Burnside's $p^\alpha q^\beta$ Theorem without Burnside

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One of the first great results in twentieth-century group theory is that if a group has order divisble by at most two primes, then it is soluble. Burnside's original proof requires characters. We will not.

1 Introduction

Recall that if G is a group and π is a set of primes, then a Hall π-subgroup is a π-subgroup whose index is coprime with any prime in π. Hall's theorem on soluble groups is the following.

Theorem 1.1 (Hall, 1928) Let G be a finite group. Then G is soluble if and only if G contains Hall π-subgroups for all sets of primes π.

Let us suppose that G is a finite group such that $\pi(G)$, the set of primes dividing the order of G, contians exactly two elements. Then this theorem becomes the following, assuming the existence of Sylow p-subgroups.

Theorem 1.2 (Burnside, 1904) Let G be a group such that $\pi(G) = \{p, q\}$ for $p \neq q$. Then G is soluble.

Hall's theorem is a generalization of Burnside's theorem; to prove that if a group G has Hall π-subgroups for all sets π then G is soluble is an induction on $|\pi(G)|$. The base case is, of course, Burnside's theorem.

During the proof of the Feit–Thompson theorem, the structure of the groups with $|\pi(G)| = 2$ became important. Burnside's proof gave basically no structural information about these groups. Hence, we will proceed differently.

2 The Broad Outline

From now on, G will denote a minimal counterexample to Burnside's theorem, of order $p^\alpha q^\beta$.

Lemma 2.1 Let H be a simple group, K be a proper subgroup of H and $A \leqslant K$. Suppose that $H = K\,\mathrm{N}_H(A)$. Then $A = 1$.

Proof: This is easy; write $h = xy$ with $x \in \mathrm{N}_H(A)$, so that $A^h = A^y \leqslant K$. Thus the normal closure of A is contained within K. Thus the normal closure is not H, so it must be trivial. □

Lemma 2.2 The following properties of G hold:

(i) G is a simple group;

(ii) $|G|$ is divisible by exactly two different primes p and q;

(iii) every proper subgroup of G is soluble; and

(iv) if P denotes a Sylow p-subgroup of G then P does not normalize any non-trivial q-subgroup of G.

Proof: (i) is obvious, as are (ii) and (iii). (iv) needs the previous lemma, vindicating its presence; if Q is a non-trivial q-subgroup, and $P \leqslant \mathrm{N}_G(Q)$, then since Q is contained in R, a Sylow q-subgroup, we have $G = PR = \mathrm{N}_Q(G)R$, a contradiction. □

The first key step is to prove the following theorem.

Theorem 2.3 Let M be a maximal subgroup of G. Then M does not have both a normal p-subgroup and a normal q-subgroup.

That M has a normal p-subgroup or a normal q-subgroup follows from the fact that minimal normal subgroups of soluble groups are elementary abelian.

The second key step is to understand the intersection of a maximal subgroup and its conjugates.

Theorem 2.4 Let M be a maximal subgroup of G, and $g \in G \setminus M$. If $\mathrm{O}_p(M) = 1$, then $M \cap M^g$ is a p-group.

From this step we can get a contradiction. Let M be a maximal subgroup containing a Sylow p-subgroup. By Lemma 2.2(iv), $\mathrm{O}_q(M) = 1$, and so $M \cap M^g$ is a q-group for any $g \notin M$. Hence the index of $M \cap M^g$ in M is at least p^α, and so

$$(p^\alpha)^2 \leqslant \frac{|M| \cdot |M^g|}{|M \cap M^g|^2} \leqslant \frac{|M| \cdot |M^g|}{|M \cap M^g|} = p^\alpha q^\beta.$$

It is also true that $q^{2\beta} \leqslant p^\alpha q^\beta$ by the same method, and we reach a contradiction.

In the next section we will prove Theorem 2.3; the proof of Theorem 2.4 is beyond the scope of this lecture, as we would have to introduce the J-subgroup as well as prove some rather delicate technical lemmas. The next section gives a flavour of the manipulations involved.

3 Theorem 2.3

The proof of this is not easy, and proceeds in stages. We begin by letting $\mathrm{F}(M)$ denote the largest normal nilpotent subgroup. To think of it another way,

$$\mathrm{F}(M) = \mathrm{O}_p(M) \times \mathrm{O}_q(M).$$

We are supposing that neither $\mathrm{O}_p(M)$ nor $\mathrm{O}_q(M)$ are trivial. We first claim that $\mathrm{F}(M)$ either has more than one subgroup of order p or more than one subgroup of order q. If this is false, then a strong theorem proves the following.

Theorem 3.1 Either $\mathrm{O}_p(M)$ is cyclic or $p = 2$ and $\mathrm{O}_p(M)$ is generalized quaternion.

The proof of this is not easy either, and will be omitted.

Assuming this, suppose that the Sylow subgroups of M are all cyclic, and that $p < q$. Then, since q doesn't divide $|\operatorname{Aut} \mathrm{O}_p(M)|$ which is $(p-1)p^a$ for some a, a Sylow q-subgroup Q of M must centralize $\mathrm{O}_p(M)$, and so

$$\mathrm{Z}(Q) \leqslant \mathrm{C}_M(\mathrm{F}(M)) \leqslant \mathrm{F}(M),$$

by a famous theorem of Fitting as M is soluble. Hence $Z = \mathrm{Z}(Q)$ is a characteristic Q-subgroup of $\mathrm{O}_q(M)$, and so is a characteristic subgroup of M. However, as G is simple, $\mathrm{N}_G(Z) = M$. However, the following is true.

Proposition 3.2 Let Q be a q-subgroup of a group G. Let Z be a characteristic subgroup of Q and suppose that Q is a Sylow q-subgroup of $\mathrm{N}_G(Z)$; then Q is a Sylow q-subgroup of G.

Proof: If Q is not a Sylow q-subgroup of G, then it is contained within one, R say. Thus $\mathrm{N}_R(Q) > Q$, and since

$$Z \operatorname{char} Q \trianglelefteqslant \mathrm{N}_R(Q),$$

and so $\mathrm{N}_R(Q) \leqslant \mathrm{N}_G(Z)$. Thus we see that $\mathrm{N}_R(Z) > Q$. This is a contradiction to the fact that Q is a Sylow q. □

Returning to our original problem, we have that Z is a characteristic q-subgroup of $M = \mathrm{N}_G(Z)$, and Q is a Sylow q-subgroup of M. Hence Q is a Sylow q-subgroup of G. By Lemma 2.2, Q cannot normalize any non-trivial p-sugbroups, but it normalizes $\mathrm{O}_p(M)$, a contradiction.

Hence we may assume that $p = 2$ and $\mathrm{O}_2(M)$ is generalized quaternion, and $\mathrm{O}_q(M)$ is cyclic. Let P be a Sylow 2-subgroup of M, and note that since $\mathrm{O}_q(M)$ is cyclic, P' must act trivially on it. Since P' is a normal subgroup, it hits the centre non-trivially, so let X be a subgroup of order 2 lying in $P' \cap \mathrm{Z}\,(P)$. Hence

$$\mathrm{Z}\,(P) \cap P' \leqslant \mathrm{C}_M(\mathrm{F}(M)) \leqslant \mathrm{F}(M),$$

just as in the last part, and so $P' \cap \mathrm{Z}\,(P) \leqslant \mathrm{O}_2(M)$, which is generalized quaternion. Hence the subgroup X is unique.

Since X is characteristic in $\mathrm{Z}\,(P) \cap P'$, it is characteristic in P, and since it is characteristic in $\mathrm{O}_2(M)$, it is characteristic in M. Hence P is a Sylow 2-subgroup of $\mathrm{N}_G(X)$, and so P is a Sylow 2-subgroup of G, so P cannot normalize any non-trivial q-subgroup. In particular, since P normalizes $\mathrm{O}_q(M)$, we must have $\mathrm{O}_q(M) = 1$.

Thus we have proven that (if we no longer assume that $p < q$) then we can assume that there are at least two subgroups of order p in $\mathrm{O}_p(M)$.

The second ingredient we need is that if $\mathrm{F}(M)$ is not of prime-power order, then if L is any other maximal subgroup with $\mathrm{Z}\,(\mathrm{F}(M)) \leqslant L$, then $\mathrm{F}(L)$ is not of prime-power order either, and $\mathrm{F}(L) \leqslant \mathrm{F}(M)$. This seems a little bizarre, so let's prove the theorem with it, before proving this part.

Let X be an elementary abelian p-subgroup of order p^2 in $\mathrm{O}_p(M)$ (which exists by the first part), and let $Z = \mathrm{Z}\,(\mathrm{F}(M))$. If x is a non-trivial element from X, then $\mathrm{C}_G(x) \neq G$, since G is simple. Let L be a maximal subgroup of G containing $\mathrm{C}_G(x)$, and note that

$$Z \leqslant \mathrm{C}_G(x) \leqslant L,$$

whence $\mathrm{F}(L)$ is not of prime-power order and $\mathrm{F}(L) \leqslant \mathrm{F}(M)$. But now, since $\mathrm{F}(L) \leqslant \mathrm{F}(M)$, we clearly have

$$\mathrm{Z}\,(\mathrm{F}(L)) \leqslant M,$$

and so $\mathrm{F}(L) \geqslant \mathrm{F}(M)$, proving that $\mathrm{F}(L) = \mathrm{F}(M)$. However, since G is simple, we must have

$$M = \mathrm{N}_G(\mathrm{F}(M)) = \mathrm{N}_G(\mathrm{F}(L)) = L,$$

and so M contains $\mathrm{C}_G(x)$ for all $x \in X \setminus \{1\}$.

From this, it can be shown that M contains any q-subgroup of G normalized by X.

Let P be a Sylow p-subgroup of M, and suppose that $g \in G$ normalizes P. The P normalizes $\mathrm{O}_q(M)^g$, and hence $\mathrm{O}_q(M)^g$ is a q-subgroup of G normalized by X, (as $X \leqslant \mathrm{O}_p(M)$). Hence by the remark above, $\mathrm{O}_q(M)^g \leqslant M$, and so g normalizes $\mathrm{O}_q(M)$. Since $\mathrm{O}_q(M)$ is non-trivial, its normalizer must be exactly M, and so

$$\mathrm{N}_G(P) = M.$$

Hence P is a Sylow p-subgroup of G, and so $\mathrm{O}_q(M) = 1$, as required.

Hence it remains to prove the statement about the second maximal subgroup above. Let Z_p and Z_q denote the Sylow p- and Sylow q-subgroups of $\mathrm{Z}(\mathrm{F}(M))$. Since M is a maximal subgroup of G, we see that $\mathrm{N}_G(Z_p) = \mathrm{N}_G(Z_q) = M$. Thus

$$\mathrm{N}_L(Z_q) \leqslant M.$$

Since $Z_p \trianglelefteq M$ and $Z_p \leqslant \mathrm{N}_L(Z_q)$; hence $Z_p \trianglelefteq \mathrm{N}_L(Z_q)$, and so

$$Z_p \leqslant \mathrm{O}_p(\mathrm{N}_L(Z_q)).$$

Now we need the notion of p-constraint to progress. Well, we don't if you assume that for all soluble groups H, and for all p-subgroups P,

$$\mathrm{O}_{p'}(\mathrm{N}_H(P)) \leqslant \mathrm{O}_{p'}(G).$$

Applying this in our situation, with $q' = \{p\}$, we see that $Z_p \leqslant \mathrm{O}_p(L)$. Hence $\mathrm{O}_q(L)$ and Z_p commute, impliying that

$$\mathrm{O}_q(L) \leqslant \mathrm{N}_G(Z_p) = M.$$

Similarly, $\mathrm{O}_p(L) \leqslant M$, and so $\mathrm{F}(L) \leqslant M$. But we need $\mathrm{F}(L) \leqslant \mathrm{F}(M)$, so we need to dig deeper. We can slightly improve this, since $\mathrm{O}_p(L)$ and $\mathrm{O}_q(L)$ commute, to $\mathrm{O}_p(L) \leqslant \mathrm{N}_M(\mathrm{O}_q(L))$.

Since $Z_p \leqslant \mathrm{O}_p(L)$, we see that $\mathrm{N}_G(\mathrm{O}_p(L)) = L$. Hence $\mathrm{O}_q(L) \trianglelefteq \mathrm{N}_M(\mathrm{O}_p(L))$, and hence

$$\mathrm{O}_q(L) \leqslant \mathrm{O}_{p'}(\mathrm{N}_M(\mathrm{O}_p(L)).$$

Thus $\mathrm{O}_q(L) \leqslant \mathrm{O}_{p'}(M) \leqslant \mathrm{F}(M)$, as we wanted.

Phew!