Representation Growth of Groups

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This lecture grew out of a theorem, which has an attached story. In September 2007, I went to two conferences involving Dan Segal: the first was an EPSRC short course organized by him, and the second was a conference in Italy for him. On both occasions, the concept of representation growth was brought up: in the first conference it was a side-line introduced by Ben Klopsch; and in the second conference it seemed to play a much more important rôle.

Let G be a group. We define $r_n(G)$ to be the number of inequivalent irreducible representations of G over the complex field, whose image is finite. Thus if $G = \mathrm{GL}_n(\mathbb{C})$, we do not take the natural n-dimensional representation. There is no guarantee that the numbers $r_n(G)$ are finite; indeed, in the case where G is the infinite cyclic group, $r_1(G) = \infty$ and $r_i(G) = 0$ for all $i > 1$.

In the next theorem, we suppose that G is a group for which all of the $r_n(G)$ are finite. Conditions that guarantee this will be discussed later.

Theorem A (Craven, Jaikin, Liebeck, Moretó, Shalev, 2006/7) Let G be a group for which $r_n(G)$ is finite for all n, and write R for the finite residual of G; that is, for the intersection of all normal subgroups of finite index. Then the following are equivalent:

(i) $|G : R|$ is finite;

(ii) there is a constant c such that $r_n(G) < c$ for all n; and

(iii) $\sum_{n\in\mathbb{N}} r_n(G)$ is finite, so that G has only finitely many irreducible finite representations.

There are a lot of names attached to this theorem, because different parts of it were proved at different times. Indeed, the theorem itself was proved in 2006, although it was not stated until 2007, during the conference in Italy. The idea of this lecture is to give an idea of the background of representation growth, and then have a look at how to prove this theorem, and where the various people come into its proof.

1 Subgroup Growth

Subgroup growth is by now an established field, and there are serious results relating the arithmetic properties of the subgroup lattice and the algebraic properties of the group. Write $a_n(G)$ for the number of subgroups of G of index n.

Lemma 1.1 Let G be a finitely generated group. Then $a_n(G)$ is finite for all n.

This is true for a free group of finite rank, and so is true for all finitely generated groups.

Lemma 1.2 Suppose that $r_n(G)$ is finite for all n. Then $a_n(G)$ is finite for all n.

This is easy, by considering the permutation representations on the infinitely many subgroups of a particular index.

2 Representation Growth

We have seen that if G is a finitely generated group then $a_n(G)$ is finite for all n, and we have seen that this is a necessary condition for $r_n(G)$ to be finite for all n. We are going to assume therefore that G is a finitely generated group. Also, since we are only considering the finite images of the group, all invariants of G reachable like this can be found in the quotient of G by the finite residual, and so we will assume that G is residually finite. The class of all finitely generated, residually finite groups is still interesting (for example, it contains all virtually polycyclic groups) and we may extend this to the class of all finitely generated profinite groups because the finite quotients are still governed in the same way.

Now suppose that an abstract group G is finitely generated. We want a necessary and sufficient condition on G such that all of the $r_n(G)$ are finite.

Proposition 2.1 Let G be a finitely generated group. Then $r_n(G)$ is finite for all n if, and only if, for every normal subgroup H of finite index, H' has finite index in H.

Proof: Suppose that H is a normal subgroup of finite index n such that H/H' is infinite. Therefore there are infinitely many inequivalent 1-dimensional representations of H, and these representations, induced to G, provide infinitely many representations of G of a degree n. Thus $r_i(G)$ is infinite for some $i \leqslant n$.

Now suppose that $r_n(G)$ is infinite for some $n \in \mathbb{N}$, and let $\{N_i : i \in I\}$ be the collection of normal subgroups that form the kernels of these representations. By a famous theorem of Jordan, there is an integer r such that each N_i has an abelian normal subgroup of index at most r. Since there are only finitely many subgroups of index at most r, we may choose an infinite subset $J \subseteq I$ such that there is a normal subgroup N with $N_j \leqslant N$ for $j \in J$, and N/N_j is abelian. Therefore N has infinite abelianization, as required. $\square$

Like the zeta functions associated with subgroup growth, we may define a zeta function associated with representation growth, by

$$\zeta_G(s) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s} = \sum_{n=1}^{\infty} r_n(G) n^{-s}.$$

Then $\zeta_{G \times H}(s) = \zeta_G(s)\zeta_H(s)$ and $\zeta_G(s) > R_n(G)n^{-s}$, where $R_n(G)$ is the partial sum of the $r_i(G)$.

One of the results that started representation growth was work of Liebeck and Shalev on the zeta functions associated with infinite families of finite simple groups.

Proposition 2.2 (Liebeck, Shalev) For each $s > 0$, we have

$$\zeta_{A_n}(s) = 1 + \mathrm{O}(n^{-s})$$

as $n \to \infty$.

Thus if for any $a > 0$, and n sufficiently large, we have that $R_k(A_n) \leqslant k^a$ for all $k \in \mathbb{N}$. However, if $m(n) = \max_k r_k(A_n)$, then as $n \to \infty$ we have that $m(n) \to \infty$.

3 Representation Growth Can Be Fast

Let S_i be a finite group for all $i \in \mathbb{N}$, and let $\mathscr{G} = \prod_{i=1}^{\infty} S_i$ be their Cartesian product. Then define a *frame subgroup* of $\mathscr{G}$ to be a finitely generated subgroup G of $\mathscr{S}$ such that

(i) G contains the direct product of the S_i, and

(ii) the natural surjection $\hat{G} \to \mathscr{G}$ is a surjection.

It is not immediately obvious that frame subgroups exist. They do, but we will not touch upon this subject.

Lemma 3.1 Let n and f be natural numbers, and take $G = A_{n+1}^f$. Then, if n is sufficiently large,

$$r_k(G) \leqslant k f^{\log_n k},$$

and $r_k(G) = 0$ for $1 < k < n$.

Let $f(n)$ be a function, and let G_f be a frame subgroup of

$$\prod_{k\geqslant 5} A_{k+1}^{f(k)}.$$

Every finite-dimensional representation of G_f factors through a finite-index subgroup, and so it is a representation of some finite collection $G_{f,N}$ of the A_{k+1} with $k < N$. We will prove that with certain growth conditions on the function f, the representation growth of G_f is the same as the growth of $F(n) = \sum_{i<n} f(i)$.

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Then f is called *admissible* if $f(n) < n!$, $f(xy) \geqslant f(x)f(y)$, and f is non-decreasing for all n.

Lemma 3.2 Let $f(n)$ be an admissible function. Then for all sufficiently large k,

$$F(k) - F(4) \leqslant R_{G_{f,k}}(k) \leqslant k^4 f(k).$$

If $f(n) = [n^b]$ for $b \geqslant 0$ and $\varepsilon > 0$, then for all sufficiently large k,

$$F(k) - F(4) \leqslant R_{G_{f,k}}(k) < k^{b+1+\varepsilon}.$$

Proof: We will only prove (b). If $s > 0$, then for all $n \geqslant 5$,

$$\zeta_{A_{n+1}}(s) \leqslant 1 + Cn^{-s}$$

for some $C = C(s) > 0$. Now

$$\log \zeta_{G_{f,N}}(s) = \sum_{n=5}^{N} n^b \log \zeta_{A_n}(s).$$

If $a \in [0, 1]$, we know that $\log(1 + a) \leqslant a$, and so

$$\log \zeta_{A_{n+1}}(s) \leqslant Cn^{-s} < 1$$

for all large n. Hence $\log \zeta_{G_{f,N}}(s) \leqslant C \sum_{n=5}^{N} n^{b-s}$. Now let $s > b+1$. It follows that the last sum is bounded for all N, and so

$$\zeta_{G_{f,N}}(s) < M$$

for some constant depending on $\varepsilon = s - b - 1$. This proves the upper bound as

$$\zeta_{G_f}(s) > R_{G_f}(k)/k^s.$$

The lower bound is obvious since A_{k+1} has an irreducible representation of degree k, and so

$$r_{G_f}(k) \geqslant [k^b].$$

□

Theorem 3.3 (Kassabov, Nikolov) Let $f : \mathbb{N} \to \mathbb{N}$ be an admissible function. Assume that $f(n)$ grows faster than any polynomial in n; that is, that $\log n = o(\log f(n))$. Then there is a finitely generated residually finite group G such that $\log r_n(G)/\log f(n) \to 1$ as $n \to \infty$. In addition, for each $b > 0$, there is a group G such that $\log R_n(G)/\log n \to b$ as $n \to \infty$.

4 Representation Growth Cannot Be Slow

Before, we denoted by $m(n)$ the maximum of the integers $r_k(A_n)$; now we will look at symmetric groups, and so we define $m(n)$ to be the maximum of $r_k(S_n)$ instead.

Theorem 4.1 (Craven, 2008) Let i be a positive integer. Then S_n has 2^i irreducible characters of the same degree if

$$n \geqslant \frac{15 - 16 \cdot 3^{i-1} + 1025 \cdot 9^{i-2} + 1584 \cdot 27^{i-2} + 576 \cdot 81^{i-2} - 8i}{32}.$$

This is indeed slower than any rational function, as proved by Liebeck and Shalev.

Theorem 4.2 (Liebeck and Shalev, 2005) For a fixed Lie type L, with Coxeter number h, there is a constant $c = c(L)$ such that

$$r_n(L(q)) < cn^{2/h}$$

for all q. Moreover, the exponent $2/h$ is best possible.

In other words, if the rank r of the groups is bounded, then there is a constant $\varepsilon = \varepsilon(r)$ such that the representation growth 'looks' like n^{ε}. Determining exactly what these growth types are is one of the aims of my current research.

Theorem 4.3 (Liebeck and Shalev, 2005) Given any $\varepsilon > 0$, there exists $r = r(\varepsilon)$ such that, if H is a classical group of rank at least r, then

$$r_n(H) < n^{\varepsilon}$$

for all n.

This means that if we allow the ranks of the classical groups to grow unboundedly, we will not get a growth type fo n^{ε}.

Finally, there is the case of p-groups and more generally soluble groups, which was settled by Andrei Jaikin. However, in his paper in 2004, it was only proved that a constant actually exists, not what it was. This is another of my aims for this research.

Work of Moretó will allow me to put all of these disparate functions together to give an explicit lower bound for the growth type of $r_n(G)$.