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The Modular Representation Theory of Finite Groups

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This dissertation will develop modular representation theory, starting from Brauer's Three Main Theorems. We will consider the Green Correspondence, then use G-algebras to unite the block-theoretic and module-theoretic approaches somewhat. We then consider some simple group theory, and finally take a cursory look at the modern-day research, and its progress on several long-standing conjectures.

Declaration

I warrant that the content of this dissertation is the direct result of my own work and that any use made in it of published or unpublished material is fully and correctly referenced.

Signed:________________ Date:________________

Preface

Representation theory itself has its origins in the work of Burnside and Fröbenius, around the turn of the twentieth century, Fröbenius being almost wholly responsible for the early development of character theory. He completely determined the character theory of the symmetric groups in 1900 and the alternating groups the following year. Schur followed these two, and many of the early developments in ordinary representations bear their names; for example, Schur's Lemma, the Fröbenius–Schur count of involutions, Fröbenius reciprocity, and Burnside's $p^\alpha q^\beta$ theorem, one of the best early applications of character theory to simple groups.

Modular representation theory, the study of representations over fields of characteristic other than zero, was started with Leonhard Eugene Dickson, who coined the term 'modular' representations. The first really major developments came with Richard Brauer, who exploited this rich and virtually untapped area of mathematics. Brauer's methods were mainly character-theoretic; one of his main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were very well-suited in the case of the so-called small groups.

The next revolution in the theory came with James Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules, rather than Brauer, who worked mostly with blocks and characters.

The 1970s saw the needs of the Classification of Finite Simple Groups shift away from character theory and toward local analysis, since the 'large' simple groups, the so-called generic groups, could not really be handled very well at all with modular characters. The new techniques in local analysis supersede many of Brauer's techniques, although for groups of small order, modular character theory is still the best way to gain considerable information relatively easily.

The work of Brauer and those who came after him left behind several outstanding problems, such as Brauer's $k(B)$-conjecture, his Height Zero Conjecture, and the Alperin–McKay Conjecture. Later other conjectures evolved, such as Olsson's Conjecture, and understanding both how to prove these conjectures, and why they are actually correct (if they are) are two of the binding concepts throughout this approach to representation theory to this day.

The module theory has also evolved, from the pioneering work of James Green, who laid the foundations of the module theory and also introduced the concept of G-algebras, through to the present day. Among the open problems in this area lies the understanding of abelian defect groups. Michel Broué's Abelian Defect Group Conjecture is a very good example of how homological algebra and category-theoretic methods have been introduced into representation theory. There are many outstanding problems in this area of representation theory, understanding the structure of the various categories associated with a representation.

This project attempts to briefly consider both approaches, looking at Brauer's methods first, then introducing the module and G-algebra approach. Chapter 1 sees the original Three Main Theorems of Brauer, relating the blocks of a group to those of its p-local subgroups in the case of the First Main Theorem, and to the centralizers of p-elements and p-groups in the case of the Second and Third Main Theorems. In this chapter we prove the First and Third Main Theorems, deferring the proof of the Second to Chapter 2. At the end of this chapter, we look at the Brauer Correspondence in the symmetric group S_7.

Chapter 2 starts by discussing relatively projective modules. This leads naturally to the concept of a vertex of a module as a minimal p-subgroup Q such that the module is relatively Q-projective. The Green Correspondence is a fundamental result in this approach to representation theory. The Green Correspondence links relatively Q-projective RG-modules to relatively Q-projective RH-modules for some H containing $\mathrm{N}_G(Q)$. After this we demonstrate the Nagao Decomposition, and use this to prove the Second Main Theorem in a module-theoretic fashion.

Chapter 3 introduces the notions of a G-algebra and an interior G-algebra. Our original interest in this chapter is to demonstrate that the notion of a G-algebra generalizes and combines the two methods of Brauer and Green. After introducing this, we define the Brauer map and defect groups for arbitrary G-algebras. We then continue with the development of G-algebras, defining the notions of pointed groups, and some of their associated definitions, eventually attaching an analogue of a defect group for every pointed group.

Chapter 4 considers some of the myriad applications of modular representation theory to finite group theory. In the first section, we examine in detail the modular representations of the alternating group A_5, the smallest non-abelian simple group. As an example of the use of the techniques of modular representation theory in the course of the mid-twentieth century during the Classification of the Finite Simple Groups, we prove two major theorems: the Brauer–Suzuki Theorem, which proves the non-existence of a simple group with quaternion Sylow 2-subgroups, and the Glauberman Z*-Theorem, which generalizes this result to any Sylow 2-subgroup with a weakly closed involution.

In Chapter 5, we examine some of the more recent developments in the field. Broué's Abelian Defect Group Conjecture concerns the module categories of a block and its Brauer correspondent in $\mathrm{N}_G(D)$. We state the conjecture, and describe some of the work done on this conjecture. We then state the Alperin–McKay Conjecture and Alperin's Conjecture (also known as Alperin's Weight Conjecture), and again describe some of the work done on this area. Two more conjectures are looked at in this chapter: Brauer's $k(B)$ Conjecture; and Brauer's Height Zero Conjecture.

Finally, in the last chapter we conclude what we have done, and give some indications of further work that can be done.

We will fix some notation throughout the course of this dissertation: R denotes a characteristic 0 complete discrete valuation ring, K its field of fractions, and k the residue field of R modulo its maximal ideal. However, often the results we quote for R work equally well for k; indeed, we will state when a result for R (or more likely, RG) does not hold for kG. We will always assume that K is a splitting field for G; that is, k contains enough roots of unity so that all indecomposable representations are absolutely indecomposable.

We will assume a reasonable amount of commutative algebra. One of the most important areas that we assume is the classical theory of lifting idempotents, which we will often use without any comment at all. Also, we assume much of the basic block theory, such as the existence of central characters, and the fact that there is only one ordinary and modular character in a block of defect zero.

Finally, I would like to thank my supervisor, Geoffrey R. Robinson, without whom this project would be non-existent, and to all my family and friends for their much-needed support and patience.

David A Craven, March 2004.

Contents

Chapter 1

Brauer's Three Main Theorems

In this chapter we will see three fundamental results, called Brauer's Three Main Theorems. They concern themselves with a correspondence between blocks with particular defect groups.

The Brauer Correspondence is defined between some of the blocks of the algebra kG and the algebra kH, for some $D\,\mathrm{C}_G(D) \leqslant H$, where D is a p-group. More precisely, it is defined between those blocks that contain D as a defect group. In particular, if we restrict our attention to blocks with D itself as a defect group and to subgroups H containing $\mathrm{N}_G(D)$, then this correspondence is in fact bijective. This is the statement of Brauer's First Main Theorem.

The First Main Theorem proves the existence of this bijection, but it doesn't really give a sufficient criterion for two blocks to be correspondents of one another. However, if the generalized decomposition number between an irreducible ordinary character of G and an irreducible modular character of $\mathrm{C}_G(x)$, (where x is p-singular) is non-zero, then the two blocks associated with the two characters correspond. This is the Second Main Theorem.

Finally, although we have a sufficient criterion for two blocks to correspond, we would like to know where particular blocks go. The only particular block we can be assured to have in every case is the block containing the trivial character. The Third Main Theorem says that if $H \geqslant D\,\mathrm{C}_G(D)$, and b is a block of kH having defect group D, then it contains the trivial character precisely when its Brauer correspondent does.

The Three Main Theorems constitute the bulk of this chapter, including the Extended First Main Theorem, which examines the Brauer Correspondence further down than $\mathrm{N}_G(D)$.

Firstly we will define the Brauer homomorphism and the Brauer Correspondence, then prove the First and Third Main Theorems, before proceeding to describe generalized decomposition numbers and state the Second Main Theorem. Its proof will be delayed until the following chapter, namely Section 2.5.

In the final section, we consider the Brauer Correspondence in a particular example, that of the symmetric group S_7. We consider the block theory for the primes 2 and 3, calculating normalizers of the p-subgroups, and using the First and Third Main Theorems to study the Brauer Correspondence.

1.1 Preliminary Lemmas

We will compile the results that we need to prove the three theorems in this section. Throughout this section G will denote a finite group with order a multiple of p, and k will denote a splitting field for G of characteristic p. The purely block-theoretic theorems that we will require will be stated without proof.

We begin with one of the definitions of a defect group of a block, which relies on the defect group of a conjugacy class. Because it includes results, we have considered it as a lemma here. Recall that the defect group of a conjugacy class $\mathscr{C}$ is a Sylow p-subgroup of $\mathrm{C}_G(x)$, for some $x \in \mathscr{C}$.

Lemma 1.1 Let B be a block of kG with associated central character ω. Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ denote the conjugacy classes of G whose class sum c_i does not vanish under ω. Let $\mathscr{D}$ be the set of all subgroups of G which are defect groups for one of the $\mathscr{C}_i$. Then each element of $\mathscr{D}$ contains a defect group of B, and in particular, the minimal elements of $\mathscr{D}$ are the defect groups of B. Also, there is a class $\mathscr{C}_t$ consisting of p-regular elements of G with defect groups the same as B.

The next lemma plays a crucial part in one step of the proof of the First Main Theorem.

Lemma 1.2 Let Q be a normal p-subgroup of G, and let I denote the k-subspace of kG generated by the class sums of conjugacy classes of G which have Q as defect group. Then Q is contained in the defect group of every block of kG, and if e is a block idempotent of kG, with associated block b, and b has Q as a defect group, then $e \in I$.

The next lemma describes how conjugacy classes split when localized to the normalizer of a p-subgroup.

Lemma 1.3 Suppose that P is a p-subgroup of G, and let $\mathscr{C}$ denote a conjugacy class of G. Then

(i) $\mathscr{C} \cap \mathrm{C}_G(P)$ is non-empty if and only if P is contained within a defect group of $\mathscr{C}$.

(ii) If P is a defect group of $\mathscr{C}$, then $\mathscr{C} \cap \mathrm{C}_G(P)$ is a single conjugacy class of $\mathrm{N}_G(P)$; i.e., two elements of $\mathrm{C}_G(P)$ both in $\mathscr{C}$ are still conjugate in $\mathrm{N}_G(P)$.

(iii) Suppose that $\bar{\mathscr{C}}$ is a conjugacy class of $\mathrm{N}_G(P)$ contained within $\mathscr{C}$. If P is a defect group of $\bar{\mathscr{C}}$, then P is a defect group of $\mathscr{C}$ and $\bar{\mathscr{C}} = \mathscr{C} \cap \mathrm{C}_G(P)$.

Proof: Suppose that $\mathscr{C} \cap \mathrm{C}_G(P) \neq \emptyset$, and let x be an element of it. Since $x \in \mathrm{C}_G(P)$, P centralizes x. This means that $P \leqslant \mathrm{C}_G(x)$. Now a p-subgroup of a group is contained in a Sylow p-subgroup P' of $\mathrm{C}_G(x)$, so $P \leqslant P'$, where P' is a defect group of $\mathscr{C}$. Conversely, if $P \leqslant P'$, where P' is a defect group of $\mathscr{C}$, then $P' \leqslant \mathrm{C}_G(y)$ for some $y \in \mathscr{C}$. Then y centralizes P, so y is in the intersection of $\mathscr{C}$ and $\mathrm{C}_G(P)$. This proves (i).

Now let P be a defect group of $\mathscr{C}$, but suppose that $\mathscr{C}$ splits up in $\mathrm{N}_G(P)$. Let x be an element of $\mathscr{C}$ with defect group P. Then x centralizes P, so $x \in \mathscr{C} \cap \mathrm{C}_G(P)$. Let y be any other element of this set, and let g be an element of G such that $x^g = y$. Now, P centralizes y, so since $y = x^g$, $P^{g^{-1}}$ centralizes $y^{g^{-1}} = x$. Thus both P and $P^{g^{-1}}$ are Sylow p-subgroups of $\mathrm{C}_G(x)$. Then P and $P^{g^{-1}}$ are conjugate in $\mathrm{C}_G(x)$, say $P^h = P^{g^{-1}}$. Then hg normalizes P, and $x^{hg} = g^{-1}h^{-1}xhg = g^{-1}xg = y$, so x and y are conjugate in $\mathrm{N}_G(P)$. This proves (ii).

Finally, let P be a defect group of $\bar{\mathscr{C}}$, and let $x \in \bar{\mathscr{C}}$ be an element with P a Sylow p-subgroup of $\mathrm{C}_G(x)$. Then x centralizes P, giving $x \in \mathscr{C} \cap \mathrm{C}_G(P)$, and in particular this intersection is non-empty. By (i) P is contained within a defect group of $\mathscr{C}$, say P'. Suppose that $P \neq P'$. Then since P' is a p-group and hence nilpotent, $P < \mathrm{N}_{P'}(P)$. So there is an element $d \notin P$ which normalizes P. Then $\langle P, d\rangle$ is a p-group, and since P' centralizes x, so does $\langle P, d\rangle$. Also $\mathrm{N}_{P'}(P) \leqslant \mathrm{N}_G(P)$, so $\langle P, d\rangle \in \mathrm{C}_{\mathrm{N}_G(P)}(x)$. This means that $\langle P, d\rangle$ is contained within a Sylow p-subgroup of $\mathrm{C}_{\mathrm{N}_G(P)}(x)$, a clear contradiction since P is such a subgroup. So $P = P'$. Note that this means that P is a defect group of $\bar{\mathscr{C}}$, so we can apply (ii) to show that $\mathscr{C} \cap \mathrm{C}_G(x) = \bar{\mathscr{C}}$, as required. □

1.2 The First Main Theorem

The aim of this section is to prove Brauer's First Main Theorem, which we can state now.

Theorem 1.4 (First Main Theorem) Suppose that k is a splitting field for the finite group G, and let D be a p-subgroup of G. Then there is a bijection between the blocks of kG with defect group D and the blocks of $k\,\mathrm{N}_G(D)$ with defect group D.

This one-to-one correspondence is called the Brauer Correspondence, and will have to be defined before we prove the First Main Theorem. Before we do this, we need to define the Brauer homomorphism.

Definition 1.5 Let G be a finite group, k be a splitting field for G, and suppose that D is a p-subgroup of G. Let $\sigma : \mathrm{Z}\,(kG) \to \mathrm{Z}\,(k\,\mathrm{N}_G(D))$ be defined by

$$\sigma\left(\sum_{x\in G} \alpha_x x\right) = \sum_{x\in \mathrm{C}_G(D)} \alpha_x x.$$

Then σ is called the *Brauer homomorphism.*

The Brauer homomorphism is a k-algebra homomorphism, as its name suggests. To prove this, we first show that σ actually maps $\mathrm{Z}\,(kG)$ to $\mathrm{Z}\,(k\,\mathrm{N}_G(D))$. Recall that the class sums of the conjugacy classes of G (and $\mathrm{N}_G(D)$) form bases for $\mathrm{Z}\,(kG)$ and $\mathrm{Z}\,(k\,\mathrm{N}_G(D))$ respectively. Let $\mathscr{C}$ be a conjugacy class of G, with class sum c. Denote by $\bar{\mathscr{C}}$ the subset $\mathscr{C} \cap \mathrm{C}_G(D)$, with sum $\bar{c}$. Then $\sigma(c) = \bar{c}$, and so we must show that $\bar{c} \in \mathrm{Z}\,(k\,\mathrm{N}_G(D))$. But $\mathrm{C}_G(D) \trianglelefteqslant \mathrm{N}_G(D)$, and $\mathscr{C}$ is a normal subset of G, so $\bar{\mathscr{C}}$ is a normal subset of $\mathrm{N}_G(D)$, and hence a union of conjugacy classes. Thus $\bar{c} \in \mathrm{Z}\,(k\,\mathrm{N}_G(D))$.

Certainly $\sigma(1) = 1$, and the Brauer homomorphism is a k-linear mapping, almost from the definition. Let c_1 and c_2 be class sums of G. Now $\sigma(c_1c_2) = \sum_{x\in \mathrm{C}_G(D)} \alpha_x x$, where α_x is the number of elements $g_1 \in \mathscr{C}_1$ and $g_2 \in \mathscr{C}_2$ such that $g_1g_2 = x$. However, $\sigma(c_1)\sigma(c_2) = \sum_{x\in \mathrm{C}_G(D)} \beta_x x$, where β_x is the number of elements $g_1 \in \bar{\mathscr{C}}_1$ and $g_2 \in \bar{\mathscr{C}}_2$ such that $g_1g_2 = x$. We need to show that $\alpha_x \equiv \beta_x \mod p$.

Notice that every pair that contributes to β_x contributes to α_x. Let A_x denote the set $\{\,(g_1, g_2) : g_1 \in \mathscr{C}_1,\ g_2 \in \mathscr{C}_2,\ g_1g_2 = x\}$, and B_x denote this set for $\bar{\mathscr{C}}_1$ and $\bar{\mathscr{C}}_2$. Then

$$A_x\backslash B_x = \{\,(g_1, g_2) : g_1 \in \mathscr{C}_1,\ g_2 \in \mathscr{C}_2\backslash\bar{\mathscr{C}}_2,\ g_1g_2 = x\,\}\cup\{\,(g_1, g_2) : g_1 \in \mathscr{C}_1\backslash\bar{\mathscr{C}}_1,\ g_2 \in \bar{\mathscr{C}}_2,\ g_1g_2 = x\,\}.$$

The second term in this union is empty, since if $g_2 \in \bar{\mathscr{C}}_2$, then g_2 (and x) both centralize D. Hence so does g_1, a contradiction since $g_1 \notin \bar{\mathscr{C}}_1$. This also deals with the case where $g_1 \in \bar{\mathscr{C}}_1$ and $g_2 \notin \bar{\mathscr{C}}_2$. This leaves the case where neither g_1 nor g_2 centralize D.

Let D act on the set $A_x \setminus B_x$ by pointwise conjugation. Since $A_x \setminus B_x$ contains only pairs of elements (g_1, g_2) which do not centralize D, no orbit of this action is trivial, and since all orbits are of size a power of p, p divides $|A_x \setminus B_x|$. Then $\alpha_x = \beta_x$, since our field has characteristic p, and the Brauer homomorphism is indeed a homomorphism.

We also need to define the correspondence between the blocks that the First Main Theorem tells us exists. This will be done with the help of central characters. Since a block has a unique central character associated with it, if we can produce a bijection between the central characters whose associated block has defect group D, then we have a 'corresponding' bijection between the blocks themselves. We may as well say that the central character has defect group D to mean that the block associated with the central character has defect group D. Then the alternative formulation we will prove is:

There is a one-to-one correspondence between the central characters of $\mathrm{N}_G(D)$ with defect group D and the central characters of G with defect group D.

To this end, we will have to find a function sending a central character of $\mathrm{N}_G(D)$ to a central character of G. Let H be a subgroup of G, and b be a block of the algebra kH, with associated character ω. Suppose that the defect group of b contains a particular p-subgroup D of G. Now, by Lemma 1.1, there is a conjugacy class $\mathscr{C}$ of G whose defect group contains that of b, and hence contains D. Thus by Lemma 1.3(i), $\bar{\mathscr{C}} = \mathscr{C} \cap \mathrm{C}_G(D)$ is non-empty. Then we can define a mapping ω^G on the k-algebra $\mathrm{Z}\,(kG)$ by

$$\omega^G(c_i) = \omega(\bar{c_i}),$$

where $\bar{\mathscr{C}}_i$ is the intersection $\mathscr{C}_i \cap \mathrm{C}_G(D)$. Notice that $\omega^G(c) \neq 0$ if and only if $\mathscr{C} \cap \mathrm{C}_G(D) \neq 0$, and so $\omega^G(c) \neq 0$ if and only if D is contained in a defect group of $\mathscr{C}$.

However, ω^G need not be a central character of G. If H contains $D\,\mathrm{C}_G(D)$ and is contained within $\mathrm{N}_G(D)$, then the function ω^G *is* a central character. We will call this the *induced central character* of G from H. We will prove this assertion now.

Theorem 1.6 Suppose that G is a finite group, and let D be a p-subgroup. Suppose that H is a subgroup of G such that $D\,\mathrm{C}_G(D) \leqslant H \leqslant \mathrm{N}_G(D)$. Let ω be a central character of kH, associated to the block b. Then $\omega^G = \omega \circ \sigma$, and is a central character of G, associated to the block B. Furthermore, both b and B have defect groups containing D.

Proof: We will first show that $\omega^G = \omega \circ \sigma$. From this, and the fact that both ω and σ are k-algebra homomorphisms, we see that ω^G is a central character of G. Since $\mathrm{C}_G(D) \trianglelefteq H$, for each conjugacy class $\mathscr{C}$ of H, either $\mathscr{C}$ is contained within $\mathrm{C}_G(D)$ or they are disjoint. Clearly then from the definition of ω^G, $\omega^G = \omega \circ \sigma$. So ω^G is a central character of kG.

Since $D \trianglelefteq H$, D is a subgroup of every defect group of every block b of kH, and so b has a defect group containing D. Let B be the block associated with ω^G. Thus there exists a class of p-regular elements $\mathscr{C}$ (with class sum c) with $\omega(c) \neq 0$ by Lemma 1.1. Then $\omega(\sigma(c)) \neq 0$, and so $\sigma(c) \neq 0$. Thus there is some conjugacy class $\bar{C}$ of H such that $\omega(\bar{\mathscr{C}}) \neq 0$, and this conjugacy class is contained within $\mathscr{C}$. Since $D \trianglelefteq G$, D is contained within every defect group of $\bar{\mathscr{C}}$, and so $D \leqslant \mathrm{C}_H(x)$ for all $x \in \bar{\mathscr{C}} \subseteq \mathscr{C}$. Thus there is some $x \in \mathscr{C}$ such that $D \leqslant \mathrm{C}_H(x) \leqslant \mathrm{C}_G(x)$. Therefore D is contained in some defect group of B. $\square$

If $D\,\mathrm{C}_G(D) \leqslant H \leqslant \mathrm{N}_G(D)$, then we can create a function between the blocks of kG with defect group containing D and the blocks of kH with defect group containing D by the map between central characters with this property. It is this function which is called the *Brauer Correspondence.*

The Brauer Correspondence will provide the bijection that we need to prove the First Main Theorem. However, for the correspondence to be bijective, we need to set $H = \mathrm{N}_G(D)$. We will prove the theorem in stages, requiring several lemmas.

Firstly suppose that D is a defect group of the conjugacy class $\bar{\mathscr{C}}$ in $\mathrm{N}_G(D)$. We know from Lemma 1.3(iii) that D is a defect group of the conjugacy class $\mathscr{C}$ of G containing $\bar{\mathscr{C}}$, and that $\bar{\mathscr{C}} = \mathscr{C} \cap \mathrm{C}_G(D)$. Since stating Lemma 1.3 we have defined the Brauer homomorphism, we can see that if c is the class sum of $\mathscr{C}$, and $\bar{c}$ is the class sum of $\bar{\mathscr{C}}$, then

$$\sigma(c) = \sum_{x \in \mathscr{C} \cap \mathrm{C}_G(D)} x = \bar{c},$$

so $\sigma(c) = \bar{c}$.

We have all of the information we need to show that the Brauer Correspondence that we defined previously actually maps blocks of $kN_G(D)$ with defect group D to blocks of kG also with defect group D. We knew before that it mapped them to blocks with defect group containing D, but this is the extra refinement that we need.

Lemma 1.7 The Brauer Correspondence maps blocks of kG with defect group D to blocks of $k\,\mathrm{N}_G(D)$ with defect group D, and vice versa.

Proof: Let Φ define the function $\omega \mapsto \omega^G$, where ω is a central character of $k\,\mathrm{N}_G(D)$. We must show that a central character of kG with defect group D is mapped to a central character of $k\,\mathrm{N}_G(D)$

with defect group D, and vice versa. Suppose that ω_N is such a central character of $k\,\mathrm{N}_G(D)$, and that ω_G is such a central character of kG.

We suppose that $\Phi(\omega) = \omega^G$ has defect group D. Then by Lemma 1.1 there is a conjugacy class $\mathscr{C}$ of G (with class sum c) such that $\mathscr{C}$ has defect group D and $\omega^G(c) \neq 0$. By Lemma 1.3(ii), $\bar{\mathscr{C}} = \mathscr{C} \cap \mathrm{C}_G(D)$ is a conjugacy class of $\mathrm{N}_G(D)$. The defect of $\bar{\mathscr{C}}$ is equal to that of $\mathscr{C}$ by Lemma 1.3(iii). Also, $\omega(\bar{c}) \neq 0$ since $\sigma(c) = \bar{c}$, so by Lemma 1.1 again, the block b of $k\,\mathrm{N}_G(D)$ associated with ω has defect group a subgroup of that of $\bar{\mathscr{C}}$. But a defect group of b must contain D, since $D \trianglelefteq \mathrm{N}_G(D)$. Hence D *is* a defect group of b. So if ω^G has defect group D, ω also has defect group D.

Now suppose that ω has defect group D. Then, using Lemma 1.1 yet again, we find a conjugacy class $\bar{\mathscr{C}}$ of $\mathrm{N}_G(D)$ with class sum $\bar{c}$ such that $\omega(\bar{c}) \neq 0$, and with D as a defect group. Denote by $\mathscr{C}$ the conjugacy class of G which contains $\bar{\mathscr{C}}$. Then by Lemma 1.3(iii) D is a defect group of $\mathscr{C}$, and $\bar{\mathscr{C}} = \mathscr{C} \cap \mathrm{C}_G(D)$. So if c denotes the class sum of $\mathscr{C}$, $\sigma(c) = \bar{c}$. Then

$$\omega^G(c) = \omega(\sigma(c)) = \omega(\bar{c}) \neq 0.$$

Again, we can use Lemma 1.1 to show that D contains every defect group of ω^G. Now D was a defect group of ω, so since the Brauer Correspondence maps central characters with defect groups containing D onto central characters with defect groups containing D, any defect group of ω^G must also contain D. Hence D is a defect group of ω^G.

We have therefore proved that the Brauer Correspondence maps central characters with defect group D onto central characters with defect group D, and vice versa. □

We now have to show that the function Φ is bijective. We prove that Φ is one-to-one first. Recall the k-subspace I from Lemma 1.2 – we will use this in the next proof.

Lemma 1.8 Let $\Phi : \omega \mapsto \omega^G$ be as before. Then Φ is injective.

Proof: Suppose that $\omega_1^G = \omega_2^G$, but ω_1 and ω_2 are distinct. Let e denote the block idempotent of b_1 (where b_1 is associated to ω_1). Then $\omega_1(e) = 1$ and $\omega_2(e) = 0$. Now Lemma 1.2 shows that e is an element of the subspace of $k\,\mathrm{N}_G(D)$ generated by the conjugacy classes of $\mathrm{N}_G(D)$ with D as defect group. By the discussion before Lemma 1.7, I is contained within the image of σ, and so we can find i such that $\sigma(i) = e$. But then $\omega_1(e) = \omega_1(\sigma(i)) = \omega_1^G(i)$, and similarly $\omega_2(e) = \omega_2^G(i)$. These are equal since $\omega_1^G = \omega_2^G$, but these are unequal since $\omega_2(e) \neq \omega_2(e)$, a contradiction. □

Finally we show that Φ is surjective.

Lemma 1.9 Let $\Phi : \omega \mapsto \omega^G$ be as before. Then Φ is surjective.

Proof: Let B be a block of kG with defect group D. We need to show that there is a block b of $k\,\mathrm{N}_G(D)$ with defect group D such that $b^G = B$. In fact, we do not need to know that b has defect group D, since this is implied from Lemma 1.7. So we only need to find a central character ω of $k\,\mathrm{N}_G(D)$ such that ω^G is the central character associated to B.

Let e be the block idempotent of B. Then $\sigma(e)$ is an idempotent of $k\,\mathrm{N}_G(D)$. So we can divide the block idempotents of $f_1, \dots, f_n$ of $k\,\mathrm{N}_G(D)$ into two, those which feature in a decomposition of $\sigma(e)$ into block idempotents and those which do not. Suppose that f_i features in such a (unique) decomposition, and let ω be the central character of the block b which has associated block idempotent f_i. Then ω^G is a central character of G, and $\omega^G(e) = (\omega \circ \sigma)(e)$, which becomes $\omega(\sum f_i) = \sum \omega(f_i) = 1$, by the definition of ω. So ω^G is the central character associated to the block B, and we have found b, a block of $k\,\mathrm{N}_G(D)$ such that $b^G = B$, as required. $\square$

We have finally proved Brauer's First Main Theorem. This proof is based on that of [79], itself based on [92]. Brauer's original proof is in [16] – he offers an alternative proof in [18] and [19]. Other proofs include [75] and [94].

To close this section, we notice that if H is any subgroup of G containing $\mathrm{N}_G(D)$, then $\mathrm{N}_H(D) = \mathrm{N}_G(D)$. So we can define the Brauer Correspondence between both G and $\mathrm{N}_G(D)$, and between H and $\mathrm{N}_H(D) = \mathrm{N}_G(D)$. Thus we have the following corollary to Theorem 1.4:

Corollary 1.10 Suppose that G is a finite group, D is a p-subgroup of G, and that k is a splitting field for G of characteristic p. Then for any subgroup H of G containing $\mathrm{N}_G(D)$, there exists a one-to-one correspondence between the blocks of kG with D as defect group, and the blocks of kH with D as defect group.

1.3 Extended First Main Theorem

We have defined the Brauer Correspondence down all the way to $D\,\mathrm{C}_G(D)$, although we have only shown that the correspondence is bijective down to $\mathrm{N}_G(D)$. This is because it fails to be bijective below this. However, considerable information can still be gleaned, since although the correspondence is not bijective, it is surjective, and bijective modulo a natural-looking equivalence relation, that of two blocks being conjugate.

Definition 1.11 Let B and B' be blocks of a group algebra kH, where $H \trianglelefteq G$. Then B and B' are *G-conjugate* if there is an element $g \in G$ such that $e^g = e'$, where e and e' are the block idempotents of B and B' respectively.

Suppose that χ is an ordinary character lying in the block B. Suppose that B is G-conjugate to another block B', say $e' = g^{-1}eg$, where e and e' are the block idempotents for B and B'. Consider the character χ' obtained from χ by the action $\chi'(x) = \chi(g^{-1}xg)$. If χ is afforded by the module M, then χ' is afforded by the module M^g, where M^g is the conjugate module with action defined by $mx = m(gxg^{-1})$. Now M lies inside B, and so $e \notin \ker M$. Therefore $M^g(g^{-1}eg)$ is non-zero, since Me is non-zero. This means that M^g lies in the block B', and so conjugate modules lie in conjugate blocks. In particular, we have also shown that e^g lies in B^g.

Now consider the block B which contains the trivial character χ_1. So χ_1^g lies in B^g. But $\chi_1^g = \chi_1$, since $\chi_1(x) = 1$ for all x, and so the B is not G-conjugate to any other block.

We will now consider the Brauer Correspondence below $\mathrm{N}_G(D)$.

Theorem 1.12 (Extended First Main Theorem) Let D be a p-subgroup, and suppose that $D\,\mathrm{C}_G(D) \leqslant H \leqslant \mathrm{N}_G(D)$. Then the Brauer Correspondence maps blocks with defect group containing D of kH to blocks of defect group containing D of kG, this map is surjective, and if $b_1, b_2 \in kH$, then $b_1^G = b_2^G$ if and only if b_1 and b_2 are G-conjugate.

In the same way as we proved the First Main Theorem, this proof will go in stages. Notice that Theorem 1.6 has proven the statement that the Brauer Correspondence maps blocks with defect group containing D to blocks of defect group containing D.

Lemma 1.13 Suppose that D is a normal p-subgroup of G, and let H be a subgroup of G containing $D\,\mathrm{C}_G(D)$. If B_1 and B_2 are blocks of kH with defect group D, then B_1 and B_2 are G-conjugate if and only if B_1 and B_2 have the same Brauer correspondent in kG.

Proof: Firstly suppose that B_1 and B_2 are G-conjugate. Let e_i be the block idempotent of B_i, and notice that $e_1^g = e_2$ for some $g \in G$. Write f for e_1^G, a central idempotent of kG. Since f is central in kG, $f^g = f$. Let ω denote the central character of B_1. Then $\sigma(f) = e_1 + \cdots + e_n$, where the e_i are central idempotents of kH.

$$
\begin{aligned}
1 = \omega^G(f) &= \omega^G(f^g) \\
&= \omega \circ \sigma(f^g) \\
&= \omega\left(\sum_{i=1}^{n} e_i^g\right) \\
&= \sum_{i=1}^{n} \omega(e_i^g).
\end{aligned}
$$

This means that $\omega(e_i^g) = 1$ for some i. So $e_2^G = (e_1^g)^G = e_1^G$, since their induced central characters are the same. We can see this because a central character is determined by when it takes 1, and we have shown that the central characters of both e_1^G and e_2^G take 1 on f. Therefore $B_1^G = B_2^G$, as required.

Now suppose that $B_1^G = B_2^G$. If ω_1 and ω_2 denote the central characters of B_1 and B_2, we know that $\omega_1^G = \omega_2^G$. But $\omega_i^G = \omega_i \circ \sigma$. This means that $\omega_1\left(\sum \alpha_x x\right)$ and $\omega_2\left(\sum \alpha_x x\right)$ (for $x \in \mathrm{C}_G(D)$) take the same value.

Now the sum of block idempotents $e_1, \ldots, e_n$ that are G-conjugate to e_1 (the block idempotent of B_1) lies in the centre of both kG and kH. Suppose that ω_1 and ω_2 agree on this region. If c

denotes the sum of the G-conjugate block idempotents,

$$\omega_i^G(c) = \omega_i \left(\sum_{j=1}^{n} e_j \right),$$

and so $\omega_1(e_i) = 1$ and $\omega_2(e_j) = 1$ for some i, j. Hence B_1 and B_2 are G-conjugate.

It remains to show that ω_1 and ω_2 agree on this region. The image of σ is $\mathrm{Z}(kG) \cap k\,\mathrm{C}_G(D)$, and so certainly ω_1 and ω_2 agree on this region. If $\mathscr{C}$ is a conjugacy class with a class sum not in $k\,\mathrm{C}_G(D)$, it must intersect $\mathrm{C}_G(D)$ trivially, since $\mathrm{C}_G(D)$ is normal in G. But since D is contained in every defect group (since $D \trianglelefteq G$), Lemmas 1.3(i) and 1.1 imply that both central characters vanish on this class sum. We have proved that ω_1 and ω_2 agree on $\mathrm{Z}(kH) \setminus \mathrm{C}_G(D)$. Then ω_1 and ω_2 agree on

$$(\mathrm{Z}(kG) \cap k\,\mathrm{C}_G(D)) \cup (\mathrm{Z}(kH) \setminus k\,\mathrm{C}_G(D)) = \mathrm{Z}(kG) \cap \mathrm{Z}(kH),$$

as required. □

This proves the first half of Theorem 1.12, since we have a bijective correspondence from G down to $\mathrm{N}_G(D)$, and thanks to this lemma, a correspondence between $\mathrm{N}_G(D)$ (in which D is, of course, normal) and H. The second part of Theorem 1.12 requires the map to be surjective. We will demonstrate this now.

Lemma 1.14 In the situation of the Extended First Main Theorem, the Brauer Correspondence defined is surjective.

Proof: Let B be a block of kG, with block idempotent e. Then $\sigma(e)$ is a central idempotent of kH, and so splits up as a sum $\sum e_i$ of primitive idempotents, with associated blocks b_i. Now if ω is the central character associated with e_j, then $\omega(e_j) = 1$. However

$$\omega^G(e) = \omega \circ \sigma(e) = \sum \omega(e_j) = 1,$$

and so ω^G is associated with the block B, and so the Brauer Correspondence is surjective, as required. □

In proving this, we actually showed that every factor of $\sigma(e)$ corresponded to B. By Lemma 1.13, this means that all factors of $\sigma(e)$ are G-conjugate. In fact, the factors of $\sigma(e)$ are the only Brauer correspondents of B. A proof of this fact is given in [79]; we do not need this fact here. Indeed, we need only the one implication of Lemma 1.13 in the next section.

Finally, remember that we proved that the block B containing the trivial character is not G-conjugate to any other block. Together with the Extended First Main Theorem, this shows that B is the unique block that corresponds to B^G. This will be necessary in the next section.

1.4 The Third Main Theorem

Definition 1.15 The *principal block* of the group algebra kG is the block containing the trivial character 1_G. We often denote it by $B_0(G)$ or $b_0(G)$.

Given this definition, we can now state Brauer's Third Main Theorem.

Theorem 1.16 (Brauer's Third Main Theorem) Let G be a finite group, and let D be a p-subgroup of G. Let H be a subgroup with $D\,\mathrm{C}_G(D) \leqslant H$. Let B be a block of kH with defect group D. Then B is the principal block of kH if and only if B^G is the principal block of kG.

We will prove this result in several stages, the first being to prove one of the implications in the case where $D \trianglelefteq H$.

Lemma 1.17 Suppose that H is a subgroup of G, with $D\,\mathrm{C}_G(D) \leqslant H \leqslant \mathrm{N}_G(D)$. Let B be a block of kH with defect group D. If B is the principal block of kH, B^G is the principal block of kG.

Proof: Suppose that B is the principal block of kH. Let ω denote the central character associated with B. Recall that if χ is an irreducible character of B, and if $\bar{c}$ is a class sum of the conjugacy class $\bar{\mathscr{C}}$ for which $\omega(\bar{c}) \neq 0$, then

$$\omega(\bar{c}) = |\bar{\mathscr{C}}|\frac{\chi(x)}{\chi(1)},$$

where $x \in \bar{\mathscr{C}}$. Applying this to $\chi = 1_G$, the trivial character, gives $\omega(\bar{c}) = |\bar{\mathscr{C}}|$ for all conjugacy classes for which $\omega(\bar{c}) \neq 0$. Consider the induced central character ω^G, and let $\mathscr{C}$ be a conjugacy class of G, with class sum c. Then if $\omega^G(c) \neq 0$,

$$\omega^G(c) = \omega(\sigma(c)) = |\mathscr{C} \cap \mathrm{C}_G(D)|,$$

since $\omega(\bar{c}) = |\bar{\mathscr{C}}|$. Let D act on $\mathscr{C}$ by conjugation. Then the fixed points of this action are those elements of $\mathscr{C}$ centralized by all of D; i.e., those elements of $\mathscr{C}$ who centralize D themselves. So, since D is a p-group acting on a set, we have $|\operatorname{Fix}_G(X)| \equiv |X| \mod p$. In this case,

$$|\mathscr{C} \cap \mathrm{C}_G(D)| \equiv |\mathscr{C}| \mod p.$$

Since k is of characteristic p, this means that $\omega^G(c) = \omega(\bar{c})$ for all conjugacy classes of G for which $\omega^G(c) \neq 0$. Thus $\omega^G(c) = |\mathscr{C}|$. Using the fact that $\omega^G(c) = |\mathscr{C}|\chi(x)/\chi(1)$, we see that χ lies in the same block as the trivial character; i.e., the principal block. So ω^G is associated to the principal block, as required. $\square$

However, proving the other way is harder, and requires the following lemma before the proof itself.

Lemma 1.18 Suppose that $D\,\mathrm{C}_G(D) \leqslant H \leqslant L \leqslant \mathrm{N}_G(D)$, and that B is a block of kH with defect group D. Then B^L is the principal block of kL if and only if B is the principal block of kH.

Proof: The one way implication is precisely that of Lemma 1.17. Since $L \leqslant \mathrm{N}_G(D)$, we have $D \trianglelefteq L$. Also, $D\,\mathrm{C}_G(D) \trianglelefteq L$. Let b_0 denote the principal block of $kD\,\mathrm{C}_G(D)$. Since B^L is the principal block of kL, we can use Lemma 1.17 to get $b_0^L = B^L$. Since the Brauer Correspondence is onto, there is a block b of $kD\,\mathrm{C}_G(D)$ such that $b^H = B$. Then by the transitivity of the Brauer Correspondence, $b^L = (b^H)^L = B^L$. So b corresponds with the principal block of kL. But in the discussion of the extension of Brauer's First Main Theorem, we said that the principal block was in its own equivalence class. Thus $b = b_0$ is the principal block itself. Then we can apply Lemma 1.17 again to b, which is the principal block of $kD\,\mathrm{C}_G(D)$, to find that $b^H = B$ is the principal block of kH. $\square$

Proof of Theorem 1.16: Now let G be a counterexample to Brauer's Third Main Theorem, so there exists a p-subgroup D of G, a subgroup H of G containing $D\,\mathrm{C}_G(D)$, a block B of kH with defect group D such that B^G is the principal block of kG but B is not the principal block of kH. Choose D to be of maximal order inside G subject to this system (G, D, H, B) being a counterexample.

Firstly notice that $D \ntrianglelefteq G$, else Lemma 1.18 would apply. This proof splits up into the cases where H contains $\mathrm{N}_G(D)$ and when it does not. Firstly suppose that $H \leqslant \mathrm{N}_G(D)$. Let $N = \mathrm{N}_G(D)$. B^N has defect group Q, say. Since Q is a defect group of N, $Q \leqslant N$. Also, any element of G that centralizes Q centralizes D, so $\mathrm{C}_G(Q) \leqslant \mathrm{C}_G(D) \leqslant N$. Thus $Q\,\mathrm{C}_G(Q) \leqslant N$. Also $(B^N)^G = B^G$, the principal block of kG. So either B^N is the principal block of kN, or we have another counterexample to Brauer's Third Main Theorem, namely $(G, Q, \mathrm{N}_G(D), B^N)$. Since D was chosen to be maximal, and $D \leqslant Q$, we must have $D = Q$. So B^N has defect group D.

By the First Main Theorem, there is a block B' of kG such that $(B^N)^G = B'$, and had D as a defect group. But $(B^N)^G = B^G$, and so is the principal block of kG. Thus D is a Sylow p-subgroup of G, and so is a Sylow p-subgroup of every subgroup of G containing it. Thus D is a defect group of the principal block b of kN, and so by Lemma 1.17, $b^G = B^G$, so $(B^N)^G = b^G$, and by the First Main Theorem, $B^N = b$. But then Lemma 1.18 shows that B is the principal block of kH, since b is the principal block of kN.

Now suppose that H properly contains $\mathrm{N}_G(D)$, and that B is a block of kH with defect group D. We also have that B^G is the principal block of kG, but B is not the principal block of kH. But the Brauer Correspondence gives a bijection between the blocks of kH of defect group D to those of kG of defect group D. Since B has defect group D, so must B^G, the principal block. This means that D is actually a Sylow p-subgroup of G, and so also a Sylow p-subgroup of H. But then if b is the principal block of kH, then b has defect group D (since D is a Sylow p-subgroup, all Sylow p-subgroups are conjugate, and the defect groups of a block form a conjugacy class of subgroups of G), so by Lemma 1.17, $b^G = B^G$, the principal block. However, Brauer's First Main Theorem quickly shows that b is equal to B, so that B is the principal block of kH, as required. $\square$

1.5 The Second Main Theorem

In the introduction to this chapter we mentioned generalized decomposition numbers, which we will now proceed to define. As we have said in the preface, suppose that R is a complete discrete valuation ring with maximal ideal $J(R)$, such that $R/J(R)$ is a finite field of characteristic p with enough roots of unity for our purposes. Let K be a field of characteristic zero with R as its ring of algebraic integers (so that K is its field of fractions and R is integrally closed in K).

We know that there exist free RG-modules $V_1, \ldots, V_l$ such that every irreducible KG-module is of the form $V_i \otimes_R K$ for precisely one of the V_i. Reduce modulo p the modules V_i, and denote these by $\bar{V}_i$. Let χ^i be the ordinary character afforded by $V_i \otimes_R K$. Then the function

$$\phi^i(x) = \begin{cases} \chi^i(x), & x \text{ is } p\text{-regular} \\ 0, & \text{otherwise} \end{cases}$$

is the modular character afforded by $\bar{V}_i$. Then the composition factors of $\bar{V}_i$ are uniquely determined by ϕ^i. Recall that the irreducible kG-modules are the quotients $W_j/J(W_j)$, where the W_j are the indecomposable submodules of kG, (which are uniquely determined up to ordering by the Krull–Schmidt Theorem). Then the number of times $W_j/J(W_j)$ appears in the decomposition of $\bar{V}_i$ is uniquely determined. We denote this by d_{ij}, and call it the *decomposition number*. The matrix D_{ij} is called the decomposition matrix. (Also important is the fact that if C_{ij} denotes the Cartan matrix, then $D_{ij}^T D_{ij} = C_{ij}$.)

The number l is the number of conjugacy classes of G, and there are m, say, of the W_j, where m is the number of conjugacy classes of p'-elements of G.

Now let t denote a p-element of G, and let $C = \mathrm{C}_G(t)$. Denote by $\chi^1, \ldots, \chi^r$ and $\psi^1, \ldots, \psi^s$ the irreducible ordinary and modular characters of C. Suppose that $\zeta^1, \ldots, \zeta^l$ are the ordinary characters of G. Then the restriction ζ^i_H can be written as a linear combination of the ordinary characters of H. But each of the ordinary characters can be written as

$$\chi^i(x) = \sum_{j=1}^{s} d_{ij}\psi^j.$$

Furthermore, since every element of C centralizes t, t is a central element of C, and so $\chi^i(t) = \mu_i$, where μ_i is a root of unity, and so is algebraic. Thus we can write

$$\chi^i(tx) = \mu_i\chi^i(x) = \sum_{j=1}^{s}(\mu_i d_{ij})\psi^j(x).$$

Now consider $\zeta^i(tx)$, which can be written as an integral linear combination of the $\chi^{i'}(tx)$, and each of these can be written as an integral linear combination of the ψ^j. Then we can write

$$\zeta^i(tx) = \sum_{j=0}^{s} d^t_{ij}\psi^j(x),$$

and the d^t_{ij}, the *generalized decomposition numbers*, are algebraic.

Given this theory, we can now state Brauer's Second Main Theorem.

Theorem 1.19 (Brauer's Second Main Theorem) Given the notation introduced in the above discussion, suppose that $d_{ij}^t \neq 0$ for some p-singular t, and $C = \mathrm{C}_G(t)$. If ζ^i lies in the block B of kG, and ψ^j lies in the block b of kC, then $b^G = B$.

The proof of this statement will not be given here, and will be delayed until Chapter 2. The reason is that although the First and Third Main Theorems are proven without much reference to modules, this theorem really is best done using the modules themselves. In particular, we will use the Nagao Decomposition, which is described in Chapter 2.

1.6 Brauer Correspondence in S_7

Denote by G the group S_7, which we consider acting on the set $\{1, \dots, 7\}$. Consider the blocks of G. There are fifteen characters of G, corresponding to the fifteen conjugacy classes of G. These are given in the table below. In this table, the conjugacy classes are labelled in Atlas notation, the classes being labelled as nX, where n refers to the order of the elements, and X is an indexer. Where there is some ambiguity, 2A refers to (1 2)(3 4), 2B refers to (1 2), 2C to (1 2)(3 4)(5 6), 3A to (1 2 3), 3B to (1 2 3)(4 5 6), 4A to (1 2 3 4)(5 6), 4B to (1 2 3 4), 6A to (1 2)(3 4)(5 6 7), 6B to (1 2 3)(4 5), and 6C to (1 2 3 4 5 6). We list the characters up to multiplication by χ_a, the alternating character. All characters except χ_3 can be multiplied by χ_a to give another irreducible character.

	1A	2A	3A	3B	4A	5A	6A	7A	2B	2C	4B	6B	6C	10A	12A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_a	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
χ_2	6	2	3	0	0	1	-1	-1	4	0	2	1	0	1	-1
χ_3	20	-4	2	2	0	0	2	-1	0	0	0	0	0	0	0
χ_5	14	2	2	-1	0	-1	2	0	6	2	0	0	-1	1	0
χ_6	14	2	-1	2	0	-1	-1	0	4	0	-2	1	0	-1	1
χ_7	15	-1	3	0	-1	0	-1	1	5	-3	1	-1	0	1	-1
χ_8	21	1	-3	0	-1	1	1	0	1	-3	-1	1	0	1	-1
χ_9	35	-1	-1	-1	1	0	-1	0	5	1	-1	-1	1	0	-1

The reason for the lack of χ_4 and for the strange ordering of the conjugacy classes is that this also serves as a character table for A_7 as well. In A_7, the character χ_3 splits as the sum of two irreducible characters, called χ_3 and χ_4, and the class 7A splits into two as well. All of the classes up to 7A are the classes of A_7, and so the restriction of this table to those classes serves as an almost complete character table (since the values of the now-split 7B and the now-split χ_3 and χ_4, although not given, are deducible).

From this character table we can see that there are no blocks of defect zero for both primes 2 and 3, since there are no characters with degrees multiples of 16 or 9. This means that we do indeed have some non-trivial block theory for 2 and 3.

Using the formula $\chi_i(\mathscr{C})|\mathscr{C}|/\chi_i(1) \equiv \chi_j(\mathscr{C})|\mathscr{C}|/\chi_j(1) \mod p$ if and only if χ_1 and χ_j are in the same block, and the fact that we cannot have a block with no characters at all in it, we find that for the prime 2 there are two blocks, and for the prime 3 there are three blocks.

For the prime 2, we have two blocks, say $B_0(G)$ and $B_1(G)$, where $B_0(G)$ is the principal 2-block. The block $B_0(G)$ contains the characters χ_1, χ_5, χ_7, χ_8, and χ_9, together with their products with χ_a. The block $B_1(G)$ contains the remaining five characters χ_2, χ_3, χ_6, together with their products with χ_a (which in the case of χ_3 is just χ_3 again).

For the prime 3, we have three blocks, say $b_0(G)$, $b_1(G)$ and $b_2(G)$, where $b_0(G)$ is the principal 3-block. The block $b_0(G)$ contains the characters χ_1, χ_3, χ_5, χ_6 and χ_9, together with their products with χ_a. In $b_1(G)$ we have the characters χ_2, χ_7 and χ_8, with their products by χ_a in the block $b_2(G)$. The reason that we now have to split these two is that the multiplication by χ_a actually alters the sign of the central character, because we are not working over a characteristic 2 field.

We would like to know the defects of the blocks involved. Certainly $B_0(G)$, the principal 2-block of G, has full defect, and so has defect groups the Sylow 2-subgroups of G. The six characters of $B_1(G)$ each have degree a multiple of 2, and four do not have degree a multiple of 4. This means that $B_1(G)$ has defect 3.

In the case $p = 3$, the principal 3-block of G has full defect, and so has defect groups the Sylow 3-subgroups of G. The three characters in $b_1(G)$ and the three in $b_2(G)$ all have degrees multiples of 3, and so both $b_1(G)$ and $b_2(G)$ have defect 1. This obviously means that they have defect group C_3.

Firstly, consider the 2-subgroups of G. Without loss of generality, we can restrict our attention to subgroups of a particular Sylow 2-subgroup, which we construct now. Note that

$$|S_7| = 2^4 \cdot 3^2 \cdot 5 \cdot 7.$$

If we can find a subgroup of G of order 16, we have found our Sylow 2-subgroup. Certainly we can find a group isomorphic with C_2^3 by taking the three involutions (1 2), (3 4) and (5 6). We can act on this set by permuting two of these transpositions, say (1 2) and (3 4). So we can adjoin the element (1 3 2 4), to get the p-subgroup P.

We know that P is the Sylow 2-subgroup generated by (1 2), (1 3 2 4) and (5 6). Then the maximal subgroups of this group are $\langle(1\ 2),(1\ 3\ 2\ 4)\rangle \cong D_8$, $\langle(1\ 3\ 2\ 4),(5\ 6)\rangle \cong C_4 \times C_2$, $\langle(1\ 2),(3\ 4),(5\ 6)\rangle \cong C_2^3$ and $\langle(1\ 4)(2\ 3),(1\ 2)(3\ 4),(5\ 6)\rangle \cong C_2^3$.

We have found all of the possible maximal subgroups of P, and the defect group of $B_1(G)$ must be one of these. We now state a useful proposition. This was originally proved in [51], although the proof we give here is due to Thompson (see [96]).

Proposition 1.20 Let D be a defect group, contained in a Sylow p-subgroup P. Then $D = P \cap P^g$ for some $g \in G$.

Proof: First we consider the case where a block B has defect zero. Then B contains an irreducible kG-module, M say, that is projective. Let P denote a Sylow p-subgroup of G. Then M_P is free, and for each element of $P \setminus \{1\}$, M_P contains all of the elements of M fixed by it. Choose $0 \neq v \in M$ such that $vx = v$ for all $x \in P$. Since M is irreducible, there is some $h \in G$ such that $vh \notin M_P$. Suppose that $g \in P \cap h^{-1}Ph$. Then, for some $y \in G$, $g = h^{-1}yh$, and

$$vhg = vh(h^{-1}yh) = vyh = xh,$$

which is clearly fixed by g. But $vh \notin M_P$, a contradiction to the fact that M_P contains all of the elements of M fixed by the element $g \in P$. Then $g = 1$, and $P \cap h^{-1}Ph = 1$, as we need.

In the general case, let B be a block with defect group D, and let $N = \mathrm{N}_G(D)$. By Brauer's First Main Theorem, there is a block b in $\mathrm{N}_G(D)$ with defect D, and so this block's correspondent in $\mathrm{N}_G(D)/D$ has defect zero. So we have already proved that there exist p-subgroups P_1 and P_2 of N such that $P_1/D \cap P_2/D = 1$, and so $P_1 \cap P_2 = D$. If Q_i denotes a Sylow p-subgroup of G containing P_i, then $Q_i \cap N = P_i$. Suppose that this is false, so that $Q_1 \cap Q_2 > D$. Since p-groups are nilpotent, $\mathrm{N}_{Q_1 \cap Q_2}(D) > D$, and so $Q_1 \cap Q_2 \cap N > D$. But

$$Q_1 \cap Q_2 \cap N = (Q_1 \cap N) \cap (Q_2 \cap N) = P_1 \cap P_2 = D,$$

a clear contradiction. Then the result is proven in the general case. $\square$

In fact, this result can be strengthened somewhat. Alperin [2] has shown that D is, in fact, the tame intersection of two Sylow p-subgroups; that is, $D = P \cap Q$, where $\mathrm{N}_Q(D)$ and $\mathrm{N}_P(D)$ are Sylow p-subgroups of $\mathrm{N}_G(D)$. This follows from a theorem of Green [53], where he shows that g in the above proposition can be taken to lie inside $\mathrm{C}_G(D)$.

We can use these results to restrict the possible choices of defect group. However, in this case it is not much help. This is because both $\langle (1\ 2), (1\ 3\ 2\ 4) \rangle \cong D_8$ and $\langle (1\ 2), (3\ 4), (5\ 6) \rangle \cong C_2^3$ are tame intersections. So we need another way to calculate the defect group of $B_1(G)$.

We construct the central character of $B_1(G)$ from an irreducible character in $B_1(G)$, say χ_2, determining the conjugacy classes whose class sum does not vanish under ω. Then we simply find the Sylow 2-subgroups of $\mathrm{C}_G(x)$ ($x \in \mathscr{C}$) for these particular conjugacy classes, and pick the smallest.

Let $\mathscr{C}$ be a conjugacy class with class sum c. Then $\omega(c) = |\mathscr{C}|\chi_i(\mathscr{C})/\chi_i(1)$ is the definition of the central character. In this case, we consider χ_2 and the conjugacy class 3A. Then $|\mathscr{C}| = 70$, and

$$\omega(c) = 70\frac{3}{6} = 35 \not\equiv 0 \mod 2.$$

So the defect group of B_1 is contained within a Sylow 2-subgroup of $\mathrm{C}_G(x)$. But if $x \in \mathscr{C}$, then $\mathrm{C}_G(x) \cong S_4 \times C_3$, and a Sylow 2-subgroup of this is isomorphic with D_8. Therefore a defect group of B_1 is isomorphic with D_8.

Now consider $\mathrm{N}_G(D_8)$, where this D_8 is equal to the one determined above. Then certainly $D_8 \times S_3$ normalizes D_8. This turns out to be equal to the normalizer. In this case, $P \leqslant \mathrm{N}_G(D_8) = N$, and so the Brauer Correspondence is defined between kG and kN. Furthermore, it is bijective between the blocks with defect group $D_8 = D$ and between the blocks with defect group P as well. Now $\mathrm{N}_G(D)$ has D as a normal p-subgroup, and so every block has defect group containing D. This means that there are two blocks of $k\,\mathrm{N}_G(D)$, one with defect group D, and one with defect group P.

Now consider the characters of N. Since N is a direct product of D_8 with S_3, the characters of N are simply the products of the characters of D_8 and S_3. Now, there are five characters of D_8, with degrees 1, 1, 1, 1, and 2, and there are three characters of S_3, with degrees 1, 1, and 2. This means that there are fifteen irreducible ordinary characters of N, eight with degree 1, six with degree 2, and one with degree 4. We will now try to work out where these ordinary characters lie.

Let $B_0(N)$ and $B_1(N)$ be the Brauer correspondents of B_0 and B_1 respectively. All eight linear characters of N must lie in $B_0(N)$. The Alperin–McKay Conjecture (see Conjecture 5.6) suggests that there are four characters of degree 2 in $B_1(N)$. This conjecture also suggests that there are two characters of degree 2 in $B_0(N)$. It seems likely that the character of degree 4 lies in $B_1(N)$. (In fact it has to, since the defect group of $B_1(N)$ is non-abelian, and Brauer's Height Zero Conjecture has been proven for p-soluble groups (see Conjecture 5.13) so all characters of a block are of height zero if and only if its defect group is abelian. This also applies to $B_0(N)$, and so at least one of the characters of degree 2 must lie in $B_0(N)$.)

D_8	1	(1 2)(3 4)	(1 2)	(1 3 2 4)	(1 3)(2 4)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

S_3	1	(5 6)	(5 6 7)
ψ_1	1	1	1
ψ_2	1	-1	1
ψ_3	2	0	-1

We can show quite easily that the degree-four character χ does not lie in the principal block: consider the conjugacy class $\mathscr{C} = \{(1\,2)(3\,4)(5\,6), (1\,2)(3\,4)(5\,7), (1\,2)(3\,4)(6\,7)\}$. Then χ takes the value 0 on this conjugacy class, and so $|\mathscr{C}|\chi(\mathscr{C})/\chi(1) \equiv 0 \mod 2$, contrary to the trivial character, which takes value $3 \equiv 1 \mod 2$. So $\chi \notin B_0(N)$.

This method also works for $\psi_3\chi_i$ for all other i, and so all five of these characters lie in $B_1(N)$. Finally, we must decide the fate of $\chi_5\psi_1$ and $\chi_5\psi_2$. Since we are working modulo 2, we need only consider one of these characters, say $\chi_5\psi_1$. In fact, rather than comparing the values of the central character for all conjugacy classes with that of the trivial character, we will simply note that on the same conjugacy class $\mathscr{C}$ that we considered before, $|\mathscr{C}|(\chi_5\psi_1)(\mathscr{C})/(\chi_5\psi_1)(1) = 3 \times (-2)/2 \equiv 1 \mod 2$, which is different from that of $\chi_5\psi_3$. This means that they do not lie in the same block.

Therefore, all eight linear characters, $\chi_5\psi_1$ and $\chi_5\psi_2$ lie in $B_0(N)$, and all of the rest lie in $B_1(N)$. This agrees with the Alperin–McKay Conjecture and the Height Zero Conjecture.

Now we consider the prime 3. We know the defect group of $b_0(G)$, and so we only need to calculate the defect groups for the other two blocks. Consider the conjugacy class 2A, which has order 105, and the character χ_2. If x lies in 2A, then $\mathrm{C}_G(x) = S_3 \times (C_2 \wr S_2)$, which has Sylow 3-subgroups generated by (up to conjugacy) (1 2 3). Now $|2\mathrm{A}|\chi_2(x)/\chi_2(1) = 35 \equiv -1 \mod 3$, and so $\langle(1\ 2\ 3)\rangle$ is a defect group for $b_1(G)$. Similarly, $|2\mathrm{A}|(\chi_2\chi_a)(x)/(\chi_2\chi_a)(1) = -35 \equiv 1 \mod 3$, and so the defect groups of $b_1(G)$ and $b_2(G)$ are equal.

We fix some notation: let Q be a Sylow 3-subgroup of G, and $R = \langle(1\ 2\ 3)\rangle$. So $\mathrm{N}_G(R) \cong S_3 \times S_4$, and has order 144, and $\mathrm{N}_G(Q) \cong S_3 \wr S_2$, and has order 72. Then Brauer's First Main Theorem says that $\mathrm{N}_G(Q)$ has only one block, naturally of full defect. It also says that $\mathrm{N}_G(R)$ has three blocks, one of full defect and two of defect 1. This is because it acts bijectively on those of defect 1, and that it also acts (not necessarily bijectively) on those of defect more than 1. But since there is only one such block, $B_0(G)$, there must be a block in $k\,\mathrm{N}_G(R)$.

Consider the characters of $\mathrm{N}_G(R) \cong S_3 \times S_4$.

S_4	1	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	-1	0	-1
χ_4	3	-1	-1	0	1
χ_5	2	0	2	-1	0

S_3	1	(5 6)	(5 6 7)
ψ_1	1	1	1
ψ_2	1	-1	1
ψ_3	2	0	-1

In the direct product $S_4 \times S_3$, there are nine characters of degree prime to 3, which have to lie in the principal block $b_0(N)$, since the other two blocks must have defect 1. So we have to decide where the other six characters – $\chi_4\psi_i$ and $\chi_5\psi_i$ for $1 \leqslant i \leqslant 3$ – lie. Now the defect groups are abelian, and so one of the consequences of the Alperin–McKay and Alperin's Weight Conjectures is that all characters of $B_0(N)$ have degrees prime to 3 (see [62]). We will verify this now.

Consider the conjugacy class $\mathscr{C}$ with representative $x = (1\ 2)(5\ 6\ 7)$, which has order 12. Then, working modulo 3, we have

$$\begin{aligned}
|\mathscr{C}|(\chi_1\psi_1)(x)/(\chi_1\psi_1)(1) &= 0,\\
|\mathscr{C}|(\chi_4\psi_1)(x)/(\chi_4\psi_1)(1) &= 1,\\
|\mathscr{C}|(\chi_4\psi_2)(x)/(\chi_4\psi_2)(1) &= 1,\\
|\mathscr{C}|(\chi_4\psi_3)(x)/(\chi_4\psi_3)(1) &= 1,\\
|\mathscr{C}|(\chi_5\psi_1)(x)/(\chi_5\psi_1)(1) &= -1,\\
|\mathscr{C}|(\chi_5\psi_2)(x)/(\chi_5\psi_2)(1) &= -1,\\
|\mathscr{C}|(\chi_5\psi_3)(x)/(\chi_5\psi_3)(1) &= -1.
\end{aligned}$$

So $b_0(N)$ contains all of the characters of degree prime to 3, $b_1(N)$ contains the characters $\chi_4\psi_i$ $(1 \leqslant i \leqslant 3)$, and $b_2(N)$ contains the characters $\chi_5\psi_i$ $(1 \leqslant i \leqslant 3)$.

Chapter 2

Green Correspondence and Module Theory

The Green Correspondence is one of the most fundamental results in the module-theoretic approach to modular representation theory. First proven in 1964 by James Green in [52], this correspondence links certain indecomposable RG modules with certain indecomposable $R\,\mathrm{N}_G(D)$ modules.

In the first section we will exhibit relatively projective modules as a natural generalization of projective modules. Recall that a module is *projective* if it is a direct summand of a free module. This is not the only characterization of projective modules: indeed, we have the following theorem, whose proof is not given here and is easily available throughout the literature. Notice that Proposition 2.3 is a variation of this theorem.

Theorem 2.1 Let R be a ring, and M an R-module. Then the following are equivalent:

(i) M is projective;

(ii) if ϕ is a homomorphism from any R-module N onto M, then $\ker \phi$ is a summand of N; and

(iii) if N and L are two R-modules, and $\phi : N \to L$ is onto and $\psi : M \to L$ is any homomorphism, then there is $\theta : M \to N$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, "\theta"'] \arrow[d, "\psi"] \\ N \arrow[r, "\phi"'] & L \end{tikzcd}$$

commutes.

Once we have finished our discussion of relatively projective modules, we consider vertices, which are minimal p-subgroups with respect to which our chosen module is relatively projective. We will prove that minimal such objects actually exist, and form a conjugacy class of p-subgroups.

With the concept of vertex behind us, we can address the word ‘certain’ in the Green Correspondence. ‘Certain’ can be replaced with ‘with a vertex in a collection of p-subgroups of G’. In fact, the correspondence holds for individual p-subgroups, so that if Q is a p-subgroup of G, there is a naturally defined one-to-one correspondence between the indecomposable RG-modules with vertex Q and the indecomposable $R\,\mathrm{N}_G(Q)$-modules with vertex Q.

We then move on to prove the Nagao Decomposition, which splits up the summands of a restriction of a module into two collections – one that consists of Brauer correspondents of the block which the original module lies in, and the other that consists of modules whose vertices are severely restricted in shape. This result can be used to give a link between the Green and Brauer Correspondences.

Then we use the Nagao Decomposition to prove the Second Main Theorem in a way that is both enlightening and easier than the character-theoretic proofs that are available, and the way that Brauer proved it in [17]. The proof here is that of Nagao, and considerably simplifies the calculations.

We need an important result, called Green's Indecomposability Criterion, first stated and proved in [50].

Theorem 2.2 (Green's Indecomposability Criterion) Let G be a group, and H a normal subgroup of index p. Suppose that M is an absolutely indecomposable RH-module. Then M^G, the induced module, is also absolutely indecomposable.

We recall that we can assume that all of our indecomposable modules are absolutely indecomposable, since we can assume that R contains enough roots of unity for our needs.

2.1 Relatively Projective Modules

Recall the definitions of free and projective modules: if A is a ring, a module M is *A-free* (or simply *free* if the context is clear) if there is a subspace U of M such that any linear transformation from U to another A-module N can be extended uniquely to an A-module homomorphism $M \to N$. A module is *A-projective* (or simply *projective* if the context is clear) if it is a direct summand of a free module. We will introduce the notation $N|M$ to mean that N is (isomorphic to) a summand of M. Then N is projective if $N|M$ where M is free.

Now a subspace of a kG-module is simply a $k1$-module, so we could say that a kG-module M is free if there exists a $k1$-module U in M such that every $k1$-module homomorphism from U to a kG-module N extends uniquely to a kG-module homomorphism $M \to N$.

We can clearly generalize this notion, and say that a module M is *relatively H-free* if there is a kH-submodule U of M such that any kH-module homomorphism from U into a kG-module N extends uniquely to a kG-module homomorphism $M \to N$. A module is *relatively H-projective* if it is a summand of a relatively H-free module.

Similarly, we can consider RG modules, and say that an RG-module is relatively H-projective and relatively H-free accordingly. Many of the results of this section apply equally well for either R or k, echoing the statement made in the preface.

The following characterization of relatively H-projective modules is due to Higman (see [55] and [56]).

Proposition 2.3 (Higman's Criterion) Let U be an RG-module and H be a subgroup of G. Then the following are equivalent.

(i) U is relatively H-projective.

(ii) $U|(U_H)^G$.

(iii) Any homomorphism ψ of an RG-module V onto U which splits as an RH-module homomorphism splits as an RG-module homomorphism.

(iv) For any two RG-modules V and W, and any two RG-module homomorphisms $\theta : V \to W$ and $\psi : U \to W$, there is an RG-module homomorphism $\rho : U \to V$, causing the diagram to commute, if and only if there is such an RH-module homomorphism.

Proof: Suppose that (iv) holds, and let θ be an RG-module homomorphism from V onto U. Let $W = U$, and ψ be the identity. Then an RG-module homomorphism $\rho : U \to V$ exists if and only if an RH-module homomorphism does, and by the commutativity of the diagram, $\rho\psi$ is the identity. This implies (iii).

Now suppose that (iii) holds. Let $\psi : (U_H)^G \to U$ be the canonical map. If this splits as an RH-module homomorphism, it can be extended to an RG-module homomorphism, since $(U_H)^G$ is relatively H-free, and so $U|(U_H)^G$, proving (ii). Certainly if $U|(U_H)^G$ then U is a summand of a relatively H-free module, and so is relatively H-projective, demonstrating (i).

Finally, suppose that U is relatively H-projective. We need to show (iv). So U is a direct summand of a relatively H-free module $F = U \oplus U'$. Let π_U be the projection onto U, and let $\bar{\psi} : F \to W$ be $\bar{\psi} = \psi\pi_U$. If there exists $\bar{\rho} : F \to V$ such that $\theta\bar{\rho} = \bar{\psi}$, then $\rho = \bar{\rho}|_U : U \to V$ will have the desired property that $\theta\rho = \psi$.

Suppose that $\rho : U \to V$ is an RH-module homomorphism causing the diagram to commute as a diagram of RH-module homomorphisms. We can let $\bar{\rho} : F \to V$ be $\bar{\rho} = \rho\pi_U$. Then this is an RH-module homomorphism, and so can be extended to an RG-module homomorphism since F is relatively H-free. This is as required, and so (iv) follows. □

Notice that every RG-module is relatively G-projective, from the definition of relative projectivity. [Every RG-module is also relatively G-free as well.] A natural question is whether every RG-module is relatively H-projective for some proper subgroup H of G.

Proposition 2.4 Suppose that P is a Sylow p-subgroup of the group G. Then if $H \geqslant P$, any RG-module is relatively H-projective.

Proof: Suppose that $\phi : V \to U$ is a homomorphism from V onto U, where U and V are RG-modules. Suppose that ϕ splits as an RH-module homomorphism. So $\ker \phi$ is a direct summand of V, when we consider V as an RH-module. Let π be the (RH-module) projection of V onto $\ker \phi = M$. We need to construct an RG-homomorphism π' which is the identity on M and is zero everywhere else. Then π' is a projection, and so M is a direct summand of V as an RG-module, ϕ splits as an RG-module homomorphism, and U is relatively H-projective.

Let G/H denote the set of right cosets of H in G, and let π' be defined by

$$\pi' = |G : H|^{-1} \sum_{s \in G/H} s\pi s^{-1},$$

where s denotes both the element of G/H and the corresponding linear transformation of V. We need to show that this is an RG-homomorphism. Since s and π are linear transformations, so is π'. The image of any element of V is in M, since

$$\pi'(V) = |G : H|^{-1} \sum_{s \in G/H} s\pi s^{-1} V \subseteq |G : H|^{-1} \sum_{s \in G/H} s\pi V = |G : H|^{-1} \sum_{s \in G/H} sM = |G : H|^{-1} \sum_{s \in G/H} M = M.$$

Also, π' is the identity on M, since $s^{-1}v \in M$ whenever V is, and so

$$\pi'(v) = |G : H|^{-1} \sum_{s \in G/H} s\pi s^{-1}(v) = |G : H|^{-1} \sum_{s \in G/H} s(s^{-1}(v)) = |G : H|^{-1} \sum_{s \in G/H} v = |G : H|^{-1}|G : H|v = v.$$

Finally, we need to show that π' is an RG-homomorphism. Let $g \in G$, and $v \in V$. Then

$$\begin{aligned} \pi'(gv) = |G : H|^{-1} \sum_{s \in G/H} s\pi s^{-1}(gv) &= |G : H|^{-1} \sum_{s \in G/H} gg^{-1}s\pi s^{-1}gv \\ &= g|G : H|^{-1} \sum_{s \in G/H} g^{-1}s\pi s^{-1}gv \\ &= g|G : H|^{-1} \sum_{r \in G/H} r\pi r^{-1}v = g\pi'(v), \end{aligned}$$

since as s runs through all elements of G/H, so does $g^{-1}s$. This means that $g^{-1}s$ represents every linear transformation of V that s does, when summed over all right cosets. So ϕ is an RG-homomorphism, as required. □

2.2 Vertices and Sources

The concept of a vertex is closely related to how projective a module is. A vertex of a module M will be a p-subgroup such that M is relatively P-projective. By Lemma 2.4 we know that Sylow p-subgroups fit this description. We choose a minimal such member amongst the collection of all p-subgroups which satisfy this criterion.

Definition 2.5 Let M be an indecomposable RG-module. A p-subgroup Q of G is a *vertex* of M if M is relatively Q-projective but not relatively P-projective for any P such that $P^g < Q$ for some $g \in G$.

Since M is projective relative to a Sylow p-subgroup, vertices of M exist. We could equivalently define a vertex as a minimal member of the partially ordered set of all p-subgroups P of G such that M is relatively projective to P. Although we are guaranteed that vertices exist, *a priori* we know nothing about their structure. In fact, all vertices of a RG-module M are conjugate. To prove this, we need the Mackey decomposition formula, or Mackey's Theorem:

Theorem 2.6 (Mackey's Theorem) Suppose that H and L are subgroups of the group G, let M be an RH-module, and let S be a set of double coset representatives for G with respect to H and L. Then

$$(M^G)_L = \bigoplus_{s\in S} ((M^s)_{L\cap sHs^{-1}})^L .$$

Theorem 2.7 Let Q be a vertex of M, an indecomposable RG-module, and let H be a subgroup of G. M is relatively H-projective if and only if H contains a conjugate of Q.

Proof: Firstly, if H contains Q, then M is a direct summand of a relatively Q-free module. Trivially from the definition of a relatively Q-free RG-module F, F is also relatively H-free, and so M is relatively H-projective. Now suppose that $Q^g \leqslant H$. Then M is relatively Q^g-projective, since in the equivalent conditions of relative projectivity, we can simply factor any homomorphism through the conjugation isomorphism to get the required map. So M is relatively Q^g-projective, and thus is relatively H-projective, by the previous part.

Now suppose that M is relatively H-projective. We must show that H contains a conjugate of Q. Since M is relatively H-projective, there is a module N with $M|N^G$. Then $M_H|(N^G)_H$. By Mackey's Theorem, we can decompose $(N^G)_H$ via (Q,H)-double cosets, and get

$$(N^G)_H = \bigoplus_{s\in S} \left((N^s)_{H\cap sQs^{-1}}\right)^H ,$$

where S is a set of double coset representatives. Then M divides this direct sum, so it divides $\left((N^s)_{H\cap sQs^{-1}}\right)^H$ for some s since M is indecomposable. Then $M|\left((N^s)_{H\cap sQs^{-1}}\right)^G$, and so this means that M is relatively $H\cap sQs^{-1}$-projective since $\left((N^s)_{H\cap sQs^{-1}}\right)^G$ is relatively $H\cap sQs^{-1}$-free. Then it is relatively $(H\cap sQs^{-1})^s$-projective by the previous part of the theorem. But

$$(H\cap sQs^{-1})^s = H^s\cap Q \leqslant Q,$$

and by virtue of the fact that Q is a vertex, $H^s\cap Q = Q$; that is, H^s contains Q, so H contains a conjugate of Q, as required. $\square$

Dual to the concept of a vertex is that of a source: the two are so closely related that they are often described together, whereas here we have separated the proofs of the existence of each. The reason behind this is so that we can compare the concept of a vertex easily in the next chapter. So now we discuss the idea of a source. In the proof above, we often needed the statement that M is a summand of some induced module. In fact, there is an indecomposable RQ-module, which is unique in $\mathrm{N}_G(Q)$, such that M is a summand of its induction to G.

Theorem 2.8 Let Q be a vertex of an indecomposable RG-module M. Then there is an indecomposable RQ-module S, called the *source*, such that $M|S^G$, and S is unique up to conjugacy in $\mathrm{N}_G(Q)$.

Proof: Since $M|(M_Q)^G$, $M|S^G$ for some summand S of M_Q, since M is indecomposable. We can clearly choose S to be indecomposable. Now suppose that V is some other indecomposable RQ-module with $M|V^G$. But $S|M_Q$, and $M|V^G$ so $S|(V^G)_Q$, and again by Mackey's Theorem and the fact that S is indecomposable, we decompose with respect to the (Q,Q)-double cosets to get $S|\left((V^s)_{Q\cap sQs^{-1}}\right)^Q$. Then since Q is a vertex, $Q = sQs^{-1}$, so $s \in \mathrm{N}_G(Q)$, and $V^s \cong V$. $\square$

2.3 The Green Correspondence

In this section we will state and prove the Green Correspondence. We fix a particular p-subgroup, Q, and write N for the normalizer in G of this subgroup. We will define a one-to-one correspondence between a subset of the indecomposable RG-modules and a similar subset of the indecomposable RN-modules. In fact, we will define this correspondence for $H \geqslant N$, but even if we only defined it for N, we could apply the same technique as we did in the case of Brauer's First Main Theorem to extend it to $H \geqslant N$. The first step is to define three types of subset that we will consider.

Definition 2.9 Let G be a finite group, Q a p-subgroup of G, and $H \geqslant \mathrm{N}_G(Q)$. Let $\mathfrak{X}$ denote the set

$$\mathfrak{X} = \{K : K \leqslant Q \cap Q^g \text{ for some } g \in G \setminus H\}.$$

[Notice that since H contains the normalizer of Q, $Q \cap Q^x \neq Q$.] Let $\mathfrak{Y}$ be defined by

$$\mathfrak{Y} = \{K : K \leqslant H \cap Q^g \text{ for some } g \in G \setminus H\}.$$

Finally, let $\mathfrak{A}$ be defined by

$$\mathfrak{A} = \{K : K \leqslant Q \text{ and } K^g \notin \mathfrak{X} \text{ for all } g \in G\}.$$

We can easily see that $\mathfrak{Y}$ contains $\mathfrak{X}$, and that since $Q \notin \mathfrak{X}$ (by the remark made after its definition), $Q \in \mathfrak{A}$. We extend the definitions of relatively H-projective to say that a module M is relatively $\mathscr{T}$-projective for some set of subgroups $\mathscr{T}$ if it is relatively H-projective for some $H \in \mathscr{T}$.

Lemma 2.10 Let K be a subgroup of Q. The following are equivalent:

(i) there exists $g \in G$ such that $K^g \in \mathfrak{X}$;

(ii) $K \in \mathfrak{X}$;

(iii) $K \in \mathfrak{Y}$; and

(iv) there exists $h \in H$ such that $K^h \in \mathfrak{Y}$.

Proof: Suppose that there is some $g \in G$ such that $K^g \in \mathfrak{X}$. So there exists $x \in G \setminus H$ such that $K^g \leqslant Q \cap Q^x$. Conjugating by g^{-1} and noting that $xg^{-1} \notin H$, we have $K \leqslant Q^{xg^{-1}}$. So $K \leqslant Q \cap Q^{xg^{-1}}$ since K is a subgroup of Q. This proves (i)$\Longrightarrow$(ii).

The statement (ii)$\Longrightarrow$(iii) follows from the fact that $\mathfrak{X} \subseteq \mathfrak{N}$. The statement (iii)$\Longrightarrow$(iv) is obvious, so it remains to show (iv)$\Longrightarrow$(i). Suppose that (iv) holds, so that there exists some $h \in H$ such that $K^h \in \mathfrak{N}$. Then $K^h \leqslant H \cap Q^x$ for some $x \in G \setminus H$. We can conjugate by h^{-1}, and notice that $xh^{-1} \in G \setminus H$, to get $K \leqslant H \cap Q^{xh^{-1}}$. Since $K \leqslant Q$, this gives $K \leqslant Q \cap Q^{xh^{-1}}$, as required. $\square$

We want to understand how RG-modules and RH-modules relate to one another. The first result is preliminary in nature, but indicates how the Green Correspondence may work.

Proposition 2.11 Let U be a RG-module with vertex Q. Then there exist RH-modules V and V', both with vertex Q, such that $U|V^G$ and $V'|U_H$.

Proof: U is relatively Q-projective, and so the source S of U is a RQ-module such that $U|S^G$. In the proof of Theorem 2.8, we showed that $S|U_Q$. Notice that $(S^H)^G = S^G$, and so since $U|S^G$, $U|V^G$ for some summand V of S^H. Since $(U_H)_Q = U_Q$ and $S|U_Q$, $S|V'_Q$ for some summand $V'|U_H$. We will show that both V and V' can be chosen to have vertex Q.

Consider V first. Now $V|S^H$, and so $V|S^G$, and V is relatively Q-projective. Let Q' be a vertex of V, which can be chosen to be contained in Q. Then there is some RQ'-module W such that $V|W^H$, and so $V^G|W^G$. Then $U|W^G$, and so U has vertex Q'. This means that $Q = Q'$, and V has vertex Q.

Next consider V'. Now $V'|U_H|(S^G)_H$, and so we can apply the Mackey decomposition, to get

$$V'|\bigoplus_{t\in\mathcal{T}}((S^t)_{H\cap tQt^{-1}})^H$$

(where $\mathcal{T}$ is a set of representatives for the (H, L)-double cosets), and so $V'|((S^t)_{H\cap tQt^{-1}})^H$ for some t. So V' is relatively $H \cap tQt^{-1}$-projective. Let Q' be a vertex of V', contained in $H \cap tQt^{-1}$, and T be a source for V', so that $V'|T^H$. Notice that $|Q'| \leqslant |Q|$. Also, we know that $S|V'_Q$, and hence $S|(T^H)_Q$, so we can decompose $(T^H)_Q$ with respect to the (Q, Q')-double cosets, yielding

$$S|((T^x)_{Q\cap xQ'x^{-1}})^Q,$$

for some $x \in H$. But this means that S is relatively $Q \cap xQ'x^{-1}$-projective, and so $|Q| \leqslant |Q \cap xQ'x^{-1}|$. Since $|Q'| \leqslant |Q|$, we can only have $xQ'x^{-1} = Q$, and V' has vertex Q, as required. $\square$

We shall now state the Green Correspondence, but its proof will require several preliminary results.

Theorem 2.12 (Green Correspondence) Up to isomorphism, there is a one-to-one correspondence between the set of all RG-modules with vertex in $\mathfrak{A}$ and the set of all RH-modules with vertex in $\mathfrak{A}$. Furthermore, it is defined by

$$U_H = V \oplus X, \qquad V^G = U \oplus Y,$$

where X is a relatively $\mathfrak{N}$-projective RH-module and Y is a relatively $\mathfrak{N}$-projective RG-module. Furthermore, if U has vertex Q, V has vertex Q as well.

We will prove this theorem in several stages, the first being to show that V is a summand of $(V^G)_H$, with an important restriction on the other summands.

Proposition 2.13 Let M be a relatively Q-projective RH-module, where $Q \leqslant H \leqslant G$. Then

$$(M^G)_H \cong M \oplus W,$$

where every indecomposable summand of W is relatively $gQg^{-1} \cap H$-projective for some $g \in G \setminus H$.

Proof: Since M is relatively Q-projective, there is some RQ-module N such that $N^Q = M \oplus M_0$, where M_0 is relatively Q-projective as well. Then

$$(N^G)_H = \bigoplus_{s \in S} \left(((N^H)^s)_{H \cap sQs^{-1}}\right)^H.$$

Now every module on the right-hand side of this expression is relatively $H \cap g^{-1}Qg$-projective for various $g \in G$. Since we can choose S to include 1, we see that N^H appears in this decomposition. We can write

$$(N^G)_H = (M^G)_H \oplus (M_0^G)_H.$$

But every term on the left-hand side is already $H \cap g^{-1}Qg$-projective, and so therefore is every term of M_H^G. Since $M|N^H|(M^G)_H$, we see that we can write

$$(M^G)_H \cong M \oplus W,$$

where W is as stated in the proposition. □

We can now prove the first half of the Green Correspondence, that of the decomposition of U_H.

Theorem 2.14 Let U be an indecomposable RG-module with vertex Q' in $\mathfrak{A}$. Then $U_H = V \oplus X$, where $U|V^G$ and X is a relatively $\mathfrak{N}$-projective module.

Proof: Proposition 2.11 implies that there exists an indecomposable RH-module V such that $U|V^G$, and V has vertex Q'. Now $U_H|(V^G)_H \cong V \oplus W$ by the previous proposition, with every summand of W relatively $Q'^g \cap H$-projective, and so W is relatively $\mathfrak{N}$-projective, from the definition of $\mathfrak{N}$. So U_H is isomorphic with a summand of $V \oplus W$, either $V \oplus X$ or X for some summand X of W. Notice that W is relatively $\mathfrak{N}$-projective, and so therefore is X. We aim to show that $U_H \ncong X$, and obtain the result.

But by using the second part of Proposition 2.11, we see that U_H has an indecomposable summand with vertex Q'. If $U_H \cong X$, a relatively $\mathfrak{Y}$-projective and hence relatively $\mathfrak{X}$-projective (by Lemma 2.10) module, this would mean that $Q' \in \mathfrak{X}$, a contradiction to the fact that $Q' \in \mathfrak{A}$. So $U_H \cong V \oplus X$, with X relatively $\mathfrak{Y}$-projective, as required. □

Theorem 2.15 Let V be an indecomposable RH-module with vertex Q' in $\mathfrak{A}$. Then $V^G = U \oplus Y$, where U is an indecomposable RG-module with $V|U_H$, and Y is a relatively $\mathfrak{Y}$-projective RG-module.

Proof: Since V is relatively Q'-projective, there is some RQ'-module U such that $U^H = V \oplus M$ (and so M is relatively Q'-projective as well). Then $U^G = V^G \oplus M^G$, and so $(U^G)_H = (V^G)_H \oplus (M^G)_H$. Now we can apply the Mackey decomposition theorem to get

$$(U^G)_H = \bigoplus_{s \in S} \left((U^s)_{H \cap sQ's^{-1}}\right)^H .$$

where s is a set of (H, Q')-double coset representatives in G. But $U^s \cong U$, and since we can choose S so that $1 \in S$, we can write

$$(U^G)_H = U_H \oplus N,$$

where N is a direct sum of RH-modules which are relatively $\mathfrak{Y}$-projective. The Krull–Schmidt Theorem now applies, yielding

$$(V \oplus M) \oplus N = (V^G)_H \oplus (M^G)_H.$$

Using Proposition 2.13 on V, and the fact that V is indecomposable, we obtain $(V^G)_H = V \oplus Y'$, where Y' is a sum of relatively $\mathfrak{X}$-projective modules, and so there is some summand U of V^G such that $U_H = V \oplus Y$, where Y is a summand of Y', and hence consists of relatively $\mathfrak{Y}$-projective modules, as required. □

From this we have enough to easily demonstrate the truth of the Green Correspondence. We have to show that the two methods of defining the correspondence are equal; i.e., if $U_H = V' \oplus Y$ or $V^G = U' \oplus X$ for some other modules U' and V' with vertex Q', then $U = U'$ and $V = V'$. But the Krull–Schmidt Theorem, together with the fact that U, U', V and V' are not relatively $\mathfrak{Y}$-projective and both X and Y are, gives that $U' = U$ and $V' = V$. Thus the Green Correspondence is proven.

We now give one final very useful result in this section, the Burry–Carlson–Puig Theorem. We do not prove it here; the reader is referred to, for example, [64] or [10].

Theorem 2.16 (Burry–Carlson–Puig Theorem) Let Q be a p-subgroup of G, and let H be a subgroup of G containing $\mathrm{N}_G(Q)$. If M is an indecomposable RG-module such that a summand N of M_H has vertex D, then M has vertex D, and M and N are Green correspondents.

2.4 Nagao Decomposition

Let us fix a p-subgroup D of G, and a subgroup H such that $D\,\mathrm{C}_G(D) \leqslant H \leqslant \mathrm{N}_G(D)$. We quote the following result of Curtis and Reiner, (see [26, 58.11]).

Theorem 2.17 Let e be a block idempotent of RG, associated with the block B. Then either $\sigma(e) \equiv 0 \mod J(RH)$ or $\sigma(e) \equiv \sum e_i \mod J(RH)$, where the sum is taken over block idempotents e_i of RH associated with blocks that are Brauer correspondents of B. Furthermore, $\sigma(e) \equiv 0 \mod J(RH)$ if and only if there are no Brauer correspondents of B in RH.

Although the statement of the Nagao Decomposition is accessible at this point, the proof requires some of the concepts from Sections 3.1 and 3.2.

Theorem 2.18 (Nagao Decomposition) Let e be an idempotent of RG, and M an RG-module such that $eM = M$. Then $e = e_1 + e_2$, where e_1 and e_2 are either orthogonal idempotents such that $e_i^h = e_i$ for arbitrary $h \in H$, or one of the e_i is zero, and the resulting decomposition of the restricted module

$$M_H = e_1 M \oplus e_2 M$$

has the following properties:

(i) $e_1 M$ is either zero or a direct sum of indecomposable RH-modules belonging to blocks that are Brauer correspondents of $e(RG)$; and

(ii) $e_2 M$ is either zero or a direct sum of indecomposable RH-modules whose vertices do not contain D.

Proof: Let B denote the block with idempotent e. Consider $\sigma(e)$. Then by Theorem 2.17, $\sigma(b)$ is congruent modulo the (unique) maximal ideal of A^H to the sum of the block idempotents of RH that are Brauer correspondents of B. So either $\sigma(e) \equiv 0$ or $\sigma(e) \equiv \bar{e}$ modulo the unique maximal ideal $J(A^H)$. If $\sigma(e) \equiv 0$, let $\bar{e} = 0$. Let $e_1 = e\bar{e}$, and $e_2 = e(e - \bar{e})$. Then $e = e_1 + e_2$, and e_1 and e_2 are orthogonal. Also, since $e_i \in A^H$, $e_i^h = e_i$ for all $h \in H$. Also,

$$M = eM = (e_1 + e_2)M = e_1 M \oplus e_2 M.$$

Since $M = eM$, $e_1 M = \bar{e} M$, and so (i) is true by construction of $\bar{e}$. We will now demonstrate (ii).

Let $\mathscr{C}$ be a conjugacy class of G, with class sum c, and write $c = c_1 = c_2$, where $c_1 = \sigma(c)$. Then c_2 is a sum of elements each of which lies outside $\mathrm{C}_G(D)$. Since $\mathrm{C}_G(D) \trianglelefteq H$, $\mathscr{C}$ splits up as $\mathscr{C}_1 \cup \mathscr{C}_2$, where c_i is the class sum of $\mathscr{C}_i$, and $\mathscr{C}_i$ is a union of conjugacy classes of H. Lemma 3.10 implies that, modulo $J(A^H)$, $A^H_{D'} \equiv \sum_i RC_i$, where each class sum C_i comes from a conjugacy class with defect group contained within D. So let $x \in \mathscr{C}_2$, and let D' be a defect group of $\mathscr{C}_2$. Then $D' \in A^H_{D'}$, and since x does not centralize D, $D \nleqslant D'$. So $c - c_1$ belongs to $A^H_{D'}$ for some $D' \leqslant H$ that does not contain D. Thus

$$c - \sigma(c) \in \sum_{D \nleqslant D'} A^H_{D'}.$$

But the block idempotent e is a linear combination of class sums, and so

$$e - \sigma(e) \in \sum_{D \nleqslant D'} A^H_{D'}.$$

Now $A^H_{D'}$ is an ideal of A^H, and so $e(e-\sigma(e)) \in \sum_{D \nleqslant D'} A^H_{D'}$. But $e(e-\sigma(e)) \equiv e_2 \mod J(A^H)$, and so

$$e_2 \in \sum_{D \nleqslant D'} A^H_{D'} + J(A^H).$$

Either $e_2 = 0$ or e_2 is a sum of primitive orthogonal idempotents that live in A^H, and so live in $\sum_{D \nleqslant D'} A^H_{D'} + J(A^H)$, since this is an ideal and e_2 lies in this ideal. We can use Rosenberg's Lemma, Theorem 3.1, to show that each of these primitive idempotents lies in one of the $A^H_{D'}$ for some $D \nleqslant D'$. So these primitive idempotents are sums of indecomposable modules that are relatively D'-projective, and so have vertices not containing D, as required. □

We will give a link between the Green and Brauer Correspondences in the next result, which can be found in [26, §59].

Corollary 2.19 Let $H = \mathrm{N}_G(D)$, and suppose that M is an indecomposable RG-module. Let M' be its Green correspondent in H, an indecomposable RH-module, and suppose that both M and M' have vertex D. Then M and M' lie inside blocks that are Brauer correspondents of one another.

Proof: Let B be the block that M' lies in. The Green Correspondence demonstrates that $M'|M_H$, and so since M' is indecomposable, it is a summand of either e_1M or e_2M, where these are given in the Nagao Decomposition. If $M'|e_2M$, then M' does not have vertex D, contrary to assumption. If $M'|e_1M$, then M' belongs to a block that is a Brauer correspondent of B, as required. □

What this result says is that the Green Correspondence behaves well with respect to the Brauer Correspondence: the Brauer Correspondence maps sets of kH-modules to sets of kG-modules, and the Green Correspondence maps individual RH-modules to individual RG-modules, but respects the partitioning that the Brauer Correspondence sets up.

2.5 Brauer's Second Main Theorem

Before we start, recall the following famous result of Green, which he proved in [51].

Theorem 2.20 (Green's Theorem on Zeros of Characters) Let Q be a p-subgroup of G, and let $H \geqslant Q$. Suppose that M is a relatively Q-projective RH-module. If x is an element of G whose p-part is not H-conjugate to an element of Q, then the character afforded by $K \otimes_R M$ is zero on x.

Proof: Let t denote the p-part of x. By assumption, t is not an element of D, so in particular is not the identity. Hence $o(x)$ is a multiple of p. Set $K = \langle x \rangle$ and $L = \langle x^p \rangle$. Then $|L : K| = p$ and $L \trianglelefteq K$. (We are planning to use Green's Indecomposability Criterion here.) We know that M is relatively Q-projective, and so there is an RQ-module N such that $M|N^H$. Then, by Mackey's Theorem,

$$(N^H)_K = \bigoplus_{s \in S} ((N^s)_{K \cap Q^s})^K,$$

where s is a set of representatives for the (K, Q)-double cosets of H. But since t is not H-conjugate to an element of Q, $t \notin Q^h$ for all $h \in H$, and so $Q^h \cap K \leqslant L$. So we can write $(N^H)_K$ as

$$(N^H)_K = \bigoplus_{s \in S} \left(((N^s)_{K \cap Q^s})^L\right)^K.$$

Now $(N_{K \cap Q^s})^L$ can be written as a direct sum of indecomposable RL-modules, say $(N_{K \cap Q^s})^L = \bigoplus V_i$. Then for each V_i, V_i^K is indecomposable by Green's Indecomposability Criterion, and so $(N_{K \cap Q^s})^K$ is a sum of indecomposable RK-modules. But K is abelian and $x \notin L$, so the trace of any induced indecomposable representation from L to K vanishes on x. Thus the representation afforded by $(N_{K \cap Q^s})^K$ takes zero on x. But M_K is a sum of such modules, so vanishes on x. Thus M vanishes on x, as required. □

The proof given here of Brauer's Second Main Theorem follows that of Curtis and Reiner (see [26]), itself based on Nagao's proof. We first give a preparatory lemma, which relates characters to the Nagao Decomposition of the previous section.

Lemma 2.21 Let e be a block idempotent in RG, and let M be an RG-module such that $eM = M$. Let $x \in G$, and write $x = vu$, where v is p-regular and u is p-singular. Set $D = \langle u \rangle$, and $H = \mathrm{C}_G(u)$. Let

$$M_H = e_1 M \oplus e_2 M$$

be the Nagao Decomposition of M_H. If χ is the character of $K \otimes_R M$ and ψ is the character of $K \otimes_R e_1 M$, then $\chi(x) = \psi(x)$.

Proof: Certainly, since $M = e_1 M \oplus e_2 M$, the trace of the representations ρ, ρ_1 and ρ_2 afforded by the modules M, $e_1 M$ and $e_2 M$, satisfy

$$\mathrm{Tr}(\rho(x)) = \mathrm{Tr}(\rho_1(x)) + \mathrm{Tr}(\rho_2(x)).$$

Now $\mathrm{Tr}(\rho(x)) = \chi(x)$, and $\mathrm{Tr}(\rho_1(x)) = \psi(x)$, so we are actually trying to show that $\mathrm{Tr}(\rho_2(x)) = 0$. So let V be an indecomposable submodule of $e_2 M$. Then V does not have vertex containing D. Let Q be a vertex of V, and notice that V is an RH-module that is relatively Q-projective. Since V does not have vertex containing D, $D \not\leqslant Q$. But this means that since $H = \mathrm{C}_G(u)$, u is not H-conjugate to an element of Q. Then Theorem 2.20 above demonstrates that the trace of V

vanishes on x. This was an arbitrary summand of e_2M, and so the result follows. $\square$

We can now give Brauer's Second Main Theorem. At this point, the reader is advised to look back at the end of Chapter 1, where we defined the generalized decomposition numbers.

Theorem 2.22 (Second Main Theorem) Given the notation introduced in the above discussion, suppose that $d^t_{ij} \neq 0$ for some p-singular t, and $C = \mathrm{C}_G(t)$. If ζ^i lies in the block B of kG, and ψ^j lies in the block b of kC, then $b^G = B$.

Proof: Since ζ^i is an irreducible ordinary character of G, it is afforded by some module $K \otimes_R M$. Then, if e is the block idempotent of B, $eM = M$. Let

$$M_C = e_1M \oplus e_2M$$

be the Nagao Decomposition of M_C. Suppose that x is an element of G with p-part t, and write $x = tu$. Then Lemma 2.21 tells us that $\zeta^i(x) = \chi(x)$, where χ is the character afforded by the module e_1M. But Theorem 2.18 says that e_1M is either zero or a sum of indecomposable RH-modules belonging to blocks that are Brauer correspondents of B. If $(\zeta_H)_{b'}$ denotes the contribution to ζ_H from irreducible characters of H lying in the block b', we have

$$\zeta_H(x) = \sum_{(b')^G = B} (\zeta_H)_{b'}(x).$$

Now, if b' does not correspond to B, then $(\zeta_H)_{b'}(x) = 0$, since the contribution to $\zeta_H(x)$ from every summand of e_2M is zero. Since each ordinary character can be written as a linear combination of modular characters in its block, and the modular characters in a block are linear independent, this implies that $d^t_{ij} = 0$ for all modular characters ψ^j for which $b^G \neq B$. Then the result follows. $\square$

Chapter 3

G-Algebras

The notions of defect groups and vertices look very different. However, they are intimately linked, via the notion of G-algebras. These were first introduced by Green [53] in 1968, although it was not until the time of Puig in the 1980s that the theory really came of age. In [23], Broué and Puig generalize the Brauer homomorphism, first considered in [16] and [17] with regards to blocks, to get the Brauer map, defined between subalgebras of fixed points of a particular G-algebra. The next development came in [76], where Puig extended the notion of subpairs created by Alperin and Broué in [6] to that of pointed groups. By considering ordering between pointed groups, we can extend the notion of defect groups to pointed groups.

The main aim of this chapter is to see how the block-theoretic and module-theoretic methods are similar. As such, we do not prove many important results in this chapter. One particularly important result is that there is a correspondence between pointed groups, called the Puig Correspondence, which mirrors that of the Green Correspondence; in fact, it is possible to derive the Green Correspondence from the Puig Correspondence. For the statement and proof of the Puig Correspondence, see [95], the book by Thévenaz.

In this chapter we will need an important result from commutative algebra.

Theorem 3.1 (Rosenberg's Lemma) Let R be an artinian ring, and $I_1, \ldots, I_n$ be a collection of ideals. If a primitive idempotent lies inside $I_1 + \cdots + I_n$, then it lies inside I_j for some j.

3.1 G-Algebras

In this short section we will give the definition of a G-algebra and interior G-algebra, and give two fundamental examples of G-algebras.

Definition 3.2 Let A be an R-algebra, and let $\phi : G \to \mathrm{Aut}(A)$ be a group homomorphism. Then A is said to be a *G-algebra.* If instead A is equipped with a map $\zeta : G \to A^*$ (the group of units of A), then A is said to be an *interior G-algebra.*

Notice that since there is always a homomorphism $\psi : A^* \to \mathrm{Aut}(A)$ by sending a to conjugation by a, an interior G-algebra is always a G-algebra because we can construct the map ζ. The existence of this homomorphism ψ is why interior G-algebras are so-called: conjugation by a is an inner automorphism. There is no uniqueness regarding ζ; there may be more than one way of making a G-algebra into an interior G-algebra. Also, there exist G-algebras that are not interior G-algebras.

Notice that G can act on A by filtering through ϕ; if $a \in A$ and $g \in G$, then $a^g = (\phi(g))(a)$, the element a is sent to by the automorphism $\phi(g)$. This action is very natural, and will be considered in the sequel. We will call this action the *left G-action* on A.

Example 3.3 As an example of an interior G-algebra, consider the group algebra RG itself. The map we need is immediately obvious: $g \mapsto 1 \cdot g$. This makes RG into an interior G-algebra. Indeed, this is one of two very important examples of interior G-algebras, which we will consider in the rest of this chapter.

This is the first of two G-algebras that we shall consider, although there are more examples around, for example, the *twisted group algebras* – see [95] for more information on these and other G-algebras.

Before we continue with the other main example, we recall how modules are connected with representation theory. An R-module M together with a map $G \to \mathrm{Aut}_R(M)$ can be easily made into an RG-module, and any RG-module can be thought of as an R-module with a map $G \to \mathrm{Aut}_R(M)$.

Example 3.4 Consider an RG-module M. So we have the map $\zeta : G \to \mathrm{Aut}_R(M)$. Let $A = \mathrm{End}_R(M)$, which is an R-module. Then $\mathrm{Aut}_R(M) = A^*$, and therefore we have a map $\zeta : G \to A^*$. This makes $\mathrm{End}_R(M)$ for any RG-module M into an interior G-algebra.

The reason that these two interior G-algebras are important lies in the Brauer homomorphism, although it does not look that way at the moment. In the next section we will define an important map Br, which we will call the Brauer map, between a certain subset A^H of A, and a quotient of A^H. It will become clear as to why this map is called the Brauer map.

3.2 Defect Groups and the Brauer Map

For this section, we let H be a subgroup of G, a finite group, and consider a G-algebra A. Recall the left G-action on A given by $a^g = (\phi(g))(a)$, and that this is an automorphism of A. It seems natural to consider the set of all elements of A fixed by this action. Then this is a subalgebra, since $a \mapsto a^g$ is an automorphism.

Definition 3.5 Let H be a subgroup of G and A be a G-algebra. The set A^H is given by

$$A^H = \{a \in A : a^h = a \text{ for all } h \in H\}.$$

Since it is the intersection of subalgebras, A^H is also a subalgebra. However, it is not necessarily a G-algebra. Now consider a conjugate $g^{-1}Hg$ of H. Since ϕ is a homomorphism, $(a^{g_1})^{g_2} = a^{g_1 g_2}$, and so if $a \in A^H$ and $h \in H$,

$$(a^g)^{g^{-1}hg} = a^{gh} = a^g,$$

and so a^g is fixed under the action of $g^{-1}Hg$. Now suppose that $g \in \mathrm{N}_G(H)$. Then a^g is fixed by $g^{-1}Hg = H$, and so $a^g \in A^H$. So A^H is preserved under the left $\mathrm{N}_G(H)$-action. This means that the map $\phi|_N : \mathrm{N}_G(H) \to \mathrm{Aut}(A^H)$ is defined, since if $g \in \mathrm{N}_G(H)$, then $\phi(g)$ acts as an automorphism which preserves A^H (because $(\phi(g))(a) = a^g \in A^H$), and so we can make A^H into an $\mathrm{N}_G(H)$-algebra. We can also make A^H into an *interior* $\mathrm{C}_G(H)$-algebra: if $g \in \mathrm{C}_G(H)$, then $\zeta(g) = g \cdot 1$ in A, and certainly $g \cdot 1 \in A^H$, since $g^h = g$ for all $h \in H$ (as g centralizes H). Thus $\zeta|_{\mathrm{C}_G(H)}$ maps into A^H.

Now suppose that K is a subgroup of H, and let T be a right transversal of K in H. Before we describe the Brauer map, we need to consider another map, called the *relative trace map*, Tr_K^H (see [53], although Green denotes this by $T_{K,H}$). This is defined as

$$\mathrm{Tr}_K^H : A^K \to A^H, \qquad \mathrm{Tr}_K^H : a \mapsto \sum_{h \in T} a^h.$$

We first ought to check that this map is well-defined. But if T' is another transversal, then for $t \in T$ and $t' \in T'$ with t, t' in the same right coset. Then $t' = kt$ for $k \in K$, and $a^{kt} = (a^k)^t = a^t$, so this map is indeed well-defined.

Since $a \mapsto a^h$ is a homomorphism, Tr_K^H is also a homomorphism. This means that the image of $\mathrm{Tr}_K^H(A^K)$, written A_K^H, is a subalgebra. In fact, it is an ideal. This follows from (i) and (ii) of the lemma below, which collects several properties of the trace map that we will need.

Lemma 3.6 Let H, K and L be subgroups of G with $L, K \leqslant H$, and let T and $\mathcal{T}$ be sets of representatives for the K cosets of H and the (K, L)-double cosets of H respectively.

(i) $b \,\mathrm{Tr}_K^H(a) = \mathrm{Tr}_K^H(ba)$ for all $a \in A^K$ and $b \in A^H$.

(ii) $\mathrm{Tr}_K^H(a)b = \mathrm{Tr}_K^H(ab)$ for all $a \in A^K$ and $b \in A^H$.

(iii) If $L \leqslant K$, $\mathrm{Tr}_K^H \mathrm{Tr}_L^K = \mathrm{Tr}_L^H$.

(iv) $\mathrm{Tr}_K^H(a) = \sum_{t \in \mathcal{T}} \mathrm{Tr}_{K^t \cap L}^L(a^t)$.

(v) $A_K^H A_L^H \leqslant \sum_{h \in H} A_{K^h \cap L}^H$.

Proof: Let $a \in A^K$, and $b \in A^H$. Then

$$b \,\mathrm{Tr}_K^H(a) = b \sum_{h \in T} a^h = \sum_{h \in T} b^h a^h = \sum_{h \in T} (ba)^h = \mathrm{Tr}_K^H(ba),$$

and

$$\mathrm{Tr}_K^H(a)b = \left(\sum_{h \in T} a^h \right) b = \sum_{h \in T} a^h b^h = \sum_{h \in T} (ab)^h = \mathrm{Tr}_K^H(ab),$$

proving (i) and (ii).

To prove that the trace map is transitive is fairly straightforward: if T is a transversal to K in H, and S is a transversal to L in K, then ST is a transversal to L in H, and

$$\operatorname{Tr}_K^H \operatorname{Tr}_L^K(a) = \sum_{t\in T}\left(\sum_{s\in S} a^s\right)^t = \sum_{t,s} a^{st} = \operatorname{Tr}_L^H a,$$

proving (iii).

Now let $\mathcal{T}$ be a set of representatives for the (K, L)-double cosets. Then for each $t \in \mathcal{T}$, we can find a $K^t \cap L$ transversal of L, say S_t. Then the union $\mathscr{T}$ of all of the transversals tS_t is a transversal of K in H, and so

$$\operatorname{Tr}_K^H(a) = \sum_{t\in\mathscr{T}} a^t = \sum_{t\in\mathcal{T}}\sum_{u\in S_t} a^{tu} = \sum_{t\in\mathcal{T}} \operatorname{Tr}_{K^t\cap L}^L(a^t),$$

proving (iv).

Now consider $\operatorname{Tr}_K^H(a)\operatorname{Tr}_L^H(b)$, for $a \in A^K$ and $b \in A^L$. Then by (i),

$$\operatorname{Tr}_K^H(a)\operatorname{Tr}_L^H(b) = \operatorname{Tr}_L^H(\operatorname{Tr}_K^H(a)b),$$

and by (ii) and (iv),

$$\operatorname{Tr}_L^H(\operatorname{Tr}_K^H(a)b) = \operatorname{Tr}_L^H\left(\sum_{t\in\mathcal{T}} \operatorname{Tr}_{K^t\cap L}^L(a^h b)\right).$$

Finally, by (iii), the transitivity of the trace map,

$$\operatorname{Tr}_L^H\left(\sum_{t\in\mathcal{T}} \operatorname{Tr}_{K^t\cap L}^L(a^h b)\right) = \sum_{t\in\mathcal{T}} \operatorname{Tr}_{K^t\cap L}^H(a^h b).$$

So $A_K^H A_L^H$ consists of elements in $\sum_{h\in H} A_{K^h\cap L}^H$, proving (v). $\square$

Parts (i) and (ii) demonstrate that indeed A_K^H is an ideal of A^H.

We now come to the definition of a defect group, given in [53]. In this paper, Green first defines defect groups of G-algebras, then shows that the two notions of defect group coincide when our G-algebra is RG (or kG), and this notion of defect group coincides with the notion of a vertex when the G-algebra is $\operatorname{End}_R(M)$, for M an indecomposable R-module.

Definition 3.7 Let A be a G-algebra over R or k, and let e be a primitive idempotent in A^G. If $e \in A_D^G$ and whenever $e \in A_H^G$ for any H of G, some conjugate of D lies in H, then D is said to be a *defect group* of the idempotent e.

In this definition, we have associated a defect group to an idempotent. This can be reconciled with the definition of defect group we already have by noting that to every block there is associated a unique primitive idempotent. We now show that these objects exist. The following result is due to Green [53].

Theorem 3.8 Let A be a G-algebra, with a primitive idempotent e. Then a defect group D of e exist, any two defect groups are conjugate, and D is a p-group.

Proof: Consider the partially ordered set of all subgroups H of G such that $e \in A_H^G$. This set is non-empty, since $e \in A_G^G$. Let D be a smallest member of this set. Then $e \in D$. Now suppose that H is another subgroup such that $e \in A_H^G$. By Lemma 3.6(v),

$$e \in A_H^G A_D^G \leqslant \sum_{g \in G} A_{H^g \cap D}^G.$$

By Rosenberg's Lemma, Theorem 3.1, e lies in one of the ideals $A_{H^g \cap D}^G$. But D was chosen to be the smallest group with $e \in A_D^G$, so H^g contains D. Then H contains D^{-1}, as required.

Finally we need to show that D is a p-group. Equivalently, we can show that for some Sylow p-subgroup P, $e \in A_P^G$. Then some conjugate of D must lie inside P, making D a p-group. Let $a \in A^G$. Recall that therefore $a \in A^P$. Then

$$\mathrm{Tr}_P^G(a) = \sum_{g \in T} a^g,$$

where T is a transversal to P in G. But $a \in A^G$, so $\mathrm{Tr}_P^G(a) = |G:P|a$. But $|G:P|$ is not divisible by p, and so an inverse to $|G:P|$ exists, and notice that as a constant, it lies inside A^G. So by Lemma 3.6(i),

$$\mathrm{Tr}_P^G\left(\frac{a}{|G:P|}\right) = |G:P|^{-1}\,\mathrm{Tr}_P^G(a) = |G:P|^{-1}|G:P|a = a.$$

Therefore $A_P^G = A^G$, and since $e \in A^G$, $e \in A_P^G$, proving that D is a p-group. $\square$

We have associated a defect group to a G-algebra. In the next section we shall prove that the two notions of defect group coincide, as well as show that in the case of the G-algebra $\mathrm{End}_R(M)$, the defect groups are the vertices. We will end this section by defining the Brauer map.

Earlier, we showed that A_K^H is an ideal of A^H. We will use this fact to define the Brauer map. Since A_K^H is an ideal of A^H for all $K \leqslant H$, we can take the sum of various of these ideals. We will also quotient out by the Jacobson radical of this quotient ring to get

$$\mathrm{Br}_P^A : A^P \to \bar{A}(P),$$

the canonical surjection given by quotienting A^P by the ideal $\sum_{Q<P} A_Q^P + J(A^P)$. This is called the *Brauer map*. Note that this is the zero map unless P is a p-group.

3.3 kG and $\mathrm{End}_R(M)$

Our first aim is to show that the G-algebra and block-theoretic definitions of defect group coincide. Because of the obvious confusion that may arise in this section, we refer to the block-theoretic defect groups as 'class defect groups', until we have shown that they do, indeed, coincide. Our first result is Osima's Theorem, which gives an equivalent definition of the class defect group of a block idempotent.

Theorem 3.9 (Osima's Theorem) Suppose that P is a p-subgroup of G. Let M_P be the submodule of $\mathrm{Z}(kG)$ generated by the class sums of conjugacy classes with defect group contained in P. Then M_P is an ideal of $\mathrm{Z}(kG)$. Let B be a block of kG, with block idempotent e. Then D is a class defect group of B if and only if $e \in M_D$ but $e \notin M_P$ for $P < D$.

We can see that this formulation of class defect group looks much more likely to be shown equivalent to that of a defect group. Before we prove this, we need a lemma, which we made use of in the proof of the Nagao Decomposition, in the previous chapter.

Lemma 3.10 Let A denote the G-algebra RG, and let H be a subgroup of G. Then A^H has a basis consisting of the class sums corresponding to the H-conjugacy classes. If $D \leqslant H \leqslant G$, and D is a p-subgroup of G, then

$$A_D^H \equiv \sum Rc_i \mod J(A^H),$$

where the terms in the sum are all H-class sums c_i whose conjugacy classes have class defect groups H-conjugate to a subgroup of D.

Now let e be a block idempotent lying in A^G. Suppose that e has defect group D. Then $e \in A_D^G$, and so may be written as

$$e \equiv \sum_{D_i = D} a_i c_i + \sum_{D_j < D} b_j c_j \mod J(A^G),$$

where the first sum consists of class sums whose conjugacy classes have class defect groups conjugate to D, and the second sum consists of class sums whose conjugacy classes have class defect groups conjugate to proper subgroups of D. If $a_i = 0$ for all i, then e would be in $\sum A_{D'}^G$ over the proper subgroups D' of D, and so lies in one of the $A_{D'}^G$ for some D' by Rosenberg's Lemma, contradicting the fact that D is a defect group for e. Thus one of the a_i is non-zero, and so by Osima's Theorem, D is a class defect group for e. By reducing modulo the Jacobson radical, the class defect groups and the defect groups coincide when the G-algebra is kG, as we have asserted. So from now on, we will simply refer to class defect groups as defect groups, as before.

Our next task is to consider the G-algebra $\mathrm{End}_R(M)$, where M is an RG-module. Recall Higman's Criterion, Proposition 2.3. At that time, we did not have the relative trace map, and so could not give one of the equivalent conditions for relative projectivity.

Proposition 3.11 (Higman's Criterion) Let U be an RG-module and H be a subgroup of G. Then the following are equivalent.

(i) U is relatively H-projective.

(ii) There exists $f \in \mathrm{End}_{RG}(U)$ such that $\mathrm{Tr}_H^G(f) = 1$.

The proof of this fact is given in [35, II.3.8].

Example 3.12 Consider the G-algebra $A = \mathrm{End}_R(M)$, where M is indecomposable. We will show that the defect groups of A are simply the vertices of M. Recall that the action of G on A worked by sending $\phi \in A$ to $g^{-1}\phi g$, which makes sense because gm is defined for $m \in M$. If H is a subgroup of G, then A^H is the subalgebra of A that is fixed by this action, so it is all of the R-endomorphisms of M that commute with $h \in H$; i.e., all RH-endomorphisms of M. Now since M is indecomposable, 1_M, the identity morphism, is a primitive idempotent of A^G (since it clearly belongs to A^G). Now if H is any subgroup of G,

$$1_M \in A_H^G \qquad \Longleftrightarrow \qquad 1 = \mathrm{Tr}_H^G(\phi),$$

for some $\phi \in A^H$, the set of all RH-endomorphisms of M. But this is precisely Higman's Criterion, and so $1_M \in A_H^G$ if and only if M is relatively H-projective. Then clearly the two concepts of defect group, as a minimal subgroup D such that $1 \in A_D^G$, and vertex, as a minimal subgroup D such that m is relatively D-projective, are equivalent.

So the defect groups of $\mathrm{End}_R(M)$ are the vertices of M, as we have asserted before.

3.4 Pointed Groups

In this section we will introduce most of the definitions and terminology that are required to study Puig's work on G-algebras. Pointed groups are fundamental tools in this approach to representation theory. First, we recall the definition of a point:

Definition 3.13 Let A be an R-algebra. A *point* of A is a conjugacy class of primitive idempotents of A.

The reasoning behind calling these conjugacy classes points is that they are in one-to-one correspondence with the maximal ideals of A, the correspondence being defined by sending the maximal ideal to the unique point that is not contained within it.

Definition 3.14 Let A be a G-algebra. A *pointed group* on A is a pair (H, α), where H is a subgroup of G and α is a point of A^H.

Following [76], we will denote a pointed group (H, α) by H_α. We will want to consider pointed groups as a generalization of the notion of subgroups, and so we need to define a partial order between pointed groups. Let H_α and K_β be pointed groups. Suppose that $K \leqslant H$. Notice that every element $a \in A^H$ is also an element of A^K. Then we write $K_\beta \leqslant H_\alpha$ if for some idempotent $e \in \alpha$, there is some $f \in \beta$ such that f appears in a decomposition of e, viewed as an element of A^K.

More formally, we can define the *restriction map*, $r_K^H : A^H \to A^K$, given by simply embedding A^H into A^K. Then for two pointed groups H_α and K_β, we say that $K_\beta \leqslant H_\alpha$ if $K \leqslant H$ and for some $e \in \alpha$ there exists $f \in \beta$ such that a decomposition of $r_K^H(e)$ includes f. Alternatively, we can write this as for some $e \in \alpha$ there exists $f \in \beta$ such that $f = efe$. In fact, if it is true for some $e \in \alpha$, it is true for every $e' \in \alpha$, since $e' = g^{-1}eg$ by the definition of a point, and so $g^{-1}fg$ appears in a decomposition of $r_K^H(e')$.

We can introduce another relation as well as $\leqslant$: we say that H_α is *relatively K_β-projective* if $K \leqslant H$ and $\alpha \subseteq \mathrm{Tr}_K^H(A^K\beta A^K)$. Let us examine what this second condition means – firstly, $A^K\beta A^K$ is an ideal of A^K (we need this rather than $A^K\beta$ since A^K is noncommutative). This means that $\mathrm{Tr}_K^H(A^K\beta A^K)$ is an ideal of A^H, by Lemma 3.6. So $\alpha \subseteq \mathrm{Tr}_K^H(A^K\beta A^K)$ simply means that one of the $e \in \alpha$ is also in this ideal, since we can conjugate to get the rest. In fact, if we write I for $\mathrm{Tr}_K^H(A^K\beta A^K)$, then $e \in I$ if and only if $\alpha \subseteq I$, and $\alpha \subseteq I$ if and only if $A^H\alpha A^H \subseteq I$. So we can see this as the fact that the principal ideal generated by β, then filtered through Tr_K^H, contains the principal ideal generated by α.

Example 3.15 Consider the G-algebra $A = RG$. Then $A^G = \mathrm{Z}(RG)$, and a point of A^G is simply a block idempotent of RG. This gives us pointed groups (G, e), where e is any block idempotent of G. Now $\mathrm{Z}(RH) \subseteq A^H$, and so corresponding to any block idempotent of RH there is a point of A^H, and so a pointed group.

We now define the concept of a local pointed group. Our final aim is to define yet another formulation of defect group, this time for pointed groups. As such, they will be called defect pointed groups. We first define the notion of a local pointed group.

Definition 3.16 A pointed group P_α of a G-algebra A is a *local pointed group* if $\mathrm{Br}_P^A(\alpha) \neq 0$. In this case α is called a *local point* of P on A.

Of course, this definition precludes the possibility that $\bar{A}(P) = 0$, so in particular P has to be a p-group, as the notation suggested. We can take the collection of all local pointed groups contained within a particular pointed group, and choose the maximal elements. These will be p-groups, of course, and so we will have picked a collection of p-groups that have been maximized with respect to inclusion. This sounds a lot like defect groups and vertices, and in fact they are the same thing.

Definition 3.17 Let A be a G-algebra, and H_α be a pointed group (so that α is a point of A^H). Let $\mathscr{S}$ denote the set of all local pointed groups P_γ for which $P_\gamma \leqslant H_\alpha$, in the sense defined above. Partially ordering this set with respect to $\leqslant$, the maximal elements of $\mathscr{S}$ are called *defect pointed groups* of H_α.

It turns out that the maximal local pointed groups contained in H_α are also the minimal groups P_γ such that H_α is relatively P_γ-projective (see [95, 18.3]).

We can already see that the definition of a defect pointed group may well be applied to the G-algebra RG, whose pointed groups G_α are simply (G, e) where e is a block idempotent. So we would essentially be associating a p-subgroup of G to a block idempotent.

We now state without proof a result regarding defect pointed groups (see [76, Theorem 1.2]).

Theorem 3.18 Let H_α be a pointed group on a G-algebra A. If P is a minimal subgroup of G such that $\alpha \subseteq A^G_P$, there exists a local point γ of P such that $\alpha \subseteq \mathrm{Tr}^G_P(A^P\gamma A^P)$.

Example 3.19 Consider the G-algebra A, and let G_α be a pointed group of A. Let P_β be a defect pointed group of G_α. Then G_α is relatively P_β-projective, and so $\alpha \subseteq \mathrm{Tr}^G_P(A^P\beta A^P) \subseteq A^G_P$. Thus α is contained in A^G_P, and every idempotent of α has a defect group contained within P. Let Q be such a defect group. Then if $e \in \alpha$, $e \in A^G_Q$, and so by the above result, there is some local point γ of Q such that

$$\alpha \subseteq \mathrm{Tr}^G_Q(A^Q\gamma A^Q),$$

and so G_α is relatively Q_γ-projective, contrary to the hypothesis that P_β is a defect pointed group of G_α. So the defects of G_α are equivalent to the defect groups of e, where $\alpha = \{e\}$.

Chapter 4

Simple Group Theory

The applications of modular representation theory were most immediate in the field of simple group theory. Richard Brauer used the modular character theory to stunning effect in the 1950s and 1960s to massively restrict the structure of simple groups with certain properties.

His character-theoretic techniques were used firstly in the Brauer–Suzuki Theorem [21], in which the two authors use modular character theory to demonstrate that there are no finite simple groups with a Sylow 2-subgroup isomorphic to a generalized quaternion group. This was one of the first steps in the Classification of the Finite Simple Groups.

After his initial successes with the Brauer–Suzuki Theorem, Brauer applied his almost magical ability with characters to play a pivotal rôle in the proof of the Alperin–Brauer–Gorenstein Theorem [5], which classifies all possible simple groups with a quasi-dihedral or wreathed Sylow 2-subgroup.

After the publication of the Brauer–Suzuki Theorem, Glauberman generalized the result, considering a very special way in which the Sylow 2-subgroup is embedded. More precisely, Glauberman considered the case where an involution in the Sylow 2-subgroup is not conjugate to any of the other involutions in that subgroup. The result is his celebrated Z*-Theorem, which is proven in Section 4.4.

Also in this chapter we consider in detail the modular representations of the smallest (non-abelian) simple group, A_5.

4.1 Modular Representations of A_5

Consider the group $G = A_5$, the smallest non-abelian simple group. It has order $60 = 2^2 \cdot 3 \cdot 5$, and has five conjugacy classes with representatives 1, $(1\,2\,3)$, $(1\,2)(3\,4)$, $(1\,2\,3\,4\,5)$ and $(1\,3\,4\,5\,2)$ of orders 1, 20, 15, 12 and 12. So there are four classes of 2-regular elements, four classes of 3-regular elements and three classes of 5-regular elements.

In characteristic 0, there are five irreducible ordinary characters, namely

χ_i	1	(1 2 3)	(1 2)(3 4)	(1 2 3 4 5)	(1 3 4 5 2)
χ_1	1	1	1	1	1
χ_2	3	0	-1	α	β
χ_3	3	0	-1	β	α
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Now consider a subgroup of the form $\mathrm{N}_G(P)$, where P is a 2-subgroup. The only 2-local subgroups of G are isomorphic to A_4, the normalizer of $C_2 \times C_2$ in A_5. This has four conjugacy classes, with representatives 1, (123), (132) and (12)(34), of cardinalities 1, 4, 4 and 3 respectively. The ordinary character table for A_4 is given by

ζ_i	1	(1 2 3)	(1 3 2)	(1 2)(3 4)
ζ_1	1	1	1	1
ζ_2	1	1	ω	ω^2
ζ_3	1	1	ω^2	ω
ζ_4	3	-1	0	0

where ω is a primitive cube root of unity.

The only non-trivial 3-subgroup is C_3, and has normalizer S_3. Its ordinary character table was given in Section 1.6. The only non-trivial 5-subgroup is C_5, which is normalized by D_{10}. If the Sylow 5-subgroup is given by $\langle x \rangle = \langle (1\,2\,3\,4\,5) \rangle$, we know that a generating involution must map x to x^{-1}. The involution (2 5)(3 4) does that. The conjugacy classes of $M = D_{10}$ are represented by 1, (1 2 3 4 5), (1 3 5 2 4) and (2 5)(3 4), and have cardinalities 1, 2, 2, and 5 respectively. The ordinary character table is given below.

θ_i	1	(2 5)(3 4)	(1 2 3 4 5)	(1 3 5 2 4)
θ_1	1	1	1	1
θ_2	1	-1	1	1
θ_3	2	0	$\alpha - 1$	$\beta - 1$
θ_4	2	0	$\beta - 1$	$\alpha - 1$

Characteristic 2

In characteristic 2, there are four irreducible Brauer characters, labelled ϕ_1, ϕ_2, ϕ_3 and ϕ_4. We know that ϕ_1 is the trivial character, and so it remains to find the other three. Let ω be a primitive cube root of unity in k. We have the isomorphism $A_5 \cong \mathrm{SL}_2(4)$, and so there is a natural representation of A_5 as 2×2 matrices over GF(4). This is given by

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Of course, GF(4) has a field automorphism of order 2, since it is field of order q^2, and so we can apply this field automorphism to get another representation

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} 1 & 0 \\ \omega^2 & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

These two representations are irreducible. The Brauer characters afforded by these two representations will be labelled ϕ_2 and ϕ_3.

Finally, we have the 4-dimensional representation of A_5 as $\Omega_4^-(2)$, as 4-dimensional matrices over GF(2),

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This is irreducible as well, and we label the Brauer character afforded by this representation ϕ_4. To calculate the Brauer character, we calculate the eigenvalues of the matrix, and express them as roots of unity inside k. Then we pull back to our field K, where we sum the eigenvalues to give the Brauer character. For example, in the 4-dimensional representation of $(1\,3\,5\,2\,4)$, which has matrix

$$(1\,3\,5\,2\,4) \sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

we find that the characteristic polynomial is $x^4 + x^3 + x^2 + x + 1$, and so the eigenvalues are the fifth roots of unity other than 1, in k. Pulling back to K, we sum the non-unital fifth roots of unity and this is equal to -1. So $\phi_4\big((1\,3\,5\,2\,4)\big) = -1$. Doing this for all of the representations, we have the table

ϕ_i	1	$(1\,2\,3)$	$(1\,2\,3\,4\,5)$	$(1\,3\,5\,2\,4)$
ϕ_1	1	1	1	1
ϕ_2	2	-1	$\alpha-1$	$\beta-1$
ϕ_3	2	-1	$\beta-1$	$\alpha-1$
ϕ_4	4	-1	-1	-1

where α and β are as before. From these two tables we can easily see that $\chi_1 = \phi_1$, $\chi_2 = \phi_1 + \phi_2$, $\chi_3 = \phi_1 + \phi_3$, $\chi_4 = \phi_4$ and $\chi_5 = \phi_1 + \phi_2 + \phi_3$. Now two irreducible ordinary characters χ_i and χ_j lie in the same block if and only if $|\mathscr{C}_s|\chi_i(x_s)/\chi_i(1) \equiv |\mathscr{C}_s|\chi_j(x_s)/\chi_j(1)$ for all conjugacy classes $\mathscr{C}_s$, where $x_s \in \mathscr{C}_s$. This means that χ_1, χ_2, χ_3 and χ_5 all lie in the principal block $B_0(G)$, and the remaining character χ_4 lies in the other block $B_1(G)$. Since $\phi_4 = \chi_4$ on G°, ϕ_4 lies in $B_1(G)$, and all of the other modular characters lie in $B_0(G)$. The principal block has defect group a Sylow 2-subgroup, and so has defect group $C_2 \times C_2$. The block $B_1(G)$ has defect zero.

The decomposition numbers for G are

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
χ_1	1	0	0	0
χ_2	1	1	0	0
χ_3	1	0	1	0
χ_4	0	0	0	1
χ_5	1	1	1	0

and we can use these decomposition numbers to get the Cartan matrix, namely

$$\Gamma = \begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The algebra kG decomposes into $\bigoplus_{i=1}^4 M_i$, where M_i is an indecomposable module with $M_i/J(M_i)$ affording the character ϕ_i. The Cartan matrix tells us the irreducible constituents in a decomposition of M_i.

We now consider the subalgebra $k\,\mathrm{N}_G(C_2 \times C_2) = kN$, and decompose it into blocks, to see the Brauer Correspondence. Since $C_2 \times C_2 \trianglelefteq A_4$, it lies in the kernel of every irreducible 2-representation, and so there are three representations which can be lifted from the abelian quotient C_3. These are therefore given by $(1\ 2\ 3) \mapsto \omega^i$, where ω is a primitive cube root of unity and $i = 0, 1, 2$. We can easily find the Brauer character of a linear representation, and the Brauer character table is given by

ψ_i	1	$(1\,2\,3)$	$(1\,3\,2)$
ψ_1	1	1	1
ψ_2	1	ω	ω^2
ψ_3	1	ω^2	ω

From this we can easily see that $\zeta_1 = \psi_1$, $\zeta_2 = \psi_2$, $\zeta_3 = \psi_3$ and $\zeta_4 = \psi_1 + \psi_2 + \psi_3$. The fact that ζ_4 is made up from all of the modular characters means that all modular and all ordinary characters lie in the principal block $B_0(N)$, which has defect 2. By Brauer's First or Third Main Theorem, $B_0(N)^G = B_0(G)$. There are no more defect groups in this characteristic. For completeness, we give the Cartan matrix for kH,

$$\Gamma_H = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Characteristic 3

In characteristic 3 there are four irreducible modular representations, of dimensions 1, 3, 3 and 4. We will denote the Brauer characters afforded by these ϕ_1, ϕ_2, ϕ_3 and ϕ_4, and let ϕ_1 be the trivial character. We have the irreducible 3-dimensional representation of G over GF(9) given by

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \lambda & \lambda & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where λ denotes a generator for GF(9), (so that $\lambda^4 = -1$ and $\lambda^8 = 1$). Since GF(9) has a field automorphism mapping λ to λ^3, we have another representation

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \lambda^3 & \lambda^3 & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Finally, we have a 4-dimensional irreducible representation over GF(3), given by

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

These give rise to the four irreducible Brauer characters, which we will calculate by other means.

Consider the blocks of kG. They have defect either 1 or 0. Now an irreducible ordinary character χ lies in a block of defect 0 if $3 \mid \zeta(1)$, and in a block of defect 1 if $3 \nmid \zeta(1)$. This means that χ_2 and χ_3 lie in blocks of defect 0, and χ_1, χ_4 and χ_5 lie in blocks of defect 1. If an ordinary character and a modular character lie in different blocks, the ordinary character does not involve the modular character in its decomposition. So χ_2 can only involve modular characters in its own block. But if a block has defect 0, then there is only one irreducible modular and ordinary character in the block. So there are at least three blocks, say $B_0(G)$ (principal block), $B_1(G)$, the block containing χ_2 (and without loss of generality, ϕ_2), and $B_2(G)$, the block containing χ_3 (and therefore ϕ_3). Given that $\phi_1 = 1_{G^\circ}$, so far we have constructed the table

ϕ_i	1	$(1\,2)(3\,4)$	$(1\,2\,3\,4\,5)$	$(1\,3\,5\,2\,4)$
ϕ_1	1	1	1	1
ϕ_2	3	-1	α	β
ϕ_3	3	-1	β	α
ϕ_4	4			

and we are left to decide where χ_4, χ_5, and ϕ_4 go, whether into the principal block or into some other block. Now ϕ_4 is the only modular character left, so there can be at most one other block. If ϕ_4 were in a different block from $B_0(G)$, then both χ_4 and χ_5 cannot be expressed as a multiple of the trivial character, and so must lie in the same block as ϕ_4. But certainly they cannot be both expressed as a multiple of *any* single Brauer character, so cannot lie in the same block as ϕ_4 on its own either. This means that ϕ_4 lies in the principal block, as does χ_4 and χ_5.

We have three blocks, $B_0(G)$ containing ϕ_1, ϕ_4, χ_1, χ_4, and χ_5, $B_1(G)$ containing ϕ_2 and χ_2, and $B_2(G)$ containing ϕ_3 and χ_3. Then χ_4 can be expressed as a non-negative linear combination of ϕ_1 and ϕ_4. This means that either $\chi_4 = 4\phi_1$ (absurd) or $\chi_4 = \phi_4$, and we have the full table, as shown below.

ϕ_i	1	$(1\,2)(3\,4)$	$(1\,2\,3\,4\,5)$	$(1\,3\,5\,2\,4)$
ϕ_1	1	1	1	1
ϕ_2	3	-1	α	β
ϕ_3	3	-1	β	α
ϕ_4	4	1	-1	-1

The decomposition numbers were found in the discussion above, and they are given below, together with the Cartan matrix they yield.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
χ_1	1	0	0	0
χ_2	0	1	0	0
χ_3	0	0	1	0
χ_4	0	0	0	1
χ_5	1	0	0	1

$$\Gamma = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

The Brauer Correspondence in this case is even less interesting than in the characteristic 2 case. Since $\mathrm{N}_G(P) \cong S_3$, there are only two irreducible modular characters, since C_3 is a normal subgroup. These are both linear, and so the ordinary character of degree 2 of S_3 must be the sum of these two. Hence again there is only one block, $B_0(\mathrm{N}_G(P))$, naturally of defect 1, and $B_0(\mathrm{N}_G(P))^G = B_0(G)$.

Characteristic 5

There are just three conjugacy classes of 5-regular elements, and so there are three irreducible 5-representations. Along with the trivial representation, we have the irreducible representations over GF(5) given by

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & 1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$(1\,2)(3\,4) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}, \qquad (1\,3\,5) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

We again calculate the modular characters without resorting to characteristic polynomials. Label the trivial character ϕ_1, the degree 3 character ϕ_2, and the degree 5 character ϕ_3. It seems likely that ϕ_1 and ϕ_2 lie in the principal block along with all but one of the ordinary characters, and ϕ_3 lies in its own block along with χ_5. This is indeed the case. The ordinary character χ_5 has degree 5, and it lies in a block of defect 0 (since the maximal power of 5 in $|G|$ divides the degree of the representation). It has to take with it a single modular character, and this modular character must have degree a multiple of the maximal power of 5 dividing $|G|$. Thus χ_5 and ϕ_3 are separated off in a block of defect 0. Thus $\phi_3 = \chi_5$ on the 5-regular elements of G.

It remains to find ϕ_2, which lies in the principal block, else we could not get all of the ordinary characters from multiples of modular characters. But χ_3 is a positive linear combination of ϕ_1 and ϕ_2, and so $\phi_2 = \chi_2$ on the 5-regular elements of G. Then $\chi_3 = \phi_2$ and $\chi_4 = \phi_1 + \phi_2$, and so we have the Brauer character table

ϕ_i	1	(1 2 3)	(1 2)(3 4)
ϕ_1	1	1	1
ϕ_2	3	0	-1
ϕ_3	5	-1	1

We have determined the decomposition numbers. These are the Cartan matrix are given below.

	ϕ_1	ϕ_2	ϕ_3
χ_1	1	0	0
χ_2	0	1	0
χ_3	0	1	0
χ_4	1	1	0
χ_5	0	0	1

$$\Gamma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Brauer Correspondence is uninteresting in this case as well. Consider the normalizer of the group $C_5 = P$ in G. Then $\mathrm{N}_G(P) = D_{10} = M$, and since $C_5 \trianglelefteq M$, any irreducible 5-representation must have kernel at least P. So there are two irreducible 5-representations of D_{10}, lifted from the two 5-representations of $M/P \cong C_2$,

$$(2\,5)(3\,4) \mapsto 1, \qquad (2\,5)(3\,4) \mapsto -1.$$

There is only one block, $B_0(M)$, which has defect 1, and $B_0(M)^G = B_0(G)$.

4.2 Preliminary Lemmas – Decomposition Numbers Revisited

In the proofs of the Brauer–Suzuki Theorem and Glauberman Z*-Theorem we will need quite a few results on decomposition numbers with respect to an arbitrary basis of $\mathrm{Char}(G^\circ)$. We will give these without proof: the interested reader is referred to [79].

Firstly we will consider the following situation. In the section on Brauer's Second Main Theorem, we defined the decomposition numbers to be the integers d_{ij} such that $\chi_i(x) = \sum_{j=1}^{n} \psi_j(x)$, where χ_i is an irreducible ordinary character and the ψ_j are the irreducible modular characters. We could write χ_i in such a way because the irreducible modular characters form a basis for $\mathrm{Char}(G^\circ)$. But there is nothing stopping us from choosing any other $\mathbb{Z}$-basis for $\mathrm{Char}(G^\circ) = M$ and decomposing χ_i relative to that. In fact we can decompose ψ_j relative to any $\mathbb{Z}$-basis, and so relative to an arbitrary basis $\bar{\psi}_1, \ldots, \bar{\psi}_n$, we have

$$\chi_i = \sum_{j=1}^{n} a_{ij} \bar{\psi}_j.$$

These are called the decomposition numbers relative to the bases $\bar{\psi}_i$. We will always place bars on top of any basis element which is not an irreducible modular character, to distinguish it from the natural basis. We define the Cartan matrix relative to the basis $\bar{\psi}_i$ as $\Delta^T \Delta$, where Δ is the matrix of decomposition numbers relative to that basis.

We can also define generalized decomposition numbers relative to any basis. Let z be a p-singular element of G, and let $H = \mathrm{C}_G(z)$. Then the irreducible ordinary characters of G χ_i, restricted to H, can be written as an algebraic combination of the irreducible modular characters of H, ψ_j, by

$$\chi_i(zx) = \sum_{j=1}^{n} d_{ij}^z \psi_j(x).$$

Now let $\bar{\psi}_j$ be a $\mathbb{Z}$-basis for $\mathrm{Char}(H^\circ)$. Then again we can write ψ_l as an integral linear combination of $\bar{\psi}_j$, and so we have

$$\chi_i(zx) = \sum_{j=1}^{n} d_{ij}^z \bar{\psi}_j(x).$$

Note that the d_{ij}^z in the two equations are different. These are called the generalized decomposition numbers relative to the basis $\bar{\psi}_i$.

We now give several results regarding generalized decomposition numbers: again, the interested reader is referred to [79], and in this case [35].

Theorem 4.1 Let the Cartan matrix of the basis $\bar{\psi}_i$ be given by c_{ij}. Then

$$\sum_{l=1}^{n} d_{li}^z \overline{d_{lj}^z} = c_{ji},$$

where the bar denotes complex conjugate. If t is a non-trivial p-singular element of G not conjugate to z, and d_{ij}^t are the generalized decomposition numbers at t relative to a $\mathbb{Z}$-basis $\bar{\psi}_i$ of $\mathrm{Char}(\mathrm{C}_G(t)^\circ)$, then

$$\sum_{l=1}^{n} d_{lj}^z \overline{d_{lj}^t} = 0.$$

Corollary 4.2 Let t be a p-singular element of G. Denote by $\chi_1, \ldots, \chi_s$ the irreducible ordinary characters in the principal block of kG, and denote by $\zeta_1, \ldots, \zeta_r$ the irreducible ordinary characters in the principal block of $k\,\mathrm{C}_G(t)$. Let $x \in \mathrm{C}_G(t)^\circ$. Then

(i) $\sum_{i=1}^{s} |\chi_i(tx)|^2 = \sum_{i=1}^{r} |\zeta_i(x)|^2$.

(ii) If y is another p-singular element not conjugate to t, then $\sum_{i=1}^{s} \chi_i(tx)\overline{\chi_i(yx)} = 0$.

(iii) If $\mathrm{C}_G(t)$ has a normal p-complement, then $\sum_{i=1}^{s} |\chi_i(tx)|^2 = |P|$, where P is a Sylow p-subgroup of $\mathrm{C}_G(t)$.

Lemma 4.3 Let t be a non-trivial 2-singular element of G, and let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two classes of involutions such that no element of $\mathscr{C}_1\mathscr{C}_2$ has t as its 2-part. Let $\chi_1, \ldots, \chi_s$ denote the ordinary characters in the principal block of G, and let $\bar{\psi}_1, \ldots, \bar{\psi}_r$ be a $\mathbb{Z}$-basis of $\mathrm{Char}(\mathrm{C}_G(t)^\circ)$ such that the first n of them form a $\mathbb{Z}$-basis for the submodule generated by modular characters lying in the principal block of $k\,\mathrm{C}_G(t)$. Let d_{ij}^t denote the generalized decomposition numbers relative to the basis above. If $x \in \mathscr{C}_1$ and $y \in \mathscr{C}_2$ we have

$$\sum_{i=1}^{s} d_{ij}^t \frac{\chi_i(x)\chi_i(y)}{\chi_i(1)} = 0,$$

for $j = 1, \ldots, n$, and

$$\sum_{i=1}^{s} \frac{\chi_i(t)\chi_i(x)\chi_i(y)}{\chi_i(1)} = 0.$$

Theorem 4.4 Let $B = B_0(G)$, the principal block of kG. If $x \in \mathrm{O}_{p'}(G)$, then $x \in \ker \phi$ for each irreducible modular character ϕ.

Lemma 4.5 Suppose that a Sylow p-subgroup P of a group G is isomorphic to $C_2 \times C_2$, and that G contains only one conjugacy class of involutions. Then there are exactly four irreducible ordinary characters in the principal block $B_0(G)$, labelled $\chi_1 = 1$, χ_2, χ_3 and χ_4, and if the χ_i are chosen in a particular order, then the restrictions of χ_1, χ_2 and either χ_3 or $-\chi_3$ form a $\mathbb{Z}$-basis for the submodule of $\mathrm{Char}(G^\circ)$ generated by the modular characters in $B_0(G)$. Furthermore, the Cartan matrix Δ with respect to this basis is $\Delta_{ij} = 1 + \delta_{ij}$, where δ_{ij} is the Kronecker delta.

4.3 Groups of 2-Rank 1

Definition 4.6 Let G be a (non-abelian) finite simple group. Then the *p-rank* of G is the rank of a maximal elementary abelian p-subgroup of G. The 2-*local p-rank* of G is the maximal p-rank of a 2-local subgroup of G.

By the Feit–Thompson Theorem, every non-abelian finite simple group is of even order, and therefore has a Sylow 2-subgroup. We can classify finite simple groups according to their 2-rank.

Lemma 4.7 Suppose that G has 2-rank 1. Then a Sylow 2-subgroup of G is either cyclic or generalized quaternion.

Suppose that a Sylow 2-subgroup P of G is cyclic, say $P = \langle x \rangle$. The automorphism group of P has order 2^{n-1}, where $|P| = 2^n$, since each automorphism sends a generator for p to one of the $\phi(2^n) = 2^{n-1}$ other generators. But $\mathrm{N}_G(P)/\mathrm{C}_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}\, P$. So since $\mathrm{C}_G(P)$ contains P (recall P is abelian), $\mathrm{N}_G(P)/\mathrm{C}_G(P)$ has odd order. Thus $\mathrm{N}_G(P) = \mathrm{C}_G(P)$. By Burnside's Transfer Theorem, G has a normal p-complement.

Now suppose that a Sylow 2-subgroup of G is generalized quaternion. The Brauer–Suzuki Theorem is as follows.

Theorem 4.8 (Brauer–Suzuki Theorem) Let G be a group with a generalized quaternion Sylow 2-subgroup of order at least 16. Then $\mathrm{Z}\left(G/\mathrm{O}_{2'}(G)\right)$ is of order 2. In particular, G is not simple.

The proof of this theorem, whilst able to be done using modular representation theory, is more often done using exceptional character theory, and is not of concern here – for a proof, see [48]. The Brauer–Suzuki Theorem also holds when the Sylow 2-subgroup has order 8, and this is the case which interests us here. The proof of this fact is best done using modular characters, although Glauberman has produced a proof without modular character theory. The proof of the Brauer–Suzuki Theorem in the case of quaternion Sylow 2-subgroups is the main aim of this section.

Theorem 4.9 Let G be a group with a Sylow 2-subgroup P which is a quaternion group. Then $\mathrm{Z}\left(G/\mathrm{O}_{2'}(G)\right)$ has order 2.

The proof of this theorem goes in stages: P is generated by two elements x and y of order 4, with $x^2 = y^2$ and $x^{-1}yx = y^{-1}$. We denote by t the (unique) involution in P. Suppose that G is a counterexample to the theorem of minimal order. We first quotient by $\mathrm{O}_{2'}(G)$. Notice that the Sylow 2-subgroup of $G/\mathrm{O}_{2'}(G)$ is also quaternion, so that $G/\mathrm{O}_{2'}(G)$ is a counterexample to the theorem whenever G is. So since G is minimal we can assume that $\mathrm{O}_{2'}(G) = 1$.

Lemma 4.10 All elements of order 4 in P are conjugate in $\mathrm{N}_G(P)$.

Proof: The six elements of order 4 are x, y, xy, x^{-1}, y^{-1} and $(xy)^{-1}$. The other elements are t (of order 2) and 1. Now $x^{-1}yx = y^{-1}$, so y and y^{-1} are conjugate. But by the symmetry of $P = Q_8$, x and x^{-1} and xy and $(xy)^{-1}$ are conjugate. (Alternatively, $y^{-1}xy = x^{-1}$ and $x^{-1}(xy)x = (xy)^{-1}$.) So there are at most three conjugacy classes of elements of order 4 in $\mathrm{N}_G(P)$, namely $\{y, y^{-1}\}$, $\{x, x^{-1}\}$, and $\{xy, (xy)^{-1}\}$. If these are not all conjugate in G, then one of them is not conjugate to either of the others. Assume without loss of generality that $\{x, x^{-1}\}$ is a conjugacy class of $\mathrm{N}_G(P)$.

Since in a Sylow p-subgroup Q, two elements of Q are conjugate in G if and only if they are conjugate in $\mathrm{N}_G(Q)$ (standard result that will not be proven here), this means that $\{x, x^{-1}\}$ is not conjugate to either of the other two sets in G either. So let $A = \{x, x^{-1}, t, 1\}$ and $B = \{y, y^{-1}, xy, (xy)^{-1}\}$. Then no element of A is conjugate to an element of B, and $A \cup B = P$. Let

$\chi : G \to K$ be the class function given by $\chi(g) = 1$ if the p-part of g lies in A and $\chi(g) = -1$ if it lies in B.

We want to show that this is a character of G. Since it is linear ($\chi(1) = 1$) it will therefore be an irreducible character of G. Consider two elements ag and $a'g'$ of G (where a, a' are the p-parts of ag and $a'g'$). Then $\chi(aga'g') = \chi(a(ga'g^{-1})gg')$. Since no element of A is conjugate to an element of B, both $\chi(a') = \chi(ga'g^{-1})$. So $\chi(aga'g') = \chi(aa')$, since $\chi(ag) = \chi(a)$ for all p'-parts g. Notice also that the product of any two elements of A is in A, as is the product of any two elements of B. However, the product of one element of B and one element of A is in B. Hence $\chi(aa') = \chi(a)\chi(a')$ for all elements $a, a' \in P$, and χ is a linear character.

Also note that $|G/\ker\chi| = 2$. Now every subgroup of P of index 2 is cyclic, so $\ker\chi$ has a cyclic Sylow 2-subgroup, and so by Burnside's Transfer Theorem $\ker\chi$ has a normal 2-complement L. But $L \operatorname{char} \ker\chi \trianglelefteq G$, so $L \trianglelefteq G$. But then $L \leqslant \mathrm{O}_{2'}(G) = 1$, so $L = 1$ and $\ker\chi$ is a 2-group, and thus so is G since $|G/\ker\chi| = 2$. Thus $G = P$, and so G satisfies the theorem. So since G is a minimal counterexample, all elements of order 4 in P are conjugate in $\mathrm{N}_G(P)$. □

We shall label the irreducible ordinary characters of G that lie in the principal 2-block $B_0(G)$ as $\chi_1, \chi_2, \ldots, \chi_s$. We now aim to determine $\chi_i(x)$ for each i.

Firstly notice that since x has order 4, the eigenvalues of any representation of x will be fourth roots of unity. Let T be a representation, and let the eigenvalues of $T(x)$ have multiplicities m_1, m_{-1}, m_i and m_{-i} respectively. Then $\chi_i(x) = m_1 - m_{-1} + (m_i - m_{-i})i$. Now $\chi_i(x^2) = \chi_i(t) = m_1 + m_{-1} - m_i - m_{-i}$ (since $(-1)^2 = 1$, etc.), and $\chi_i(1) = m_1 + m_{-1} + m_i + m_{-i}$. Also $\chi_i(x^{-1}) = m_1 - m_{-1} - i(m_i - m_{-i})$, and since $\chi_i(x) = \chi_i(x^{-1})$, we must have $m_3 - m_4 = 0$. Then $\chi_i(x) = m_1 - m_{-1}$, $\chi_i(t) = m_1 + m_{-1}$, and $\chi_i(1) = m_1 + m_{-1} + 2m_{-i}$, and so $\chi_i(x)$ is congruent to $\chi_i(t) \mod 2$, and $\chi_i(1) \mod 2$. This is true for all characters χ_i in the principal 2-block.

Now $x \notin \mathrm{Z}(P)$, and so $\mathrm{C}_G(x) < G$. Since x is of order 4, the Sylow 2-subgroup of $\mathrm{C}_G(x)$ must be $\langle x\rangle$. But this means that $\mathrm{C}_G(P)$ has a cyclic Sylow 2-subgroup, and so has a normal 2-complement. Then by Corollary 4.2(ii), $\sum_{i=1}^{s} |\chi_i(x)| = |\langle x\rangle| = 4$.

Now $\chi_1(x) = 1_G(x) = 1$, so the sum of the squares of the remaining characters χ_i for $2 \leqslant i \leqslant s$ is 3. This must mean that $\chi_i(x) = \pm 1$ for three of the characters, and $\chi_i(x) = 0$ for the remaining characters. In this case $\chi_i(1)$ is odd for three of the characters, and even for the rest.

Denote by H a subgroup $\mathrm{C}_G(t)$, and let $\bar{H} = H/\langle t\rangle$. If α is a $2'$-element of H, then it is the only one in its coset of $\langle t\rangle$ (since this coset is $\{\alpha, t\alpha\}$). Conversely, if $\{\beta, t\beta\}$ is a $2'$-element in $\bar{H}$, then $\beta^n = 1$ or $(t\beta)^n = 1$, where n is odd, providing us with a $2'$-element of H. So there is a bijection between the set of $2'$-elements of H and the $2'$-elements of $\bar{H}$. Let $\bar{P}$ denote the image of P under this quotient. We know that $Q_8/\mathrm{Z}(Q_8) \cong C_2^2$, the direct product of two C_2 groups. Any element of G that normalizes P must centralize t since $\{t\}$ is a characteristic subset of P. Since all elements of order 4 are conjugate in $\mathrm{N}_G(P) \subseteq \mathrm{C}_G(t)$, their images – all three involutions in $\bar{P}$ – are conjugate in $\bar{H}$. By Lemma 4.5, the submodule M of $\mathrm{Char}(H^\circ)$ corresponding to the principal block $B_0(\bar{H})$ has a basis (as a $\mathbb{Z}$-module) consisting of three generalized modular characters, which we denote $\bar{\psi}_1$, $\bar{\psi}_2$ and $\bar{\psi}_3$. Let $\bar{\psi}_1$ be the trivial modular character of $\bar{H}$. The Cartan matrix relative to this basis has the form $c_{ij} = 1+\delta_{ij}$. Now $\langle t\rangle$ is a normal 2-subgroup of H, and so every irreducible representation of H over a field of characteristic 2 has kernel including $\langle t\rangle$. So we can link the irreducible modular characters of H to those of $\bar{H}$ by $\psi(x) \mapsto \bar{\psi}(x\langle t\rangle)$. In particular, we can map upwards the three generalized modular characters $\bar{\psi}_1$, $\bar{\psi}_2$ and $\bar{\psi}_3$ to three generalized modular characters of H, which generate the submodule of $\mathrm{Char}(H^\circ)$ corresponding to the principal block $B_0(H)$. We have

$$\langle \psi_i, \psi_j\rangle_{H^\circ} = \frac{1}{2}\langle \psi_i, \psi_j\rangle_{\bar{H}^\circ},$$

since $|H^\circ| = |\bar{H}^\circ|$, but $|H| = 2|\bar{H}|$. This means that the Cartan matrix for ψ_1, ψ_2, ψ_3 is twice that of $\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3$, namely $c_{ij} = 2 + 2\delta_{ij}$

Finally, the principal block of $k\bar{H}$ has defect 2, since it has Sylow C_2^2. This means that the degrees of the irreducible modular characters in $B_0(H)$ have odd degree. By this correspondence just described, the degrees of the irreducible modular characters in $B_0(\bar{H})$ are also odd.

Now we shall apply the Second and Third Main Theorems to obtain information about the decomposition relative to the basis ψ_1, ψ_2 and ψ_3.

Proposition 4.11 With the notation introduced previously in this section, t lies in the kernel of χ_2.

Proof: We know that ψ_1, ψ_2 and ψ_3 form a basis for $\mathrm{Char}(H^\circ)$, and so by the Second and Third Main Theorems

$$\chi_i(th) = \sum_{j=1}^{3} a_{ij}\psi_j(h),$$

for all 2-regular $h \in H$. Now t has order 2, so the λ_i in the definition of the (ordinary) generalized decomposition numbers are ± 1, and so in particular are integers. Also, since ψ_1, ψ_2, ψ_3 form a $\mathbb{Z}$-basis of $\mathrm{Char}(H^\circ)$, each modular character in $B_0(H)$ can be expressed as an integer combination of these three modular characters. This means that a_{ij} is an integer for all i and j.

By Theorem 4.1, with the Cartan matrix $c_{ij} = 2 + 2\delta_{ij}$,

$$\sum_{i=1}^{s} a_{ij}^2 = 4,\ \ (1 \leqslant j \leqslant 3), \qquad \sum_{l=1}^{s} a_{li}a_{lj} = 2\ \ (1 \leqslant i \neq 2 \leqslant 3).$$

We know that the degrees of the irreducible modular characters in $B_0(H)$ have odd degree, and so

$$\chi_i(1) \equiv \chi_i(t) = \sum_{j=1}^{3} a_{ij}\psi_j(1) \equiv \sum_{j=1}^{3} a_{ij} \mod 2,$$

since $\chi_i(t) \equiv \chi_i(1) \mod 2$ and $\psi_j(1) \equiv 1 \mod 2$.

Next, the ordinary orthogonality relations give

$$0 = \sum_{i=1}^{s} \chi_i(1)\chi_i(th) = \sum_{j=1}^{3}\sum_{i=1}^{s} a_{ij}\chi_i(1)\psi_j(h),$$

from the expression we have for $\chi_i(th)$. Since the ψ_j are linearly independent, for a linear combination of them to be 0, each co-efficient must be 0, and so

$$\sum_{i=1}^{s} a_{ij}\chi_i(1) = 0.$$

Consider the matrix $A = [a_{ij}]$. Suppose a row of A is full of zeros; i.e., $a_{ij} = 0$ for $j = 1, 2, 3$. Then $\chi_i(t) = 0$ and so $\chi_i(1)$ is even since it is even whenever $\chi_i(t)$ is. Since $\chi_i(1)$ is even, we have shown before that $\chi_i(x) = 0$, and so the only element $g \in P$ for which $\chi_i(g) \neq 0$ is $g = 1$. So

$$\langle \chi_i, 1_P \rangle_P = 8^{-1}\chi_i(1).$$

Thus $\chi_i(1)$ is a multiple of 8, and so χ_i lies in a block of defect 0 since 8 is the highest power of 2 dividing $|G|$. But $\chi_i \in B_0(G)$, a contradiction. So no row of A is zero.

Since the sum of the squares of a_{ij} down the columns is 4, if any of the a_{ij} were not equal to 0 or ± 1, $a_{ij} = \pm 2$ and then all other entries in that column would be zero. Then $\sum a_{ij}\chi_i(1) \neq 0$, contradicting a previous assertion. So each entry in A is 0 or ± 1. Also, each column has four non-zero entries since the sum of their squares is 4.

So far we have not used the fact that $\sum_{l=1}^{s} a_{li}a_{lj} = 2$ for $i \neq j$. We can use this to show that the non-zero entries of any row are the same sign. Indeed, suppose that for column i and column j, $a_{li} = 1$ and $a_{lj} = -1$. Then their product is -1, and since there are only four non-zero entries in each column, for all other rows l', $a_{l'i} = a_{l'j}$. But then the sum $\sum_{l'=1}^{t} a_{l'i}\chi_{l'}(1)$ is not the same as $\sum_{l'=1}^{t} a_{l'j}\chi_{l'}(1)$, since $a_{li} \neq a_{lj}$ and all other terms are equal. But both of these are meant to be zero, a clear contradiction.

How many rows of the matrix a_{ij} are there; that is, what is s? Well, each row has either one, two, or three non-zero entries, and so we can denote the number of rows with i non-zero entries by r_i. We need three simultaneous equations for these three unknowns. Firstly,

$$\sum_{i=1}^{s} a_{ij}^2 = 4 \Longrightarrow r_1 + 2r_2 + 3r_3 = 12, \qquad (1)$$

by adding together a_{ij}^2 for all i and j. On the one hand, it is equal to 3×4, and on the other, it is equal to $r_1 + 2r_2 + 3r_3$.

Next,

$$\sum_{l=1}^{s} a_{li}a_{lj} = 2 \Longrightarrow r_2 + 3r_3 = 6, \qquad (2)$$

since summing $i = 1$ and $j = 2$, $i = 1$ and $j = 3$, and $i = 2$ and $j = 3$, and noticing that the non-zero entries in each row are of the same sign, we count each row with two non-zero entries once and each row with three non-zero entries three times. This is also equal to $2 \times 3 = 6$.

Finally,

$$\sum_{j=1}^{3} a_{ij} \equiv \chi_i(1) \mod 2 \Longrightarrow r_1 + r_3 = 4, \qquad (3)$$

since exactly four of the $\chi_i(1)$ are odd, and exactly $r_1 + r_3$ of the sums across j of a_{ij} are odd.

There is a unique solution to these three simultaneous equations, namely $r_1 = 3$, $r_2 = 3$ and $r_3 = 1$. [(2) − (3) gives $r_2 + r_3 = 4$, which combined with (1) gives $r_3 = 1$. Substituting this into (3) and (1) gives $r_2 = 3$ and $r_1 = 3$.] Thus $s = 7$. $\chi_1 = \psi_1$, and so there are three other rows with $a_{i1} = \pm 1$, say $i = 2, 3, 4$. One of these, say $i = 2$, is the unique row with three non-zero entries. Since two distinct columns i and j have non-zero entries in exactly two of the same rows, we have χ_3 involving ψ_1 and ψ_2, χ_4 involving ψ_1 and ψ_3, and χ_5 involving ψ_2 and ψ_3. This leaves χ_6 and χ_7 as involving only ψ_2 and only ψ_3 respectively, since each column has to have four non-zero entries. So the table is as below.

χ_i	ψ_1	ψ_2	ψ_3
χ_1	1	0	0
χ_2	α_2	α_2	α_2
χ_3	α_3	α_3	0
χ_4	α_4	0	α_4
χ_5	0	α_5	α_5
χ_6	0	α_6	0
χ_7	0	0	α_7

Now consider $\chi_i(x)$. We know that it is equal to 0 for three of the characters, and for these characters $\chi_i(1)$ is even – we need to determine which ones. Well, we know that $\sum_{j=1}^{3} a_{ij} \equiv \chi_i(1) \mod 2$, and we can evaluate the left-hand side of this easily. Then $\chi_i(1)$ is even for $i = 3, 4, 5$ and odd for the rest. So $\chi_i(x) = 0$ for $i = 3, 4, 5$ and for the remaining characters $\chi_i(x) = \pm 1$. Let β_i denote the value that $\chi_i(x)$ takes.

Consider $\mathrm{C}_G(x)$. Now a Sylow 2-subgroup of $\mathrm{C}_G(x)$ must be contained in a Sylow 2-subgroup of G, and if so must be either equal to $\langle x \rangle$ or a quaternion group. But then x would centralize a quaternion group, so lie in its centre, which is not possible. So $\mathrm{C}_G(x)$ has a cyclic Sylow 2-subgroup, and therefore has a normal 2-complement. [Notice that this means that $\mathrm{C}_G(x)$ is actually the direct product of $\langle x \rangle$ with a group of odd order.]

Putting this together, a basis for $\mathrm{Char}(\mathrm{C}_G(x)^\circ)$ consists of one element, the trivial character χ_1. Then the generalized decomposition numbers d^z_{ij} are simply the values $\chi_i(x)$. Applying Theorem 4.1 to the sets of generalized decomposition numbers d^t_{li} and d^z_{l1} gives (recalling $\beta_i = 0$ for $i = 3, 4, 5$)

$$1 + \alpha_2\beta_2 = 0, \quad (i = 1), \qquad \alpha_2\beta_2 + \alpha_6\beta_6 = 0, \quad (i = 2), \qquad \alpha_2\beta_2 = \alpha_7\beta_7 = 0, \quad (i = 3).$$

Remembering that $\alpha_i, \beta_i = 0, \pm 1$, we get $\alpha_2 = -\beta_2$ from the first equation, and then $\alpha_6 = \beta_6$, $\alpha_7 = \beta_7$ for the second and third equations.

Now if a and b are involutions in G, then ab cannot be 2-singular (and not equal to 1). To see this, suppose that ab is, in fact, 2-singular. Then $\langle a, b \rangle = 2o(ab)$ is a power of 2, and so since $\langle a, b \rangle$ is a dihedral group, it is a subgroup of Q_8. But Q_8 contains only one involution, so $a = b$ and $ab = 1$.

By Lemma 4.3, with $\mathscr{C}_1 = \mathscr{C}_2$ the unique class of involutions of G, we have for t,

$$\sum_{i=1}^{7} a_{ij} \frac{\chi_i(t)^2}{\chi_i(1)} = 0, \tag{1,2,3}$$

and for the second equation

$$1 + \beta_2 \frac{\chi_2(t)^2}{\chi_2(1)} + \beta_6 \frac{\chi_6(t)^2}{\chi_6(1)} + \beta_7 \frac{\chi_7(t)^2}{\chi_7(1)} = 0.$$

The a_{ij} are given in the table above, and replacing the β_i with the corresponding expressions involving α_i gives

$$1 - \alpha_2 \frac{\chi_2(t)^2}{\chi_2(1)} + \alpha_6 \frac{\chi_6(t)^2}{\chi_6(1)} + \alpha_7 \frac{\chi_7(t)^2}{\chi_7(1)} = 0. \tag{4}$$

We have four equations, the three (1), (2) and (3) depending on $j = 1, 2, 3$ and Equation (4). Consider the sum (1) − (2) − (3) + (4). Then this is

$$\sum_{i=1}^{7} (a_{i1} - a_{i2} - a_{i3} + \chi_i(x)) \frac{\chi_i(t)^2}{\chi_i(1)} = 0.$$

It remains to determine the terms in this sum. Substituting in our expressions for α_i (and dividing by 2) gives

$$1 - \alpha_2 \frac{\chi_2(t)^2}{\chi_2(1)} - \alpha_5 \frac{\chi_5(t)^2}{\chi_5(1)} = 0. \tag{5}$$

Consider Lemma 4.3 again, but instead we use 1 rather than t. So we have the equations

$$\sum_{i=1}^{7} a_{ij} \chi_i(1) = 0, \tag{6,7,8}$$

and

$$1 - \alpha_2\chi_2(1) + \alpha_6\chi_6(1) + \alpha_7\chi_7(1) = 0. \tag{9}$$

(Here we have substituted in the expressions for β_i in terms of α_i.) In fact, these are exactly the same equations as before, except with $\chi_i(1)$ instead of $\chi_i(t)^2/\chi_i(1)$. So (6) − (7) − (8) + (9) gives

$$1 - \alpha_2\chi_2(1) - \alpha_5\chi_5(1) = 0. \tag{10}$$

Finally consider the decomposition numbers: $\chi_2(t) = \alpha_2(1 + \psi_2(t) + \psi_3(t))$ and $\chi_5(t) = \alpha_5(\psi_2(t) + \psi_3(t))$. So $\alpha_5\chi_2(t) = \alpha_2\alpha_5 + \alpha_2\chi_5(t)$, and since $\alpha_i = \pm 1$, we can divide by $\alpha_2\alpha_5$ to get

$$1 - \alpha_2\chi_2(t) + \alpha_5\chi_5(t) = 0. \tag{11}$$

Multiplying (11) by $\chi_5(t)/\chi_5(1)$ and adding it to (5) gives

$$\left(1 + \frac{\chi_5(t)}{\chi_5(1)}\right) - \alpha_2\chi_2(t)\left(\frac{\chi_2(t)}{\chi_2(1)} + \frac{\chi_5(t)}{\chi_5(1)}\right) = 0.$$

Multiplying (10) by $\chi_5(t)/\chi_5(1)$ and adding it to (11) gives

$$\left(1 + \frac{\chi_5(t)}{\chi_5(1)}\right) - \alpha_2\chi_2(1)\left(\frac{\chi_2(t)}{\chi_2(1)} + \frac{\chi_5(t)}{\chi_5(1)}\right) = 0.$$

These two equations clearly imply $\chi_2(t) = \chi_2(1)$, and $t \in \ker \chi_2$. $\square$

Proof of Theorem 4.9: Consider the kernel of the character χ_2. Denote this by N. Suppose firstly that P is contained in N. Since $N \neq G$ (χ_2 is not the trivial character), and G is a minimal counterexample to the theorem, N must satisfy it, and so $\mathrm{Z}(N) = \{1, u\}$. So $\langle u \rangle$ is characteristic in N.

Now suppose that $P \not\leqslant N$. Then the Sylow 2-subgroup of N is $P \cap N$, and is thus cyclic of order 2 or 4 (else N is a normal $2'$-group, contrary to the statement that $\mathrm{O}_{2'}(G) = 1$). In either case, by Burnside's Transfer Theorem, N has a normal 2-complement, say M. Then M lies inside $\mathrm{O}_{2'}(N)$, which itself is contained within $\mathrm{O}_{2'}(G) = 1$. Then $N = P \cap N$, and so $N \leqslant P$. Then $\mathrm{Z}(N) = \langle u \rangle$ and so $\langle u \rangle$ is again characteristic in N.

In either case, $\{1, u\} \trianglelefteq G$, and so u is a central element of G. Now $\{1, u\} = \mathrm{Z}(P)$. Since $\mathrm{O}_{2'}(G) = 1$, every central element of G is a central element of P, and so $\mathrm{Z}(G) = \langle u \rangle$, agreeing with the theorem. $\square$

The proof of this theorem does not require modular characters – in 1974, Glauberman proved Theorem 4.9 using only ordinary character theory. This proof is in [41]. The original proof, by Brauer and Suzuki, was briefly given in [21].

4.4 Glauberman's Z*-Theorem

Before we begin, we will state without proof the following lemma about dihedral groups.

Lemma 4.12 Let G be a dihedral group of order $2n$, generated by the involutions x and y. Then $o(xy) = n$, and n is odd if and only if x and y are conjugate in G.

The proof of Glauberman's Z*-Theorem requires a specific lemma regarding the values of characters, which we will state and prove now.

Lemma 4.13 Suppose that x and y are distinct involutions, contained in conjugacy classes $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively. Let χ be a character, and assume that for each $a \in \mathscr{C}_2$, $\chi(xy) = \chi(xa)$. Then

$$\chi(x)\chi(y) = \chi(1)\chi(xy).$$

Proof: Let c_1 and c_2 be the class sums of $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively. Suppose that T is a representation which affords the character χ. Now $T(c_1)$ and $T(c_2)$ are scalars, and so by taking traces we find (since $x \in \mathscr{C}_1$ and $y \in \mathscr{C}_2$)

$$T(c_1) = |\mathscr{C}_1| \frac{\chi(x)}{\chi(1)}, \qquad T(c_2) = |\mathscr{C}_2| \frac{\chi(y)}{\chi(1)}.$$

Consider a product ab where $a \in \mathscr{C}_1$ and $b \in \mathscr{C}_2$. Then $a^g = x$ for some $g \in G$, and so $(ab)^g = xb^g$. So every product ab is conjugate to xd, where $d \in \mathscr{C}_2$. Now $\chi(ab) = \chi(xd) = \chi(xy)$, the first equality since conjugate elements have the same character values, and the second by assumption. Now

$$T(c_1)T(c_2) = \sum_{a\in\mathscr{C}_1, b\in\mathscr{C}_2} T(ab).$$

Evaluating both sides, we have

$$\left(|\mathscr{C}_1|\frac{\chi(x)}{\chi(1)}\right)\left(|\mathscr{C}_2|\frac{\chi(y)}{\chi(1)}\right) = |\mathscr{C}_1|\,|\mathscr{C}_2|\left(\frac{\chi(xy)}{\chi(1)}\right).$$

Cancelling $|\mathscr{C}_1|\,|\mathscr{C}_2|/\chi(1)$ from both sides, we have

$$\chi(x)\chi(y) = \chi(1)\chi(xy),$$

as required. □

Recall that a subset S of a group is *weakly closed* in a subgroup H if the only conjugates of S in H are already in S.

Theorem 4.14 (Glauberman's Z*-Theorem) Let G be a finite group and t an involution in G such that t is weakly closed in a Sylow 2-subgroup P. Then $t^* \in \mathrm{Z}(G^*)$, where $G^* = G/\mathrm{O}_{2'}(G)$.

Proof: Suppose that G is a minimal counterexample to the theorem, and suppose that $\mathrm{O}_{2'}(G)$ is non-trivial. Let t be an involution weakly closed in a Sylow 2-subgroup P. Now $P\,\mathrm{O}_{2'}(G)/\mathrm{O}_{2'}(G)$ is a Sylow 2-subgroup containing $t\,\mathrm{O}_{2'}(G)$, so if $t\,\mathrm{O}_{2'}(G)$ is weakly closed in $P\,\mathrm{O}_{2'}(G)/\mathrm{O}_{2'}(G)$ then since $G/\mathrm{O}_{2'}(G)$ is of strictly lower order $t\,\mathrm{O}_{2'}(G) \in \mathrm{Z}(G/\mathrm{O}_{2'}(G))$, a contradiction to the fact that G is a minimal counterexample.

Since we are quotienting by $\mathrm{O}_{2'}(G)$, our cosets are of the form $x\,\mathrm{O}_{2'}(G)$, where $x \in P\,\mathrm{O}_{2'}(G)$. By expressing x as a 2-element multiplied by a $2'$-element zq, we can write

$$x\,\mathrm{O}_{2'}(G) = zq\,\mathrm{O}_{2'}(G) = z\,\mathrm{O}_{2'}(G),$$

and so we can assume that our coset representatives are all 2-singular, so that they come from P.

Suppose that $t\,\mathrm{O}_{2'}(G)$ is conjugate to $x\,\mathrm{O}_{2'}(G) \subseteq P\,\mathrm{O}_{2'}(G)/\mathrm{O}_{2'}(G)$. We can assume that x is 2-singular, and

$$t = g^{-1}xgh,$$

where $g \in G$ and $h \in \mathrm{O}_{2'}(G)$. But the decomposition into 2-part and $2'$-part is unique, and so $h = 1$ and $t = x^g$. But $x \in P$ and t is weakly closed in P, and so $t = x$. Thus $t\,\mathrm{O}_{2'}(G) = x\,\mathrm{O}_{2'}(G)$ as required, and $t\,\mathrm{O}_{2'}(G)$ is weakly closed in its Sylow 2-subgroup. This is a contradiction, and so we can assume that $\mathrm{O}_{2'}(G) = 1$. This also means that $\mathrm{Z}(G) \leqslant \mathrm{O}_2(G)$.

Suppose that t lies inside a proper normal subgroup N of G. Now $\mathrm{O}_{2'}(N) = 1$, since $\mathrm{O}_{2'}(N) \leqslant \mathrm{O}_{2'}(G) = 1$. Also, if P is a Sylow 2-subgroup of G containing t, then $P \cap N$ is a Sylow 2-subgroup of N containing t. Clearly t is weakly closed in $P \cap N$, and so since G is a minimal counterexample to the theorem, t is central in N. Since $\mathrm{O}_{2'}(N) = 1$, there are no 2-regular elements in $\mathrm{Z}(N)$. So

$\mathrm{Z}(N) \leqslant \mathrm{O}_2(N) \leqslant P \cap N$. Also, any conjugate of t by an element of G lies inside N, and so by the same argument this conjugate also lies in $\mathrm{Z}(N)$. But t is weakly closed in $P \geqslant \mathrm{Z}(N)$, and so $\{t\}$ is a conjugacy class of G. So $t \in \mathrm{Z}(G)$, contrary to the fact that G is a minimal counterexample. So t lies outside every proper normal subgroup of G.

Next we will show that $\mathrm{Z}(G) = 1$, by proving $\mathrm{O}_2(G) = 1$. Then $\mathrm{O}_2(G) \leqslant P$, and for any $x \in \mathrm{O}_2(G)$, $x^{-1}tx \in P$, and so t centralizes $\mathrm{O}_2(G)$. Since the centralizer of $\mathrm{O}_2(G)$ is a normal subgroup containing t, G centralizes $\mathrm{O}_2(G)$, whence $\mathrm{O}_2(G) \leqslant \mathrm{Z}(G)$. [This means $\mathrm{Z}(G) = \mathrm{O}_2(G)$.] Hence $\mathrm{O}_{2'}(G/\mathrm{Z}(G))$ is trivial. If $t\,\mathrm{Z}(G)$ is conjugate to some other element of $P\,\mathrm{Z}(G)$, then $g^{-1}tg$ is an element of P, contrary to the fact that t is weakly closed in P. Then $t\,\mathrm{Z}(G)$ is weakly closed in $G/\mathrm{Z}(G)$, and the fact that G is a minimal counterexample means that $t\,\mathrm{Z}(G) \in \mathrm{Z}(G/\mathrm{Z}(G))$, so $t \in \mathrm{Z}(G)$. But then $\mathrm{Z}(G) = G$, and so G is not a counterexample. So $\mathrm{Z}(G) = \mathrm{O}_2(G) = 1$.

We know from the Brauer–Suzuki Theorem, Burnside's p-Complement Theorem and Lemma 4.7, P contains more than one involution. Label another of these involutions x. Note that t and x are not conjugate. We consider the principal 2-block B of G. Let χ be an irreducible ordinary character in B.

Let g be an arbitrary element of G. We will show that $\chi(tx) = \chi(tx^g)$. If this is true, then Lemma 4.13 shows that $\chi(tx) = \chi(t)\chi(x)/\chi(1)$. Similarly, $\chi(x) = \chi(t(tx)) = \chi(t)\chi(tx)/\chi(1)$. Then

$$\chi(tx) = \frac{\chi(t)\chi(x)}{\chi(1)} = \frac{\chi(1)\chi(x)}{\chi(t)},$$

and so $\chi(x) = [\chi(t)^2/\chi(1)^2]\chi(x)$, for all χ in the principal 2-block. Since t is not in the centre of G, by assumption, for a non-trivial χ, $\chi(t) \neq \chi(1)$. So either $\chi(x) = 0$ or $\chi(t)^2 = \chi(1)^2$, which implies $\chi(t) = -\chi(1)$. We have, since $\chi(1) + \chi(t) = 0$ unless χ is the trivial character,

$$\sum_{\chi \in B} \chi(1)\chi(x) + \sum_{\chi \in B} \chi(t)\chi(x) = \sum_{\chi \in B} (\chi(1) + \chi(t))\chi(x) = 2.$$

However, by Corollary 4.2(ii), with $y = 1$ (noting that t is not conjugate to x),

$$\sum_{\chi \in B} \chi(x)\chi(t) = 0,$$

and from the block orthogonality relations (since x is a 2-element and 1 is a $2'$-element),

$$\sum_{\chi \in B} \chi(x)\chi(1) = 0.$$

This contradiction will prove the theorem.

It remains to show that $\chi(tx) = \chi(tx^g)$ for all $g \in G$. Write y for tx^g. If y has odd order, then x^g and t are conjugate in the dihedral group $\langle t, x^g \rangle$, and so y has even order $2n$. Then y^n is an involution. Consider the subgroup $\mathrm{C}_G(y^n)$. Since t and x^g both invert y, they centralize $y^n = y^{-n}$ and so lie inside $\mathrm{C}_G(y^n)$.

Suppose that b is a 2-block of $\mathrm{C}_G(y^n)$ with defect group D. Then since y^n is a normal element of $\mathrm{C}_G(y^n)$, $y^n \in D$. Thus $D\,\mathrm{C}_G(D) \leqslant \mathrm{C}_G(y^n)$, and so b is the principal block of $\mathrm{C}_G(y^n)$ whenever

b^G is the principal block of G by Brauer's Third Main Theorem. Now we use the Second Main Theorem: if $\psi_1, \dots, \psi_s$ denote the irreducible modular characters, then for all p-regular elements z,

$$\chi(y^n z) = \sum_{i=1}^{s} d_i^{y^n} \psi_i(z),$$

where $d_j^{y^n}$ denote the generalized decomposition numbers of χ.

Since $\mathrm{Z}(G) = 1$, $\mathrm{C}_G(y^n) \neq G$, and so by the minimality of G, $t \in \mathrm{Z}^*(\mathrm{C}_G(y^n))$. So $tx^g t^{-1}(x^g)^{-1} \in \mathrm{O}_{2'}(H)$. But $t^{-1} = t$ and $(x^g)^{-1} = x^g$, so $(tx^g)^2 = y^2 \in \mathrm{O}_{2'}(H)$. This means that n is odd, since $o(y) = 2n$. So y^{n+1} is an even power of y, and thus is a power of y^2 and so in $\mathrm{O}_{2'}(H)$. Consider the equation above with $z = y^{n+1}$, we have

$$\chi(y) = \chi(y^n y^{n+1}) = \sum_{i=1}^{s} d_i^{y^n} \psi_i(y^{n+1}) = \sum_{i=1}^{s} d_i^{y^n} \psi_i(1),$$

because $y^{n+1} \in \ker \phi$ by Theorem 4.4. But

$$\chi(y^n) = \sum_{i=1}^{s} d_i^{y^n} \psi_i(1),$$

by letting $z = 1$ in the equation above. So $\chi(y) = \chi(y^n)$.

We must now show that tx is conjugate to y^n, and the proof is complete. Firstly notice that since t is weakly closed in P, $t \in \mathrm{Z}(P)$. Since x is also in P, t centralizes x. This fact will be important. Also $t \in \mathrm{C}_G(y^n)$ and since both t and y^n are involutions, ty^n is an involution as well. However, we also have

$$ty^n = (ty)(y^{n-1}) = x^g y^{n-1} \in x^g \langle y^2 \rangle \subseteq \mathrm{O}_{2'}(H).$$

This means that $ty^n x^g \in \mathrm{O}_{2'}(H)$, and so has odd order. By Lemma 4.12(i), ty^n and x^g are conjugate in G, and so ty^n is also conjugate to x. Therefore, there is an element $z \in G$ such that $z^{-1}xz = ty^n$.

Recall that $t \in \mathrm{C}_G(x)$, and so

$$z^{-1}tz \in z^{-1}\,\mathrm{C}_G(x) z = \mathrm{C}_G(z^{-1}xz) = \mathrm{C}_G(ty^n).$$

Now t centralizes both t and y^n, and so centralizes ty^n. Thus both involutions $z^{-1}tz$ and t lie in $\mathrm{C}_G(ty^n)$. But $\mathrm{Z}(G) = 1$, and so $\mathrm{C}_G(ty^n)$ satisfies the theorem. Then $t \in \mathrm{Z}^*(\mathrm{C}_G(ty^n))$, and so $zt\,\mathrm{O}_{2'}(\mathrm{C}_G(ty^n)) = tz\,\mathrm{O}_{2'}(\mathrm{C}_G(ty^n))$. But this means that $z^{-1}tzt$ is 2-regular and so by Lemma 4.12, $z^{-1}tz$ is conjugate to t in the dihedral group generated by t and $z^{-1}tz$. Thus there exists $a \in \mathrm{C}_{ty^n}$ such that $a^{-1}z^{-1}tza = t$. Then

$$(za)^{-1}x(za) = a^{-1}\left(z^{-1}xz\right)a = a^{-1}ty^n a = ty^n = \left((za)^{-1}t(za)\right)y^n,$$

since a centralizes ty^n and za centralizes t. But this clearly means $x = t(za)y^n(za)^{-1}$, or in other words

$$tx = (za)y^n(za)^{-1},$$

and so tx is conjugate to y^n and hence to tx^g. So $\chi(tx) = \chi(tx^g)$, as required, and the resault follows. $\square$

Sometimes an alternative form of the Z*-Theorem is required, and this is given now.

Theorem 4.15 (Glauberman's Z*-Theorem) Let G be a finite group, and t an involution, contained in a Sylow 2-subgroup P. Then the following are equivalent:

(i) t is weakly closed in P;

(ii) $[g,t]$ is 2-regular for all $g \in G$; and

(iii) $\mathrm{O}_{2'}(G)t$ lies in the centre of $G/\mathrm{O}_{2'}(G)$.

Proof: The previous theorem dealt with (i)$\Longrightarrow$(iii). We will now show (i) $\Longleftrightarrow$ (ii). Suppose that $[g,t]$ is a p-regular element of G, for every g in G. Let $g^{-1}tg$ be an element of P. Then so is $g^{-1}tgt = [g,t]$. But this is a p-regular element of G, and so since it lies in a 2-group, must be the identity. So $g^{-1}tg = t$, and t is weakly closed in P.

Conversely, suppose that t is weakly closed in P. Let $g \in G$, and suppose that $[g,t]$ has even order. Now the element t^g is also an involution, and so $H = \langle t, t^g \rangle$ is a dihedral group. Now $\langle t^g t \rangle$ has index 2 in a dihedral group. But $t^g t = [g,t]$, and so $|H|$ is a multiple of 4. Write u for t^g. Then $t^{-1}u^i t = u^{-i}$, and so the only power of u in $\mathrm{Z}(H)$ is $u^{o(u)/2}$ (which exists since $u = [t, g^t]$ has even order). Write z for this central involution. Notice that $z \neq t, t^g$.

Let S be a Sylow 2-subgroup of H that contains t. Then $z \in S$ since it lies in $\mathrm{O}_2(H)$, and so $|S| > 2$. Also S contains a non-central (and hence reflectional) involution of H, along with the central (rotational) involution. So they generate a (finite) group of reflections and rotations which fix a $|H|$-gon, and so therefore S is dihedral.

Now t^g and t generate H, and $(t^g t)^2 \in H'$. Since $t^g t$ has order $|H|/2$, and $\langle (tg^t)^2 \rangle \trianglelefteq H$, this means that $|H : H'| = 4$. Also $tH' \neq t^g H'$, since neither t nor t^g are elements of H', and so t and t^g are not conjugate in H, and so certainly not in S. Since t^g is an involution, it is contained within its own Sylow 2-subgroup of H. But all Sylow subgroups are conjugate in H, and so there exists $h \in H$ such that $t^{gh} \in S$. Now $t^{gh} \neq t$, by the argument earlier in this paragraph.

Finally, since P is a Sylow 2-subgroup of G, $P \cap H$ is a Sylow 2-subgroup of H, and so is conjugate to S. Thus there is some element $k \in H$ such that $S^k = P \cap H$. Then $t^k \in T$, since $t \in S$. But $t^k = t$ since t is weakly closed in T. Also, $(t^{gh})^k \in P$, since $t^{gh} \in S$. Thus $t^{ghk} = t$ as well. But then

$$\left(t^{gh}\right)^x = t = t^x,$$

so that $t^{gh} = t$. But we have already said that $t^{gh} \neq t$, a contradiction. So (i)$\Longrightarrow$(ii).

Lastly, we need to show that (iii) implies either (i) or (ii). But if $\mathrm{O}_{2'}(G)t$ commutes with all other cosets of $\mathrm{O}_{2'}(G)$, this means that for all $g \in G$, $tg\,\mathrm{O}_{2'}(G) = gt\,\mathrm{O}_{2'}(G)$, or that $g^{-1}t^{-1}gt =$

$[g,t] \in \mathrm{O}_{2'}(G)$, as required. $\square$

Proven in 1966 in [39], Glauberman's Z*-Theorem is of fundamental importance in the Classification. As an application of the Z*-Theorem outside of the Classification, Glauberman proved a special case of the Schreier Conjecture. Recall that the Schreier Conjecture is that $\mathrm{Out}(G)$ is soluble for any finite simple group G. This conjecture is a corollary of the Classification, although it is interesting that the following theorem is the only real progress made on this conjecture outside of the Classification (see [49]).

Theorem 4.16 Let G be a finite group with $\mathrm{O}_{2'}(G) = 1$ (particularly G simple) and let P be a Sylow 2-subgroup of G. Then $\mathrm{C}_{\mathrm{Aut}(G)}(P)$ has abelian Sylow 2-subgroups and a normal 2-complement; in particular, $\mathrm{C}_{\mathrm{Aut}(G)}(S)$ is soluble.

The proof is described briefly in [49, p223]; Glauberman's original proof of this theorem is [40].

The Z*-Theorem has been generalized, in particular by Timmesfeld (see [97]), who considered the case of an elementary abelian 2-group. We state it here, following [49], for simple groups.

Theorem 4.17 (Timmesfeld) Let G be a simple group and A a (non-trivial) elementary abelian 2-subgroup of G. If A is weakly closed in $\mathrm{C}_G(a)$ with respect to G for each non-trivial $a \in A$, then

(i) $G \cong \mathrm{L}_n(q)$, $\mathrm{Sz}(q)$, $\mathrm{U}_3(q)$, $q = 2^n$;

(ii) $G \cong A_n$, $6 \leqslant n \leqslant 9$; or

(iii) $G \equiv M_{22}$, M_{23}, M_{24} or He.

Chapter 5

Frontiers in the Theory

In this chapter we will take a broad look at some of the recent progress in representation theory. We examine several long-standing conjectures, and look at some of the work that has been done to prove them.

Firstly, we consider Broué's Abelian Defect Group Conjecture. This conjecture says that if D is an abelian defect group of a block B, then B and its Brauer correspondent in $k\,\mathrm{N}_G(D)$ are derived equivalent. This conjecture has been verified in a number of situations, most notably for blocks with cyclic defect groups and for blocks with defect group $C_2 \times C_2$.

We next examine two of the most famous conjectures in modular representation theory: the Alperin–McKay and Alperin's Weight Conjectures. The first has its origins in simple group theory: it was becoming apparent that for certain simple groups, the number of irreducible ordinary characters of odd degree in the principal block $B_0(G)$ was equal to that of $B_0(\mathrm{N}_G(D))$, in characteristic 2. The Alperin–McKay Conjecture generalizes this property, using the definition of height zero, a concept which coincides with having odd degree in the case of the principal block over characteristic 2.

Finally, we consider two other famous conjectures – Brauer's $k(B)$ and Height Zero Conjectures. The first conjecturally bounds the number of irreducible ordinary characters lying in a block by the order of its defect groups. That the number of irreducible ordinary characters is bounded above by a function of the defect of the block was proven by Brauer and Feit, in the same paper as the $k(B)$ Conjecture itself. The second concerns itself with the powers of p dividing the degree of a character lying in a block B with $|D| = p^d$. It says that all of the ordinary characters lying in B have degree $\chi(1) \equiv 0 \mod p^d$ but $\chi(1) \not\equiv 0 \mod p^{d+1}$ if and only if D is abelian.

For the relevant category theory and homological algebra, the books [70] and [99] are recommended. We will assume a familiarity in particular with the notions of complexes and Morita and derived equivalences.

5.1 Broué's Abelian Defect Group Conjecture

Broué's Abelian Defect Group Conjecture first appeared in [22]. It relates to the structure of blocks with abelian defect group, as its name suggests.

Conjecture 5.1 (Abelian Defect Group Conjecture) Let G be a finite group, and B a block of RG with defect group D. Suppose that D is abelian. Let b be the (unique) block of $R\,\mathrm{N}_G(D)$ that is the Brauer correspondent of B. Then B and b are derived equivalent.

If the blocks B and b are derived equivalent, then this has many consequences; for example, the two blocks will have the same numbers of irreducible ordinary and modular characters.

The Abelian Defect Group Conjecture is known to be true for blocks with a cyclic defect group. This was proven in [81] and [67]. In the first paper, Rickard proves the following.

Theorem 5.2 (Rickard) Let G be a finite group, and B a block of kG with abelian defect group D. Let $E = \mathrm{N}_G(D)/\,\mathrm{C}_G(D)$. Then the derived categories of B and $k(D : E)$ are equivalent, where $D : E$ denotes the semidirect product of D by E.

This theorem is proven by the method of tilting complexes (see later). In the second paper, Linckelmann lifts the tilting complex up to the local ring, demonstrating the result for a block of RG.

Another major area in which this conjecture is proven is for defect groups $C_2 \times C_2$. The final step in this proof was given by Linckelmann [68], building on work of Erdmann. In this paper, Linckelmann actually proves more:

Theorem 5.3 (Erdmann, Linckelmann) Let B be a block of the group algebra RG, having $D = C_2 \times C_2$ as its defect group. The source algebra is isomorphic to one of the interior D-algebras $\Omega^n_D(RD)$, $\Omega^n_D(RA_4)$ or $\Omega^n_D(B_0(RA_5))$ for some integer n.

Now the source algebra is Morita equivalent to the block algebra, and so this gives a list of the possible isomorphism types of the block algebra. Now all of the Heller translates are Morita equivalent, and so the block algebra B is Morita equivalent to either RD, RA_4, or $B_0(RA_5)$. Rickard has shown that the algebras RA_4 and $b_0(RA_5)$ are derived equivalent, and so the block algebra B is derived equivalent to either RD or RA_4. Thus Broué's Conjecture is proven in this case.

In the case of p-soluble groups, it is also known to be true. This fact was proven in 2000, by Harris and Linckelmann [54], working on a result of Dade. Let B denote a block of RG, and b its Brauer correspondent in $R\,\mathrm{N}_G(D)$. In [28], Dade proves that B and b are Morita equivalent. Harris and Linckelmann use this result to show that this Morita equivalence induces a derived equivalence, which is isomorphic with a splendid derived equivalence, as defined by Rickard.

Koshitani and Kunugi in [63] prove the Broué Conjecture in the case where $D = C_3 \times C_3$ is the defect group of the principal block. Their method is to reduce the case to that where $\mathrm{O}_{3'}(G) = 1$, and then use a theorem of Yoshiara, which says that any finite group with elementary abelian Sylow

3-subgroup of order 9 (and $O_{3'}(G) = 1$) is either the direct product of two finite simple groups with cyclic Sylow 3-subgroups, or a finite simple group itself. Their proof then follows by checking each possible case, and so this relies heavily on the Classification of the Finite Simple Groups.

There are also results for all blocks of a particular type of group. One of the best examples is the symmetric group, where the Chuang–Rouquier Theorem gives considerable information about the structure of blocks in the symmetric group. One of the consequences of this theorem is that the Abelian Defect Group Conjecture is proven true for all blocks of the symmetric group. In [71], Andrei Marcus uses this result and some Clifford theory to show that the conjecture is also true for all blocks of the alternating group.

There are also proofs for particular groups, normally simple, or connected with a simple group. In this case it is often shown that the two blocks are derived equivalent directly, rather than in any general theorem. What has appeared as the standard method of proving that two rings are derived equivalent is the method of so-called *tilting complexes*. These are given in the following theorem (see [84] and [82]).

Theorem 5.4 Two rings R and S are derived equivalent if and only if S is isomorphic to the endomorphism algebra, in $D^b(R)$, of an object T such that

(i) T is a bounded complex of finitely generated projective R-modules,

(ii) $\mathrm{Hom}_{D^b(R)}(T, T[i]) = 0$ for $i \neq 0$, and

(iii) if X is an object of $D^b(R)$ such that $\mathrm{Hom}_{D^b(R)}(T, X[i]) = 0$ for all $i \in \mathbb{Z}$, then $X \cong 0$.

The object T described above is a one-sided tilting complex. There is also a two-sided tilting complex, with the following theorem (see [83] and [60]).

Theorem 5.5 Two rings R and S are derived equivalent if and only if there is a bounded complex X of (R, S)-bimodules and a bounded complex $Y = \mathrm{Hom}_R(X, R)$ of (S, R)-bimodules such that

(i) all the terms of X and Y are finitely generated and projective as left modules and as right modules,

(ii) as a complex of R-bimodules, $X \otimes_R Y \cong R \oplus C$ for some acyclic complex C, and

(iii) as a complex of S-bimodules, $Y \otimes_R X \cong R \oplus C'$ for some acyclic complex C'.

In this case X is called a two-sided tilting complex. Tilting complexes are seen to be one of the best hopes for proving the Abelian Defect Group Conjecture for particular cases; for example, Gollan and Okuyama [47] prove the conjecture in the Janko group J_1, $(p = 2)$, with defect group C_2^3.

Work on the simple groups of Lie type includes, for example, Landrock's and Michler's proof that all of the principal blocks of the Ree groups ${}^2G_2(3^{2n+1})$ are Morita equivalent, which reduces the Broué Conjecture to the case of ${}^2G_2(3)$, for which it is known to be true.

5.2 Alperin–McKay Conjecture and Alperin's Conjecture

Recall that if G is a finite group, the character χ lies in a block of defect d, and the Sylow p-subgroups of G have order p^a, then the *height* of χ is $\chi(1)_p - (a-d)$, where $\chi(1)_p$ denotes the power of p dividing $\chi(1)$. Particularly important are the height zero characters, for which $\chi(1)_p = a-d$; since $p^{a-d}|\chi(1)$ for all characters χ in a block of defect d, these are called height zero because they have the smallest power of p dividing their degree.

We fix some notation: let $\mathrm{ht}(\chi)$ be the height of the character χ; if B is a block, write $k(B)$ for the number of irreducible ordinary characters that lie in B, and write $k_0(B)$ for the number of irreducible ordinary characters of height zero lying in B. The Alperin–McKay has its roots in [72], and its modern formulation appeared in [1].

Conjecture 5.6 (Alperin–McKay Conjecture) Suppose that B is a block of the group algebra kG with defect group D, and let b be its Brauer correspondent in $\mathrm{N}_G(D)$. Then $k_0(B) = k_0(b)$.

We next describe the other conjecture in this section. Following [62], let $l(B)$ denote the number of isomorphism types of simple B-modules, and $f_0(B)$ be the number of isomorphism types of projective simple B-modules. Alperin's Conjecture is given now (see [3]).

Conjecture 5.7 (Alperin's Conjecture) For any finite group G,

$$l(G) = \sum_{P \in \mathcal{P}} f_0(\mathrm{N}_G(P)/P),$$

where $\mathcal{P}$ is a set of representatives for the conjugacy classes of p-subgroups of G.

This conjecture has been tightened to deal with individual blocks – if B is a block of G, and has correspondents $b_1, \ldots, b_n$ in $\mathrm{N}_G(P)$, then the left-hand side becomes $l(B)$, and the right-hand side becomes the sum of the number of all projective simple $\mathrm{N}_G(P)/P$-modules that lie in one of the b_i when viewed as a $k\,\mathrm{N}_G(P)$-module.

Okuyama [74] has proven Alperin's Conjecture for p-soluble groups, proceeding by induction on $|G|$. Firstly, he supposes that $\mathrm{O}_p(G) \neq 1$. Then quotienting out by this normal subgroup sets up an induction, and the result follows easily in this case. So he is reduced to considering the case where $\mathrm{O}_p(G) = 1$. Then, since G is p-soluble, $\mathrm{O}_{p'}(G) = E \neq 1$. Again, he splits into cases, whether or not ED is a normal subgroup of G. If it is not, then $H = \mathrm{N}_G(D)E < G$, and so he can perform an induction argument to show that the result holds for H and $\mathrm{N}_G(D)$.

If the inertia subgroup of a particular kE-module is not equal to G, then a result of Fong and Cliff is used to get the needed result. If this inertia subgroup is G, Okuyama shows that $\mathrm{O}_{p'}(G)$ is central and $\mathrm{O}_p(G) \neq 1$, reducing to the earlier case. So only the case where $DE \trianglelefteq G$ remains, and a similar argument to the case $DE \ntrianglelefteq G$ with the inertia subgroup of a particular block reduces to the case that D is a Sylow p-subgroup of G, which is much easier to deal with and Okuyama quickly proves the result in that case.

It is also known that both Alperin's and the Alperin–McKay Conjectures are true in the case of finite groups with T.I. Sylow p-subgroups. Recall that a subgroup H is *T.I.* (or *trivial intersection*) if $H^g \cap H = 1$ for $g \in G \setminus \mathrm{N}_G(H)$. In this case, the structure of the defect groups are very easy to determine: by Proposition 1.20 any defect group is the intersection of two Sylow p-subgroups, which in this case is either the Sylow p-subgroup itself or the trivial group.

Blau and Michler [12] have proven these two conjectures, as well as Brauer's Height Zero Conjecture, for all finite groups with T.I. Sylow p-subgroups. Their method relies heavily on the Classification, just as Koshitani and Kunugi did in their proof of a special case of Broué's Conjecture. For now, suppose that G is a finite group with a T.I. Sylow p-subgroup, and let P denote a Sylow p-subgroup of G. Blau and Michler rely heavily on Proposition 1.20, so a block either had defect P or defect zero.

The first stage is to restrict the possibilities for G and P. If X is a finite group, and $\mathrm{Z}(X) = \mathrm{O}_{p'}(X) = \mathrm{Z}\left(\mathrm{O}^{p'}(X)\right) = \mathrm{C}_X(\mathrm{O}^{p'}(X))$ and $\mathrm{O}^{p'}(X)/\mathrm{Z}\left(\mathrm{O}^{p'}(X)\right)$ is a non-abelian simple group, then Blau and Michler call X *almost simple* with respect to p. They then show, in a lengthy derivation, that a minimal counterexample G to the theorem must be almost simple. But in this definition $\mathrm{O}^{p'}(X)$ is quasisimple, and with the Classification of Finite Simple Groups, all of the Schur multipliers and hence all of the quasisimple groups are known.

All of the non-abelian simple groups with T.I. Sylow p-subgroups are enumerated:

(i) $\mathrm{PSL}_2(q)$ for $q = p^n$, $n \neq 1$;

(ii) $\mathrm{PSU}_3(q^2)$ for $q = p^n$;

(iii) $p = 2$, ${}^2B_2(2^{2m+1})$;

(iv) $p = 3$, ${}^2G_2(3^{2m+1})$ for $m \geqslant 1$, $\mathrm{PSL}_3(4)$ and M_{11};

(v) $p = 5$, ${}^2F_4(2)'$ and M^cL; and

(vi) $p = 11$, J_4.

They then check in turn each of the almost simple groups whose simple factors are the groups on this list: if B is a block of kG, and b is a block of $k\,\mathrm{N}_G(P)$, then $k(B) = k(b)$ and $k_0(B) = k_0(b)$. With this verification, these equations hold for all finite groups with a T.I. Sylow p-subgroup. Since the only non-trivial defect group is P, if $z(G)$ denotes the number of defect zero characters of G, then $k(G) = k(\mathrm{N}_G(P)) + z(G)$. Finally, by [35, IV.6.6], and the fact that $\mathrm{C}_G(x) \leqslant \mathrm{N}_G(P)$ for every non-identity $x \in P$, $k(B) - l(B) = k(b) - l(b)$, and so $l(B) = l(b)$.

So they prove the following theorem.

Theorem 5.8 (Blau, Michler) Let G be a finite group with T.I. Sylow p-subgroup P. Let B be a block of kG with defect group P, and let b be its Brauer correspondent in $\mathrm{N}_G(P)$. Then $k(B) = k(b)$, $k_0(B) = k_0(b)$, $l(B) = l(b)$ and $k(G) = k(\mathrm{N}_G(P)) + z(G)$.

This demonstrates Alperin's Conjecture and the Alperin–McKay Conjecture in one go, as well as Brauer's Height Zero Conjecture (see Section 5.3). These two cases are in some sense 'easy', because in the one case the presence of normal p- and p'-subgroups is very useful, and in the second case there is a great restriction placed on the possible defect groups. In general, Alperin's Weight Conjecture in this form is far from being understood. However, there are alternative forms of this conjecture, which may well be more amenable to solution.

In [62], Knörr and Robinson introduce four types of simplicial complex: $\mathscr{P}$, which consists of all chains of p-subgroups; $\mathscr{E}$, which consists of those of $\mathscr{P}$ which consist of elementary abelian subgroups of G; $\mathscr{N}$, which consists of those of $\mathscr{P}$ with the added condition that each term is normal in the following; and $\mathscr{U}$, which consists of those chains of $\mathscr{P}$ such that each term Q_i has the property that $Q_i = \mathrm{O}_p(\mathrm{N}_G(Q_i))$. If C is the chain $Q_0 < Q_1 < \cdots < Q_n$, then let $|C| = n$. Denote by G_C the stabilizer of the chain C (under conjugation), and $B_C = \mathrm{Br}_{Q_n}(1_B)kG_c$, which is either 0 or a sum of blocks of kG_c.

The main result of this paper is the following:

Theorem 5.9 (Knörr–Robinson) Let B be a block of the group algebra kG, with G a finite group, and write $l_0(B)$ for $\dim_k(\mathrm{Tr}_1^H(B))$. Then

$$\sum_{C\in\Delta_1/G}(-1)^{|C|}k(B_c) = \sum_{C\in\Delta_2/G}(-1)^{|C|}l(B_c) = \sum_{C\in\Delta_3/G}(-1)^{|C|}l_0(B_c),$$

where the Δ_i are any of the four complexes defined above.

Moreover, Alperin's Conjecture is valid for the prime p if and only if each of these sums is zero for every block B of non-zero defect in kG.

The proof of the first statement requires first to show that the sums do not depend on the choice of simplicial complex. Then they prove $\sum_{C\in\mathscr{P}/G}(-1)^{|C|}k(B_c) = \sum_{C\in\mathscr{P}/G}(-1)^{|C|}l_0(B_c)$; they consider the Lefschetz conjugation module of B, and show that this virtual module is virtually projective in the Green ring. This then provides two ways of counting the fixed points, giving the result. After demonstrating this, the other half of the equality $\left(\sum_{C\in\mathscr{P}/G}(-1)^{|C|}k(B_c) = \sum_{C\in\mathscr{P}/G}(-1)^{|C|}l(B_c)\right)$ is proven again making use of the virtual projectivity of the Lefschetz conjugation module.

Earlier in the paper, Knörr and Robinson had shown that Alperin's Conjecture was equivalent to the statement

$$\sum_{C\in\mathscr{N}/G}(-1)^{|C|}l(B_c) = 0,$$

thus completing the proof of Theorem 5.9.

Since this article appeared in 1989, many variants of Alperin's Weight Conjecture have appeared, due to Theorem 5.9. The first variant appears in [90] (it is easily seen to imply Alperin's Conjecture, and was actually proved equivalent to it in [66]), and other formulations appeared soon after that. Another formulation, due to Robinson, (see [86]) is the so-called Ordinary Weight Conjecture.

Again working from [62], the Dade Conjecture is actually a series of conjectures: we refrain from presenting them here. In [29] he presents his ‘Ordinary Conjecture’, and shows that it is true for blocks with cyclic defect groups and a couple of sporadic simple groups. In [30] he present what is known as Dade’s Projective Conjecture (Charles Eaton has proven that Dade’s ‘Projective Conjecture’ is equivalent to Robinson’s Ordinary Weight Conjecture in [34]), and in [32] he presents other forms. In [31], Dade proves that his ‘Invariant Projective Conjecture’, which implies the normal Projective Conjecture, holds for blocks with cyclic defect groups. The most general of Dade’s Conjectures, the ‘Inductive Conjecture’, has the very useful property that if it holds for all of the finite simple groups (and their automorphism groups and covering groups), then it holds for all groups. However, a proof of this fact has not been published, and even the statement of the conjecture in [32] is not fully correct.

Dade believes that this reduction to the case of simple groups is the best hope that there is for proving Alperin’s Conjecture, and since Dade’s Projective Conjecture (and hence all further forms of Dade’s Conjectures and the Ordinary Weight Conjecture) implies the Alperin–McKay Conjecture, a proof of Dade’s Inductive Conjecture would significantly alter the field of modular representation theory. However, a purely number-crunching approach to these conjectures will not provide the insight into the way that p-local subgroups influence representation theory that we need.

The body of evidence supporting all of these conjectures is enough to convince many of their truth. To quote Jonathan Alperin in his review of [30], ‘Proofs of all these results elude us still but the evidence for them is overwhelming and includes proofs of special cases and examples, derivation of known results from the conjectures as well as connections between all the conjectures. If the subject were physics and not mathematics all these special conjectures would be accepted truths.’

5.3 Brauer’s Conjectures

Brauer’s name is associated with two major conjectures, Brauer’s $k(B)$ Conjecture and the Height Zero Conjecture. We will deal firstly with the $k(B)$ Conjecture, which appeared in 1959 in a paper by Brauer and Feit [20].

Conjecture 5.10 (Brauer’s $k(B)$ Conjecture) Let B be a block of kG, with defect d. Then $k(B) \leqslant p^d$.

In [20], Brauer and Feit prove $k(B) \leqslant p^{2d}/4 + 1$, so it is known that $k(B)$ is bounded by some function of the defect. However, the $k(B)$ Conjecture bounds this rather tighter than $p^{2d}/4 + 1$.

The $k(B)$ Conjecture has recently been verified for all p-soluble groups. The first results in this area were due to Nagao [73], who proved the following statement.

Theorem 5.11 (Nagao) The Brauer $k(B)$ Conjecture is valid for p-soluble groups if and only if whenever $G = SP$, where P is a normal elementary abelian p-group of order p^n, and S is a p'-group acting faithfully and irreducible on P, then G has at most p^n conjugacy classes.

We can think of P in this case as a vector space of dimension n, and so for p-soluble groups the conjecture is equivalent to the so-called $k(GV)$ Conjecture, which considers a p'-group G acting on a vector space, and counting the number of conjugacy classes of the semidirect product GV.

Reinhard Knörr introduced an important technique in [61], which led to a proof of the $k(GV)$ Conjecture when $|G|$ is odd by Gluck [42] – of course, since V is assumed to be a 2-group, this means that GV is 2-soluble – and so it remained to consider the cases where V has odd order.

The next breakthrough came in the paper [91], where Robinson and Thompson proved the $k(GV)$ Conjecture for all primes over 5^{30} – the presence of 5^{30} is because of a result of Liebeck's which determines the structure of certain $\mathrm{GF}(p)G$ modules, where G is a p'-group (see [91] for more information). In fact, their paper goes much further than that – it proves the following theorem.

Theorem 5.12 (Robinson–Thompson) Let p be a prime such that for all finite p'-groups X, finite fields $\mathrm{GF}(p^n)$, and faithful $\mathrm{GF}(p^n)X$-modules M, either

(i) for some prime q, $\mathrm{O}_q(X)$ acts absolutely irreducibly on M, and every characteristic abelian subgroup of $\mathrm{O}_q(X)$ is central; or

(ii) if $E(X)$ is quasisimple and acts absolutely irreducibly on U, then there is a vector $\boldsymbol{v} \in M$ such that $\mathrm{Res}^X_{\mathrm{C}_X(\boldsymbol{v})}(M)$ contains as a summand a faithful permutation module.

Then the $k(B)$ Conjecture holds for this prime, so for all finite p-soluble groups G and p-blocks B with defect group D, $k(B) \leqslant |D|$.

This allowed the problem to be assaulted, because not only had all but finitely many primes been checked, but there were now two possible ways to attack a prime p. The first of these two conditions became known as the 'symplectic case', and the second the 'quasisimple case'. The work on the quasisimple case by Riese in [98], and Gluck and Magaard in [43], solves the $k(GV)$ problem for all primes except 3, 5, 7, 11, 13, 19, and 31. The problem now became very tractable, and only seven cases needed to be decided.

All cases except for $p = 5$ were solved, mostly by Riese, Schmid, and Gluck, by 2002, and finally the case $p = 5$ was settled last year in [44]: in the end they had to actually count conjugacy classes to get the solution. This means that for p-soluble groups, the $k(B)$ Conjecture is true. We are still far from a solution in the general case, however, since no reductions of the form of Nagao are known in this case, so the entire structure we have built up here no longer applies.

The second of the conjectures in this section is called Brauer's Height Zero Conjecture. It is easy to state.

Conjecture 5.13 (Brauer's Height Zero Conjecture) Let B be a block of a finite group G. Then $k(B) = k_0(B)$ if and only if B has abelian defect group.

So the conjecture is that a block has only height zero characters precisely when its defect group is abelian. The first real result in this area is in [36], where Fong proves the following theorem.

Theorem 5.14 (Fong) Let G be a p-soluble group, and B a block of kG, with defect group D. If D is abelian, then all characters of B have height zero.

So this proves one direction of the Height Zero Conjecture for p-soluble groups. Fong also proves the converse for the principal block.

The full converse had to wait until 1984. First, Gluck and Wolf in [46] prove the Height Zero Conjecture for soluble groups. They then extend this result in [45] to p-soluble groups, proving:

Theorem 5.15 (Fong, Gluck–Wolf) Let G be a p-soluble group. Every character in a block B of kG has height zero if and only if the block's defect group is abelian.

As Gluck and Wolf do, we introduce the notation $\mathrm{Irr}(G|\psi)$ to mean the set of all irreducible characters of G that are involved in a decomposition of ψ^G into irreducible constituents. Gluck and Wolf prove the following theorem in both [46] and [45], with 'soluble' changed to 'p-soluble'.

Theorem 5.16 (Gluck–Wolf) Suppose that $N \trianglelefteqslant G$, that G/N is (p-)soluble, that $\psi \in \mathrm{Irr}(N)$, and that $p \nmid (\chi(1)/\psi(1))$ for all $\chi \in \mathrm{Irr}(G|\psi)$. Then the Sylow p-subgroups of G/N are abelian.

Proof of Theorem 5.15: Proceed by induction on $|G : \mathrm{O}_{p'}(G)|$. Since G is (p-)soluble, so is $G/\mathrm{O}_{p'}(G)$, and so Theorem 5.16 applies. We use an important result in [37], which states that if G is a p-block of a p-soluble group G, then there is a subgroup H of G, and a block b of H, such that there is a height-preserving bijection from $\mathrm{Irr}(B)$ onto $\mathrm{Irr}(b)$, and such that either H contains $\mathrm{O}_{p'}(G)$ or $H/\mathrm{O}_{p'}(H) \cong G/\mathrm{O}_{p'}(G)$, and $\mathrm{Irr}(b) = \mathrm{Irr}(H|\psi)$ for some irreducible character ψ of $\mathrm{O}_{p'}(H)$, and b has full defect in H.

The first possibility falls to induction easily, and so we can assume that $\mathrm{Irr}(b) = \mathrm{Irr}(H|\psi)$. We can apply induction again to get $\mathrm{Irr}(B) = \mathrm{Irr}(G|\phi)$ for some irreducible character ϕ of $\mathrm{O}_{p'}(G)$, and the defect groups of B are Sylow p-subgroups of G. Since every character of B is of height zero, $p \nmid \chi(1)$ for all $\chi \in B$. Then Theorem 5.16 shows that the defect groups of B are abelian, as required. □

The proof of Theorem 5.16 for p-soluble groups requires the Classification, because it has to pin down the possible structure of a p-soluble group that is not soluble, and consider automorphism groups of simple groups.

Considering the Height Zero Conjecture for arbitrary finite groups, in 1963 Reynolds [80] proves (both directions of) the conjecture when D is a normal subgroup of G. In [11], Berger and Knörr prove another reduction of the Height Zero Conjecture, possibly making it accessible to exhaustive search methods.

Theorem 5.17 (Berger, Knörr) The conjecture 'every irreducible ordinary character in B has height zero if its defect group is abelian' holds for all finite groups if it holds for quasisimple groups.

Including the previously mentioned results of Blau and Michler proving the Height Zero Conjecture for groups with T.I. Sylow p-subgroups, this is broadly the current progress of directly tackling this conjecture to date. The Berger–Knörr Theorem allows us to check every single quasisimple group, and so prove one half of the conjecture. But this sounds like a daunting task, and it has not been seriously attempted since the publication of the result in 1988.

However, there is interplay between the various conjectures: Knörr and Robinson in [62] state that, for abelian defect groups, any two of Alperin's Conjecture, the Alperin–McKay Conjecture and the 'if' direction of Brauer's Height Zero Conjecture imply the third. Given that if Dade's Inductive Conjecture is finally proved, it will simultaneously prove both Alperin's Weight Conjecture and the Alperin–McKay Conjecture, it will also prove one direction of Brauer's Height Zero Conjecture. At the moment, this really does seem the best way to prove this conjecture. Of course, a solution to the Broué Conjecture would also yield a solution, but it seems that a proof even of the Height Zero Conjecture is more likely than that.

Chapter 6

Conclusions and Further Topics

In this short final chapter, we conclude this dissertation, and also give some suggestions for further exploration of this topic.

6.1 Conclusions

The main aim of this dissertation was to examine the modular representation theory of finite groups, and particularly to exhibit some non-trivial results from both the block-based and module-based approaches.

We have seen Brauer's approach, using block theory. The Three Main Theorems, and the Brauer Correspondence in general, are the fundamental concepts in this way of doing representation theory.

In the module theory, we proved the Green Correspondence and the Nagao Decomposition, certainly two non-trivial results in this area. The Green Correspondence can be seen as the module equivalent of the Brauer Correspondence, since both lie at the centres of their respective fields.

Corollary 2.19, although not a major result in itself, hints at the subtle interplay of the two correspondences, and of the two approaches. We used a module approach to prove Brauer's Second Main Theorem, mainly because the proof is cleaner and more illuminating, but it also illustrates the co-dependency of the modules and of the blocks.

To demonstrate this relationship Green developed the concept of G-algebra, and we examined this straight after the two approaches. There is some argument to place the applications of block theory, Chapter 4, straight after Chapter 1, because nothing from the following two chapters is required. However, this would break up the comparison between the two methods, and so was left until Chapter 4.

To conclude, the comparison between the two methods was at least partially successful, although the author believes that more weight could have been placed on the block theory, by perhaps describing Brauer's theory of blocks of defect one. On the whole, especially with the chapter on G-algebras connecting the two, the author considers the main aim to have been fulfilled.

It was also important to stress how this theory is used to extract information about particular groups, and applications to group theory in general.

Ordinary character theory can be used to gain considerable knowledge about groups: we can easily check if a group is simple, by examining the kernel of each character; and we can find out the quantity $|G : G'|$ by counting the number of linear characters, to name just two of the many applications of ordinary character theory. In the preface we mentioned Burnside's $p^\alpha q^\beta$ theorem, a major application of character theory to finite groups.

It seems important to stress the usefulness of modular representation theory, given that it requires considerably more effort to understand the characteristic p representation theory of a group than the characteristic 0 theory. We have tried to accomplish this in Chapter 4, giving two theorems – the Brauer–Suzuki Theorem and Glauberman's Z*-Theorem – that are of fundamental importance in finite group theory.

Finally, we tried to see some of the representation theory that is being done now, to place this work in context. The applications that we gave of block theory were both simple group theory related, indicative of how modular representations have been used in the past. However, with the Classification of the Finite Simple Groups over, the field of modular representation theory, like many fields of group theory, has had to change direction. It is perhaps not surprising that when modular character theory stopped being useful in the Classification – about 1975 – the new methods of representation theory started to appear.

The goal of Chapter 5 was to make clear this change in impetus and, in the author's opinion, it has succeeded. There is a definite sense of progress and vitality in representation theory, and it is hoped that some of this has come across. Perhaps a more in-depth study of one or two results would have been useful, but then considerably more methods and techniques would have had to be described to make the jump between the first four chapters and the current research. It is the author's opinion that what has been done in Chapter 5 is all that could have reasonably been achieved in an MSci project.

In summary, generally the aims have been achieved, although the author has concerns over the depth of study in the block theory.

6.2 Further Topics

In the block theory, the major outstanding problems of Brauer, Alperin and McKay, Olsson, and others, dominate the current research. In Chapter 5 we gave a brief description of the current state of affairs on some of the conjectures, although the processes by which this knowledge is reached are not given. For a better understanding of this research, the reader should consult the references given in that chapter.

We have not really mentioned the theory of cyclic defect groups in any detail at all, and this theory is very important in modern block theory. The blocks with cyclic defect groups are much better understood than arbitrary blocks, and this theory is possibly the next logical step for a better understanding of this subject.

The blocks of defect zero have a very simple structure, having only one irreducible ordinary and modular character, having trivial Cartan matrices and decomposition numbers, and so on. The blocks of defect one were studied by Brauer in [13], [14] and [15]. Of course, a block of defect one has a cyclic defect group. Brauer's methods do not seem to generalize, and the theory of blocks with arbitrary cyclic defect group was really started with Dade in [27], building on results of Thompson. Later work by several mathematicians has lead to the current theory of these blocks. For an introduction, the reader is referred to, for example, [35], which also gives a brief chronology of this theory.

Clifford theory is concerned with how the representation theory is affected by the presence of normal subgroups. Clifford's Theorem itself describes how a simple module breaks up when restricted to a normal subgroup; [26, §11] describes some of the results in Clifford theory, including Clifford's Theorem, and is recommended for further reading on this topic.

Moving on to the module-theoretic viewpoint, the books by David Benson [10] offer a concise introduction to this area. This approach to representation theory is inextricably intertwined with cohomology, and so a necessary background in cohomology is essential.

Auslander–Reiten theory is one of the central areas in this approach; the book [8] introduces the subject. An *Auslander–Reiten sequence* (sometimes *almost split sequence*) of modules over the ring R is one of the form

$$0 \longrightarrow M \longrightarrow E \xrightarrow{\sigma} N \longrightarrow 0,$$

where M and N are indecomposable, σ does not split, and given any R-module L and homomorphism $\theta : L \to N$ that is not split onto, there is a homomorphism ϕ such that the diagram

$$\begin{tikzcd} & & & L \arrow[dl, dashed, "\phi"'] \arrow[d, "\theta"] & \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r, "\sigma"] & N \arrow[r] & 0 \end{tikzcd}$$

commutes.

In 1975 Auslander and Reiten proved that if R is finitely generated when viewed as a module over its centre, and its centre is an artinian ring, then such sequences exist for any non-projective indecomposable module N, and they are essentially unique. Auslander–Reiten sequences are connected to objects known as *quivers*, which are broadly a type of directed graph. For a very readable introduction to quivers see [85].

Finally, although we have given some of the foundations of the block-theoretic and module-theoretic approaches, there is also the theory of integral representations, that is not considered at all in this dissertation. For a reasonably comprehensive treatment of this topic, the book of Curtis and Reiner [26] is recommended.

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