Perverse Equivalences and Broué's Conjecture

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- K is a field of characteristic ℓ,
- P is a Sylow ℓ-subgroup of G, and
- Q is a general ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

This talk is joint work with Raphaël Rouquier.

From $\mathbb{C}$-representations to K-representations

Maschke's theorem says that every $\mathbb{C}$-representation of a finite group G is a sum of simple representations. This is equivalent to $\mathbb{C}G$ being a direct sum of matrix algebras, each of degree that of a representation, with one $\mathbb{C}G$-module associated to each matrix algebra.

If $K = \bar{\mathbb{F}}_\ell$ (with $\ell \mid |G|$) then this is not true. However, write KG as a sum of indecomposable 2-sided ideals, called blocks. Each indecomposable KG-module is associated to a block, but this time **more than one** KG-module is associated to a given block (in general). If an indecomposable module is associated to a block B, then so are all of its composition factors. Hence every block has at least one simple module associated to it.

The number of simple KG-modules in a block B is denoted $\ell(B)$.

Representation Theory is Local

The deepest and most difficult conjectures in representation theory tend to relate the representation theory of G in characteristic ℓ with that of (ℓ-)local subgroups $N_G(Q)$, where Q is an ℓ-subgroup of G.

To every block is attached a defect group D (an ℓ-subgroup of G up to conjugacy), which 'controls' the representation theory of B. The local conjectures are localized further to relate B with a block b of $KN_G(D)$, called the Brauer correspondent.

Alperin's weight conjecture gives a precise conjecture about the number of simple B-modules, $\ell(B)$, in terms of local information. If D is abelian, the conjecture reduces to

$$\ell(B) = \ell(b).$$

Broué's Conjecture

[B is a block of KG, defect group D, b its Brauer correspondent in $N_G(D)$.]

If D is abelian, Alperin's weight conjecture states that

$$\ell(B) = \ell(b);$$

is there a structural/geometric reason for B and b having the same number of simple modules?

Conjecture (Broué, 1990)

Let G be a finite group, and let B be a ℓ-block of G with abelian defect group D. Let b be the Brauer correspondent in $N_G(D)$. Then B and b are derived equivalent.

When Is Broué's Conjecture Known?

Broué's conjecture is known for quite a few groups:

- A_n, S_n (Chuang–Rouquier, Marcus);
- $\mathrm{GL}_n(q)$, $\ell \nmid q$ (Chuang–Rouquier);
- D cyclic, $C_2 \times C_2$ (Rouquier, Erdmann, Rickard);
- G finite, $\ell = 2$, B principal;
- G finite, $\ell = 3$, $|P| = 9$, B principal (Koshitani, Kunugi, Miyachi, Okuyama, Waki);
- $\mathrm{SL}_2(q)$, $\ell \mid q$ (Chuang, Kessar, Okuyama);
- various low-rank Lie type groups $L(q)$ with $\ell \nmid q$ and sporadic groups. (Okuyama, Holloway, Robbins, etc.)

The Principal Block

If $B_1, \ldots, B_r$ are the blocks of KG, then the simple KG-modules are exactly the union of the simple B_i-modules.

The block contributing the trivial module is called the principal block, and denoted by $B_0(KG)$. Its defect group is always the Sylow ℓ-subgroup P, so its Brauer correspondent is a block of $KN_G(P)$.

Theorem (Brauer's third main theorem)

The Brauer correspondent of $B_0(KG)$ is $B_0(KN_G(P))$.

Thus if we are considering principal blocks, we need to relate the principal block of KG with the principal block of $KN_G(P)$.

Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use their classification.

In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

Theorem

Let G be a finite group, and suppose that P is abelian. Then there are normal subgroups $H \leq L$ such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian ℓ-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L (compatible with automorphisms of the simple components) passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups, it holds for all groups.

How Do You Find Derived Equivalences?

There are four main methods to prove that B and b are derived equivalent.

1. Okuyama deformations: using many steps, deform the Green correspondents of the simple modules for B into the simple modules for b. This works well for small groups.
2. Rickard's Theorem: randomly find complexes in the derived category of b related to the Green correspondents of the simple modules for B, and if they 'look' like simple modules (i.e., Homs and Exts behave nicely) then there is a derived equivalence $B \to b$.
3. More structure: if B and b are more closely related (say Morita or Puig equivalent) then they are derived equivalent. More generally, find another block B' for some other group, an equivalence $B \to B'$, and a (previously known) equivalence $B' \to b$.
4. Perverse equivalence: build a derived equivalence up step by step in an algorithmic way.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = \textit{mod-}A$, $\mathcal{B} = \textit{mod-}B$

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \dots, S_r$, T_1, $T_2, \dots$, T_r, and
- a function $\pi : \{1, \dots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \dots, S_i$, and $D_i^b(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support modules in $\mathcal{A}_i$, then

- F induces equivalences $D_i^b(\mathcal{A}) \to D_i^b(\mathcal{B})$, and
- $F[\pi(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{B}_i/\mathcal{B}_{i-1}$.

Note that mod-B is determined, up to equivalence, by A, π, and the ordering of the S_i.

What is a Perverse Equivalence?

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An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, $T_1, T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, for all i, the composition factors of $H^{-j}(F(S_i))$ are S_j for $j < i$ for $j \neq \pi(i)$ and S_j for $j \leq i$ for $j = \pi(i)$.

In other words, the cohomology of $F(S_i)$ only involves S_j for $j < i$, except for one copy of S_i in degree $-\pi(i)$.

Benefits of a Perverse Equivalence

The perverse equivalence is ‘better’ than a general derived equivalence.

- Has an underlying geometric interpretation (for Lie-type groups).
- The π-function ‘comes from’ Lusztig's A-function (so is approximately known).
- There is an algorithm that gives us a perverse equivalence from $B_0(KN)$ to **some** algebra, so only need to check that the target is $B_0(KG)$. (This is simply checking that the Green correspondents are the last terms in the complexes.)

This algorithm is very useful!

An Example

Let $G = M_{11}$, $\ell = 3$.

π	Ord. Char.	S_1	S_3	S_7	S_2	S_4	S_6	S_5
0	1	1						
2	10		1					
3	10			1				
4	16	1	1		1			
5	11	1			1	1		
6	44			1	1	1	1	
7	55	1	1		1	1	1	1
	10							1
	16	1				1		1

The cohomology of the complexes gives the rows of the decomposition matrix.

Which Groups Have Perverse Equivalences?

- All groups, D cyclic or $C_2 \times C_2$
- $\mathrm{PSL}_3(q)$, $\ell = 3 \mid (q-1)$, P abelian
- $\mathrm{PSL}_4(q)$, $\mathrm{PSL}_5(q)$, $\ell = 3 \mid (q+1)$, $P = C_3 \times C_3$
- $\mathrm{PSU}_3(q)$, $\ell = 3 \mid (q+1)$, P abelian
- $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$, $\ell = 3 \mid (q-1)$
- $\mathrm{PSp}_4(q)$, $\ell = 3 \mid (q-1)$ or $(q+1)$, $P = C_3 \times C_3$
- (almost) $\mathrm{PSp}_8(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- (almost) $\Omega_8^+(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- $G_2(q)$, $\ell = 5 \mid (q+1)$, $P = C_5 \times C_5$
- S_6, A_7, A_8, $\ell = 3$ (A_6 does not)
- M_{11}, $M_{22}.2$, M_{23}, HS, $\ell = 3$ (M_{22} does not)
- $\mathrm{SL}_2(8)$, J_1, ${}^2G_2(q)$, $\ell = 2$ in two steps
- S_n, A_n, $\mathrm{GL}_n(q)$ in multiple steps

An Example: $\mathrm{PSL}_3(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char	S_1	S_5	S_2	S_3	S_4
0	1	1				
2	$q(q+1)$	1	1			
3	$(q+1)(q^2+q+1)/3$	1	1	1		
3	$(q+1)(q^2+q+1)/3$	1	1		1	
3	$(q+1)(q^2+q+1)/3$	1	1			1

		H^{-3}	H^{-2}	H^{-1}	Total
X_5:	$0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to C_5 \to 0.$		$1/5$	11	$5-1$
X_2:	$0 \to \mathcal{P}(2) \to \mathcal{P}(34) \to \mathcal{P}(5) \to C_2 \to 0.$	$1/5/2$	1		$2-5$
X_3:	$0 \to \mathcal{P}(3) \to \mathcal{P}(24) \to \mathcal{P}(5) \to C_3 \to 0.$	$1/5/3$	1		$3-5$
X_4:	$0 \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(5) \to C_4 \to 0.$	$1/5/4$	1		$4-5$

An Example: $\mathrm{PSp}_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char	S_1	S_5	S_2	S_3	S_4
0	1	1				
3	$q(q-1)^2/2$		1			
3	$q(q^2+1)/2$	1		1		
3	$q(q^2+1)/2$	1			1	
4	q^4	1	1	1	1	1

$$X_5: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0.$$
$$X_2: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0.$$
$$X_3: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(5) \to \mathcal{P}(2) \oplus M_{1,1} \to C_3 \to 0.$$
$$X_4: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(5) \to C_4 \to 0.$$

An Example: $\mathrm{PSL}_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
3	$q(q^2+q+1)$	1	1			
4	$q^2(q^2+1)$		1	1		
5	$q^3(q^2+q+1)$	1	1	1	1	
6	q^6	1			1	1

$$X_2 : \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0.$$
$$X_5 : \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0.$$
$$X_3 : \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0.$$
$$X_4 : \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.$$

Some Remarks

- Since $\pi(-)$, the ordering and the first category determine the perverse equivalence, it is a very compact way of defining a (type of) derived equivalence.
- Computationally, this reduces finding a derived equivalence to finding the Green correspondents of the simple modules for G, a much simpler task.
- For groups of Lie type, it seems as though the complexes above do not really depend on ℓ, and only on d, where $\ell \mid \Phi_d(q)$. It might be possible to use these perverse equivalences to prove real results in this direction.

Complex Reflection Groups

Generically, the automizer H of a Φ_d-torus in a group of Lie type is a complex reflection group with $|H|$ and $\Phi_d(q)$ coprime. The action of H on the torus gives a representation of H over $\mathbb{F}_\ell^n$ for $\ell \mid \Phi_d(q)$.

It is believed (but not proved) that this representation is invariant of ℓ and q, and only dependent on d and G (maybe even only on H !)

The principal block of the normalizer is Morita equivalent to the group algebra of $k\mathrm{N}_G(P)/\mathrm{O}_{\ell'}(\mathrm{C}_G(P))$, which is $\mathbb{F}_\ell^n \rtimes H$.

This focuses attention on (in particular complex reflection) groups acting on $\mathbb{F}_\ell$-modules M, and $M \rtimes H$.

ℓ-Extended Finite Groups

Let H be a finite group, and let ρ be a faithful complex representation of H. It is well known that there exists an algebraic number field K, with ring of integers $\mathcal{O} = \mathcal{O}_K$, such that $H \leq \mathrm{GL}_n(\mathcal{O})$ and this induces ρ.

Let $\ell \nmid |H|$ be a prime such that the map $\mathcal{O} \to \mathbb{F}_\ell$ induces a faithful representation of H over $\mathbb{F}_\ell$ via ρ. Write M for the $\mathbb{F}_\ell H$-module, and $G_\ell = M \rtimes H$.

- $k(G_\ell)$ is a polynomial in ℓ, and $k(G_\ell) \cdot |H|$ is a monic polynomial in ℓ with integer coefficients.
- If H is a reflection group and ρ is its natural representation over $\mathbb{Z}$, then the second coefficient of $k(G_\ell) \cdot |H|$ is $3N$, where N is the number of reflections in H. (A similar formula exists for complex reflection groups.)