A Group of Order 8 That's Hard: Indecomposable String Modules for Dihedral Groups

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In this talk, K is a field of characteristic 2, and G is the dihedral group D_{4q}, where q is a power of 2. We will describe the indecomposable modules for this group, and analyze some tensor products of these modules. To do this, we will have to introduce some representation theory, which we will explain as we go. Recall that a module is algebraic if

$$M^{\otimes n} = \bigoplus_{i<n} a_i M^{\otimes i}$$

where the coefficients a_i may be positive or negative integers.

We start with the group V_4.

1 Modules for V_4

Let $G = \langle x, y \rangle$ be a Klein four-group. There are four types of indecomposable module for G, called A_n, B_n, $C_n(\pi)$ and D. Write $X = 1 + x$ and $Y = 1 + y$. Let I_n denote the $n \times n$ identity matrix. Write α_X and α_Y for the matrices

$$\alpha_X = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \qquad \alpha_Y = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Write $\beta_X = \alpha_X^T$ and $\beta_Y = \alpha_Y^T$. Then A_n (of dimension $2n+1$) is given by

$$X \mapsto \begin{pmatrix} 0 & \alpha_X \\ 0 & 0 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & \alpha_Y \\ 0 & 0 \end{pmatrix}$$

and B_n (again of dimension $2n+1$) is given by

$$X \mapsto \begin{pmatrix} 0 & \beta_X \\ 0 & 0 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & \beta_Y \\ 0 & 0 \end{pmatrix}.$$

[Note that B_n is the dual of A_n.]

The periodic modules are slightly more complicated. Let n be a natural number, and let π by an irreducible polynomial over K, given by

$$T^m + r_{m-1}T^{m-1} + \cdots + r_1T + r_0.$$

Let R be given by

$$R = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ r_0 & r_1 & \cdots & r_{m-2} & r_{m-1} \end{pmatrix}$$

and S be $m \times m$ matrix with 0 everywhere except the bottom-left entry, which is 1. Finally, let $C_n(\pi)$ be the module

$$X \mapsto \begin{pmatrix} 0 & I_{mn} \\ 0 & 0 \end{pmatrix}, \qquad Y \mapsto \begin{pmatrix} 0 & \gamma_Y \\ 0 & 0 \end{pmatrix}$$

where γ_Y is given by the $n \times n$ block matrix, with R along the diagonal, S along the leading subdiagonal, and 0 everywhere else.

The last non-projective module is denoted by $C_n(\infty)$, and is given by

$$X \mapsto \begin{pmatrix} 0 & \gamma_X \\ 0 & 0 \end{pmatrix}, \qquad Y \mapsto \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$$

where γ_X is an $n \times n$ matrix with 1 down the leading superdiagonal and 0 everywhere else.

Let D denote the projective indecomposable module, which can be given by for example

$$X \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad Y \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The multiplication table for these modules is as follows. (Since the product of D with any module is projective, this row has been omitted.)

$n \leqslant n'$	A_n	B_n	$C_n(\pi)$	$C_n(\infty)$
$A_{n'}$	$nn'D \oplus A_{n+n'}$	$n(n'+1)D \oplus A_{n'-n}$	$nn'mD \oplus C_n(\pi)$	$nn'D \oplus C_n(\infty)$
$B_{n'}$	$n(n'+1)D \oplus B_{n'-n}$	$nn'D \oplus B_{n+n'}$	$nn'mD \oplus C_n(\pi)$	$nn'D \oplus C_n(\infty)$
$C_{n'}(\pi')$	$nn'mD \oplus C_{n'}(\pi')$	$nn'mD \oplus C_{n'}(\pi')$	X	$nn'mD$
$C_{n'}(\infty)$	$nn'D \oplus C_{n'}(\infty)$	$nn'D \oplus C_{n'}(\infty)$	$nn'mD$	$n(n'-1)D \oplus 2C_n(\infty)$

The remaining entry X is $nmn'm'D$ if $\pi \neq \pi'$, and $nm(n'm-1)D \oplus 2C_n(\pi)$ if $\pi = \pi'$.

We therefore see that, modulo projective modules,

$$\begin{aligned} A_n^{\otimes i} &= A_{in}, \\ B_n^{\otimes i} &= B_{in}, \\ C_n(\pi)^{\otimes 2} &= 2C_n(\pi), \text{ and} \\ C_n(\infty)^{\otimes 2} &= 2C_n(\infty). \end{aligned}$$

2 Modules for D_{4q}

There are two classes of non-projective indecomposable KG-module; the string modules and band modules. The string modules are easier to describe, and we do this first. Write $G = D_{4q}$, where q is a power of 2.

Introduce symbols a and b, and let $\mathscr{W}$ denote the set of all finite strings of symbols a, b, a^{-1} and b^{-1}, which we will call *words*, with the proviso that a symbol of the form $a^{\pm 1}$ is followed by one of the form $b^{\pm 1}$, and vice versa. Hence there are 2^n strings of length n that start with $a^{\pm 1}$. If w is a word in $\mathscr{W}$, then the inverse of w will be given by

$$(w_1 w_2 \dots w_n)^{-1} = w_n^{-1} w_{n-1}^{-1} \dots w_1^{-1},$$

so that for example if $w = aba^{-1}b^{-1}a$, then $w^{-1} = a^{-1}bab^{-1}a^{-1}$. If w is a word with n symbols, then let $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$ be two $(n+1)$-dimensional matrices given by the following procedure:

(i) set $\alpha_{ii} = \beta_{ii} = 1$, and $\alpha_{ij} = \beta_{ij} = 0$ for $j \neq i$;

(ii) running through the symbols of w, if the ith symbol in w is an a, then set $\alpha_{i+1,i} = 1$, and if it is a^{-1}, then set $\alpha_{i,i+1} = 1$; and

(iii) running through the symbols of w, if the ith symbol in w is an b, then set $\beta_{i+1,i} = 1$, and if it is b^{-1}, then set $\beta_{i,i+1} = 1$.

This procedure is best seen by example: if $w = ab^{-1}aba^{-1}$, then the two matrices α and β for $M(w)$ acting on the space V with basis $\{v_1, \dots, v_6\}$ are given by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This can be represented by a diagram. Write $x' = x-1$ and $y' = y-1$. Then if $w = ab^{-1}aba^{-1}$, as before, the elements x' and y' act as the diagram below, with all other actions on the v_i being 0.

$$v_1 \xleftarrow{x'} v_2 \xrightarrow{y'} v_3 \xleftarrow{x'} v_4 \xleftarrow{y'} v_5 \xrightarrow{x'} v_6$$

If $G = \langle x, y : x^2 = y^2 = (xy)^{2q} = 1 \rangle$, then let $M(w)$ denote the function $G \to \mathrm{GL}_n(2)$ defined by $x \mapsto \alpha$ and $y \mapsto \beta$. This will be a representation of the dihedral group D_{4q} whenever no instance of $(ab)^q$, $(ba)^q$, $(a^{-1}b^{-1})^q$, or $(b^{-1}a^{-1})^q$ occurs. For the subset of $\mathscr{W}$ so defined, we use the symbol $\mathscr{W}_q$.

There are two important points to be made about the representations $M(w)$: firstly, they are always indecomposable representations; and secondly, $M(w)$ and $M(w')$ are isomorphic if and only if $w' = w$ or $w' = w^{-1}$. This latter point is important, and we will often blur the distinction between the words w and w^{-1}. The modules $M(w)$ are called *string modules*. An important fact is that any odd-dimensional indecomposable module is a string module for some string of even length.

For the group V_4, the modules A_n and B_n, and $C_n(0)$ and $C_n(\infty)$ are the string modules, of odd and even dimension respectively.

The remaining modules are the band modules: let $\mathscr{W}_q'$ denote the subset of words, all of whose powers lie in $\mathscr{W}_q$, but that are not non-trivial powers of smaller words, so that for example $ab^{-1}ab^{-1}$ is not in $\mathscr{W}_q'$, but also ab is not in $\mathscr{W}_q'$ because a large power of this word does not lie in $\mathscr{W}_q$. A consequence of this is that all words in $\mathscr{W}_q'$ are of even length. If w is a word in $\mathscr{W}_q'$, we will not make the distinction between w and w^{-1}, and between w and the word got from w by moving the first letter to the end of w, so that $abab^{-1}$ is equivalent to $b^{-1}aba$. More formally, we may take equivalence classes of words in $\mathscr{W}_q'$ under this equivalence relation.

Let w be a word of even length n, and let V denote an m-dimensional vector space, equipped with an indecomposable linear transformation ϕ. By cycling the letters of w and by inverting, we may assume that w begins with either a or b. We intend to construct matrices similar to those for string modules.

Let α' and β' denote square matrices of size n, initially equal to the zero matrix. We will associate a pair of numbers to this: if it is direct, associate $(i+1, i)$, and if it is inverse, associate $(i, i+1)$.

Next, we place an I in all positions (i, j) of α' where (i, j) is associated to some $a^{\pm 1}$, and in positions (i, j) of β' where (i, j) is associated to some $b^{\pm 1}$. (These entries should be taken modulo n, so that $n+1$ becomes 1.) The exception is the position $(1, 2)$ or $(2, 1)$, which should have a ϕ placed in this position. Finally, add I to the diagonal entries of both α' and β'.

The matrices α and β are square matrices of size mn, considered as block matrices, whose n^2 blocks are given by the entries of α' and β'. For this, regard I as the $m \times m$ identity matrix and ϕ as the matrix representing the automorphism ϕ. Finally, associate x with α and y with β; this produces a representation of G, which is denoted by $M(w, \phi)$.

In the example $w = aba^{-1}b$ and ϕ is some map on an m-dimensional vector space, the matrices α' and β' are

$$\alpha' = \begin{pmatrix} I & 0 & 0 & 0 \\ \phi & I & 0 & 0 \\ 0 & 0 & I & I \\ 0 & 0 & 0 & I \end{pmatrix}, \qquad \beta' = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & I & I & 0 \\ I & 0 & 0 & I \end{pmatrix}.$$

The matrices α and β are block matrices represented by α' and β', where I is the $m \times m$ identity matrix.

Similarly to the string modules, the modules $M(w, \phi)$ are all indecomposable, and $M(w, \phi)$ and $M(w', \phi')$ are isomorphic if and only if w and w' are the same word modulo inverses and cycling letters, and ϕ and ϕ' are equivalent transformations. The $M(w, \phi)$ are called *band modules.*

3 Tensor Products and Restrictions

Lemma 3.1 Let M be an indecomposable KG-module.

(i) If M is odd-dimensional then $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are both the sum of a trivial module and projective modules.

(ii) If M is an even-dimensional string module then either $M \downarrow_{\langle x \rangle}$ is projective and $M \downarrow_{\langle y \rangle}$ is the direct sum of two copies of K and a projective, or vice versa.

(iii) If M is a band module, then both $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are projective.

Proof: Let w be a word of even length $2n$, beginning with $X^{\pm 1}$ say, and let v_i denote the standard basis, for $1 \leqslant i \leqslant 2n+1$. Then the submodules generated by v_i and v_{i+1} for $1 \leqslant i < 2n+1$ and i odd form a copy of the projective module, which therefore splits off. Hence $M \downarrow_{\langle x \rangle}$ is the sum of n projective modules and a trivial module. The same occurs for $M \downarrow_{\langle y \rangle}$, proving (i).

If M is an even-dimensional string module then it is defined by a word w of odd length $2n-1$, with first and last letters $X^{\pm 1}$ without loss of generality. Then $M \downarrow_{\langle y \rangle}$ has $n-$

1 submodules $\langle v_i, v_{i+1}\rangle$ (for i even) isomorphic with the projective indecomposable $K\langle y\rangle$-module, and two trivial submodules, $\langle v_1\rangle$ and $\langle v_{2n}\rangle$. Similarly, $\langle v_i, v_{i+1}\rangle$ is a projective submodule of $M\downarrow_{\langle x\rangle}$ for eacvh odd i, and so $M\downarrow_{\langle x\rangle}$ is projective, proving (ii).

It remains to discuss the band modules. By cycling, we may assume that the module begins with X, and then we again see easily that the matrix corresponding to the action of y on M is a sum of projective modules, and this is true for any band module for a word beginnning $X^{\pm 1}$. However, by cycling the word we find that M is isomorphic with a band module for a word beginning $Y^{\pm 1}$, and hence $M\downarrow_{\langle x\rangle}$ must also be projective, as required. $\square$

We collect the following basic facts about odd-dimensional string modules: for this, we will need to know the length of a word, and we write $\ell(w)$ for this quantity.

Lemma 3.2 Let $w, w' \in \mathscr{W}$ be words, and suppose that $\ell(w) = 2n$ and $\ell(w') = 2m$ are even. Write $M = M(w)$ and $M' = M(w')$.

(i) Either w or w^{-1} begins with the symbol $X^{\pm 1}$.

(ii) The restrictions $M\downarrow_{\langle x\rangle}$ and $M\downarrow_{\langle y\rangle}$ are the sum of the trivial module and n projective 2-dimensional modules.

(iii) The restrictions of the tensor product $M \otimes M'$ to $\langle x\rangle$ and $\langle y\rangle$ are both the sum of a trivial module and $2nm + n + m$ projective 2-dimensional modules.

(iv) The tensor product $M \otimes M'$ is the direct sum of a string module $M(w'')$ for some w'' of even length and various even-dimensional modules.

Proof: The proof of (i) is obvious, as is (ii) from the description of the matrices given in the construction of string modules; (iii) follows from the fact that $(M \otimes M')\downarrow_H = M\downarrow_H \otimes M'\downarrow_H$ for any subgroup H; and (iv) follows from the fact that at least one odd-dimensional summand must occur in $M \otimes M'$, since it has odd dimension, and each odd-dimensional summand would contribute one copy of K to $(M \otimes M')\downarrow_{\langle x\rangle}$, which only contains one trivial summand. $\square$

Lemma 3.3 Let $w, w' \in \mathscr{W}$ be words, and suppose that $\ell(w) = 2n - 1$ and $\ell(w') = 2m - 1$ are odd. Write $M = M(w)$ and $M' = M(w')$.

(i) The word w begins with $a^{\pm 1}$ if and only if it ends with $a^{\pm 1}$.

(ii) If w begins with $a^{\pm 1}$, then the restriction $M\downarrow_{\langle x\rangle}$ is projective, and the restriction $M\downarrow_{\langle y\rangle}$ is the sum of a $2(m-1)$-dimensional projective module and a 2-dimensional trivial module.

(iii) If w begins with $a^{\pm 1}$ and w' begins with $b^{\pm 1}$, then $M \otimes M'$ contains no summands that are string modules.

(iv) If both w and w' begin with $a^{\pm 1}$, then $M \otimes M'$ contains exactly two even-dimensional string module summands.

Proof: (i) is obvious, and (ii) easily follows from the construction of string modules, since the only place that a trivial summand can occur is at the beginning or end of a word. The proof of (iii) comes from the fact that if $M \otimes M'$ contains a string module, there must be a trivial summand of either $(M \otimes M') \downarrow_{\langle x \rangle}$ or $(M \otimes M') \downarrow_{\langle y \rangle}$, which is impossible since both $M \downarrow_{\langle x \rangle}$ and $M' \downarrow_{\langle y \rangle}$ are projective. The proof of (iv) is similar: if M and M' both begin with $a^{\pm 1}$, then both $M \downarrow_{\langle y \rangle}$ and $M' \downarrow_{\langle y \rangle}$ contain two trivial summands, proving that $(M \otimes M') \downarrow_{\langle y \rangle}$ contains four trivial summands. Since band modules restrict to projective modules, and no odd-dimensional summand can occur by Theorem 4.1(ii) below, the tensor product must contain two even-dimensional string modules as summands. $\square$

4 The Odd-Dimensional Modules

We begin with the following theorem.

Theorem 4.1 (Benson–Carlson, 1986) Let G be a finite group and M and N be absolutely indecomposable KG-modules.

(i) $K | M \otimes N$ if and only if $p \nmid \dim M$ and $M \cong N^*$, in which case $2 \cdot K$ is not a summand of $M \otimes N$.

(ii) If $p \mid \dim M$, then every summand of $M \otimes N$ has dimension a multiple of p.

Lemma 3.2(iv) yields the following corollary.

Corollary 4.2 Let w and w' be words in $\mathscr{W}$ of even length, and let $M = M(w)$ and $M' = M(w')$. If N is a module with a unique odd-dimensional summand, write $\bar{N}$ for this summand. Then the set of all odd-dimensional indecomposable KG-modules form a group under the operation

$$M \circ M' = \overline{M \otimes M'}.$$

Proof: That this is a binary operation comes from Lemma 3.2, so we need to check that $\circ$ is associative, that there is an identity, and that there is an inverse. The associativity of $\circ$

follows immediately from the associativity of $\otimes$; the trivial module clearly forms an identity; and Theorem 4.1 implies that, if M is an odd-dimensional indecomposable module, then

$$\overline{M \otimes M^*} = K,$$

and so therefore M^* is the inverse of M. $\square$

Theorem 4.3 (Archer, 2005) If M is an odd-dimensional indecomposable KD_{4q}-module, then $\bigoplus_{i \geqslant 1} M^{\otimes i}$ contains infinitely many non-isomorphic string modules.

To prove this, notice that $a(KG)$ modulo the ideal of even-dimensional modules forms a group under tensor multiplication. We will prove that this group is torsion-free. If $M^{\otimes n} = K$ modulo even-dimensional modules, then $M^{\otimes(n-1)} = M^*$ modulo even-dimensional modules. We may assume (by taking duals if necessary) that $M = M(w)$, where w begins with X. Notice that the dual module is represented by a string beginning with X^{-1}. We will prove more generally that if M and N begin with X, so does $M \otimes N$. Write m_i and n_i for the elements of the natural bases for M and N. Then $Xm_2 = m_1$ and $Xn_2 = n_1$, and so $X(m_2 \otimes n_2) = m_1 \otimes n_1$. Hence

$$m_1 \otimes n_1 \in \ker X \cap \ker Y \cap \operatorname{im} X \setminus \operatorname{im} Y.$$

Hence there is some element a of $M \otimes N$ such that $a \in \ker X \cap \ker Y \cap \operatorname{im} X$ and $a \notin \operatorname{im} Y$. Write a as a linear combination of α_i, where the α_i are elements of the natural basis for $M \otimes N$, which is the sum of a string module, band modules and projective modules.

Since $a \in \operatorname{im} X$, we can write $a = \sum_i X\alpha_i$. Examining the band modules, string modules beginning with X^{-1}, and projective modules, we can see that for each basis element v, $v \in \operatorname{im} X$ implies either $v \in \operatorname{im} Y$ or $v \notin \ker Y$. Since $v \notin \operatorname{im} Y$ and $v \in \ker Y$, there must be some basis element of $\sum X\alpha_i$ that does not come from a band module, a projective module, or a string module beginning with X^{-1}, proving the result.

Corollary 4.4 Let M be an odd-dimensional KG-module (whose odd-dimensional summands are not all trivial). Then M is not algebraic.

5 The Ω Operator

Given a string module w, write it in so-called 'generation form', $C_1 C_2^{-1} \dots C_{2n}^{-1}$, where no inverses appear in the C_i. Let D_i be the string such that $D_i C_i$ is the string $(XY)^{2q}$. Denote by $K(w)$ the word $D_1^{-1} D_2 \dots D_{2n}$. Then $\Omega(w) = K(\alpha w \beta^{-1})$, where α and β are either X or Y.

The operator Ω^2 is much easier to define. We will do so via two more independently-useful operators, L_q and R_q; write $A = (XY)^{q-1}X$ and $B = (YX)^{q-1}Y$. The operator L_q is defined by either adding or removing a string at the left of the word w, and R_q is defined by either adding or removing a string at the right of w. The strings that are added/removed are given below.

L_q	Add	$A^{-1}Y$	$B^{-1}X$
	Remove	AY^{-1}	BX^{-1}
R_q	Add	$X^{-1}B$	$Y^{-1}A$
	Remove	XB^{-1}	YA^{-1}

Then $\Omega^2(M(w)) = M(wL_qR_q)$. The following are fixed points under both Ω^2: $(AY^{-1})^iA$ and $(BX^{-1})^iB$ for any $i \in \mathbb{N}$.

Lemma 5.1 $\Omega(M \oplus N) = \Omega(M) \oplus \Omega(N)$, and $\Omega(M \otimes N) = \Omega(M) \otimes N$ modulo projectives. Also, $\Omega(M \uparrow^G) = \Omega(M) \uparrow^G$, and $\Omega(M \downarrow_H) = \Omega(M) \downarrow_H$.

Proposition 5.2 Suppose that M is a non-periodic indecomposable module. Then at most one of the $\Omega^i(M)$ is algebraic.

6 The Auslander–Reiten Quiver

We first introduce Auslander–Reiten sequences. These are particular short exact sequences that, whilst not split, are 'close' to being split.

Definition 6.1 Let $A \longrightarrow B \xrightarrow{\psi} C$ be a short exact sequence. This sequence is an *Auslander–Reiten sequence* if

(i) A and C are both indecomposable;

(ii) the sequence is not split; and

(iii) if $\rho : D \to C$ is a non-split epimorphism, then there is $\phi : D \to B$ such that $\rho = \psi\phi$.

For group algebras, it turns out that for any C there is an AR-sequence terminating in C, and that $A = \Omega^2(C)$ in this case. Furthermore, this sequence is unique.

Proposition 6.2 Suppose that M is an indecomposable module, and that H does not contain a vertex of M. Then the AR-sequence

$$0 \to \Omega^2(M) \to X \to M \to 0$$

splits upon restriction to H.

Given these AR-sequences, we can draw the (stable) Auslander–Reiten quiver. The vertices are the non-projective indecomposable modules, and two vertices are connected by an arrow starting at M and terminating at N if M appears as a term in the middle of the AR-sequence ending in N.

The almost split sequences for string modules are

$$0 \to M(wL_qR_q) \to M(wL_q) \oplus M(wR_q) \to M(w) \to 0,$$

unless $w = AB^{-1}$, in which case it is

$$0 \to M(wL_qR_q) \to M(wL_q) \oplus M(wR_q) \oplus P \to M(w) \to 0,$$

where P is the projective indecomposable module KG.

A component of the Auslander–Reiten quiver is given on the accompanying slide. The following theorem can be proved about these components.

Theorem 6.3 (Craven, 2007) Let Γ be a component of the Auslander–Reiten quiver of D_{4q}, and suppose that the modules on Γ are not periodic. Then at most one module on Γ is algebraic.