bar{b} + \bar{e} + \bar{f} \rangle, \qquad D_2 = \langle a + \bar{b} + c' + d + \bar{e} + f', a' + b + \bar{c} + \bar{d} + e' + f \rangle.$$

We can make identifications $x = \bar{a}+\bar{c}+\bar{d}+\bar{f}$ and $y = \bar{a}+\bar{b}+\bar{e}+\bar{f}$, giving $D_1 \cong D$, and $x = a+\bar{b}+c'+d+\bar{e}+f'$ and $y = a' + b + \bar{c} + \bar{d} + e' + f$, giving $D_2 \cong D$. The direct sum $N_1 \oplus D_1 \oplus D_2$ is a 5-dimensional module, forming the socle

$$\mathrm{soc}(M) = \langle \bar{a} + \bar{f}, \bar{b} + \bar{e}, \bar{c} + \bar{d}, a + c' + d + f', a + b' + e + f' \rangle.$$

Modulo the socle, there are two fixed points, $a + b + c + d + e + f$ and $\bar{a} + \bar{b} + \bar{c}$, and so we have two 6-dimensional modules

$$N_{6,1} = \langle N_5, a + b + c + d + e + f \rangle, \qquad N_{6,2} = \langle N_5, \bar{a} + \bar{b} + \bar{c} \rangle.$$

There is also another submodule, N_7, given by

$$N_7 = \langle N_5, \bar{a} + \bar{c}, \bar{a} + \bar{b} \rangle;$$

letting $x = N_5 + \bar{a} + \bar{c}$, and $y = N_5 + \bar{a} + \bar{b}$, we get the isomorphism with the simple module D given above. The sum of these three submodules gives the second socle layer,

$$\mathrm{soc}^2(M) = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, a + f, b + e, c + d \rangle.$$

The remaining quotient is generated by $\mathrm{soc}^2(M)+a$, $\mathrm{soc}^2(M)+b$ and $\mathrm{soc}^2(M)+c$: clearly $\mathrm{soc}^2(M)+a+b+c$ is a fixed point, and also $\langle \mathrm{soc}^2(M) + a + c, \mathrm{soc}^2(M) + a + b \rangle \cong D$.

Thus the socle series of M is given by

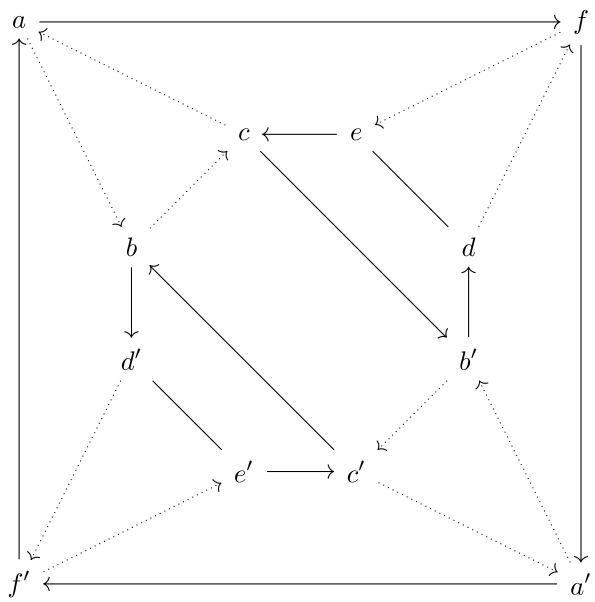
$$\begin{array}{c} D \oplus 1_G \\ 1_G \oplus D \oplus 1_G. \\ D \oplus 1_G \oplus D \end{array}$$

[Of course, since M is self-dual, this is *not* the radical series of M.]

Finally for the non-normal subgroups, consider $H = \langle (1,2) \rangle$, and $M = S(G, H)$ again. We have the cosets

$$\begin{aligned}
H &= \{1, (1,2)\} = a, & H(3,4) &= \{(3,4), (1,2)(3,4)\} = a', \\
H(1,3) &= \{(1,3), (1,2,3)\} = b, & H(1,3,4) &= \{(1,3,4), (1,2,3,4)\} = b', \\
H(2,3) &= \{(2,3), (1,3,2)\} = c, & H(2,3,4) &= \{(2,3,4), (1,3,4,2)\} = c', \\
H(1,4,3) &= \{(1,4,3), (1,2,4,3)\} = d, & H(1,4) &= \{(1,4), (1,2,4)\} = d', \\
H(2,4) &= \{(2,4), (1,4,2)\} = e, & H(2,4,3) &= \{(2,4,3), (1,4,3,2)\} = e', \\
H(1,3,2,4) &= \{(1,3,2,4), (1,4)(2,3)\} = f, & H(1,4,2,3) &= \{(1,4,2,3), (1,3)(2,4)\} = f'.
\end{aligned}$$

The action of g and h is given by the diagram



[Here, a solid line is the g-action, and a dotted line is the h-action. In the two cases where there is no arrowhead, g and h act in the opposite directions, and those are obvious from the picture and the fact that they act as cycles.] Again, we let $\bar{a} = a + a'$ and so on.

This is, in fact, decomposable: it splits up into an 8-dimensional and a 4-dimensional module. Consider the two modules N and P, given by

$$N = \langle a + b + c, b' + d' + f, c' + e + f', a' + d + e' \rangle$$

and

$$P = \langle a + d', a' + b', b + f, a' + c', c + e', c + f, b + d, d' + e \rangle;$$

these intersect trivially, in fact. Since N clearly contains the sum of all permutation basis elements, N is the Scott module. We will describe the structure of both modules, however.

Firstly, consider N: we have a 1-dimensional submodule N_1, generated by the sum of all permutation basis elements. We have another submodule

$$N_3 = \langle N_1, a' + c' + d + \bar{e} + f', a' + b' + \bar{d} + e' + f \rangle.$$

Then $x = a' + c' + d + \bar{e} + f'$ and $y = a' + b' + \bar{d} + e' + f$ gives $N_3/N_1 \cong D$. This clearly gives N as

$$\begin{matrix} 1_G \\ D \\ 1_G \end{matrix}.$$

Now consider the 8-dimensional module P: we have a 2-dimensional submodule P_2 given by

$$P_2 = \langle \bar{a} + \bar{c} + \bar{d} + \bar{f}, \bar{a} + \bar{b} + \bar{e} + \bar{f} \rangle,$$

which is the socle of P, and isomorphic with D.

We can construct two simple submodules of $P/\operatorname{soc}(P)$, namely

$$P_3 = \langle \bar{a} + \bar{c} + \bar{e}, \bar{d} + \bar{e} + \bar{f}, \bar{a} + \bar{b} + \bar{d} \rangle,$$

and

$$P_4 = \langle P_2, b + c + d' + e + \bar{f}, a' + c' + d' + \bar{e} + f \rangle.$$

Then the second socle layer is given by

$$\operatorname{soc}^2(M) = \langle \bar{a} + \bar{c} + \bar{e}, \bar{d} + \bar{e} + \bar{f}, \bar{a} + \bar{b} + \bar{d}, b + c + d + e', a + c + d' + f \rangle.$$

The third socle layer is simply a fixed point, which is given by

$$\operatorname{soc}^3(M) = \langle \operatorname{soc}^2(M), b + \bar{d} + e \rangle.$$

This quotient is 2-dimensional, and is isomorphic with D, giving the socle series for P as

$$\begin{matrix} 2_G \\ 1_G \\ 1_G \oplus 2_G \\ 2_G \end{matrix}.$$

The only normal case is where $H = V_4$. In this case, since $S_4 = V_4 \rtimes S_3$, we have an easy transversal, namely

$$T = \{1, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}:$$

this makes it much easier to do calculations. Make the following assignments:

$$1 = a, \qquad (1,2,3) = b, \qquad (1,3,2) = c, \qquad (1,2) = f, \qquad (1,3) = d, \qquad (2,3) = e.$$

Then $g = (1,3,2,4)$ and $h = (1,2,3)$ act on this transversal as

$$\begin{tikzcd}
d \arrow[rr, "h"] \arrow[ddd, leftrightarrow, "g"'] & & e \arrow[dl, "h"] \arrow[ddd, leftrightarrow, "g"] \\
& f \arrow[ul, "h"] \arrow[d, leftrightarrow, "g"] & \\
& a \arrow[dr, "h"] & \\
c \arrow[ur, "h"] & & b \arrow[ll, "h"]
\end{tikzcd}$$

Notice that there exists a submodule $N_2 = \langle a+b+c, d+e+f \rangle$, which contains the trivial submodule. There is also a submodule $N_4 = \langle a+b, b+c, d+e, d+f \rangle$, whose intersection with N_2 is trivial, and so N_2 is a summand of M; this implies that N_2 is the Scott module $S(S_4, V_4)$. This is consistent with Proposition 2.25, which said that $S(S_4, V_4)$ would be isomorphic with $\mathcal{P}(1_{S_3})$, which is indeed uniserial of dimension 2. [For completeness, the module N_4 has Loewy length 2, and has a submodule $M_2 = \langle a+b+d+f, b+c+d+e \rangle$, and $M_2 \cong N_4/M_2 \cong D$, the simple module of degree 2.] Another way to prove this is to remember the isomorphism

$$S(S_4, V_4) \cong S(S_4, A_4),$$

and note that $S(S_4, A_4)$ is indecomposable by Green's Indecomposability Criterion, and is 2-dimensional: Thus is has to be uniserial of length 2, with two copies of the trivial module.

This means that we have finally found all Scott modules over all fields of characteristic 2 (and 3), since all other Scott modules are either 1_G in the case of $H = D_8$ and $H = C_3$, or $\mathcal{P}(1_G)$ in the case $H = 1$.

As an aside, consider the case $G = S_6$: let 1_G denote the trivial module, let 4_1 denote the heart of the permutation module on a point stabilizer, and 4_2 denote this simple module twisted under the outer automorphism of S_6. Then the Scott modules are given in the tables below.

H	$\langle (1,2) \rangle$	$\langle (1,2)(3,4) \rangle$	$\langle (1,2)(3,4)(5,6) \rangle$	$\langle (1,2),(3,4)(5,6) \rangle$
$S(G,H)$	1_G $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ 1_G	1_G $4_1 \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 4_2$ 1_G	1_G $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ 1_G	$4_1 \oplus 4_2 \oplus 4_2 \oplus 4_1$ $1_G \oplus 1_G \oplus 1_G \oplus 1_G$ $4_1 \oplus 4_2 \oplus 4_2 \oplus 4_1$ $1_G \oplus 1_G \oplus 1_G \oplus 1_G$ $4_1 \oplus 4_2 \oplus 1_G \oplus 4_1 \oplus 4_2$ $1_G \oplus 4_1 \oplus 4_2 \oplus 1_G$ $4_1 \oplus 4_2 \oplus 1_G \oplus 1_G \oplus 4_1 \oplus 4_2$ $1_G \oplus 1_G \oplus 4_1 \oplus 4_2 \oplus 1_G \oplus 1_G$ $4_1 \oplus 4_2 \oplus 1_G \oplus 4_1 \oplus 4_2$

H	$\langle(1,2,3,4)(5,6)\rangle$	$\langle(1,2,3,4),(5,6)\rangle$	$\langle(1,2),(3,4),(5,6)\rangle$	$\langle(1,2,3,4),(1,3)\rangle$
$S(G,H)$	1_G $4_1 \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 4_2$ 1_G	$4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 4_2$	4_2 1_G 4_1 1_G 4_2 1_G $1_G \oplus 4_1$ $1_G \oplus 4_2$ $1_G \oplus 4_2$	1_G 4_1 1_G

H	$\langle(1,2,3,4)(5,6),(1,3)\rangle$	$\langle(1,2,3,4),(1,3)(5,6)\rangle$	$\langle(1,2,3,4)(5,6),(1,3)(5,6)\rangle$
$S(G,H)$	1_G $4_1 \oplus 4_2$ 1_G	1_G 4_2 1_G	1_G 1_G

H	1	$\langle(1,2),(3,4)\rangle$	$\langle(1,2,3,4)\rangle$	$\langle(1,2)(5,6),(3,4)(5,6)\rangle$
$S(G,H)$	1_G $4_1 \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 1_G \oplus 4_2$ $4_1 \oplus 1_G \oplus 4_2$ 1_G	1_G $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ 1_G	1_G $4_1 \oplus 4_2$ $1_G \oplus 1_G$ $4_1 \oplus 4_2$ 1_G	1_G $4_2 \oplus 1_G$ $4_2 \oplus 1_G$ 1_G

Conclusion

To understand precisely *why* Scott modules and lower defect groups are the same thing is something that Burry and Green, in their breakthrough papers, do not explicitly address. This appears to be a very difficult task, certainly using Green's proof, as it is very long. Burry's proof really only shows an inequality, that $m_B(P) \leqslant M_B(P)$, and only in the case where $P \trianglelefteq G$: he then notes that the sums of $m_B(P)$ and $M_B(P)$ across all blocks are equal, and hence $m_B(P) = M_B(P)$, and that both quantities are locally controlled, and hence the result.

We saw Brauer's definition of lower defect groups in the first chapter. This is a rather convoluted definition, and perhaps it is this that obscures what lower defect groups are. However, Proposition 1.9 exposed lower defect groups for what they really were: simply defect groups of conjugacy classes lying in a block. This makes obvious most of Brauer's theorems on lower defect groups; of course defect groups form the maximal elements of the set of lower defect groups, as lower defect groups are just defect groups of conjugacy classes, and the defect group of a block was *defined* as a maximal member of this set. Of course the dimension of $\mathrm{Z}(B)$ is equal to the sum of the multiplicities of all lower defect groups, since this number is just the number of conjugacy classes in B.

The first step then should be to try to understand why lower defect group multiplicities are characterized in this way. But this is not all that difficult: $m_B(P)$ is simply the difference in the dimensions of the subspaces of $\mathrm{Z}(B)$ spanned by the class sums with defect group less than P and those with defect group strictly less than P. Now let us read Brauer's definition: he has an f with $\bar{C}f \neq 0$ (where C has defect group P), but for which $(\bar{C}')f = 0$ when C' has defect group strictly less than that of C. Then we find a maximal subspace of $\mathrm{Z}(B)^*$ with this property. Previously we found a maximal subspace of $\mathrm{Z}(B)$ with basically exactly the same property. The fact that the two notions coincide is because $V \cong V^*$, and they are isomorphic in a special way, in that you can define an isomorphism in terms of the basis, which in this case just happens to consist of class sums!

The author believes that Brauer's original definition of lower defect groups should be rejected in favour of the definition implicit in Proposition 1.9. This is much more tractable. However, it does also entail a possible reason as to why lower defect groups have not found as many applications as defect groups, which are special elements of the set of all defect groups of conjugacy classes. Lower defect groups are simply all of them thrown together.

We should therefore really abandon the notion of lower defect group as defined by Brauer, and just think of a lower defect group as a defect group of a conjugacy class lying in $\mathrm{Z}(B)$. The next hurdle is comparing that to Scott modules. The result says that there are $m_B(P)$ copies of the Scott module $S(G, P)$ lying in B. We know that there are $m_B(P)$ class sums, whose conjugacy classes have defect group P, lying in $\mathrm{Z}(B)$, so there must be a way of relating to each class sum a Scott module, and to each Scott module a class sum. The author is, unfortunately, unaware of such a way.

Continued Research

In [1], Alperin asks the following question: if G is a p-group acting transitively on a set Ω, what is the structure of $\mathrm{End}(K\Omega)$, where $K\Omega$ is the permutation module? He quotes the example where G is dihedral of order 8 and H is a non-central subgroup of order 2. We saw in the previous chapter that $S(G,H)$ is uniserial of length 4; however, Alperin asserts that the Hecke algebra $E = \mathrm{End}(S(G,H))$ is 3-dimensional with $J(E)$ of dimension 2 and $J^2(E) = 0$. The structure of endomorphism algebras of transitive permutation modules is reasonably well understood, and this behaviour may well be explainable given the current machinery available on Hecke algebras and endomorphism algebras in particular.

In [20], Darren Semmen proves an analogon of Jennings' Theorem for so-called p-split groups – i.e., finite groups whose Sylow p-subgroup is normal – which naturally extends Alperin's work, along with Shalev's (in [21]) and Quillen's (in [19]). However, Semmen casts this work in the light of the universal enveloping algebra, and so loses a lot of the clarity and computability that Alperin's work had. It would be interesting to attempt to produce a theorem in the style of Alperin's for groups with a normal Sylow p-subgroup.

In general, it is a very difficult problem understanding the structure of Scott modules for arbitrary finite groups, even if one is interested merely in producing superficial statements on their structure; indeed, Lemma 2.5 and Proposition 2.24 are about the most that one can say, except for one result in Section 2.5: Proposition 2.25 is, the author believes, of major importance in working with Scott modules. It is perhaps surprising then that this result does not appear in the literature at all; perhaps this is because of its rather trivial nature, but it is an important observation nevertheless.

The fact that projective covers of trivial modules are Scott modules lends credence to the statement that Scott modules are difficult: the calculation of $\mathcal{P}(1_G)$ is difficult in the majority of groups, and relatively little can be said. In the case of the groups of the form $P \rtimes Q$, where P is a p-group and Q is an abelian p'-group (so that the algebra is basic), the structure of $\mathcal{P}(1_G)$ is governed by the Sylow p-subgroup of G, due to Lemma 5.8 of [2], which states that $J(M)$ and $J(M\downarrow_P)$ are equal for any KG-module M, if P is a normal Sylow p-subgroup. For general groups the structure is not known.

The study of lower defect groups has not been taken up in the same way as defect groups have been: despite their elegance, perhaps lower defect groups do not have the importance and applications that defect groups do. However, Burry's Theorem demonstrates that they certainly play some part in the module theory. Trying to understand this relationship, and the application of lower defect groups more generally, might be an interesting avenue of further study, if only to demonstrate that yet another of Brauer's insights was useful to the subject. The remarks of the last section, however, do little to elucidate this relationship.

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